

Original citation:

Greenlees, John (2016) Ausoni–Bökstedt duality for topological Hochschild homology. *Journal of Pure and Applied Algebra*, 220 (4). pp. 1382-1402. doi:10.1016/j.jpaa.2015.09.007

Permanent WRAP URL:

<http://wrap.warwick.ac.uk/100096>

Copyright and reuse:

The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher's statement:

© 2017. This manuscript version is made available under the CC-BY-NC-ND 4.0 license
<http://creativecommons.org/licenses/by-nc-nd/4.0/>

A note on versions:

The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP URL' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

AUSONI-BÖKSTEDT DUALITY FOR TOPOLOGICAL HOCHSCHILD HOMOLOGY

J.P.C.GREENLEES

ABSTRACT. We consider the Gorenstein condition for topological Hochschild homology, and show that it holds remarkably often. More precisely, if R is a commutative ring spectrum and $R \rightarrow k$ is a map to a field of characteristic p then, provided k is small as an R -module, $THH(R; k)$ is Gorenstein in the sense of [11]. In particular, this holds if R is a (conventional) regular local ring with residue field k of characteristic p .

Using only Bökstedt's calculation of $THH(k)$, this gives a non-calculational proof of dualities observed by Bökstedt [9] and Ausoni [3], Lindenstrauss-Madsen [17], Angeltveit-Rognes [4] and others.

CONTENTS

1. Introduction	1
2. Hochschild homology and cohomology	3
3. Two spectral sequences	3
4. Gorenstein ring spectra	5
5. Some known calculations	9
6. Gorenstein ascent	10
7. Ausoni-Bökstedt duality.	13
8. Examples	15
Appendix A. Thom spectra	18
Appendix B. Dwyer-Miller and Kontsevich duality	19
References	20

1. INTRODUCTION

The present work was stimulated by calculations of topological Hochschild homology by Bökstedt [9] and Ausoni [3]. Given a map of commutative ring spectra $R \rightarrow k$, we may view k as an R -bimodule and hence define $THH_*(R; k)$. Identifying conventional rings with Eilenberg-MacLane ring spectra, we may take k to be a field, and the calculations give striking examples where $THH_*(R; k)$ is Gorenstein. These and other calculations are summarized in Section 5, and the reader unfamiliar with them may wish to glance at them before proceeding.

The purpose of the present paper is to give a non-calculational explanation of this duality. Most are instances of the following result which covers many cases where there is currently no complete calculation. The basic definitions of THH are described in Section 2 and the Gorenstein apparatus is described in Section 4. The following theorem appears below as Corollary 7.5.

Theorem 1.1. *If R is a connective commutative ring spectrum with a map $R \rightarrow k$ where k is a field of characteristic $p > 0$ then provided (i) k is small as a R -module and (ii) R is Gorenstein of shift a then $\mathrm{THH}(R; k)$ is Gorenstein of shift $-a - 3$ and has Noetherian homotopy groups.*

This should be contrasted with the algebraic case (i.e., working under k rather than under the sphere spectrum). In this case, if R is a k -algebra, under the same hypotheses we expect $H\mathrm{H}_*(R|k; k)$ to be Gorenstein a shift of $-a$ (see Remark 7.6 and Appendix B).

It is essential to the argument that we are working in characteristic p , and the only calculational input is Bökstedt's result that $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[\mu_2]$ (this is Gorenstein of shift -3 , which explains the -3 in the statement of the theorem). The two technical ingredients are (A) a cofibre sequence conjectured to explain the Gorenstein calculations and proved by Dundas and (B) an extension of the usual Gorenstein ascent theorem.

There is a strong precedent for calculations based on Gorenstein ascent. Indeed if $S \rightarrow R \rightarrow Q$ is a cofibre sequence (i.e., $Q \simeq R \otimes_S k$)¹, the Gorenstein property often behaves well in the sense that if S and Q are Gorenstein then so is R . To illustrate its use we show how this, together with Morita invariance, lets us generate most Gorenstein rings from from 0-dimensional ones. To start with, exterior algebras E are Poincaré duality algebras (and hence 0-dimensional Gorenstein rings) and since any polynomial algebra P is Morita equivalent to an exterior algebra E , polynomial algebras are Gorenstein. Next, by Noether normalization any Noetherian k -algebra R is finitely generated as a module over a polynomial subring P so that the cofibre $Q = R \otimes_P k$ is finite dimensional. Since P is Gorenstein, R is Gorenstein if and only if Q is a Poincaré duality algebra, so Gorenstein rings are constructed from a polynomial algebra and a Poincaré duality algebra. Altogether, Gorenstein rings are constructed from Poincaré duality algebras using Morita equivalences and cofibre sequences.

Duality phenomena are also ubiquitous in topology, starting with Poincaré duality and moving on to coefficient rings of many equivariant cohomology theories [11]. Once again it seems these all come from a rather small collection of basic examples, namely the chains on a group or the cochains on a manifold. Using Morita equivalences and cofibre sequences one can generate a wide variety of further examples, perhaps most notably $C^*(BG)$ for compact Lie groups G whose adjoint representation is orientable [11]. The present paper shows that Bökstedt's calculation provides a new source of Gorenstein examples.

The rest of the paper is organized as follows. Sections 2, 3 and 4 provide summaries of relevant background. Section 5 gives summaries of various calculations from the literature. Section 6 proves the Gorenstein Ascent result we need, and Section 7 proves the main result. We finish in Section 8 by discussing the implications of the result for a number of examples. There are two appendices which describe similar results. In Appendix A we consider THH of Thom spectra via the work of Blumberg-Cohen-Schlichtkrull [8], and in Appendix B we consider algebraic Hochschild homology (i.e., under k rather than under the sphere spectrum) where a result of Dwyer-Miller gives analogous duality statements.

During the genesis of the paper, I have discussed my speculations with many people, and I am grateful to V.Angeltveit, C.Ausoni, D.Benson, B.Dundas, W.G.Dwyer and A.Lindenstrauss

¹These are often called *fibre* sequences in the algebra literature because of the fact that cochains on topological fibre sequences give examples, but this would lead to confusion in the present context.

for their patience and sharing their ideas. I am particularly grateful to B.Dundas for providing a proof of the critical conjectured cofibration described in Lemma 7.1, and allowing me to publish it here. I would like to thank the University of Lille for inviting me to give lectures on duality in 2012, when these ideas started to make progress, and MSRI for providing an excellent environment for completing this account.

2. HOCHSCHILD HOMOLOGY AND COHOMOLOGY

We suppose given maps $S \rightarrow R \rightarrow k$ of ring spectra; as usual we include the case of conventional rings through the use of Eilenberg-MacLane spectra. We write $R^e = R_S^e = R \otimes_S R$, and we write P for an R^e -module, which we refer to as an (R, R) -bimodule over S . Thus we may talk of the Hochschild homology spectrum

$$HH_\bullet(R|S; P) = R \otimes_{R^e} P$$

and the Hochschild cohomology spectrum

$$HH^\bullet(R|S; P) = \text{Hom}_{R^e}(R, P).$$

The Hochschild homology and cohomology groups are obtained by taking homotopy groups

$$HH_*(R|S; P) = \pi_* HH_\bullet(R|S; P) \text{ and } HH^*(R|S; P) = \pi_* HH^\bullet(R|S; P).$$

If R and S are conventional rings, R is flat over S and P is a conventional module, this agrees with the standard definitions in algebra.

In the examples of most concern to us here, $S = \mathbb{S}$ is the sphere spectrum, and we have we have topological Hochschild homology $THH(R; P) := HH_\bullet(R|\mathbb{S}; P)$.

3. TWO SPECTRAL SEQUENCES

We are going to be concerned with cofibre sequences of commutative ring spectra, $S \rightarrow R \rightarrow Q$, in the sense that $Q \simeq R \otimes_S k$. In the topological context one is used to having spectral sequences as basic calculational tools. It is convenient to have them available more generally.

3.A. The connective case. This is the situation when the ring spectra S, R and Q are all connective. The first example of this is analogous to what happens when we have a short exact sequence of compact Lie groups $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$. We take $S = C_*(N)$, $R = C_*(G)$ and $Q = C_*(G/N)$ and in this case we have the homological Serre spectral sequence

$$E_{*,*}^2 = H_*(G/N; H_*(N)) \Rightarrow H_*(G)$$

of the fibration $N \rightarrow G \rightarrow G/N$.

Lemma 3.1. *If $S \rightarrow R \rightarrow Q$ is a cofibre sequence of connective commutative algebras augmented over k and $\pi_0(S) = k$, and R is upward finite type as an S module (for example [11, 3.13] if $\pi_n(R)$ is finite dimensional for each n) then there is a multiplicative spectral sequence*

$$E_{s,t}^2 = \pi_s(Q) \otimes_k \pi_t(S) \Rightarrow \pi_{s+t}(R),$$

with differentials

$$d^r : E_{s,t}^r \longrightarrow E_{s-r, t+r-1}^r.$$

Proof: We construct a tower

$$S \longrightarrow \cdots \longrightarrow S^{(n)} \longrightarrow S^{(n-1)} \longrightarrow \cdots \longrightarrow S^{(1)} \longrightarrow S^{(0)} = k$$

of commutative rings by killing homotopy groups, where $S \longrightarrow S^{(n)}$ is an isomorphism of $\pi_{\leq n}$ and $\pi_i(S^{(n)}) = 0$ for $i > n$. Taking $R^{(n)} = R \otimes_S S^{(n)}$ we obtain a corresponding multiplicative filtration of R , and consider the resulting spectral sequence. In view of the cofibre sequence $\Sigma^t \pi_t S \longrightarrow S^{(t)} \longrightarrow S^{(t-1)}$ of S -modules it is easy to write down an exact couple, and we grade it so that

$$D_{s,t}^2 = \pi_{s+t}(R \otimes_S S^{(t)}) \text{ and } E_{s,t}^2 = \pi_{s+t}(R \otimes_S \Sigma^t \pi_t(S)).$$

Now the action of S on $\pi_t S$ factors through $S^{(0)} = k$, and hence

$$R \otimes_S \Sigma^t \pi_t(S) \simeq \Sigma^t \pi_t(S) \otimes_k Q.$$

The d^2 differential is induced by

$$R \otimes_S \Sigma^t \pi_t(S) \longrightarrow R \otimes_S S^{(t)} \longrightarrow R \otimes_S \Sigma^{t+2} \pi_{t+1}(S).$$

The convergence of the spectral sequence is the statement that the natural map

$$\kappa : R = R \otimes_S S \simeq R \otimes_S [\operatorname{holim}_n S^{(n)}] \longrightarrow \operatorname{holim}_n [R \otimes_S S^{(n)}]$$

is an equivalence. Since the $S^{(n)}$ are uniformly bounded below (by -1) the result follows (as summarized in Lemma 6.3 below). \square

3.B. The coconnective case. This is the situation when the ring spectra S, R and Q are all coconnective. This does not play a role in our main applications and is included for comparison. Because free commutative algebras are not usually coconnective we will need to add a significant hypothesis.

One example of this arises if we start from a fibration $F \longrightarrow E \longrightarrow B$ with B simply connected and take $S = C^*(B)$, $R = C^*(E)$ and $Q = C^*(F)$. This obviously satisfies the stringent additional hypothesis identified below, so the construction generalizes the Serre spectral sequence

$$E_2^{*,*} = H^*(B; H^*(F)) \Rightarrow H^*(E).$$

Lemma 3.2. *Suppose $S \longrightarrow R \longrightarrow Q$ is a cofibre sequence of coconnective commutative algebras augmented over k and $\pi_0(S) = k$ and R is downward finite type as an S module (for example [11, 3.14] if $\pi_*(S)$ is simply coconnected (i.e., $\pi_0(S) = k$ and $\pi_{-1}(S) = 0$) and $\pi_n(R)$ is finite dimensional for each n). If in addition that there is a tower*

$$S \longrightarrow \cdots \longrightarrow S_{(n)} \longrightarrow S_{(n-1)} \longrightarrow \cdots \longrightarrow S_{(1)} \longrightarrow S_{(0)} = k$$

of coconnective commutative rings with $\pi_i(S_{(n)}) = 0$ for $i < n$. then there is a multiplicative spectral sequence

$$E_2^{s,t} = \pi_{-s}(S) \otimes_k \pi_{-t}(Q) \Rightarrow \pi_{-s-t}(R),$$

with differentials

$$d_r : E_r^{s,t} \longrightarrow E_r^{s-r, t+r-1}.$$

Proof: By hypothesis, there is a tower

$$S \longrightarrow \cdots \longrightarrow S_{(n)} \longrightarrow S_{(n-1)} \longrightarrow \cdots \longrightarrow S_{(1)} \longrightarrow S_{(0)} = k$$

of commutative rings. Thus the map $S \rightarrow S_{(n)}$ is an isomorphism of $\pi_{\geq -n}$ and we have cofibre sequences of S -modules

$$\Sigma^{-s}\pi_{-s}S \longrightarrow S_{(s)} \longrightarrow S_{(s-1)}.$$

We then get a spectral sequence

$$D_2^{s,t} = \pi_{-s-t}(R \otimes_S S_{(s)}) \text{ and } E_2^{s,t} = \pi_{-s-t}(R \otimes_S \Sigma^{-s}\pi_{-s}S)$$

The differentials then take the familiar cohomological form

$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r, t-r+1}.$$

The convergence of the spectral sequence is the statement that the natural map

$$R = R \otimes_S S \simeq R \otimes_S \operatorname{holim}_{\leftarrow n} S_{(n)} \longrightarrow \operatorname{holim}_{\leftarrow n} R \otimes_S S_{(n)}$$

is an equivalence, and by Lemma 6.3 below this holds if R is of downward finite type over S . \square

4. GORENSTEIN RING SPECTRA

We are considering duality phenomena modelled on those in commutative algebra of Noetherian rings, namely those associated to Gorenstein local rings. For ring spectra there is a corresponding development, starting by restricting the class of rings by a finiteness condition and then the core Gorenstein condition followed by a duality statement. We recall some definitions from [11].

4.A. Finiteness conditions. In a triangulated category if N can be *finitely built* from M using cofibre sequences, finite sums and retracts, we write $M \models N$; if N can be *built* from M using cofibre sequences and arbitrary sums we write $M \vdash N$.

We consider a map $R \rightarrow k$ of rings. The terminology comes from the special case when R is a commutative local ring with residue field k . The first requirement is a finiteness condition, which plays the role of the Noetherian condition from classical commutative algebra. The Auslander-Buchsbaum-Serre theorem in commutative algebra states that if R is a Noetherian local ring, k is small if and only if R is regular. This is far too strong a condition for us to assume, but there is a much weaker and more practical one in the same vein. Indeed, for commutative Noetherian local rings, we can always form the Koszul complex K associated to a finite set of generators of the maximal ideal; this has the properties (i) K is small ($R \models K$) (ii) K is finitely built from k ($k \models K$) and (iii) k is built from K ($K \vdash k$). In the context of more general ring objects, we say R is *proxy-regular* if there is an R -module K so that (i), (ii) and (iii) hold, and we think of this as a finiteness condition playing a similar role to that of being Noetherian.

4.B. The Gorenstein condition. We now say that $S \rightarrow k$ is *Gorenstein* of shift a (and write $\text{shift}(S) = \text{shift}(k|S) = a$) if we have an equivalence

$$\text{Hom}_S(k, S) \simeq \Sigma^a k$$

of R -modules. More generally, we say that $S \rightarrow R$ is *relatively Gorenstein of shift a* (and write $\text{shift}(R|S) = a$) if

$$\text{Hom}_S(R, S) \simeq \Sigma^a R.$$

Analogously to the classical case, we are interested in proxy-regular rings which satisfy the Gorenstein condition.

4.C. Gorenstein duality. Although the Gorenstein condition itself is convenient to work with, the real reason for considering it is the duality property that it implies.

In classical local commutative algebra the Gorenstein duality property is that all local cohomology is in a single cohomological degree, where it is the injective hull $I(k)$ of the residue field. To give a formula, we write $\Gamma_{\mathfrak{m}} M$ for the \mathfrak{m} -power torsion in an R -module M , and $H_{\mathfrak{m}}^*(M)$ for the local cohomology of M , recalling Grothendieck's theorem that if R is Noetherian, $H_{\mathfrak{m}}^*(M) = R^* \Gamma_{\mathfrak{m}}(M)$. The Gorenstein duality statement for a local ring of Krull dimension r therefore states

$$H_{\mathfrak{m}}^*(R) = H_{\mathfrak{m}}^r(R) = I(k).$$

If R is a k -algebra, $I(k) = R^\vee = \Gamma_{\mathfrak{m}} \text{Hom}_k(R, k)$.

Turning to ring spectra, if R is a k -algebra we may again define $R^\vee = \text{cell}_k(\text{Hom}_k(R, k))$ and observe this has the Matlis lifting property

$$\text{Hom}_R(T, R^\vee) \simeq \text{Hom}_k(T, k)$$

for any T built from k . The case when R is not a k -algebra is more complicated, but will not be needed here.

In particular, if R is Gorenstein of shift a we have equivalences of R -modules

$$\text{Hom}_R(k, \text{cell}_k R) \simeq \text{Hom}_R(k, R) \xrightarrow{(g)} \Sigma^a k \xrightarrow{(m)} \text{Hom}_R(k, \Sigma^a R^\vee),$$

where the equivalence (g) is the Gorenstein property and the equivalence (m) is the Matlis lifting property. We would like to remove the $\text{Hom}_R(k, \cdot)$ to deduce

$$\text{cell}_k R \simeq \Sigma^a R^\vee.$$

Such an equivalence is known as *Gorenstein duality*, since $\text{cell}_k(R)$ is a covariant functor of R and R^\vee is a contravariant functor of R .

Morita theory [11] says that if R is proxy-regular we may make this deduction provided R is orientably Gorenstein in the sense that the right actions of $\mathcal{E} = \text{Hom}_R(k, k)$ on $\Sigma^a k$ implied by the two equivalences (g) and (m) agree. This is automatic when the ring spectrum is both a k -algebra and connected.

Proposition 4.1. *Suppose R is a proxy-regular, connected k -algebra and $\pi_*(R)$ is Noetherian with $\pi_0(R) = k$ and maximal ideal \mathfrak{m} of positive degree elements. If R is Gorenstein of shift a , then R is automatically orientable and so has Gorenstein duality. Accordingly there is a local cohomology spectral sequence*

$$H_{\mathfrak{m}}^*(R_*) \Rightarrow \Sigma^a R_*^\vee.$$

Proof: First we argue that if R is Gorenstein, it is automatically orientable. Indeed, we show that \mathcal{E} has a unique action on k . Since R is a k -algebra, the action of \mathcal{E} on k factors through

$$\mathcal{E} = \text{Hom}_R(k, k) \longrightarrow \text{Hom}_k(k, k) = k,$$

so since k is an Eilenberg-MacLane spectrum, the action is through $\pi_0(\mathcal{E})$. Now we observe that since R is connected, $\text{Ext}_{R_*}^s(k, k)$ is in degrees $\leq -s$, so that the spectral sequence for calculating $\pi_*(\text{Hom}_R(k, k))$ shows \mathcal{E} is coconnective with $\pi_0(\mathcal{E}) = k$ which must act trivially on k . \square

We note that if the coefficient ring $\pi_*(R)$ is Gorenstein and R is connective then R is Gorenstein. Indeed, the spectral sequence

$$\text{Ext}_{R_*}^{*,*}(k, R_*) \Rightarrow \pi_*(\text{Hom}_R(k, R))$$

collapses, to show $\pi_*(\text{Hom}_R(k, R)) = \Sigma^a k$ for some a . The R -module k is characterised by its homotopy, so $\text{Hom}_R(k, R) \simeq \Sigma^a k$. Conversely, if R is Gorenstein, this shows that the ring $\pi_*(R)$ has very special properties (even if it falls short of being Gorenstein). The following statement corrects a typographical error in [12, 6.2].

Corollary 4.2. [12] Suppose R has Gorenstein duality of shift a , that $\pi_*(R)$ is Noetherian of Krull dimension r and Hilbert series $p(t) = \sum_s \dim_k(R_s)t^s$.

(1) If $\pi_*(R)$ is Cohen-Macaulay it is also Gorenstein, and the Hilbert series satisfies

$$p(1/t) = (-1)^r t^{r-a} p(t).$$

(2) If $\pi_*(R)$ is almost Cohen-Macaulay it is also almost Gorenstein, and the Hilbert series satisfies

$$p(1/t) - (-1)^r t^{r-a} p(t) = (-1)^{r-1}(1+t)q(t) \text{ and } q(1/t) = (-1)^{r-1}t^{a-r+1}q(t).$$

In any case $\pi_*(R)$ is Gorenstein in codimension 0 and almost Gorenstein in codimension 1.

4.D. The relatively Gorenstein case. We make the elementary observation that for any ring map $\theta : S \longrightarrow R$

$$\text{Hom}_R(k, \text{Hom}_S(R, S)) \simeq \text{Hom}_S(k, S).$$

Thus we conclude that if $S \longrightarrow R$ is relatively Gorenstein then R is Gorenstein if and only if S is Gorenstein, and in that case

$$\text{shift}(k|S) = \text{shift}(k|R) + \text{shift}(R|S).$$

Example 4.3. The ring map $S = ko \longrightarrow ku = R$ is relatively Gorenstein of shift 2. Indeed, the connective version of Wood's theorem states that there is an equivalence $ku \simeq ko \wedge (S^0 \cup_{\eta} e^2)$ of ko -modules, so that

$$\text{Hom}_{ko}(ku, ko) \simeq \Sigma^{-2}ku.$$

Since $ku_* = \mathbb{Z}[v]$ we see that ku is Gorenstein of shift -4 over \mathbb{F}_2 , and it follows that ko is Gorenstein of shift -6 over \mathbb{F}_2 .

Example 4.4. Precisely similar statements hold for tmf . This is based on results of Hopkins-Mahowald [14], with an improved formal context of Hill-Lawson [13] giving maps in the category of commutative tmf algebras. The results about finite cell complexes are proved by Mathew [21].

As background we note that at primes $p \geq 5$, we have $tmf_* = \mathbb{Z}_{(p)}[c_4, c_6]$ with c_4 of degree 8 and c_6 of degree 12. It is therefore immediate from the coefficients that tmf is Gorenstein of shift -23 over \mathbb{F}_p . The primes 3 and 2 are more interesting.

(i) At the prime 3, we consider the map $tmf \rightarrow tmf_1(2)$ of commutative tmf -algebras [13, Theorem 6.1]. There is an equivalence of tmf -modules

$$tmf_1(2) \simeq tmf \wedge (S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8)$$

([21, Theorem 7.7] gives an equivalence of spectra. Writing $T = S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$, a map $f : T \rightarrow tmf_1(3)$ determines a map $tmf \wedge T \rightarrow tmf_1(3)$ of tmf -modules. To see that the map is an equivalence we may check it is an isomorphism in mod 3 cohomology, and for this we only need to check it is an epimorphism in mod 3 cohomology. Since $tmf_1(3) \simeq BP\langle 2 \rangle \vee \Sigma^8 BP\langle 2 \rangle$ only two generators are required over the Steenrod algebra, and it suffices to choose f so that generators in degrees 0 and 8 are in the image). It follows that

$$\mathrm{Hom}_{tmf}(tmf_1(2), tmf) \simeq \Sigma^{-8} tmf_1(2).$$

Since $tmf_1(2)_* = \mathbb{Z}_{(3)}[c_2, c_4]$ (where $|c_i| = 2i$) we see that $tmf_1(2)$ is Gorenstein of shift -15 . Hence we deduce by Gorenstein descent that $tmf \rightarrow \mathbb{F}_3$ is Gorenstein of shift -23 .

(ii) At the prime 2, we consider the map $tmf \rightarrow tmf_1(3)$ [13, Theorem 6.1] of commutative tmf -algebras. Here $tmf_1(3)$ is a form of $BP\langle 2 \rangle$ (previously proved to have a commutative model by Lawson-Naumann [15, 16]) and there is an equivalence of tmf -module spectra

$$tmf_1(3) \simeq tmf \wedge DA(1)$$

(again, the equivalence of spectra is given in [21, Theorem 6.6]. A map $DA(1) \rightarrow tmf_1(3)$ determines a map of tmf -modules, and as above it suffices to show the resulting tmf -module map $tmf \wedge DA(1) \rightarrow tmf_1(3)$ is an epimorphism in mod 2 cohomology. Since the mod 2 cohomology is generated in degree 0 over the Steenrod algebra, this is easily arranged). It follows that

$$\mathrm{Hom}_{tmf}(tmf_1(3), tmf) \simeq \Sigma^{-12} tmf_1(3).$$

Since $tmf_1(3)_* = \mathbb{Z}_{(2)}[\alpha_1, \alpha_3]$ (where $|\alpha_i| = 2i$) we see that $tmf_1(3)$ is Gorenstein of shift -11 . Hence we deduce by Gorenstein descent that $tmf \rightarrow \mathbb{F}_2$ is Gorenstein of shift -23 .

In general, it can be difficult to decide if $S \rightarrow R$ is relatively Gorenstein, and we prefer to give conditions depending on Q .

4.E. Gorenstein Ascent. In effect the Gorenstein Ascent theorem will state that under suitable hypotheses (see Section 6) there is an equivalence

$$\mathrm{Hom}_R(k, R) \simeq \mathrm{Hom}_Q(k, \mathrm{Hom}_S(k, S) \otimes_k Q).$$

When this holds, it follows that if S and Q are Gorenstein, so is R and

$$\mathrm{shift}(R) = \mathrm{shift}(S) + \mathrm{shift}(Q).$$

4.F. Arithmetic of shifts. We summarize the behaviour of Gorenstein shifts in the ideal situation when ascent and descent both hold. If all rings and maps are Gorenstein of the indicated shifts

$$S \xrightarrow{a} R \xrightarrow{\lambda} Q \xrightarrow{b} R \xrightarrow{\mu} Q \xrightarrow{c}$$

then $b = a + c$, $\lambda = -c$ and $\mu = a$

5. SOME KNOWN CALCULATIONS

The paper is motivated by several calculations when $S = \mathbb{S}$ is the sphere spectrum.

Example 5.1. (The map $R = \mathbb{F}_p \longrightarrow \mathbb{F}_p = k$.) We consider the homotopy of $THH(\mathbb{F}_p)$. Bökstedt [9] has calculated $THH_*(\mathbb{F}_p) = \mathbb{F}_p[\mu_2]$. The ring $THH_*(\mathbb{F}_p)$ is Gorenstein of shift -3 .

Example 5.2. (The map $R = \mathbb{Z} \longrightarrow \mathbb{F}_p = k$.) We consider the mod p homotopy of $THH(\mathbb{Z})$. Bökstedt [9] has calculated $THH_*(\mathbb{Z}; \mathbb{F}_p) = \mathbb{F}_p[\mu_{2p}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_{2p-1})$. The ring $THH_*(\mathbb{Z}; \mathbb{F}_p)$ is Gorenstein of shift $(-2p-1) + (2p-1) = -2$ (Bökstedt Duality). The ring but not the shift depends on p .

The calculations of Lindenstrauss and Madsen [17, 4.4] have a similar pattern. Indeed, if \mathcal{O} is a ring of integers in a number field, which is either unramified or wildly ramified, $THH_*(\mathcal{O}, \mathcal{O}/p)$ is polynomial tensor exterior (over \mathcal{O}/p) on generators differing in degree by 1 . In the tamely ramified case the ring $THH_*(\mathcal{O}, \mathcal{O}/p)$ is more complicated, but it is still Gorenstein of shift -2 .

Example 5.3. (The map $R = lu \longrightarrow \mathbb{F}_p = k$.) We consider mod v_1, p homotopy of $THH(lu)$ where lu is the Adams summand of p -local connective K -theory with coefficients $lu_* = \mathbb{Z}_{(p)}[v_1]$. McClure-Staffeldt [19] (see also Ausoni-Rognes [4]) have calculated $THH_*(lu; \mathbb{F}_p) = \mathbb{F}_p[\mu_{2p^2}] \otimes \Lambda_{\mathbb{F}_p}(\lambda_{2p-1}, \lambda_{2p^2-1})$. The ring $THH_*(lu; \mathbb{F}_p)$ is Gorenstein of shift $(-2p^2-1) + (2p-1+2p^2-1) = 2p-3$.

Example 5.4. (The map $R = ku \longrightarrow ku/(p, v_1) = k$.) For primes $p > 2$, Ausoni calculates the mod p, v_1 homotopy of $THH(ku)$ and shows that $THH_*(ku; ku/(p, v_1)) = \Lambda(\lambda_{2p-1}) \otimes \mathbb{F}_p[\mu_{2p^2}] \otimes Q$, where Q is Poincaré duality algebra of formal dimension $2p^2-1$ [3, 9.15]. Although $\pi_*(ku/(p, v_1)) = \mathbb{F}_p[v]/(v^{p-1})$ is not a field, we may make still consider duality properties over \mathbb{F}_p . The ring $THH_*(ku; ku/(p, v_1))$ has Gorenstein duality over \mathbb{F}_p with shift $(-2p^2-1) + (2p-1) + (2p^2-1) = 2p-3$ (Ausoni Duality). This striking example stimulated the author to investigate Gorenstein duality for THH .

Example 5.5. (The map $R = ko \longrightarrow H\mathbb{F}_2 = k$.) Angeltveit and Rognes [4] show that $THH_*(ko; \mathbb{F}_2) = \Lambda(\lambda_5, \lambda_7) \otimes \mathbb{F}_2[\mu_8]$. The ring $THH_*(ko; \mathbb{F}_2)$ is Gorenstein of shift $5+7-8-1=3$.

Example 5.6. (The map $R = tmf \longrightarrow H\mathbb{F}_2 = k$.) It is easily deduced from the calculations of Angeltveit and Rognes [4] that $THH_*(tmf; \mathbb{F}_2) = \Lambda(\lambda_9, \lambda_{13}, \lambda_{15}) \otimes \mathbb{F}_2[\mu_{16}]$. The ring $THH_*(tmf; \mathbb{F}_2)$ is Gorenstein of shift $9+13+15-16-1=20$.

6. GORENSTEIN ASCENT

We have begun to see the value of understanding the behaviour of the Gorenstein condition in cofibre sequences, and we turn to a more systematic discussion.

We suppose that $S \rightarrow R \rightarrow Q$ is a cofibre sequence of commutative algebras with a map to k , and we now consider the Gorenstein ascent question. When does the fact that S is Gorenstein imply that R is Gorenstein? It is natural to assume that Q is Gorenstein, but it is known this is not generally sufficient. We identify a number of circumstances in which it is sufficient, and in characteristic p we give a useful general result. Before we do this, we look at the finiteness conditions.

6.A. Proxy-regularity. We provide a tool for proxy-regular ascent. It seems that some hypothesis is necessary and we give one in a form applying to cases of interest here.

First, we should introduce notation for the standard Koszul complex associated to a sequence r_1, r_2, \dots, r_n of elements of $\pi_*(R)$. For $x \in \pi_*(R)$ we define $K(R; x)$ by the cofibre sequence

$$R \xrightarrow{x} R \longrightarrow K(R; x),$$

and now we take

$$K(R; r_1, \dots, r_n) = K(R; r_1) \otimes_R \cdots \otimes_R K(R; r_n).$$

In the usual way, a concrete realization requires the choice of specific cocycle representatives, but the homotopy type does not depend on these choices. If R is a classical ring, this gives the standard construction $K(R; x) = [R \xrightarrow{x} R]$ with the copies of R in degrees 0 and 1.

Lemma 6.1. *Suppose that S is proxy-regular with Koszul complex K_S and that Q has a Koszul complex of the special form $K_Q = K(Q; q_1, \dots, q_n)$ where $q_i \in \pi_*(Q)$ lifts to $r_i \in \pi_*(R)$ for $i = 1, \dots, n$. Then R is proxy-regular with Koszul complex $K_R := K_S \otimes_S K(R; r_1, \dots, r_n)$.*

Proof: There are three things to prove.

Since $S \models K_S$ and $R \models K(R; r_1, \dots, r_n)$, it follows that

$$R = S \otimes_S R \models K_S \otimes_S K(R; r_1, \dots, r_n) = K_R.$$

Since $Q \simeq k \otimes_S R$, we find firstly

$$k \models K_Q = k \otimes_S K(R; r_1, \dots, r_n) \models K_S \otimes_S K(R; r_1, \dots, r_n) = K_R$$

and secondly

$$K_R = K_S \otimes_S K(R; r_1, \dots, r_n) \vdash k \otimes_S K(R; r_1, \dots, r_n) = K_Q \vdash k.$$

This completes the proof. \square

6.B. Good approximation implies ascent. The core of our results about ascent come from [11]. Indeed, the proof of [11, 8.6] gives a sufficient condition for Gorenstein ascent in the commutative context.

Lemma 6.2. *If S and R are commutative and the natural map $\nu : \mathrm{Hom}_S(k, S) \otimes_S R \rightarrow \mathrm{Hom}_S(k, R)$ is an equivalence then*

$$\mathrm{Hom}_R(k, R) \simeq \mathrm{Hom}_Q(k, \mathrm{Hom}_S(k, S) \otimes_k Q).$$

In this case, if S and Q are Gorenstein, so is R , and the shifts add up: $\text{shift}(R) = \text{shift}(S) + \text{shift}(Q)$. \square

Now that we have a sufficient condition for Gorenstein ascent, we want to identify cases in which ν is an equivalence. The most familiar case is when R is small over S (or equivalently, when Q is finitely built from k). We emphasize that the hypothesis on ν in Lemma 6.2 only depends on R as a module over S , and we will obtain a useful generalization by approximating R by S -modules for which ν is an equivalence. The approximation will be as an inverse limit, and to see the approximation is accurate we need to impose hypotheses to ensure inverse limits and tensor products commute.

Lemma 6.3. Suppose M and N are S -modules and $N \simeq \underset{\leftarrow}{\lim}_n N_n$, and consider the natural map

$$\kappa : M \otimes_S [\underset{\leftarrow}{\lim}_n N_n] \longrightarrow \underset{\leftarrow}{\lim}_n [M \otimes_S N_n].$$

The map κ is an equivalence in either of the following circumstances

- S is connective, M is of upward finite type and the modules N_n are uniformly bounded below. The hypothesis on M holds if $\pi_0(S) = k$, $\pi_*(M)$ is bounded below and $\pi_n(M)$ is finite dimensional over k for all n .
- S is coconnective, M is of downward finite type and the modules N_n are uniformly bounded above. The hypothesis on M holds if S is simply coconnected, $\pi_*(M)$ is bounded above and $\pi_n(M)$ is finite dimensional over k for all n .

Proof: In the first part the fact that M is of upward finite type and the N_n are uniformly bounded below is enough to see that the limit is achieved in each degree. It is proved as [11, 3.13] that the homotopy level condition ensures M is of upward finite type.

The proof of the second part is precisely similar, with a reference to [11, 3.14]. \square

Lemma 6.4. Suppose that $\pi_*(S)$ is Noetherian and that $\pi_*(\text{Hom}_S(k, S))$ is a finitely generated module over $\pi_*(S)$ and that $R \simeq \underset{\leftarrow}{\lim}_n R_n$ for small S -modules R_n . The hypothesis of Lemma 6.2 applies in either of the following circumstances

- S is connected and the R_n are uniformly bounded below
- S is simply coconnected and the R_n are uniformly bounded above

In this case,

$$\text{Hom}_R(k, R) \simeq \text{Hom}_Q(k, \text{Hom}_S(k, S) \otimes_k Q)$$

and Gorenstein ascent holds for the cofibre sequence $S \longrightarrow R \longrightarrow Q$.

Proof: First,

$$\text{Hom}_S(k, R) \simeq \text{Hom}_S(k, \underset{\leftarrow}{\lim}_n R_n) \simeq \underset{\leftarrow}{\lim}_n \text{Hom}_S(k, R_n).$$

Now, since R_n is a small S -module, $\text{Hom}_S(k, R_n) \simeq \text{Hom}_S(k, S) \otimes_S R_n$.

It therefore remains to show that the natural map

$$\kappa : M \otimes_S [\underset{\leftarrow}{\lim}_n R_n] \longrightarrow \underset{\leftarrow}{\lim}_n [M \otimes_S R_n]$$

is an equivalence when $M = \text{Hom}_S(k, S)$ so the conclusion follows from Lemma 6.3. \square

6.C. Building good approximations. We give criteria under which R may be approximated in this way. First, we assume that R is a k -algebra, and it is convenient to introduce some further terminology.

Definition 6.5. We say that a map $R \rightarrow Q$ is π_* -finite if $\pi_*(Q)$ is finitely generated as a module over $k[x_1, \dots, x_n]$ for some finite set of elements x_1, \dots, x_n of $\pi_*(R)$. A cofibration sequence $S \rightarrow R \rightarrow Q$ is π_* -finite if the map $R \rightarrow Q$ is π_* -finite.

Remark 6.6. If $\pi_*(R)$ is Noetherian, it is equivalent to ask that $\pi_*(Q)$ is finitely generated over $\pi_*(R)$.

Proposition 6.7. Suppose that $\pi_*(S)$ is Noetherian, $\pi_*(R)$ and $\pi_*(\text{Hom}_S(k, S))$ are finitely generated $\pi_*(S)$ -modules, R is a k -algebra and the cofibration is π_* -finite, and suppose that either (i) S, R and Q are all connected or (ii) that S, R and Q are all coconnected and S is simply coconnected. Under these conditions,

$$\text{Hom}_R(k, R) \simeq \text{Hom}_Q(k, \text{Hom}_S(k, S) \otimes_k Q)$$

and Gorenstein ascent holds for the cofibre sequence $S \rightarrow R \rightarrow Q$.

Proof: From the π_* -finite hypothesis, by the Noether normalization argument, there is a polynomial subring $R(1)_*$ of $\pi_*(R)$ over which $\pi_*(Q)$ is finitely generated. Now let $R(2)_*, R(3)_*, \dots$ be the subrings generated by the 2nd, 4th, 8th powers of generators of $R(1)_*$.

Next we construct (non-commutative) k -algebra spectra $R(n)$ with $\pi_* R(n) = R(n)_*$. For a polynomial ring on a single generator of degree d , we can consider the James construction $J_k(S^d)$ on a sphere over k . This is the free associative k -algebra spectrum on the d -sphere and has homotopy $k[X_d]$. If $R(n)_* = k[x_1, \dots, x_s]$, we form

$$J_k(S^{d_1}) \otimes_k \cdots \otimes_k J_k(S^{d_s})$$

where x_i is of degree d_i . If A is a commutative k -algebra then we may construct a ring map taking X_i to x_i as the composite

$$J_k(S^{d_1}) \otimes_k \cdots \otimes_k J_k(S^{d_s}) \longrightarrow A \otimes_k \cdots \otimes_k A \longrightarrow A.$$

The first map takes X_i to x_i in the i th factor, and is a map of associative rings. The second map is multiplication in A , and this is a ring map since A is commutative.

Using these ring spectra $R(n)$, using tensor products of the single variable case as above, we may construct maps

$$\dots \longrightarrow R(2) \longrightarrow R(1) \longrightarrow R \longrightarrow Q$$

realizing the algebras we took in homotopy. Now by construction $Q_n = Q \otimes_{R(n)} k$ is finitely built from k since its homotopy is a finite dimensional k -vector space. Since the polynomial generators were in increasingly large degrees, $Q = \underset{\leftarrow}{\text{holim}}_n Q_n$. Similarly, if we write $R_n = R \otimes_{R(n)} k$ we have $R \simeq \underset{\leftarrow}{\text{holim}}_n R_n$, and $k \otimes_S R_n \simeq k \otimes_S \overset{n}{R} \otimes_{R(n)} k \simeq Q \otimes_{R(n)} k = Q_n$. Thus we have sequences $S \longrightarrow \overset{n}{R} \longrightarrow Q_n$ and R_n is small as an S -module.

The result now follows from Lemma 6.4. \square

To apply this, we first note that in characteristic p the π_* -finite condition is automatic when $\pi_*(S)$ is in a finite range of degrees.

Lemma 6.8. *Suppose $S \rightarrow R \rightarrow Q$ is a cofibre sequence, either connected and satisfying the hypotheses of Lemma 3.1 or coconnected and satisfying the hypotheses of Lemma 3.2. Suppose in addition that the cofibre sequence is one of k -algebras, where k is a field of characteristic $p > 0$ and $\pi_*(S)$ is Noetherian and in a finite range of degrees, then the cofibre sequence is π_* -finite.*

Proof: Using the spectral sequence of Lemma 3.1 or Lemma 3.2 as appropriate we see that if $x \in \pi_*(Q)$ survives to the r th page then $d_r(x^p) = 0$, so that x^p survives to the $(r+1)$ st page. If $\pi_*(S)$ is in a finite range of degrees, the spectral sequence collapses at the N th stage for some N and the p^{N-1} th powers of all elements survive, and therefore lie in the image of $\pi_*(R) \rightarrow \pi_*(Q)$ so that the cofibration is π_* -finite. \square

We may now apply Proposition 6.7 to give a useful characteristic p Gorenstein ascent theorem.

Corollary 6.9. *Consider a cofibre sequence $S \rightarrow R \rightarrow Q$ of k -algebras as in Lemma 6.8. Suppose that $\pi_*(S)$ is Noetherian and either connected or simply coconnected and k is a field of characteristic $p > 0$. If $\pi_*(R)$ and $\pi_*(\text{Hom}_S(k, S))$ are finitely generated $\pi_*(S)$ -modules, and $\pi_*(S)$ is concentrated in a finite range of degrees then*

$$\text{Hom}_R(k, R) \simeq \text{Hom}_Q(k, \text{Hom}_S(k, S) \otimes_k Q)$$

and Gorenstein ascent holds for the cofibre sequence.

7. AUSONI-BÖKSTEDT DUALITY.

We now have the necessary ingredients to state and prove our duality result. The idea is that if we are given maps $C \rightarrow B \rightarrow k$ of commutative ring spectra with the cofibre ring spectrum $A = B \otimes_C k$ Gorenstein then (at least under some hypotheses on k and A) if $\overline{T}B = THH(B; k)$ is Gorenstein then $\overline{TC} = THH(C; k)$ is also Gorenstein. Since we are deducing the domain \overline{TC} is Gorenstein from the fact that $\overline{T}B$ is Gorenstein, we think of this as a *descent* theorem, even though the principal ingredient is an *ascent* theorem for a suitable cofibration.

7.A. Gorenstein descent for THH . The key to method is the existence of a suitable cofibration sequence conjectured on the basis of the examples and proved by Dundas.

Lemma 7.1. *(Dundas) Given a cofibre sequence $C \rightarrow B \rightarrow A$ of commutative ring spectra over k (i.e., B has a map to k and $A = B \otimes_C k$) there is a cofibre sequence of commutative k -algebra spectra*

$$A \rightarrow \overline{TC} \rightarrow \overline{TB}$$

where $\overline{TC} = C \otimes_{C^e} k = THH(C; k)$ and $\overline{TB} = B \otimes_{B^e} k = THH(B; k)$.

Remark 7.2. Dundas's lemma makes $THH(\cdot; k)$ remarkably computable. For example, Lindenstrauss points out that if R is k -algebra, we may apply the Dundas Lemma to the cofibre sequence $k \rightarrow R \rightarrow R$ to deduce

$$THH_*(R; k) \cong THH_*(k) \otimes \text{Tor}_*^R(k, k).$$

In particular, this allows one to deduce from Bökstedt's calculation that $THH_*(k) = k[\mu_2]$ for any field k of characteristic p .

Remark 7.3. Dundas's lemma applies also to $HH_\bullet(\cdot|S; k)$ when S is less complicated than the sphere spectrum. For example we may take $S = C^*(Z)$ for a space Z , and suppose given a map $Y \leftarrow X$ of simply connected spaces over Z with fibre F , giving

$$(C \rightarrow B \rightarrow A) = (C^*(Y) \rightarrow C^*(X) \rightarrow C^*(F)).$$

We see

$$HH_\bullet(C^*(Y)|C^*(Z); k) = C^*(\text{fibre}(Y \rightarrow Y \times_Z Y))$$

provided $Y \times_Z Y$ is simply connected, so that if X also satisfies the corresponding hypothesis, Dundas's lemma gives a cofibre sequence

$$C^*(F) \rightarrow C^*(\text{fibre}(Y \rightarrow Y \times_Z Y)) \rightarrow C^*(\text{fibre}(X \rightarrow X \times_Z X)).$$

If $Z = *$ this comes from the fibre sequence

$$F \leftarrow \Omega Y \leftarrow \Omega X$$

obtained from the Puppe sequence generated by $Y \leftarrow X$.

Proof: For a C bimodule M , according to the original definition, the topological Hochschild homology $THH(C; M)$ is a realization of the Hochschild simplicial spectrum with n th term $M \otimes_S C^{\otimes n}$, where the tensor power is for \otimes_S ; this is natural for maps of rings and of bimodules. In particular, the map $C \rightarrow B$ of ring spectra vertically and the map $B \otimes_S k \rightarrow k$ of B -bimodules horizontally give a commutative square

$$\begin{array}{ccc} B \otimes_S k \otimes_S C^{\otimes n} & \rightarrow & k \otimes_S C^{\otimes n} \\ \downarrow & & \downarrow \\ B \otimes_S k \otimes_S B^{\otimes n} & \rightarrow & k \otimes_S B^{\otimes n} \end{array}$$

which is evidently a pushout square of commutative ring spectra. Taking geometric realizations we obtain the pushout square

$$\begin{array}{ccc} THH(C; B \otimes_S k) & \rightarrow & THH(C; k) \\ \downarrow & & \downarrow \\ THH(B; B \otimes_S k) & \rightarrow & THH(B; k). \end{array}$$

Now we use the fact that THH with coefficients in a bimodule of the form $M = X \otimes_S Y$ simplifies:

$$THH(C; X \otimes_S Y) = Y \otimes_C X.$$

The pushout square now gives the required result. \square

We now want to take the cofibre sequence $A \rightarrow \overline{T}C \rightarrow \overline{T}B$ and deduce that when A and $\overline{T}B$ are Gorenstein, so is $\overline{T}C$. We need only verify that the hypotheses of Lemma 6.1 and Corollary 6.9 are satisfied.

Theorem 7.4. (Gorenstein descent for THH) Suppose $C \rightarrow B \rightarrow A$ is a cofibre sequence of connective commutative ring spectra with maps to k and that

- (1) A and $\overline{T}B$ are proxy-regular and Gorenstein
- (2) $\pi_*(A)$ and $\pi_*(\overline{T}B)$ are Noetherian
- (3) the map $\overline{T}C \rightarrow \overline{T}B$ is π_* -finite

then \overline{TC} is proxy-regular and Gorenstein with

$$\text{shift}(\overline{TC}) = \text{shift}(\overline{TB}) + \text{shift}(A).$$

Proof: We consider the cofibration $A \rightarrow \overline{TC} \rightarrow \overline{TB}$ of Lemma 7.1.

By the π_* -finite hypothesis we may choose a finite number of elements of $\pi_*(\overline{TC})$ so that $\pi_*(\overline{TB})$ is finitely generated over the k -algebra they generate, and we may form a Koszul complex by using these generators. This verifies the hypotheses necessary to see that \overline{TC} is proxy-regular by Lemma 6.1.

The hypotheses for Gorenstein ascent from A to \overline{TC} are stated explicitly. \square

Corollary 7.5. *If C is Gorenstein of shift a and augmented over a field k of characteristic p and if C is regular then $\text{THH}(C; k) \rightarrow k$ is Gorenstein of shift $-a - 3$.*

Proof: We apply the theorem to the cofibre sequence $C \rightarrow k \rightarrow A$. Since C is regular, $\pi_* A$ is finite dimensional and hence Noetherian. Now note that k is Gorenstein of shift 0, so by Gorenstein Ascent for $C \rightarrow k \rightarrow A$, the ring A is Gorenstein and we have $\text{shift}(A) = -a$.

From Bökstedt's calculation $\overline{TB} = \text{THH}(k)$ has Noetherian homotopy $k[\mu_2]$, so it is Gorenstein of shift -3 . Now observe that by Lemma 6.8, the cofibration $A \rightarrow \overline{TC} \rightarrow \overline{TB}$ is π_* -finite. Thus the hypotheses of Theorem 7.4 are satisfied, and we may apply Gorenstein Ascent to the cofibration $A \rightarrow \overline{TC} \rightarrow \overline{TB}$ to obtain the conclusion. \square

Remark 7.6. As in Remark 7.3, the same argument applies if S less complicated than the sphere spectrum. In particular if we have a map $C \rightarrow B$ of augmented k -algebras, Dundas's Lemma supplies a cofibre sequence

$$A \rightarrow \text{HH}_\bullet(C|k; k) \rightarrow \text{HH}_\bullet(B|k; k).$$

If $B = k$ we find

$$k \otimes_C k \simeq \text{HH}_\bullet(C|k; k).$$

For instance if $C = C^*(X)$ this corresponds to the fact that ΩX is the fibre of the diagonal $X \rightarrow X \times X$.

8. EXAMPLES

We observe that Theorem 7.4 gives a non-calculational proof of several of the dualities we observed in Section 5 above, as well as giving many new examples where the coefficient rings are not known. In each case we specify $C \rightarrow B$ and k and then discuss the resulting cofibre sequence $A \rightarrow \text{THH}(C; k) \rightarrow \text{THH}(B; k)$.

8.A. Known examples revisited. We do not add to the explicit calculations described in Section 5 above, but we emphasize that the only calculational input is Bökstedt's theorem. Interesting structural relationships are highlighted by this approach.

Example 8.1. (*Example 5.1 revisited: \mathbb{F}_p .*) If we take $C \rightarrow B$ to be $\mathbb{F}_p \rightarrow \mathbb{F}_p$ and $k = \mathbb{F}_p$, we find $A = \mathbb{F}_p$. It is immediate that \mathbb{F}_p is small over \mathbb{F}_p and $\mathbb{F}_p \rightarrow \mathbb{F}_p$ is Gorenstein of shift 0. Corollary 7.5 shows $\text{THH}(\mathbb{F}_p)$ is Gorenstein of shift $-0 - 3 = -3$.

Example 8.2. (*Example 5.2 revisited: \mathbb{Z}*) If we take $C \rightarrow B$ to be $\mathbb{Z} \rightarrow \mathbb{F}_p$ and $k = \mathbb{F}_p$, we find $A \sim C_*(S^1)$ (where \sim means that the coefficient rings are isomorphic). It is immediate that \mathbb{F}_p is small over \mathbb{Z} and easy to check that $\mathbb{Z} \rightarrow \mathbb{F}_p$ is Gorenstein of shift -1 . Corollary 7.5 shows that $THH(\mathbb{Z}; \mathbb{F}_p)$ is Gorenstein of shift $1 + (-3) = -2$.

The spectral sequence of Lemma 3.1 gives an alternative approach to the calculational proof. The only necessary input would be to know that the differential $d_2(\mu_2) \neq 0$. This then shows that $d_2(\mu_2^n) \neq 0$ unless n is a multiple of p , so that the $E_3 = E_\infty$ term is generated by μ_2^p (giving μ_{2p}) and $\mu_2^{p-1}\tau$ (giving λ_{2p-1}).

The Lindenstrauss-Madsen example in the unramified case can be treated in the same way, since $\mathcal{O}/p = k$. However, the ramified case is not covered by our analysis since $THH_*(\mathcal{O}/p)$ is not Noetherian.

Example 8.3. (*Example 5.3 revisited: lu*) If we take $C \rightarrow B$ to be $lu \rightarrow \mathbb{Z}$ and $k = \mathbb{F}_p$, we find $A \sim C_*(S^{2p-1})$. It is easy to check that \mathbb{F}_p is small over lu and that $lu \rightarrow \mathbb{F}_p$ is Gorenstein of shift $-(2p-2) - 1 - 1 = -2p$. From Bökstedt duality for $THH(\mathbb{Z}; \mathbb{F}_p)$, Theorem 7.4 shows $THH(lu; \mathbb{F}_p)$ is Gorenstein of shift $(2p-1) + (-2) = 2p-3$.

The spectral sequence of Lemma 3.1 gives an alternative approach to the proof. The only necessary input would be to know that the differential $d_{2p}(\mu_{2p}) \neq 0$. This then shows that $d_{2p}(\mu_{2p}^n) \neq 0$ unless n is a multiple of p , so that the $E_{2p+1} = E_\infty$ term is generated by μ_{2p}^p (giving μ_{2p^2}), $\mu_{2p}^{p-1}\tau_{2p-1}$ (giving λ_{2p^2-1}) and λ_{2p-1} (which survives as it is).

Of course we get the same conclusion by taking $C \rightarrow B$ to be $lu \rightarrow \mathbb{F}_p$ and $k = \mathbb{F}_p$. In that case we find $A \sim C_*(S^1 \times S^{2p-1})$ so that by Corollary 7.5 $THH(lu; \mathbb{F}_p)$ is Gorenstein of shift $[(2p-1) + 1] - 3 = 2p-3$.

Example 8.4. (*Example 5.4 revisited: ku*) If we take $C \rightarrow B$ to be $ku \rightarrow ku/(p, v_1)$ and $k = ku/(p, v_1)$, we find $A \sim C_*(S^1 \times S^{2p-1}; ku/(p, v_1))$ (a Poincaré duality algebra of formal dimension $4p-4$). It is easy to see that $ku/(p, v_1)$ is small over ku and $ku \rightarrow ku/(p, v_1)$ is Gorenstein of shift $-2p$.

In order to proceed we would need to know that $ku/(p, v_1)$ has a commutative ring model and that $THH(ku/(p, v_1))$ is Gorenstein of shift x . We would then deduce $THH(ku; ku/(p, v_1))$ is Gorenstein of shift $4p-4+x$.

Example 8.5. (*Example 5.5 revisited: ko*) We take $C \rightarrow B$ to be $ko \rightarrow ku$ and $k = \mathbb{F}_2$. As in the discussion of the relatively Gorenstein condition (Example 4.3) Wood's Theorem gives the cofibre sequence $\Sigma ko \xrightarrow{\eta} ko \rightarrow ku$, showing that ku is small over ko and $A \sim C_*(S^2)$.

Since $THH(ku; \mathbb{F}_2)$ is Gorenstein of shift 1 by Example 8.8, and the complex in Wood's Theorem is self dual of dimension 2 , Theorem 7.4 shows $THH(ko; \mathbb{F}_2)$ is Gorenstein of shift $1+2=3$.

More directly, using Example 4.3 we could take $C \rightarrow \mathbb{F}_2$ to be $ko \rightarrow \mathbb{F}_2$, and apply Corollary 7.5 to conclude that $THH(ko; \mathbb{F}_2)$ is Gorenstein of shift $-(-6) - 3 = 3$.

Example 8.6. (*Example 5.6 revisited at $p=3$: tmf localized at 3*) We take $C \rightarrow B$ to be $tmf \rightarrow tmf_1(2)$ at the prime 3 and $k = \mathbb{F}_3$. As in the discussion of the relatively Gorenstein condition (Example 4.4(i)) the tmf -module $tmf_1(2)$ is tmf extended by a three cell complex, so it is small. We have also seen $tmf_1(2)$ is Gorenstein of shift -15 , so that by Theorem 7.4 the ring $THH(tmf_1(2); \mathbb{F}_3)$ is Gorenstein of shift 12 . Since the three cell complex is self-dual of dimension 8 , Theorem 7.4 shows that $THH(tmf; \mathbb{F}_3)$ is Gorenstein of shift 20 .

More directly, using Example 4.4 (i) we observe that since $tmf_1(2)$ is small over tmf and \mathbb{F}_3 is small over $tmf_1(2)$ then \mathbb{F}_3 is small over tmf . Since tmf is Gorenstein of shift -23 we may apply Corollary 7.5 to deduce $THH(tmf; \mathbb{F}_3)$ is Gorenstein of shift 20 .

Example 8.7. (*Example 5.6 revisited at $p = 2$: tmf localized at 2*) We take $C \rightarrow B$ to be $tmf \rightarrow tmf_1(3)$ at the prime 2 and $k = \mathbb{F}_2$. As in the discussion of the relatively Gorenstein condition (Example 4.4(ii)) $tmf_1(3)$ is tmf extended by a finite complex, it is small. We have also seen $tmf_1(3)$ is Gorenstein of shift -11 , so that by Theorem 7.4, the ring $THH(tmf_1(3); \mathbb{F}_2)$ is Gorenstein of shift 8 . Since the complex is self-dual of dimension 12 , Theorem 7.4 shows that $THH(tmf; \mathbb{F}_2)$ is Gorenstein of shift 20 .

More directly, using Example 4.4 (ii) we observe that since $tmf_1(3)$ is small over tmf and \mathbb{F}_2 is small over $tmf_1(3)$ then \mathbb{F}_2 is small over tmf . Since $tmf_1(3)$ is Gorenstein of shift -23 we may apply Corollary 7.5 to deduce $THH(tmf; \mathbb{F}_2)$ is Gorenstein of shift 20 .

8.B. New examples. The possibilities are innumerable, but we select three for illustration.

Example 8.8. (*Example 5.4 revisited again: ku*) If we take $C \rightarrow B$ to be $ku \rightarrow \mathbb{Z}$ and $k = \mathbb{F}_p$, we find $A \sim C_*(S^3)$, and from the fact that $THH(\mathbb{Z}; \mathbb{F}_p)$ is Gorenstein of shift -2 we deduce from Theorem 7.4 that the ring $THH(ku; \mathbb{F}_p)$ is Gorenstein of shift $3 + (-2) = 1$.

The spectral sequence of Lemma 3.1 gives a calculation. If p is odd, there can be no differentials and $THH(ku; \mathbb{F}_p) = \mathbb{F}_p[\mu_{2p}] \otimes \Lambda(\lambda_{2p-1}, \lambda_3)$.

If $p = 2$ the only necessary input would be to know whether the differential $d_4(\mu_4)$ is zero or not. If it is zero then $THH_*(ku; \mathbb{F}_2) = \mathbb{F}_2[\mu_4] \otimes \Lambda(\lambda_3, \lambda'_3)$. If it is non-zero then $THH_*(ku; \mathbb{F}_2) = \mathbb{F}_2[\mu_8] \otimes \Lambda(\lambda_3, \lambda_7)$. The second of these is what actually happens, as one may see from Dundas's Lemma applied to $ko \rightarrow ku$ together with the result for ko described in Example 8.5.

Of course we get the same conclusion by working with $C \rightarrow B$ to be $ku \rightarrow \mathbb{F}_p$ and $k = \mathbb{F}_p$ we find $A \sim C_*(S^1 \times S^3)$ so that by Corollary 7.5 the ring $THH(ku; \mathbb{F}_p)$ is Gorenstein of shift $[3+1]-3=1$.

Example 8.9. We take $C \rightarrow B$ to be $e_n \rightarrow \mathbb{F}_p$, where e_n is the connective Lubin-Tate commutative ring spectrum with homotopy $W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]] [u]$. From its homotopy we see that it is Gorenstein of shift $-n-3$. From Corollary 7.5 we conclude $THH(e_n; \mathbb{F}_p)$ is Gorenstein of shift n .

These examples are iterable.

Example 8.10. Veen's calculation [22] of the homotopy ring of the double THH of k , as $k[\mu'_2, \mu''_2] \otimes \Lambda(\lambda_3)$ gives an example whose coefficient ring is Noetherian and of Krull dimension 2 .

If we are given map $R \rightarrow k$ we may take $C \rightarrow B$ to be $THH(R; k) \rightarrow THH(k)$, and take A to be its cofibre. Now apply Dundas's Lemma to obtain a cofibre sequence

$$A \rightarrow THH(THH(R; k); THH(k)) \rightarrow THH(THH(k)).$$

Unfortunately there seems to be no obvious example for which A is finite dimensional, or even Noetherian. For example, if we take $R = \mathbb{Z}$ it seems $\pi_* A = \mathbb{F}_p[\mu_2]/(\mu_2^p) \otimes \Gamma(\gamma_{2p})$.

8.C. Discussion. The main obstacle to finding more examples is the need to ensure that the coefficient rings should be Noetherian, which seems rather rare for THH . In the cases with complete calculations and Noetherian rings, the coefficient rings are all themselves

Gorenstein. We have not yet found an example where the ring spectrum is proxy regular and Gorenstein except when the coefficients are already Gorenstein.

Our analysis is based on the cofibre sequence $A \longrightarrow \overline{T}C \longrightarrow \overline{T}B$ and requires that it is π_* -finite. We have relied on the fact that if A is finite dimensional and k is of characteristic p , the cofibre sequence is π_* -finite. In this case $\overline{T}C$ will have the same Krull dimension as $\overline{T}B$.

Appendices

We end with two appendices describing closely related phenomena. They do not form part of the argument, and are included for comparison.

APPENDIX A. THOM SPECTRA

Given a 3-fold loop map $f : X \longrightarrow BF$, Blumberg-Cohen-Schlichtkrull [8] prove

$$THH(Mf) \simeq Mf \wedge BX_+.$$

Since f is a 3-fold loop map, X is a 3-fold loop space and $BX \simeq \Omega BBX$. We expect that $THH(Mf) = C_*(BX; Mf)$ being Gorenstein over Mf will be related to $C^*(BBX; Mf) = F(BBX_+, Mf)$ being Gorenstein over Mf .

Example A.1. (\mathbb{F}_p revisited.) This example is closely related to the fact that $THH(\mathbb{F}_p)$ is Gorenstein of shift -3 .

Mahowald [18] showed that the Eilenberg-MacLane spectrum \mathbb{F}_p is the Thom spectrum of a map $\Omega^2 S^3 \longrightarrow BF$. Although this is only a double loop space map, it is shown in [8, 1.3] that

$$THH(\mathbb{F}_p) \simeq \mathbb{F}_p \wedge \Omega S^3_+ =: C_*(\Omega S^3),$$

but this is not an equivalence of ring spectra, since the ring structures in homotopy groups are different.

Since S^3 is a 3-manifold, $C^*(S^3)$ is Gorenstein of shift -3 , and by Morita invariance of the Gorenstein condition [11], we conclude that $C_*(\Omega S^3)$ is Gorenstein of shift -3 . Although it is precisely parallel, this doesn't directly imply anything about $THH(\mathbb{F}_p)$.

Example A.2. (\mathbb{Z} revisited.) This is closely related to the fact that $THH(\mathbb{Z}; \mathbb{Z}/p)$ is Gorenstein of shift -2 .

Mahowald [18] proved that the Eilenberg-MacLane spectrum \mathbb{Z} is the Thom spectrum of a map $\Omega^2 S^3 \langle 3 \rangle \longrightarrow BF$. Although it is only a double loop map it is shown as [8, 1.4] that $THH(\mathbb{Z}) \simeq \mathbb{Z} \wedge \Omega S^3 \langle 3 \rangle_+$, and that

$$THH(\mathbb{Z}; \mathbb{Z}/p) \simeq \mathbb{F}_p \wedge \Omega S^3 \langle 3 \rangle_+ = C_*(\Omega S^3 \langle 3 \rangle).$$

Applying cochains to the fibration

$$K(\mathbb{Z}, 2) \longrightarrow S^3 \langle 3 \rangle \longrightarrow S^3$$

we get a cofibre sequence of ring spectra, and a standard calculation shows this is π_* -finite. By Corollary 6.9, the cochains on the total space is Gorenstein if the cochains on the base and the cochains on the fibre are, and that the shifts add. Since S^3 is a 3-manifold, $C^*(S^3)$ is Gorenstein of shift -3 . On the other hand $C_*(\Omega BU(1)) = C_*(U(1))$ is Gorenstein of shift 1 since $U(1)$ is a 1-dimensional compact Lie group, and hence by Morita invariance of the Gorenstein condition [11], we find $C^*(BU(1))$ is Gorenstein of shift 1 . By Gorenstein ascent,

we deduce that $C^*(S^3\langle 3 \rangle)$ is Gorenstein of shift $-2 = -3 + 1$. By Morita invariance of the Gorenstein condition [11] we conclude that $C_*(\Omega S^3\langle 3 \rangle)$ is Gorenstein of shift -2 .

APPENDIX B. DWYER-MILLER AND KONTSEVICH DUALITY

This section describes some known dualities for Hochschild homology and cohomology with a similar flavour. The analogue to bear in mind is the case $R = k[x]$ on a polynomial generator of even degree d . This is Gorenstein of shift $-d - 1$. Now we calculate $HH^*(R) = k[x, \alpha]$ where α is of degree $-d - 1$, and $HH_*(R) = k[x, \beta]$ with β of degree $d + 1$, so we have $HH^*(R) = \Sigma^{-d-1} HH_*(R)$.

In this section we work under k rather than under the sphere spectrum \mathbb{S} . In particular, we assume that our rings are k -algebras so the situation is very different to the one considered in the body of the paper. The discussion developed from the work of Cohen-Jones [10].

The following assumption is more often satisfied by objects like group rings than the commutative rings we have been concerned with in the body of the paper.

Assumption B.1. We assume that $S = k$, so that $R^e = R \otimes_k R$ and that there is a ring map $R \rightarrow R^e$ of R^e -modules. Finally, we assume that the bimodule R is induced from an R -module: $R = R^e \otimes_R k$.

Proposition B.2. (*Dwyer-Miller*) *If Assumption B.1 holds, k is small over R and $R \rightarrow k$ is Gorenstein of shift a then*

$$HH^*(R; P) \cong \Sigma^a HH_*(R; P)$$

for all bimodules P .

Remark B.3. There seems no prospect of a result like this for THH since (as in the case with $P = k$ for instance, $THH^\bullet(k)$ (for example) is not bounded below.

Proof: We argue as follows, where the third equivalence requires k to be small, and where P^{ad} is the restriction of P along the map $R \rightarrow R^e$.

$$\begin{aligned} HH^*(R; P) &= \text{Hom}_{R^e}(R, P) \\ &= \text{Hom}_R(k, P^{ad}) \\ &\simeq \text{Hom}_R(k, R) \otimes_R P^{ad} \\ &= \Sigma^a k \otimes_R P^{ad} \\ &= \Sigma^a R \otimes_{R^e} P \\ &= \Sigma^a HH_*(R; P) \end{aligned}$$

□

Remark B.4. Take $R = C_*(\Omega X)$ for a simply connected d -manifold X . Since X can be given the structure of a finite CW complex, $C^*(X)$ is finitely built from k , and applying $\text{Hom}_{C^*(X)}(\cdot, k)$ we see that k is small over $C_*(\Omega X)$ and the hypotheses of Proposition B.2 hold.

By the Morita invariance of the Gorenstein condition [11, Proposition 8.5] this is Gorenstein of shift $a = -d$. Finally, note that $HH_*(R) = H_*(\Lambda X)$, so taking $P = R$, we see $\Sigma^d H_*(\Lambda X) = HH^*(R)$, showing that the shifted homology of the free loop space has a ring structure; Malm [20] shows this corresponds to the Chas-Sullivan product.

Corollary B.5. (*Kontsevich duality*) If Assumption B.1 holds, k is small over R and $R \rightarrow k$ is Gorenstein with shift a then $R^e \rightarrow R$ is relatively Gorenstein with shift a .

Remark B.6. The connection between the Gorenstein conditions on R and R^e in commutative algebra is of great interest [5, 6].

Proof: Taking $P = R^e$ in the Proposition B.2 we find

$$\text{Hom}_{R^e}(R, R^e) = HH^*(R|S; R^e) = \Sigma^a HH_*(R|S; R^e) = \Sigma^a R \otimes_{R^e} R^e = \Sigma^a R.$$

□

Now consider the $(R|S)$ -bimodule $P = k$, and note that if R and k are commutative rings, then so is

$$HH_*(R|S; k) = R \otimes_{R^e} k.$$

Furthermore R^e is an R -algebra, so we have an algebra map

$$R \otimes_{R^e} k \rightarrow R^e \otimes_{R^e} k \xrightarrow{\cong} k$$

Corollary B.7. (*Algebraic Ausoni-Bökstedt duality*) If Assumption B.1 holds, k is small over R and $R \rightarrow k$ is Gorenstein with shift a then $HH_*(R|S; k) \rightarrow k$ is Gorenstein with shift $-a$.

Proof: Taking $P = k$ in the above we find

$$HH^\bullet(R|S; k) = \Sigma^a HH_\bullet(R|S; k).$$

Now we calculate

$$\begin{aligned} \text{Hom}_{HH_\bullet(R|S; k)}(k, HH_\bullet(R|S; k)) &= \Sigma^{-a} \text{Hom}_{HH_\bullet(R|S; k)}(k, HH^\bullet(R|S; k)) \\ &= \Sigma^{-a} \text{Hom}_{HH_\bullet(R|S; k)}(k, \text{Hom}_{R^e}(R, k)) \\ &= \Sigma^{-a} \text{Hom}_{HH_\bullet(R|S; k)}(R \otimes_{R^e} k, k) \\ &= \Sigma^{-a} k \end{aligned}$$

□

REFERENCES

- [1] V.Angeltveit, M.Hill and T.Lawson “Topological Hochschild homology of ℓ and ko .” American J Math **132** (2010) 297-330
- [2] V.Angeltveit and J. Rognes “Hopf algebra structure on topological Hochschild homology.” Algebraic and Geometric Topology **5** (2005) 1223-1290
- [3] C.Ausoni “Topological Hochschild homology of connective complex K -theory.” Amer. J. Math. 127 (2005), no. 6, 12611313.
- [4] C. Ausoni and J. Rognes, “Algebraic K-theory of topological K-theory”, Acta Mathematica **188** (2002), 1-39.
- [5] L.Avramov and S.B.Iyengar “Gorenstein algebras and Hochschild cohomology” Michigan Math. J. **57** (2008) 17-35
- [6] L.Avramov, S.B.Iyengar and J.Lipman “Reflexivity and rigidity for complexes, I” arXiv 0904.4695
- [7] H. Bass “On the ubiquity of Gorenstein rings.” Math. Z. **82** (1963) 828.
- [8] A. Blumberg, R.Cohen and C.Schlichtkrull “The topological Hochschild homology of Thom spectra and the free loop space.” Geom Topol **14** (2010) 1165-1242
- [9] M. Bökstedt, “Topological Hochschild homology of F_p , and \mathbb{Z} ”, Preprint, Bielefeld University.

- [10] R.Cohen and J.D.S.Jones “A homotopy theoretic realization of string topology.” *Math. Ann.* **324** (2002), no. 4, 773-798.
- [11] W.G.Dwyer, J.P.C.Greenlees and S.B.Iyengar “Duality in algebra and topology” *Advances in Maths.* **200** (2006) 357-402
- [12] J.P.C.Greenlees and G.Lyubeznik “Rings with a local cohomology theorem and applications to cohomology rings of groups.” *J. Pure and Applied Algebra* **149** (2000) 267-285.
- [13] M.Hill and T.Lawson “Topological modular forms with level structure” arXiv: 1312.7394, 53pp.
- [14] M.J.Hopkins and M.E.Mahowald “From elliptic curves to homotopy theory” AMS Mathematical Surveys and Mongraphs, **201** (2015) 261-285
- [15] T.Lawson and N.Naumann Commutativity conditions for truncated Brown-Peterson spectra of height 2. (English summary) *J. Topol.* **5** (2012), no. 1, 137-168.
- [16] T.Lawson and N.Naumann “Strictly commutative realizations of diagrams over the Steenrod algebra and topological modular forms at the prime 2.” *Int. Math. Res. Not. IMRN* 2014, no. 10, 2773-2813.
- [17] A.Lindenstrauss and I. Madsen “Topological Hochschild homology of number rings.” *Trans AMS* **352** (2000) 2179-2204
- [18] M.E.Mahowald “Ring spectra which are Thom complexes.” *Duke Math. J.*, **46** (1979) 549559
- [19] J.E.McClure and R.E.Staffeldt “On the topological Hochschild homology of bu I” *AJM* **118** (1993) 1005-1012
- [20] E.J.Malm “String topology and the based loop space” Preprint (2011) arXiv:1103.6198
- [21] Akhil Matthew “The homology of tmf ” Preprint (2013) arXiv:1305.6100
- [22] T.Veen “Detecting Periodic Elements in Higher Topological Hochschild Homology” Preprint (2014) arXiv:1312.5699

SCHOOL OF MATHEMATICS AND STATISTICS, HICKS BUILDING, SHEFFIELD S3 7RH. UK.

E-mail address: j.greenlees@sheffield.ac.uk