A Hamiltonian Cycle in the Square of a 2-connected Graph in Linear Time*†

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Abstract

Fleischner's theorem says that the square of every 2-connected graph contains a Hamiltonian cycle. We present a proof resulting in an $O(|E|)$ algorithm for producing a Hamiltonian cycle in the square G^2 of a 2-connected graph $G = (V, E)$. The previous best was $O(|V|^2)$ by Lau in 1980. More generally, we get an $O(|E|)$ algorithm for producing a Hamiltonian path between any two prescribed vertices, and we get an $O(|V|^2)$ algorithm for producing cycles $C_3, C_4, \ldots, C_{|V|}$ in G^2 of lengths $3, 4, \ldots, |V|$, respectively.

1 Introduction

Fleischner [5] proved in 1974 that the square of every 2-connected graph is Hamiltonian, solving a conjecture from 1966 by Nash-Williams. This remarkable result has stimulated much work on paths and cycles in the square of a finite graph, e.g [2], [6], [12] and [20]. Fleischner's theorem has also been extended to infinite locally finite graphs with at most two ends by Thomassen [21], and to (compactifications of) all locally finite graphs by Georgakopoulos [8].

A short proof of Fleischner's theorem was obtained by Říha (1991) [22]. The technique in that proof has some resemblance with the technique in [21]. More recently, a simpler proof was presented by Georgakopoulos (2009) [9], see also [3].

Lau (1980) [15] was the first to give an efficient constructive algorithm, more precisely an $O(|V|^2)$ algorithm, for finding a Hamiltonian cycle in the square of a 2-connected graph.

In this paper, we present a simple proof of Fleischner's theorem based on the ideas of [9], which results in a linear time algorithm for finding a Hamiltonian cycle in the square G^2 of a 2-connected graph G.

Finding a Hamiltonian cycle in a graph is fundamental and used in many graph algorithms, but is also known to be NP-complete (as shown by Karp [13], see also Garey and Johnson [7]). Finding a Hamiltonian cycle in the square of a 2-connected graph is used explicitly in the Bottleneck Travelling Salesman Problem, see Parker and Rardin [17], but can also be used to compact implicit representations of distances in general graphs (labelling schemes), see Alstrup et al. [1].

Combining this algorithm with the proof in [2] (which we reproduce) that G^2 is *Hamiltonian connected*, that is, it has a Hamiltonian path between any two prescribed vertices, we get an $O(|E|)$ algorithm for producing such a path. We also apply the algorithm to the result in [12, 20] that G^2 is *pancyclic*, that is, it has a collection of cycles $C_3, C_4, \ldots, C_{|V|}$ of lengths $3, 4, \ldots, |V|$, respectively. More precisely, we combine the algorithm with the proof in [20] (which we reproduce in the present paper as well) and obtain thereby an $O(|V|^2)$ algorithm for producing cycles of all lengths in the square of a 2-connected graph. In fact, we may, in $O(|V|^2)$ time, produce such cycles C_i with nested vertex sets, that is, $V(C_i) \subset V(C_{i+1})$, such that $x \in V(C_3)$ for any prescribed vertex x of a graph whose block-cutvertex tree is a path.

These results follow from our main theorem:

THEOREM 1.1. *There is a linear time algorithm finding a Hamiltonian cycle in the square of any 2-connected graph.*

Proof. The algorithm comprises the following steps:

1. Find a minimally 2-connected spanning subgraph G of the 2-connected graph under consideration.
2. Find a proper ear decomposition of G.
3. Pick a vertex of degree 2 in each ear, which exists by Lemma 3.2.
4. Modify the proof of [9] such that it results in a linear time algorithm.

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A linear time algorithm for step 1 above was found by Han et. al. [10, Theorem 12]. A linear time algorithm for step 2 was found by Schmidt [18]. Given an ear in our decomposition, one can check the degrees of all vertices and choose one of degree 2 for each ear, ensuring linear running time of step 3 above. Finally, we will show in Section 4 why step 4 runs in linear time.

2 Preliminaries

A *graph* has no loops or multiple edges. In the proofs we shall double some edges resulting in a *multigraph*, and we shall also introduce orientations of edges. An edge between the vertices x, y in an undirected graph is denoted xy.

A *proper ear decomposition* of an undirected graph G is a partition of its set of edges into a sequence $C^0, \ldots, C^k$ where C^0 is a cycle and C^i is a path for every $i \geq 1$, such that for every $i \geq 1$, $C^i \cap \bigcup_{j<i} C^j$ consists of the two endvertices of C^i.

A graph is *k-connected* if no deletion of $k-1$ vertices disconnects the graph. An edge e of a graph G is *k-essential*, if $G - e$ is not k-connected. A *minimally k-connected* graph is one where every edge is k-essential.

An *Euler tour* (respectively *Euler walk*) of a multigraph is a walk which uses every edge exactly once, and has a last vertex which is the same (respectively not the same) as the first vertex. A multigraph has an Euler tour if and only if all components except one are isolated vertices, and every vertex of the exceptional component has even degree. An Euler tour may be found in linear time using Hierholzer's algorithm [11]. A multigraph is *Eulerian* if and only if it has an Euler tour.

The *square* of a graph $G = (V, E)$ is the graph $G^2 = (V, E')$ where $uv \in E'$ if and only if u and v are connected by a path of length at most 2 in G. The *G-degree*, or just *degree* of a vertex v in a graph G is the number of edges in G incident with v. A *chord* of a path or cycle is an edge which is not in the path or cycle and which joins two vertices of the path or cycle. Note that Dirac [4, Definition 5] uses the term chord with a different meaning.

A *Hamiltonian cycle* is a cycle that contains all vertices of the graph. A graph G is *pancyclic* if it contains a cycle of length i for every $i \in \{3, 4, \ldots, |V(G)|\}$. A graph is *vertex-pancyclic* if, for every vertex x, these cycles can be chosen to pass through x.

3 Every ear has a vertex of degree 2

Dirac [4] gave a detailed investigation of the minimally 2-connected graphs. His work inspired deep results on minimally k-connected graphs, see e.g. [14, 16]. We use the definition introduced by Dirac [4, Definition 6]: Given two vertices of a minimally 2-connected graph, they are *compatible* if no path between them has a chord. We also use Dirac's observation that every 2-connected subgraph of a minimally 2-connected graph is minimally 2-connected.

LEMMA 3.1. *Let G be a minimally 2-connected graph, and let u and v be vertices of G. Suppose there are three internally disjoint paths P_1, P_2, P_3 between u and v in G. Then each of the three paths contains at least one vertex of G-degree 2.*

Proof. We claim that u and v must be compatible. To prove this claim, let P_4 be any path between u and v, and consider the union $G' = P_1 \cup P_2 \cup P_3 \cup P_4$ which is a 2-connected subgraph of G, and hence minimally 2-connected by the above observation. Assume for contradiction that e is a chord of P_4. Then e lies on at most one of P_1, P_2, P_3. But then, $G' - e$ is still 2-connected, contradicting the minimality. Now [4, Corollary 2 to Theorem 6] says that every path from u to v has a vertex of degree 2 in G.

LEMMA 3.2. *Let $C^0, \ldots, C^k$ be a proper ear decomposition of a minimally 2-connected graph G. Then every C^i contains a vertex of G-degree 2, and C^0 contains at least two vertices of G-degree 2.*

Proof. Consider a proper ear decomposition of a minimally 2-connected graph as stated. Then the union of the first i ears $C^0 \cup \ldots \cup C^{i-1}$ forms a 2-connected graph. If u, v are the endvertices of C^i, there exist two internally disjoint paths between them in $C^0 \cup \ldots \cup C^{i-1}$. Together with C^i they form three internally disjoint paths, each of which contains a vertex of G-degree 2 by Lemma 3.1. Thus, C^i contains a vertex of G-degree 2.

For C^0 we have a stronger statement. If $k = 0$, then all vertices of $C^0 = G$ are of G-degree 2. Otherwise, let u, v be the endvertices of C^1. Then there are two internally disjoint paths in C^0 between u, v. Together with C^1 we have three internally disjoint paths, as before, and each must contain a vertex of G-degree 2. Two of these are in C^0.

4 A Hamiltonian cycle in linear time

Let G be a minimally 2-connected graph. In this section, we use the ear decomposition found above in order to construct a Hamiltonian cycle in the square of G. This part of our algorithm draws heavily from the proof of [9].

Let $C^0, C^1, \ldots, C^k$ be a proper ear decomposition of G, where C^0 is a cycle and each other C^i has both its endvertices in ears with smaller indices. By Lemma 3.2, every C^i contains an interior vertex y^i of G-degree 2, and it is easy to pick such a vertex for each i in linear time. Furthermore, by Lemma 3.2, C^0 contains two vertices of G-degree 2, say x and y^0.

We enumerate the vertices of each C^i as $x_0^i, x_1^i, \ldots, x_{\ell_i}^i$ in the order they appear on C^i, starting with the endvertex lying in the ear with the smallest index. For C^0 we just start the enumeration at $x_0^0 = x = x_{\ell_0}^0$.

We start our procedure by turning G into an Eulerian multigraph $G_{\between}$ by adding parallel edges to some existing edges of G and deleting some edges of G as follows.

For each $i = k, k-1, \ldots, 0$, we define the graph $G_i = C^k \cup C^{k-1} \cup \ldots C^i$. For $i = k, k-1, \ldots, 0$, we define the multigraph G_i^+ as follows. First we put $G_k^+ = G_k = C^k$. Suppose we have defined G_{i+1}^+. We now traverse the vertices and edges of $C^i - x_0^i$, starting with x_1^i. When we traverse the edge $e = x_j^i x_{j+1}^i$ we either delete e or add an edge e^+ parallel with e, or we leave e unchanged. Suppose we have traversed $x_1^i x_2^i \ldots x_j^i$. If the degree $d(x_j^i)$ is odd in the current graph, we introduce a new edge e^+ parallel to $e = x_j^i x_{j+1}^i$ so that $d(x_j^i)$ becomes even. If the last edge $e = x_{\ell_{i-1}}^i x_{\ell_i}^i$ of C^i is doubled by this procedure, we delete both of e, e^+ (see Figure 1a). If the last edge is not doubled, but one (and hence both) of the edges incident with y^i is doubled, then we delete the pair of parallel edges incident with y^i which succeedes y^i as we move along C^i from x_0^i (see Figure 1b). Figure 1c shows the situation where possibly some edges of $C^i - x_0^i$ are doubled but none are deleted. The procedure terminates when we have defined G_0^+. Clearly, this procedure for defining G_0^+ has a linear running time.

We claim that every vertex v distinct from x has even G_0^+-degree. To see this we first observe that there is a smallest i such that v is in C_i. Then v is interior in C_i (as v is distinct from x). The degree of v is made even when we form G_i^+, and the degree of v remains even after that because v is not contained in any C_j with $j < i$. More precisely, the G_0^+-degree of v equals the G_i^+-degree of v. The only vertex we never consider in this procedure is x. But x, too, has even G_0^+-degree by the handshaking lemma. Moreover, G_0^+ is connected because every vertex of C^i is still connected to $C^0 \cup C^1 \cup \ldots \cup C^{i-1}$ after any edge deletion, because at most one edge of C^i is deleted. Thus G_0^+ is Eulerian. We denote G_0^+ by $G_{\between}$.

Next, we orient the edges of $G_{\between}$ as follows. We orient any pair e, e^+ of parallel edges in $G_{\between}$ in opposite directions. We go through the ears C^i of G again (in fact, this step of our algorithm can be combined with the previous step). If the last edge of C^i has been deleted, we orient all edges of $C^i \cap G_{\between}$ (that is, the edges that have not been doubled) from x_0^i towards $x_{\ell_i-1}^i$ (see Figure 1a). Otherwise, we orient all edges of $C^i \cap G_{\between}$ (that have not been doubled) towards y^i (see Figure 1b and c).

Note that after we are done, though vertices may have arbitrarily high out-degree, every vertex v has at most 2 incoming edges, since only edges of the ear C^i containing v as an interior vertex can be directed towards v by construction; here we used the fact that the first edge of each C^i is never doubled, and if the last one is doubled it is immediately deleted. Moreover, if $v \neq x$, then v has at least 1 incoming edge.

We now describe an Euler tour J of the underlying undirected graph $G_{\between}$ such that for every vertex v having two incoming edges vw, vz, these edges are consecutive in J. This can easily be achieved by first replacing these two edges vw, vz by a single wz edge for every vertex v having two incoming edges, then finding an Euler tour in the resulting auxiliary graph G^-, and finally replacing the new edge wz by the pair wv, vz for every v as above. Note that v becomes an isolated vertex if v has $G_{\between}$ -degree 2. The fact that G^- has only vertices of even degree follows immediately from the construction of $G_{\between}$. It only remains to be proved that G^- consists of a connected graph and possibly a set of isolated vertices. When the ear is as Figure 1c, its special vertex y^i may become an isolated vertex. But, all other vertices in the ear are still part of the same connected component. More precisely, for each vertex $v \in C^i$ $(v \neq y^i)$, if it has indegree 2, it has at least one outneighbour on C^i, and thus, the deletion of its two incoming edges does not disconnect v from C^i.

Finally, we transform the Euler tour J into a Hamiltonian cycle, which we call H, by replacing some pairs of edges of $G_{\between}$ by single edges of G^2. More precisely, when we traverse the Euler tour J, we replace every 2-edge subwalk $u \leftarrow w \rightarrow v$ in J by the single edge uv. We make a single exception in that we keep the unique subwalk having x as its middle vertex as it is. Note that this operation is well-defined, as whenever we have the subwalk $u \leftarrow w \rightarrow v$, the edges in question are incoming to both u and v. So an edge cannot be part of two such subwalks.

We claim that H is indeed a Hamiltonian cycle of G^2. When we replace pairs of edges by single edges, we clearly transform an Euler tour of the original graph into an Euler tour of the resulting graph. One problem that may occur, is that a vertex becomes isolated. The discussion below shows that this problem will not occur; every vertex is indeed traversed precisely once.

Clearly, H traverses x precisely once, because J traverses x once, and we keep the two edges incident

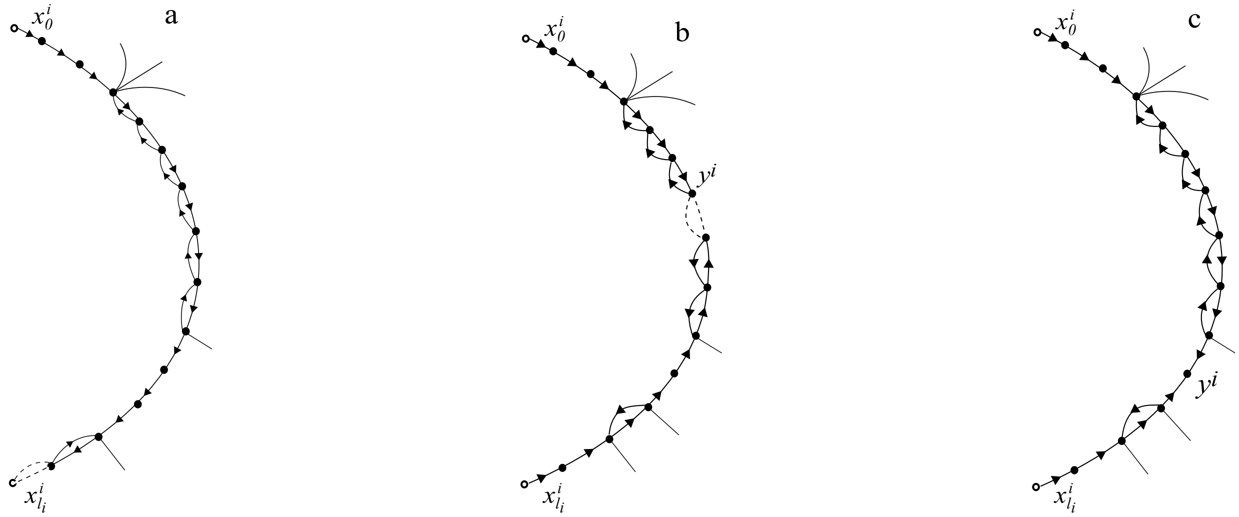

Figure 1: We go through the ears in decreasing order, double some edges and delete at most one (dotted line), thus, turning the graph into an Eulerian graph.

with x when we form H. For every vertex $w \neq x$, the number of times that H traverses w equals the number of subwalks uwv in J containing an incoming edge of w. We claim that there is exactly one such subwalk. This is clear if w has indegree 1. As each vertex $w \neq x$ has indegree either 1 or 2 in $G_{\Diamond}$, we consider the case where w has indegree 2. By the construction of J, the two edges entering w form a subwalk of J, and those two edges are part of H. All other pairs of edges incident with w are replaced by single edges not containing w when we transform J into H.

5 A Hamiltonian path in linear time

Given vertices u, v of a 2-connected graph G, it was shown by Chartrand, Hobbs, Jung, Kapoor, and Nash-Williams [2] that G^2 contains a Hamiltonian path from u to v. We shall now describe an efficient algorithm to find it.

Theorem 5.1. *There exists a linear time algorithm for finding a Hamiltonian path between any two prescribed vertices u, v in the square of a 2-connected graph G.*

Proof. We use the following trick from [2]. Take the union of five disjoint copies of G. Add two new vertices x, y. Let x be joined to the five copies of u, and let y be joined to the five copies of v. The resulting graph H is 2-connected, and therefore our algorithm can produce, in linear time, a Hamiltonian cycle C in H^2. One of the five copies of G does not contain a neighbor of x, y in C. The intersection of that copy with C is a Hamiltonian path from u to v in G^2.

6 Cycles of all lengths in quadratic time

Hobbs [12] proved that the square of a 2-connected graph is pancyclic. Thomassen [20] proved the same under the weaker assumption that the block-cutvertex tree is a path.

A *block* of a graph is either a bridge together with its two ends, or a maximal 2-connected subgraph. If G is a graph, then its *block-cutvertex tree* is the tree whose vertices are the blocks and cutvertices of G. There is an edge between a block and a cutvertex if and only if the cutvertex is contained in the block. The block-cutvertex tree can be found in linear time [19].

The first part of the following result was first proven in [20].

Lemma 6.1. *If G is a graph whose block-cutvertex tree is a path, then G^2 has a Hamiltonian cycle. Moreover, there exists a linear time algorithm for finding a Hamiltonian cycle in G^2.*

Proof. If G is 2-connected we use Theorem 1.1. So assume that G is not 2-connected. We let u, v be two non-cutvertices in distinct end-blocks of G. We now use the following trick from [20]. Take the union of four disjoint copies G_1, G_2, G_3, G_4 of G. Add two new vertices x, y. Let x be joined to the two copies of u in G_1, G_2, and let y be joined to the two copies of v in G_3, G_4. Let the copy of v in G_1 (respectively G_2) be joined to the copy of u in G_3 (respectively G_4). The resulting graph H is 2-connected, and therefore our algorithm can produce, in linear time, a Hamiltonian path P between x, y in H^2. As proved in [14], the intersection of P with one of the four copies of G gives rise to a Hamiltonian cycle in G^2.

Theorem 6.1. *There exists an $O(n^2)$ algorithm for producing cycles $C_3, C_4, \dots, C_n$ of lengths $3, 4, \dots n$, respectively in the square of a graph G on n vertices whose block-cutvertex tree is a path. Moreover, if x_0 is any vertex in G, then the cycles can be chosen such that $x_0 \in V(C_3) \subset V(C_4) \subset \dots \subset V(C_n)$.*

Proof. Again, we use the idea in [20]. First, we may find the block-cutvertex graph in linear time using [19], and use the linear time algorithm of [10] on each block to obtain a spanning subgraph such that every block is minimally 2-connected. Dirac [4] proved that such a graph has at most $2n-4$ edges. Then, it follows from Lemma 6.1 that we may find a Hamiltonian cycle C_n in in G^2 in linear time. If G has only one block, we delete any edge of G and use the linear algorithm in [19] to find the block-cutvertex tree which is a path. As pointed out by Dirac [4], each block is minimally 2-connected. If G is not 2-connected, we let x be a cutvertex contained in an end-block B of G. If x has degree at least 2 in B we delete any edge in B incident with x, and use the linear algorithm in [19] to find the block-cutvertex tree which is a path. Finally, if B has only two vertices x, y, then we delete y. (If $y = x_0$, we consider the other end-block of the current graph instead of B.) Then we use the algorithm in Lemma 6.1 to find a Hamiltonian cycle C_{n-1} in $(G-y)^2$. We repeat the argument.

We now discuss the complexity. We first spend $O(m)$, $m = |E|$, time obtaining a subgraph in which each block is minimally 2-connected. Then we successively delete an edge in an end-block which is incident with a cutvertex. When an isolated vertex appears, we delete that, too. Immediately after we delete an isolated vertex we find a Hamiltonian cycle in the square of the current graph. Thus there are less than $2n-4$ edge-deletions, by an afore-mentioned result of Dirac [4], and between two of these edge-deletions, it takes only $O(n)$ time to find a Hamiltonian cycle in the square of the current graph, by Theorem 1.1. Thus, the total time consumption is $O(m+n^2) = O(n^2)$. □

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