NUMBERS WITH SIMPLY NORMAL β-EXPANSIONS

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ABSTRACT. In [6] the first author proved that for any $\beta \in (1, \beta_{KL})$ every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion, where $\beta_{KL} \approx 1.78723$ is the Komornik-Loreti constant. This result is complemented by an observation made in [22], where it was shown that whenever $\beta \in (\beta_T, 2]$ there exists an $x \in (0, \frac{1}{\beta-1})$ with a unique β-expansion, and this expansion is not simply normal. Here $\beta_T \approx 1.80194$ is the unique zero in $(1, 2]$ of the polynomial $x^3 - x^2 - 2x + 1$. This leaves a gap in our understanding within the interval $[\beta_{KL}, \beta_T]$. In this paper we fill this gap and prove that for any $\beta \in (1, \beta_T]$, every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion. For completion, we provide a proof that for any $\beta \in (1, 2)$, Lebesgue almost every x has a simply normal β-expansion. We also give examples of x with multiple β-expansions, none of which are simply normal.

Our proofs rely on ideas from combinatorics on words and dynamical systems.

1. Introduction

Expansions in non-integer bases were first introduced and studied in the papers of Parry [31] and Rényi [32]. These representations are obtained by taking the usual integer base representations of the positive real numbers, and replacing the base by some non-integer. Despite being a simple generalisation of an idea that is well known to high school students, these representations exhibit many fascinating properties. One of these properties is the fact that typically a number has infinitely many representations. Consequently, one might ask whether amongst the set of representations there exists an expansion that satisfies a certain additional property. Properties we might be interested in could be combinatorial, number theoretic, or statistical. These ideas motivate this paper, wherein we study the existence of an expansion satisfying a certain statistical property, namely being simply normal.

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Let $\beta \in (1,2]$ and $I_\beta := [0, \frac{1}{\beta-1}]$. Given $x \in I_\beta$ we call a sequence $(\epsilon_i) \in \{0,1\}^{\mathbb{N}}$ a β-expansion of x if

$$x = \pi_\beta((\epsilon_i)) := \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}.$$

It is a straightforward exercise to show that every $x \in I_\beta$ has at least one β-expansion. When $\beta = 2$ then modulo a countable set every $x \in [0,1]$ has a unique binary expansion. Moreover, within this exceptional set every x has precisely two expansions. However, when $\beta \in (1,2)$ the situation is very different. Below we recall some results that exhibit these differences.

(1) Let $\beta \in (1, \frac{1+\sqrt{5}}{2})$. Then every $x \in (0, \frac{1}{\beta-1})$ has a continuum of β-expansions [20].
(2) Let $\beta \in (1,2)$. Then Lebesgue almost every $x \in I_\beta$ has a continuum of β-expansions [13, 33].
(3) For any $k \in \mathbb{N} \cup \{\aleph_0\}$ there exist $\beta \in (1,2)$ and $x \in I_\beta$ with exactly k β-expansions [8, 9, 18, 19, 24, 34].

We emphasise that the endpoints of I_β have a unique β-expansion for any $\beta \in (1,2)$. Consequently, most of the statements we make will relate to its interior $(0, \frac{1}{\beta-1})$.

Given a sequence $(\epsilon_i) \in \{0,1\}^{\mathbb{N}}$, we define the *frequency of zeros of* (ϵ_i) to be the limit

$$\mathrm{freq}_0(\epsilon_i) := \lim_{n\to\infty} \frac{\#\{1 \le i \le n : \epsilon_i = 0\}}{n}.$$

Assuming the limit exists, where $\#A$ denotes the cardinality of a set A. We say that (ϵ_i) is *simply normal* if $\mathrm{freq}_0(\epsilon_i) = 1/2$. In [6] the first author proved the following theorem.

Theorem 1.1. *(1) Let $\beta \in (1, \beta_{KL})$. Then every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion.*
(2) Let $\beta \in (1, \frac{1+\sqrt{5}}{2})$. Then every $x \in (0, \frac{1}{\beta-1})$ has a β-expansion for which the frequency of zeros does not exist.
(3) Let $\beta \in (1, \frac{1+\sqrt{5}}{2})$. Then there exists $c = c(\beta) > 0$ such that for every $x \in (0, \frac{1}{\beta-1})$ and $p \in [1/2 - c, 1/2 + c]$, there exists a β-expansion of x with frequency of zeros equal to p.

The quantity $\beta_{KL} \approx 1.78723$ appearing in statement (1) of Theorem 1.1 is the *Komornik-Loreti constant* introduced in [26]. Both statements (2) and (3) appearing in Theorem 1.1 are sharp. For any $\beta \in [\frac{1+\sqrt{5}}{2}, 2)$, there exists an $x \in (0, \frac{1}{\beta-1})$ such that for any β-expansion of x its frequency of zeros exists and is equal to either 0 or $1/2$. It is natural to wonder whether the parameter space described in statement (1) of Theorem 1.1 is optimal. In [22] Jordan, Shmerkin, and Solomyak proved the following result.

Theorem 1.2. *If $\beta \in (\beta_T, 2]$. Then there exists $x \in (0, \frac{1}{\beta-1})$ with a unique β-expansion, and this expansion is not simply normal.*

Here $\beta_T \approx 1.80194$. We will elaborate more on how β_T and β_{KL} are defined later. Theorem 1.1 and Theorem 1.2 leave an interval $[\beta_{KL}, \beta_T]$ for which we do not know whether every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion. In this paper we fill this gap and prove the following theorem.

Theorem 1.3. *Let $\beta \in (1, \beta_T]$. Then every $x \in (0, \frac{1}{\beta-1})$ has a simply normal β-expansion.*

With Theorem 1.2 in mind it is natural to ask whether it is possible for an x to have multiple β-expansions, none of which are simply normal. In this paper we include several explicit examples which demonstrate that this behaviour is possible.

The rest of this paper is arranged as follows. In Section 2 we recall some necessary preliminaries. We prove Theorem 1.3 in Section 3. We conclude in Section 4 with our aforementioned examples, and we also provide a short proof that for any $\beta \in (1, 2)$, Lebesgue almost every $x \in I_\beta$ has a simply normal β-expansion. At the end of the paper we pose some questions.

2. Preliminaries

The proof of Theorem 1.3 will make use of a dynamical interpretation of β-expansions, along with some properties of unique expansions. We start by detailing the relevant dynamical preliminaries.

2.1. **Dynamical preliminaries.** Given $\beta \in (1, 2)$ and $x \in I_\beta$, we denote the set of β-expansions of x as follows

$$\Sigma_\beta(x) := \Big\{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : x = \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i}\Big\}.$$

Now let us fix the maps $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$. Notice that the maps T_0 and T_1 depend on the parameter β. Given $\beta \in (1, 2)$ and $x \in I_\beta$, let

$$\Omega_\beta(x) := \Big\{(a_i) \in \{T_0, T_1\}^{\mathbb{N}} : (a_n \circ \cdots \circ a_1)(x) \in I_\beta \text{ for all } n \in \mathbb{N}\Big\}.$$

The following lemma was proved in [7] (see also, [12]). It shows how one can interpret a β-expansion dynamically as a sequence of maps that do not map a point out of I_β.

Lemma 2.1. *For any $x \in I_\beta$ we have Card $\Sigma_\beta(x) =$ Card $\Omega_\beta(x)$. Moreover, the map which sends (ϵ_i) to (T_{ϵ_i}) is a bijection between $\Sigma_\beta(x)$ and $\Omega_\beta(x)$.*

We refer the reader to Figure 1 for a graph of the functions T_0 and T_1. One observes that these graphs overlap on the interval

$$S_\beta := \Big[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\Big].$$

If $x \in S_\beta$ then both T_0 and T_1 map x back into I_β. In which case, by Lemma 2.1, x has a β-expansion that begins with a 0 and a β-expansion that begins with a 1. More generally, if x can be mapped into S_β under a finite sequence of T_0's and T_1's, then x has at least two β-expansions. In the literature S_β is commonly referred to as the *switch region*. An understanding of how orbits are mapped into S_β, and how orbits can avoid S_β, often proves to be profitable when studying a variety of problems. The main technical innovation of this paper is Proposition 2.5, which gives a thorough description of how orbits are mapped into S_β.

By Lemma 2.1, one can reinterpret Theorem 1.3 in terms of the existence of a sequence of maps with limiting frequency of T_0's equal to $1/2$. We make use of this interpretation in our proof. With this in mind we introduce the following notation. Let $\{T_0, T_1\}^* := \cup_{n=1}^{\infty}\{T_0, T_1\}^n$. Given $a \in \{T_0, T_1\}^*$ let $|a|$ denote the length of a. Moreover, given $a \in \{T_0, T_1\}^*$ let

$$|a|_0 := \#\{1 \le i \le |a| : a_i = T_0\}$$

and

$$|a|_1 := \#\{1 \le i \le |a| : a_i = T_1\}.$$

We will use the same notation to denote the analogous quantities for finite sequences of zeros and ones. Whether we are referring to a finite sequence of maps or a finite sequence of zeros and ones should be clear from the context.

It is useful at this point to introduce the following interval. Given $\beta \in (1, 2)$, let

$$O_\beta := [\pi_\beta((01)^\infty), \pi_\beta((10)^\infty)] = \Big[\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}\Big].$$

Here and throughtout we use ω^∞ to denote the element of $\{0,1\}^{\mathbb{N}}$ obtained by infinitely concatenating a finite sequence ω. Notice that $T_0(\frac{1}{\beta^2-1}) = \frac{\beta}{\beta^2-1}$ and $T_1(\frac{\beta}{\beta^2-1}) = \frac{1}{\beta^2-1}$. What is more, T_0 and T_1 expand distances between points by a factor β, and have their unique fixed points at 0 and $\frac{1}{\beta-1}$ respectively. It is a consequence of these observations that given $x \in (0, \frac{1}{\beta-1}) \setminus O_\beta$, there exists $k \in \mathbb{N}$ and $i \in \{0,1\}$ such that $T_i^k(x) \in O_\beta$. Therefore all orbits are eventually mapped into O_β, and thus O_β can be thought of as an attractor for this system.

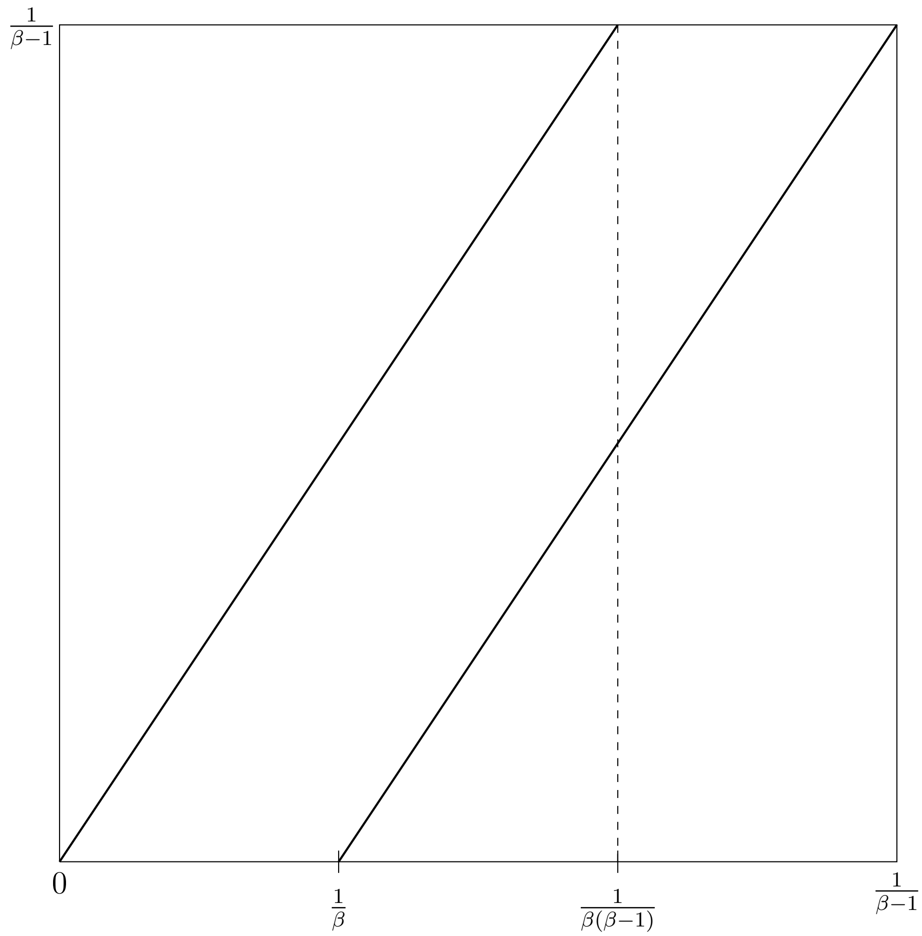

FIGURE 1. The overlapping graphs of T_0 and T_1.

2.2. **Univoque preliminaries.** A classical object of study within expansions in non-integer bases is the set of x with a unique expansion. Fixing notation, given $\beta \in (1,2)$ let

$$U_\beta := \left\{x \in \left[0, \frac{1}{\beta-1}\right] : x \text{ has a unique } \beta\text{-expansion}\right\}$$

and

$$\widetilde{U}_\beta := \left\{(\epsilon_i) \in \{0,1\}^{\mathbb{N}} : \sum_{i=1}^{\infty} \frac{\epsilon_i}{\beta^i} \in U_\beta\right\}.$$

We call U_β the *univoque set* and $\widetilde{U}_\beta$ the set of *univoque sequences*. By definition there is a bijection between these two sets. For more on these sets we refer the reader to [2, 15, 25] and the survey papers [16, 23].

The lexicographic ordering on $\{0,1\}^{\mathbb{N}}$ is a useful tool for studing the univoque set. This ordering is defined as follows. Given $(\epsilon_i),(\delta_i)\in\{0,1\}^{\mathbb{N}}$ we say that $(\epsilon_i)\prec(\delta_i)$ if $\epsilon_1<\delta_1$, or if there exists $n\in\mathbb{N}$ such that $\epsilon_{n+1}<\delta_{n+1}$ and $\epsilon_i=\delta_i$ for all $1\le i\le n$. We define $\preccurlyeq,\succ,\succcurlyeq$ in the obvious way. These definitions also have the obvious interpretation for finite sequences. We define the *reflection* of a word by $\overline{\epsilon_1\ldots\epsilon_n}=(1-\epsilon_1)\cdots(1-\epsilon_n)$, and the reflection of a sequence by $\overline{(\epsilon_i)}=(1-\epsilon_1)(1-\epsilon_2)\cdots$.

Many properties of $\widetilde{U}_\beta$ and consequently U_β are encoded in the quasi-greedy expansion of 1. The *quasi-greedy* expansion of 1 is the lexicographically largest β-expansion of 1 that does not end in 0^∞ (cf. [14]). Given a $\beta\in(1,2)$ we denote the quasi-greedy expansion of 1 by $\alpha(\beta)=(\alpha_i(\beta))$. The following description of $\alpha(\beta)$ is well-known (cf. [27]).

Lemma 2.2. *The map $\beta\mapsto\alpha(\beta)$ is a strictly increasing bijection between the interval $(1,2]$ and the set of sequence $(\gamma_i)\in\{0,1\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying*

$$\gamma_{n+1}\gamma_{n+2}\ldots\preccurlyeq\gamma_1\gamma_2\ldots\quad\text{for all}\quad n\ge 0.$$

Furthermore, the map $\beta\mapsto\alpha(\beta)$ is left continuous with respect to the order topology on $\{0,1\}^{\mathbb{N}}$ induced by the metric $\rho((\epsilon_i),(\delta_i))=2^{-\inf\{j\ge1:c_j\ne d_j\}}$.

Based on the notation $\alpha(\beta)$ we give the lexicographical characterization of $\widetilde{U}_\beta$ (cf. [15]).

Lemma 2.3. *Let $\beta\in(1,2]$. Then $(\epsilon_i)\in\widetilde{U}_\beta$ if and only if the sequence (ϵ_i) satisfies*

$$\begin{aligned}\epsilon_{n+1}\epsilon_{n+2}\ldots&\prec\alpha(\beta)\quad\text{whenever}\quad\epsilon_n=0,\\ \epsilon_{n+1}\epsilon_{n+2}\ldots&\succ\overline{\alpha(\beta)}\quad\text{whenever}\quad\epsilon_n=1.\end{aligned}$$

Note by Lemma 2.2 that the map $\beta\mapsto\alpha(\beta)$ is strictly increasing. Then by Lemma 2.3 it follows that $\widetilde{U}_{\beta_1}\subseteq\widetilde{U}_{\beta_2}$ whenever $\beta_1<\beta_2$.

The aforementioned constants β_{KL} and β_T are defined by their quasi-greedy expansions. The Komornik-Loreti constant β_{KL} is the unique $\beta\in(1,2)$ whose quasi-greedy expansion is the shifted Thue-Morse sequence. This sequence is defined as follows. Let $\tau^0=0$, we define τ^1 to be τ^0 concatenated with its reflection, in other words $\tau^1=\tau^0\overline{\tau^0}$. We then define τ^2 to be the concatentation of τ^1 with its reflection. We repeat this process in the natural way, given τ^k let τ^{k+1} be the concatenation of τ^k with its reflection. The first few words built using this procedure are listed below

$$\tau^0=0,\quad\tau^1=01,\quad\tau^2=0110,\quad\tau^3=01101001.$$

Repeating this reflection and concatenation process indefinitely gives rise to an infinite sequence $(\tau_i)_{i=0}^\infty$. This sequence is called the *Thue-Morse sequence*. The Komornik-Loreti constant β_{KL} satisfies $\alpha(\beta_{KL})=(\tau_i)_{i=1}^\infty$. The Komornik-Loreti constant first appeared in

[26] where it was shown to be the smallest $\beta \in (1,2)$ for which 1 has a unique β-expansion. It has since been shown to be important for a variety of other reasons, see [21]. In [3] it was shown that β_{KL} is transcendental. For more on the Thue-Morse sequence we refer the reader to [5].

The quantity β_T is the unique $\beta \in (1,2)$ such that $\alpha(\beta) = 1(10)^\infty$. Alternatively, β_T is the unique root of $x^3 - x^2 - 2x + 1 = 0$ that lies within the interval $(1,2)$. We emphasise here that β_T is not a Pisot number. β_T although not as exotic as β_{KL} is still of importance when it comes to studying U_β and $\widetilde{U}_\beta$. β_T is the smallest $\beta \in (1,2)$ for which the attractor of $\widetilde{U}_\beta$ is transitive under the usual shift map, see [1, 2]. Moreover, it is a consequence of the work done in [4] that $\widetilde{U}_\beta$ contains a periodic orbit of odd length if and only if $\beta \in (\beta_T, 2)$. Observe that this result in fact implies Theorem 1.2. For completion we provide a short proof that if $\beta \in (\beta_T, 2)$ then $\widetilde{U}_\beta$ contains a periodic orbit of odd length. For any $j \in \mathbb{N}$ there exists $\beta_j \in (\beta_T, 2)$ such that $\alpha(\beta_j) = (1(10)^j)^\infty$. It can be shown that $\beta_j \searrow \beta_T$ as $j \to \infty$. It follows from an application of Lemma 2.3 that for any $j \in \mathbb{N}$ the sequence $(1(10)^{j+1})^\infty$ is contained in $\widetilde{U}_{\beta_j}$. Notice that the periodic block of $(1(10)^{j+1})^\infty$ has odd length. Now for any $\beta \in (\beta_T, 2)$ there exists $\beta_j \in (\beta_T, \beta)$, so by our previous observation and the fact that $\widetilde{U}_{\beta_j} \subseteq \widetilde{U}_\beta$, it follows that $(1(10)^{j+1})^\infty \in \widetilde{U}_\beta$.

In [22] the following useful technical result was proved.

Lemma 2.4. *Let $\beta \in (1, \beta_T]$. If $(\epsilon_i) \in \widetilde{U}_\beta \setminus \{0^\infty, 1^\infty\}$ then (ϵ_i) is simply normal.*

In our proofs we will also require the notion of a Thue-Morse chain and a Thue-Morse interval. We define these now. Let $\omega^0 \in \{0,1\}^*$ be a finite word beginning with zero. We then let $\omega^1 = \omega^0\overline{\omega^0}$. More generally, suppose that ω^k has been defined for some $k \in \mathbb{N}$. We then let $\omega^{k+1} = \omega^k\overline{\omega^k}$. Note that $|\omega^k| \to \infty$ as $k \to \infty$ and ω^{k+1} coincides with ω^k in the first $|\omega^k|$ entries. Consequently, we can consider the componentwise limit of the sequence (ω^k). We denote this infinite sequence by ω^{TM}. We call the sequence (ω^k) a *Thue-Morse chain.* The Thue-Morse sequence is obtained by taking $\omega^0 = 0$. In this case $\omega^{TM} = (\tau_i)_{i=0}^\infty$. Given a $\beta \in (1,2)$ and a Thue-Morse chain (ω^k), we say that the interval

$$I_{\omega^0} := [\pi_\beta((\omega^0)^\infty), \pi_\beta((\omega^{TM}))]$$

is a *Thue-Morse interval* if the following inequalities hold:

$$\pi_\beta((\omega^0)^\infty) < \pi_\beta((\omega^1)^\infty) < \cdots < \pi_\beta((\omega^k)^\infty) < \pi_\beta((\omega^{k+1})^\infty) < \cdots < \pi_\beta(\omega^{TM}).$$

Similarly, we say that the interval

$$J_{\omega^0} := [\pi_\beta(\overline{\omega^{TM}}), \pi_\beta((\overline{\omega^0})^\infty)]$$

is a Thue-Morse interval if the following inequalities hold:

$$\pi_\beta(\overline{\omega^{TM}}) < \cdots < \pi_\beta((\overline{\omega^{k+1}})^\infty) < \pi_\beta((\overline{\omega^{k}})^\infty) < \cdots < \pi_\beta((\overline{\omega^{1}})^\infty) < \pi_\beta((\overline{\omega^{0}})^\infty).$$

Note that I_{ω^0} is a Thue-Morse interval if and only if J_{ω^0} is a Thue-Morse interval. The following proposition will be used to understand the possible itineraries of an x that is mapped into the switch region.

Proposition 2.5. *For any $\beta \in (1, \beta_T]$ there exists a set of words $\{\omega^\theta\}_{\theta\in\Theta}$ such that the following properties are satisfied:*

(1) For each $\theta \in \Theta$ the intervals I_{ω^θ} and J_{ω^θ} are Thue-Morse intervals. Furthermore, the intervals $I_{\omega^\theta}, J_{\omega^\theta}$ with $\theta \in \Theta$ are pairwise disjoint.

(2)
$$\Big[\frac{1}{\beta^2-1}, \frac{1}{\beta}\Big) \setminus \bigcup_{\theta\in\Theta} I_{\omega^\theta} \subseteq U_\beta.$$

(3)
$$\Big(\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1}\Big] \setminus \bigcup_{\theta\in\Theta} J_{\omega^\theta} \subseteq U_\beta.$$

(4) Each ω^θ satisfies
$$\frac{\#\{1 \le i \le |\omega^\theta| : \omega_i^\theta = 0\}}{|\omega^\theta|} = \frac{1}{2}.$$

(5) There exists $C > 0$ such that for any $\theta \in \Theta$ and $1 \le n \le |w^\theta|$
$$\Big|\#\{1 \le i \le n : \omega_i^\theta = 0\} - \#\{1 \le i \le n : \omega_i^\theta = 1\}\Big| \le C.$$

We remark that statements (1), (2), and (3) in Proposition 2.5 in fact hold for any $\beta \in (1, 2)$. Before proving this proposition we recall the following. Let $\mathcal{U}$ be the set of $\beta \in (1, 2]$ such that $1 \in U_\beta$. Then $\beta_{KL} = \min \mathcal{U}$ and its topological closure $\overline{\mathcal{U}}$ is a Cantor set (cf. [27]). Furthermore,

$$\Big[\frac{1+\sqrt{5}}{2}, 2\Big] \setminus \mathcal{U} = \bigcup [\beta_0, \beta_*), \tag{2.1}$$

where the union on the right hand side is countable and pairwise disjoint. Indeed, even the closed intervals $[\beta_0, \beta_*]$ are pairwise disjoint. For each connected component $[\beta_0, \beta_*) \subset [\frac{1+\sqrt{5}}{2}, 2]$ the left endpoint β_0 satisfies that $\alpha(\beta_0)$ is periodic, say $\alpha(\beta_0) = (\alpha_1 \ldots \alpha_m)^\infty$ with period m. Then $m \ge 2$ and $\alpha_m = 0$. The right endpoint β_* is called a *de Vries-Komornik number* in [28] and satisfies $\beta_* \in \mathcal{U}$. The quasi-greedy expansion $\alpha(\beta_*)$ is a Thue-Morse type sequence defined as follows. Let $\alpha^0 = \alpha_1 \ldots \alpha_m$. Then we set $\alpha^1 = \alpha_1 \ldots \alpha_m^+ \overline{\alpha_1 \ldots \alpha_m^+} = (\alpha^0)^+ \overline{(\alpha^0)^+}$. Here for a word $\epsilon_1 \ldots \epsilon_n$ with $\epsilon_n = 0$ we write $\epsilon_1 \ldots \epsilon_n^+ =$

$\epsilon_1 \dots \epsilon_{n-1}(\epsilon_n + 1)$. More generally, suppose α^k has been defined for some $k \geq 0$. Then we set $\alpha^{k+1} = (\alpha^k)^+\overline{(\alpha^k)^+}$. Thus, $\alpha(\beta_*)$ is the component-wise limit of the sequence (α^k). In this case $[\beta_0, \beta_*)$ is called the connected component generated by $\alpha^0 = \alpha_1 \dots \alpha_m$ and denoted by $C_{\alpha_1 \dots \alpha_m} = C_{\alpha^0}$. Note by Lemma 2.2 that for each $k \geq 1$ there exists a unique $\beta_k \in (\beta_0, \beta_*)$ such that $\alpha(\beta_k) = (\alpha^k)^\infty$ (cf. [15]). From the definition of α^k it follows that

$$\alpha(\beta_0) \prec \alpha(\beta_1) \prec \cdots \prec \alpha(\beta_k) \prec \alpha(\beta_{k+1}) \prec \cdots \prec \alpha(\beta_*),$$

and $\alpha(\beta_k)$ converges to $\alpha(\beta_*)$ as $k \to \infty$. By Lemma 2.2 this implies that

$$\beta_0 < \beta_1 < \cdots < \beta_k < \beta_{k+1} < \cdots < \beta_* \quad \text{and} \quad \beta_k \nearrow \beta_* \text{ as } k \to \infty. \tag{2.2}$$

Observe that $\alpha(\frac{1+\sqrt{5}}{2}) = (10)^\infty$. Set $\alpha^0 = 10$. Then $\alpha^1 = 1100, \alpha^2 = 11010010, \dots$, and $\alpha(\beta_{KL})$ is the component-wise limit of the sequence (α^k). So the interval $[\frac{1+\sqrt{5}}{2}, \beta_{KL})$ is indeed the first connected component generated by $\alpha^0 = 10$, i.e., $C_{10} = [\frac{1+\sqrt{5}}{2}, \beta_{KL})$.

Proof of Proposition 2.5. Fix $\beta \in (1, \beta_T]$. Let $\{\alpha^\theta\}_{\theta \in \Theta}$ be the set of words such that for any $\theta \in \Theta$ the connected component $C_{\alpha^\theta} = [\beta_0^\theta, \beta_*^\theta)$ intersects $[\frac{1+\sqrt{5}}{2}, \beta)$. Then by (2.1) it follows that

$$\left[\frac{1+\sqrt{5}}{2}, \beta\right) \setminus \bigcup_{\theta \in \Theta} \overline{C_{\alpha^\theta}} \subseteq \left[\frac{1+\sqrt{5}}{2}, \beta\right) \cap \mathcal{U}, \tag{2.3}$$

where $\overline{C_{\alpha^\theta}} = [\beta_0^\theta, \beta_*^\theta]$ denotes the topological closure of C_{α^θ}. We emphasize that the closed intervals $[\beta_0^\theta, \beta_*^\theta], \theta \in \Theta$ are pairwise disjoint. We first construct for each connected component C_{α^θ} a unique Thue-Morse interval I_{ω^θ}.

Take $\theta \in \Theta$ and let $C_{\alpha^\theta} = [\beta_0^\theta, \beta_*^\theta)$ be the connected component generated by $\alpha^\theta = \alpha^{\theta,0} = \alpha_1 \dots \alpha_m$. Then $\alpha(\beta_0^\theta) = (\alpha^{\theta,0})^\infty = (\alpha_1 \dots \alpha_m)^\infty$ with $\alpha_m = 0$. Furthermore, for each $k \geq 0$ let $\alpha^{\theta,k+1}$ be recursively defined by $\alpha^{\theta,k+1} = (\alpha^{\theta,k})^+\overline{(\alpha^{\theta,k})^+}$. Then for any $k \geq 0$ there exists a unique $\beta_k^\theta \in (\beta_0^\theta, \beta_*^\theta)$ such that $\alpha(\beta_k^\theta) = (\alpha^{\theta,k})^\infty$. So we obtain a sequence of strictly increasing bases (β_k^θ) as described in (2.2). In the following we construct the Thue-Morse chain $(\omega^{\theta,k})$ in terms of the bases (β_k^θ).

Let $(\omega^{\theta,k})$ be the Thue-Morse chain generated by $\omega^\theta = \omega^{\theta,0} = \alpha_m \alpha_1 \dots \alpha_{m-1}$. We claim that

$$(\omega^{\theta,k})^\infty = 0\alpha(\beta_k^\theta) \tag{2.4}$$

for all $k \geq 0$. We will prove the claim by induction on k. First we consider $k = 0$. Note that $\alpha_m = 0$. Then

$$(\omega^{\theta,0})^\infty = 0(\alpha_1 \dots \alpha_m)^\infty = 0(\alpha^{\theta,0})^\infty = 0\alpha(\beta_0^\theta).$$

So (2.4) holds for $k=0$. Now suppose (2.4) holds for some $k\ge 0$. Then

$$(\omega^{\theta,k})^\infty = 0\alpha(\beta_k^\theta) = 0(\alpha^{\theta,k})^\infty. \tag{2.5}$$

Note that the word $\omega^{\theta,k}$ begins with a 0 and the word $\alpha^{\theta,k}$ ends with a 0. Furthermore, the two words $\omega^{\theta,k}$ and $\alpha^{\theta,k}$ have the same length $2^k m$. Then by (2.5) and the definitions of $(\omega^{\theta,i})$, $(\alpha^{\theta,i})$ it follows that

$$(\omega^{\theta,k+1})^\infty = (\omega^{\theta,k}\overline{\omega^{\theta,k}})^\infty = 0((\alpha^{\theta,k})^+\overline{(\alpha^{\theta,k})^+})^\infty = 0(\alpha^{\theta,k+1})^\infty = 0\alpha(\beta_{k+1}^\theta).$$

This implies that (2.4) also holds for $k+1$. By induction this proves the claim. Hence, by (2.4) we conclude that

$$\pi_\beta((\omega^{\theta,k})^\infty) = \pi_\beta(0\alpha(\beta_k^\theta)) \quad \text{for all} \quad k\ge 0. \tag{2.6}$$

Notice that $\beta_k^\theta \nearrow \beta_*^\theta$ as $k\to\infty$. Thus, letting $k\to\infty$ in (2.6) and by Lemma 2.2 it follows that

$$\pi_\beta(\omega^{\theta,TM}) = \pi_\beta(0\alpha(\beta_*^\theta)). \tag{2.7}$$

Note by Lemma 2.2 that the map $q\mapsto\alpha(q)$ is strictly increasing and left continuous. This implies that the following map

$$\Phi_\beta: \quad \left[\frac{1+\sqrt5}{2},\beta\right) \longrightarrow \left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right); \quad q \mapsto \pi_\beta(0\alpha(q))$$

is also strictly increasing and left continuous. Indeed, for any $p,q\in[\frac{1+\sqrt5}{2},\beta)$ with $p<q$, by Lemma 2.2 it follows that

$$\sigma^n(0\alpha(p)) \preccurlyeq \alpha(p) \prec \alpha(\beta), \quad \sigma^n(0\alpha(q)) \preccurlyeq \alpha(q) \prec \alpha(\beta)$$

for all $n\ge 0$. This implies that $0\alpha(p)$ and $0\alpha(q)$ are the lexicographically largest (*greedy*) β-expansions of $\pi_\beta(0\alpha(p))$ and $\pi_\beta(0\alpha(q))$ respectively (cf. [31]). In [31] it is also shown that π_β preserves the lexicographic ordering on $\{0,1\}^{\mathbb N}$ when restricted to the set of greedy β-expansions. Therefore, since $0\alpha(p)\prec 0\alpha(q)$ by Lemma 2.2, it follows that $\Phi_\beta(p) = \pi_\beta(0\alpha(p)) < \pi_\beta(0\alpha(q)) = \Phi_\beta(q)$.

Therefore, by (2.2), (2.6) and (2.7) it follows that

$$\pi_\beta((\omega^{\theta,0})^\infty) < \pi_\beta((\omega^{\theta,1})^\infty) < \cdots < \pi_\beta(\omega^{\theta,TM}).$$

This implies that $I_{\omega^\theta} = [\pi_\beta((\omega^{\theta,0})^\infty), \pi_\beta(\omega^{\theta,TM})]$ is a Thue-Morse interval. Furthermore, by (2.6), (2.7) and the monotonicity of Φ_β it follows that

$$I_{\omega^\theta} = [\Phi_\beta(\beta_0^\theta), \Phi_\beta(\beta_*^\theta)] \quad \text{for any} \quad C_{\alpha^\theta} = [\beta_0^\theta, \beta_*^\theta). \tag{2.8}$$

Note by (2.3) that the closed intervals $\{[\beta_0^\theta, \beta_*^\theta]\}_{\theta\in\Theta}$ are pairwise disjoint. By (2.8) and the monotonicity of Φ_β it follows that the Thue-Morse intervals $\{I_{\omega^\theta}\}_{\theta\in\Theta}$ are also pairwise disjoint. This proves statement (1).

In order to prove (2) we need the following inclusion:

$$\left[\frac{1}{\beta^2-1}, \frac{1}{\beta}\right) \setminus \bigcup_{\theta\in\Theta} I_{\omega^\theta} \subseteq \Phi_\beta\left(\left[\frac{1+\sqrt{5}}{2}, \beta\right) \setminus \bigcup_{\theta\in\Theta} \overline{C_{\alpha^\theta}}\right). \tag{2.9}$$

Note that the function Φ_β is left-continuous. Unfortunately Φ_β is not in general right-continuous, however it is continuous at any point of $\mathcal{U}$ (cf. [27]). For this reason we consider the following continuous function $\Psi_\beta : [\frac{1+\sqrt{5}}{2}, \beta) \to [\frac{1}{\beta^2-1}, \frac{1}{\beta})$ which coincides with Φ_β on $\mathcal{U}$ and is affine on each closed interval $\overline{C_{\alpha^\theta}} = [\beta_0^\theta, \beta_*^\theta]$. To be more precise,

$$\Psi_\beta(q) = \Phi_\beta(q) \quad \text{for any} \quad q \in \left[\frac{1+\sqrt{5}}{2}, \beta\right) \cap \mathcal{U}, \tag{2.10}$$

and

$$\Psi_\beta(q) = \frac{\Phi_\beta(\beta_*^\theta) - \Phi_\beta(\beta_0^\theta)}{\beta_*^\theta - \beta_0^\theta}(q - \beta_0^\theta) + \Phi_\beta(\beta_0^\theta) \quad \text{for any } q \in [\beta_0^\theta, \beta_*^\theta] \tag{2.11}$$

if $\beta \notin (\beta_0^\theta, \beta_*^\theta]$, and

$$\Psi_\beta(q) = \frac{\Phi_\beta(\beta - 0) - \Phi_\beta(\beta_0^\theta)}{\beta - \beta_0^\theta}(q - \beta_0^\theta) + \Phi_\beta(\beta_0^\theta) \quad \text{for any } q \in [\beta_0^\theta, \beta) \tag{2.12}$$

if $\beta \in (\beta_0^\theta, \beta_*^\theta]$. Here $\Phi_\beta(\beta - 0) := \lim_{q\nearrow\beta} \Phi_\beta(q) = \frac{1}{\beta}$ by the left-continuity of Φ_β.

We claim that Ψ_β is continuous and strictly increasing on the interval $[\frac{1+\sqrt{5}}{2}, \beta)$. Clearly, by (2.10)–(2.12) and the monotonicity of Φ_β it follows that Ψ_β is strictly increasing. As for the continuity of Ψ_β we consider the following four cases.

I. $q \in [\frac{1+\sqrt{5}}{2}, \beta) \cap \bigcup_{\theta\in\Theta}(\beta_0^\theta, \beta_*^\theta)$. Then by (2.11) and (2.12) it follows that Ψ_β is continuous at q.

II. $q \in [\frac{1+\sqrt{5}}{2}, \beta) \setminus \bigcup_{\theta\in\Theta}[\beta_0^\theta, \beta_*^\theta]$. Then by (2.3) it follows that $q \in \mathcal{U}$. Furthermore, there exists a sequence $(\beta_0^{\theta_j})_{j=1}^\infty$ with each $\beta_0^{\theta_j}$ the left endpoint of a connected component $C_{\alpha^{\theta_j}}$ such that $\beta_0^{\theta_j} \nearrow q$ as $j \to \infty$. By (2.11), (2.12) and the continuity of Φ_β in $\mathcal{U}$ we obtain

$$\lim_{j\to\infty} \Psi_\beta(\beta_0^{\theta_j}) = \lim_{j\to\infty} \Phi_\beta(\beta_0^{\theta_j}) = \Phi_\beta(q) = \Psi_\beta(q).$$

Since Ψ_β is strictly increasing, this implies that Ψ_β is left-continuous at q. Similarly, we could also find a sequence $(\beta_0^{\tilde{\theta}_k})$ such that $\beta_0^{\tilde{\theta}_k} \searrow q$ as $k \to \infty$. By a similar argument we conclude that Ψ_β is also right-continuous at q.

III. $q=\beta_0^\theta\in[\frac{1+\sqrt{5}}{2},\beta)$. By (2.11) and (2.12) it follows that Ψ_β is right-continuous at q. Furthermore, by (2.3) there exists a sequence $(\beta_0^{\theta_j})_{j=1}^\infty$ such that $\beta_0^{\theta_j}\nearrow q$ as $j\to\infty$. By a similar argument as in Case II we conclude that Ψ_β is also left-continuous at q.

IV $q=\beta_*^\theta\in[\frac{1+\sqrt{5}}{2},\beta)$. By (2.11) and (2.12) it follows that Ψ_β is left-continuous at q. Furthermore, by (2.3) there exists a sequence $(\beta_0^{\tilde\theta_k})_{k=1}^\infty$ such that $\beta_0^{\tilde\theta_k}\searrow q$ as $k\to\infty$. By a similar argument as in Case II we could prove that Ψ_β is also right-continuous at q.

Note by (2.11) and (2.12) that $\Psi_\beta(\frac{1+\sqrt{5}}{2})=\Phi_\beta(\frac{1+\sqrt{5}}{2})=\frac{1}{\beta^2-1}$ and $\lim_{q\nearrow\beta}\Psi_\beta(q)=\Phi_\beta(\beta-0)=\frac{1}{\beta}$. Therefore, by the monotonicity and continuity of Ψ_β it follows that

$$\Psi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\right)=\left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right).\tag{2.13}$$

Furthermore, by (2.8) and (2.11) it follows that if $\beta\notin(\beta_0^\theta,\beta_*^\theta]$ then the interval $[\beta_0^\theta,\beta_*^\theta]$ and the Thue-Morse interval I_{ω^θ} satisfy

$$I_{\omega^\theta}=[\Phi_\beta(\beta_0^\theta),\Phi_\beta(\beta_*^\theta)]=[\Psi_\beta(\beta_0^\theta),\Psi_\beta(\beta_*^\theta)]=\Psi_\beta([\beta_0^\theta,\beta_*^\theta]).\tag{2.14}$$

Similarly, by (2.8) and (2.12) it follows that if $\beta\in(\beta_0^\theta,\beta_*^\theta]$, then the interval $[\beta_0^\theta,\beta)$ and the truncated Thue-Morse interval $I_{\omega^\theta}\cap[\frac{1}{\beta^2-1},\frac{1}{\beta})$ satisfy

$$I_{\omega^\theta}\cap\left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right)=\Psi_\beta([\beta_0^\theta,\beta)).\tag{2.15}$$

Therefore, by (2.10) and (2.13)–(2.15) it follows that

$$\begin{aligned}\left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right)\setminus\bigcup_{\theta\in\Theta}I_{\omega^\theta}&=\Psi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\right)\setminus\bigcup_{\theta\in\Theta}\Psi_\beta([\beta_0^\theta,\beta_*^\theta])\\&\subseteq\Psi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\setminus\bigcup_{\theta\in\Theta}[\beta_0^\theta,\beta_*^\theta]\right)\\&=\Phi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\setminus\bigcup_{\theta\in\Theta}\overline{C_{\alpha^\theta}}\right).\end{aligned}$$

This proves (2.9).

Hence, by (2.3) and (2.9) it follows that

$$\left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right)\setminus\bigcup_{\theta\in\Theta}I_{\omega^\theta}\subseteq\Phi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\setminus\bigcup_{\theta\in\Theta}\overline{C_{\alpha^\theta}}\right)$$
$$\subseteq\Phi_\beta\left(\left[\frac{1+\sqrt{5}}{2},\beta\right)\cap\mathcal{U}\right)\subseteq U_\beta,$$

where the last inclusion holds by the following observation. Note that for any $q\in[\frac{1+\sqrt{5}}{2},\beta)\cap\mathcal{U}$ the quasi-greedy expansion $\alpha(q)\in\widetilde{U}_q$. Since $\alpha(q)\prec\alpha(\beta)$, by Lemma 2.3 it follows that $0\alpha(q)\in\widetilde{U}_\beta$, and hence $\Phi_\beta(q)\in U_\beta$. This proves statement (2).

Observe by symmetry that

$$\left(\frac{1}{\beta(\beta-1)},\frac{\beta}{\beta^2-1}\right]\setminus\bigcup_{\theta\in\Theta}J_{\omega^\theta}=\frac{1}{\beta-1}-\left(\left[\frac{1}{\beta^2-1},\frac{1}{\beta}\right)\setminus\bigcup_{\theta\in\Theta}I_{\omega^\theta}\right),$$

and $\frac{1}{\beta-1}-U_\beta=U_\beta$. Therefore statement (3) follows from statement (2). Note that each word ω^θ corresponds to a unique connected component $C_{\alpha^\theta}=[\beta_0^\theta,\beta_*^\theta)$. By (2.4) we have $(\omega^\theta)^\infty=0\alpha(\beta_0^\theta)$. For the first connected component $[\frac{1+\sqrt{5}}{2},\beta_{KL})$ we have $\alpha(\frac{1+\sqrt{5}}{2})=(10)^\infty$, and then statement (4) holds in this case since $(\omega^\theta)=(01)^\infty$. For the other connected components $C_{\alpha^\theta}=[\beta_0^\theta,\beta_*^\theta]$ we have $\beta_0^\theta\in[\frac{1+\sqrt{5}}{2},\beta)\cap\overline{\mathcal{U}}$. Hence, by using $\alpha(\beta_0^\theta)\prec\alpha(\beta)$ in Lemma 2.3 we conclude that (cf. [27])

$$(\omega^\theta)^\infty=0\alpha(\beta_0^\theta)\in\widetilde{U}_\beta.$$

So statement (4) follows from Lemma 2.4. Finally statement (5) follows from the proof of [22, Lemma 2.3]. It is a consequence of the proof of this lemma that every ω^θ is the concatenation of words from the sets

$$\{1(10)^j0:j=0,1,\ldots\}\quad\text{and}\quad\{0(01)^j1:j=0,1,\ldots\}.$$

Consequently we can take the constant $C=2$. □

We emphasise that the C appearing in property (5) from Proposition 2.5 is a uniform bound over all $\theta\in\Theta$ and n. Note it follows from the construction of the Thue-Morse chain that every word $\omega^{\theta,k}$ appearing in the Thue-Morse chain $(\omega^{\theta,k})$ also satisfies

$$\frac{\#\{1\le i\le|\omega^{\theta,k}|:\omega_i^{\theta,k}=0\}}{|\omega^{\theta,k}|}=\frac{1}{2}$$

and

$$\left|\#\{1\le i\le n:\omega_i^{\theta,k}=0\}-\#\{1\le i\le n:\omega_i^{\theta,k}=1\}\right|\le C$$

for all $\theta \in \Theta$ and $1 \leq n \leq |w^{\theta,k}|$. Moreover, it is a straightforward exercise to show that

$$\lim_{n\to\infty} \frac{\#\{1 \leq i \leq n : \omega_i^{\theta,TM} = 0\}}{n} = \frac{1}{2}.$$

We also highlight the following equalities. To a finite sequence $\omega = (\omega_1, \ldots, \omega_n) \in \{0,1\}^n$ we associate the concatenation of maps $T_\omega := (T_{\omega_n} \circ \cdots \circ T_{\omega_1})$. The following holds for any $\beta \in (1,2)$ and Thue-Morse chain (ω^k):

$$T_{\omega^k}(\pi_\beta((\omega^k)^\infty)) = \pi_\beta((\omega^k)^\infty) \quad \text{and} \quad T_{\overline{\omega^k}}(\pi_\beta((\overline{\omega^k})^\infty)) = \pi_\beta((\overline{\omega^k})^\infty) \tag{2.16}$$

for all ω^k. Moreover using $\omega^{k+1} = \omega^k\overline{\omega^k}$ it follows that

$$T_{\omega^k}(\pi_\beta((\omega^{k+1})^\infty)) = \pi_\beta((\overline{\omega^{k+1}})^\infty), \tag{2.17}$$

and

$$T_{\overline{\omega^k}}(\pi_\beta((\overline{\omega^{k+1}})^\infty)) = \pi_\beta((\omega^{k+1})^\infty) \tag{2.18}$$

for all ω^k.

3. Proof of Theorem 1.3

Equipped with the preliminaries detailed in Section 2, we are now in a position to prove Theorem 1.3. We split our proof into two parameter spaces. Note by Theorem 1.1 it suffices to consider the interval $[\beta_{KL}, \beta_T]$. First we examine the case where $\beta \in [\beta_{KL}, \beta_T)$ before moving on to the specific case where $\beta = \beta_T$. Our proof in either case involves splitting S_β into a left interval, a centre interval, and a right interval (see Figure 2 for $\beta \in [\beta_{KL}, \beta_T)$ and Figure 3 for $\beta = \beta_T$). Loosely speaking, in our proofs we will see that if a point is contained in the left interval or the right interval, then there is a specific sequence of transformations that map our point back into S_β, where importantly the frequency of T_0's within these maps is approximately $1/2$. If a point is contained in the centre interval then we have a choice between a sequence of maps that increase the frequency of T_0's and map our point back to S_β, or a sequence of maps that decrease the frequency of T_0's and map our point back to S_β. Importantly, in this case we will have strong bounds on how much the frequency can change. In each case we return to S_β. By carefully choosing which maps we perform we can construct the desired simply normal expansion.

Proof of Theorem 1.3 for $\beta \in [\beta_{KL}, \beta_T)$. Fix $\beta \in [\beta_{KL}, \beta_T)$. Let us start by making several observations. First of all, by Lemma 2.4 it suffices to prove that every $x \in S_\beta$ has a simply normal β-expansion. What is more, it is a consequence of Lemma 2.4 that one may assume that there exists no $a \in \{T_0, T_1\}^*$ such that

$$a(x) \in U_\beta \setminus \Big\{0, \frac{1}{\beta-1}\Big\}. \tag{3.1}$$

Similarly, adopting the notation used in Proposition 2.5, one may assume that there exists no $a \in \{T_0, T_1\}^*$ such that

$$a(x) \in \bigcup_{\theta\in\Theta}\bigcup_{k=0}^{\infty}\{\pi_\beta((\omega^{\theta,k})^\infty), \pi_\beta((\overline{\omega^{\theta,k}})^\infty)\} \cup \bigcup_{\theta\in\Theta}\{\pi_\beta(\omega^{\theta,TM}), \pi_\beta(\overline{\omega^{\theta,TM}})\}. \tag{3.2}$$

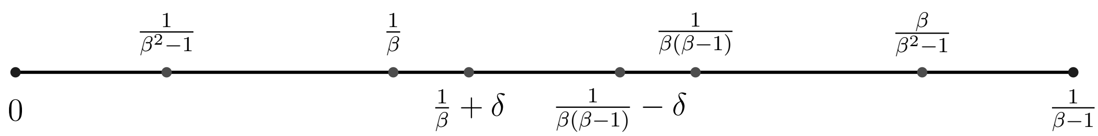

FIGURE 2. The attractor $O_\beta = [\frac{1}{\beta^2-1}, \frac{\beta}{\beta^2-1}]$. The switch region $S_\beta = [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$ is partitioned into three subintervals by the two points $\frac{1}{\beta}+\delta$ and $\frac{1}{\beta(\beta-1)}-\delta$.

It is a consequence of $\beta \in [\beta_{KL}, \beta_T)$ that

$$(T_1\circ T_0)\Big(\frac{1}{\beta}\Big) \in \Big(\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1}\Big) \text{ and } (T_0\circ T_1)\Big(\frac{1}{\beta(\beta-1)}\Big) \in \Big(\frac{1}{\beta^2-1}, \frac{1}{\beta}\Big).$$

Therefore, there exists $\delta(\beta) := \delta > 0$ such that if

$$x \in \Big[\frac{1}{\beta}, \frac{1}{\beta}+\delta\Big) \text{ then } (T_1\circ T_0)(x) \in \Big(\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1}\Big), \tag{3.3}$$

and if

$$x \in \Big(\frac{1}{\beta(\beta-1)}-\delta, \frac{1}{\beta(\beta-1)}\Big] \text{ then } (T_0\circ T_1)(x) \in \Big(\frac{1}{\beta^2-1}, \frac{1}{\beta}\Big). \tag{3.4}$$

We also recall from [6] that there exists a parameter $K := K(\beta) \in \mathbb{N}$ such that if

$$x \in \Big[\frac{1}{\beta}+\delta, \frac{1}{\beta(\beta-1)}-\delta\Big] \text{ then } (T_1^j\circ T_0)(x) \in O_\beta \tag{3.5}$$

for some $1 \leq j \leq K$. Similarly, for the same parameter K if

$$x \in \Big[\frac{1}{\beta}+\delta, \frac{1}{\beta(\beta-1)}-\delta\Big] \text{ then } (T_0^j\circ T_1)(x) \in O_\beta \tag{3.6}$$

for some $1 \leq j \leq K$. The existence of the K appearing in (3.5) and (3.6) is essentially a consequence of the fact that T_0 and T_1 scale distances between arbitrary points and their unique fixed points by a factor β.

Equipped with the above observations we now fix an $x \in S_\beta$ and describe an algorithm which constructs an element of $\Omega_\beta(x)$ that corresponds to a simply normal expansion via Lemma 2.1. As mentioned above it is useful to partition S_β into three intervals (see Figure 2). This we do now.

Case 1. If $x \in [\frac{1}{\beta}, \frac{1}{\beta} + \delta)$ then

$$(T_1 \circ T_0)(x) \in \Big(\frac{1}{\beta(\beta-1)}, \frac{\beta}{\beta^2-1}\Big)$$

by (3.3). By our assumptions we know that $(T_1 \circ T_0)(x) \notin U_\beta$. Therefore by Proposition 2.5 we have $(T_1 \circ T_0)(x) \in J_{\omega^{\theta_1}}$ for some $\theta_1 \in \Theta$. By (3.2) we know that

$$\pi_\beta((\overline{\omega^{\theta_1,k_1+1}})^\infty) < (T_1 \circ T_0)(x) < \pi_\beta((\overline{\omega^{\theta_1,k_1}})^\infty)$$

for some $k_1 \geq 0$. By (2.16) we know that $\pi_\beta((\overline{\omega^{\theta_1,k_1}})^\infty)$ is the unique fixed point of the map $T_{\overline{\omega^{\theta_1,k_1}}}$. Importantly this map expands distances by a factor $\beta^{|\overline{\omega^{\theta_1,k_1}}|}$. Therefore it follows from the monotonicity of our maps and (2.16)–(2.18) that there must exist $n_1 \in \mathbb{N}$ such that

$$T^{n_1}_{\overline{\omega^{\theta_1,k_1}}}\big((T_1 \circ T_0)(x)\big) \in [\pi_\beta((\omega^{\theta_1,k_1+1})^\infty), \pi_\beta((\overline{\omega^{\theta_1,k_1+1}})^\infty)].$$

Here $T^{n_1}_{\overline{\omega^{\theta_1,k_1}}}$ stands for the n_1 times composition of the map $T_{\overline{\omega^{\theta_1,k_1}}}$. In the above inclusion it is not important that this image point is contained in this particular interval parameterized by ω^{θ_1,k_1+1}. What is important is that it is contained in O_β. This means we can reuse Proposition 2.5.

At this point in our algorithm we stop and consider where

$$T^{n_1}_{\overline{\omega^{\theta_1,k_1}}}\big((T_1 \circ T_0)(x)\big)$$

lies within O_β. If it is contained in S_β we stop and let

$$a^1 := (T_0, T_1, (T_{\overline{\omega^{\theta_1,k_1}}})^{n_1}).$$

If this image is not contained in S_β, then we know by (3.1) and Proposition 2.5 that it must be contained in a Thue-Morse interval. In which case, repeating the above argument, there must exist θ_2, k_2, and n_2 such that

$$T^{n_2}_{\omega^{\theta_2,k_2}}\big(T^{n_1}_{\overline{\omega^{\theta_1,k_1}}}((T_1 \circ T_0)(x))\big) \in O_\beta \quad \text{or} \quad T^{n_2}_{\overline{\omega^{\theta_2,k_2}}}\big(T^{n_1}_{\overline{\omega^{\theta_1,k_1}}}((T_1 \circ T_0)(x))\big) \in O_\beta.$$

If this image point is in S_β we stop and let

$$a^1 := (T_0, T_1, (T_{\overline{\omega^{\theta_1,k_1}}})^{n_1}, (T_{\omega^{\theta_2,k_2}})^{n_2}) \quad \text{or} \quad a^1 := (T_0, T_1, (T_{\overline{\omega^{\theta_1,k_1}}})^{n_1}, (T_{\overline{\omega^{\theta_2,k_2}}})^{n_2})$$

accordingly. We can repeat this process indefinitely. If we are never mapped into the switch region then it follows from Proposition 2.5 properties (4) and (5) that we've constructed an element of $\Omega_\beta(x)$ with limiting frequency of zeros 1/2. Which by Lemma 2.1 proves our result. Alternatively, if this process eventually maps x into S_β, then the corresponding sequence $a^1 \in \{T_0, T_1\}^*$ satisfies $a^1(x) \in S_\beta$ and has the following useful properties as a consequence of Proposition 2.5:

$$|a^1|_1 = |a^1|_0$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^1 = T_0\} - \#\{1 \leq i \leq n : a_i^1 = T_1\}\Big| \leq C$$

for all $1 \leq n \leq |a^1|$.

Case 2. The case where $x \in (\frac{1}{\beta(\beta-1)} - \delta, \frac{1}{\beta(\beta-1)}]$ is handled in the same way as Case 1. The difference being in this case, instead of intially applying the map $T_1 \circ T_0$ we apply $T_0 \circ T_1$. Our orbit then travels through successive Thue-Morse intervals before landing in the switch region S_β, or our image never maps into S_β and then we have immediately constructed a simply normal expansion. In the first case we construct a sequence $a^1 \in \{T_0, T_1\}^*$ which satisfies $a^1(x) \in S_\beta$,

$$|a^1|_1 = |a^1|_0$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^1 = T_0\} - \#\{1 \leq i \leq n : a_i^1 = T_1\}\Big| \leq C$$

for all $1 \leq n \leq |a^1|$.

Case 3. When $x \in [\frac{1}{\beta} + \delta, \frac{1}{\beta(\beta-1)} - \delta]$ we can initially apply T_0 or T_1. By (3.5) and (3.6) we then successively apply either T_1 or T_0 until $(T_1^j \circ T_0)(x) \in O_\beta$ or $(T_0^j \circ T_1)(x) \in O_\beta$. Once x is mapped into O_β we then proceed as in Case 1. We travel through successive Thue-Morse intervals before being eventually mapped into S_β, or x is never mapped into S_β and we then automatically have a simply normal expansion. In the case where we initially apply T_0, by (3.5) we will have constructed a sequence $a^1 \in \{T_0, T_1\}^*$ that satisfies $a^1(x) \in S_\beta$,

$$0 \leq |a^1|_1 - |a^1|_0 \leq K, \tag{3.7}$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^1 = T_0\} - \#\{1 \leq i \leq n : a_i^1 = T_1\}\Big| \leq C + K$$

for all $1 \leq n \leq |a^1|$. If we initially applied T_1, then by (3.6) we will have constructed a sequence $a^1 \in \{T_0, T_1\}^*$ that satisfies $a^1(x) \in S_\beta$,

$$-K \leq |a^1|_1 - |a^1|_0 \leq 0, \tag{3.8}$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^1 = T_0\} - \#\{1 \leq i \leq n : a_i^1 = T_1\}\Big| \leq C + K$$

for all $1 \leq n \leq |a^1|$.

Now suppose we've constructed a finite sequence $a^m \in \{T_0, T_1\}^*$ such that $a^m(x) \in S_\beta$,

$$\big||a^m|_1 - |a^m|_0\big| \leq K, \tag{3.9}$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^m = T_0\} - \#\{1 \leq i \leq n : a_i^m = T_1\}\Big| \leq C + K \tag{3.10}$$

for all $1 \leq n \leq |a^m|$. We now construct a sequence a^{m+1} that has a^m as a prefix and satisfies (3.9) and (3.10). If $a^m(x) \in [\frac{1}{\beta}, \frac{1}{\beta} + \delta)$ or $a^m(x) \in (\frac{1}{\beta(\beta-1)} - \delta, \frac{1}{\beta(\beta-1)}]$ then we repeat the arguments as in Case 1 or Case 2 respectively. In either case we construct a sequence of transformations $a^{m+1} \in \{T_0, T_1\}^*$ that begins with a^m and satisfies $a^{m+1}(x) \in S_\beta$,

$$\big||a^{m+1}|_1 - |a^{m+1}|_0\big| \leq K,$$

and

$$\Big|\#\{1 \leq i \leq n : a_i^{m+1} = T_0\} - \#\{1 \leq i \leq n : a_i^{m+1} = T_1\}\Big| \leq C + K$$

for all $1 \leq n \leq |a^{m+1}|$. If $a^m(x) \in [\frac{1}{\beta}+\delta, \frac{1}{\beta(\beta-1)}-\delta]$ then we consider the sign of $|a^m|_1 - |a^m|_0$. If $0 \leq |a^m|_1 - |a^m|_0 \leq K$ then we repeat the arguments given in Case 3 when we initally apply T_1. In this case (3.8) guarantees that

$$\big||a^{m+1}|_1 - |a^{m+1}|_0\big| \leq K.$$

We also have $a^{m+1}(x) \in S_\beta$ and

$$\Big|\#\{1 \leq i \leq n : a_i^{m+1} = T_0\} - \#\{1 \leq i \leq n : a_i^{m+1} = T_1\}\Big| \leq C + K$$

for all $1 \leq n \leq |a^{m+1}|$. If $|a^m|_1 - |a^m|_0$ is negative then we repeat the above argument except we use Case 3 where we first apply T_0.

Clearly we can repeat the above steps indefinitely. In doing so we construct an infinite sequence in $\Omega_\beta(x)$. It is a consequence of (3.10) that this sequence has the desired frequency. Therefore by Lemma 2.1 we know that x has a simply normal expansion. □

Proof of Theorem 1.3 for $\beta = \beta_T$. We start with an observation. For any $J \in \mathbb{N}$ there exists $\delta_J > 0$ such that if $x \in [\frac{1}{\beta_T}, \frac{1}{\beta_T} + \delta_J)$ then

$$((T_1 \circ T_0)^J \circ T_1^2 \circ T_0)(x) \in O_{\beta_T}. \tag{3.11}$$

This is because $(T_1^2 \circ T_0)(\frac{1}{\beta_T}) = \frac{1}{\beta_T^2-1}$. Similarly, if $x \in (\frac{1}{\beta_T(\beta_T-1)} - \delta_J, \frac{1}{\beta_T(\beta_T-1)}]$ then

$$((T_0 \circ T_1)^J \circ T_0^2 \circ T_1)(x) \in O_{\beta_T}. \tag{3.12}$$

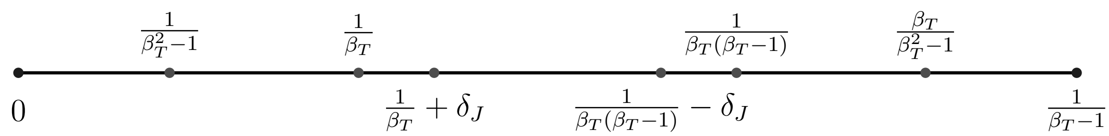

FIGURE 3. The attractor $O_{\beta_T} = [\frac{1}{\beta_T^2-1}, \frac{\beta_T}{\beta_T^2-1}]$. The switch $S_{\beta_T} = [\frac{1}{\beta_T}, \frac{1}{\beta_T(\beta_T-1)}]$ is partitioned into three subintervals by the two points $\frac{1}{\beta_T}+\delta_J$ and $\frac{1}{\beta_T(\beta_T-1)} - \delta_J$.

Moreover, for each $J \in \mathbb{N}$ there exists $K_J \in \mathbb{N}$ such that if $x \in [\frac{1}{\beta_T} + \delta_J, \frac{1}{\beta_T(\beta_T-1)} - \delta_J]$, then

$$(T_1^i \circ T_0)(x) \in O_{\beta_T} \tag{3.13}$$

for some $1 \leq i \leq K_J$, and

$$(T_0^i \circ T_1)(x) \in O_{\beta_T} \tag{3.14}$$

for some $1 \leq i \leq K_J$. As in our proof for $\beta \in [\beta_{KL}, \beta_T)$ it is useful to partition S_{β_T} into three intervals (see Figure 3). This time however our partition will depend upon J.

Case 1. If $x \in [\frac{1}{\beta_T}, \frac{1}{\beta_T} + \delta_J)$ then by (3.11) we know that $((T_1 \circ T_0)^J \circ T_1^2 \circ T_0)(x) \in O_{\beta_T}$. Repeating arguments given in our proof for $\beta \in [\beta_{KL}, \beta_T)$, we may assume that we may concatenate $(T_0, T_1, T_1, (T_0, T_1)^J)$ with a sequence of maps that map x back into S_{β_T}, and satisfy Properties (4) and (5) of Proposition 2.5. Letting $a \in \{T_0, T_1\}^*$ be the concatenation of $(T_0, T_1, T_1, (T_0, T_1)^J)$ with this second sequence of maps, we can assert by Proposition 2.5 and (3.11) that $a(x) \in S_{\beta_T}$ and

$$\frac{\#\{1 \leq i \leq |a| : a_i = T_0\}}{|a|} \in \Big[\frac{J+1}{2J+3}, \frac{1}{2}\Big] \tag{3.15}$$

Case 2. If $x \in (\frac{1}{\beta_T(\beta_T-1)} - \delta_J, \frac{1}{\beta_T(\beta_T-1)}]$, then by (3.12) and a similar analysis to that done in Case 1, except this time first applying the sequence of maps $(T_1, T_0, T_0, (T_1, T_0)^J)$, implies the existence of a sequence $a \in \{T_0, T_1\}^*$ such that $a(x) \in S_{\beta_T}$ and

$$\frac{\#\{1 \leq i \leq |a| : a_i = T_0\}}{|a|} \in \Big[\frac{1}{2}, \frac{J+2}{2J+3}\Big]. \tag{3.16}$$

Case 3. If $x \in [\frac{1}{\beta_T} + \delta_J, \frac{1}{\beta_T(\beta_T - 1)} - \delta_J]$ then by (3.13) and (3.14) we know that $(T_1^i \circ T_0)(x) \in O_{\beta_T}$ for some $1 \leq i \leq K_J$, and $(T_0^j \circ T_1)(x) \in O_{\beta_T}$ for some $1 \leq j \leq K_J$. Repeating the arguments given in Case 3 of our proof for $\beta \in [\beta_{KL}, \beta_T)$ where we appealed to Proposition 2.5, we may assert that for such an x there exists a sequence $a \in \{T_0, T_1\}^*$ such that $a(x) \in S_{\beta_T}$ and a satisfies

$$0 \leq |a|_1 - |a|_0 \leq K_J \tag{3.17}$$

if we initially applied T_0, or if we initially applied T_1 then a satisfies

$$-K_J \leq |a|_1 - |a|_0 \leq 0. \tag{3.18}$$

Having described the maps we can perform in each of the three subintervals of S_{β_T}, let us now fix an $x \in S_\beta$. Moreover, let $\varepsilon_n = n^{-1}$ and let (J_n) be a strictly increasing sequence of natural numbers such that

$$\frac{1}{2} - \varepsilon_n < \frac{J_n + 1}{2J_n + 3} \quad \text{and} \quad \frac{J_n + 2}{2J_n + 3} < \frac{1}{2} + \varepsilon_n \tag{3.19}$$

for all $n \geq 1$.

We now show how to construct a simply normal expansion of $x \in S_{\beta_T}$. By repeatedly applying the maps detailed in Cases 1, 2, and 3, we can construct an arbitrarily long sequence of maps a^1 that satisfies $a^1(x) \in S_{\beta_T}$ and

$$\frac{\#\{1 \leq i \leq |a^1| : a_i^1 = T_0\}}{|a^1|} \in \Big(\frac{1}{2} - \varepsilon_1, \frac{1}{2} + \varepsilon_1\Big). \tag{3.20}$$

To construct such an a^1 the strategy is as follows. Consider the partition of S_{β_T} given by J_1. If our point is mapped into either of the intervals described in Cases 1 and 2 then we always perform the sequence of maps that satisfy (3.15) or (3.16). If we are mapped into the interval covered by Case 3 we have a choice. If the number of T_0's appearing in the sequence of maps we have constructed so far exceeds the number of T_1's, then we apply the sequence of maps corresponding to (3.17). If the number of T_1's appearing in the sequence of maps we have constructed so far exceeds the number of T_0's, then we apply the sequence of maps corresponding to (3.18). Since each of the sequences of maps described in Cases 1, 2, and 3 map us back into S_{β_T}, we can clearly repeat this process indefinitely. Since the maps described by Case 3 increase or decrease the difference between the number of T_0's and T_1's by at most K_{J_1}, it follows that any sufficiently large sequence of maps constructed using the above steps satisfies $a^1(x) \in S_{\beta_T}$ and (3.20) by (3.19)

Now we repeat the same process but with x replaced by $a^1(x)$ and J_1 replaced by J_2. We may assert that there exists a^2 that extends a^1 such that $a^2(x) \in S_{\beta_T}$, and

$$\frac{\#\{1 \leq i \leq |a^2| : a_i^2 = T_0\}}{|a^2|} \in \Big(\frac{1}{2} - \varepsilon_2, \frac{1}{2} + \varepsilon_2\Big)$$

by (3.19). It is a consequence of property (5) of Proposition 2.5, and the fact that a^1 may be made arbitrarily long, that we may also assume that a^2 satisfies

$$\frac{\#\{1 \leq i \leq n : a_i^2 = T_0\}}{n} \in \Big(\frac{1}{2} - 2\varepsilon_1, \frac{1}{2} + 2\varepsilon_1\Big)$$

for all $|a^1| \leq n < |a^2|$. Importantly a^2 can also be made to be arbitrarily long.

Now assume that we have constructed $a^1, \ldots, a^N$ such that a^N is arbitrarily long, $a^N(x) \in S_{\beta_T}$,

$$\frac{\#\{1 \leq i \leq |a^N| : a_i^N = T_0\}}{|a^N|} \in \Big(\frac{1}{2} - \varepsilon_N, \frac{1}{2} + \varepsilon_N\Big),$$

and for all $|a^j| \leq n < |a^{j+1}|$ with $1 \leq j < N$ we have

$$\frac{\#\{1 \leq i \leq n : a_i^N = T_0\}}{n} \in \Big(\frac{1}{2} - 2\varepsilon_j, \frac{1}{2} + 2\varepsilon_j\Big). \tag{3.21}$$

By repeating the above arguments, this time considering $a^N(x)$ and J_{N+1}, we may construct an arbitrarily long sequence a^{N+1} that extends a^N and satisfies $a^{N+1}(x) \in S_{\beta_T}$,

$$\frac{\#\{1 \leq i \leq |a^{N+1}| : a_i^{N+1} = T_0\}}{|a^{N+1}|} \in \Big(\frac{1}{2} - \varepsilon_{N+1}, \frac{1}{2} + \varepsilon_{N+1}\Big),$$

and for all $|a^N| \leq n < |a^{N+1}|$ we have

$$\frac{\#\{1 \leq i \leq n : a_i^{N+1} = T_0\}}{n} \in \Big(\frac{1}{2} - 2\varepsilon_N, \frac{1}{2} + 2\varepsilon_N\Big). \tag{3.22}$$

Continuing indefinitely we construct an element of $\Omega_{\beta_T}(x)$. This sequence corresponds to a simply normal expansion by Lemma 2.1, (3.21), and (3.22). □

4. Non-simply normal numbers and examples

For $\beta \in (1, 2]$ let

$$\mathcal{N}_\beta := \left\{x \in \Big(0, \frac{1}{\beta - 1}\Big) : x \text{ does not have a simply normal } \beta\text{-expansion}\right\}.$$

By Theorems 1.2 and 1.3 it follows that $\mathcal{N}_\beta = \emptyset$ for any $\beta \in (1, \beta_T]$, and $\mathcal{N}_\beta \neq \emptyset$ for any $\beta \in (\beta_T, 2]$. Indeed, by [22, Lemma 2.3] it follows that $\dim_H \mathcal{N}_\beta > 0$ for any $\beta \in (\beta_T, 2]$. In [6] the first author showed that $\dim_H \mathcal{N}_\beta \to 1$ as $\beta \to 2$. Furthermore, when $\beta = 2$ it is a consequence of the well known work of Besicovich and Eggleston [10, 17], and Borel

[11], that $\mathcal{N}_2$ is a Lebesgue null set of full Hausdorff dimension. In the following theorem we show that the set $\mathcal{N}_\beta$ is indeed a Lebesgue null set for all $\beta \in (1,2)$.

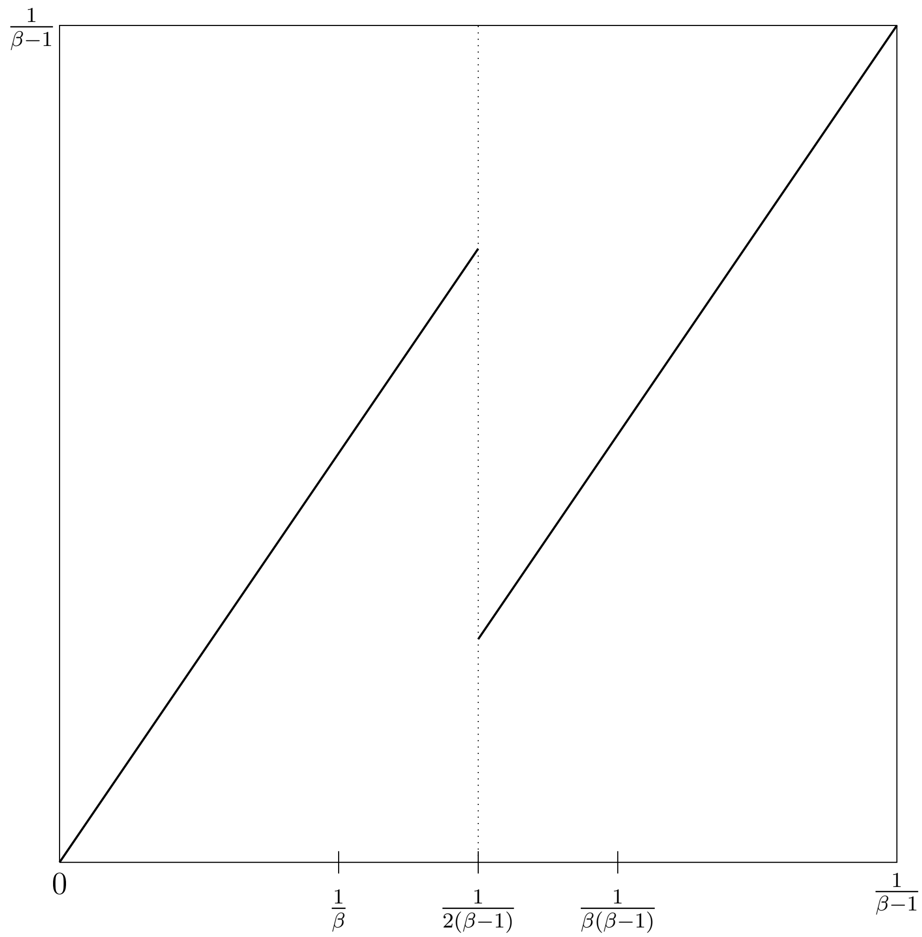

FIGURE 4. The graph of M_β

Theorem 4.1. *Let $\beta \in (1,2)$. Then Lebesgue almost every $x \in I_\beta$ has a simply normal β-expansion.*

Proof. Consider the following map $M_\beta : I_\beta \to I_\beta$:

$$M_\beta(x) = \begin{cases} T_0(x) & \text{if } x \in [0, \frac{1}{2(\beta-1)}) \\ T_1(x) & \text{if } x \in [\frac{1}{2(\beta-1)}, \frac{1}{\beta-1}]. \end{cases}$$

We include a graph of the function M_β in Figure 4. One can verify that the map M_β eventually maps elements of $(0, \frac{1}{\beta-1})$ into the interval

$$A_\beta := \Big[\frac{\beta}{2(\beta-1)} - 1, \frac{\beta}{2(\beta-1)}\Big].$$

Moreover, once an element is mapped into A_β it is never mapped out. The map M_β is a piecewise linear expanding map, so we can employ the results of [29] and [30] to assert that there exists a unique M_β-invariant probability measure which is ergodic and absolutely continuous with respect to the Lebesgue measure. We call this measure μ. We remark that as long as x is never mapped onto the discontinuity point of M_β then the following equality holds for all $n \in \mathbb{N}$:

$$M_\beta^n(x) = \frac{1}{\beta-1} - M_\beta^n\Big(\frac{1}{\beta-1} - x\Big). \tag{4.1}$$

In [29] the author gives an explicit formula for the density of μ. We do not state this formula here but merely remark that it is strictly positive on A_β. This observation implies that there exists $x^* \in A_\beta$ such that its orbit under M_β equidistributes in A_β with respect to μ, and the orbit of $\frac{1}{\beta-1} - x^*$ also equidistributes in A_β with respect to μ. Without loss of generality we may also assume that x^* is not a preimage of the discontinuity point of M_β. Therefore, by the Birkhoff ergodic theorem and (4.1) we have

$$\begin{aligned}
\mu\Big(\Big[0, \frac{1}{2(\beta-1)}\Big]\Big) &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[0, \frac{1}{2(\beta-1)}]} M_\beta^k(x^*) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_{[\frac{1}{2(\beta-1)}, \frac{1}{\beta-1}]} M_\beta^k\left(\frac{1}{\beta-1} - x^*\right) \\
&= \mu\Big(\Big[\frac{1}{2(\beta-1)}, \frac{1}{\beta-1}\Big]\Big).
\end{aligned}$$

It follows therefore that $\mu([0, \frac{1}{2(\beta-1)}]) = \mu([\frac{1}{2(\beta-1)}, \frac{1}{\beta-1}]) = 1/2$. Recall that we perform the map T_0 whenever an image point is in the interval $[0, \frac{1}{2(\beta-1)})$, and we perform the map T_1 whenever our point is within the interval $[\frac{1}{2(\beta-1)}, \frac{1}{\beta-1}]$. Consequently, by Lemma 2.1 and the Birkhoff ergodic theorem, μ almost every x has a simply normal β-expansion. Since μ has strictly positive density on A_β, it follows that Lebesgue almost every $x \in A_\beta$ has a simply normal β-expansion. Extending this statement to Lebesgue almost every $x \in I_\beta$ follows by considering preimages. □

Until now the only elements we know in $\mathcal{N}_\beta$ are numbers with a unique β-expansion. In the following we construct examples which show that there also exist $\beta \in (\beta_T, 2]$ and

$x \in (0, \frac{1}{\beta-1})$, such that x has precisely k different β-expansions, and none of them are simply normal, where $k = 2, 3, \ldots, \aleph_0$ or $2^{\aleph_0}$. The following example was motivated by Erdős and Joó [19].

Example 4.2. Let $\beta \approx 1.92756$ be a multinacci number which is the root of $\beta^4 - \beta^3 - \beta^2 - \beta - 1 = 0$. Then $\alpha(\beta) = (1110)^\infty$. We claim that for any $k \geq 1$

$$x_k := \pi_\beta(01^{4k-1}(011)^\infty)$$

has precisely k different β-expansions. We will prove this by induction on k.

When $k = 1$ we have $x_1 = \pi_\beta(01^3(011)^\infty)$. Then

$$\overline{\alpha(\beta)} = (0001) \prec \sigma^n(01^3(011)^\infty) \prec (1110)^\infty = \alpha(\beta)$$

for all $n \geq 0$. By Lemma 2.3 it follows that $x_1 \in U_\beta$. Now suppose x_k has precisely k different β-expansions. We consider x_{k+1}. Since $\pi_\beta(10^\infty) = \pi_\beta(01^4 0^\infty)$, we have the word substitution $10^4 \sim 01^4$. So,

$$x_{k+1} = \pi_\beta(01^{4k+3}(011)^\infty) = \pi_\beta(10^4 1^{4k-1}(011)^\infty) = \frac{1}{\beta} + \frac{x_k}{\beta^4}.$$

By the inductive hypothesis it follows that x_{k+1} has at least $k+1$ different β-expansions: one is $01^{4k+3}(011)^\infty$ and the others begin with 10^3. Furthermore, one can verify that x_{k+1} has precisely $k+1$ different β-expansions by verifying that $T_0(x_{k+1}) \in U_\beta$ and $(T_0^i \circ T_1)(x_{k+1}) \notin [\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$ for all $i \in \{0, 1, 2, 3\}$.

Therefore, x_k has precisely k-different β-expansions, all of which end with $(011)^\infty$. Therefore, all β-expansions of x_k are not simply normal. Letting $k \to \infty$ we conclude that $x_\infty = \pi_\beta(01^\infty)$ has a countable infinity of β-expansions, all of which end with 1^∞, i.e., all β-expansions of x_∞ are not simply normal.

Now we construct an example of an x which has a continuum of β-expansions, none of which are simply normal.

Example 4.3. Let $\beta \approx 1.84408$ be the unique root in $(1, 2]$ of

$$\pi_\beta((10^3(110)^4)^\infty) = \pi_\beta((01^5)^\infty).$$

By observing the substitution $10^3(110)^4 \sim 01^5$ it follows that $x = \pi_\beta((01^5)^\infty) \approx 0.628296$ has a continuum of β-expansions. We claim that all β-expansions of x are of the form

$$\mathbf{x}_1\mathbf{x}_2\cdots, \tag{4.2}$$

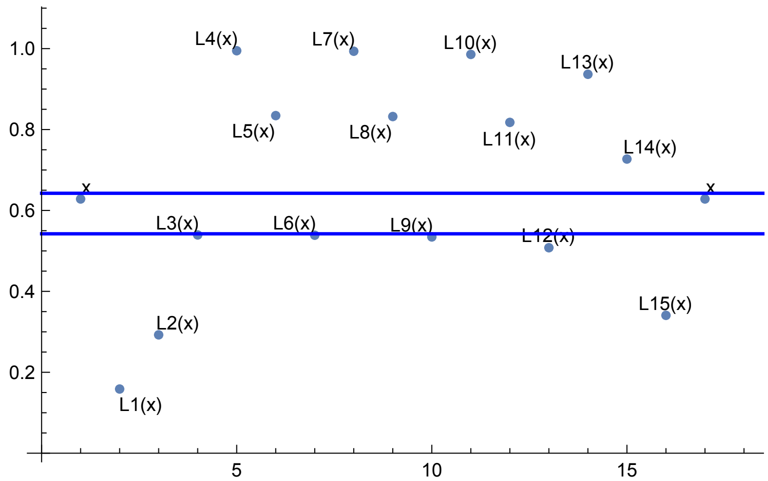

FIGURE 5. The graph for the orbits $\{L_i(x)\}_{i=1}^{15}$. The region between the two horizontal lines is the switch region $[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$.

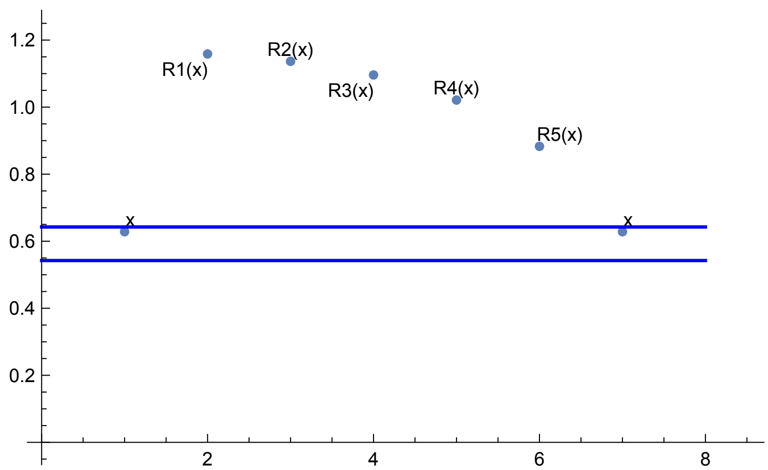

FIGURE 6. The graph for the orbits $\{R_i(x)\}_{i=1}^{5}$. The region between the two horizontal lines is the switch region $[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}]$.

where the words $\mathbf{x}_i = 10^3(110)^4$ or $\mathbf{x}_i = 01^5$ for all $i \geq 1$. Write $c_1 \dots c_{16} = 10^3(110)^4$ and $d_1 \dots d_6 = 01^5$. To prove this claim it suffices to show that the orbits

$$\{L_i(x) = T_{c_1 \dots c_i}(x) : i \in \{1, \dots, 15\}\} \quad \text{and} \quad \left\{R_j(x) = T_{d_1 \dots d_j}(x) : j \in \{1, \dots, 5\}\right\}$$

do not fall into the switch region $[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}] \approx [0.542276, 0.642445]$. This can be verified by some numerical calculation as described in Figure 5 for the orbits $\{L_i(x)\}_{i=1}^{15}$ and in Figure 6 for the orbits $\{R_i(x)\}_{i=1}^{5}$.

Hence, all β-expansions of x are of the form in (4.2), and none of them are simply normal.

At the end of this section we pose some questions related to the set $\mathcal{N}_\beta$. In terms of Theorem 4.1 it is natural to ask about the Hausdorff dimension of the set $\mathcal{N}_\beta$ for $\beta \in (\beta_T, 2)$.

Q1. For each $\beta \in (\beta_T, 2)$ can we calculate the Hausdorff dimension of $\mathcal{N}_\beta$?

Q2. Is it true that $\dim_H \mathcal{N}_\beta < 1$ for any $\beta < 2$? This question was first raised in [6].

Q3. Is the function $\beta \mapsto \dim_H \mathcal{N}_\beta$ continuous?

In this paper we study numbers with a simply normal β-expansion where $\beta \in (1, 2]$ and the digit set is $\{0, 1\}$. It would be interesting to extend the results obtained in this paper to a larger digit set. To be more precise, study numbers with a simply normal β-expansion where $\beta \in (1, m+1]$ and the digit set is $\{0, 1, \ldots, m\}$ for some $m \in \mathbb{N}$. Denote by $\mathcal{N}_\beta(m)$ the set of all $x \in (0, \frac{m}{\beta-1})$ which do not have a simply normal β-expansion. We ask the following.

Q4. Does there exist a critical value $\beta_c = \beta_c(m)$ such that $\mathcal{N}_\beta(m) = \emptyset$ for any $\beta \in (1, \beta_c)$ and $\mathcal{N}_\beta(m) \neq \emptyset$ for any $\beta \in (\beta_c, m+1]$? Furthermore, if such a β_c exists what can one say about $\mathcal{N}_{\beta_c}(m)$?

Q5. What can we say about the Hausdorff dimension of $\mathcal{N}_\beta(m)$ as in **Q1**–**Q3**?

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