Conditioning a Markov chain upon the behaviour of an additive functional

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Declaration

I hereby declare that this thesis is my own work completed under the guidance of my supervisors Saul D Jacka and Jon Warren and has not been submitted for a degree at another university.

Abstract

We consider a finite statespace continuous-time irreducible Markov chain $(X_t)_{t\geq 0}$ together with some fluctuating additive functional $(\varphi_t)_{t\geq 0}$. The objective is to condition the Markov process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative. There are three possible types of behaviour of the process $(\varphi_t)_{t\geq 0}$: it can drift to $+\infty$, oscillate, or drift to $-\infty$, and in each of these cases we condition the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative.

In the positive drift case, the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is of positive probability and the process $(X_t, \varphi_t)_{t\geq 0}$ can be conditioned on it in the standard way. In the oscillating and the negative drift cases, the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is of zero probability and we cannot condition the process $(X_t, \varphi_t)_{t\geq 0}$ on it in the standard way. Instead, we look at the limits of laws of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on the event that the process $(\varphi_t)_{t\geq 0}$ hits large levels before it crosses zero, and of laws of the process $(X_t, \varphi_t)_{t\geq 0}$ conditioned on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative for a large time. In the oscillating case both limits exists and are equal to the same probability law. In the negative drift case, under certain conditions, both limits exist but give distinct probability laws.

In addition, in the negative drift case, conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$ and then further conditioning on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative yields the same result as the limit of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ hits large levels before it crosses zero. Similarly, conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ oscillates and then further conditioning on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative yields the same result as the limit of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative for a large time.

Frequently Used Notation

Q	The Q-matrix of the process $(X_t)_{t\geq 0}$ (page 14).
E	The statespace of the process $(X_t)_{t\geq 0}$ (page 14).
μ	The invariant measure of the process $(X_t)_{t\geq 0}$ (page 14).
V	The diagonal matrix $diag(v(e))$ (page 14).
E^+, E^-	The sets $v^{-1}(0,+\infty)$ and $v^{-1}(-\infty,0)$ (page 14).
E_y^+, E_y^-	The halfspaces $(E\times(y,+\infty))\bigcup(E^+\times\{y\})$ and $(E\times(-\infty,y))\bigcup(E^-\times\{y\})$ (page 15).
H_y	The first crossing time of the level y by the process $(\varphi_t)_{t\geq 0}$ (page 15).
Γ, G	The matrices given by the Wiener-Hopf factorization of the matrix $V^{-1}Q$ (page 18).
Π^-, Π^+	The components of the matrix Γ (page 18).
G^+, G^-	The components of the matrix G (page 18).
Γ_α, G_α	The matrices given by the Wiener-Hopf factorization of the matrix $V^{-1}(Q-\alpha I)$, $\alpha>0$ (page 16).
Π_α^-, Π_α^+	The components of the matrix Γ_α, $\alpha>0$ (page 16).
G_α^+, G_α^-	The components of the matrix G_α, $\alpha>0$ (page 16).
$\alpha_j,\ j=1,\ldots,n$	The eigenvalues of the matrix $V^{-1}Q$ with non-positive real parts (page 26).
$\beta_k,\ k=1,\ldots,m$	The eigenvalues of the matrix $V^{-1}Q$ with non-negative real parts (page 26).

α_{max}	The eigenvalue of the matrix $V^{-1}Q$ with maximal non-positive real part (page 26).
β_{min}	The eigenvalue of the matrix $V^{-1}Q$ with minimal non-negative real part (page 26).
$f_j,\ j = 1, \ldots, n$	Vectors associated with the eigenvalues of the matrix $V^{-1}Q$ with non-positive real parts (page 26).
$g_k,\ k = 1, \ldots, m$	Vectors associated with the eigenvalues of the matrix $V^{-1}Q$ with non-negative real parts (page 27).
f_{max}	The eigenvector of the matrix $V^{-1}Q$ associated with the eigenvalue α_{max} (page 28).
g_{min}	The eigenvector of the matrix $V^{-1}Q$ associated with the eigenvalue β_{min} (page 28).
Γ_2, J, $F(y)$,	Matrices defined on pages 29 and 30.

Introduction

The problem of conditioning a stochastic process to stay forever in a certain region has been extensively studied in the literature. Many authors have addressed the essentially same problem by conditioning a process with a possibly finite lifetime to live forever. An interesting case is when the event that the process remains in some region is of zero probability, or in terms of the lifetime of the process, when the process has a finite lifetime with probability one. In that case we cannot condition the process to stay in the region forever in the standard way. Instead, we can look at the limit of conditioning the process to stay in the region for a large time or at the limit of conditioning the process to stay away from the boundary of the region.

There are many well-known examples of such conditionings. For instance, Knight [24] in 1969 showed that the standard Brownian motion conditioned not to cross zero for a large time converges weakly to a three-dimensional Bessel process; Iglehart [16] in 1974 considered a general random walk conditioned to stay non-negative for a large time and showed that it converges weakly; Williams [34] in 1974 showed that Brownian motion conditioned to hit level y before hitting zero converges weakly as $y \to +\infty$ to a three-dimensional Bessel process, which is the same limit as of Brownian motion conditioned not to cross zero for a large time. Williams also showed that Brownian motion with a negative drift conditioned to hit level y converges weakly as $y \to \infty$ to Brownian motion with a positive drift; Pinsky [27] in 1985 showed that under certain conditions, a homogeneous diffusion on $\mathbb{R}^d$ conditioned to remain in an open connected bounded region for a large time converges weakly to a homogeneous diffusion; Jacka and Roberts [36] in 1988 proved weak convergence of an Ito diffusion conditioned to remain in an interval (a, b) until a large time.

However, weak convergence of the approximations does not always occur. There are counterexamples in which a process conditioned to stay in a region for a large time does not converge at all or it does converge but to a dishonest limit. Jacka and Warren [21] in 2002 gave two examples of such processes.

Bertoin and Doney [4] in 1994 considered a real-valued random walk $\{S_n, n \geq 0\}$ and discussed these two ways of conditioning it to stay non-negative. Namely, they looked at the limit when $n \to \infty$ of conditioning $\{S_n, n \geq 0\}$ on the event

$$\Lambda_n^{(1)} = \Big\{\{S_n, n \geq 0\} \text{ hits } [n, +\infty) \text{ before it hits } (-\infty, 0)\Big\},$$

that is the event that $\{S_n, n \geq 0\}$ hits at least level n before crossing zero, and at the limit when $n \to \infty$ of conditioning $\{S_n, n \geq 0\}$ on the event

$$\Lambda_n^{(2)} = \Big\{S_k \geq 0 \text{ for all } 0 \leq k \leq n\Big\},$$

that is the event that $\{S_n, n \geq 0\}$ stays non-negative until time n. They showed that

- when the random walk oscillates, these two ways of conditioning yield the same honest limit;
- when the random walk drifts to $-\infty$, then under a certain condition the two ways of conditioning yield distinct honest limits, but if the upper tail of the step distribution is slowly varying, then the two ways of conditioning yield the same dishonest limit;
- under a certain condition, the random walk $\{S_n, n \geq 0\}$ with a negative drift conditioned on the event $\{\{S_n, n \geq 0\} \text{ hits } [n, +\infty)\}$ converges weakly as $n \to +\infty$ to a random walk with a positive drift and then further conditioning the resulting random walk with a positive drift on the event that it stays non-negative yields the same result as the limit as $n \to +\infty$ of conditioning the random walk $\{S_n, n \geq 0\}$ on the event $\Lambda_n^{(1)}$;
- under a certain condition, the random walk $\{S_n, n \geq 0\}$ with a negative drift conditioned on the event $\{S_k \geq 0 \text{ for some } k \geq n\}$ converges weakly as $n \to +\infty$

to an oscillating random walk, and further conditioning this oscillating random walk on the event that it stays non-negative yields the same result as the limit as $n \to +\infty$ of conditioning the random walk $\{S_n, n \geq 0\}$ on the event $\Lambda_n^{(2)}$.

These results by Bertoin and Doney for a random walk were the motivation and the starting point for our work. Instead of considering a random walk $\{S_n, n \geq 0\}$ we study a finite statespace continuous time Markov chain and an associated fluctuating additive functional, and want to condition the Markov chain on the event that the fluctuating functional stays non-negative.

More precisely, let $X = (X_t)_{t \geq 0}$ be an irreducible Markov chain with statespace E and let v be a map $v : E \to \mathbb{R}\backslash\{0\}$. Suppose that both $E^+ = v^{-1}(0, \infty)$ and $E^- = v^{-1}(-\infty, 0)$ are non-empty. Define the process $(\varphi_t)_{t \geq 0}$ by

$$\varphi_t = \varphi + \int_0^t v(X_s)ds,$$

where $\varphi \in \mathbb{R}$ is some non-random initial value for φ_0.

The objective is to condition the process $(X_t, \varphi_t)_{t \geq 0}$ starting at $(e, \varphi) \in E \times (0, +\infty)$ or $(e, \varphi) \in E^+ \times \{0\}$, on the event that the process $(\varphi_t)_{t \geq 0}$ stays non-negative. We distinguish between three possible cases: when the process $(\varphi_t)_{t \geq 0}$ drifts to $+\infty$, when it oscillates and when it drifts to $-\infty$ and perform conditioning in each of the cases separately.

The behaviour of the process $(\varphi_t)_{t \geq 0}$ is completely determined by the matrix Q and the function v and is related to the processes obtained from the process $(X_t)_{t \geq 0}$ via time substitutions based on the process $(\varphi_t)_{t \geq 0}$. Namely, for stopping times τ^+ and τ^- given by

$$\begin{aligned}
\tau_y^+ &= \inf\{t > 0 : \varphi_t > y\} \\
\tau_y^- &= \inf\{t > 0 : \varphi_t < -y\}.
\end{aligned}$$

we define processes Y^+ and Y^- by $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ and $Y^- = (X_{\tau_y^-})_{y \geq 0}$. The relation between the chain $(X_t)_{t \geq 0}$ and the chains Y^+ and Y^- is given by what is known as the Wiener-Hopf factorization for Markov chains. The Wiener-Hopf factorization has various meanings for various processes. For instance, the Wiener-Hopf factorization for Levy processes (see Rogers [29]) gives the relation between a Levy process and its running maximum and minimum; the Wiener-Hopf factorization for diffusions (see Rogers [29]) involves processes obtained from a diffusion by time changing; the Wiener-Hopf factorization for random walks has many formulations (see Alili and Doney [1]) but it always involves the ladder times and the ladder heights processes associated with a random walk.

The Wiener-Hopf factorization for matrices and its probabilistic interpretation in the theory of Markov chains (see Williams [36], Barlow, Rogers, Williams [3], London, McKean, Rogers, Williams [26]) is a very powerful tool for obtaining and proving results for Markov chains. As will be seen, the Wiener-Hopf factorization of the matrix $V^{-1}Q$, where the matrix V is the diagonal matrix $diag(v(e))$ and the matrix Q is the Q-matrix of the process $(X_t)_{t\geq 0}$, plays a prominent role in our work since most of the results are based on it and its consequences. Because of that, a whole section (Section 1.4) is dedicated to the study of the Wiener-Hopf factorization of the matrix $V^{-1}Q$ and its implications. Other major techniques and tools used are the theory of martingales, the theory of Laplace transforms, the Tauberian theorems and the Perron-Frobenius theorems.

The thesis is organized into six chapters. Chapter 1 is preliminary and is intended to introduce the notation and prepare for the results in the following chapters. The first two sections in Chapter 1 contain matrix definitions and some auxiliary matrix lemmas. The process $(X_t, \varphi_t)_{t\geq 0}$ which is the basic object in our work is introduced in Section 1.3. Section 1.4 is, as previously mentioned, concerned with the Wiener-Hopf factorization for matrices and it also, importantly, introduces certain matrix and vector notation. In Section 1.5, the hitting probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$ are discussed, in Section

1.6 the generator of the process $(X_t, \varphi_t)_{t\geq 0}$, and in Section 1.7 the behaviour of the process $(\varphi_t)_{t\geq 0}$. Results in Sections 1.8 and 1.9, about the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$, $\alpha \geq 0$, and h-transforms of the process $(X_t, \varphi_t)_{t\geq 0}$, are used in Chapters 3 and 4. In Section 1.10 we start with conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative. The first and the easiest case, when the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$, is discussed in Section 1.10. In that case, the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is of positive probability and the process $(X_t, \varphi_t)_{t\geq 0}$ can be conditioned on it in the standard way.

Chapter 2 deals with the Green's functions of the process $(X_t, \varphi_t)_{t\geq 0}$ and of the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero. We present several ways for calculating them and show the variety of ideas and techniques that are used.

In Chapter 3 we look at conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative when the process $(\varphi_t)_{t\geq 0}$ oscillates. In that case, the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is of zero probability and conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on it is not possible in any elementary way. Instead, we approximate the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative by some events of positive probabilities, and look at the limit of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on those events. We discuss conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on two approximations of the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative: in Section 3.1 on the approximation by the events that the process $(\varphi_t)_{t\geq 0}$ hits large levels before it crosses zero, and in Section 3.2 on the approximation by the events that the process $(\varphi_t)_{t\geq 0}$ stays non-negative for a large time.

In Chapter 4 we finally look at the most interesting case of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative, that is the case when the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$. Again, as in the oscillating case, the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is of zero probability and instead of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on it, we look (in Section 4.1) at the limit of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ hits large levels before it crosses zero, and (in Section 4.3) at the limit of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative for a large time. In addition, in Sections 4.2 and 4.4, we make two more transformations of the process $(X_t, \varphi_t)_{t\geq 0}$ in order to change the behaviour of the process $(\varphi_t)_{t\geq 0}$, and we look at relations between the processes obtained in these two transformations and the original process $(X_t, \varphi_t)_{t\geq 0}$. Our objective is to obtain the results analogous to those obtained by as Bertoin and Doney (1994) for a random walk (which have been listed above).

Chapter 5 summarises the results from Section 1.10 and Chapters 3 and 4. Appendix A contains the statements of the Perron-Frobenius theorems and the references for their proofs and Appendix B contains the proofs of the auxiliary lemmas in Sections 1.8 and 2.3.

Chapter 1

Preliminaries

In this chapter we introduce the notation and review some results and prove some other results that will be used in following chapters.

1.1 Conventions and some matrix lemmas

By positive we mean " > 0 ". By negative we mean " < 0 ". By non-positive we mean " ≤ 0 ". By non-negative we mean " ≥ 0 ".

We denote the $d \times d$ identity matrix by I where the dimension d varies from line to line and is meant to be clear from context.

All equalities and inequalities between vectors or matrices are meant componentwise.

Let A be a square matrix and α its eigenvalue. We say that a non-zero vector g is associated with the eigenvalue α if there exists $n \in \mathbb{N}$ such that

$$(A - \alpha I)^n g = 0$$

if g is a column vector, or

$$g(A - \alpha I)^n = 0$$

if g is a row vector. By Jordan normal form theory, the number of independent column vectors (or row vectors) associated with the same eigenvalue is equal to the algebraic multiplicity of the eigenvalue.

Lemma 1.1.1 *Let α and β, $\alpha \neq \beta$, be eigenvalues of a square matrix M.*

(i) Let g be a row eigenvector of the matrix M associated with the eigenvalue α, and let f be a column vector associated with the eigenvalue β. Then $gf = 0$;

If g is a column eigenvector of the matrix M associated with the eigenvalue α, and f is a row vector associated with the eigenvalue β, then $fg = 0$;

(ii) Let α be a simple eigenvalue of M. If g and f are left and right eigenvectors, respectively, of M associated with the eigenvalue α, then $gf \neq 0$.

Proof: (i) Let $k \in \mathbb{N}$ such that $(M - \beta I)^k f = 0$. Then, because $g(M - \alpha I) = 0$,

$$\begin{aligned} 0 &= g(M - \beta I)^k f = g((M - \alpha I) + (\alpha - \beta)I)^k f \\ &= g \sum_{j=0}^{k} \binom{k}{j} (M - \alpha I)^j \, (\alpha - \beta)^{k-j} \, f \\ &= (\alpha - \beta)^k \, gf, \end{aligned}$$

and because $\alpha \neq \beta$, $g \neq 0$ and $f \neq 0$, $gf = 0$. the statement for a column vector g and a row vector f can be proved in the same way.

(ii) By Jordan normal form theory, there exists a basis S in the space of all vectors on $\mathbb{R}^n$ which consists only of vectors associated with the eigenvalues of the matrix M. If α is a simple eigenvalue, then there is only one vector in the basis S associated with α and that is its associated right eigenvector f. By (i), the vector g, a left eigenvector of M associated with α, is orthogonal to all vectors in the basis S which are not equal to f. If also $gf = 0$, then g is orthogonal to all vectors in the basis S which implies that $g = 0$. But $g \neq 0$ since it is a left eigenvector of M. Therefore, $gf \neq 0$. □

A matrix with all positive entries is called positive. A matrix with all non-negative entries is called non-negative. A square matrix with non-negative off-diagonal entries is called essentially non-negative.

A Q-matrix is an essentially non-negative matrix with non-positive row sums. If all row sums are equal to zero then the Q-matrix is called conservative.

A square non-negative matrix is called substochastic if all row sums are less than or equal to 1, strictly substochastic if it is substochastic and at least one row sum is strictly less than 1, and stochastic if all row sums are equal to 1.

A square non-negative matrix is called primitive if there exists $k \in \mathbb{N}$ such that T^k is a positive matrix.

Let i be arbitrary index from them index set $\{1, 2, \ldots n\}$ of the non-negative matrix T. Suppose that there exists $m \in \mathbb{N}$ such that $T^m_{i,i}$ is positive. Then, the period $d(i)$ of the index i is the greatest common divisor of those k for which $T^k_{i,i}$ is positive.

A square non-negative matrix T is called irreducible if for every pair i, j of its index set, there exists $k \in \mathbb{N}$ such that $T^k_{i,j}$ is positive. An irreducible matrix is said to be cyclic with period d if the period of any one (and so of each one) of its indices satisfies $d > 1$, and is said to be acyclic if $d = 1$.

Lemma 1.1.2 *A square non-negative matrix is irreducible and acyclic if and only if it is primitive.*

Proof: See Theorem 1.4. in Seneta [31]. □

An essentially non-negative matrix B is associated with a non-negative matrix T through the relation

$$T = B + cI,$$

for some positive real constant c. An essentially non-negative matrix B is called irreducible if its associated non-negative matrix T is irreducible.

The Perron-Frobenius theorems for primitive matrices and for irreducible essentially non-negative matrices are very useful and powerful tools and will be frequently used throughout our work. Their statements and references for the proofs are given in Appendix A.

One of the implications of the Perron-Frobenius theorems for primitive matrices and for irreducible essentially non-negative matrices is that there exist simple eigenvalues of such matrices with which can be associated positive right and left eigenvectors. This fact together with Lemma 1.1.1 proves

Lemma 1.1.3 *The Perron-Frobenius left and right eigenvectors of a primitive matrix are the only positive vectors associated with the eigenvalues of the matrix.*

The same statement is true for an irreducible essentially non-negative matrix.

Proof: Let T be a primitive matrix. By the Perron-Frobenius theorems for primitive matrices there exists a simple eigenvalue α of T such that left g^{left} and right g^{right} eigenvectors associated with α are positive. Let f^{left} and f^{right} be a row and a column vector, respectively, associated with an eigenvalue β of T where $\beta \neq \alpha$. Then, by Lemma 1.1.1 (i), $g^{left} f^{right} = 0$ and $f^{left} g^{right} = 0$, and because g^{left} and g^{right} are positive, it follows that the vectors f^{left} and f^{right} cannot be positive. Hence, the only positive vectors associated with the eigenvalues of T are its Perron-Frobenius eigenvectors.

The only property of the primitive matrix T that is used in the proof is that it has a simple eigenvalue with which can be associated positive right and left eigenvectors. Since an irreducible essentially non-negative matrix has the same property, it follows that the statement in the lemma which was proved for primitive matrices is also valid for irreducible essentially non-negative matrices. □

We give three more lemmas, the first of which is proved in Seneta [31].

Lemma 1.1.4 *An essentially non-negative matrix Q is irreducible iff e^{tQ} is positive for all $t > 0$.*

Proof: See Theorem 2.7. in Seneta [31]. □

Lemma 1.1.5 *Let Q be an irreducible Q-matrix. Then*

$$e^{tQ}1 = 1 \ \text{for some } t > 0 \quad \text{iff} \quad e^{tQ}1 = 1 \ \text{for all } t \geq 0.$$

In addition,

$$\begin{aligned} Q \text{ is conservative} \quad &\text{iff} \quad e^{tQ} \text{ is stochastic for all } t \geq 0 \\ Q \text{ is not conservative} \quad &\text{iff} \quad e^{tQ}1 < 1 \quad \text{for all } t > 0. \end{aligned}$$

Proof: For the first part of the lemma, it is enough to show that if $e^{tQ}1 = 1$ for some $t > 0$ then $e^{tQ}1 = 1$ for all $t \geq 0$.

Differentiating $e^{tQ}1$ over t we obtain

$$\frac{de^{tQ}1}{dt} = e^{tQ}Q1.$$

Since $Q1 \leq 0$ and, by Lemma 1.1.4, e^{tQ} is positive for all $t > 0$, the last equation implies that $\frac{de^{tQ}1}{dt} \leq 0$ which means that the function $t \mapsto e^{tQ}1$ is decreasing.

Suppose that $e^{t_0Q}1 = 1$ for some $t_0 > 0$. Then, $e^{sQ}1 = 1$ for all $0 \leq s < t_0$. If $t > t_0$ then there exists $k \in \mathbb{N}$ such that $(k-1)t_0 < t \leq kt_0$. Thus, for such t,

$$1 = (e^{t_0Q})^k 1 = e^{kt_0Q}1 \leq e^{tQ}1 \leq e^{(k-1)t_0Q}1 = (e^{t_0Q})^{k-1}1 = 1.$$

Therefore, $e^{tQ}1 = 1$ for all $t \geq 0$.

For the second part of the lemma, we first notice that

$$Q1 = 0 \quad \text{iff} \quad e^{tQ}1 = 1 \text{ for all } t \geq 0.$$

Hence, all we have to prove is that if Q is not conservative then $e^{tQ}1 < 1$ for all $t > 0$.

Suppose that Q is not conservative. Then, e^{tQ} is strictly substochastic for some $t > 0$, which, by the first part of the lemma, implies that e^{tQ} is strictly substochastic for all $t \geq 0$. By Lemma 1.2.1, $e^{tQ} > 0$ for all $t > 0$. Hence, for any $t > 0$,

$$e^{tQ}1 = e^{\frac{t}{2}Q}e^{\frac{t}{2}Q}1 < e^{\frac{t}{2}Q}1 \leq 1,$$

which proves that $e^{tQ}1 < 1$ for all $t > 0$. □

Lemma 1.1.6 *Let Q be an irreducible essentially non-negative matrix, V a diagonal matrix and $\beta \in \mathbb{R}$. Then the matrix $(Q - \beta V)$ is essentially non-negative matrix and irreducible.*

Proof: The matrices Q and $(Q - \beta V)$ are essentially non-negative which implies that there exists sufficiently large real constant c such that the matrices

$$T = Q + cI \qquad \text{and} \qquad S = Q - \beta V + cI$$

are non-negative and have positive entries on the main diagonal. Since the matrices T and S have positive entries and zero entries in same positions, it is also valid for all their powers T^k and S^k, $k \in \mathbb{N}$. Thus the matrix T is primitive if and only if the matrix S is primitive.

Since the matrix Q is irreducible, the matrix T is also irreducible, and because all its diagonal entries are positive it is also acyclic. Thus, by Lemma 1.1.2, the matrix T is primitive. We conclude that the matrix S is primitive which, again by Lemma 1.1.2, implies that the matrix S is irreducible. Therefore, by the definition, the matrix $(Q - \beta V)$ is irreducible. □

1.2 Irreducible Markov chains

Let $(X_t)_{t \geq 0}$ be a Markov chain on a finite statespace E and let Q be its Q-matrix, that is, for all $t \geq 0$,

$$P_e(X_t = e') = e^{tQ}(e, e'), \quad e, e' \in E,$$

where $P_e(X_t = e')$ denotes the probability that the process $(X_t)_{t \geq 0}$ starting at the state e is at the state e' at time t.

A Markov chain is called irreducible if the entire statespace forms a single communicating class, that is if all states can be reached from each other.

Lemma 1.2.1 *Let Q be the Q-matrix of a Markov chain $(X_t)_{t\geq 0}$ on a statespace E. Then*

$$(X_t)_{t\geq 0} \text{ is irreducible} \quad \text{iff} \quad \text{the matrix } Q \text{ is irreducible.}$$

Proof: By the definition, the Markov chain $(X_t)_{t\geq 0}$ is irreducible if for every $e, e' \in E$ there exists $t > 0$ such that

$$P_e(X_t = e') = e^{tQ}(e, e') > 0.$$

-the 'if' part: if the matrix Q is irreducible, then by Lemma 1.1.4 the matrix e^{tQ} is positive for all $t > 0$ and therefore the chain $(X_t)_{t\geq 0}$ is irreducible.

-the 'only if' part: let $e, e' \in E$ and suppose that $e^{t_0 Q}(e, e') > 0$ for some t_0.Let

$$T = Q + cI$$

for some constant c such that the matrix T is non-negative. Then

$$\begin{aligned} e^{t_0 Q}(e, e') &= e^{-ct_0} e^{t_0 T}(e, e') && (1.1) \\ &= e^{-ct_0} \sum_{k=0}^{infty} t_0^k \frac{T^k(e, e')}{k!} > 0. && (1.2) \end{aligned}$$

Since $e^{-ct_0} > 0$ and $t_0^k > 0$, $k \geq 0$, there exists $k \in \mathbb{N}$ such that $T^k(e, e') > 0$. Thus, for any $e, e' \in E$, there exists $k \in \mathbb{N}$ such that $T^k(e, e') > 0$, which by definition means that the matrix T is irreducible and therefore, the matrix Q is also irreducible. □

1.3 The process $(X_t, \varphi_t)_{t\geq 0}$

Let $(X_t)_{t\geq 0}$ be an irreducible honest Markov chain on a probability space $(\Omega, \mathcal{F}, P)$ and let the probability space $(\Omega, \mathcal{F}, P)$ be equiped with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the process $(X_t)_{t\geq 0}$.

Let E, Q and μ denote a finite statespace, the conservative irreducible Q-matrix and the unique invariant probability measure, respectively, of the Markov chain $(X_t)_{t\geq 0}$.

Let v be a map $v: E \to \mathbb{R}\backslash\{0\}$ and let V be the diagonal matrix $\mathrm{diag}(v(e))$. Suppose that both $E^+ = v^{-1}(0,\infty)$ and $E^- = v^{-1}(-\infty,0)$ are non-empty and that $|E^+| = n$ and $|E^-| = m$ for some $n, m \in \mathbb{N}$.

Define the process $(\varphi_t)_{t\geq 0}$ by

$$\varphi_t = \varphi + \int_0^t v(X_s)ds,$$

where $\varphi \in \mathbb{R}$ is the starting point of the process $(\varphi_t)_{t\geq 0}$.

By the definition, the process $(\varphi_t)_{t\geq 0}$ is increasing when the process $(X_t)_{t\geq 0}$ is in E^+ and decreasing when the process $(X_t)_{t\geq 0}$ is in E^-. A typical path of the process $(X_t, \varphi_t)_{t\geq 0}$ is as shown in the diagram in Figure 1.1.

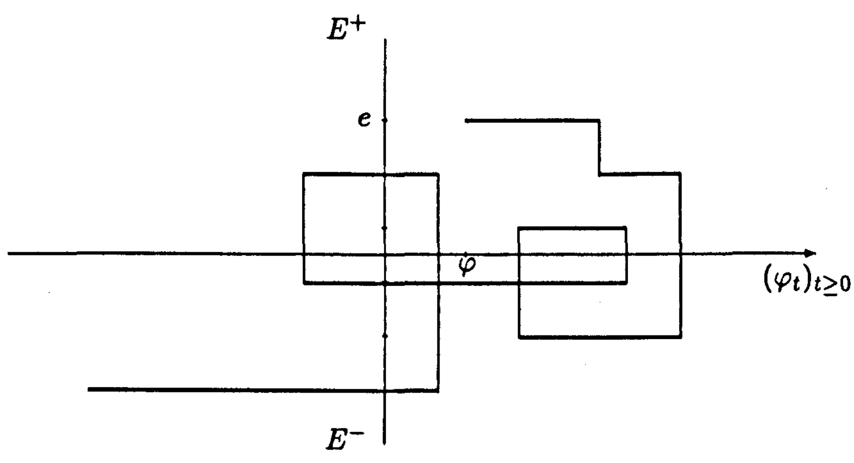

Figure 1.1: A typical path of the process $(X_t, \varphi_t)_{t\geq 0}$ starting at $(e, \varphi) \in E_0^+$ (the values of the process $(\varphi_t)_{t\geq 0}$ are given on the x-axis and the states of the process $(X_t)_{t\geq 0}$ are given on the y-axis)

In order to simplify the notation, let $P_{(e,\varphi)}$ denote the law of the process $(X_t, \varphi_t)_{t\geq 0}$

starting at (e,φ), that is

$$P_{(e,\varphi)}(\,\cdot\,) = P(\,\cdot \mid X_0 = e, \varphi_0 = \varphi),$$

and let $E_{(e,\varphi)}$ denote the expectation operator associated with the probability measure $P_{(e,\varphi)}$.

For $y \geq 0$ define the stopping times

$$\begin{aligned}
\tau_y^+ &= \inf\{t > 0 : \varphi_t > y\} \\
\tau_y^- &= \inf\{t > 0 : \varphi_t < -y\}.
\end{aligned}$$

For the process $(\varphi_t)_{t\geq 0}$ starting at zero define the processes $Y^+ = (Y_y^+)_{y\geq 0}$ and $Y^- = (Y_y^-)_{y\geq 0}$ by

$$Y_y^+ = X_{\tau_y^+}, \qquad Y_y^- = X_{\tau_y^-}.$$

The irreducibility of the process $(X_t)_{t\geq 0}$ implies that the processes Y^+ and Y^- are also irreducible Markov chains, which will be proved in the following section.

Let E_y^+ and E_y^-, $y \in \mathbb{R}$, be the halfspaces

$$\begin{aligned}
E_y^+ &= (E \times (y, +\infty)) \bigcup (E^+ \times \{y\}), \\
E_y^- &= (E \times (-\infty, y)) \bigcup (E^- \times \{y\}).
\end{aligned}$$

In the sequel we shall always assume that the process $(X_t, \varphi_t)_{t\geq 0}$ starts at $(e,\varphi) \in E_0^+$, unless otherwise stated.

Let H_y, $y \in \mathbb{R}$, be the first crossing time of the level y by the process $(\varphi_t)_{t\geq 0}$ defined by

$$H_y = \begin{cases} \inf\{t > 0 : \varphi_t < y\} & \text{if} \quad (X_t, \varphi_t)_{t\geq 0} \quad \text{starts in} \quad E_y^+ \\ \inf\{t > 0 : \varphi_t > y\} & \text{if} \quad (X_t, \varphi_t)_{t\geq 0} \quad \text{starts in} \quad E_y^-. \end{cases}$$

If the process $(X_t, \varphi_t)_{t\geq 0}$ starts at $(e,\varphi) \in E_0^+$, then because the process $(\varphi_t)_{t\geq 0}$ is continuous, the only possible way for the process $(\varphi_t)_{t\geq 0}$ to enter the negative half-line $(-\infty, 0)$ is by crossing zero. Hence, the events $\{\textit{the process } (\varphi_t)_{t\geq 0} \textit{stays non-negative}\}$

and $\{H_0 = +\infty\}$ are equal. In the sequel we shall use this equality without further comment.

Before we close the section, we introduce vector notation that will be constantly in use in the sequel. For any vector g on E, let g^+ and g^- denote its restrictions to E^+ and E^- respectively. We write the column vector g as $g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix}$ and the row vector μ as $\mu = (\,\mu^+ \quad \mu^-\,)$.

The constant vector $g(e) = 1$, $e \in E$, is denoted by 1 and for fixed $e \in E$, 1_e denotes the indicator of e.

1.4 Wiener-Hopf factorization for matrices

The main tool in approaching the problems and obtaining results in the presented work is Wiener-Hopf theory. In this section we introduce the Wiener-Hopf factorizations of the matrices $V^{-1}(Q - \alpha I)$, $\alpha \geq 0$, and discuss some of their implications.

Let α be a positive real number. We state two lemmas which were proved in Barlow *et al.* [3]:

Lemma 1.4.1 *For fixed $\alpha > 0$, there exists a unique pair $(\Pi_\alpha^+, \Pi_\alpha^-)$, where Π_α^+ is an $E^- \times E^+$ matrix and Π_α^- is an $E^+ \times E^-$ matrix, and there exist Q-matrices G_α^+ and G_α^- on $E^+ \times E^+$ and $E^- \times E^-$, respectively, such that, if*

$$\Gamma_\alpha = \begin{pmatrix} I & \Pi_\alpha^- \\ \Pi_\alpha^+ & I \end{pmatrix} \quad \text{and} \quad G_\alpha = \begin{pmatrix} G_\alpha^+ & 0 \\ 0 & -G_\alpha^- \end{pmatrix},$$

then Γ_α is invertible and

$$\Gamma_\alpha^{-1}\, V^{-1}(Q - \alpha I)\, \Gamma_\alpha = G_\alpha.$$

Moreover, Π_α^+ and Π_α^- are strictly substochastic.

Lemma 1.4.2 *Let $\alpha > 0$ be fixed. Then*

$$
\begin{array}{rcll}
E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0} = e'\}) & = & \Pi_\alpha^+(e,e'), & (e,e') \in E^- \times E^+, \\
E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0} = e'\}) & = & \Pi_\alpha^-(e,e'), & (e,e') \in E^+ \times E^-, \\
E_{(e,0)}(e^{-\alpha H_y} I\{X_{H_y} = e'\}) & = & e^{yG_\alpha^+}(e,e'), & (e,e') \in E^+ \times E^+, \quad y > 0, \\
E_{(e,0)}(e^{-\alpha H_{-y}} I\{X_{H_{-y}} = e'\}) & = & e^{yG_\alpha^-}(e,e'), & (e,e') \in E^- \times E^-, \quad y > 0.
\end{array}
$$

Lemma 1.4.1 is said to yield the Wiener-Hopf factorization of the matrix $V^{-1}(Q - \alpha I)$.

The following results were also shown in Barlow *et al.* [3]: The matrix $V^{-1}(Q - \alpha I)$ cannot have strictly imaginary eigenvalues and there exists a basis $\mathcal{B}(\alpha)$ in the space of all vectors on E which consists only of vectors associated with the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$, that is if $g(\alpha)$ is a vector in $\mathcal{B}(\alpha)$, then

$$(V^{-1}(Q - \alpha I) - \lambda(\alpha)\, I)^k g(\alpha) = 0, \tag{1.3}$$

for some eigenvalue $\lambda(\alpha)$ of $V^{-1}(Q - \alpha I)$ and some $k \in \mathbb{N}$. The number of vectors in the basis $\mathcal{B}(\alpha)$ associated with the same eigenvalue is equal to the algebraic multiplicity of that eigenvalue.

Let $\mathcal{N}(\alpha)$ and $\mathcal{P}(\alpha)$ be the sets of vectors $g(\alpha) \in \mathcal{B}(\alpha)$ associated with eigenvalues with positive and with negative real parts, respectively. Then, if the vector $g(\alpha)$ is in $\mathcal{N}(\alpha)$, it is of the form

$$g(\alpha) = \begin{pmatrix} g^+(\alpha) \\ \Pi_\alpha^+ g^+(\alpha) \end{pmatrix}, \tag{1.4}$$

and if the vector $g(\alpha)$ is in $\mathcal{P}(\alpha)$, it is of the form

$$g(\alpha) = \begin{pmatrix} \Pi_\alpha^- g^-(\alpha) \\ g^-(\alpha) \end{pmatrix}. \tag{1.5}$$

The set $\mathcal{N}(\alpha)$ contains exactly $|E^+| = n$ vectors and the vectors $g^+(\alpha)$ for all $g(\alpha) \in \mathcal{N}(\alpha)$ form a basis in the space of all vectors on E^+, and the set $\mathcal{P}(\alpha)$ contains exactly $|E^-| = m$ vectors and the vectors $g^-(\alpha)$ for all $g(\alpha) \in \mathcal{P}(\alpha)$ form a basis in the space of all vectors on E^-. The eigenvalues of $V^{-1}(Q - \alpha I)$ with strictly negative real part coincide with the eigenvalues of G_α^+, and the eigenvalues of $V^{-1}(Q - \alpha I)$ with strictly positive real part coincide with the eigenvalues of $-G_\alpha^-$.

The case $\alpha = 0$ in Lemmas 1.4.1 and 1.4.2 has also been discussed in Barlow *et al.* [3] and the following lemma, which yields the Wiener-Hopf factorization of the matrix $V^{-1}Q$, has been proved.

Lemma 1.4.3 *There exists a unique pair* (Π^+, Π^-), *where* Π^+ *is an* $E^- \times E^+$ *matrix and* Π^- *is an* $E^+ \times E^-$ *matrix, and there exist Q-matrices* G^+ *on* $E^+ \times E^+$ *and* G^- *on* $E^- \times E^-$ *such that*

$$(V^{-1}Q)\,\Gamma = \Gamma\, G, \tag{1.6}$$

where

$$\Gamma = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix}.$$

Moreover, Π^+ *and* Π^- *are substochastic and*

$$\begin{aligned}
P_{(e,0)}(X_{H_0} = e') &= \Pi^+(e,e'), \qquad (e,e') \in E^- \times E^+, \\
P_{(e,0)}(X_{H_0} = e') &= \Pi^-(e,e'), \qquad (e,e') \in E^+ \times E^-, \\
P_{(e,0)}(X_{H_y} = e') &= e^{yG^+}(e,e'), \qquad (e,e') \in E^+ \times E^+, \quad y \geq 0, \\
P_{(e,0)}(X_{H_{-y}} = e') &= e^{yG^-}(e,e'), \qquad (e,e') \in E^- \times E^-, \quad y \geq 0.
\end{aligned}$$

The irreducibility of the chain $(X_t)_{t\geq 0}$ implies that the matrices Π^+ and Π^-, defined in the previous lemma, are positive. This is proved in the next lemma.

Lemma 1.4.4 *(i) The matrices* Π^- *and* Π^+ *are positive.*

(ii) The matrices $(\Pi^+\Pi^- - I)$ *and* $(\Pi^-\Pi^+ - I)$ *are essentially non-negative and irreducible.*

Proof: (i) By Lemma 1.4.3, the matrices Π^- and Π^+ are substochastic which implies that they are non-negative. Let $(e,e') \in E^+ \times E^-$. Then, by Lemma 1.4.3,

$$P_{(e,0)}(X_{H_0} = e') = \Pi^-(e,e').$$

We shall prove that the matrix Π^- is positive.

Let the process $(X_t)_{t\geq 0}$ start at the state e and leave it at time t. Then during time s such that $0 < s < \frac{tv(e)}{|\min\{v(e), e\in E^-\}|}$ the process $(\varphi_t)_{t\geq 0}$ cannot reach zero. On the other hand, during the same time s the process $(X_t)_{t\geq 0}$ can jump into the state e' and after the jump it can stay in e' long enough for the process $(\varphi_t)_{t\geq 0}$ to hit zero. Hence,

$$\begin{aligned}
P_{(e,0)}(X_{H_0} = e') \quad &\geq \quad \int_0^\infty \int_0^{\frac{tv(e)}{|\min\{v(e),e\in E^-\}|}} P_{(e,0)}\Big((X_t)_{t\geq 0} \text{ leaves the state} \\
&\qquad e \text{ in time } dt, \text{ it is in the state } e' \text{ at time } s \text{ and after} \\
&\qquad \text{time } s \text{ it stays in } e' \text{ at least for the time} \\
&\qquad \frac{tv(e) + s\max\{v(e), e\in E^+\}}{|v(e')|}\Big) ds\ dt \\
&= \int_0^\infty \int_0^{\frac{tv(e)}{|\min\{v(e),e\in E^-\}|}} (-Q(e,e))\ e^{tQ(e,e)}\ e^{sQ}(e,e') \\
&\qquad e^{\frac{tv(e)+s\max\{v(e),e\in E^+\}}{|v(e')|}Q(e',e')} ds\ dt > 0.
\end{aligned}$$

Hence, for all $(e,e') \in E^+ \times E^-$, $P_{(e,0)}(X_{H_0} = e') = \Pi^-(e,e') > 0$.

It can be proved in the same way that the matrix Π^+ is positive.

(ii) By (i), the matrices $\Pi^+\Pi^-$ and $\Pi^-\Pi^+$ are positive. It folows that they are primitive, and, by Lemma 1.1.2, irreducible. Therefore, the matrices $(\Pi^+\Pi^- - I)$ and $(\Pi^-\Pi^+ - I)$ are essentially non-negative and irreducible. □

The matrix Γ defined in Lemma 1.4.3 is not necessarily invertible. More precisely,

Lemma 1.4.5 *If the matrices Π^+ and Π^- are both stochastic, then the matrices $(I - \Pi^-\Pi^+)$, $(I - \Pi^+\Pi^-)$ and Γ are not invertible.*

If at least one of the matrices Π^+ and Π^- is strictly substochastic, then the matrices $(I - \Pi^-\Pi^+)$, $(I - \Pi^+\Pi^-)$ and Γ are invertible and the inverse Γ^{-1} is given by

$$\Gamma^{-1} = \begin{pmatrix} (I - \Pi^-\Pi^+)^{-1} & -\Pi^-(I - \Pi^+\Pi^-)^{-1} \\ -\Pi^+(I - \Pi^-\Pi^+)^{-1} & (I - \Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

Moreover, the matrices $(I - \Pi^-\Pi^+)^{-1}$ and $(I - \Pi^+\Pi^-)^{-1}$ are positive.

Proof: Suppose that Π^+ and Π^- are stochastic matrices. Then $(I - \Pi^-\Pi^+)1^+ = 0$, $(I - \Pi^+\Pi^-)1^- = 0$ and

$$\Gamma\begin{pmatrix} 1^+ \\ -1^- \end{pmatrix} = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix}\begin{pmatrix} 1^+ \\ -1^- \end{pmatrix} = \begin{pmatrix} 1^+ - \Pi^-1^- \\ \Pi^+1^+ - 1^- \end{pmatrix} = 0,$$

which shows that none of the matrices $(I - \Pi^-\Pi^+)$, $(I - \Pi^+\Pi^-)$ and Γ is invertible.

Suppose now that at least one of the matrices Π^+ and Π^- is strictly substochastic, that is $\Pi^+1^+ \leq 1^-$ or $\Pi^-1^- \leq 1^+$ with strict inequality in at least one entry. By Lemma 1.4.4 (i), the matrices Π^+ and Π^- are positive which implies that $(\Pi^-\Pi^+ - I)1^+ \leq 0$ and $(\Pi^+\Pi^- - I)1^- \leq 0$ with strict inequality in at least one entry. In addition, By Lemma 1.4.4 (ii), $(\Pi^-\Pi^+ - I)$ and $(\Pi^+\Pi^- - I)$ are irreducible essentially non-negative matrices. Thus, the Perron-Frobenius theorem for irreducible essentially non-negative matrices implies that the Perron-Frobenius eigenvalues of the matrices $(\Pi^-\Pi^+ - I)$ and $(\Pi^+\Pi^- - I)$ are negative. Thus, the matrices $(\Pi^-\Pi^+ - I)$ and $(\Pi^+\Pi^- - I)$ do not have zero as an eigenvalue and therefore they are invertible. Moreover, by the same theorem, the inverses $(I - \Pi^-\Pi^+)^{-1}$ and $(I - \Pi^+\Pi^-)^{-1}$ are positive.

Finally, directly checking shows that the matrix

$$\begin{pmatrix} (I - \Pi^-\Pi^+)^{-1} & -\Pi^-(I - \Pi^+\Pi^-)^{-1} \\ -\Pi^+(I - \Pi^-\Pi^+)^{-1} & (I - \Pi^+\Pi^-)^{-1} \end{pmatrix}$$

is the inverse of the matrix Γ. □

Another immediate consequence of Lemma 1.4.3 is

Lemma 1.4.6 *(i) The matrices G^+ and G^- are irreducible Q-matrices.*

(ii) G^+ *is conservative iff* Π^+ *is stochastic*
G^- *is conservative iff* Π^- *is stochastic.*

Proof: (i) By Lemma 1.4.3, the matrices G^+ and G^- are the Q-matrices. In order to prove that they are irreducible, by Lemma 1.1.4 it is enough to prove that the matrices e^{yG^+} and e^{yG^-} are positive for all $y > 0$.

We shall show first that the matrix e^{yG^+} is positive for all $y > 0$. By Lemma 1.4.3, for $y > 0$ and $e, e' \in E^+$,

$$e^{yG^+}(e,e') = P_{(e,0)}(X_{H_y} = e').$$

Therefore, the matrix e^{yG^+} is non-negative.

Choose s such that $s \max\{v(e), e \in E^+\} < y$. During time s the process $(\varphi_t)_{t\geq 0}$ starting at zero cannot hit the level y and the furthest that it can get into the negative side is $s \min\{v(e), e \in E^-\}$. Hence, for any $e, e' \in E^+$,

$$\begin{aligned}
P_{(e,0)}(X_{H_y} = e') &\geq E_{(e,0)}\Big(I\{X_s = e'\} I\{(X_t)_{t\geq 0} \text{ stays after time } s \text{ in the} \\
&\qquad \text{state } e' \text{ at least for the time } \frac{y - s\min\{v(e), e \in E^-\}}{v(e')}\}) \\
&\geq P_{(e,0)}(X_s = e')\ P_{(e', s\min\{v(e), e \in E^-\})}((X_t)_{t\geq 0} \text{ stays in the} \\
&\qquad \text{state } e' \text{ at least for the time } \frac{y - s\min\{v(e), e \in E^-\}}{v(e')}\Big)) \\
&= e^{sQ}(e,e')\ e^{\frac{y - s\min\{v(e), e \in E^-\}}{v(e')} Q(e',e')} > 0,
\end{aligned}$$

because the matrix Q is irreducible and by Lemma 1.1.4, for any $e, e' \in E$ and any $t > 0$,

$$P_{(e,0)}(X_t = e') = e^{tQ}(e,e') > 0.$$

Therefore, for any $y > 0$ and $e, e' \in E^+$, $P_{(e,0)}(X_{H_y} = e') = e^{yG^+}(e,e') > 0$. Since tha matrix e^{yG^+} is positive for all $y > 0$, it follows from Lemma 1.1.4 that the matrix G^+ is irreducible.

It can be proved in the same way that the matrix G^- is irreducible.

(ii) From Lemma 1.4.3 we have that

$$(V^{-1}Q)\,\Gamma \begin{pmatrix} 1^+ \\ 0 \end{pmatrix} = \Gamma \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix} \begin{pmatrix} 1^+ \\ 0 \end{pmatrix}.$$

Since

$$(V^{-1}Q)\,\Gamma \begin{pmatrix} 1^+ \\ 0 \end{pmatrix} = (V^{-1}Q) \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \begin{pmatrix} 1^+ \\ 0 \end{pmatrix} = (V^{-1}Q) \begin{pmatrix} 1^+ \\ \Pi^+ 1^+ \end{pmatrix},$$

and

$$\Gamma \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix} \begin{pmatrix} 1^+ \\ 0 \end{pmatrix} = \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \begin{pmatrix} G^+1^+ \\ 0 \end{pmatrix} = \begin{pmatrix} G^+1^+ \\ \Pi^+G^+1^+ \end{pmatrix},$$

we get that

$$(V^{-1}Q) \begin{pmatrix} 1^+ \\ \Pi^+1^+ \end{pmatrix} = \begin{pmatrix} G^+1^+ \\ \Pi^+G^+1^+ \end{pmatrix}.$$

Suppose that G^+ is conservative, that is $G^+1^+ = 0$. Then, from the last equation we get that

$$Q \begin{pmatrix} 1^+ \\ \Pi^+1^+ \end{pmatrix} = 0,$$

which implies that $\Pi^+1^+ = 1^-$ because the only eigenvector of the matrix Q associated with the eigenvalue zero is 1. Hence, the matrix Π^+ is stochastic.

Suppose now that Π^+ is stochastic. Then

$$\begin{pmatrix} G^+1^+ \\ \Pi^+G^+1^+ \end{pmatrix} = V^{-1}Q\, 1 = 0.$$

It follows that $G^+1^+ = 0$.

Similarly,

$$(V^{-1}Q)\, \Gamma \begin{pmatrix} 0 \\ 1^- \end{pmatrix} = \Gamma \begin{pmatrix} G^+ & 0 \\ 0 & -G^- \end{pmatrix} \begin{pmatrix} 0 \\ 1^- \end{pmatrix}$$

implies that

$$(V^{-1}Q) \begin{pmatrix} \Pi^-1^- \\ 1^- \end{pmatrix} = \begin{pmatrix} -\Pi^-G^-1^- \\ -G^-1^- \end{pmatrix}.$$

In the same way as in the case of the matrices G^+ and Π^+ we conclude that $G^-1^- = 0$ iff $\Pi^-1^- = 1^+$. □

We recall the processes $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ and $Y^- = (X_{\tau_y^-})_{y \geq 0}$ introduced in the previous section. It was said there that they are irreducible Markov chains. Now we shall prove that.

Lemma 1.4.7 *The processes $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ and $Y^- = (X_{\tau_y^-})_{y \geq 0}$ are irreducible Markov chains.*

Proof: Since the process $(X_t)_{t\geq 0}$ is strong Markov and τ_y^+ and τ_y^- are stopping times, the processes Y^+ and Y^- are Markov.

By Lemma 1.4.3, the matrices G^+ and G^- are the Q-matrices of the processes Y^+ and Y^-, respectively. Since, by Lemma 1.4.6 (i), the matrices G^+ and G^- are irreducible, it follows from Lemma 1.2.1 that the processes Y^+ and Y^- are irreducible. □

It can also be shown that, for $\alpha > 0$, the matrices Π_α^+ and Π_α^- are positive and that the matrices G_α^+ and G_α^- are irreducible.

Lemma 1.4.8 *For fixed $\alpha > 0$,*

(i) the matrices Π_α^+ and Π_α^- are positive;

(ii) the matrices G_α^+ and G_α^- are irreducible Q-matrices;

(iii) Let $f_{max}^+(\alpha)$ and $g_{min}^-(\alpha)$ be the Perron- Frobenius eigenvectors of G_α^+ and G_α^-, respectively. Then, the vectors

$$f_{max}(\alpha) = \begin{pmatrix} f_{max}^+(\alpha) \\ \Pi_\alpha^+ f_{max}^+(\alpha) \end{pmatrix} \text{ and } g_{min}(\alpha) = \begin{pmatrix} \Pi_\alpha^- g_{min}^-(\alpha) \\ g_{min}^-(\alpha) \end{pmatrix}$$

are the only positive eigenvectors of the matrix $V^{-1}(Q - \alpha I)$.

Proof: (i) By Lemma 1.4.1, the matrices Π_α^+ and Π_α^- are strictly substochastic which implies that they are non-negative. Suppose that for some $(e, e') \in E^+ \times E^-$, $\Pi_\alpha^-(e, e') = 0$. Then, by Lemma 1.4.2,

$$E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0} = e'\}) = \Pi_\alpha^-(e, e') = 0,$$

which implies that $e^{-\alpha H_0} I\{X_{H_0} = e'\} = 0$ a.s., and that

$$P_{(e,0)}(X_{H_0} = e') = \Pi^-(e, e') = 0.$$

But, that is a contradiction because, by Lemma 1.4.4 (i), the matrix Π^- is positive. It follows that the matrix Π_α^- is positive.

It can be proved in the same way that the matrix Π_α^+ is positive.

(ii) By Lemma 1.4.1, the matrices G_α^+ and G_α^- are Q-matrices. In order to prove that they are irreducible, by Lemma 1.1.4 it is enough to prove that the matrices $e^{yG_\alpha^+}$ and $e^{yG_\alpha^-}$ are positive for all $y > 0$.

By Lemma 1.4.2, for $y > 0$ and $e, e' \in E^+$,

$$e^{yG_\alpha^+}(e,e') = E_{(e,0)}(e^{-\alpha H_y} I\{X_{H_y} = e'\}).$$

Therefore, the matrix $e^{yG_\alpha^+}$ is non-negative.

Suppose that there exists $y_0 > 0$ and $(e,e') \in E^+ \times E^+$ such that $e^{y_0 G_\alpha^+}(e,e') = 0$. Then, by Lemma 1.4.2,

$$E_{(e,0)}(e^{-\alpha H_{y0}} I\{X_{H_{y0}} = e'\}) = e^{y_0 G_\alpha^+}(e,e') = 0,$$

which implies that $e^{-\alpha H_{y0}} I\{X_{H_{y0}} = e'\} = 0$ a.s., and that, by Lemma 1.4.3

$$P_{(e,0)}(X_{H_{y0}} = e') = e^{y_0 G^+}(e,e') = 0.$$

But that is not possible since, by Lemma 1.4.6 (i), the matrix G^+ is irreducible Q-matrix which, by Lemma 1.2.1, implies that the matrix e^{yG^+} is positive for all $y > 0$.

By Lemma 1.4.7, the process $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ is an irreducible which implies that for all $y > 0$ and $e, e' \in E^+$, $P_{(e,0)}(X_{H_y} = e') > 0$. Therefore, $e^{yG_\alpha^+}$ is positive for all $y > 0$.

It can be shown in the same way that the matrix $e^{yG_\alpha^-}$ is positive for all $y > 0$. Hence, it follows from Lemma 1.1.4 that the matrices G_α^+ and G_α^- are irreducible.

(iii) By (ii), G_α^+ and G_α^- are irreducible essentially non-negative matrices. Hence, by the Perron-Frobenius theorem for irreducible essentially non-negative matrices and by Lemma 1.1.3, the only positive eigenvectors of G_α^+ and G_α^- associated with their eigenvalues are their Perron-Frobenius eigenvectors, $f_{max}^+(\alpha)$ and $g_{min}^-(\alpha)$, respectively.

Let f be a positive eigenvector of the matrix $V^{-1}(Q-\alpha I)$. Then, by (1.4) and (1.5) either

$$f = \begin{pmatrix} f^+ \\ \Pi_\alpha^+ f^+ \end{pmatrix} \text{ and } f^+ \text{ is an eigenvector of } G_\alpha^+$$

or

$$f = \begin{pmatrix} \Pi_\alpha^- f^- \\ f^- \end{pmatrix} \text{ and } f^- \text{ is an eigenvector of } G_\alpha^-.$$

The only positive eigenvectors of G_α^+ and G_α^- are $f_{max}^+(\alpha)$ and $g_{min}^-(\alpha)$, respectively. Hence, because by (i), Π_α^+ and Π_α^- are positive matrices, $f = f_{max}(\alpha)$ or $f = g_{min}(\alpha)$.

□

The following lemma establishes the relation between the matrices Γ_α, $\alpha > 0$, and the matrix Γ.

Lemma 1.4.9 $\lim_{\alpha\to 0}\Gamma_\alpha = \Gamma$.

Proof: By Lemmas 1.4.2 and 1.4.3, for any $(e,e') \in E^- \times E^+$,

$$\Pi_\alpha^+(e,e') = E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0} = e'\}) \quad \text{and} \quad \Pi^+(e,e') = E_{(e,0)}(I\{X_{H_0} = e'\}),$$

and, for any $(e,e') \in E^+ \times E^-$,

$$\Pi_\alpha^-(e,e') = E_{(e,0)}(e^{-\alpha H_0} I\{X_{H_0} = e'\}) \quad \text{and} \quad \Pi^-(e,e') = E_{(e,0)}(I\{X_{H_0} = e'\}).$$

Since $e^{-\alpha H_0} I\{X_{H_0} = e'\}$ is a bounded random variable, we have that,

$$\lim_{\alpha\to 0}\Pi_\alpha^+(e,e') = \Pi^+(e,e'), \qquad (e,e') \in E^- \times E^+$$
$$\lim_{\alpha\to 0}\Pi_\alpha^-(e,e') = \Pi^-(e,e'), \qquad (e,e') \in E^+ \times E^-.$$

Hence, $\lim_{\alpha\to 0}\Gamma_\alpha = \Gamma$. □

In the rest of this section we look closely at the eigenvalues of the matrix $V^{-1}Q$ and vectors associated with them and introduce some more notation that will be constantly in use in following sections.

The Wiener-Hopf factorization (1.6) of the matrix $V^{-1}Q$ implies that

1) $G^+f^+ = \alpha f^+$ for some $\alpha \in \mathbb{R}$ and some vector f^+ on E^+ iff

$$V^{-1}Q\begin{pmatrix} f^+ \\ \Pi^+ f^+ \end{pmatrix} = \alpha \begin{pmatrix} f^+ \\ \Pi^+ f^+ \end{pmatrix}; \tag{1.7}$$

2) $G^-g^- = -\beta g^-$ for some $\beta \in \mathbb{R}$ and some vector g^- on E^- iff

$$V^{-1}Q\begin{pmatrix} \Pi^- g^- \\ g^- \end{pmatrix} = \beta \begin{pmatrix} \Pi^- g^- \\ g^- \end{pmatrix}. \tag{1.8}$$

Let α_j, $j = 1, \dots, n$, be the eigenvalues (not necessarily distinct) of the matrix G^+, and $-\beta_k$, $k = 1, \dots, m$, be the eigenvalues (not necessarily distinct) of the matrix G^-. Since, by Lemma 1.4.6 (i), G^+ and G^- are irreducible Q-matrices, it follows from the Perron-Frobenius theorem for irreducible essentially non-negative matrices that

$$\alpha_{max} \equiv \max_{1 \le j \le n} Re(\alpha_j)$$

and

$$-\beta_{min} \equiv \max_{1 \le k \le m} Re(-\beta_k) = -\min_{1 \le k \le m} Re(\beta_k),$$

(where " $\equiv$ " means "defined to be") are simple eigenvalues of G^+ and G^-, respectively, and that $\alpha_{max} \le 0$ and $-\beta_{min} \le 0$. Hence, from (1.7) it follows that all eigenvalues of $V^{-1}Q$ with negative real part coincide with the eigenvalues of G^+ and from (1.8) that all eigenvalues of $V^{-1}Q$ with positive real part coincide with the eigenvalues of $-G^-$.

Jordan normal form theory implies that the space of all vectors on E has a basis $\mathcal{B}$ such that every vector $g \in \mathcal{B}$ satisfies the equation

$$(V^{-1}Q - \lambda I)^k g = 0 \tag{1.9}$$

for some eigenvalue λ of $V^{-1}Q$ and some $k \in \mathbb{N}$.

It can be shown (see Barlow *et al.* [3]) that there exist exactly $n = |E^+|$ vectors $\{f_1, f_2, \dots, f_n\}$ in the basis $\mathcal{B}$ such that

$$f_j = \begin{pmatrix} f_j^+ \\ \Pi^+ f_j^+ \end{pmatrix}, \quad j = 1, \dots, n,$$

and that for each vector f_j, $j = 1, \dots, n$, there exists an eigenvalue α_j of $V^{-1}Q$, $Re(\alpha_j) \leq 0$, and $c_j \in \mathbb{N}$ such that

$$(V^{-1}Q - \alpha_j I)^{c_j} f_j = 0, \qquad (G^+ - \alpha_j I)^{c_j} f_j^+ = 0.$$

The vectors $\{f_1^+, f_2^+, \dots, f_n^+\}$ are independent and form a basis $\mathcal{N}^+$ in the space of all vectors on E^+.

The vectors $\{f_1, f_2, \dots, f_n\}$ are determined uniquely up to a constant multiple, but because the choice of their normalization does not affect any result in the presented work, we shall refer to them as if they were fixed.

From the last equation it follows that, for $j = 1, \dots, n$,

$$\begin{array}{ll} (i) & Re(\alpha_j) < 0 \ \Rightarrow \ e^{yV^{-1}Q} f_j = e^{y\alpha_j}\ e^{y(V^{-1}Q - \alpha_j I)} f_j \to 0, \quad y \to +\infty, \\ (ii) & Re(\alpha_j) < 0 \ \Rightarrow \ e^{yG^+} f_j^+ = e^{y\alpha_j}\ e^{y(G^+ - \alpha_j I)} f_j^+ \to 0, \quad y \to +\infty. \end{array} \tag{1.10}$$

Similarly, there exist exactly $m = |E^-|$ vectors $\{g_1, g_2, \dots, g_m\}$ in the basis $\mathcal{B}$ such that

$$g_k = \begin{pmatrix} \Pi^- g_k^- \\ g_k^- \end{pmatrix}, \quad k = 1, \dots, m,$$

and that for each vector g_k, $k = 1, \dots, m$, there exists an eigenvalue β_k of $V^{-1}Q$, $Re(\beta_k) \geq 0$, and $d_k \in \mathbb{N}$ such that

$$(V^{-1}Q - \beta_k I)^{d_k} g_k = 0, \qquad (G^- + \beta_k I)^{d_k} g_k^- = 0.$$

The vectors $\{g_1^-, g_2^-, \dots, g_m^-\}$ form a basis $\mathcal{P}^-$ in the space of all vectors on E^-.

The vectors $\{g_1, g_2, \dots, g_m\}$ are determined uniquely up to a constant multiple, but because the choice of their normalization does not affect any result in the presented work, we shall refer to them as if they were fixed.

The last equation implies that, for $k = 1, \dots, m$,

$$\begin{array}{ll} (i) & Re(\beta_k) > 0 \ \Rightarrow \ e^{-yV^{-1}Q} g_k = e^{-y\beta_k}\ e^{-y(V^{-1}Q - \beta_k I)} g_k \to 0, \quad y \to +\infty, \\ (ii) & Re(\beta_k) > 0 \ \Rightarrow \ e^{yG^-} g_k^- = e^{-y\beta_k}\ e^{y(G^- + \beta_k I)} g_k^- \to 0, \quad y \to +\infty. \end{array} \tag{1.11}$$

Let f_{max} and g_{min} be the eigenvectors of the matrix $V^{-1}Q$ associated with its eigenvalues α_{max} and β_{min}, respectively. Then, f^+_{max} and g^-_{min} are the Perron-Frobenius eigenvectors of the matrices G^+ and G^-, respectively, and, by Lemma 1.1.3, they are the only positive eigenvectors of G^+ and G^-, respectively. Moreover, analogously to Lemma 1.4.8 (iii)

Lemma 1.4.10 *The vectors f_{max} and g_{min} are the only positive eigenvectors of the matrix $V^{-1}Q$.*

Proof: Let f be a positive eigenvector of the matrix $V^{-1}Q$. Then, by (1.7) and (1.8) either

$$f = \begin{pmatrix} f^+ \\ \Pi^+ f^+ \end{pmatrix} \text{ and } f^+ \text{ is an eigenvector of } G^+$$

or

$$f = \begin{pmatrix} \Pi^- f^- \\ f^- \end{pmatrix} \text{ and } f^- \text{ is an eigenvector of } G^-.$$

The only positive eigenvectors of G^+ and G^- are f^+_{max} and g^-_{min}, respectively. Hence,

$$f = \begin{pmatrix} f^+_{max} \\ \Pi^+ f^+_{max} \end{pmatrix} = f_{max} \quad \text{or} \quad f = \begin{pmatrix} \Pi^- g^-_{min} \\ g^-_{min} \end{pmatrix} = g_{min}.$$

Since, by Lemma 1.4.4 (i), the matrices Π^+ and Π^- are positive, we have that f_{max} and g_{min} are positive which finishes the proof. $\square$

Interesting properties of non-negative vectors on E^+ and non-negative vectors on E^- are given in the following lemma.

Lemma 1.4.11 *There are no non-negative vectors on E^+ which are linearly independent of the vector f^+_{max}.*

There are no non-negative vectors on E^- which are linearly independent of the vector g^-_{min}.

Proof: We shall prove only the first part of the lemma about non-negative vectors on E^+. The second statement in the lemma can be proved in the same way.

Let f^+ be a non-negative vector on E^+. Since $\mathcal{N}^+ = \{f_1^+, f_2^+, \ldots, f_n^+\}$ is a basis in the space of all vectors on E^+, the vector f^+ has a decomposition

$$f^+ = \sum_{j=1}^{n} a_j f_j^+ \tag{1.12}$$

for some coefficients a_j, $j = 1, \ldots, n$.

Let a_{max} be a coefficient in linear combination (1.12) associated with f_{max}^+. Suppose that $a_{max} = 0$. Then, for any $t \geq 0$,

$$e^{tG^+} f^+ = \sum_{f_j^+ \neq f_{max}^+} a_j \, e^{tG^+} f_j^+.$$

Let $f_{max}^{left,+}$ be the left Perron-Frobenius eigenvector of G^+. Then $f_{max}^{left,+} e^{tG^+} = e^{\alpha_{max} t} f_{max}^{left,+}$ and, by Lemma 1.1.1 (i), $f_{max}^{left,+} f_j^+ = 0$ for all $f_j^+ \neq f_{max}^+$, $j = 1, \ldots, n$. Thus,

$$\begin{aligned} f_{max}^{left,+} e^{tG^+} f^+ &= \sum_{f_j^+ \neq f_{max}^+} a_j \, f_{max}^{left,+} e^{tG^+} f_j^+ \\ &= \sum_{f_j^+ \neq f_{max}^+} a_j \, e^{\alpha_j t} f_{max}^{left,+} f_j^+ = 0, \end{aligned}$$

but that is a contradiction because f^+ and $f_{max}^{left,+}$ are non-negative and

$$f_{max}^{left,+} e^{tG^+} f^+ = e^{\alpha_{max} t} f_{max}^{left,+} f^+ > 0.$$

Therefore, $a_{max} \neq 0$ and the vectors f^+ and f_{max}^+ are not linearly independent. $\square$

Before we close the section we introduce some matrix notation.

Let Γ_2 and J be the matrices

$$\Gamma_2 = \begin{pmatrix} I & -\Pi^- \\ -\Pi^+ & I \end{pmatrix} \qquad J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then

$$\Gamma_2 = J\ \Gamma\ J,$$

and since the matrix J is invertible,

$$\Gamma_2 \text{ is invertible} \quad \text{iff} \quad \Gamma \text{ is invertible.} \tag{1.13}$$

The processes Y^+ and Y^- play an important role in our work. Therefore we define the $E \times E$ matrix, $F(y)$, $y \neq 0$, by

$$F(y)(e,e') = \begin{cases} P_{(e,0)}(Y_y^+ = e'), & y > 0 \\ P_{(e,0)}(Y_y^- = e'), & y < 0 \end{cases} = \begin{cases} \begin{pmatrix} e^{yG^+} & 0 \\ 0 & 0 \end{pmatrix}(e,e'), & y > 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & e^{-yG^-} \end{pmatrix}(e,e'), & y < 0, \end{cases}$$

to contain the information about the transition probabilities of the processes Y^+ and Y^-.

Let J_1 and J_2 be the $E \times E$ matrices

$$J_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad J_2 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Then

$$\begin{aligned} J_1\Gamma + \Gamma_2 J_2 = I, &\qquad \Gamma J_1 + J_2\Gamma_2 = I, \\ J_2\Gamma + \Gamma_2 J_1 = I, &\qquad \Gamma J_2 + J_1\Gamma_2 = I, \end{aligned} \tag{1.14}$$

and

$$F(y) \;=\; \begin{cases} J_1\ e^{yG} = e^{yG}\ J_1, & y > 0 \\ J_2\ e^{yG} = e^{yG}\ J_2, & y < 0, \end{cases} \tag{1.15}$$

1.5 The hitting probabilities of $(X_t, \varphi_t)_{t \geq 0}$

The event that the process $(\varphi_t)_{t \geq 0}$, starting at $\varphi \geq 0$, stays non-negative is essential for our work. Naturally, we want first to calculate its probability.

We recall the stopping time H_0 which is, by the definition, the first crossing time of zero by the process $(\varphi_t)_{t\geq 0}$.

Before we find the probability of the event $\{H_0 = +\infty\}$, we find the probability of the event that the process $(X_t)_{t\geq 0}$ is in a certain state at the moment when the process $(\varphi_t)_{t\geq 0}$ crosses zero for the first time.

Lemma 1.5.1 *For any $e, e' \in E$,*

$$P_{(e,0)}(X_{H_0} = e', H_0 < +\infty) = (I - \Gamma_2)(e,e') = \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix}(e,e'),$$

and for any $e, e' \in E$ and $\varphi \neq 0$,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty) &= \Gamma\, F(-\varphi)(e,e') \\ &= \begin{cases} \begin{pmatrix} 0 & \Pi^- e^{\varphi G^-} \\ 0 & e^{\varphi G^-} \end{pmatrix}, & \varphi > 0, \\ \begin{pmatrix} e^{\varphi G^+} & 0 \\ \Pi^+ e^{\varphi G^+} & 0 \end{pmatrix}, & \varphi < 0. \end{cases} \end{aligned}$$

In addition, for any $(e,\varphi) \in E_0^+$ and $e' \in E^-$, or any $(e,\varphi) \in E_0^-$ and $e' \in E^+$,

$$P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty) > 0.$$

Proof: The first equality in the lemma follows directly from the definitions of Π^+, Π^- and H_0.

For the proof of the second equality, let $\varphi > 0$ and $f \in E^-$. Then, by Lemma 1.4.3, for $e \in E^-$,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) &= P_{(e,0)}(X_{H_{-\varphi}} = f, H_{-\varphi} < +\infty) \\ &= e^{\varphi G^-}(e,f), \end{aligned}$$

and for $e \in E^+$,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) &= \sum_{e' \in E^-} P_{(e,\varphi)}(X_{H_\varphi} = e')\, P_{(e',\varphi)}(X_{H_0} = f) \\ &= \sum_{e' \in E^-} \Pi^-(e,e')\, e^{\varphi G^-}(e',f) = \Pi^- e^{\varphi G^-}(e,f). \end{aligned}$$

Thus, if $\left(P_{(\cdot,\varphi)}(X_{H_0} = \cdot, H_0 < +\infty)\right)_{E\times E}$ denotes an $E \times E$ matrix with entries

$$P_{(\cdot,\varphi)}(X_{H_0} = \cdot, H_0 < +\infty)(e,f) = P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty),$$

then for $\varphi > 0$,

$$\left(P_{(\cdot,\varphi)}(X_{H_0} = \cdot, H_0 < +\infty)\right) = \begin{pmatrix} 0 & \Pi^- e^{\varphi G^-} \\ 0 & e^{\varphi G^-} \end{pmatrix} = \Gamma\ F(-\varphi). \tag{1.16}$$

Similarly, if $\varphi < 0$ and $f \in E^+$, then, by Lemma 1.4.3, for $e \in E^+$,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) &= P_{(e,o)}(X_{H_{-\varphi}} = f, H_{-\varphi} < +\infty) \\ &= e^{-\varphi G^+}(e,f), \end{aligned}$$

and for $e \in E^-$,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) &= \sum_{e' \in E^+} P_{(e,\varphi)}(X_{H_\varphi} = e')\ P_{(e',\varphi)}(X_{H_0} = f) \\ &= \sum_{e' \in E^+} \Pi^+(e,e')\ e^{-\varphi G^+}(e',f) = \Pi^+ e^{-\varphi G^+}(e,f). \end{aligned}$$

Hence, for $\varphi < 0$,

$$\left(P_{(\cdot,\varphi)}(X_{H_0} = \cdot, H_0 < +\infty)\right) = \begin{pmatrix} e^{-\varphi G^+} & 0 \\ \Pi^+ e^{-\varphi G^+} & 0 \end{pmatrix} = \Gamma\ F(-\varphi). \tag{1.17}$$

Therefore, (1.16) and (1.17) prove the second equality of the lemma.

Furthermore, by Lemma 1.4.6 (i), the matrices G^+ and G^- are irreducible Q-matrices which, by Lemma 1.2.1, implies that the matrices e^{yG^+} and e^{yG^-} are positive for all $y > 0$. By Lemma 1.4.4 (i) the matrices Π^+ and Π^- are positive. Therefore, we conclude that for any $(e,\varphi) \in E_0^+$ and $e' \in E^-$, or $(e,\varphi) \in E_0^-$ and $e' \in E^+$,

$$P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty) > 0.$$

$\square$

As a consequence of the previous lemma, we deduce the probability of the event $\{H_0 = +\infty\}$.

Lemma 1.5.2 *The probability of the event $\{H_0 = +\infty\}$ is given by*

$$\begin{aligned} P_{(e,\varphi)}(H_0 = +\infty) &= \Big(I - \Gamma F(-\varphi)\Big)1(e), \quad e \in E,\ \varphi \neq 0, \\ P_{(e,0)}(H_0 = +\infty) &= \Gamma_2 1(e). \end{aligned}$$

Proof: By Lemma 1.5.1, we have that, for $\varphi \neq 0$,

$$\begin{aligned} P_{(e,\varphi)}(H_0 = +\infty) &= 1 - P_{(e,\varphi)}(H_0 < +\infty) \\ &= 1 - \sum_{f \in E} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) \\ &= 1 - \Gamma F(-\varphi)1(e) = (I - \Gamma F(-\varphi))1(e), \end{aligned}$$

and, for $\varphi = 0$,

$$\begin{aligned} P_{(e,0)}(H_0 = +\infty) &= 1 - P_{(e,0)}(H_0 < +\infty) \\ &= 1 - \sum_{f \in E} P_{(e,\varphi)}(X_{H_0} = f, H_0 < +\infty) \\ &= 1 - (I - \Gamma_2)1(e) = \Big(I - (I - \Gamma_2)\Big)1(e) \\ &= \Gamma_2 1(e), \end{aligned}$$

which proves the lemma. □

If the process $(X_t, \varphi_t)_{t \geq 0}$ starts in E_0^+ (E_0^- respectively) then with a positive probability the process $(\varphi_t)_{t \geq 0}$ crosses level y, $y > 0$ ($y < 0$ respectively), before it crosses zero. We prove this in the following lemma:

Lemma 1.5.3 *For any $y > 0$ and $(e, \varphi) \in E_0^+ \cap E_y^-$,*

$$P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0) > 0, \qquad e' \in E^+.$$

For any $y < 0$ and $(e, \varphi) \in E_0^- \cap E_y^+$,

$$P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0) > 0, \qquad e' \in E^-.$$

In addition, for any $y > 0$ and $(e,\varphi) \in E_0^+ \cap E_y^-$, or any $y < 0$ and $(e,\varphi) \in E_0^- \cap E_y^+$,

$$0 < P_{(e,\varphi)}(H_y < H_0) < 1.$$

Proof: We shall prove the statements in the lemma for $(e,\varphi) \in E_0^+ \cap E_y^-$, $y > 0$. The statements for $(e,\varphi) \in E_0^- \cap E_y^+$, $y < 0$, can be proved in the same way.

Let $y > 0$. Let the process $(X_t, \varphi_t)_{t\geq 0}$ start at $(e,\varphi) \in E_0^+ \cap E_y^-$ and let $e' \in E^+$. Let T be defined by

$$T = \begin{cases} \min\{\frac{\varphi}{|\min\{v(e), e\in E^-\}|}, \frac{y-\varphi}{\max\{v(e), e\in E^+\}}\}, & \varphi \neq 0, y \\ \frac{y}{\max\{v(e), e\in E^+\}}, & \varphi = 0 \\ \frac{y}{|\min\{v(e), e\in E^-\}|} & \varphi = y. \end{cases}$$

Then during time s such that $0 < s < T$ the process $(\varphi_t)_{t\geq 0}$ cannot exit the interval $(0, y)$. However, during the same time s the process $(X_t)_{t\geq 0}$ can jump into the state $e' \in E^+$ and after the jump it can stay in e' long enough for the process $(\varphi_t)_{t\geq 0}$ to hit y. Hence,

$$\begin{aligned} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0) \ \geq & \int_0^T P_{(e,\varphi)}\Big((X_t)_{t\geq 0} \text{ is in the state } e' \text{ at time } s \\ & \text{and after time } s \text{ it stays in } e' \text{ at least for} \\ & \text{the time } \frac{y - \varphi - s\min\{v(e), e \in E^-\}}{v(e')}\Big) ds \\ = & \int_0^T e^{sQ}(e, e') e^{\frac{y-\varphi-s\min\{v(e), e\in E^-\}}{v(e')} Q(e',e')} ds > 0, \end{aligned}$$

because the matrix Q is irreducible and, by Lemma 1.2.1, $e^{sQ} > 0$ for all $s > 0$.

It follows that, for any $(e,\varphi) \in E_0^+ \cap E_y^-$,

$$P_{(e,\varphi)}(H_y < H_0) = \sum_{e' \in E^+} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0) > 0.$$

In the same way we can show that, for any $y < 0$ and $(e,\varphi) \in E_0^- \cap E_y^+$,

$$P_{(e,\varphi)}(H_y < H_0) > 0.$$

Moreover, $E_0^+ \cap E_y^- = E_{-y}^+ \cap E_0^-$ and $P_{(e,\varphi)}(H_0 < H_y) = P_{(e,\varphi-y)}(H_{-y} < H_0)$ for any $\varphi, y \in \mathbb{R}$ and $e \in E$. Thus, for $y > 0$ and $(e,\varphi) \in E_0^+ \cap E_y^-$,

$$P_{(e,\varphi)}(H_0 < H_y) = P_{(e,\varphi-y)}(H_{-y} < H_0) > 0,$$

which implies that for $y > 0$ and $(e,\varphi) \in E_0^+ \cap E_y^-$, $P_{(e,\varphi)}(H_y < H_0) < 1$. □

As a consequence of Lemma 1.5.3 we prove

Lemma 1.5.4 *For any $(e,\varphi) \in E \times \mathbb{R}$ and $T > 0$, $P_{(e,\varphi)}(H_0 > T) > 0$.*

Proof: Suppose that, for some $T > 0$, $P_{(e,\varphi)}(H_0 > T) = 0$, that is the process $(\varphi_t)_{t\geq 0}$ almost surely crosses zero before time T. If $(e,\varphi) \in E_0^+$, then the process $(\varphi_t)_{t\geq 0}$ cannot pass beyond the level $\varphi + T\ max_{e\in E^+} v(e)$ before it crosses zero, and if $(e,\varphi) \in E_0^-$, then the process $(\varphi_t)_{t\geq 0}$ cannot pass below the level $\varphi - T\ max_{e\in E^-}|v(e)|$ before it crosses zero. Hence,

$$\begin{aligned} P_{(e,\varphi)}(H_y < H_0) = 0, &\quad (e,\varphi) \in E_0^+, \quad y \geq \varphi + T \max\nolimits_{e\in E^+} v(e) \\ P_{(e,\varphi)}(H_y < H_0) = 0, &\quad (e,\varphi) \in E_0^-, \quad y \geq \varphi - T \max\nolimits_{e\in E^-} |v(e)|, \end{aligned}$$

which is in contradiction with Lemma 1.5.3. Therefore, for any (e,φ) and any $T > 0$, $P_{(e,\varphi)}(H_0 > T) > 0$. □

1.6 The generators of $(X_t, \varphi_t, t)_{t\geq 0}$ and $(X_t, \varphi_t)_{t\geq 0}$

Let $\mathcal{A}$ be the infinitesimal generator of the process $(X_t, \varphi_t, t)_{t\geq 0}$ and let $\mathcal{D}_{\mathcal{A}}$ denote its domain. Then $f \in \mathcal{D}_{\mathcal{A}}$ if the limit

$$\mathcal{A}f(e,\varphi,s) = \lim_{t\to 0} \frac{E_{(e,\varphi)} f(X_t, \varphi_t, s+t) - f(e,\varphi,s)}{t} \tag{1.18}$$

exists for any $(e,\varphi,s) \in E \times \mathbb{R} \times [0,+\infty)$. We shall prove that the class of functions continuously differentiable in φ and t is in the domain of $\mathcal{A}$.

Lemma 1.6.1 *Let a function $f(e,\varphi,t)$ on $E\times\mathbb{R}\times[0,+\infty)$ be continuously differentiable in φ and t. Then $f\in\mathcal{D}_{\mathcal{A}}$ and*

$$\mathcal{A}f=\Big(Q+V\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial t}\Big)f,$$

where

$$\begin{aligned}Qf(e,\varphi,t) &= \textstyle\sum_{e'\in E}Q(e,e')f(e',\varphi,t)\\ V\tfrac{\partial f}{\partial\varphi}(e,\varphi,t) &= V(e,e)\tfrac{\partial f}{\partial\varphi}(e,\varphi,t).\end{aligned}$$

Proof: First we have to check if the limit (1.18) exists:

We can rewrite $E_{(e,\varphi)}f(X_t,\varphi_t,s+t)-f(e,\varphi,s)$ as

$$\begin{aligned}&E_{(e,\varphi)}\,f\,(X_t,\varphi_t,s+t)-f(e,\varphi,s)\\
&\quad= E_{(e,\varphi)}\Big((f(X_t,\varphi_t,s+t)-f(e,\varphi,s))\ I\{(X_s)_{s\geq 0}\text{ stays in } e \text{ for time } t\}\Big)\\
&\quad+\sum_{e'\in E\setminus\{e\}}E_{(e,\varphi)}\Big((f(X_t,\varphi_t,s+t)-f(e,\varphi,s))\ I\{(X_s)_{s\geq 0}\text{ jumps exactly}\\
&\qquad\text{once during } [0,t] \text{ and it jumps to } e'\}\Big)\\
&\quad+E_{(e,\varphi)}\Big((f(X_t,\varphi_t,s+t)-f(e,\varphi,s))I\{(X_s)_{s\geq 0}\text{ makes more than}\\
&\qquad\text{one jump in time } t\}\Big).\end{aligned}\tag{1.19}$$

If $q_{ef}=Q(e,f)$, $e,f\in E$, then we have that

$$P_{(e,\varphi)}\Big((X_s)_{s\geq 0}\text{ stays in } e \text{ for time } t\Big)=e^{-q_e t},$$

where $q_e=-q_{ee}=\sum_{f\in E\setminus\{e\}}q_{ef}$, and

$$\begin{aligned}&P_{(e,\varphi)}\ \Big(\ (X_s)_{s\geq 0}\text{ jumps exactly once during time } t \text{ and it jumps to } e'\Big)\\
&\quad=\int_0^t e^{-q_e s}\ q_{ee'}\ e^{-q_{e'}(t-s)}\ ds\ =\ q_{ee'}\ e^{-q_{e'}t}\ \frac{1-e^{-(q_e-q_{e'})t}}{q_e-q_{e'}}.\end{aligned}$$

Thus,

$$\begin{aligned}&\lim_{t\to 0}\frac{P_{(e,\varphi)}\Big((X_s)_{s\geq 0}\text{ makes more than one jump during time } t\Big)}{t}\\
&=\lim_{t\to 0}\Big(\frac{1-e^{-q_e t}}{t}-\sum_{f\in E\setminus\{e\}}q_{ef}\ e^{-q_f t}\ \frac{1-e^{-(q_e-q_f)t}}{(q_e-q_f)t}\Big)\ =\ -q_e-\sum_{f\in E\setminus\{e\}}q_{ef}=0.\end{aligned}\tag{1.20}$$

The first term on the right-hand side of (1.19) is equal to

$$\begin{aligned}\Big(f(e,\varphi+tv(e),s+t)-f(e,\varphi,s)\Big)\ & P_{(e,\varphi)}\Big((X_s)_{s\geq 0} \text{ stays in } e \text{ during time } t\Big)\\ &= \Big(f(e,\varphi+tv(e),s+t)-f(e,\varphi,s)\Big)e^{-q_e t}\end{aligned}$$

and then

$$\begin{aligned}&\lim_{t\to 0}\frac{f(e,\varphi+tv(e),s+t)-f(e,\varphi,s)}{t}P_{(e,\varphi)}\Big((X_s)_{s\geq 0} \text{ stays in } e \text{ during time } t\Big)\\ &=\lim_{t\to 0}\Big(\frac{f(e,\varphi+tv(e),s+t)-f(e,\varphi,s+t)}{t}+\frac{f(e,\varphi,s+t)-f(e,\varphi,s)}{t}\Big)\ e^{-q_e t}\\ &=v(e)\ \frac{\partial f}{\partial\varphi}(e,\varphi,s)+\frac{\partial f}{\partial t}(e,\varphi,s)=\Big(V\frac{\partial f}{\partial\varphi}+\frac{\partial f}{\partial t}\Big)(e,\varphi,s). \qquad (1.21)\end{aligned}$$

The second term on the right-hand side of (1.19) can be written as

$$\begin{aligned}&\sum_{e'\in E\setminus\{e\}}\int_0^t (f(e',\varphi+uv(e)+(t-u)v(e'),s+t)-f(e,\varphi,s))\ \ e^{-q_e u}\ q_{ee'}\ e^{-q_{e'}(t-u)}\ du\\ &\quad=\sum_{e'\in E\setminus\{e\}} e^{-q_e t}\ q_{ee'}\Big(\int_0^t (f(e',\varphi,s+t)-f(e',\varphi,s))e^{-(q_e-q'_e)u}\ du\\ &\quad+\sum_{e'\in E\setminus\{e\}} e^{-q_e t}\ q_{ee'}\Big(\int_0^t (f(e',\varphi,s)-f(e,\varphi,s))e^{-(q_e-q'_e)u}\ du\\ &\quad+\int_0^t (f(e',\varphi+uv(e)+(t-u)v(e'),s+t)-f(e',\varphi,s+t))e^{-(q_e-q'_e)u}\ du\Big).\end{aligned}$$

The function $f(e,\varphi,s)$ is continuously differentiable in φ and s and so the derivatives $\frac{\partial f}{\partial\varphi}$ and $\frac{\partial f}{\partial s}$ are bounded on every closed interval. If B_φ is the bound for $|\frac{\partial f}{\partial\varphi}(e',y,0)|$ when $|y-\varphi|\leq\max_{e\in E}|v(e)|$, then

$$|f(e',\varphi+uv(e)+(t-u)v(e'),s+t)-f(e',\varphi,s+t)|\ \leq\ B_\varphi\ \max_{e\in E}|v(e)|,$$

and if C is the bound for $|\frac{\partial f}{\partial s}(e',\varphi,u)|$ when $0\leq u\leq t$, then

$$|f(e',\varphi,s+t)-f(e',\varphi,s)|\ \leq\ C\ t.$$

Since

$$\lim_{t\to 0}e^{-q_e t}\ q_{ee'}\int_0^t e^{-(q_e-q'_e)u}\ du=0$$

and

$$\lim_{t\to 0} \frac{1}{t}\, e^{-q_e t}\, q_{ee'} \int_0^t e^{-(q_e - q'_e)u}\, du = q_{ee'},$$

we obtain

$$\lim_{t\to 0} \frac{1}{t}\, e^{-q_e t} q_{ee'} \int_0^t (f(e', \varphi + uv(e) + (t-u)v(e'), s+t) - f(e', \varphi, s)) e^{-(q_e - q'_e)u}\, du = 0,$$

and

$$\lim_{t\to 0} \frac{1}{t}\, e^{-q_e t} q_{ee'} \int_0^t (f(e', \varphi, s+t) - f(e', \varphi, s)) e^{-(q_e - q'_e)u}\, du = 0,$$

and therefore

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t} \sum_{e' \in E\setminus\{e\}} \int_0^t & (f(e', \varphi + uv(e) + (t-u)v(e'), s+t) - f(e, \varphi, s)) \times \\
& e^{-q_e u}\, q_{ee'}\, e^{-q_{e'}(t-u)}\, du \\
= & \sum_{e' \in E\setminus\{e\}} q_{ee'}(f(e', \varphi, s) - f(e, \varphi, s)) \\
= & \sum_{e' \in E\setminus\{e\}} q_{ee'} f(e', \varphi, s) - q_e f(e, \varphi, s) = Qf(e, \varphi, s). \qquad (1.22)
\end{aligned}$$

For the third term on the right-hand side of (1.19) it is enough to note that, because the statespace E is finite and the function $f(e, \varphi, t)$ is continuous, $|f(e', y, t) - f(e, \varphi, 0)|$ is bounded on every closed interval and therefore, from (1.20) it follows that

$$\begin{aligned}
\lim_{t\to 0} \frac{1}{t}\, E_{(e,\varphi)}\Big(& (f(X_t, \varphi_t, s+t) - f(e, \varphi, s)) I\{(X_s)_{s\geq 0} \text{ makes} \\
& \text{more than one jump}\}\Big) = 0. \qquad (1.23)
\end{aligned}$$

Finally, from (1.19), (1.21), (1.22) and (1.23) we get that

$$\lim_{t\to 0} \frac{E_{(e,\varphi)} f(X_t, \varphi_t, s+t) - f(e, \varphi, s)}{t} = (V\frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial s} + Qf)(e, \varphi, s).$$

Therefore, we conclude that $f \in \mathcal{D}_\mathcal{A}$ and that

$$\mathcal{A}f = Qf + V\frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial t}.$$

□

Let $\mathcal{G}$ be the infinitesimal generator of the process $(X_t, \varphi_t)_{t\geq 0}$ and $\mathcal{D}_\mathcal{G}$ its domain. Then $f \in \mathcal{D}_\mathcal{G}$ if the limit

$$\mathcal{G}f(e,\varphi) = \lim_{t\to 0} \frac{E_{(e,\varphi)} f(X_t, \varphi_t) - f(e,\varphi)}{t}$$

exists for any $(e,\varphi) \in E \times \mathbb{R}$.

Similarly to Lemma 1.6.1, we prove that all functions on $E \times \mathbb{R}$ continuously differentiable in φ are in $\mathcal{D}_\mathcal{G}$.

Lemma 1.6.2 *Let a function $f(e,\varphi)$ on $E\times\mathbb{R}$ be continuously differentiable in φ. Then $f \in \mathcal{D}_\mathcal{G}$ and*

$$\mathcal{G}f = \Big(Q + V\frac{\partial}{\partial\varphi}\Big)f.$$

Proof: Let a function $\tilde{f}$ be defined on $E \times \mathbb{R} \times [0,+\infty)$ by

$$\tilde{f}(e,\varphi,t) = f(e,\varphi) \qquad (e,\varphi,t) \in E \times \mathbb{R} \times [0,+\infty).$$

The function $\tilde{f}$ is continuously differentiable in φ and t and, by Lemma 1.6.1, $\mathcal{A}\tilde{f}$ exists and is equal to $(Q + V\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t})\tilde{f}$. Hence, for any $(e,\varphi) \in E \times \mathbb{R}$,

$$\begin{aligned}
\mathcal{G}f(e,\varphi) &= \lim_{t\to 0} \frac{E_{(e,\varphi)} f(X_t, \varphi_t) - f(e,\varphi)}{t} \\
&= \lim_{t\to 0} \frac{E_{(e,\varphi)} \tilde{f}(X_t, \varphi_t, t) - \tilde{f}(e,\varphi,0)}{t} \\
&= \mathcal{A}\tilde{f}(e,\varphi,0) = (Q + V\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t})\tilde{f}(e,\varphi,0) \\
&= (Q + V\frac{\partial}{\partial\varphi})f(e,\varphi),
\end{aligned}$$

which proves the lemma. □

The next lemma gives the general solution of the differential equation $\mathcal{G}h = 0$.

Lemma 1.6.3 *The general solution of the differential equation*

$$\mathcal{G}h = \Big(Q + V\frac{\partial}{\partial\varphi}\Big)h = 0$$

is of the form

$$h(e,\varphi) = e^{-\varphi V^{-1}Q}\, g(e), \qquad (e,\varphi) \in E \times \mathbb{R},$$

where g is a vector on E.

Proof: We recall the basis $\mathcal{B}$ in the space of all vectors on E, which consists of the vectors associated with the eigenvalues of the matrix $V^{-1}Q$.

It is easy to check that for any vector g on E, a function

$$h(e,\varphi) = e^{-\varphi V^{-1}Q}\, g(e), \qquad (e,\varphi) \in E \times \mathbb{R},$$

is a solution of the differential equation

$$Qh + V\frac{\partial h}{\partial\varphi} = 0. \tag{1.24}$$

Hence, for any $u \in \mathcal{B}$, a function $h_u(e,\varphi)$ defined by

$$h_u(e,\varphi) = e^{-\varphi V^{-1}Q}\, u(e), \qquad (e,\varphi) \in E \times \mathbb{R}$$

is a solution of differential equation (1.24).

By Theorem 2.1. in Coddington, Levinson [10], all solutions of differential equation (1.24) form a $(n+m)$-dimensional vector space, where $(n+m)$ is the dimension of the matrix Q. We shall show that the system of functions $\{h_u, u \in \mathcal{B}\}$ is a basis in this vector space.

Firstly, because the basis $\mathcal{B}$ contains $(n+m)$ vectors, the system $\{h_u, u \in \mathcal{B}\}$ contains also $(n+m)$ functions. Secondly, we have to show that the functions h_u, $u \in \mathcal{B}$, are linearly independent. Suppose that

$$\sum_{u\in\mathcal{B}} \alpha_u\, h_u(e,\varphi) = 0$$

for any $(e, \varphi) \in E \times \mathbb{R}$ and some scalars α_u, $u \in \mathcal{B}$. Substituting $\varphi = 0$ in the previous equation we obtain

$$0 = \sum_{u \in \mathcal{B}} \alpha_u \; h_u(e, 0) = \sum_{u \in \mathcal{B}} \alpha_u \; u(e),$$

and because the vectors u, $u \in \mathcal{B}$, are linearly independent, it folows that $\alpha_u = 0$, $u \in \mathcal{B}$. Therefore, the functions $\{h_u, u \in \mathcal{B}\}$ are linearly independent.

It follows that the system $\{h_u, u \in \mathcal{B}\}$ is a basis in the vector space of all solutions of differential equation (1.24). Hence any solution is a linear combination of $\{h_u, u \in \mathcal{B}\}$ which implies that the general solution of (1.24) is of the form

$$h(e, \varphi) = e^{-\varphi V^{-1} Q} \; g(e), \qquad (e, \varphi) \in E \times \mathbb{R},$$

where g is a vector on E. $\square$

1.7 The behaviour of the process $(\varphi_t)_{t \geq 0}$

In Section 1.5 we were concerned with the hitting probabilities of the process $(X_t, \varphi_t)_{t \geq 0}$ and in particular we found the probability of the event that the process $(\varphi_t)_{t \geq 0}$ stays non-negative. However, in order to condition the process $(X_t, \varphi_t)_{t \geq 0}$ on the event that the process $(\varphi_t)_{t \geq 0}$ stays non-negative, we need more information about the behaviour of the process $(\varphi_t)_{t \geq 0}$. More importantly, we need to know how the behaviour of the process $(\varphi_t)_{t \geq 0}$ is determined by the matrices Q, V, Γ and G.

We start with the connection between the process $(\varphi_t)_{t \geq 0}$ and the matrices G^+ and G^-. By Lemma 1.4.3 the matrices G^+ and G^- are the Q-matrices of the processes $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ and $Y^- = (X_{\tau_y^-})_{y \geq 0}$. Hence, by the definition of the processes Y^+ and Y^- and the stopping times τ_y^+ and τ_y^-, $y > 0$, it follows that the process Y^+ has infinite lifetime and the matrix G^+ is conservative iff $sup_{t \geq 0} \varphi_t = +\infty$. Similarly, the process Y^- has infinite lifetime and the matrix G^- is conservative iff $inf_{t \geq 0} \varphi_t = -\infty$.

There are four cases depending on which of the matrices G^+ and G^- are conservative. One of them, that neither of the matrices G^+ and G^- is conservative, is not possible, as we show in the following lemma.

Lemma 1.7.1 *At least one of the matrices G^+ and G^- is conservative.*

Moreover, if both matrices G^+ and G^- are conservative, then zero is an eigenvalue of the matrix $V^{-1}Q$ with algebraic multiplicity two and geometric multiplicity one.

Proof: Suppose that neither of the matrices G^+ and G^- is conservative. Then, by Lemma 1.4.6 (ii), both matrices Π^+ and Π^- are strictly substochastic, which, by Lemma 1.4.5 implies that the matrix Γ is invertible. Hence, the Wiener-Hopf factorization (1.6) of the matrix $V^{-1}Q$ can be written as

$$V^{-1}Q = \Gamma\, G\, \Gamma^{-1},$$

which implies that the matrices $V^{-1}Q$ and G are similar.

Since, by Lemma 1.4.6 (i), the matrices G^+ and G^- are irreducible and also neither of them is conservative, by the Perron-Frobenius theorem for irreducible essentially non-negative matrices, both matrices G^+ and G^- have their Perron-Frobenius eigenvalues negative. Therefore, the matrix G does not have a zero eigenvalue which implies that $V^{-1}Q$ does not have a zero eigenvalue. But that is a contradiction because $V^{-1}Q1 = 0$. Thus at least one of the matrices G^+ and G^- is conservative.

Suppose now that both matrices G^+ and G^- are conservative. Then zero is a Perron-Frobenius and therefore simple eigenvalue of both G^+ and G^-. In addition, by (1.7) and (1.8), the eigenvalues of G^+ and G^- are also eigenvalues of the matrix $V^{-1}Q$. Hence, zero is an eigenvalue of $V^{-1}Q$ with algebraic multiplicity two.

Let f be an eigenvector of $V^{-1}Q$ associated with zero eigenvalue, that is $V^{-1}Qf = 0$. Then $Qf = 0$. Since Q is a conservative Q- matrix, its Perron-Frobenius eigenvalue is zero and, by the Perron-Frobenius theorem for irreducible essentially non-negative matrices, the eigenvector associated with zero eigenvalue of Q is unique up to a constant multiple.

Hence, there do not exists linearly independent eigenvectors of the matrix $V^{-1}Q$ associated with its zero eigenvalue which implies that the geometric multiplicity of zero eigenvalue is one. $\square$

By the previous lemma, either exactly one of the matrices G^+ and G^- is conservative or both of them are conservative. If both of the matrices G^+ and G^- are conservative, then $sup_{t\geq 0}\varphi_t = +\infty$ and $inf_{t\geq 0}\varphi_t = -\infty$, and because the process $(\varphi_t)_{t\geq 0}$ is continuous, we say that in this case the process $(\varphi_t)_{t\geq 0}$ oscillates.

We can also investigate the connection between the process $(\varphi_t)_{t\geq 0}$ and the matrix $V^{-1}Q$. The ergodic theorem for Markov chains (see Chung [9]) implies that

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t I\{X_s = e\}\;ds = \mu(e), \text{ a.s.}$$

where μ is the invariant measure of the process $(X_t)_{t\geq 0}$.

Hence,

$$\begin{aligned}
\lim_{t\to+\infty}\frac{\varphi_t}{t} &= \lim_{t\to+\infty}\frac{1}{t}\,(\varphi + \int_0^t v(X_s))\;ds \\
&= \lim_{t\to+\infty}\frac{1}{t}\int_0^t\sum_{e\in E} v(e)I\{X_s = e\}\;ds \\
&= \sum_{e\in E} v(e)\lim_{t\to+\infty}\frac{1}{t}\int_0^t I\{X_s = e\}\;ds \\
&= \sum_{e\in E} v(e)\mu(e) = \mu V1, \text{ a.s.} \qquad (1.25)
\end{aligned}$$

It follows from the last equation that if $\mu V1 > 0$ then $\lim_{t\to+\infty}\varphi_t = +\infty$ and we say that in this case the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$. Similarly, if $\mu V1 < 0$ then $\lim_{t\to+\infty}\varphi_t = -\infty$ and we say that in that case the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$.

The following theorem establishes a "one-to-one" correspondence between the sign of $\mu V1$ and the conservative property of the Q-matrices G^+ and G^-.

Theorem 1.7.1

$$\begin{array}{llll} (i) & \mu V1 > 0 & \textit{iff} & G^+ \textit{ is conservative}, G^- \textit{ is not conservative} \\ (ii) & \mu V1 = 0 & \textit{iff} & G^+ \textit{ and } G^- \textit{ are conservative}, \\ (iii) & \mu V1 < 0 & \textit{iff} & G^- \textit{ is conservative}, G^+ \textit{ is not conservative} \end{array}$$

Proof: First we prove the "only if" parts of all three statements in the lemma.

- proof of the (i) "only if" part: Suppose that $\mu V1 > 0$. From (1.25) we have that, for any $e \in E$,

$$P_{(e,0)}\big(\lim_{t\to+\infty} \frac{\varphi_t}{t} > 0\big) = 1,$$

which implies that

$$P_{(e,0)}\Big((\varphi_t)_{t\geq 0} \text{ is eventually positive}\Big) = 1.$$

In addition, by Lemma 1.4.3,

$$\begin{aligned} e^{yG^+}1^+(e) &= P_{(e,0)}(H_y < +\infty), \quad e \in E^+,\ y > 0 \\ e^{yG^-}1^-(e) &= P_{(e,0)}(H_{-y} < +\infty), \quad e \in E^-,\ y > 0. \end{aligned}$$

Hence, for fixed $y_0 > 0$ and any $e \in E^+$,

$$\begin{aligned} (1^+ - e^{y_0 G^+}1^+)(e) &= P_{(e,0)}(H_{y_0} = +\infty) \\ &\leq P_{(e,0)}\Big((\exists y > 0)\ H_y = +\infty\Big) \\ &= P_{(e,0)}\Big((\exists y > 0)\ (\forall t)\ \varphi_t < y\Big) \\ &= P_{(e,0)}\Big((\exists y > 0)\ (\forall t)\ \frac{\varphi_t}{t} < \frac{y}{t}\Big) \\ &\leq P_{(e,0)}\Big(\lim_{t\to+\infty} \frac{\varphi_t}{t} \leq 0\Big) = 0, \end{aligned}$$

which implies that $e^{y_0 G^+}1^+ = 1^+$. Hence, there exists $y > 0$ such that $e^{yG^+}1^+ = 1^+$ and since, by Lemma 1.4.6 (i), the matrix G^+ is an irreducible Q-matrix, it follows from Lemma 1.1.5 that the matrix G^+ is conservative.

On the other hand, for $e \in E^-$,

$$\begin{aligned} 1 &= P_{(e,0)}\Big((\varphi_t)_{t\geq 0} \text{ is eventually positive}\Big) \\ &\leq P_{(e,0)}\Big((\varphi_t)_{t\geq 0} \text{ spends finite time in } (-\infty,0)\Big) \\ &\leq P_{(e,0)}\Big((\exists y > 0)\ H_{-y} = +\infty\Big) \\ &\leq P_{(e,0)}\Big((\exists y \in \mathbb{N})\ H_{-y} = +\infty\Big) \\ &\leq \sum_{y=1}^{+\infty} P_{(e,0)}(H_{-y} = +\infty). \end{aligned}$$

It follows that there exists $y \in \mathbb{N}$ such that

$$(1^- - e^{yG^-}1^-)(e) = P_{(e,0)}(H_{-y} = +\infty) > 0.$$

Therefore, there exists $y > 0$ such that $e^{yG^-}1^- < 1^-$, and by Lemma 1.4.6 (i), the matrix G^- is an irreducible Q-matrix. Hence, by Lemma 1.1.5, the matrix G^- is not conservative.

- proof of the (ii) "only if" part, proof by contradiction: We shall show that if at least one of the matrices G^+ and G^- is not conservative then $\mu V 1 \neq 0$. Suppose that at least one of the matrices G^+ and G^- is not conservative. Since by Lemma 1.7.1 at least one of the matrices G^+ and G^- is conservative, we conclude that exactly one of the matrices G^+ and G^- is conservative. Then, by Lemma 1.4.6 (ii), exactly one of the matrices Π^+ and Π^- is strictly substochastic, which, by Lemma 1.4.5 implies that the matrix Γ is invertible. Hence, the Wiener-Hopf factorization (1.6) of the matrix $V^{-1}Q$ can be written as

$$V^{-1}Q = \Gamma\, G\, \Gamma^{-1},$$

which implies that the matrices $V^{-1}Q$ and G are similar.

Since exactly one of the matrices G^+ and G^- is conservative, the matrix G has a simple zero eigenvalue and therefore the matrix $V^{-1}Q$ has a simple zero eigenvalue. By Lemma 1.1.1 (ii), the product of the left and right eigenvectors of a matrix which are associated with the same simple eigenvalue cannot be zero. Thus, since μV and 1 are the left and right eigenvectors, respectively, of the matrix $V^{-1}Q$ associated with the zero eigenvalue, we conclude that $\mu V 1 \neq 0$.

- the (iii) "only if" part can be proved in the same way as the (i)'only if' part

Hence, we have proved the "only if" parts of all three statements in the lemma. The only three possible cases of the sign of $\mu V 1$ imply the only three possible combinations of the matrices G^+ and G^- being conservative. Therefore, the "if" parts of the statements in the lemma are also proved. □

From the discussion before the previous theorem and by the theorem itself it follows that there are only three possible cases of the behaviour of the process $(\varphi_t)_{t\geq 0}$. More precisely,

$$\begin{aligned} (\varphi_t)_{t\geq 0} \text{ drifts to } +\infty \quad &\text{iff} \quad \mu V 1 > 0 \\ (\varphi_t)_{t\geq 0} \text{ oscillates} \quad &\text{iff} \quad \mu V 1 = 0 \\ (\varphi_t)_{t\geq 0} \text{ drifts to } -\infty \quad &\text{iff} \quad \mu V 1 < 0. \end{aligned} \tag{1.26}$$

We shall discuss conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 = +\infty\}$ in each of these three cases separately. In future we shall refer to them as the positive drift case, the oscillating case and the negative drift case.

Recalling the notation introduced in Section 1.4, we point out consequences of Lemma 1.7.1, Theorem 1.7.1 and (1.26) that will be constantly in use in the sequel:

1) in the positive drift case, the matrix G^+ is conservative and the matrix G^- is not conservative; $\alpha_{max} = 0$ and $-\beta_{min} < 0$ and zero is a simple eigenvalue of $V^{-1}Q$; (1.27)

2) in the oscillating case, both matrices G^+ and G^- are conservative; $\alpha_{max} = \beta_{min} = 0$ and zero is an eigenvalue of $V^{-1}Q$ with algebraic multiplicity two and geometric multiplicity one; by Jordan normal form theory the equation $V^{-1}Qx = 1$ has a solution. (1.28)

3) in the negative drift case, the matrix G^- is conservative and the matrix G^+ is not conservative; $\alpha_{max} < 0$ and $-\beta_{min} = 0$ and zero is a simple eigenvalue of $V^{-1}Q$; (1.29)

Let $f_1 = f_{max}$ and $g_1 = g_{min}$ be the eigenvectors of $V^{-1}Q$ associated with the eigenvalues α_{max} and β_{min}, respectively. Then, in the positive drift case, $f_{max} = 1 \neq g_{min}$, and in the negative drift case, $g_{min} = 1 \neq f_{max}$, and in both cases the basis $\mathcal{B}$ in the space of all vectors on E is equal to $\{f_j, j = 1,\dots,n,\ g_k, k = 1,\dots,m\}$. In the oscillating case, $f_{max} = g_{min} = 1$ and by (1.28), the equation

$$V^{-1}Qx = 1 \tag{1.30}$$

has a solution. More precisely, the previous equation has infinitely many solutions since, for any solution r, the vector $r + c1$, $c \in \mathbb{R}$, is also a solution. If r is a solution of equation (1.30), then, by Jordan normal form theory, r is linearly independent from the vectors $\{f_j, j = 1,\dots,n,\ g_k, k = 1,\dots,m\}$ and $\mathcal{B} = \{1, r, f_j, j = 2,\dots,n,\ g_k, k = 2,\dots,m\}$ is a basis in the space of all vectors on E.

1.8 The maximal negative and the minimal positive eigenvalues of $V^{-1}(Q - \alpha I)$

In Section 1.4 we have looked more closely at the eigenvalues of the matrix $V^{-1}Q$ and vectors associated with them. However, for some of our results we need information about the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$, $\alpha > 0$, and their associated vectors. By Lemma 1.4.1 and the discussion immediately after it, for $\alpha > 0$, all eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ have either positive or negative real part, the eigenvalue with maximal negative real part is simple, real and is the Perron-Frobenius eigenvalue of the matrix G_α^+, and the eigenvalue with minimal positive real part is simple and real and has the same absolute value but the opposite sign of the Perron-Frobenius eigenvalue of the matrix G_α^-. In addition, by Lemma 1.4.8 (iii), for $\alpha > 0$, the eigenvectors of the matrix $V^{-1}(Q - \alpha I)$ which are associated with its eigenvalues with maximal negative and minimal positive real parts are the only positive eigenvectors of $V^{-1}(Q - \alpha I)$. Our aim in this section is to find the behaviour for small $\alpha > 0$ of the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ with maximal negative and minimal positive real parts and their associated eigenvectors.

Theorem 1.8.3 given below establishes this behaviour in each of the three possible cases of the behaviour of the process $(\varphi_t)_{t\geq 0}$. But before we prove Theorem 1.8.3, we prove several lemmas. Lemmas 1.8.2 and 1.8.3 are auxiliary lemmas only needed for the results in this section and their proofs are given in Appendix B. Lemmas 1.8.1, 1.8.4 and 1.8.5 will be used in sections that follow and their proofs are given in full in this section.

Let $\alpha \geq 0$ and let $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ be the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ with maximal negative and minimal positive real part, respectively. In order to understand their behaviour for small $\alpha > 0$ we shall study their behaviour for small $|\alpha|$. The Wiener-Hopf factorizations of the matrices $V^{-1}(Q - \alpha I)$, $\alpha \geq 0$, given in Lemmas 1.4.1 and 1.4.3 provide us with information about $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ only for $\alpha \geq 0$. For $\alpha < 0$, the Wiener-Hopf factorization of the matrix $V^{-1}(Q-\alpha I)$ and its probabilistic interpretation are not valid any longer and we are left with no information about the eigenvalues of $V^{-1}(Q - \alpha I)$. However, the equivalence

$$V^{-1}(Q - \alpha I)u = \beta u \qquad \text{iff} \qquad (Q - \beta V)u = \alpha u \tag{1.31}$$

is true for all $\alpha \in \mathbb{R}$. In order to stress its importance, we put (1.31) into words: β is an eigenvalue of the matrix $V^{-1}(Q - \alpha I)$ if and only if α is an eigenvalue of the matrix $(Q - \beta V)$. Vaguely speaking, the values α and β appear to be inverses of each other.

By Lemma 1.1.6, the matrix $(Q - \beta V)$ is essentially non-negative and irreducible for all $\beta \in \mathbb{R}$. Hence, (1.31) implies that α is the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$ if and only if β is an eigenvalue of the matrix $V^{-1}(Q - \alpha I)$ whose associated eigenvector is positive. By Lemma 1.4.8 (iii), for fixed $\alpha > 0$, the only eigenvalues of $V^{-1}(Q - \alpha I)$ which have associated positive eigenvectors are $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$. Then, by (1.31), for fixed $\alpha > 0$, the eigenvalues $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ of the matrix $V^{-1}(Q - \alpha I)$ are the only values of β for which the matrix $(Q - \beta V)$ has the Perron-Frobenius eigenvalue equal to α. This relation is the key argument in the proof of Theorem 1.8.3 about the behaviour of $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ for small $|\alpha|$.

The advantage of looking at the matrix $(Q-\beta V)$ and its Perron-Frobenius eigenvalue α, instead of the matrix $V^{-1}(Q - \alpha I)$ and its eigenvalues, is that we are not limited to non-negative values of α.

For any $\beta \in \mathbb{R}$, let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$ and let $u^{left}(\beta)$ and $u^{right}(\beta)$ be the associated left and right eigenvectors such that $\|u^{left}(\beta)\| = \|u^{right}(\beta)\| = 1$ in some norm in the space $\mathbb{R}^E$. The choice of the norm $\|\cdot\|$ is arbitrary since the only reason why we need the vectors $u^{left}(\beta)$ and $u^{right}(\beta)$ to be unit is to prove their continuity in Lemma 1.8.3.

A striking property of the eigenvalue $\alpha(\beta)$ is that it is a convex function of β. We prove that in the following lemma.

Lemma 1.8.1 *Let $\beta \in \mathbb{R}$ and let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$. Then, $\alpha(\beta)$ is a convex function of β and therefore continuous. It attains its global minimum and has two zeros, $\alpha_{max} \leq 0$ and $\beta_{min} \geq 0$, not necessarily distinct.*

Proof: For any $\beta \in \mathbb{R}$, by Lemma 1.1.6, $(Q - \beta V)$ is an irreducible essentially non-negative matrix. In Cohen [11] it was proved that the Perron-Frobenius eigenvalue of an essentially nonnegative matrix A is a convex function of the main diagonal of A. In other words, if $r(A)$ is the Perron-Frobenius eigenvalue of A, then for any diagonal matrix D and any t, $0 < t < 1$,

$$r((1-t)A + t(A+D)) \leq (1-t)\, r(A) + t\, r(A+D). \tag{1.32}$$

Let

$$\alpha(\beta) = r(Q - \beta V).$$

Then, for fixed $x, y \in \mathbb{R}$, by substituting in (1.32) $(Q - xV)$ and $(x-y)V$ for A and D, respectively, we get that, for $0 < t < 1$,

$$\begin{aligned}
\alpha((1-t)\, x + t\, y) &= r(Q - ((1-t)\, x + t\, y)V) \\
&= r((1-t)(Q - xV) + t(Q - yV)) \\
&\leq (1-t)\, r(Q - xV) + t\, r(Q - yV) \\
&= (1-t)\, \alpha(x) + t\, \alpha(y),
\end{aligned}$$

which implies that the Perron-Frobenius eigenvalue $\alpha(\beta)$ of $(Q - \beta V)$ is a convex function of β. Since every finite convex function is continuous, $\alpha(\beta)$ is also a continuous function of β.

In addition, from the Perron-Frobenius theorem for irreducible essentially non-negative matrices it follows that if $\alpha(\beta) \leq 0$, then there exists a positive vector f such that

$$(Q - \beta V)f \leq 0, \tag{1.33}$$

and $\alpha(\beta) = 0$ if and only if there is equality in (1.33).

Suppose that

$$\beta < \min\{-\frac{q_e}{v(e)} :\ e \in E\} \tag{1.34}$$

or

$$\beta > \max\{-\frac{q_e}{v(e)} :\ e \in E\}, \tag{1.35}$$

where $q_e = -Q(e,e)$, $e \in E$. Then some diagonal elements of $(Q - \beta V)$ are non-negative (for instance, if β satisfies (1.34) then for all e for which $v(e) > 0$, $-q_e - \beta v(e) > 0$), which implies that $(Q - \beta V)$ has some non-negative non-zero rows and that inequality (1.33) is not possible for any positive vector f. Therefore, for those β which satisfy (1.34) or (1.35), or in other words, for $|\beta|$ sufficiently large, $\alpha(\beta) > 0$.

Suppose that $\alpha(\beta) = 0$. Then there exists a positive vector f such that

$$(Q - \beta V)f = 0$$

or, equivalently,

$$V^{-1}Qf = \beta\, f.$$

By Lemma 1.4.10, there exist exactly two eigenvalues of $V^{-1}Q$ (not necessarily distinct) whose associated eigenvectors are positive: they are α_{max} and β_{min}. Thus, α_{max} and β_{min} are the only zeros of $\alpha(\beta)$. In addition, since α_{max} and $-\beta_{min}$ are the Perron-Frobenius eigenvalues of the matrices G^+ and G^-, respectively, $\alpha_{max} \leq 0$ and $\beta_{min} \geq 0$, and because, by Lemma 1.7.1, at least one of the matrices G^+ and G^- is conservative, at least one of the values α_{max} and β_{min} is equal to zero.

Therefore, the function $\alpha(\beta)$ is continuous because it is convex, for $|\beta|$ sufficiently large it is positive and it has either one or two zeros. All of these together imply that $\alpha(\beta)$ attains its minimum. □

For the further study of properties of the eigenvalue $\alpha(\beta)$, we need two technical lemmas whose proofs are given in Appendix B.

Lemma 1.8.2 *If f is a convex function on $\mathbb{R}$ and attains a local minimum at x_0, then this minimum is a global minimum. In addition, if f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof: See Appendix B. □

Lemma 1.8.3 *The vector $u^{right}(\beta)$ is a continuous function of β.*

Proof: See Appendix B. □

Now we are able to prove the differentiability of $\alpha(\beta)$.

Lemma 1.8.4 *Let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue and let $u^{left}(\beta)$ and $u^{right}(\beta)$ be the unit Perron-Frobenius left and right eigenvectors of the matrix $(Q-\beta V)$. Then $\alpha(\beta)$ is a differentiable function of β and*

$$\frac{d\alpha}{d\beta}(\beta) = -\frac{u^{left}(\beta)\ V\ u^{right}(\beta)}{u^{left}(\beta)\ u^{right}(\beta)}.$$

In addition, there is a unique $\beta_0 \in (\alpha_{max}, \beta_{min})$ such that $\frac{d\alpha}{d\beta}(\beta_0) = 0$ and $\alpha(\beta_0)$ is the global minimum of the function $\alpha(\beta)$ and

$$\frac{d\alpha}{d\beta}(\beta) \begin{cases} < 0, & \text{if } \beta < \beta_0 \\ = 0, & \text{if } \beta = \beta_0 \\ > 0, & \text{if } \beta > \beta_0, \end{cases}$$

Proof: By multiplying from the left by $u^{left}(\beta)$ the equality

$$\begin{aligned}(Q-\beta V)\ u^{right}(\beta+h) - h\ V\ u^{right}(\beta+h)& \\ &= (\alpha(\beta+h)-\alpha(\beta))\ u^{right}(\beta+h)\ +\ \alpha(\beta)\ u^{right}(\beta+h),\end{aligned}$$

we obtain

$$-\,h\ u^{left}(\beta)\ V\ u^{right}(\beta+h) = (\alpha(\beta+h)-\alpha(\beta))\ u^{left}(\beta)\ u^{right}(\beta+h). \tag{1.36}$$

For any $\beta \in \mathbb{R}$, the eigenvectors $u^{left}(\beta)$ and $u^{right}(\beta)$ are positive because they are the Perron-Frobenius eigenvectors of $(Q-\beta V)$, and also by Lemma 1.8.3, $u^{right}(\beta)$ is a continuous function of β. Thus $\lim_{h\to 0} u^{left}(\beta)u^{right}(\beta+h) = u^{left}(\beta)u^{right}(\beta) > 0$ and by dividing (1.36) by h and letting $h \to 0$, we obtain

$$\lim_{h\to 0}\frac{\alpha(\beta+h)-\alpha(\beta)}{h} = -\frac{u^{left}(\beta)\ V\ u^{right}(\beta)}{u^{left}(\beta)\ u^{right}(\beta)}.$$

The last equation shows that the Perron-Frobenius eigenvalue $\alpha(\beta)$ is a differentiable function of β and since by Lemma 1.8.1 it is also convex and attains its minimum, we conclude, by Lemma 1.8.2, that there exists unique β_0 such that $\alpha(\beta_0)$ is the global minimum of $\alpha(\beta)$ and that $\frac{d\alpha}{d\beta}(\beta_0) = 0$. By Lemma 1.8.1, $\alpha(\beta)$ has two zeros, $\alpha_{max} \leq 0$ and $\beta_{min} \geq 0$. Hence, $\beta_0 \in (\alpha_{max}, \beta_{min})$ when $\alpha_{max} \neq \beta_{min}$ and $\beta_0 = \alpha_{max} = \beta_{min}$ when $\alpha_{max} = \beta_{min}$.

It remains to show that $\alpha(\beta)$ is strictly monotone on the intervals $(-\infty, \beta_0]$ and $[\beta_0, +\infty)$, which will imply that $\frac{d\alpha}{d\beta}$ is negative on $(-\infty, \beta_0]$ and positive on $[\beta_0, +\infty)$.

Let α and $u(\beta)$ be the Perron-Frobenius eigenvalue and eigenvector, respectively, of the matrix $(Q - \beta V)$ and let $\alpha_0 = \alpha(\beta_0)$ be the global minimum of the Perron-Frobenius eigenvalue of $(Q - \beta V)$. Then, $\alpha \geq \alpha_0$ and, by (1.31), $u(\beta)$ is a positive eigenvector of the matrix $V^{-1}(Q - \alpha I)$.

(i) Suppose that $\beta_0 = 0$. Then $\alpha_0 = 0$ since α_0 is the Perron-Frobenius eigenvalue of the matrix $(Q - \beta_0 V) = Q$, and therefore $\alpha \geq \alpha_0 = 0$. By Lemma 1.4.8 (iii), for $\alpha > 0$, the only positive eigenvectors of the matrix $V^{-1}(Q - \alpha I)$ are $f_{max}(\alpha)$ and $g_{min}(\alpha)$ associated with the eigenvalues $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$, respectively. It follows that, for fixed $\alpha \geq \alpha_0$, there exist only two values of β, $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$, such that α is the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$. Since $\alpha_{max}(\alpha) \leq 0$ and $\beta_{min}(\alpha) \geq 0$, we conclude that $\alpha(\beta)$ is strictly monotone on both intervals $(-\infty, 0]$ and $[0, +\infty)$.

(ii) Let now $\beta_0 \in \mathbb{R}$ and let

$$Q_0 = Q - \beta_0 V - \alpha_0 I.$$

The matrix Q_0 is essentially non-negative and, by Lemma 1.1.6, irreducible, and so is the matrix $(Q_0 - \beta V)$ for any $\beta \in \mathbb{R}$. Let $\alpha_0(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q_0 - \beta V)$. then

$$\alpha_0(\beta) = \alpha(\beta + \beta_0) - \alpha_0. \tag{1.37}$$

Since the function $\alpha(\beta)$ attains its global minimum α_0 at $\beta = \beta_0$, it follows from (1.37) that the function $\alpha_0(\beta)$ attains its global minimum zero at $\beta = 0$. Therefore, by (i), $\alpha_0(\beta)$ is strictly monotone on the intervals $(-\infty, 0]$ and $[0, +\infty)$, which implies that $\alpha(\beta)$ is strictly monotone on the intervals $(-\infty, \beta_0]$ and $[\beta_0, +\infty)$. □

The sign of the unique argument β_0 of the global minimum of the function $\alpha(\beta)$, whose existence has been proved in the previous lemma, is found to depend on the behaviour of the process $(\varphi_t)_{t\geq 0}$. Namely,

Lemma 1.8.5

In the positive drift case $\beta_0 > 0$ *and* $\alpha_0 < 0$.

In the oscillating case $\beta_0 = 0$ *and* $\alpha_0 = 0$.

In the negative drift case $\beta_0 < 0$ *and* $\alpha_0 < 0$.

Proof: In the drift cases, $\alpha_{max} \neq \beta_{min}$ and therefore, by Lemma 1.8.4, $\beta_0 \in (\alpha_{max}, \beta_{min})$. In the positive drift case, by (1.27), $\alpha_{max} = 0$ and $\beta_{min} > 0$, and therefore $\beta_0 > 0$. In the negative drift case, by (1.29), $\beta_{min} = 0$ and $\alpha_{max} < 0$, and therefore $\beta_0 < 0$. Since in both cases the function $\alpha(\beta)$ has two distinct zeros, its global minimum α_0 is negative.

Finally, in the oscillating case, by (1.28), $\alpha_{max} = \beta_{min} = 0$ and then $\beta_0 = 0$. Thus, the function $\alpha(\beta)$ has exactly one zero at $\beta = 0$ and, since by 1.8.1, it attains a global minimum, it follows that $\alpha(\beta)$ attains its global minimum at $\beta_0 = 0$ and that $\alpha_0 = \alpha(\beta_0) = 0$. □

Therefore, by Lemmas 1.8.1, 1.8.4 and 1.8.5, the function $\alpha(\beta)$ is differentiable and convex, it has two zeros, α_{max} and β_{min}, not necessarily distinct, and it attains its global minimum α_0 at β_0, where in the drift cases $\beta_0 \in (\alpha_{max}, \beta_{min})$ and in the oscillating case $\alpha_{max} = \beta_{min} = \beta_0 = \alpha_0 = 0$. In addition, $\alpha(\beta)$ is strictly monotone on the intervals $(-\infty, \beta_0]$ and $[\beta_0, +\infty)$. The graph of the function $\alpha(\beta)$ is shown in Figure 1.2.

The final step, before we establish the behaviour for sufficiently small $|\alpha|$ of the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ with maximal negative and minimal positive

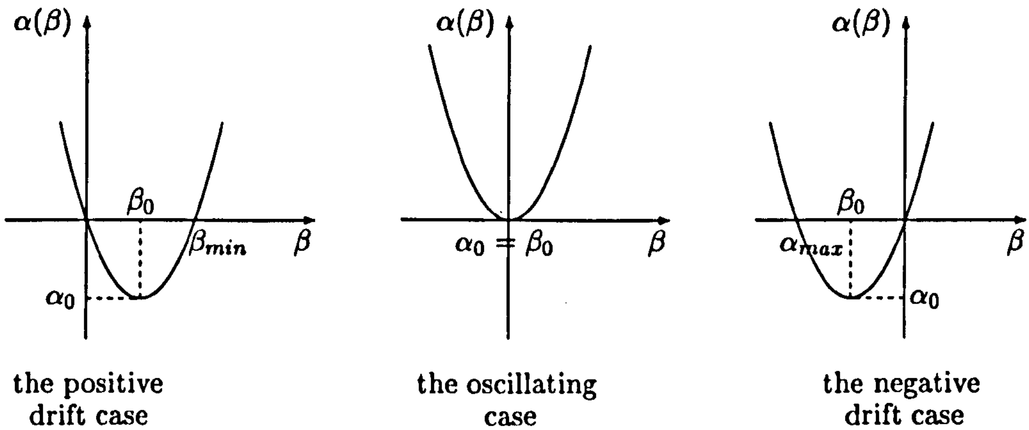

Figure 1.2: The Perron-Frobenius eigenvalue $\alpha(\beta)$ of the matrix $(Q - \beta V)$

real parts and their associated eigenvectors, is to review some results from the theory of algebraic functions.

Let $f(x, y)$ be defined by

$$f(x,y) = y^n + p_{n-1}(x)y^{n-1} + p_{n-2}(x)y^{n-2} + \ldots + p_0(x),$$

where the $p_i(x)$ are polynomials in x. For any value of x the equation $f(x, y) = 0$ has n roots $y_1(x), y_2(x), \ldots, y_n(x)$, not necessarily distinct.

The following two theorems are given in Wilkinson [33].

Theorem 1.8.1 *Suppose $y_i(0)$ is a simple root of $f(0, y) = 0$. Then there exists a positive δ_i such that there is a simple root $y_i(x)$ of $f(x, y) = 0$ defined by*

$$y_i(x) = y_i(0) + p_{i1}x + p_{i2}x^2 + \ldots,$$

where the series on the right of the previous equation is absolutely convergent for $|x| < \delta_i$.

Theorem 1.8.2 *If $y(0)$ is a root of multiplicity m of $f(0,y) = 0$ then there exists a positive δ such that there are exactly m roots of $f(x,y) = 0$ when $|x| < \delta$ having the following properties:*

The roots fall into r groups of $m_1, m_2, \ldots, m_r$ roots where

$$\sum_{i=1}^{r} m_i = m,$$

and those roots in the i^{th} group of are the values of

$$y(0) + p_{i1}z + p_{i2}z^2 + \ldots$$

evaluated at the m_i^{th} roots of x.

Now we are able to prove the main result in this section:

Theorem 1.8.3 *For $\alpha \geq 0$, Let $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ be the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ with maximal negative and minimal positive real parts, respectively, and let $f_{max}(\alpha)$ and $g_{min}(\alpha)$ be their associated eigenvectors, respectively.*

Then, in the oscillating case, there exists $\varepsilon > 0$ such that, for $0 < \alpha < \varepsilon$, and some constants d_n, $n = 2, 3, \ldots$ and $c > 0$,

$$\begin{aligned} \alpha_{max}(\alpha) &= -\frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + d_2\alpha + d_3\alpha^{\frac{3}{2}} + \ldots = -\frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + \Theta_{max}(\alpha^{\frac{1}{2}}) \\ \beta_{min}(\alpha) &= \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + d_2\alpha - d_3\alpha^{\frac{3}{2}} + \ldots = \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}} + \Theta_{min}(\alpha^{\frac{1}{2}}), \end{aligned}$$

and

$$|\Theta_{max}(\alpha^{\frac{1}{2}})| < c\ \alpha \quad \textit{and} \quad |\Theta_{min}(\alpha^{\frac{1}{2}})| < c\ \alpha.$$

The vectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be chosen to be

$$\begin{aligned} f_{max}(\alpha) &= 1 - \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \alpha\ v_2 + \ldots = 1 - \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \Xi_{max}(\alpha^{\frac{1}{2}}) \\ g_{min}(\alpha) &= 1 + \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \alpha\ w_2 + \ldots = 1 + \frac{1}{\sqrt{-\mu V r}}\ \alpha^{\frac{1}{2}}\ r + \Xi_{min}(\alpha^{\frac{1}{2}}), \end{aligned}$$

where r is a vector on E such that $V^{-1}Qr = 1$, and

$$|\Xi_{max}(\alpha^{\frac{1}{2}})| < \alpha\, v \quad \text{and} \quad |\Xi_{min}(\alpha^{\frac{1}{2}})| < \alpha\, w$$

for some positive vectors v and w on E.

In the negative drift case, there exists $\varepsilon > 0$ such that, for $0 < \alpha < \varepsilon$ and some constants a_n and b_n, $n \in mathbbN$,

$$\alpha_{max}(\alpha) = \alpha_{max} + a_1\alpha + a_2\alpha^2 + \ldots \quad \text{and} \quad \beta_{min}(\alpha) = b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \ldots,$$

and the vectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be chosen to be

$$f_{max}(\alpha) = f_{max} + \alpha v_1 + \alpha^2 v_2 + \ldots \quad \text{and} \quad g_{min}(\alpha) = 1 + \alpha w_1 + \alpha^2 w_2 + \ldots,$$

where v_n and w_n, $n \in mathbbN$, are some constant vectors, α_{max} is the eigenvalue of the matrix $V^{-1}Q$ with maximal negative real part and f_{max} is the eigenvector associated with it.

In the positive drift case, there exists $\varepsilon > 0$ such that, for $0 < \alpha < \varepsilon$ and some constants a_n and b_n, $n \in mathbbN$,

$$\alpha_{max}(\alpha) = a_1\alpha + +a_2\alpha^2 + \ldots \quad \text{and} \quad \beta_{min}(\alpha) = \beta_{min} + b_1\alpha + b_2\alpha^2 + \ldots,$$

and the vectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be chosen to be

$$f_{max}(\alpha) = 1 + \alpha v_1 + \alpha^2 v_2 + \ldots \quad \text{and} \quad g_{min}(\alpha) = g_{min} + \alpha w_1 + \alpha^2 w_2 + \ldots,$$

where v_n and w_n, $n \in mathbbN$, are some constant vectors, β_{min} is the eigenvalue of the matrix $V^{-1}Q$ with minimal positive real part and g_{min} is the eigenvector associated with it.

Proof: By Theorem 1.8.2 the eigenvalues of the matrix $V^{-1}(Q - \alpha I)$ converge to the eigenvalues of the matrix $V^{-1}Q$ as $\alpha \to 0$. Thus, all eigenvalues of $V^{-1}(Q - \alpha I)$ with negative real part converge to all eigenvalues of $V^{-1}Q$ with non-positive real part and all eigenvalues of $V^{-1}(Q-\alpha I)$ with positive real part converge to all eigenvalues of $V^{-1}Q$ with non-negative real part. Since, for all $\alpha > 0$, $\alpha_{max}(\alpha)$ is the real eigenvalue of the matrix $V^{-1}(Q-\alpha I)$ with maximal negative real part, it converges to the eigenvalue of $V^{-1}Q$ with maximal non-positive real part, which is α_{max}. Therefore, $\alpha_{max}(\alpha) \to \alpha_{max}$ as $\alpha \to 0$. Similarly, $\beta_{min}(\alpha)$ is the real eigenvalue of the matrix $V^{-1}(Q-\alpha I)$ with minimal positive real part, and it converges to the eigenvalue of $V^{-1}Q$ with minimal non-negative real part, which is β_{min}. Therefore, $\beta_{min}(\alpha) \to \beta_{min}$ as $\alpha \to 0$.

We recall that α_{max} and β_{min} are simple eigenvalues of the matrices G^+ and G^-, respectively, and that they are also eigenvalues of the matrix $V^{-1}Q$.

In the drift cases, by (1.27) and (1.29), $\alpha_{max} \neq \beta_{min}$. Hence, by the Wiener-Hopf factorization (1.6) of the matrix $V^{-1}Q$, α_{max} and β_{min} are simple eigenvalues of $V^{-1}Q$. Since $\alpha_{max}(\alpha) \to \alpha_{max}$ and $\beta_{min}(\alpha) \to \beta_{min}$ as $\alpha \to 0$, by Theorem 1.8.1, for sufficiently small $\alpha > 0$, $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ can be represented by convergent power series. Similarly, it can be shown (see Wilkinson [14]) that for $|\alpha| < \varepsilon$, each component of the eigenvectors $f_{max}(\alpha)$ and $g_{min}(\alpha)$ can be represented by a convergent power series.

In addition, in the positive drift case, $\alpha_{max} = 0$ and $f_{max} = 1$ and in the negative drift case $\beta_{min} = 0$ and $g_{min} = 1$. Therefore, the part of the theorem for the drift cases is proved.

In the oscillating case, by (1.28), zero is an eigenvalue of the matrix $V^{-1}Q$ with algebraic multiplicity two. Hence, by Theorem 1.8.2 there exists $\varepsilon > 0$ such that for $0 < |\alpha| < \varepsilon$ there exist two eigenvalues of $V^{-1}(Q-\alpha I)$ which converge to zero as $\alpha \to 0$. We know that they are $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ and, again by Theorem 1.8.2, for $|\alpha| < \varepsilon$ one of the following is valid:

either

$$\begin{aligned} \alpha_{max}(\alpha) &= a_1\alpha + a_2\alpha^2 + a_3\alpha^3 + \dots \\ \beta_{min}(\alpha) &= b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \dots, \end{aligned} \tag{1.38}$$

for some constants a_k, b_k, $k \in \mathbb{N}$, or

$$\begin{aligned} \alpha_{max}(\alpha) &= d_1\alpha^{\frac{1}{2}} + d_2\alpha + d_3\alpha^{\frac{3}{2}} + \dots \\ \beta_{min}(\alpha) &= -d_1\alpha^{\frac{1}{2}} + d_2\alpha - d_3\alpha^{\frac{3}{2}} + \dots, \end{aligned} \tag{1.39}$$

for some constants d_k, $k \in \mathbb{N}$.

We shall show that (1.38) is not possible.

For any $\alpha > 0$, the equation

$$V^{-1}(Q - \alpha I)f_{max}(\alpha) = \alpha_{max}(\alpha)f_{max}(\alpha)$$

can be rewritten as

$$(Q - \alpha_{max}(\alpha)V)f_{max}(\alpha) = \alpha f_{max}(\alpha). \tag{1.40}$$

Thus, for small $\alpha > 0$, $f_{max}(\alpha)$ is a positive eigenvector of the matrix $(Q - \alpha_{max}V)$. By Lemma 1.1.6, the matrix $(Q - \alpha_{max}V)$ is irreducible and essentially non-negative, and by Lemma 1.1.3, the only positive eigenvector of an irreducible essentially non-negative matrix is its Perron-Frobenius eigenvector. Hence, $f_{max}(\alpha)$ is the Perron-Frobenius eigenvector of the matrix $(Q - \alpha_{max}V)$ which implies that α is the Perron-Frobenius eigenvalue of $(Q - \alpha_{max}V)$. In the same way we conclude that α is the Perron-Frobenius eigenvalue of $(Q - \beta_{min}V)$.

Let $\beta \in \mathbb{R}$ and consider the matrix $(Q - \beta V)$ and its Perron-Frobenius eigenvalue $\alpha(\beta)$. The eigenvalue $\alpha(\beta)$ is simple and it converges to a simple eigenvalue of the matrix Q as $\beta \to 0$. Then, by Theorem 1.8.1, there exists $\delta > 0$ such that for any $|\beta| < \delta$, $\alpha(\beta)$ has a representation by a convergent power series. Thus, for $|\beta| < \delta$, we may write

$$\alpha(\beta) = c_0 + c_1\beta + c_2\beta^2 + \dots,$$

for some constants c_k, $k \in \mathbb{N} \cup \{0\}$. Similarly, (see Wilkinson [14]), for $|\beta| < \delta$ each component of the vector $u(\beta)$ can be represented by a convergent power series in β. Hence, because $u(0) = 1$,

$$u(\beta) = 1 + \beta v_1 + \beta^2 v_2 + \dots, \tag{1.41}$$

for some vectors v_k, $k \in \mathbb{N}$, on E.

Suppose that the process $(\varphi_t)_{t\geq 0}$ oscillates. By Lemmas 1.8.4 and 1.8.5 the eigenvalue $\alpha(\beta)$ attains its global minimum 0 at $\beta = 0$. Hence, $\alpha(0) = \frac{d\alpha}{d\beta}(0) = 0$, which gives that $c_0 = c_1 = 0$, and therefore

$$\alpha(\beta) = c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + \dots . \tag{1.42}$$

Since $\alpha(\beta)$ and $u(\beta)$ are represented by convergent power series (1.42) and (1.41), respectively, by substituting them into the equation

$$(Q - \beta V)u(\beta) = \alpha(\beta)u(\beta)$$

and by equating terms in β and β^2 on each side of the previous equation, we obtain

$$V^{-1}Qv_1 = 1 \tag{1.43}$$

$$Qv_2 - Vv_1 = c_2 1. \tag{1.44}$$

In the oscillating case, by (1.28) there exists vector r such that $V^{-1}Qr = 1$. Hence, (1.43) implies that $v_1 = r$ which together with (1.41) gives that, for $|\beta| < \delta$,

$$u(\beta) = 1 + \beta r + \beta^2 v_2 + \dots . \tag{1.45}$$

Equation (1.44) implies that $c_2 \neq 0$. To show that, suppose that $c_2 = 0$. Then, from (1.44), for $v_1 = r$, we obtain

$$V^{-1}Qv_2 = r.$$

Since $\mathcal{B} = \{1, r, f_j, j = 2, \dots n, g_k, k = 2, \dots m\}$ is a basis in the space of all vectors on E,

$$v_2 = p_1\, 1 + p_2\, r + \sum_{j=2}^{n} q_j\, f_j + \sum_{k=2}^{m} d_k\, g_k,$$

for some constants $p_1, p_2, q_j, j = 2, \dots n$ and $d_k, k = 2, \dots m$. Then, because $V^{-1}Qr = 1$,

$$r = V^{-1}Qv_2 = p_2\, 1 + \sum_{j=2}^{n} q_j\, V^{-1}Qf_j + \sum_{k=2}^{m} d_k\, V^{-1}Qg_k. \tag{1.46}$$

By (1.9), for any $j = 2, \dots n$, $V^{-1}Qf_j$ is a linear combination only of vectors $\{1, f_j, j = 2, \dots n\}$, and for any $k = 2, \dots m$, $V^{-1}Qg_k$ is a linear combination only of vectors $\{1, g_k, k = 2, \dots m\}$. Then, it follows from (1.46) that the vector r is a linear combination of the vectors from the basis $\mathcal{B}$ which are independent from r itself, but that is not possible. Therefore $c_2 \neq 0$.

Moreover, by Lemmas 1.8.4 and 1.8.5, $\alpha(0) = 0$ is the minimum of the function $\alpha(\beta)$ which implies that $\alpha(\beta) > 0$ for all $\beta \in \mathbb{R}$, and therefore from (1.42) we get that $c_2 > 0$. By multiplying (1.44) by μ from the left, we obtain (because $\mu 1 = 1$),

$$c_2 = \frac{-\mu V r}{\mu 1} = -\mu V r. \tag{1.47}$$

Suppose that (1.38) is true. Then, it follows from (1.38) and (1.40) that, for $|\alpha| < \varepsilon$,

$$\begin{aligned} \alpha = \alpha(\alpha_{max}) &= c_2\, \alpha^2_{max}(\alpha) + c_3\, \alpha^3_{max}(\alpha) + \dots \\ &= c_2(a_1\alpha + a_2\alpha^2 + \dots)^2 + c_3(a_1\alpha + a_2\alpha^2 + \dots)^3 + \dots \\ &= c_2 a_1^2 \alpha^2 + const.\alpha^3 + \dots, \end{aligned}$$

which is not possible for every $|\alpha| < \varepsilon$. Hence, (1.38) is not true and we conclude that (1.39) holds.

Substituting $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ from (1.39) into (1.42) gives

$$d_1^2 = \frac{1}{c_2}. \tag{1.48}$$

Thus, (1.47), (1.39) and (1.48) imply that, for $|\alpha| < \varepsilon$,

$$\begin{aligned} \alpha_{max}(\alpha) &= -\tfrac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}} + d_2\alpha + d_3\alpha^{\frac{3}{2}} + \dots \\ \beta_{min}(\alpha) &= \tfrac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}} + d_2\alpha - d_3\alpha^{\frac{3}{2}} + \dots . \end{aligned}$$

By substituting $\alpha_{max}(\alpha)$ and $\beta_{min}(\alpha)$ into (1.45) we obtain, for $|\alpha| < \varepsilon$,

$$\begin{aligned} f_{max}(\alpha) &= 1 - \tfrac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, r + \alpha\, v_2 + \dots \\ g_{min}(\alpha) &= 1 + \tfrac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, r + \alpha\, w_2 + \dots . \end{aligned}$$

□

1.9 h-transforms

We have seen in Section 1.7 that the process $(\varphi_t)_{t\geq 0}$ can drift to $+\infty$, oscillate or drift to $-\infty$. An interesting question is how we can change the behaviour of the process $(\varphi_t)_{t\geq 0}$ from one to another. In particular, we are interested in transformations of the process $(X_t, \varphi_t)_{t\geq 0}$ which preserve the Markov property of the process $(X_t)_{t\geq 0}$ and change the behaviour of the process $(\varphi_t)_{t\geq 0}$. In addition, we restrict ourselves to such transformations among h-transforms of the processes $(X_t, \varphi_t)_{t\geq 0}$ and $(X_t, \varphi_t, t)_{t\geq 0}$.

In Chapter 4 we shall use the results of this section and see that changing the behaviour of the process $(\varphi_t)_{t\geq 0}$ is related to conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative.

We begin with the definition of an h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$. Let $h(e, \varphi, t)$ be a positive function on $E\times\mathbb{R}\times[0, +\infty)$ such that the process $(h(X_t, \varphi_t, t))_{t\geq 0}$ is a martingale. For fixed $(e, \varphi) \in E \times \mathbb{R}$, define a probability measure $P^h_{(e,\varphi)}$ by

$$P^h_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h(X_t, \varphi_t, t)\Big)}{h(e, \varphi, 0)}, \qquad A \in \mathcal{F}_t. \tag{1.49}$$

The martingale property of the process $(h(X_t, \varphi_t))_{t\geq 0}$ ensures that the consistency condition is satisfied, that is

$$\frac{E\Big(I(A)\ h(X_s, \varphi_s, s)\Big)}{h(e, \varphi, 0)} = \frac{E\Big(I(A)\ h(X_t, \varphi_t, t)\Big)}{h(e, \varphi, 0)}, \quad A \in \mathcal{F}_s,\ 0 \leq s \leq t.$$

The process $(X_t, \varphi_t, t)_{t\geq 0}$ under the measure $P^h_{(e,\varphi)}$ (and more generally any other process having the same law) is called the h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$ with the function h.

Let $E^h_{(e,\varphi)}$ denote the expectation operator associated with the probability measure $P^h_{(e,\varphi)}$.

By the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$ and by the positivity of the function h, we have that, for $0 < u < t$ and any bounded measurable function f on

$E \times \mathbb{R} \times [0, +\infty)$,

$$\begin{aligned}
E^h_{(e,\varphi)}\Big(f(X_t, \varphi_t, t) \mid \mathcal{F}_u\Big) = & \frac{E_{(e,\varphi)}\Big(f(X_t, \varphi_t, t)\ h(X_t, \varphi_t, t) \mid \mathcal{F}_u\Big)}{h(X_u, \varphi_u, u)} \\
= & \frac{E_{(X_u,\varphi_u)}\Big(f(X_{t-u}, \varphi_{t-u}, t-u)\ h(X_{t-u}, \varphi_{t-u}, t-u)\Big)}{h(X_u, \varphi_u, u)} \\
= & E^h_{(X_u,\varphi_u)}\Big(f(X_{t-u}, \varphi_{t-u}, t-u)\Big)
\end{aligned}$$

which shows that the process $(X_t, \varphi_t, t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov.

Let $\mathcal{A}^h$ denote the infinitesimal generator of the process $(X_t, \varphi_t, t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ and let $\mathcal{D}^h_{\mathcal{A}}$ denote its domain. Then, $f \in \mathcal{D}^h_{\mathcal{A}}$ if the limit

$$\mathcal{A}^h f(e, \varphi, 0) = \lim_{t\to 0} \frac{E^h_{(e,\varphi)} f(X_t, \varphi_t, t) - f(e, \varphi, 0)}{t}$$

exists for all $(e, \varphi) \in E \times \mathbb{R}$.

By the definitions of the measure $P^h_{(e,\varphi)}$ and the infinitesimal generator $\mathcal{A}$ of the process $(X_t, \varphi_t, t)_{t\geq 0}$,

$$\begin{aligned}
\mathcal{A}^h f(e, \varphi, 0) &= \lim_{t\to 0} \frac{E^h_{(e,\varphi)} f(X_t, \varphi_t, t) - f(e, \varphi, 0)}{t} \\
&= \lim_{t\to 0} \frac{E_{(e,\varphi)}(f(X_t, \varphi_t, t)\ h(X_t, \varphi_t, t)) - f(e, \varphi, 0) h(e, \varphi, 0)}{t\ h(e, \varphi, 0)} \\
&= \frac{\mathcal{A}(hf)(e, \varphi, 0)}{h(e, \varphi, 0)}.
\end{aligned}$$

Lemma 1.6.1 gives a representation formula for the generator $\mathcal{A}$ on the set of functions defined on $E \times \mathbb{R} \times [0, +\infty)$ which are continuously differentiable on $\mathbb{R} \times [0, +\infty)$. Thus, for a function $f(e, \varphi, t)$ on $E \times \mathbb{R} \times [0, +\infty)$ continuously differentiable in φ and t, it follows from Lemma 1.6.1 and from the previous equation that

$$\mathcal{A}^h f = \frac{\mathcal{A}(hf)}{h} = \frac{Qhf + V\frac{\partial h}{\partial \varphi} f + \frac{\partial h}{\partial t} f}{h} + V\frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial t}, \tag{1.50}$$

where

$$\begin{aligned}
Qf(e, \varphi, t) &= \textstyle\sum_{e' \in E} Q(e, e') f(e', \varphi, t) \\
V\tfrac{\partial f}{\partial \varphi}(e, \varphi, t) &= V(e, e) \tfrac{\partial f}{\partial \varphi}(e, \varphi, t).
\end{aligned}$$

For the rest of this section we restrict our attention to functions $h(e,\varphi,t)$ defined on $E \times \mathbb{R} \times [0,+\infty)$ which are continuously differentiable in φ and t.

For fixed $\varphi \in \mathbb{R}$ and $s \in [0,+\infty)$, let $Q^h_{\varphi,s}$ be the $E \times E$ matrix with coefficients

$$Q^h_{\varphi,s}(e,e') = \frac{h(e',\varphi,s)}{h(e,\varphi,s)}Q(e,e') + \frac{\frac{\partial h}{\partial \varphi}(e,\varphi,s)}{h(e,\varphi,s)}V(e,e') + \frac{\frac{\partial h}{\partial t}(e,\varphi,s)}{h(e,\varphi,s)}I(e,e'). \tag{1.51}$$

The function h is positive and the matrices I and V are diagonal. Hence, the matrix $Q^h_{\varphi,s}$ is essentially non-negative. Since the process $(h(X_t,\varphi_t,t))_{t\geq 0}$ is a martingale, $\mathcal{A}h = (Q + V\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t})h = 0$. Therefore,

$$Q^h_{\varphi,s}1(e) = \frac{1}{h(e,\varphi,s)}(Q + V\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t})h(e,\varphi,s) = 0$$

which shows that $Q^h_{\varphi,s}$ is a conservative Q-matrix.

Substituting the matrix $Q^h_{\varphi,s}$ into generator equation (1.50) we obtain

$$\mathcal{A}^h f(e,\varphi,s) = Q^h_{\varphi,s} f(e,\varphi,s) + V\frac{\partial f}{\partial\varphi}(e,\varphi,s) + \frac{\partial f}{\partial t}(e,\varphi,s).$$

It follows that the jump intensities of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ are given by the matrix $Q^h_{\varphi,s}$. The matrix $Q^h_{\varphi,s}$ in general depends on φ and s. Thus, the jump probabilities of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ depend on the present value of the process $(\varphi_t)_{t\geq 0}$, which depends on the past of the process $(X_t)_{t\geq 0}$ and therefore the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is not in general Markov. However, if the matrix $Q^h_{\varphi,s}$ does not depend on φ and s then the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov.

Suppose now that the function h in (1.49) does not depend on t. In the same way as for the process $(X_t,\varphi_t,t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$, we conclude that the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov with the infinitesimal generator $\mathcal{G}^h$ given by

$$\mathcal{G}^h f = \frac{\mathcal{G}(hf)}{h},$$

where $\mathcal{G}$ is the infinitesimal generator of the process $(X_t,\varphi_t)_{t\geq 0}$. By Lemma 1.6.2, for any function $f(e,\varphi)$ defined on $E \times \mathbb{R}$ which is continuously differentiable in φ,

$$\mathcal{G}^h f = \frac{\mathcal{G}(hf)}{h} = \frac{Qhf + V\frac{\partial h}{\partial\varphi}f}{h} + V\frac{\partial f}{\partial\varphi}.$$

For fixed $\varphi \in \mathbb{R}$ define Q^h_φ to be the $E \times E$ matrix with coefficients

$$Q^h_\varphi(e,e') = \frac{h(e',\varphi)}{h(e,\varphi)}Q(e,e') + \frac{\frac{\partial h}{\partial \varphi}(e,\varphi)}{h(e,\varphi)}V(e,e'). \tag{1.52}$$

The matrix Q^h_φ is a special case of the matrix $Q^h_{\varphi,s}$ given in (1.51). Hence, it follows that the matrix Q^h_φ is a conservative Q-matrix, that the jump intensities of the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ are given by Q^h_φ and that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is not in general Markov.

Suppose that for some function h the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and let μ^h be its invariant measure. Then, Theorem 1.7.1 and (1.26) applied to the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ imply that

$$\mu^h V1 \begin{cases} > 0 & \text{iff} \quad (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } +\infty \\ = 0 & \text{iff} \quad (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ oscillates} \\ < 0 & \text{iff} \quad (\varphi_t)_{t\geq 0} \text{ under } P^h_{(e,\varphi)} \text{ drifts to } -\infty. \end{cases} \tag{1.53}$$

Our aim is to find all positive functions $h(e,\varphi,t)$ on $E \times \mathbb{R} \times [0,+\infty)$ continuously differentiable in φ and t for which the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and for each such function h to determine the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$.

First we look at h-transforms of the process $(X_t, \varphi_t)_{t\geq 0}$. We recall the eigenvalues α_{max} and β_{min} of the matrix $V^{-1}Q$ and their associated eigenvectors f_{max} and g_{min}, respectively.

Theorem 1.9.1 *There exist only two functions $h(e,\varphi)$ on $E \times \mathbb{R}$ continuously differentiable in φ such that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and they are*

$$h_{max}(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e) \qquad \text{and} \qquad h_{min}(e,\varphi) = e^{-\beta_{min}\varphi} g_{min}(e).$$

Moreover,

1) if the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$ then $h_{max} = 1$ and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$ drifts to $-\infty$;

2) if the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$ then $h_{min} = 1$ and the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$;

3) if the process $(\varphi_t)_{t\geq 0}$ oscillates then $h_{max} = h_{min} = 1$.

Before we prove the previous theorem, we prove an auxiliary lemma:

Lemma 1.9.1 *Let $h(e,\varphi)$ be a function on $E \times \mathbb{R}$ continuously differentiable in φ. If, for any $e, e' \in E$, $\frac{h(e',\varphi)}{h(e,\varphi)}$ and $\frac{\frac{dh}{d\varphi}(e,\varphi)}{h(e,\varphi)}$ do not depend on φ, then h is of the form*

$$h(e,\varphi) = e^{-\lambda\varphi} g(e),$$

for some $\lambda \in \mathbb{R}$ and some function g on E.

Proof: If $\frac{\frac{dh}{d\varphi}(e,\varphi)}{h(e,\varphi)}$ does not depend on φ, then

$$\frac{\frac{dh}{d\varphi}(e,\varphi)}{h(e,\varphi)} = v(e)h(e,\varphi),$$

for some function v on E, and therefore

$$h(e,\varphi) = g(e)\ e^{v(e)\varphi},$$

for some function g on E.

Since for any $e, e' \in E$, $\frac{h(e',\varphi)}{h(e,\varphi)}$ does not depend on φ, it follows from the last equation that $v(e) = \lambda = const.$, $e \in E$. Hence,

$$h(e,\varphi) = e^{-\lambda\varphi} g(e).$$

□

Now we prove Theorem 1.9.1:

Proof of Theorem 1.9.1: Suppose that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov. Then its Q-matrix Q^h_φ given by (1.52) does not depend on φ which implies that $\frac{h(e',\varphi)}{h(e,\varphi)}$ and $\frac{\frac{dh}{d\varphi}(e,\varphi)}{h(e,\varphi)}$ do not depend on φ. In that case, by Lemma 1.9.1 the function h is of the form

$$h(e,\varphi) = e^{-\lambda\varphi} g(e)$$

for some $\lambda \in \mathbb{R}$ and some function g on E.

Furthermore, the process $(h(X_t, \varphi_t))_{t\geq 0}$ is a positive martingale. Hence, the function h is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t, \varphi_t)_{t\geq 0}$ and $\mathcal{G}h = 0$. Since, by Lemma 1.6.2,

$$\mathcal{G} = Q + V\frac{d}{d\varphi},$$

we have that

$$(Q + V\frac{d}{d\varphi})h = 0 \quad \text{and} \quad h(e,\varphi) = e^{-\lambda\varphi}g(e),$$

which imply that λ and g must satisfy the equation

$$V^{-1}Qg = \lambda g.$$

In other words, λ is an eigenvalue of $V^{-1}Q$ and g is its associated right eigenvector.

Finally, the vector g has to be positive. By Lemma 1.4.10, the only positive eigenvectors of the matrix $V^{-1}Q$ are f_{max} and g_{min}. Therefore, there exist only two positive functions $h(e,\varphi)$ on $E \times \mathbb{R}$ continuously differentiable in φ such that the process $(h(X_t, \varphi_t))_{t\geq 0}$ is a martingale and that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov, and they are

$$h_{max}(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e) \qquad \text{and} \qquad h_{min}(e,\varphi) = e^{-\beta_{min}\varphi} g_{min}(e).$$

This proves the first part of the theorem.

For the second part, we shall prove only the case when the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$. The case when the process $(\varphi_t)_{t\geq 0}$ drifts to $-\infty$ can be proved in the same way, and the case when the process $(\varphi_t)_{t\geq 0}$ oscillates is trivial because $\alpha_{max} = \beta_{min} = 0$ and $h_{max} = h_{min} = 1$.

Suppose now that the process $(\varphi_t)_{t\geq 0}$ drifts to $+\infty$. Then $\alpha_{max} = 0$ and $f_{max} = 1$, hence making the h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$ with $h_{max} = 1$ does not change the process. But $\beta_{min} > 0$ and we can make the h-transform by using h_{min}. By (1.53) the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$ is determined by the sign of $\mu^{h_{min}}V1$, where $\mu^{h_{min}}$ is the invariant measure of the process $(X_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$. Hence, we shall look for the sign of $\mu^{h_{min}}V1$.

By (1.52), the Q-matrix $Q^{h_{min}}$ of the Markov process $(X_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$ is given by

$$Q^{h_{min}}(e,e') = \frac{g_{min}(e')}{g_{min}(e)}(Q - \beta_{min}V)(e,e'), \qquad e,e' \in E.$$

The matrix $(Q-\beta_{min}V)$ is, by Lemma 1.1.6, irreducible and essentially nonnegative and g_{min} is its right Perron-Frobenius eigenvector. Let g^{left}_{min} be the left Perron-Frobenius eigenvector of the matrix $(Q - \beta_{min}V)$ and let μ_{min} be a vector on E defined by

$$\mu_{min}(e) = g^{left}_{min}(e)\; g_{min}(e), \quad e \in E.$$

Then μ_{min} is positive because g^{left}_{min} and g_{min} are positive as Perron-Frobenius eigenvectors of $(Q - \beta_{min}V)$, and

$$\begin{aligned}
\mu_{min}Q^{h_{min}}(e) &= \sum_{e'\in E} \mu_{min}(e')\; Q^{h_{min}}(e',e) \\
&= \sum_{e'\in E} g^{left}_{min}(e')\; g_{min}(e')\; \frac{g_{min}(e)}{g_{min}(e')}\; (Q-\beta_{min}V)(e',e) \\
&= g^{left}_{min}(Q-\beta_{min}V)(e)\; g_{min}(e) = 0.
\end{aligned}$$

Therefore, the invariant measure $\mu^{h_{min}}$ of the process $(X_t)_{t\geq 0}$ under $P^{h_{min}}_{(e,\varphi)}$ is proportional to μ_{min}. In addition,

$$\mu^{h_{min}}V1 = \mu_{min}V1 = \sum_{e\in E} g^{left}_{min}(e)V(e,e)g_{min}(e) = g^{left}_{min}Vg_{min}.$$

Let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$. Then, by Lemma 1.8.4,

$$\frac{d\alpha}{d\beta}(\beta_{min}) = -\frac{\mu^{h_{min}}V1}{g^{left}_{min}\; g_{min}} > 0,$$

because, by the same lemma, $\beta_{min} > \beta_0$ and $\frac{d\alpha}{d\beta} > 0$ for $\beta > \beta_0$.

Since $g_{min}^{left}\, g_{min}$ is positive, we conclude from the last equation that $\mu^{h_{min}} V1 < 0$ which, by (1.53) implies that the process $(\varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^{h_{min}}$ drifts to $-\infty$. □

From Theorem 1.9.1 it follows that making an h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^h$ such that the process $(X_t)_{t\geq 0}$ under $P_{(e,\varphi)}^h$ is again Markov can never give the oscillating process $(\varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^h$. However, the next theorem shows that making an h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$ can produce the oscillating process $(\varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^h$.

Theorem 1.9.2 *All functions $h(e,\varphi,t)$ on $E \times \mathbb{R} \times [0,+\infty)$ continuously differentiable in φ and t for which the process $(X_t)_{t\geq 0}$ under $P_{(e,\varphi)}^h$ is Markov are of the form*

$$h(e,\varphi,t) = e^{-\alpha t} e^{-\beta\varphi} g(e), \quad (e,\varphi,t) \in E \times \mathbb{R} \times [0,+\infty),$$

where, for fixed $\beta \in \mathbb{R}$, α is the Perron-Frobenius eigenvalue and g is the right Perron-Frobenius eigenvector of the matrix $(Q - \beta V)$.

Moreover, there exists unique $\beta_0 \in \mathbb{R}$ such that

$$\begin{array}{rcl} (\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)}^h \text{ drifts to } +\infty & \text{iff} & \beta < \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)}^h \text{ oscillates} & \text{iff} & \beta = \beta_0 \\ (\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)}^h \text{ drifts to } -\infty & \text{iff} & \beta > \beta_0, \end{array}$$

and β_0 is determined by the equation $\alpha'(\beta_0) = 0$, where $\alpha(\beta)$ is the Perron-Frobenius eigenvalue of $(Q - \beta V)$.

Again, before we prove the theorem we prove an auxiliary lemma.

Lemma 1.9.2 *Let $h(e,\varphi,t)$ be a function on $E \times \mathbb{R} \times [0,\infty)$ continuously differentiable in φ and t and let v be a non-constant function on E such that $v(e) \neq 0$, $e \in E$. If, for any $e, e' \in E$, $\frac{h(e',\varphi,s)}{h(e,\varphi,s)}$ and $v(e)\frac{\frac{\partial h}{\partial \varphi}(e,\varphi,s)}{h(e,\varphi,s)} + \frac{\frac{\partial h}{\partial t}(e,\varphi,s)}{h(e,\varphi,s)}$ do not depend on φ and s, then h is of the form*

$$h(e,\varphi,t) = e^{-\alpha t} e^{-\beta\varphi} g(e),$$

for some $\alpha, \beta \in \mathbb{R}$ and some function g on E.

Proof: Suppose that $\frac{h(e',\varphi,s)}{h(e,\varphi,s)}$ does not depend on φ and s. Then there exists a function g on $E \times E$ such that

$$\frac{h(e',\varphi,s)}{h(e,\varphi,s)} = g(e',e), \quad e,e' \in E,$$

or equivalently,

$$h(e',\varphi,s) = g(e',e)\ h(e,\varphi,s), \quad e,e' \in E.$$

For fixed $e \in E$, let functions u_e on $\mathbb{R} \times [0,+\infty)$ and v_e on E be given by $u_e(\varphi,s) = h(e,\varphi,s)$ and $v_e(e') = g(e',e)$. Then

$$h(e',\varphi,s) = u_e(\varphi,s)\ v_e(e'), \quad e' \in E.$$

If $v(e')\frac{\frac{\partial h}{\partial \varphi}(e',\varphi,s)}{h(e',\varphi,s)} + \frac{\frac{\partial h}{\partial s}(e',\varphi,s)}{h(e',\varphi,s)}$ does not depend on φ and s, then $v(e')\frac{\frac{\partial u_e}{\partial \varphi}(\varphi,s)}{u_e(\varphi,s)} + \frac{\frac{\partial u_e}{\partial s}(\varphi,s)}{u_e(\varphi,s)}$ is a function only of e', which implies that

$$\frac{\frac{\partial u_e}{\partial \varphi}(\varphi,s)}{u_e(\varphi,s)} = const. \quad \text{and} \quad \frac{\frac{\partial u_e}{\partial s}(\varphi,s)}{u_e(\varphi,s)} = const.$$

Thus,

$$u_e(\varphi,s) = c(e)\ e^{-\alpha s}\ e^{-\beta\varphi}$$

for some $\alpha, \beta \in \mathbb{R}$ and some function c on E.

Therefore,

$$h(e',\varphi,s) = c(e)\ e^{-\alpha s}\ e^{-\beta\varphi}\ v_e(e'), \quad e' \in E.$$

which implies that there exists function g on E such that

$$h(e',\varphi,s) = e^{-\alpha s}\ e^{-\beta\varphi}\ g(e').$$

□

Now we prove Theorem 1.9.2:

Proof of Theorem 1.9.2: We want to find all positive functions $h(e, \varphi, t)$ defined on $E \times \mathbb{R} \times [0, \infty)$ which are continuously differentiable in φ and t and for which the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov.

Suppose that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov. Then its Q-matrix $Q^h_{\varphi,s}$ given by (1.51) does not depend on φ and s which implies that, for all $e, e' \in E$, $\frac{h(e',\varphi,s)}{h(e,\varphi,s)}$ and $v(e)\frac{\frac{\partial h}{\partial \varphi}(e,\varphi,s)}{h(e,\varphi,s)} + \frac{\frac{\partial h}{\partial t}(e,\varphi,s)}{h(e,\varphi,s)}$ do not depend on φ and s. Hence, by Lemma 1.9.2, all such functions h are of the form

$$h(e, \varphi, t) = e^{-\alpha t} e^{-\beta \varphi} g(e) \tag{1.54}$$

for some $\alpha, \beta \in \mathbb{R}$ and some function g on E.

Since the process $(h(X_t, \varphi_t, t))_{t\geq 0}$ is a martingale, the function h satisfies the equation $\mathcal{A}h = 0$. Hence, by Lemma 1.6.1,

$$\mathcal{A}h = Qh + V\frac{\partial h}{\partial \varphi} + \frac{\partial h}{\partial t} = 0,$$

which together with (1.54) gives that

$$(Q - \beta V)g = \alpha g. \tag{1.55}$$

Therefore, for fixed $\beta \in \mathbb{R}$, α is an eigenvalue of the matrix $(Q - \beta V)$ and g is its associated eigenvector. Since the function h is positive, the vector g is positive. By Lemma 1.1.6, the matrix $(Q - \beta V)$ is irreducible and essentially non-negative, and by Lemma 1.1.3, the only positive eigenvector of an irreducible essentially non-negative matrix is its Perron-Frobenius eigenvector. Thus, α and g in (1.54) and (1.55) must be the Perron-Frobenius eigenvalue and eigenvector, respectively, of the matrix $(Q - \beta V)$. This finishes the proof of the first part of the theorem.

For the proof of the second part of the theorem, let $\alpha(\beta)$ and $g(\beta)$ be the Perron-Frobenius eigenvalue and eigenvector, respectively, of the matrix $(Q - \beta V)$. For fixed $\beta \in \mathbb{R}$, let a function h_β be given by

$$h_\beta(e, \varphi, t) = e^{-\alpha(\beta)t}\, e^{-\beta\varphi}\, g(\beta)(e).$$

Then, by the first part of the theorem, the process $(X_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$ is Markov, where $P^{h_\beta}_{(e,\varphi)}$ is $P^{h}_{(e,\varphi)}$ with $h = h_\beta$. In that case, by (1.53), the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$ is determined by the sign of $\mu^{h_\beta}V1$, where μ^{h_β} is the invariant measure of the process $(X_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$. Hence, in order to find the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$, we shall look for the sign of $\mu^{h_\beta}V1$. First, we need to find μ^{h_β}.

By (1.51), the Q-matrix Q^{h_β} of the Markov process $(X_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$ is given by

$$Q^{h_\beta}(e,e') = \frac{g(\beta)(e')}{g(\beta)(e)}(Q - \alpha(\beta) - \beta V)(e,e'), \qquad e,e' \in E.$$

Let $g^{left}(\beta)$ be the left Perron-Frobenius eigenvector of the matrix $(Q - \beta V)$ and let μ_β be a vector on E defined by

$$\mu_\beta(e) = g^{left}(\beta)(e)\; g(\beta)(e), \quad e \in E.$$

Then μ_β is positive because $g^{left}(\beta)$ and $g(\beta)$ are positive as Perron-Frobenius eigenvectors of $(Q - \beta V)$, and

$$\begin{aligned}
\mu_\beta Q^{h_\beta}(e) &= \sum_{e'\in E} \mu_\beta(e')\; Q^{h_\beta}(e',e) \\
&= \sum_{e'\in E} g^{left}(\beta)(e')\; g(\beta)(e')\; \frac{g(\beta)(e)}{g(\beta)(e')}\; (Q - \beta V)(e',e) \\
&= g^{left}(\beta)(Q - \beta V)(e)\; g(\beta)(e) = 0.
\end{aligned}$$

Therefore, the invariant measure μ^{h_β} of the process $(X_t)_{t\geq 0}$ under $P^{h_\beta}_{(e,\varphi)}$ is equal to μ_β. Moreover,

$$\mu^{h_\beta}V1 = \mu_\beta V1 = \sum_{e\in E} g^{left}(\beta)(e)V(e,e)g(\beta)(e) = g^{left}(\beta)Vg(\beta).$$

By Lemma 1.8.4,

$$\frac{d\alpha}{d\beta}(\beta) = -\frac{\mu^{h_\beta}V1}{g^{left}(\beta)g(\beta)} \begin{cases} < 0, & \text{if } \beta < \beta_0 \\ = 0, & \text{if } \beta = \beta_0 \\ > 0, & \text{if } \beta > \beta_0. \end{cases}$$

Since $g^{left}(\beta)g(\beta)$ is positive for all $\beta \in \mathbb{R}$, we conclude that

$$\mu^{h_\beta}V1 \begin{cases} > 0, & \text{if } \beta < \beta_0 \\ = 0, & \text{if } \beta = \beta_0 \\ < 0, & \text{if } \beta > \beta_0, \end{cases}$$

which together with (1.53) proves the second part of the theorem. □

Before we end the section, we recall the stopping time H_0 defined as the first crossing time of zero by the process $(\varphi_t)_{t\geq 0}$ and we define an h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$ killed at time H_0.

For the process $(X_t, \varphi_t)_{t\geq 0}$ run up to time H_0 we say the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero or, shortly, the killed process $(X_t, \varphi_t)_{t\geq 0}$. In the sequel, we shall use these expressions without further notice.

Let h be a positive function on E_0^+ such that the process $\{h(X_t, \varphi_t, t)I\{t < H_0\}, t \geq 0\}$, starting at $(e, \varphi) \in E_0^+$, is a martingale. Define a probability measure $\widetilde{P}^h_{(e,\varphi)}$ by

$$\widetilde{P}^h_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h(X_t, \varphi_t, t)I\{t < H_0\}\Big)}{h(e, \varphi, 0)}, \qquad A \in \mathcal{F}_t, \quad t \geq 0.$$

The martingale property of the process $\{h(X_t, \varphi_t, t)I\{t < H_0\}, t \geq 0\}$ ensures that the consistency condition is satisfied, that is, for $0 \leq s \leq t$ and $A \in \mathcal{F}_s$,

$$\frac{E\Big(I(A)\ h(X_s, \varphi_s, s)I\{s < H_0\}\Big)}{h(e, \varphi, 0)} = \frac{E\Big(I(A)\ h(X_t, \varphi_t, t)I\{t < H_0\}\Big)}{h(e, \varphi, 0)}.$$

The process $(X_t, \varphi_t, t)_{t\geq 0}$ under the measure $\widetilde{P}^h_{(e,\varphi)}$ (and more generally any other process having the same law) is called the h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero. If the function h does not depend on t, then the process $(X_t, \varphi_t)_{t\geq 0}$ under the measure $\widetilde{P}^h_{(e,\varphi)}$ is called the h-transform with the function h of the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

Let $\widetilde{E}^h_{(e,\varphi)}$ denote the expectation operator associated with the probability measure $\widetilde{P}^h_{(e,\varphi)}$. Then, by the properties of the conditional expectation, for any $0 < s < t$, $A \in \mathcal{F}_s$

and any bounded function f,

$$\begin{aligned}
&\widetilde{E}^h_{(e,\varphi)}\Big(I(A)I\{s < H_0\}f(X_t,\varphi_t,t)\Big)\\
&= \frac{E_{(e,\varphi)}\Big(I(A)I\{s < H_0\}f(X_t,\varphi_t,t)h(X_t,\varphi_t,t)I\{t < H_0\}\Big)}{h(e,\varphi,0)}\\
&= \frac{E_{(e,\varphi)}\Big(I(A)I\{s \le H_0\}E_{(e,\varphi)}(f(X_t,\varphi_t,t)h(X_t,\varphi_t,t)I\{t < H_0\} \mid \mathcal{F}_s)\Big)}{h(e,\varphi,0)}\\
&= \widetilde{E}^h_{(e,\varphi)}\Big(I(A)\ \frac{E_{(e,\varphi)}\Big(f(X_t,\varphi_t,t)h(X_t,\varphi_t,t)I\{t < H_0\} \mid \mathcal{F}_s\Big)}{h(X_s,\varphi_s,s)}\Big).
\end{aligned}$$

Hence,

$$\begin{aligned}
&I\{s < H_0\}\widetilde{E}^h_{(e,\varphi)}\Big(f(X_t,\varphi_t,t) \mid \mathcal{F}_s\Big)\\
&\qquad = \frac{E_{(e,\varphi)}\Big(f(X_t,\varphi_t,t)h(X_t,\varphi_t,t)I\{t < H_0\} \mid \mathcal{F}_s\Big)}{h(X_s,\varphi_s,s)}.
\end{aligned}$$

By the Markov property of the process $(X_t,\varphi_t)_{t\ge 0}$ and by the positivity of the function $h(e,\varphi,s)$, it folows from the last equation that, for $0 < u < t$ and any bounded measurable function f on $E \times \mathbb{R} \times [0,+\infty)$,

$$\begin{aligned}
&I\{s < H_0\}\ \widetilde{E}^h_{(e,\varphi)}\Big(f(X_t,\varphi_t,t) \mid \mathcal{F}_s\Big)\\
&= I\{s < H_0\}\frac{E_{(X_s,\varphi_s)}\Big(f(X_{t-s},\varphi_{t-s},t-s)h(X_{t-s},\varphi_{t-s},t-s)I\{t-s < H_0\}\Big)}{h(X_s,\varphi_s,s)}\\
&= I\{s < H_0\}\ \widetilde{E}^h_{(X_s,\varphi_s)}\Big(f(X_{t-s},\varphi_{t-s},t-s)\Big).
\end{aligned}$$

Therefore, the process $(X_t,\varphi_t,t)_{t\ge 0}$ under $\widetilde{P}^h_{(e,\varphi)}$ is Markov.

1.10 Conditioning the process $(X_t,\varphi_t)_{t\ge 0}$ on the event $\{H_0 = +\infty\}$ in the positive drift case

Throughout this section we assume that the process $(\varphi_t)_{t\ge 0}$ drifts to $+\infty$. Our aim is to condition the process $(X_t,\varphi_t)_{t\ge 0}$, starting in E_0^+, on the event that the process $(\varphi_t)_{t\ge 0}$

stays non-negative, that is on the event $\{H_0 = +\infty\}$.

First we find the probability of the event $\{H_0 = +\infty\}$.

By Lemma 1.5.2, for $\varphi > 0$,

$$P_{(\cdot,\varphi)}(H_0 = +\infty) = \Big(I - \Gamma F(-\varphi)\Big)1 = \begin{pmatrix} 1^+ - \Pi^- e^{\varphi G^-} 1^- \\ 1^- - e^{\varphi G^-} 1^- \end{pmatrix},$$

and for $e \in E^+$, $\varphi = 0$,

$$P_{(e,0)}(H_0 = +\infty) = (1^+ - \Pi^- 1^-)(e).$$

By Lemma 1.4.6 (i), the matrix G^- is an irreducible Q-matrix. In addition, in the positive drift case, by (1.27), the matrix G^- is not conservative. Therefore, G^- is an irreducible non-conservative Q-matrix which by Lemma 1.1.5 implies that e^{yG^-} is strictly substochastic for all $y > 0$ and by Lemma 1.4.6 (ii), that the matrix Π^- is strictly substochastic. Therefore,

$$P_{(e,\varphi)}(H_0 = +\infty) > 0, \qquad (e,\varphi) \in E_0^+.$$

and we can condition the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_0 = +\infty\}$ in the standard way.

For fixed $t \geq 0$ and any $A \in \mathcal{F}_t$, we have that

$$\begin{aligned} P_{(e,\varphi)}(A|H_0 = \infty) &= \frac{E_{(e,\varphi)}(I(A)I\{H_0 = \infty\})}{P_{(e,\varphi)}(H_0 = \infty)} \\ &= \frac{E_{(e,\varphi)}(E_{(e,\varphi)}((I(A)I\{H_0 = \infty\}|\mathcal{F}_t))}{P_{(e,\varphi)}(H_0 = \infty)} \\ &= \frac{E_{(e,\varphi)}\Big(I(A)E_{(e,\varphi)}(I\{H_0 = \infty\}|\mathcal{F}_t)\Big)}{P_{(e,\varphi)}(H_0 = \infty)}. \end{aligned} \tag{1.56}$$

By the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$, we have that

$$E_{(e,\varphi)}(I\{H_0 = \infty\}|\mathcal{F}_t) = P_{(X_t,\varphi_t)}(H_0 = +\infty)I\{t < H_0\}. \tag{1.57}$$

Let $h(e,\varphi)$ be a function on $E \times \mathbb{R}$ defined by

$$h(e,\varphi) = P_{(e,\varphi)}(H_0 = \infty).$$

Then from (1.56) and (1.57) we obtain

$$P_{(e,\varphi)}(A|H_0 = \infty) = \frac{E_{(e,\varphi)}\Big(I(A)h(X_t,\varphi_t)I\{t < H_0\}\Big)}{h(e,\varphi)}.$$

Moreover, by (1.57) the process $\{h(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Putting the pieces together, we deduce the law of the process $(X_t,\varphi_t)_{t\geq 0}$ conditioned on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative. Before that, we recall from Section 1.9 the definition of an h-transform of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero: for a positive function h on E_0^+ such that the process $\{h(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$, starting at $(e,\varphi) \in E_0^+$, is a martingale, and for a probability measure $\widetilde{P}^h_{(e,\varphi)}$ be defined by

$$\widetilde{P}^h_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h(X_t,\varphi_t)I\{t < H_0\}\Big)}{h(e,\varphi)}, \qquad A \in \mathcal{F}_t, \quad t \geq 0,$$

the process $(X_t,\varphi_t)_{t\geq 0}$ under the measure $\widetilde{P}^h_{(e,\varphi)}$ is called the h-transform with the function h of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

Theorem 1.10.1 *The process $(X_t,\varphi_t)_{t\geq 0}$ conditioned on the event $\{H_0 = +\infty\}$ is an honest Markov process with the law $\widetilde{P}_{(e,\varphi)}$, $(e,\varphi) \in E_0^+$, where $\widetilde{P}_{(e,\varphi)}$ is such that the process $(X_t,\varphi_t)_{t\geq 0}$ under $\widetilde{P}_{(e,\varphi)}$ is the $h-$transform of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero with the function $h(e,\varphi) = P_{(e,\varphi)}(H_0 = +\infty)$, $(e,\varphi) \in E \times \mathbb{R}$.*

More precisely, for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\widetilde{P}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h(X_t,\varphi_t)I\{t < H_0\}\Big)}{h(e,\varphi)}.$$

Proof: In the discussion which precedes the theorem we have seen that $h(e,\varphi) > 0$ on E_0^+ and that the process $\{h(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Thus, the measure $\widetilde{P}_{(e,\varphi)}$ is a probability measure.

In addition, it follows from (1.56) and (1.57) that

$$P_{(e,\varphi)}(A|H_0 = \infty) = \widetilde{P}_{(e,\varphi)}(A),$$

and, by the discussion from the end of Section 1.9, the process $(X_t, \varphi_t)_{t\geq 0}$ under $\widetilde{P}_{(e,\varphi)}$ is Markov. $\square$

Chapter 2

The Green's function

2.1 A first approach to the Green's function

The Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$, denoted by $G((e,\varphi),(f,y))$, for any $(e,\varphi),(f,y) \in E \times \mathbb{R}$, is defined as the expected number of times that the process $(X_t, \varphi_t)_{t\geq 0}$, starting at (e,φ), hits (f,y), that is

$$G((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y)\Big),$$

noting that the process $(X_t, \varphi_t)_{t\geq 0}$ hits any fixed state at discrete times.

For simplicity of notation, let $G(\varphi, y)$ denote the matrix $(G((\cdot,\varphi),(\cdot,y)))_{E\times E}$.

The Green's function $G(\varphi, y)$ can be decomposed as

$$\begin{aligned} G((e,\varphi),(f,y)) &= \sum_{e'\in E} P_{(e,\varphi)}(X_{H_y} = e', H_y < +\infty)\ G((e',y),(f,y)) \\ &= \sum_{e'\in E} P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty)\ G((e',0),(f,0)), \qquad (2.1) \end{aligned}$$

where the last equality follows from the fact that $G(y,y) = G(0,0)$ for any $y \in \mathbb{R}$.

Therefore, for any $\varphi, y \in \mathbb{R}$, the function $G(\varphi, y)$ is determined by the hitting probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$, which are given in Lemma 1.5.1, and by the special case of the Green's function, $G(0,0)$.

We calculate $G(0,0)$ in the following theorem:

Theorem 2.1.1 *In the drift cases,*

$$G(0,0) = \Gamma_2^{-1} = \begin{pmatrix} (I - \Pi^-\Pi^+)^{-1} & \Pi^-(I - \Pi^+\Pi^-)^{-1} \\ \Pi^+(I - \Pi^-\Pi^+)^{-1} & (I - \Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

In the oscillating case,

$$G(0,0) = +\infty.$$

Proof: For any $e, e' \in E$, the Green's function $G((e,y),(e',y))$ can be decomposed as

$$G((e,y),(e',y)) = \sum_{n=0}^{\infty} P_{(e,y)}\Big((X_t)_{t\geq 0} \text{ hits } e' \text{ on the } n^{th} \text{ return of } (\varphi_t)_{t\geq 0} \text{ to } y\Big).$$

Since

$$P_{(e,y)}\Big((X_t)_{t\geq 0} \text{ hits } e' \text{ on the } 1^{st} \text{ return of } (\varphi_t)_{t\geq 0} \text{ to } y\Big)$$
$$= \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix}(e,e') = (I - \Gamma_2)(e,e'),$$

by the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$ we obtain

$$P_{(e,y)}\Big((X_t)_{t\geq 0} \text{ hits } e' \text{ on the } n^{th} \text{ return of } (\varphi_t)_{t\geq 0} \text{ to } y\Big) = (I - \Gamma_2)^n(e,e').$$

Suppose that the process $(\varphi_t)_{t\geq 0}$ drifts either to $+\infty$ of $-\infty$. Then by (1.27) and (1.29), exactly one of the matrices G^+ and G^- is conservative which, by Lemma 1.4.6 (ii) implies that exactly one of the matrices Π^+ and Π^- is strictly substochastic. Hence, the matrices $\Pi^-\Pi^+$ and $\Pi^+\Pi^-$ are strictly substochastic, which implies that $(\Pi^-\Pi^+ - I)1 \leq 0$ with strict inequality in at least one entry. By Lemma 1.4.4 (ii), $(\Pi^-\Pi^+ - I)$ is an irreducible essentially non-negative matrix. If λ is the Perron-Frobenius eigenvalue of $(\Pi^-\Pi^+ - I)$, then, by the Perron-Frobenius theorem for irreducible essentially non-negative matrices, $\lambda < 0$. On the other hand, by Lemma 1.4.4 (i), the matrix $\Pi^-\Pi^+$ is positive and therefore primitive. By the Perron-Frobenius theorem for primitive matrices, the Perron-Frobenius eigenvalue of $\Pi^-\Pi^+$ is positive. Since the Perron-Frobenius eigenvalue of $\Pi^-\Pi^+$ is equal to $1+\lambda$, we obtain $0 < 1+\lambda$. Hence, $0 < 1+\lambda < 1$. Moreover, again by the Perron-Frobenius theorem for primitive matrices,

$$\lim_{n\to\infty} \frac{(\Pi^-\Pi^+)^n}{(1+\lambda)^n} = const. \neq 0.$$

Therefore, $(\Pi^-\Pi^+)^n \to 0$ elementwise as $n \to +\infty$. In the same way we conclude that $(\Pi^+\Pi^-)^n \to 0$ elementwise as $n \to +\infty$.

Hence,

$$(I-\Gamma_2)^n = \begin{pmatrix} 0 & \Pi^- \\ \Pi^+ & 0 \end{pmatrix}^n \to 0, \qquad n \to +\infty.$$

Since

$$I - (I-\Gamma_2)^{n+1} = (I-(I-\Gamma_2))\sum_{k=0}^{n}(I-\Gamma_2)^k = \Gamma_2\sum_{k=0}^{n}(I-\Gamma_2)^k,$$

and, by Lemma 1.4.5, Γ_2^{-1} exists, by letting $n \to +\infty$ in the previous equation we obtain

$$\sum_{n=0}^{\infty}(I-\Gamma_2)^n = \Gamma_2^{-1}. \tag{2.2}$$

Therefore,

$$G(y,y) = \sum_{n=0}^{\infty}(I-\Gamma_2)^n = \Gamma_2^{-1},$$

and by the definition of the matrix Γ_2 and and Lemma 1.4.5,

$$\Gamma_2^{-1} = \begin{pmatrix} (I-\Pi^-\Pi^+)^{-1} & \Pi^-(I-\Pi^+\Pi^-)^{-1} \\ \Pi^+(I-\Pi^-\Pi^+)^{-1} & (I-\Pi^+\Pi^-)^{-1} \end{pmatrix}.$$

Suppose now that the process $(\varphi_t)_{t\geq 0}$ oscillates. Then by (1.28), both matrices G^+ and G^- are conservative, which, by Lemma 1.4.6 (ii) implies that the matries Π^+ and Π^- are stochastic. Thus, $(I-\Gamma_2)1 = 1$ and

$$G(0,0)1 = \sum_{n=0}^{\infty}(I-\Gamma_2)^n 1 = \sum_{n=0}^{\infty} 1 = +\infty. \tag{2.3}$$

Hence, row sums of the matrix $G(y,y)$ are infinite which means that in each row there is at least one entry that is infinite. Suppose that $G((e,y),(e',y)) = +\infty$ for some

$e, e' \in E$. Then, for any $e'' \in E$,

$$\begin{aligned} G((e,0),(e'',0)) &\geq \sum_{n=0}^{\infty} P_{(e,0)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } (e',0) \text{ exactly n times} \\ &\qquad \text{and then hits } (e'',0) \text{ before returning to } (e',0)\Big) \\ &= \sum_{n=0}^{\infty} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } (e',0) \text{ exactly n times}\Big) \\ &\qquad P_{(e',0)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } (e'',0) \text{ before returning to } (e',0)\Big) \\ &= G((e,0),(e',0))\; P_{(e',0)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } (e'',0) \text{ before} \\ &\qquad \text{returning to } (e',0)\Big) \end{aligned} \tag{2.4}$$

By Lemma 1.4.4 (i) the matrices Π^+ and Π^- are positive which implies that

$$\begin{aligned} P_{(e',y)}\Big((X_t,\varphi_t)_{t\geq 0} &\text{ hits } (e'',y) \text{ before returning to } (e',y)\Big) \\ &\geq \begin{pmatrix} \Pi^-\Pi^+ & \Pi^- \\ \Pi^+ & \Pi^+\Pi^- \end{pmatrix}(e,e') > 0. \end{aligned}$$

Hence, from (2.4) it follows that $G((e,y),(e'',y)) = +\infty$ for all $e'' \in E$.

Therefore, if one entry in a row of the matrix $G(y,y)$ is infinite, then all entries in that row are infinite which together with (2.3) gives that in the oscillating case $G(y,y) = +\infty$.

□

Theorem 2.1.1 and equality (2.1) imply that in the drift cases the process $(X_t,\varphi_t)_{t\geq 0}$ is transient and the Green's function $G(\varphi,y)$ is finite, and that in the oscillating case the process $(X_t,\varphi_t)_{t\geq 0}$ is recurrent and the Green's function $G(\varphi,y)$ is infinite.

Hence, we can calculate the Green's function $G(\varphi,y)$ only in the drift cases.

Theorem 2.1.2 *In the drift cases, the Green's function $G((e,\varphi),(f,y))$ of the process $(X_t,\varphi_t)_{t\geq 0}$ is given by the $E \times E$ matrix $G(\varphi,y)$, where*

$$G(\varphi,y) = \begin{cases} \Gamma\, F(y-\varphi)\, \Gamma_2^{-1}, & \varphi \neq y \\ \Gamma_2^{-1}, & \varphi = y. \end{cases}$$

Proof: By Theorem 2.1.1, $G(y,y) = \Gamma_2^{-1}$.

On the other hand, by Lemma 1.5.1,

$$P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty) = \Gamma\ F(y-\varphi)(e,e'), \qquad \varphi \neq y.$$

Thus, (2.1) and Theorem 2.1.1 imply that, for $\varphi \neq y$,

$$G(\varphi,y) = \Gamma\ F(y-\varphi)\ G(0,0) = \Gamma\ F(y-\varphi)\ \Gamma_2^{-1}.$$

□

Therefore, whenever we deal with the Green's function $G(\varphi, y)$, we make the assumption that the process $(\varphi_t)_{t\geq 0}$ drifts either to $+\infty$ of $-\infty$.

In the following section we present another way of calculating $G(\varphi, y)$ by considering its Laplace transform first.

2.2 The Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$ and its Laplace transform

In the previous section we have calculated the Green's function $G(\varphi, y)$ of the process $(X_t, \varphi_t)_{t\geq 0}$ via hitting probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$. In this section we present another way for calculating the Green's function $G(\varphi, y)$ which involves its Laplace transform. At first, the approach via Laplace transform might look cumbersome. However it appears to be interesting as it employs nice mathematical ideas and techniques.

We shall first find the Laplace transform of the Green's function $G(\varphi, y)$. Then we shall construct a function whose Laplace transform is identical to the Laplace transform of the Green's function $G(\varphi, y)$. Finally, by the uniqueness of the inverse Laplace transform, we shall conclude that the Green's function $G(\varphi, y)$ has to be equal to the constructed function.

We recall the definition of the Green's function $G((e,\varphi),(e',y))$, $(e,\varphi),(e',y) \in E\times\mathbb{R}$:

$$G((e,\varphi),(e',y)) = E_{(e,\varphi)}\Big(\sum_{0\le s<\infty} I\{X_s = e', \varphi_s = y\}\Big).$$

Let $e, e' \in E$ and $\varphi \in \mathbb{R}$ be fixed. The two-sided Laplace transform of the Green's function $G((e,\varphi),(e',y))$ is given by

$$\int_{-\infty}^{+\infty} e^{-\beta y} G((e,\varphi),(e',y))\, dy = \int_{-\infty}^{+\infty} e^{-\beta y} E_{(e,\varphi)}\Big(\sum_{0\le s<\infty} I\{X_s = e', \varphi_s = y\}\Big)\, dy.$$

Thus, we have to calculate the integral on the right-hand side of the previous equation. First we shall show that

$$\int_0^\infty e^{-\beta\varphi_s} I\{X_s = e'\}\, ds = \frac{1}{|v(e')|}\int_{-\infty}^{+\infty} e^{-\beta y} \sum_{0\le s<\infty} I\{X_s = e', \varphi_s = y\}\, dy. \qquad (2.5)$$

Let T_n, $n \ge 0$, be the jump times of the process $(X_t)_{t\ge 0}$ defined by

$$T_0 = 0, \qquad T_{n+1} = \inf\{t > T_n : X_t \ne X_{T_n}\},\ n \ge 1.$$

Then, for fixed $n \ge 0$ and $T_n \le s < T_{n+1}$,

$$X_s = X_{T_n}, \quad \varphi_s = \varphi_{T_n} + v(X_{T_n})(s - T_n), \quad d\varphi_s = v(X_{T_n})ds.$$

Therefore,

$$\begin{aligned}
\int_0^\infty e^{-\beta\varphi_s} I\{X_s = e'\}\, ds &= \sum_{n=0}^{\infty}\int_{T_n}^{T_{n+1}} e^{-\beta\varphi_s} I\{X_s = e'\}\, ds \\
&= \sum_{n=0}^{\infty}\int_{T_n}^{T_{n+1}} e^{-\beta\varphi_s} I\{X_{T_n} = e'\}\, ds \\
&= \sum_{n=0}^{\infty} I\{X_{T_n} = e'\}\int_{T_n}^{T_{n+1}} e^{-\beta\varphi_s}\, \frac{d\varphi_s}{v(X_{T_n})}.
\end{aligned}$$

For fixed $n \ge 0$, suppose that $X_{T_n} \in E^+$. Then the process $(\varphi_t)_{t\ge 0}$ is strictly increasing on the interval $[T_n, T_{n+1})$ and, by the change-of-variable formula,

$$\begin{aligned}
\int_{T_n}^{T_{n+1}} e^{-\beta\varphi_s}\, \frac{d\varphi_s}{v(X_{T_n})} &= \frac{1}{v(X_{T_n})}\int_{\varphi_{T_n}}^{\varphi_{T_{n+1}}} e^{-\beta y}\, dy \\
&= \frac{1}{v(X_{T_n})}\int_{-\infty}^{+\infty} e^{-\beta y} I\{y \in (\varphi_{T_n}, \varphi_{T_{n+1}})\}\, dy.
\end{aligned}$$

Hence, by Fubini's theorem because $I\{X_{T_n} = e'\}I\{y \in (\varphi_{T_n}, \varphi_{T_{n+1}})\}$ is non-negative, for $e' \in E^+$,

$$\begin{aligned}\int_0^{\infty} e^{-\beta\varphi_s} \; I\{X_s = e'\} \; ds &= \sum_{n=0}^{\infty} \frac{1}{v(X_{T_n})} \; I\{X_{T_n} = e'\} \int_{-\infty}^{+\infty} e^{-\beta y} I\{y \in (\varphi_{T_n}, \varphi_{T_{n+1}})\} \; dy \\ &= \frac{1}{v(e')} \int_{-\infty}^{+\infty} e^{-\beta y} \sum_{n=0}^{\infty} I\{X_{T_n} = e'\} I\{y \in (\varphi_{T_n}, \varphi_{T_{n+1}})\} \; dy.\end{aligned}$$

For $X_{T_n} = e'$ and $y \in (\varphi_{T_n}, \varphi_{T_{n+1}})$, since the process $(\varphi_t)_{t \geq 0}$ is continuous and strictly monotone on $[T_n, T_{n+1})$, there exists exactly one $s \in [T_n, T_{n+1})$ such that $X_s = e'$ and $\varphi_s = y$. Thus,

$$\sum_{n=0}^{\infty} I\{X_{T_n} = e'\} I\{y \in (\varphi_{T_n}, \varphi_{T_{n+1}})\} = \sum_{0 \leq s < \infty} I\{X_s = e', \varphi_s = y\},$$

and finally, for $e' \in E^+$,

$$\int_0^{\infty} e^{-\beta\varphi_s} I\{X_s = e'\} \; ds = \frac{1}{v(e')} \int_{-\infty}^{+\infty} e^{-\beta y} \sum_{0 \leq s < \infty} I\{X_s = e', \varphi_s = y\} \; dy. \tag{2.6}$$

Similarly, if $X_{T_n} \in E^-$, the process $(\varphi_t)_{t \geq 0}$ is strictly decreasing on the interval $[T_n, T_{n+1})$ and $\varphi_{T_{n+1}} < \varphi_{T_n}$. Thus,

$$\begin{aligned}\int_{T_n}^{T_{n+1}} e^{-\beta\varphi_s} \; \frac{d\varphi_s}{v(X_{T_n})} &= \frac{1}{v(X_{T_n})} \int_{\varphi_{T_n}}^{\varphi_{T_{n+1}}} e^{-\beta y} \; dy \\ &= \frac{-1}{v(X_{T_n})} \int_{-\infty}^{+\infty} e^{-\beta y} I\{y \in (\varphi_{T_{n+1}}, \varphi_{T_n})\} \; dy.\end{aligned}$$

In the same way as in the previous case $e' \in E^+$, we deduce that, for $e' \in E^-$,

$$\int_0^{\infty} e^{-\beta\varphi_s} I\{X_s = e'\} \; ds = \frac{-1}{v(e')} \int_{-\infty}^{+\infty} e^{-\beta y} \sum_{0 \leq s < \infty} I\{X_s = e', \varphi_s = y\} \; dy. \tag{2.7}$$

Therefore, (2.6) and (2.7) imply that (2.5) is valid.

Taking the expectation in (2.5), by Fubini's theorem, because $e^{-\beta\varphi_s} I\{X_s = e'\}$ and $\sum_{0 \leq s < \infty} I\{X_s = e', \varphi_s = y\}$ are non-negative, we obtain

$$\int_{-\infty}^{+\infty} e^{-\beta y} \; G((e, \varphi), (e', y)) \; dy = |v(e')| \int_0^{+\infty} E_{(e,\varphi)}(e^{-\beta\varphi_s} 1_{e'}(X_s)) \; ds. \tag{2.8}$$

Now we need to find $E_{(e,\varphi)}(e^{-\beta\varphi_s}1_{e'}(X_s))$. For fixed $e' \in E$, let $h_{e'}(e,\varphi,t)$ be a function on $E \times \mathbb{R} \times [0,\infty)$ defined by

$$h_{e'}(e,\varphi,t) = e^{-\beta\varphi}\, e^{-(Q-\beta V)t}\, 1_{e'}(e).$$

The function $h_{e'}(e,\varphi,t)$ is continuous and differentiable in φ and t, and also

$$(Q + V\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial t})h_{e'} = 0.$$

Thus, by Lemma 1.6.1, $\mathcal{A}h_{e'} = 0$ where $\mathcal{A}$ is the infinitesimal generator of the process $(X_t, \varphi_t, t)_{t\geq 0}$, which implies that $(h_{e'}(X_t,\varphi_t,t))_{t\geq 0}$ is a local martingale. But since $h_{e'}$ is bounded on every finite interval, the process $(h_{e'}(X_t,\varphi_t,t))_{t\geq 0}$ is a martingale. Hence,

$$E_{(e,\varphi)}(e^{-\beta\varphi_s}1_{e'}(X_s)) = e^{-\beta\varphi}e^{(Q-\beta V)s}1_{e'}(e). \tag{2.9}$$

Therefore, from (2.8) and (2.9) it follows that

$$\int_{-\infty}^{+\infty} e^{-\beta y}\, G((e,\varphi),(e',y))\, dy = |v(e')|\, e^{-\beta\varphi} \int_0^{+\infty} e^{(Q-\beta V)s}1_{e'}(e)\, ds. \tag{2.10}$$

Our next step is to calculate the integral

$$\int_0^{+\infty} e^{(Q-\beta V)s}1_{e'}(e)\, ds.$$

We recall the Perron-Frobenius eigenvalues α_{max} and $-\beta_{min}$ of the matrices G^+ and G^- and that, in the drift cases, by (1.27) and (1.29), exactly one of the matrices G^+ and G^- is conservative and $\alpha_{max} < \beta_{min}$, and that in the oscillating case, by (1.28), both matrices G^+ and G^- are conservative and $\alpha_{max} = \beta_{min} = 0$.

Lemma 2.2.1 *For any $e' \in E$*

$$\int_0^{+\infty} e^{(Q-\beta V)s}1_{e'}\, ds = \begin{cases} (Q-\beta V)^{-1}1_{e'}, & \alpha_{max} < \beta < \beta_{min}, \\ +\infty, & \textit{otherwise}. \end{cases}$$

Proof: From (2.10) we have that the integral

$$\int_0^{+\infty} e^{(Q-\beta V)s} 1_{e'} \, ds \tag{2.11}$$

converges for those values $\beta \in \mathbb{R}$ for which the integral on the left-hand side of (2.10) converges. Since $e^{-\beta y} G(\varphi, y) \geq 0$, the integral on the left-hand side of (2.10) converges in some interval and diverges on its complement, and therefore, the integral in (2.11) converges in the same interval and diverges on its complement. We shall show that the interval of convergence of the integral in (2.11) is $(\alpha_{max}, \beta_{min})$.

Let $\alpha(\beta)$ be the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$, which is, by Lemma 1.1.6, irreducible and essentially non-negative. Since in the drift cases, $\alpha_{max} \neq \beta_{min}$, it follows from Lemma 1.8.1 that $\alpha(\beta)$ is negative for $\beta \in (\alpha_{max}, \beta_{min})$ and positive or equal to zero for $\beta \notin (\alpha_{max}, \beta_{min})$.

Let $\beta \in (\alpha_{max}, \beta_{min})$. Then $\alpha(\beta)$ is negative which implies that all other eigenvalues of $(Q - \beta V)$ have negative real part. Hence, the integral in (2.11) converges, and since $(Q - \beta V)$ is invertible,

$$\int_0^{+\infty} e^{(Q-\beta V)s} 1_{e'} = (Q - \beta V)^{-1} 1_{e'}.$$

Let $\beta = \alpha_{max}$ or $\beta = \beta_{min}$. Then $\alpha(\beta) = 0$ and all other eigenvalues of $(Q - \beta V)$ have negative real part. Hence, if the vector $1_{e'}$ is not linearly independent of the Perron-Frobenius eigenvector of $(Q - \beta V)$, then the integral in (2.11) diverges, and if the vector $1_{e'}$ is linearly independent of the Perron-Frobenius eigenvector of $(Q - \beta V)$, then the integral in (2.11) converges.

We recall the basis $\mathcal{B} = \{f_j, j = 1 \ldots. n, g_k, k = 1, \ldots m\}$ of all vectors on E. Let

$$1_{e'} = \sum_{j=1}^{n} a_j(e') \, f_j \; + \; \sum_{k=1}^{m} b_k(e') \, g_k$$

for some coefficients $a_j(e')$, $j = 1 \ldots. n$, and $b_k(e')$, $k = 1, \ldots m$. Let $a_{max}(e')$ and $b_{min}(e')$ be the coefficients in the previous linear combination which are associated with the vectors f_{max} and g_{min}, and let f_{max}^{left} and g_{min}^{left} be the left eigenvectors of the matrix $V^{-1}Q$ associated with the eigenvalues α_{max} and β_{min}, respectively. Since α_{max} and β_{min} are simple eigenvalues of $V^{-1}Q$, it follows from Lemma 1.1.1 (i) that the vector f_{max}^{left} is orthogonal to all the vectors in the basis $\mathcal{B}$ except to the vector f_{max}, and that the vector g_{min}^{left} is orthogonal to all the vectors in the basis $\mathcal{B}$ except to the vector g_{min}. Hence,

$$f_{max}^{left}\, 1_{e'} = a_{max}(e') f_{max}^{left} f_{max} \quad \text{and} \quad g_{min}^{left}\, 1_{e'} = b_{min}(e') g_{min}^{left} g_{min}. \tag{2.12}$$

It follows from the Wiener-Hopf factorisation (1.6) that the vector f_{max}^{left} is of the form

$$f_{max}^{left} = \Big((f_{max}^{left})^{+} \quad - (f_{max}^{left})^{+}\Pi^{-} \Big),$$

where the vector $(f_{max}^{left})^{+}(I - \Pi^{-}\Pi^{+})$ is the left Perron-Frobenius vector of the matrix G^{+} associated with its Perron-Frobenius eigenvalue α_{max}, and therefore positive. Thus, $x = (f_{max}^{left})^{+}(I - \Pi^{-}\Pi^{+}) > 0$ implies

$$(f_{max}^{left})^{+} = (I - \Pi^{-}\Pi^{+})^{-1} x > 0,$$

because, by Lemma 1.4.5, the matrix $(I - \Pi^{-}\Pi^{+})^{-1}$ is positive. Since, by Lemma 1.4.4 (i), the matrix Π^{-} is positive, we conclude that for any $e \in E$, $f_{max}^{left} \neq 0$. Similarly, it can be shown that $g_{min}^{left}(e) \neq 0$ for any $e \in E$.

By Lemma 1.1.1 (ii) $f_{max}^{left} f_{max} \neq 0$ and $g_{min}^{left} g_{min} \neq 0$. Therefore, it follows from (2.12) that for any $e' \in E$, $a_{max}(e') \neq 0$ and $b_{min}(e') \neq 0$, which implies that the vector $1_{e'}$, $e' \in E$, is not linearly independent of the vectors f_{max} and g_{min}.

Hence, the integral in (2.11) diverges for $\beta = \alpha_{max}$ and $\beta = \beta_{min}$ and, by the discussion from the beginning of the proof, the integral in (2.11) diverges for $\beta \notin (\alpha_{max}, \beta_{min})$.

□

From Lemma 2.2.1 and (2.10), it follows that, for $\alpha_{max} < \beta < \beta_{min}$,

$$\int_{-\infty}^{+\infty} e^{-\beta y}\, G((e,\varphi),(e',y))\, dy = -e^{-\beta\varphi}|v(e')|(Q - \beta V)^{-1}(e,e'). \tag{2.13}$$

Recall the matrix J introduced in Section 1.4 as

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Then

$$\begin{aligned} -(Q-\beta V)^{-1}(e,e')|v(e')| &= -(Q-\beta V)^{-1}|V|(e,e') \\ &= -(Q-\beta V)^{-1}VJ(e,e') \\ &= -(V^{-1}Q-\beta I)^{-1}J(e,e'). \end{aligned}$$

The previous equation together with (2.13) proves the following theorem:

Theorem 2.2.1 *The Laplace transform of the Green's function $G(\varphi, y)$ of the process $(X_t, \varphi_t)_{t\geq 0}$ is given by*

$$\int_{-\infty}^{+\infty} e^{-\beta y} G(\varphi, y)\, dy = \begin{cases} -e^{-\beta\varphi}(V^{-1}Q-\beta I)^{-1}J, & \alpha_{max} < \beta < \beta_{min}, \\ +\infty, & otherwise, \end{cases}$$

where α_{max} and $-\beta_{min}$ are the Perron-Frobenius eigenvalues of the matrices G^+ and G^-.

Our task now is to construct a function whose Laplace transform is identical to the Laplace transform of the Green's function $G(\varphi, y)$ given in Theorem 2.2.1.

We recall the basis $\mathcal{B}$ in the space of all vectors on E which is in the drift cases equal to $\{f_j, j=1,\ldots,n, g_k, k=1,\ldots,m\}$ where the vectors f_j^+, $j=1,\ldots,n$, are associated with the eigenvalues α_j, $j=1,\ldots,n$, of the matrix G^+ and the vectors g_k^-, $k=1,\ldots,m$, are associated with the eigenvalues β_k, $k=1,\ldots,m$, of the matrix G^-. In addition, $Re(\alpha_j)\leq 0$ and $f_j^- = \Pi^+ f_j^+$, $j=1,\ldots,n$, and $Re(\beta_k)\geq 0$ and $g_k^+ = \Pi^- g_k^-$, $k=1,\ldots,m$.

Let X be a non-negative $E\times E$ matrix such that for any $e\in E$, the vector $X1_e$, which is actually the column of the matrix X corresponding to the element $e\in E$, is a linear combination only of the vectors $\{g_1, g_2, \ldots, g_m\}$ from the basis $\mathcal{B}$. In that case, the matrix X must be of the form

$$X = \begin{pmatrix} \Pi^- X_1 & \Pi^- X_2 \\ X_1 & X_2 \end{pmatrix} \tag{2.14}$$

for some matrices X_1 and X_2. For fixed $e \in E$, let

$$X1_e = \sum_{k=1}^{m} b_k(e)\; g_k, \tag{2.15}$$

for some coefficients $b_k(e)$, $k = 1, \ldots, m$, depending on $e \in E$. Then the integral

$$\begin{aligned} \int_{-\infty}^{0} e^{-\beta y}\; e^{yV^{-1}Q} X1_e\; dy &= \sum_{k=1}^{m} b_k(e) \int_{-\infty}^{0} e^{-\beta y}\; e^{yV^{-1}Q} g_k\; dy \\ &= \sum_{k=1}^{m} b_k(e) \int_{-\infty}^{0} e^{-(\beta-\beta_k)y}\; e^{y(V^{-1}Q-\beta_k I)} g_k\; dy \end{aligned} \tag{2.16}$$

is the Laplace transform of the function $y \mapsto e^{yV^{-1}Q} X1_e$ and it converges in some interval $(-\infty, \bar{\beta})$. From Section 1.4 we know that for fixed k, $k = 1, \ldots, m$, there exists $d_k \in \mathbb{N}$ such that

$$(V^{-1}Q - \beta_k I)^{d_k} g_k = 0.$$

Hence, for $|y| > 1$,

$$|e^{y(V^{-1}Q-\beta_k I)} g_k| \le const.\; |y|^{d_k}.$$

which implies that for $\beta < \beta_k$ the integral

$$\int_{-\infty}^{0} e^{-(\beta-\beta_k)y}\; e^{y(V^{-1}Q-\beta_k I)} g_k\; dy$$

converges absolutely. Thus, the sum in (2.16) is finite for $\beta < \beta_{min}$. Let now $\beta = \beta_{min}$. Then all terms in the sum in (2.16) still converge except the term which corresponds to g_{min}, since

$$\int_{-\infty}^{0} e^{y(V^{-1}Q-\beta_{min} I)} g_{min}\; dy = \int_{-\infty}^{0} g_{min}\; dy = -\infty.$$

Hence, the sum in (2.16) in finite for $\beta = \beta_{min}$ if and only if the coefficient $b_{min}(e)$ which corresponds to g_{min} in (2.15) is equal to zero, that is if the vector $X1_e$ is linearly independent of g_{min}. But, since X is non-negative, the vector $X1_e$ is also non-negative which by Lemma 1.4.11 implies that it is not linearly independent of g_{min}. Thus, $b_{min}(e) \neq 0$ and the sum in (2.16) is infinite for $\beta = \beta_{min}$.

Therefore, if X is a non-negative matrix of the form (2.14) then

$$\int_{-\infty}^{0} e^{-\beta y} \, e^{yV^{-1}Q} X \, dy \begin{cases} < +\infty, & \beta < \beta_{min}, \\ = \infty, & \beta \geq \beta_{min}. \end{cases} \tag{2.17}$$

Similarly, let Y be a non-negative $E \times E$ matrix whose columns are vectors in the space spanned only by the vectors $\{f_1, f_2, \dots, f_n\}$ from the basis $\mathcal{B}$. Then the matrix Y is of the form

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ \Pi^+ Y_1 & \Pi^+ Y_2 \end{pmatrix} \tag{2.18}$$

for some matrices Y_1 and Y_2. By arguments analogous to those just used to determine the matrix X, it follows that

$$\int_{0}^{+\infty} e^{-\beta y} \, e^{yV^{-1}Q} Y \, dy \begin{cases} < +\infty, & \beta > \alpha_{max}, \\ = \infty, & \beta \leq \alpha_{max}. \end{cases} \tag{2.19}$$

Let X and Y be non-negative matrices of the forms (2.14) and (2.18), respectively, and let $G_{X,Y}(0, y)$, $y \neq 0$, be defined for such matrices by

$$G_{X,Y}(0, y) = \begin{cases} e^{yV^{-1}Q} Y, & y > 0 \\ e^{yV^{-1}Q} X, & y < 0. \end{cases} \tag{2.20}$$

Since the function $y \mapsto G_{X,Y}(0, y)$ has finite left and right limits at zero, we can integrate it over the interval $(-\infty, +\infty)$. Then, by (2.17) and (2.19),

$$\int_{-\infty}^{+\infty} e^{-\beta y} \, G_{X,Y}(0, y) \, dy \begin{cases} < +\infty, & \alpha_{max} < \beta < \beta_{min}, \\ = \infty, & \text{otherwise.} \end{cases}$$

Moreover, for $\beta \in (\alpha_{max}, \beta_{min})$ the matrix $(V^{-1}Q - \beta I)$ is invertible. Thus, for $\beta \in (\alpha_{max}, \beta_{min})$,

$$\int_{-\infty}^{+\infty} e^{-\beta y} \, G_{X,Y}(0, y) \, dy \;=\; \int_{-\infty}^{0} e^{-\beta y} \, e^{yV^{-1}Q} X + \int_{0}^{+\infty} e^{-\beta y} \, e^{yV^{-1}Q} Y \, dy$$

$$\begin{aligned} &= \int_{-\infty}^{0} e^{y(V^{-1}Q-\beta I)}X + \int_{0}^{+\infty} e^{y(V^{-1}Q-\beta I)}Y\, dy \\ &= (V^{-1}Q-\beta I)^{-1}(X-Y). \end{aligned}$$

Therefore,

$$\int_{-\infty}^{+\infty} e^{-\beta y}\, G_{X,Y}(0,y)\, dy = \begin{cases} (V^{-1}Q-\beta I)^{-1}(X-Y), & \alpha_{max} < \beta < \beta_{min}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.21}$$

We remind ourselves that our aim is to construct a function whose Laplace transform is identical to the Laplace transform of the Green's function $G(\varphi, y)$ given in Theorem 2.2.1. By comparing the last equation with Theorem 2.2.1 we see that if $X - Y = -J$, then the Laplace transform of $G_{X,Y}(0,y)$ is identical to the Laplace transform of $G(0,y)$.

By (1.27), (1.28) and (1.29), $\alpha_{max} < \beta_{min}$ is possible only in the drift cases. Hence, in the oscillating case we put $G_{X,Y}(0,y) = +\infty$, and in the drift cases we look for non-negative matrices X and Y of the forms (2.14) and (2.18), respectively, which satisfy $X - Y = -J$.

From the equation $X - Y = -J$, (2.14) and (2.18), we get two systems

$$\begin{cases} \Pi^- X_1 &= Y_1 - I \\ X_1 &= \Pi^+ Y_1 \end{cases} \quad \text{and} \quad \begin{cases} \Pi^- X_2 &= Y_2 \\ X_2 &= \Pi^+ Y_2 + I. \end{cases}$$

By (1.27) and (1.29), in the drift cases exactly one of the matrices G^+ and G^- is conservative, which implies, by Lemma 1.4.6 (ii), that exactly one of the matrices Π^+ and Π^- is stochastic. Then it follows from Lemma 1.4.5 that the matrices $(I - \Pi^+\Pi^-)$ and $(I - \Pi^-\Pi^+)$ are invertible and therefore the solutions of the previous two systems of matrix equations are given by

$$\begin{cases} X_1 &= \Pi^+\, (I - \Pi^-\, \Pi^+)^{-1} \\ Y_1 &= (I - \Pi^-\, \Pi^+)^{-1} \end{cases} \quad \text{and} \quad \begin{cases} X_2 &= (I - \Pi^+\, \Pi^-)^{-1} \\ Y_2 &= \Pi^-\, (I - \Pi^+\, \Pi^-)^{-1}. \end{cases}$$

Hence, the matrices X and Y which are of the forms (2.14) and (2.18), respectively, and which satisfy $X - Y = -J$ are given by

$$\begin{aligned} X &= \begin{pmatrix} \Pi^-\Pi^+ (I - \Pi^- \Pi^+)^{-1} & \Pi^-(I - \Pi^+ \Pi^-)^{-1} \\ \Pi^+ (I - \Pi^- \Pi^+)^{-1} & (I - \Pi^+ \Pi^-)^{-1} \end{pmatrix} = \Gamma J_2 \Gamma_2^{-1} \\ Y &= \begin{pmatrix} (I - \Pi^- \Pi^+)^{-1} & \Pi^- (I - \Pi^+ \Pi^-)^{-1} \\ \Pi^+(I - \Pi^- \Pi^+)^{-1} & \Pi^+\Pi^- (I - \Pi^+ \Pi^-)^{-1} \end{pmatrix} = \Gamma J_1 \Gamma_2^{-1}. \end{aligned} \tag{2.22}$$

We need to check if the matrices X and Y are non-negative. By Lemma 1.4.5 the matrices $(I - \Pi^-\Pi^+)^{-1}$ and $(I - \Pi^+\Pi^-)^{-1}$ are positive, and since, by Lemma 1.4.4 (i), the matrices Π^+ and Π^- are positive, we conclude that the matrices X and Y are positive.

Therefore, the matrices X and Y given in (2.22) are positive, they are of the forms (2.14) and (2.18), respectively, and $X - Y = -J$. Hence, by (2.20), for the matrices X and Y given in the drift cases by (2.22), the function $G_{X,Y}(0, y)$ is given by

$$G_{X,Y}(0,y) = \begin{cases} e^{yV^{-1}Q}\, \Gamma\, J_1\, \Gamma_2^{-1}, & y > 0 \\ e^{yV^{-1}Q}\, \Gamma\, J_2\, \Gamma_2^{-1}, & y < 0. \end{cases}$$

By $V^{-1}Q\, \Gamma = \Gamma\, G$ and equalities (1.14) and (1.15), $G_{X,Y}(0, y)$ can be rewritten as

$$G_{X,Y}(0,y) = \Gamma\, F(y)\, \Gamma_2^{-1}, \quad y \neq 0.$$

Finally, from (2.21), Theorem 2.2.1 and the previous equation we conclude that, for $y \neq 0$,

$$G(0,y) = \begin{cases} \Gamma\, F(y)\, \Gamma_2^{-1}, & \text{in the drift cases} \\ +\infty, & \text{in the oscillating case.} \end{cases} \tag{2.23}$$

It remains to calculate $G(0, y)$ for $y = 0$. By the definition of the Green's function $G((e, \varphi), (f, y))$, the function $y \mapsto G((e, \varphi), (f, y))$ is at $y = \varphi$ right-continuous for $(e, f) \in E^+ \times E^+$, left-continuous for $(e, f) \in E^- \times E^-$ and continuous for $(e, f) \in E^- \times E^+$ and $(e, f) \in E^+ \times E^-$. Hence,

$$G(0,0) = \begin{cases} \lim_{\substack{y \to 0 \\ y>0}} G(0,y), & f \in E^+ \\ \lim_{\substack{y \to 0 \\ y<0}} G(0,y), & f \in E^- \end{cases} = \begin{cases} Y, & f \in E^+ \\ X, & f \in E^-, \end{cases}$$

or explicitly,

$$G(0,0) = \begin{pmatrix} (I - \Pi^- \Pi^+)^{-1} & \Pi^+(I - \Pi^- \Pi^+)^{-1} \\ \Pi^-(I - \Pi^+ \Pi^-)^{-1} & (I - \Pi^+ \Pi^-)^{-1} \end{pmatrix} = \Gamma_2^{-1}. \tag{2.24}$$

Finally, since $G(\varphi, y) = G(0, y - \varphi)$, from (2.23) and (2.24) we conclude that, in the drift cases,

$$G(\varphi, y) = \begin{cases} \Gamma\, F(y - \varphi)\, \Gamma_2^{-1}, & \varphi \neq y \\ \Gamma_2^{-1}, & \varphi = y. \end{cases}$$

which is equal to the previously obtained formula for the Green's function $G(\varphi, y)$ in Theorem 2.1.2.

2.3 The Green's function of the process killed on reaching zero

We recall that by the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero or, shortly, by the killed process $(X_t, \varphi_t)_{t\geq 0}$, we mean the process $(X_t, \varphi_t)_{t\geq 0}$ run up to time H_0. In this section we shall calculate its Green's function.

The Green's function $G_0((e,\varphi),(f,y))$, $(e,\varphi),(f,y) \in E \times \mathbb{R}$, (in matrix notation $G_0(\varphi, y)$) of the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero is defined as the expected number of times that the process $(X_t, \varphi_t)_{t\geq 0}$, starting at (e, φ), hits (f, y) before $(\varphi_t)_{t\geq 0}$ crosses zero, that is

$$G_0((e,\varphi),(f,y)) = E_{(e,\varphi)}\Big(\sum_{0 \leq s < H_0} I(X_s = f, \varphi_s = y) \Big).$$

It follows directly from the definition of $G_0(\varphi, y)$ that $G_0(\varphi, y) = 0$ if φ and y have the opposite signs, that $G_0(\varphi, 0) = 0$ if $\varphi \neq 0$, and that $G_0(0,0) = I$. Hence, we have to calculate $G_0(\varphi, y)$ for $\varphi y \geq 0$, $y \neq 0$.

The Green's function $G_0(\varphi, y)$ for $|\varphi| > |y|$, $\varphi y > 0$, can be decomposed as

$$G_0((e,\varphi),(f,y)) = \sum_{e'\in E} P_{(e,\varphi)}(X_{H_y} = e')\ G_0((e',y),(f,y)).$$

Since, by Lemma 1.5.1,

$$P_{(e,\varphi)}(X_{H_y} = e') = \Gamma F(y-\varphi)(e,e'),$$

it follows that

$$G_0((e,\varphi),(f,y)) = \Gamma F(y-\varphi)\ G_0(y,y)(e,f), \qquad |\varphi| > |y|,\ \varphi y > 0.$$

Therefore, all that we need to calculate is the Green's function $G_0(\varphi, y)$ for $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$.

For fixed $(f, y) \in E \times \mathbb{R}$ and $0 < \varphi < y$, let $(M_t)_{t\geq 0}$ be a process defined by

$$\begin{aligned} M_t &= E_{(e,\varphi)}\Big(\sum_{0\leq s<H_0} I(X_s = f, \varphi_s = y) \mid \mathcal{F}_t\Big) \\ &= I\{t < H_0\}\Big(\sum_{0\leq s<t} I(X_s = f, \varphi_s = y) + E_{(X_t,\varphi_t)}\big(\sum_{0\leq s<H_0} I(X_s = f, \varphi_s = y)\big)\Big) \\ &\quad + I\{t \geq H_0\} \sum_{0\leq s<H_0} I(X_s = f, \varphi_s = y). \end{aligned}$$

The process $(M_t)_{t\geq 0}$ is a uniformly integrable martingale and $t\wedge H_0 \wedge H_y$ is a bounded stopping time. Thus, the process

$$M_{t\wedge H_0\wedge H_y} = G_0((X_{t\wedge H_0\wedge H_y}, \varphi_{t\wedge H_0\wedge H_y}),(f,y)),$$

starting at $G_0((e,\varphi),(f,y))$, $0 \leq \varphi < y$ or $y < \varphi \leq 0$, is a uniformly integrable martingale. Moreover, the function $(e,\varphi) \mapsto G_0((e,\varphi),(f,y))$ is continuous on E^+ and right-continuous on E^-, and satisfies the boundary conditions $G_0((e,0),(f,y)) = 0$, $e \in E^-$, $y > 0$ and $G_0((e,0),(f,y)) = 0$, $e \in E^+$, $y < 0$. This is enough information to calculate $G_0(\varphi, y)$ for $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$, as the calculations will be based on the following lemma.

Lemma 2.3.1 *Let $(f,y) \in E^+ \times (0,+\infty)$ be fixed and let the process $(X_t,\varphi_t)_{t\geq 0}$ start at $(e,\varphi) \in E \times (0,y)$. Let $(e,\varphi) \mapsto h((e,\varphi),(f,y))$ be a bounded function on $E \times (0,y)$ such that the process $(h((X_{t\wedge H_0 \wedge H_y}, \varphi_{t\wedge H_0\wedge H_y}),(f,y)))_{t\geq 0}$ is a uniformly integrable martingale and that*

$$h((e,0),(f,y)) = 0, \qquad e \in E^- \tag{2.25}$$

$$h((e,y),(f,y)) = G_0((e,y),(f,y)). \tag{2.26}$$

Then

$$h((e,\varphi),(f,y)) = G_0((e,\varphi),(f,y)), \quad (e,\varphi) \in E \times (0,y).$$

Proof: The proof is given in Appendix B. □

Let A_y, B_y, C_y and D_y be components of the matrix $e^{-yV^{-1}Q}$ such that, for any $y \in \mathbb{R}$,

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}. \tag{2.27}$$

Now we calculate the Green's function $G_0(\varphi,y)$ for $|\varphi| \leq |y|$, $\varphi y \geq 0$, $y \neq 0$.

Theorem 2.3.1 *The Green's function $G_0((e,\varphi),(f,y))$, $|\varphi| < |y|$, $\varphi y \geq 0$, $y \neq 0$, $e,f \in E$, of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero is given by the $E \times E$ matrix $G_0(\varphi,y)$ with the components*

$$G_0(\varphi,y) = \begin{cases} \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & A_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & C_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix}, & 0 \leq \varphi < y \\ \begin{pmatrix} B_\varphi(D_y - \Pi^+ B_y)^{-1}\Pi^+ & B_\varphi(D_y - \Pi^+ B_y)^{-1} \\ D_\varphi(D_y - \Pi^+ B_y)^{-1}\Pi^+ & D_\varphi(D_y - \Pi^+ B_y)^{-1} \end{pmatrix}, & y < \varphi \leq 0, \end{cases}$$

$$G_0(y,y) = \begin{cases} \begin{pmatrix} (I - \Pi^- C_y A_y^{-1})^{-1} & \Pi^-(I - C_y A_y^{-1}\Pi^-)^{-1} \\ C_y A_y^{-1}(I - \Pi^- C_y A_y^{-1})^{-1} & (I - C_y A_y^{-1}\Pi^-)^{-1} \end{pmatrix}, & y > 0 \\ \begin{pmatrix} (I - B_y D_y^{-1}\Pi^+)^{-1} & ByD_y^{-1}(I - \Pi^+ B_y D_y^{-1})^{-1} \\ \Pi^+(I - B_y D_y^{-1}\Pi^+)^{-1} & (I - \Pi^+ B_y D_y^{-1})^{-1} \end{pmatrix}, & y < 0, \end{cases}$$

$$= \begin{cases} \begin{pmatrix} I & -\Pi^- \\ -C_y A_y^{-1} & I \end{pmatrix}^{-1}, & y > 0 \\ \begin{pmatrix} I & -B_y D_y^{-1} \\ -\Pi^+ & I \end{pmatrix}^{-1}, & y < 0. \end{cases}$$

In addition, the Green's function $G_0(\varphi, y)$ is positive for all $\varphi, y \in \mathbb{R}$ except for $y = 0$ and $\varphi y < 0$.

Proof: We shall prove the theorem for $y > 0$. The case $y < 0$ can be proved in the same way.

Let $y > 0$. First we shall calculate the Green's function $G_0(y, y)$.

For any $e, e' \in E$, the Green's function $G_0((e, y), (e', y))$ can be decomposed as

$$G_0((e,y),(e',y)) = \sum_{n=0}^{\infty} P_{(e,y)}\Big((X_t)_{t\geq 0} \text{ hits } e' \text{ on the } n^{th} \text{ return of } (\varphi_t)_{t\geq 0} \text{ to } y \text{ before crossing zero}\Big).$$

Let Y_y denote a matrix on $E^- \times E^+$ with entries

$$Y_y(e, e') = P_{(e,y)}(X_{H_y} = e', H_y < H_0).$$

By the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$ we have that, for $e, e' \in E^+$,

$$G_0((e,y),(e',y)) = \sum_{n=0}^{\infty} (\Pi^- Y_y)^n (e, e'),$$

and that, for $e, e' \in E^-$,

$$G_0((e,y),(e',y)) = \sum_{n=0}^{\infty} (Y_y \Pi^-)^n (e, e').$$

Hence, the Green's function $G_0(y, y)$ is given by

$$G_0(y,y) = \begin{pmatrix} I & \Pi^- \\ Y_y & I \end{pmatrix} \begin{pmatrix} \sum_{n=0}^{\infty}(\Pi^- Y_y)^n & 0 \\ 0 & \sum_{n=0}^{\infty}(Y_y \Pi^-)^n \end{pmatrix}.$$

Now we shall show that the series $\sum_{n=0}^{\infty}(\Pi^{-}Y_y)^n$ and $\sum_{n=0}^{\infty}(Y_y\Pi^{-})^n$ converge and find their limits.

By Lemma 1.5.3, the matrix Y_y is positive. Hence, the matrices $\Pi^{-}Y_y$ and $Y_y\Pi^{-}$ are positive which implies that they are primitive and, by Lemma 1.1.2, irreducible. Thus, the matrices $(\Pi^{-}Y_y - I)$ and $(Y_y\Pi^{-} - I)$ are essentially non-negative and irreducible. Since, by Lemma 1.5.3, $0 < Y_y 1^{+} < 1^{-}$, it follows that $(\Pi^{-}Y_y - I)1^{+} < 0$ and $(Y_y\Pi^{-} - I)1^{-} < 0$. If λ denotes the Perron-Frobenius eigenvalue of the matrix $(\Pi^{-}Y_y - I)$, then $1+\lambda$ is the Perron-Frobenius eigenvalue of the matrix $\Pi^{-}Y_y$. It follows from the Perron-Frobenius theorem for irreducible essentially non-negative matrices that $\lambda < 0$ and from the Perron-Frobenius theorem for primitive matrices that $1+\lambda > 0$. Therefore, $0 < 1+\lambda < 1$ and, again by the Perron-Frobenius theorem for primitive matrices,

$$\lim_{n\to\infty} \frac{(\Pi^{-}Y_y)^n}{(1+\lambda)^n} = const. \neq 0.$$

Hence, $(\Pi^{-}Y_y)^n \to 0$ elementwise as $n \to +\infty$. We can show in the same way that $(Y_y\Pi^{-})^n \to 0$ elementwise as $n \to +\infty$.

In addition, the matrices $(\Pi^{-}Y_y - I)$ and $(Y_y\Pi^{-} - I)$ are invertible because their Perron-Frobenius eigenvalues are negative and, by the Perron-Frobenius theorem for irreducible non- negative matrices, the inverses $(I - \Pi^{-}Y_y)^{-1}$ and $(I - Y_y\Pi^{-})^{-1}$ are positive.

$$\begin{aligned} I - (\Pi^{-}Y_y)^{n+1} &= (I - \Pi^{-}Y_y)\ \textstyle\sum_{k=0}^{n}(\Pi^{-}Y_y)^k \\ I - (Y_y\Pi^{-})^{n+1} &= (I - Y_y\Pi^{-})\ \textstyle\sum_{k=0}^{n}(Y_y\Pi^{-})^k, \end{aligned}$$

we have that

$$\begin{aligned} \textstyle\sum_{k=0}^{n}(\Pi^{-}Y_y)^k &= (I - \Pi^{-}Y_y)^{-1}\,(I - (\Pi^{-}Y_y)^{n+1}) \\ \textstyle\sum_{k=0}^{n}(Y_y\Pi^{-})^k &= (I - Y_y\Pi^{-})^{-1}\,(I - (Y_y\Pi^{-})^{n+1}). \end{aligned}$$

By letting $n \to +\infty$ into the previous two equations we finally obtain

$$\sum_{n=0}^{\infty}(\Pi^{-}Y_y)^n = (I - \Pi^{-}Y_y)^{-1} \qquad \sum_{n=0}^{\infty}(Y_y\Pi^{-})^n = (I - Y_y\Pi^{-})^{-1},$$

and the Green's function $G_0(y,y)$

$$\begin{aligned} G_0(y,y) &= \begin{pmatrix} I & \Pi^- \\ Y_y & I \end{pmatrix} \begin{pmatrix} (I-\Pi^- Y_y)^{-1} & 0 \\ 0 & (I-Y_y\Pi^-)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} (I-\Pi^- Y_y)^{-1} & \Pi^-(I-Y_y\Pi^-)^{-1} \\ Y_y(I-\Pi^- Y_y)^{-1} & (I-Y_y\Pi^-)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I & -\Pi^- \\ -Y_y^{-1} & I \end{pmatrix}^{-1}. \end{aligned} \tag{2.28}$$

By Lemma 1.4.4 (i), the matrix Π^- is positive and by Lemma 1.5.3 the matrix Y_y is positive. Since the matrices $(I-\Pi^- Y_y)^{-1}$ and $(I-Y_y\Pi^-)^{-1}$ are also positive, we conclude that $G_0(y,y)$, $y>0$ is positive.

Now we shall calculate the Green's function $G_0(\varphi,y)$ for $0 \le \varphi < y$. Let $(f,y) \in E^+ \times (0,+\infty)$ be fixed and let the process $(X_t,\varphi_t)_{t\ge 0}$ start in $E \times (0,y)$. We shall find a function which satisfies all the conditions of Lemma 2.3.1 and then the result will follow from the lemma.

In order to find a function h such that $(h((X_{t\wedge H_0\wedge H_y},\varphi_{t\wedge H_0\wedge H_y}),(f,y)))_{t\ge 0}$ is a uniformly integrable martingale, we look at the solutions of the equation $\mathcal{G}h=0$ where $\mathcal{G}$ is the infinitesimal generator of the process $(X_t,\varphi_t)_{t\ge 0}$. If h is continuously differentiable in φ, then by Lemma 1.6.1 it satisfies the equation

$$(Q+V\frac{\partial}{\partial\varphi})h=0,$$

and therefore, by Lemma 1.6.3, it has to be of the form

$$h((e,\varphi),(f,y)) = e^{-\varphi V^{-1}Q} g_{f,y}(e), \tag{2.29}$$

for some vector $g_{f,y}$ on E.

Since $\mathcal{G}h=0$, the process $(h((X_t,\varphi_t),(f,y)))_{t\ge 0}$ is a local martingale, and because the function h is bounded on every finite interval, $(h((X_t,\varphi_t),(f,y)))_{t\ge 0}$ is a martingale. In addition, the process $(h((X_{t\wedge H_0\wedge H_y},\varphi_{t\wedge H_0\wedge H_y}),(f,y)))_{t\ge 0}$ is a bounded martingale and therefore a uniformly integrable martingale.

We need the function h also to satisfy the boundary conditions.

Let $h_y(\varphi)$ be an $E \times E^+$ matrix with entries

$$h_y(\varphi)(e,f) = h((e,\varphi),(f,y)).$$

Then, from (2.29) and the boundary condition (2.25),

$$h_y(\varphi) = \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} M_y \\ 0 \end{pmatrix} = \begin{pmatrix} A_\varphi M_y \\ C_\varphi M_y \end{pmatrix}, \quad 0 \le \varphi < y,$$

for some $E^+ \times E^+$ matrix M_y.

From the boundary condition (2.26) we get that

$$A_y M_y = (I - \Pi^- Y_y)^{-1} \tag{2.30}$$

$$C_y M_y = Y_y (I - \Pi^- Y_y)^{-1}. \tag{2.31}$$

Multiplying (2.31) by Π^- and substracting this from (2.30) we obtain

$$(A_y - \Pi^- C_y) M_y = I,$$

which implies that the matrix M_y is invertible and that

$$M_y = (A_y - \Pi^- C_y)^{-1}.$$

Hence,

$$h_y(\varphi) = \begin{pmatrix} A_\varphi (A_y - \Pi^- C_y)^{-1} \\ C_\varphi (A_y - \Pi^- C_y)^{-1} \end{pmatrix}, \qquad 0 \le \varphi < y$$

and the function $h((e,\varphi),(f,y))$ satisfies the boundary conditions (2.25) and (2.26) in Lemma 2.3.1. Therefore, by Lemma 2.3.1, for $0 \le \varphi < y$,

$$G_0(\varphi, y) = h_y(\varphi) \quad \text{on } E \times E^+.$$

Let now $f \in E^-$. Then

$$\begin{aligned} G_0((e,\varphi),(f,y)) &= \sum_{e' \in E^+} G_0((e,\varphi),(e',y)) \, \Pi^-(e',f) \\ &= \sum_{e' \in E^+} h((e,\varphi),(e',y)) \, \Pi^-(e',f), \end{aligned}$$

and finally,

$$G_0(\varphi, y) = \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & A_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & C_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix}, \qquad 0 \le \varphi < y.$$

Multiplying (2.30) by Y_y and substracting this from (2.31) we obtain

$$(C_y - Y_y A_y) M_y = 0. \tag{2.32}$$

Since by (2.30) the matrices M_y and A_y are invertible, by multiplying the previous equation by M_y^{-1} and A_y^{-1} we obtain

$$Y_y = C_y A_y^{-1},$$

which together with (2.28) yields the formula for the Green's function $G_0(y, y)$.

To finish the proof we have to show that $G_0(\varphi, y)$ is positive. Let $0 \le \varphi < y$. Then

$$G_0((e,\varphi),(e',y)) = \sum_{e'' \in E^+} P_{(e,\varphi)}(X_{H_y} = e'', H_y < H_0)\ G_0((e',y),(e',y)).$$

By Lemma 1.5.3, the exit probabilities on the right-hand side of the previous equation are positive, and since the Green's function $G_0(y, y)$ is positive, we conclude that $G_0(\varphi, y)$ for $0 \le \varphi < y$ is also positive.

Similarly, for $0 < y < \varphi$,

$$G_0((e,\varphi),(e',y)) = \sum_{e'' \in E^-} P_{(e,\varphi)}(X_{H_y} = e'')\ G_0((e',y),(e',y)).$$

Since $G_0(y, y)$ is positive and by Lemma 1.5.1, all exit probabilities on the right-hand side of the previous equation are positive, it follows that $G_0(\varphi, y)$, $0 < y < \varphi$, is positive.

□

It is more convenient to have the Green's function $G_0(\varphi, y)$ expressed in terms of the familiar matrices Q, V, Γ and G. Therefore, we shall rewrite the representation formula for $G_0(\varphi, y)$ which is given in the previous theorem.

From the Wiener-Hopf factorization (1.6) of the matrix $V^{-1}Q$ it follows that

$$e^{-yV^{-1}Q}\Gamma = \Gamma e^{-yG},$$

which written componentwise is

$$\begin{aligned} A_y + B_y\Pi^+ &= e^{-yG^+} \\ A_y\Pi^- + B_y &= \Pi^- e^{yG^-} \\ C_y + D_y\Pi^+ &= \Pi^+ e^{-yG^+} \\ C_y\Pi^- + D_y &= e^{yG^-}. \end{aligned} \tag{2.33}$$

From the previous system of equations it follows that

$$\begin{aligned} B_y - \Pi^- D_y &= -(A_y - \Pi^- C_y)\Pi^- \\ C_y - \Pi^+ A_y &= -(D_y - \Pi^+ B_y)\Pi^+ \\ (A_y - \Pi^- C_y)(I - \Pi^-\Pi^+) &= (I - \Pi^-\Pi^+)e^{-yG^+} \\ (D_y - \Pi^+ B_y)(I - \Pi^+\Pi^-) &= (I - \Pi^+\Pi^-)e^{yG^-} \end{aligned} \tag{2.34}$$

The equation

$$e^{-yV^{-1}Q}\, e^{yV^{-1}Q} = I \quad \Leftrightarrow \quad \begin{aligned} A_yA_{-y} + B_yC_{-y} &= I \\ A_yB_{-y} + B_yD_{-y} &= 0 \\ C_yA_{-y} + D_yC_{-y} &= 0 \\ C_yB_{-y} + D_yD_{-y} &= I \end{aligned} \tag{2.35}$$

and (2.34) imply that, for any $y \in \mathbb{R}$,

$$\begin{aligned} (A_y - \Pi^- C_y)(A_{-y} - \Pi^- C_{-y}) &= I \\ (D_y - \Pi^+ B_y)(D_{-y} - \Pi^+ B_{-y}) &= I. \end{aligned} \tag{2.36}$$

From the previous equation and Theorem 2.3.1 it follows that, for $0 \le \varphi < y$,

$$\begin{aligned} G_0(\varphi, y) &= \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} A_{-y} - \Pi^- C_{-y} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \\ &= \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} I & -\Pi^- \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{-y} & 0 \\ C_{-y} & 0 \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \\ &= e^{-\varphi V^{-1}Q} J_1 \Gamma_2\, e^{yV^{-1}Q} J_1 \Gamma, \end{aligned}$$

and that, for $y < \varphi \leq 0$,

$$\begin{aligned} G_0(\varphi, y) &= \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D_{-y} - \Pi^+ B_{-y} \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \\ &= \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\Pi^+ & I \end{pmatrix} \begin{pmatrix} 0 & B_{-y} \\ 0 & D_{-y} \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ \Pi^+ & I \end{pmatrix} \\ &= e^{-\varphi V^{-1} Q} J_2 \Gamma_2 \, e^{y V^{-1} Q} J_2 \Gamma. \end{aligned}$$

Then, (2.33) and (2.35) imply that

$$\begin{aligned} (I - C_y A_y^{-1} \Pi^-)(C_y \Pi^- + D_y) D_{-y} = I \qquad & (A_y \Pi^- + B_y) = \Pi^-(C_y \Pi^- + D_y) \\ (I - B_y D_y^{-1} \Pi^+)(B_y \Pi^+ + A_y) A_{-y} = I \qquad & (D_y \Pi^+ + C_y) = \Pi^+(B_y \Pi^+ + A_y). \end{aligned}$$

Hence, for $y > 0$,

$$\begin{aligned} G_0(y, y) &= \begin{pmatrix} A_y(A_{-y} - \Pi^- C_{-y}) & (A_y \Pi^- + B_y) D_{-y} \\ C_y(A_{-y} - \Pi^- C_{-y}) & (C_y \Pi^- + D_y) D_{-y} \end{pmatrix} \\ &= \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \begin{pmatrix} A_{-y} - \Pi^- C_{-y} & 0 \\ 0 & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \begin{pmatrix} 0 & \Pi^- \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & B_{-y} \\ 0 & D_{-y} \end{pmatrix} \\ &= e^{-y V^{-1} Q} J_1 \Gamma_2 \, e^{y V^{-1} Q} J_1 + e^{-y V^{-1} Q} \Gamma J_2 \, e^{y V^{-1} Q} J_2 \\ &= e^{-y V^{-1} Q} (J_1 \Gamma_2 \, e^{y V^{-1} Q} J_1 + \Gamma J_2 \, e^{y V^{-1} Q} J_2), \end{aligned}$$

and, for $y < 0$,

$$\begin{aligned} G_0(y, y) &= \begin{pmatrix} (B_y \Pi^+ + A_y) A_{-y} & B_y(D_{-y} - \Pi^+ B_{-y}) \\ (D_y \Pi^+ + C_y) A_{-y} & D_y(D_{-y} - \Pi^+ B_{-y}) \end{pmatrix} \\ &= \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \begin{pmatrix} I & 0 \\ \Pi^+ & 0 \end{pmatrix} \begin{pmatrix} A_{-y} & 0 \\ C_{-y} & 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & D_{-y} - \Pi^+ B_{-y} \end{pmatrix} \\ &= e^{-y V^{-1} Q} \Gamma J_1 \, e^{y V^{-1} Q} J_1 + e^{-y V^{-1} Q} J_2 \Gamma_2 \, e^{y V^{-1} Q} J_2 \\ &= e^{-y V^{-1} Q} (\Gamma J_1 \, e^{y V^{-1} Q} J_1 + J_2 \Gamma_2 \, e^{y V^{-1} Q} J_2). \end{aligned}$$

Therefore, we summarize all the results from this section in the following theorem.

Theorem 2.3.2 *The Green's function $G_0((e,\varphi),(f,y))$ of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when $(\varphi_t)_{t\geq 0}$ crosses zero is given by the $E\times E$ matrix $G_0(\varphi,y)$, where*

$$G_0(\varphi,y)=\begin{cases} e^{(y-\varphi)V^{-1}Q}\Gamma J_1 e^{-yV^{-1}Q}(\Gamma J_1 e^{yV^{-1}Q}J_1+J_2\Gamma_2 e^{yV^{-1}Q}J_2), & \varphi<y<0\\ e^{-yV^{-1}Q}(\Gamma J_1\, e^{yV^{-1}Q}J_1+J_2\Gamma_2\, e^{yV^{-1}Q}J_2), & \varphi=y<0\\ e^{-\varphi V^{-1}Q}J_2\Gamma_2\, e^{yV^{-1}Q}J_2\Gamma, & y<\varphi\leq 0\\ 0, & \varphi y\leq 0,\varphi\neq 0\\ I, & \varphi=y=0\\ e^{-\varphi V^{-1}Q}J_1\Gamma_2\, e^{yV^{-1}Q}J_1\Gamma, & 0\leq\varphi<y\\ e^{-yV^{-1}Q}(J_1\Gamma_2\, e^{yV^{-1}Q}J_1+\Gamma J_2\, e^{yV^{-1}Q}J_2), & \varphi=y>0\\ e^{(y-\varphi)V^{-1}Q}\Gamma J_2 e^{-yV^{-1}Q}(J_1\Gamma_2 e^{yV^{-1}Q}J_1+\Gamma J_2 e^{yV^{-1}Q}J_2), & 0<y<\varphi. \end{cases}$$

2.4 Alternative way of calculating the Green's function of the killed process in the drift cases

In the previous section we have calculated the Green's function $G_0(\varphi,y)$ of the killed process $(X_t,\varphi_t)_{t\geq 0}$ regardless of the behaviour of the process $(\varphi_t)_{t\geq 0}$. However, the Green's function $G_0(\varphi,y)$ can also be calculated directly from the Green's function $G(\varphi,y)$ in those cases in which $G(\varphi,y)$ is finite. In Section 2.1 we have seen that the Green's function $G(\varphi,y)$ is finite only in the drift cases. Therefore, throughout this section we shall assume that the process $(\varphi_t)_{t\geq 0}$ drifts either to $+\infty$ or $-\infty$ and we shall calculate the Green's function $G_0(\varphi,y)$ out of the Green's function $G(\varphi,y)$.

As in the previous section, we conclude from the definition of $G_0(\varphi,y)$ that $G_0(\varphi,0)=0$, $\varphi\neq 0$, and that $G_0(0,0)=I$. Hence we have to calculate $G_0(\varphi,y)$ for $y\neq 0$.

By the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$, we have that, for any $t \geq 0$,

$$E_{(e,\varphi)}\Big(\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y) \mid \mathcal{F}_t\Big) =$$
$$\textstyle\sum_{0\leq s<t} I(X_s = f, \varphi_s = y) + E_{(X_t,\varphi_t)}\Big(\displaystyle\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y)\Big).$$

Since in the drift cases, by Theorem 2.1.2, $G((e,\varphi),(f,y)) < +\infty$, the random variable $\sum_{0\leq s<\infty} I(X_s = f, \varphi_s = y)$ is integrable. Then it follows from the previous equation that the process $(M_t)_{t\geq 0}$ defined by

$$M_t = \sum_{0\leq s<t} I(X_s = f, \varphi_s = y) + G((X_t,\varphi_t),(f,y))$$

is a uniformly integrable martingale. Then, the process $(M_{t\wedge H_0})_{t\geq 0}$ is a uniformly integrable martingale and by taking the limit of $E(M_{t\wedge H_0})$ as $t \to +\infty$ we get that

$$\begin{aligned} \lim_{t\to+\infty} E(M_{t\wedge H_0}) &= G((e,\varphi),(f,y)) \\ &= E_{(e,\varphi)}\Big(\sum_{0\leq s<H_0} I(X_s = f, \varphi_s = y)\Big) \\ &\quad + E_{(e,\varphi)} G((X_{H_0},\varphi_{H_0}),(f,y)). \end{aligned}$$

Hence,

$$G_0((e,\varphi),(f,y)) = G((e,\varphi),(f,y)) - E_{(e,\varphi)} G((X_{H_0},\varphi_{H_0}),(f,y)). \tag{2.37}$$

We need to calculate $E_{(e,\varphi)} G((X_{H_0},\varphi_{H_0}),(f,y))$. From Lemma 1.5.1 and Theorem 2.1.2, we have that, for $\varphi, y \neq 0$,

$$\begin{aligned} E_{(e,\varphi)} G((X_{H_0},\varphi_{H_0}),(f,y)) &= \sum_{e'\in E} P_{(e,\varphi)}(X_{H_0} = e', H_0 < +\infty)\; G((e',0),(f,y)) \\ &= \Gamma F(-\varphi)\; \Gamma F(y) \Gamma_2^{-1}(e,f), \end{aligned}$$

and for $\varphi = 0, y \neq 0$,

$$\begin{aligned} E_{(e,0)} G((X_{H_0},\varphi_{H_0}),(f,y)) &= \Big(I - \Gamma_2\Big)\; G(0,y)(e,f) \\ &= \Big(I - \Gamma_2\Big)\; \Gamma F(y) \Gamma_2^{-1}(e,f). \end{aligned}$$

Thus, by Theorem 2.1.2 and (2.37), the Green's function $G_0(\varphi, y)$ can be written as

$$\begin{aligned} G_0(\varphi,y) &= G(\varphi,y) - E_{(\cdot,\varphi)}G((X_{H_0},\varphi_{H_0}),(\cdot,y)) \\ &= \begin{cases} \Gamma\Big(F(y-\varphi) - F(-\varphi)\Gamma F(y)\Big)\Gamma_2^{-1}, & \varphi \neq y,\ \varphi y > 0 \\ \Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1}, & \varphi = y \neq 0 \\ \Gamma_2\Gamma F(y)\Gamma_2^{-1}, & \varphi = 0, y \neq 0 \\ 0, & \varphi y \leq 0, \varphi \neq 0 \\ I, & \varphi = y = 0. \end{cases} \end{aligned}$$

It can be easily shown that

$$F(y-\varphi) - F(-\varphi)\Gamma F(y) = \begin{cases} e^{-\varphi G}\,\Gamma_2\,F(y), & 0 < \varphi < y \quad \text{or} \quad y < \varphi < 0, \\ F(-\varphi)\,\Gamma_2\,e^{yG}, & 0 < y < \varphi \quad \text{or} \quad \varphi < y < 0. \end{cases}$$

Therefore, we have proved the following theorem:

Theorem 2.4.1 *In the drift cases, the Green's function $G_0((e,\varphi),(f,y))$ of the process $(X_t,\varphi_t)_{t\geq 0}$ killed when $(\varphi_t)_{t\geq 0}$ crosses zero is given by the $E \times E$ matrix $G_0(\varphi,y)$, where*

$$G_0(\varphi,y) = \begin{cases} \Gamma\,e^{-\varphi G}\,\Gamma_2\,F(y)\,\Gamma_2^{-1}, & 0 \leq \varphi < y \quad \text{or} \quad y < \varphi \leq 0 \\ \Gamma\,F(-\varphi)\,\Gamma_2\,e^{yG}\,\Gamma_2^{-1}, & 0 < y < \varphi \quad \text{or} \quad \varphi < y < 0 \\ \Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1}, & \varphi = y \neq 0 \\ 0, & \varphi y \leq 0, \varphi \neq 0 \\ I, & \varphi = y = 0. \end{cases}$$

Remark: The representation formula for $G_0(\varphi,y)$ in the drift cases given in Theorem 2.4.1 agrees with the general formula given in Theorem 2.3.2. To show that, we first remind ourselves of the equalities

$$\begin{array}{lll} \Gamma\Gamma_2 = \Gamma_2\Gamma & J_1\Gamma\Gamma_2 = \Gamma\Gamma_2 J_1 & J_2\Gamma\Gamma_2 = \Gamma\Gamma_2 J_2 \\ J_1 G = G J_1 & J_2 G = G J_2 & J_1 J_2 = J_2 J_1 = 0, \end{array}$$

which imply that, for any $y \in \mathbb{R}$,

$$J_1\Gamma_2\, e^{yV^{-1}Q}\Gamma J_2 = 0 \qquad J_1\Gamma_2\, e^{yV^{-1}Q} = J_1\Gamma_2\, e^{yV^{-1}Q}J_1\Gamma_2$$
$$J_2\Gamma_2\, e^{yV^{-1}Q}\Gamma J_1 = 0 \qquad J_2\Gamma_2\, e^{yV^{-1}Q} = J_2\Gamma_2\, e^{yV^{-1}Q}J_2\Gamma_2.$$

Hence, for $y > 0$,

$$\begin{aligned}
\Gamma\,\Gamma_2\, F(y)\,\Gamma_2^{-1} &= \Gamma\Gamma_2 J_1 e^{yG}\Gamma_2^{-1} = J_1\Gamma_2\Gamma e^{yG}\,\Gamma_2^{-1} \\
&= J_1\Gamma_2\, e^{yV^{-1}Q}\Gamma\,\Gamma_2^{-1} = J_1\Gamma_2\, e^{yV^{-1}Q}\Gamma_2^{-1}\,\Gamma \\
&= J_1\Gamma_2\, e^{yV^{-1}Q}J_1\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
\Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1} &= \Big(I - \Gamma e^{-yG}J_2\Gamma e^{yG}J_1\Big)\Gamma_2^{-1} \\
&= \Big(I - e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q}\Gamma J_1\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big(e^{yV^{-1}Q} - \Gamma J_2\, e^{yV^{-1}Q}\Gamma J_1\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big(e^{yV^{-1}Q} - \Gamma J_2\, e^{yV^{-1}Q}(I - J_2\Gamma_2)\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big((I - \Gamma J_2)\, e^{yV^{-1}Q} + \Gamma J_2\, e^{yV^{-1}Q}J_2\Gamma_2\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big(J_1\Gamma_2\, e^{yV^{-1}Q}\Gamma_2^{-1} + \Gamma J_2\, e^{yV^{-1}Q}J_2\Big) \\
&= e^{-yV^{-1}Q}\Big(J_1\Gamma_2\, e^{yV^{-1}Q}J_2 + \Gamma J_2\, e^{yV^{-1}Q}J_2\Big),
\end{aligned}$$

and, for $y < 0$,

$$\begin{aligned}
\Gamma\,\Gamma_2\, F(y)\,\Gamma_2^{-1} &= \Gamma\Gamma_2 J_2 e^{yG}\Gamma_2^{-1} = J_2\Gamma_2\Gamma e^{yG}\,\Gamma_2^{-1} \\
&= J_2\Gamma_2\, e^{yV^{-1}Q}\Gamma\,\Gamma_2^{-1} = J_2\Gamma_2\, e^{yV^{-1}Q}\Gamma_2^{-1}\,\Gamma \\
&= J_2\Gamma_2\, e^{yV^{-1}Q}J_2\Gamma,
\end{aligned}$$

and

$$\begin{aligned}
\Big(I - \Gamma F(-y)\Gamma F(y)\Big)\Gamma_2^{-1} &= \Big(I - \Gamma e^{-yG}J_1\Gamma e^{yG}J_2\Big)\Gamma_2^{-1} \\
&= \Big(I - e^{-yV^{-1}Q}\Gamma J_1\, e^{yV^{-1}Q}\Gamma J_2\Big)\Gamma_2^{-1}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-yV^{-1}Q}\Big(e^{yV^{-1}Q} - \Gamma J_1\, e^{yV^{-1}Q}\Gamma J_2\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big(e^{yV^{-1}Q} - \Gamma J_1\, e^{yV^{-1}Q}(I - J_1\Gamma_2)\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big((I - \Gamma J_1)\, e^{yV^{-1}Q} + \Gamma J_1\, e^{yV^{-1}Q} J_1\Gamma_2\Big)\Gamma_2^{-1} \\
&= e^{-yV^{-1}Q}\Big(J_2\Gamma_2\, e^{yV^{-1}Q}\Gamma_2^{-1} + \Gamma J_1\, e^{yV^{-1}Q} J_1\Big) \\
&= e^{-yV^{-1}Q}\Big(J_2\Gamma_2\, e^{yV^{-1}Q} J_2 + \Gamma J_1\, e^{yV^{-1}Q} J_1\Big).
\end{aligned}
$$

2.5 Some relations between $G(\varphi, y)$ and $G_0(\varphi, y)$

The Green's functions $G(\varphi, y)$ and $G_0(\varphi, y)$ are closely related. For instance, in Section 2.4 we have showed how to calculate the Green's function $G_0(\varphi, y)$ from the Green's function $G(\varphi, y)$. In this section our aim is to show two more relations between $G(\varphi, y)$ and $G_0(\varphi, y)$.

By directly comparing Theorems 2.1.2 and 2.4.1 we deduce that

$$G(\varphi, y) = G(\varphi, \varphi)\ G_0(0, y - \varphi), \qquad \varphi \neq y,\ \varphi y > 0, \tag{2.38}$$

and from Theorem 2.3.2 we deduce that

$$G_0(\varphi, y) = G_0(\varphi, \varphi)\ G_0(0, y - \varphi), \qquad 0 < \varphi < y \ \text{ or } \ y < \varphi < 0. \tag{2.39}$$

Theorems 2.1.2 and 2.4.1 hold only in the drift cases. Hence, (2.38) holds in the drift cases. However, in the oscillating case $G(\varphi, y) = G(\varphi, \varphi) = +\infty$, and therefore (2.38) again holds.

Now we shall give the probabilistic explanations of these two relations.

Let the process $(X_t, \varphi_t)_{t \geq 0}$ start at $(e, \varphi) \in E \times \mathbb{R}$ and let $\varphi y > 0$, $\varphi \neq y$. The Green's function $G((e, \varphi), (f, y))$ is, by its definition, expected number of visits to (f, y) by the process $(X_t, \varphi_t)_{t \geq 0}$. The overall number of visits to (f, y) from (e, φ) by $(X_t, \varphi_t)_{t \geq 0}$

can be split into numbers of times that $(X_t, \varphi_t)_{t\geq 0}$ visits (f, y) between every two successive hittings of $E \times \{\varphi\}$. Let $N_\varphi((e', \varphi), (f, y))$ be the number of visits to (f, y) by $(X_t, \varphi_t)_{t\geq 0}$ starting at (e', φ) and without hitting $E \times \{\varphi\}$. Then $N_\varphi((e', \varphi), (f, y)) = N_0((e', 0), (f, y - \varphi))$, and therefore, $E_{(e',\varphi)}(N_\varphi((e', \varphi), (f, y))) = G_0((e', 0), (f, y - \varphi))$. Thus, by the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$,

$$
\begin{aligned}
G((e\,,\varphi),(f,y)) &= G_0((e,0),(f,y-\varphi)) \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits once } E\times\{\varphi\} \text{ at } (e',\varphi)\Big) \\
&\quad G_0((e',0),(f,y-\varphi)) \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits two times } E\times\{\varphi\}, \text{ last time at } (e',\varphi)\Big) \\
&\quad G_0((e',0),(f,y-\varphi)) \\
&+ \dots \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } n \text{ times } E\times\{\varphi\}, \text{ last time at } (e',\varphi)\Big) \\
&\quad G_0((e',0),(f,y-\varphi)) \\
&+ \dots \\
&= \sum_{e'\in E}\left(\sum_{n=0}^{\infty} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } n \text{ times } E\times\{\varphi\}, \text{last time at } (e',\varphi)\Big)\right) \\
&\quad G_0((e',0),(f,y-\varphi)) \\
&= \sum_{e'\in E} G((e,\varphi),(e',\varphi))\ G_0((e',0),(f,y-\varphi)),
\end{aligned}
$$

which is, written in matrix notation,

$$G(\varphi, y) = G(\varphi, \varphi)\ G_0(0, y - \varphi), \qquad \varphi \neq y,\ \varphi y > 0.$$

By using the same argument, we show that (2.39) holds. Instead of looking at the process $(X_t, \varphi_t)_{t\geq 0}$ and $G(\varphi, y)$ in the above discussion, we look at the killed process $(X_t, \varphi_t)_{t\geq 0}$ and $G_0(\varphi, y)$. Let again the process $(X_t, \varphi_t)_{t\geq 0}$ start at $(e, \varphi) \in E \times \mathbb{R}$ and let $0 < \varphi < y$ or $y < \varphi < 0$. We split the overall number of visits to (f, y) from (e, φ)

by the process $(X_t, \varphi_t)_{t\geq 0}$ without hitting $E \times \{0\}$ into numbers of visits to (f, y) by $(X_t, \varphi_t)_{t\geq 0}$ starting at (e, φ) between two successive hittings of $E \times \{\varphi\}$ and without hitting $E \times \{0\}$. Let $N_{\varphi,0}((e', \varphi), (f, y))$ be the number of visits to (f, y) by $(X_t, \varphi_t)_{t\geq 0}$ starting at (e', φ) and without hitting $E \times \{\varphi\}$ and $E \times \{0\}$. Then $N_{\varphi,0}((e', \varphi), (f, y)) = N_\varphi((e', \varphi), (f, y-\varphi))$, and therefore, $E_{(e',\varphi)}(N_{\varphi,0}((e', \varphi), (f, y))) = G_0((e', 0), (f, y-\varphi))$. Thus, by the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$, we obtain

$$\begin{aligned}
G_0((e,\varphi),(f,y)) &= G_0((e,0),(f,y-\varphi)) \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } E \times \{\varphi\} \text{ at } (e',\varphi) \text{ without hitting } E \times \{0\}\Big) \\
&\quad G_0((e',0),(f,y-\varphi)) \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits two times } E \times \{\varphi\} \text{ without hitting } E \times \{0\}, \\
&\quad \text{last time at } (e',\varphi)\Big)\ G_0((e',0),(f,y-\varphi)) \\
&+ \cdots \\
&+ \sum_{e'\in E} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } n \text{ times } E \times \{\varphi\} \text{ without hitting } E \times \{0\}, \\
&\quad \text{last time at } (e',\varphi)\Big)\ G_0((e',0),(f,y-\varphi)) \\
&+ \cdots \\
&= \sum_{e'\in E} \Bigg(\sum_{n=0}^{\infty} P_{(e,\varphi)}\Big((X_t,\varphi_t)_{t\geq 0} \text{ hits } n \text{ times } E \times \{\varphi\} \text{ without hitting } E \times \{0\}, \\
&\quad \text{last time at } (e',\varphi)\Big)\Bigg)\ G_0((e',0),(f,y-\varphi)) \\
&= \sum_{e'\in E} G_0((e,\varphi),(e',\varphi))\ G_0((e',0),(f,y-\varphi)),
\end{aligned}$$

which is, in matrix notation,

$$G_0(\varphi, y) = G_0(\varphi,\varphi)\ G_0(0, y-\varphi), \qquad \varphi \neq y,\ \varphi y > 0.$$

2.6 The two-sided exit probabilities of $(X_t, \varphi_t)_{t\geq 0}$

Let $y > 0$ and let the process $(X_t, \varphi_t)_{t\geq 0}$ start in $(e, \varphi) \in E \times (0, y)$. We know from Lemma 1.5.3 that for any $e \in E$ and $0 < \varphi < y$, $P_{(e,\varphi)}(H_y < H_0) > 0$ and that $P_{(e,\varphi)}(H_0 < H_y) > 0$. Thus the process $(\varphi_t)_{t\geq 0}$ has a positive probability of exiting the interval $(0, y)$ on either of the sides. By knowing the Green's function of the process $(X_t, \varphi_t)_{t\geq 0}$ we are able to find these exit probabilities explicitly.

Lemma 2.6.1 *For $y \neq 0$ and any $(e, f) \in E \times E$*

$$\begin{aligned} P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) &= G_0(\varphi, y)(G_0(y, y))^{-1}(e, f), \qquad 0 < |\varphi| < |y| \\ P_{(e,y)}(X_{H_y} = f, H_y < H_0) &= \Big(I - (G_0(y, y))^{-1}\Big)(e, f). \end{aligned}$$

Proof: For $\varphi \neq y \neq 0$ the Green's function $G_0(\varphi, y)$ can be decomposed as

$$G_0((e,\varphi),(f,y)) = \sum_{e'\in E} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0)\; G_0((e', y),(f, y)), \tag{2.40}$$

and for $y \neq 0$ the Green's function $G_0(y, y)$ can be decomposed as

$$G_0((e,y),(f,y)) = I(e,f) + \sum_{e'\in E} P_{(e,y)}(X_{H_y} = e', H_y < H_0) G_0((e', y),(f, y)), \tag{2.41}$$

Since, by Theorem 2.3.1, the matrix $G_0(y, y)$ is invertible for any $y \neq 0$, (2.40) and (2.41) prove the lemma. □

From Lemma 2.6.1 we deduce a componentwise expression for the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$ which sometimes appear to be more useful in applications.

Lemma 2.6.2 *For $0 \leq \varphi < y$ and any $(e, f) \in E \times E$,*

$$P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) = \begin{pmatrix} A_\varphi A_y^{-1} & 0 \\ C_\varphi A_y^{-1} & 0 \end{pmatrix} (e, f),$$

and for $y < \varphi \leq 0$ and any $(e,f) \in E \times E$,

$$P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) = \begin{pmatrix} 0 & B_\varphi D_y^{-1} \\ 0 & D_\varphi D_y^{-1} \end{pmatrix} (e,f),$$

where, for any $y \in \mathbb{R}$ the matrices A_y, B_y, C_y and D_y are given by

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}.$$

Proof: By Lemma 2.6.1 and Theorem 2.3.1 we have that, for $0 \leq \varphi < y$,

$$\begin{aligned}
&P_{(e,\varphi)}(X_{H_y} = f, H_y < H_0) \\
&= G_0(\varphi,y)(G_0(y,y))^{-1}(e,f) \\
&= \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & A_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & C_\varphi(A_y - \Pi^- C_y)^{-1}\Pi^- \end{pmatrix} \begin{pmatrix} I & -\Pi^- \\ -C_y A_y^{-1} & I \end{pmatrix} (e,f) \\
&= \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & 0 \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & 0 \end{pmatrix} \begin{pmatrix} I & \Pi^- \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & -\Pi^- \\ -C_y A_y^{-1} & I \end{pmatrix} (e,f) \\
&= \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1} & 0 \\ C_\varphi(A_y - \Pi^- C_y)^{-1} & 0 \end{pmatrix} \begin{pmatrix} (I - \Pi^- C_y A_y^{-1}) & 0 \\ 0 & 0 \end{pmatrix} (e,f) \\
&= \begin{pmatrix} A_\varphi(A_y - \Pi^- C_y)^{-1}(A_y - \Pi^- C_y)A_y^{-1} & 0 \\ C_\varphi(A_y - \Pi^- C_y)^{-1}(A_y - \Pi^- C_y)A_y^{-1} & 0 \end{pmatrix} (e,f) \\
&= \begin{pmatrix} A_\varphi A_y^{-1} & 0 \\ C_\varphi A_y^{-1} & 0 \end{pmatrix} (e,f).
\end{aligned}$$

The proof for the case $y < \varphi \leq 0$ follows in the same way from Lemma 2.6.1 and Theorem 2.3.1. □

Chapter 3

The oscillating case

In this section we assume that the process $(\varphi_t)_{t\geq 0}$ oscillates. Our aim is to condition the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative.

Let us first recall an essential property of the matrix $V^{-1}Q$ in the oscillating case. The zero eigenvalue of $V^{-1}Q$ is with the algebraic multiplicity two and the geometric multiplicity one which implies that there exists vector r independent of the vector 1 such that

$$V^{-1}Qr = 1,$$

and that the system $\{r, f_j, j = 1, \ldots, n, g_k, k = 1, \ldots, m\}$ is a basis in the space of all vectors on E. In addition, for any $c \in \mathbb{R}$, the vector $r + c1$ also satisfies the last equation. We shall see that neither the choice nor the normalization of the vector r which satisfies the last equation affects the results in this and following sections. Therefore we shall refer to it as if it was fixed.

Now we investigate the probability of the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative, that is $P_{(e,\varphi)}(H_0 = +\infty)$, $(e, \varphi) \in E_0^+$.

In the oscillating case, by (1.28), the matrix G^- is conservative which, by Lemma

1.4.6 (ii), implies that the matrix Π^- is stochastic. Hence, by Lemma 1.5.2, for $\varphi > 0$,

$$P_{(\cdot,\varphi)}(H_0 = +\infty) = \Big(I - \Gamma F(-\varphi)\Big)1 = \begin{pmatrix} 1^+ - \Pi^- e^{\varphi G^-} 1^- \\ 1^- - e^{\varphi G^-} 1^- \end{pmatrix} = 0,$$

and for $e \in E^+$, $\varphi = 0$,

$$P_{(e,0)}(H_0 = +\infty) = (1^+ - \Pi^- 1^-)(e) = 0.$$

Therefore, the event $\{H_0 = +\infty\}$ is of zero probability and we cannot condition the process $(X_t, \varphi_t)_{t\geq 0}$ on it in the standard way. Instead, we can condition the process $(X_t, \varphi_t)_{t\geq 0}$ on some events of positive probability that approximate the event $\{H_0 = +\infty\}$.

We consider two approximations of the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative: one approximation is by the events that the process $(\varphi_t)_{t\geq 0}$ hits large positive levels y before it crosses zero, that is by the events $\{H_y < H_0\}$, $y > 0$, and another approximation is by the events that the process $(\varphi_t)_{t\geq 0}$ stays non-negative for long times, that is by the events $\{H_0 > T\}$, $T > 0$. In Section 3.1 we look at the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < H_0\}$, and in Section 3.2 we look at the limit as $T \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 > T\}$. We shall see that these limits coincide which implies that the two mentioned approximations of the event $\{H_0 = +\infty\}$ yield the same result.

3.1 Conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < H_0\}$

For fixed $y > 0$, let $P^y_{(e,\varphi)}$ be the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e, \varphi) \in E_0^+$, conditioned on $\{H_y < H_0\}$. Let $t \geq 0$ be fixed and let $P^y_{(e,\varphi)}|_{\mathcal{F}_t}$ denote the restriction of $P^y_{(e,\varphi)}$ to $\mathcal{F}_t$. We are interested in weak convergence of the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ as $y \to +\infty$.

Let $t \geq 0$ be fixed and let $A \in \mathcal{F}_t$. We start by looking at the limit of $P^y_{(e,\varphi)}(A)$ as $y \to +\infty$. For $(e,\varphi) \in E_0^+$ and $y > \varphi$, by Lemma 1.5.3, the event $\{H_y < H_0\}$, $y > 0$, is of positive probability and we can condition the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$ in the standard way.

By the Markov property of the process $(X_t, \varphi_t)_{t\geq 0}$, we have that, for any $(e,\varphi) \in E_0^+$ and any $A \in \mathcal{F}_t$,

$$\begin{aligned}
P^y_{(e,\varphi)}(A) &= P_{(e,\varphi)}(A \mid H_y < H_0)\\
&= \frac{E_{(e,\varphi)}\Big(I(A)I\{H_y < H_0\}\Big)}{P_{(e,\varphi)}(H_y < H_0)}\\
&= \frac{1}{P_{(e,\varphi)}(H_y < H_0)}\, E_{(e,\varphi)}\Big(I(A)(I\{t < H_0 \wedge H_y\}P_{(X_t,\varphi_t)}(H_y < H_0)\\
&\quad + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\})\Big). \qquad (3.1)
\end{aligned}$$

In the following lemma we show that the limit $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}$ exists and calculate it. After that, we will be able to deduce weak limit of the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ as $y \to +\infty$.

Lemma 3.1.1 *Let r be a vector such that $V^{-1}Qr = 1$. Then*

$$\begin{aligned}
&(i) \qquad h_r(e,\varphi) \equiv e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e) > 0, \quad (e,\varphi) \in E_0^+,\\
&(ii) \quad \lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1 \Gamma_2 r(e')}{e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)}, \quad (e,\varphi),(e',\varphi') \in E_0^+,
\end{aligned}$$

(where "$\equiv$" means "defined to be").

Proof: (i) We recall the matrices A_{-y}, B_{-y}, C_{-y} and D_{-y}, which are given by (2.27) as

$$e^{yV^{-1}Q} = \begin{pmatrix} A_{-y} & B_{-y}\\ C_{-y} & D_{-y}\end{pmatrix}.$$

Then, the function h_r can be rewritten as

$$h_r(\cdot,\varphi) = e^{-\varphi V^{-1}Q} J_1\Gamma_2 r = \begin{pmatrix} A_\varphi(r^+ - \Pi^- r^-) \\ C_\varphi(r^+ - \Pi^- r^-) \end{pmatrix}.$$

The outline of the proof is the following: first we show that, for fixed vector r which satisfies $V^{-1}Qr = 1$, the vector $A_\varphi(r^+ - \Pi^- r^-)$ is positive. Then, we note that

$$C_\varphi(r^+ - \Pi^- r^-) = C_\varphi A_\varphi^{-1}\, A_\varphi(r^+ - \Pi^- r^-),$$

and that the matrix $C_\varphi A_\varphi^{-1}$ is positive because, by Lemma 2.6.2, its entries are the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$, which are, by Lemma 1.5.3, strictly positive. Thus, it follows that the vector $C_\varphi(r^+ - \Pi^- r^-)$ is also positive and that the function h_r is positive for fixed solution r of the equation $V^{-1}Qx = 1$. Since any other solution of the same equation is of the form $r + c1$ for some $c \in \mathbb{R}$, and because $J_1\Gamma_2 1 = 0$ and

$$J_1\Gamma_2(r + c1) = J_1\Gamma_2 r,$$

we conclude that the function h_r does not depend on the choice of the solution of the equation $V^{-1}Qx = 1$.

Therefore, all we have to prove is that the vector $A_\varphi(r^+ - \Pi^- r^-)$ is positive for any $\varphi \in \mathbb{R}$. We shall show that it is a Perron-Frobenius eigenvector of a positive matrix which will imply that it is positive.

Let r be fixed vector which satisfies $V^{-1}Qr = 1$. Then

$$e^{yV^{-1}Q} r = r + y1,$$

which written componentwise is

$$\begin{aligned} A_{-y} r^+ + B_{-y} r^- &= r^+ + y1^+ \\ C_{-y} r^+ + D_{-y} r^- &= r^- + y1^-. \end{aligned}$$

Thus, because $1^+ = \Pi^- 1^-$,

$$(A_{-y} - \Pi^- C_{-y}) r^+ + (B_{-y} - \Pi^- D_{-y}) r^- = r^+ - \Pi^- r^-,$$

and because, by (2.34), $(B_{-y} - \Pi^- D_{-y}) = -(A_{-y} - \Pi^- C_{-y})\Pi^-$,

$$(A_{-y} - \Pi^- C_{-y})(r^+ - \Pi^- r^-) = r^+ - \Pi^- r^-.$$

By (2.36), $(A_{-y} - \Pi^- C_{-y}) = (A_y - \Pi^- C_y)^{-1}$ and, by (2.30), the matrix A_φ is invertible. Hence

$$\left(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right) A_\varphi (r^+ - \Pi^- r^-) = A_\varphi (r^+ - \Pi^- r^-).$$

The matrix $A_\varphi (A_y - \Pi^- C_y)^{-1}$ is, by Theorem 2.3.1, positive because it is the restriction to $E^+ \times E^+$ of the Green's function $G_0(\varphi, y)$ of the killed process $(X_t, \varphi_t)_{t\geq 0}$. The matrix A_φ^{-1} is also strictly positive since, by Lemma 2.6.2, its entries are the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$, which are, by Lemma 1.5.3, positive. Hence, the matrix $A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ is positive and it has the Perron-Frobenius eigenvector which is also positive.

If the vector $A_\varphi (r^+ - \Pi^- r^-) \neq 0$ then the last equation implies that $A_\varphi (r^+ - \Pi^- r^-)$ is the eigenvector of the matrix $(A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1})$ associated with its eigenvalue 1.

We shall now prove that $A_\varphi (r^+ - \Pi^- r^-) \neq 0$. Suppose that $A_\varphi (r^+ - \Pi^- r^-) = 0$. Then, because A_φ is invertible, $(r^+ - \Pi^- r^-) = 0$. If $r^+ = \Pi^- r^-$ then the vector r is a linear combination of the vectors g_k, $k = 1, \dots, m$ in the basis $\mathcal{B}$, but that is not possible because the vector r is also in the basis $\mathcal{B}$ and therefore independent from g_k, $k = 1, \dots, m$. Hence, the vector $A_\varphi (r^+ - \Pi^- r^-)$ is a non-zero vector and it is the eigenvector of the matrix $A_\varphi (A_{-y} - \Pi^- C_{-y}) A_\varphi^{-1}$ which corresponds to its eigenvalue 1.

In order to show that the eigenvector $A_\varphi (r^+ - \Pi^- r^-)$ of the matrix $A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ associated with the eigenvalue 1 is positive, it is enough to show that it is its Perron-Frobenius eigenvector, or that 1 is its Perron-Frobenius eigenvalue. Hence, we shall look for other eigenvalues of the matrix $A_\varphi (A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ and check if 1 is its eigenvalue with maximal real part.

By the second equation in (2.34) and (2.36)

$$\left(A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}\right) A_\varphi(I - \Pi^-\Pi^+) = A_\varphi(I - \Pi^-\Pi^+)\, e^{yG^+}. \tag{3.2}$$

We shall show that if α is a non-zero eigenvalue of the matrix G^+ with some algebraic multiplicity, then $e^{\alpha y}$ is an eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ with the same algebraic multiplicity.

Let α be an non-zero eigenvalue of G^+. It follows from (3.2) and Jordan normal form theory, that if $A_\varphi(I - \Pi^-\Pi^+)f^+ \neq 0$ for any vector f^+ associated with the eigenvalue α of G^+, then $e^{\alpha y}$ is an eigenvalue of $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ and $A_\varphi(I - \Pi^-\Pi^+)f$ is a vector associated with $e^{\alpha y}$.

Suppose that f^+ is a vector associated with an non-zero eigenvalue of G^+ and that $A_\varphi(I - \Pi^-\Pi^+)f^+ = 0$. Then, $(I - \Pi^-\Pi^+)f^+ = 0$, or, equivalently, $f^+ = \Pi^-\Pi^+ f^+$. In addition, by (1.7), $f^- = \Pi^+ f^+$. Therefore, $f^+ = \Pi^- f^-$. Since the vectors g_k^-, $k = 1, \dots, m$ form a basis in the space of all vectors on E^-, the vector f^- is a linear combination of them. It follows that f^+ is a linear combination of $\Pi^- g_k^- = g_k^+$, $k = 1, \dots, m$, and that f is a linear combination of g_k, $k = 1, \dots, m$. But that is not possible since f is associated with an eigenvalue of $V^{-1}Q$ with negative real part and therefore independent of g_k, $k = 1, \dots, m$. Hence, for a vector f^+ associated with an eigenvalue of the matrix G^+ with a negative real part, the vector $A_\varphi(I - \Pi^-\Pi^+)f^+$ is non-zero.

Let α_j, $j = 2, \dots, n$, be the non-zero eigenvalues of G^+ and f_j^+, $j = 2, \dots, n$, the vectors associated with them. Then, the vectors $A_\varphi(I - \Pi^-\Pi^+)f_j^+$ are non-zero which implies that for all $\alpha_j \neq 0$, $j = 1, \dots, n$, $e^{\alpha_j y}$ is an eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ with the same algebraic multiplicity as the algebraic multiplicity of the eigenvalue α_j of the matrix G^+. Since all non-zero eigenvalues of G^+ are with negative real parts, all eigenvalues $e^{\alpha_j y}$, $\alpha_j \neq 0$, $j = 1, \dots, n$, of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ have real parts strictly less than 1.

Therefore, the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ has $n - 1$ eigenvalues with real parts strictly less than 1 and, by the previous discussion, it has the eigenvalue 1 with the associated eigenvector $A_\varphi(r^+ - \Pi^- r^-)$. Thus, 1 is the Perron-Frobenius eigenvalue of the matrix $A_\varphi(A_y - \Pi^- C_y)^{-1} A_\varphi^{-1}$ and the vector $A_\varphi(r^+ - \Pi^- r^-)$ is its Perron-Frobenius eigenvector, and therefore positive.

(ii) Let $X(\varphi, y)$ be an $E \times E$ matrix with entries

$$X(\varphi,y)(e,e') = P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0), \qquad e, e' \in E.$$

We shall show that, for any $(e,\varphi), (e',\varphi') \in E_0^+$,

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \lim_{y\to+\infty} \frac{X(\varphi',y)1(e')}{X(\varphi,y)1(e)} = \lim_{y\to+\infty} \frac{G_0(\varphi',y)1(e')}{G_0(\varphi,y)1(e)} \tag{3.3}$$

and that

$$G_0(\varphi,y)1 = \sum_{j,\alpha_j\neq 0} a_j'\, e^{-\varphi V^{-1}Q} J_1 \Gamma_2\, e^{yV^{-1}Q} f_j + c'\, e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r, \tag{3.4}$$

for some constants a_j', $j = 1, \dots n$, and $c' \neq 0$. Then, because $Re(\alpha_j) < 0$, $j = 1, \dots, n$, and, by (1.10) (i), $e^{yV^{-1}Q} f_j \to 0$ as $y \to +\infty$, it will follow from (3.3) and (3.4) that

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1}Q} J_1 \Gamma_2 r(e')}{e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)}.$$

Thus, we first show that (3.3) is valid. By decomposing the Green's function $G_0(\varphi, y)$ for the killed process $(X_t, \varphi_t)_{t\geq 0}$, we get that, for any $e, e' \in E$ and any $0 < \varphi < y$,

$$G_0((e,\varphi),(e',y)) = \sum_{e''\in E} P_{(e,\varphi)}(X_{H_y} = e'', H_y < H_0)\; G_0((e'',y),(e',y)),$$

or, in matrix notation,

$$G_0(\varphi,y) = X(\varphi,y)\; G_0(y,y). \tag{3.5}$$

For fixed $e, e' \in E$,

$$0 \leq X(\varphi,y)(e,e') \leq \sum_{e'\in E} X(\varphi,y)(e,e') = P_{(e,\varphi)}(H_y < H_0),$$

and

$$\lim_{y\to+\infty} P_{(e,\varphi)}(H_y < H_0) = P_{(e,\varphi)}(H_0 = +\infty) = 0,$$

Thus, the limit $\lim_{y\to+\infty} X(\varphi, y)(e, e')$ exists and is equal to zero.

The next step is to show that $\lim_{y\to+\infty} \frac{G_0(y,y)\,1}{y} = const.\ 1 \neq 0$. Then it will folow from (3.5) that

$$\lim_{y\to+\infty} \frac{G_0(\varphi, y)\,1}{y} = \lim_{y\to+\infty} X(\varphi, y)\,\frac{G_0(y,y)\,1}{y} = c \lim_{y\to+\infty} X(\varphi, y)1, \quad c \neq 0,$$

which will enable us to deduce (3.3).

By Theorem 2.3.2,

$$\begin{aligned} G_0(y,y)1 &= e^{-yV^{-1}Q}\Big(J_1\Gamma_2\, e^{yV^{-1}Q}J_1 + \Gamma J_2\, e^{yV^{-1}Q}J_2\Big)1 \\ &= e^{-yV^{-1}Q}\Big((I - \Gamma J_2)\, e^{yV^{-1}Q}J_1 + \Gamma J_2\, e^{yV^{-1}Q}J_2\Big)1 \\ &= \Big(J_1 + e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q}(J_2 - J_1)\Big)1. \end{aligned}$$

Since the system $\{r, f_j, j = 1,\ldots,n, g_k, k = 1,\ldots,m\}$ forms a basis in the space of all vectors on E, there exist constants a_j, $j = 1,\ldots,n$, b_k, $k = 1,\ldots,m$, and c such that

$$(J_2 - J_1)1 = \sum_{j,\alpha_j\neq 0} a_j f_j + \sum_{k=1}^{m} b_k g_k + c\, r.$$

Then,

$$\begin{aligned} G_0(y,y)1 &= J_1 1 + \sum_{j,\alpha_j\neq 0} a_j e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q} f_j \\ &\quad + \sum_{k=1}^{m} b_k e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q} g_k + c\, e^{-yV^{-1}Q}\Gamma J_2 e^{yV^{-1}Q} r \\ &= J_1 1 + \sum_{j,\alpha_j\neq 0} a_j \Gamma e^{-yG} J_2\, e^{yV^{-1}Q} f_j \\ &\quad + \sum_{k=1}^{m} b_k e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q} g_k + c\, \Gamma e^{-yG} J_2 (r + y1). \end{aligned} \tag{3.6}$$

First we shall show that $c \neq 0$. Suppose that $c = 0$. Then the vector $(J_2 - J_1)1$ is a linear combination only of vectors f_j, $j = 1, \dots, n$, and g_k, $k = 1, \dots, m$, which implies that

$$(J_2 - J_1)1 = \begin{pmatrix} -1^+ \\ 1^- \end{pmatrix} = \begin{pmatrix} x^+ \\ \Pi^+ x^+ \end{pmatrix} + \begin{pmatrix} \Pi^- y^- \\ y^- \end{pmatrix},$$

for some vectors x^+ on E^+ and y^- on E^-. The previous vector equation is equivalent to the system

$$x^+ + \Pi^- y^- = -1^+ \qquad \Pi^+ x^+ + y^- = 1^-,$$

which implies that

$$(\Pi^-\Pi^+ - I)(\frac{1}{2}x^+) = 1^+ \qquad (\Pi^+\Pi^- - I)(-\frac{1}{2}y^-) = 1^-. \tag{3.7}$$

By Lemma 1.4.4 (ii), the matrices $(\Pi^-\Pi^+ - I)$ and $(\Pi^+\Pi^- - I)$ are essentially non-negative and irreducible. In addition, by (1.28), in the oscillating case the matrices G^+ and G^- are conservative, which, by Lemma 1.4.6 (ii) implies that the matries Π^+ and Π^- are stochastic. Thus, $(\Pi^-\Pi^+ - I)1^+ = 0$ and $(\Pi^+\Pi^- - I)1^- = 0$. It follows from the Perron-Frobenius theorem for essentially non-negative matrices that both matrices $(\Pi^-\Pi^+ - I)$ and $(\Pi^+\Pi^- - I)$ have the Perron-Frobenius eigenvalue equal to zero and that their Perron-Frobenius eigenvectors are 1^+ and 1^-, respectively. Since the Perron-Frobenius eigenvalue is always simple, there do not exist vectors x^+ and y^- such that (3.7) is valid. Therefore, $c \neq 0$.

For the third term on the right-hand side of (3.6) we have that

$$\begin{aligned} e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q} g_k &= e^{-yV^{-1}Q}(I - J_1\Gamma_2)\, e^{yV^{-1}Q} g_k \\ &= g_k - e^{-yV^{-1}Q} J_1 \Gamma_2\, e^{yV^{-1}Q} g_k \\ &= g_k - e^{y\beta_k} e^{-yV^{-1}Q} J_1\Gamma_2\, e^{y(V^{-1}Q - \beta_k I)} g_k. \end{aligned}$$

By Jordan normal form theory, for fixed l, $1 \le l \le m$, the vector $e^{y(V^{-1}Q - \beta_l I)} g_l$ is a linear combination of vectors associated with the eigenvalue β_l of the matrix $V^{-1}Q$. Since the vectors associated with the eigenvalue β_l are among the vectors $\{g_k, k = 1, \dots m\}$ and $J_1\Gamma_2\, g_k = 0$ for all $k = 1, \dots, m$, it follows that

$$J_1\Gamma_2\, e^{y(V^{-1}Q-\beta_k I)} g_k = 0, \qquad k = 1, \dots, m. \tag{3.8}$$

Thus, we conclude that

$$e^{-yV^{-1}Q}\Gamma J_2\, e^{yV^{-1}Q} g_k = g_k.$$

Hence, because $\Gamma e^{-yG} J_2 1 = 1$, it follows form (3.6) that

$$\begin{aligned} \frac{G_0(y,y)\ 1}{y} &= c\ 1 + \frac{J_1 1 + \sum_{k=1}^{m} b_k g_k}{y} + \frac{c\, \Gamma e^{-yG} J_2 r}{y} \\ &\quad + \sum_{j,\alpha_j \neq 0} a_j \frac{\Gamma e^{-yG} J_2\, e^{yV^{-1}Q} f_j}{y}. \end{aligned} \tag{3.9}$$

By Lemma 1.4.3, the matrix e^{yG^-} is the matrix of transition probabilities of the process Y^- which implies that it is bounded for all $y > 0$. Thus, for any vector g on E,

$$\Gamma e^{-yG} J_2 g = \begin{pmatrix} \Pi^- e^{yG^-} g^- \\ e^{yG^-} g^- \end{pmatrix},$$

is bounded. Hence, the third term on the right-hand side of (3.9) tends to zero as $y \to +\infty$. In addition, by (1.10) (i), for $Re(\alpha_j) < 0$, $j = 1, \dots, n$, $e^{yV^{-1}Q} f_j$ is bounded. Thus, for $Re(\alpha_j) < 0$, $j = 1, \dots, n$, $\Gamma e^{-yG} J_2 e^{yV^{-1}Q} f_j$ is bounded and

$$\lim_{y \to +\infty} \frac{\Gamma e^{-yG} J_2 e^{yV^{-1}Q} f_j}{y} = 0.$$

Hence, the last term on the right-hand side of (3.6) also tends to zero when $y \to \infty$.

Therefore, (3.9) implies that

$$\lim_{y \to +\infty} \frac{G_0(y,y)\ 1}{y} = c\ 1.$$

Finally, by (3.5),

$$\begin{aligned} \lim_{y \to +\infty} \frac{G_0(\varphi,y)\ 1(e)}{y} &= \sum_{e' \in E} \lim_{y \to +\infty} X(\varphi,y)(e,e') \lim_{y \to +\infty} \frac{G_0(y,y)\ 1(e')}{y} \\ &= c \lim_{y \to +\infty} X(\varphi,y) 1(e), \end{aligned}$$

which implies that (3.3) is valid.

Now we shall show that (3.4) is valid as well. Let

$$J_1\Gamma 1 = \sum_{j,\alpha'_j\neq 0} a'_j f_j + \sum_{k=1}^{m} b'_k g_k + c'\ r$$

for some constants a'_j, $j = 1,\dots,n$, b'_k, $k = 1,\dots,m$, and c'.

From Theorem 2.3.2 we obtain

$$\begin{aligned} G_0(\varphi,y)1 &= e^{-\varphi V^{-1}Q} J_1\Gamma_2\, e^{yV^{-1}Q} J_1\Gamma 1 \\ &= \sum_{j,\alpha_j\neq 0} a'_j\ e^{-\varphi V^{-1}Q} J_1\Gamma_2\, e^{yV^{-1}Q} f_j \\ &\quad + \sum_{k=1}^{m} b'_k\ e^{-\varphi V^{-1}Q} J_1\Gamma_2\, e^{y\beta_k}\ e^{y(V^{-1}Q-\beta_k I)}\ g_k \\ &\quad + c'\ e^{-\varphi V^{-1}Q} J_1\Gamma_2(r + y1). \end{aligned}$$

and because $J_1\Gamma_2 1 = 0$ and, by (3.8), $J_1\Gamma_2\ e^{y(V^{-1}Q-\beta_k I)} g_k = 0$, $k = 1,\dots,m$,

$$G_0(\varphi,y)1 = \sum_{j,\alpha_j\neq 0} a'_j\ e^{-\varphi V^{-1}Q} J_1\Gamma_2\, e^{yV^{-1}Q} f_j +\ c'\ e^{-\varphi V^{-1}Q} J_1\Gamma_2 r.$$

Suppose that $c' = 0$. Then, for some vectors x^+ on E^+ and y^- on E^-,

$$J_1\Gamma 1 = \begin{pmatrix} 1^+ \\ 0 \end{pmatrix} = \begin{pmatrix} x^+ \\ \Pi^+ x^+ \end{pmatrix} + \begin{pmatrix} \Pi^- y^- \\ y^- \end{pmatrix},$$

which implies that

$$(I - \Pi^-\Pi^+)x^+\ =\ 1^+.$$

By the same argument as in (3.7), we conclude that such vector x^+ does not exist. Hence, $c' \neq 0$ and (3.4) is valid.

As it was previously said, the statement in the lemma follows from (3.3) and (3.4).

□

We proceed with preparing for the proof of weak convergence of the measures $(P^y_{(e,\varphi)}|\mathcal{F}_t)_{y\geq 0}$. We need two more lemmas.

Lemma 3.1.2 *(i) Let $\{f_n, n \in \mathbb{N}\}$ and f be non-negative random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $Ef_n = Ef = 1$, where expectation is taken with respect to the probability measure P. If $f_n \to f$ a.s. as $n \to +\infty$, then $f_n \to f$ in $L^1(\Omega, \mathcal{F}, P)$ as $n \to +\infty$.*

(ii) Let $\{P_n, n \in \mathbb{N}\}$ and P be probability measures on a measurable space $(\Omega, \mathcal{F})$ such that, for any $A \in \mathcal{F}$, $P_n(A) \to P(A)$ as $n \to +\infty$. Then the measures $\{P_n, n \in \mathbb{N}\}$ converge weakly to P on $\mathcal{F}$.

Proof: (i) Since $\{f_n, n \in \mathbb{N}\}$ and f are non-negative and $Ef_n = Ef = 1$, the functions $\{f_n(\omega), n \in \mathbb{N}\}$ and $f(\omega)$, $\omega \in \Omega$, are densities with respect to the measure P. In addition, $f_n \to f$ a.s. as $n \to +\infty$ and so $f_n \to f$ in probability as $n \to +\infty$. Therefore, by Theorem 2.2. from Jacka, Roberts [20], $f_n \to f$ in $L^1(\Omega, \mathcal{F}, P)$ as $n \to +\infty$.
(ii) Let for any $A \in \mathcal{F}$, $P_n(A) \to P(A)$ as $n \to +\infty$. Then, by the definition of strong convergence in Jacka, Roberts, [20], P_n converges strongly to P which, by Theorem 2.1. in [20], implies that $\{P_n, n \in \mathbb{N}\}$ converge weakly to P. $\square$

Lemma 3.1.3 *Let $h_r(e, \varphi)$ be a function on $E \times \mathbb{R}$ defined by*

$$h_r(e, \varphi) = e^{-\varphi V^{-1} Q} J_1 \Gamma_2 r(e).$$

Then the process $\{h_r(X_t, \varphi_t) I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.

Proof: The function h_r is continuously differentiable in φ which by Lemma 1.6.2 implies that h_r is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t, \varphi_t)_{t \geq 0}$. In addition, $\mathcal{G} h_r = 0$. Thus, the process $(h_r(X_t, \varphi_t))_{t \geq 0}$ is a local martingale under $P_{(e,\varphi)}$ and therefore the stopped process $(h_r(X_{t \wedge H_0}, \varphi_{t \wedge H_0}))_{t \geq 0}$ is also a local martingale under $P_{(e,\varphi)}$. By the definition of the function h_r, if the process $(X_t, \varphi_t)_{t \geq 0}$ starts in E_0^+ then $h_r(X_{H_0}, \varphi_{H_0}) = 0$. Hence,

$$h_r(X_t, \varphi_t) I\{t < H_0\} = h_r(X_{t \wedge H_0}, \varphi_{t \wedge H_0}), \quad t \geq 0,$$

and we conclude that the process $\{h_r(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a local martingale under $P_{(e,\varphi)}$. Since it is also bounded on every finite interval, it follows that the process $\{h_r(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. □

Finally, we prove the main result in the section.

Theorem 3.1.1 *For fixed $(e, \varphi) \in E_0^+$, let a measure $P^r_{(e,\varphi)}$ be such that the process $(X_t, \varphi_t)_{t\geq 0}$ under the measure $P^r_{(e,\varphi)}$ is the $h-$transform with the function h_r of the killed process $(X_t, \varphi_t)_{t\geq 0}$. The function h_r is given by*

$$h_r(e', y) = e^{-yV^{-1}Q} J_1 \Gamma_2 r(e'), \qquad (e', y) \in E \times \mathbb{R},$$

and the killed process $(X_t, \varphi_t)_{t\geq 0}$ is the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

More precisely, for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^r_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_r(X_t, \varphi_t) I\{t < H_0\}\Big)}{h_r(e, \varphi)}.$$

For fixed $t \geq 0$, let $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ be the restriction of $P^r_{(e,\varphi)}$ to $\mathcal{F}_t$. Then, the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the probability measure $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$.

Proof: First we prove that $P^r_{(e,\varphi)}$ is a probability measure. By Lemma 3.1.1 (i) $h_r(e, \varphi)$ is positive. Hence, $P^r_{(e,\varphi)}$ is well-defined and it is a measure. Since, by Lemma 3.1.3, the process $\{h_r(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$, $P^r_{(e,\varphi)}$ is a probability measure.

Now we prove weak convergence of the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ by the use of Lemma 3.1.2.

For fixed $(e, \varphi) \in E_0^+$, $t \in [0, +\infty)$ and $y \geq 0$, let $h_y(e, \varphi, t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$\begin{aligned} h_y(e, \varphi, t) &= \frac{1}{P_{(e,\varphi)}(H_y < H_0)} \Big(I\{t < H_0 \wedge H_y\} P_{(X_t, \varphi_t)}(H_y < H_0) \\ &\quad + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\} \Big). \end{aligned}$$

The random variables h_y, $y \geq 0$, are non-negative and, since $H_y \to +\infty$ as $y \to +\infty$, by Lemma 3.1.1 (ii), for fixed $t \geq 0$,

$$\begin{aligned} \lim_{y \to +\infty} h_y(e, \varphi, t) &= \lim_{y \to +\infty} I\{t < H_0 \wedge H_y\} \frac{P_{(X_t, \varphi_t)}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} \\ &= I\{t < H_0\} \frac{h_r(X_t, \varphi_t)}{h_r(e, \varphi)} \quad a.s.. \end{aligned}$$

In addition, by (3.1),

$$E_{(e,\varphi)}\Big(h_y(e, \varphi, t)\Big) = P_{(e,\varphi)}\Big(\Omega \mid H_y < H_0\Big) = 1, \quad t \geq 0,$$

and because, by Lemma 3.1.3, the process $\{h_r(X_t, \varphi_t) I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$,

$$E_{(e,\varphi)}\Big(\frac{h_r(X_t, \varphi_t)}{h_r(e, \varphi)} I\{t < H_0\}\Big) = 1, \quad t \geq 0.$$

Therefore, by applying Lemma 3.1.2 (i) to the random variables $\{h_y(e, \varphi, t), y \geq 0\}$, we get that $h_y(e, \varphi, t)$ converge to $\frac{h_r(X_t, \varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $y \to +\infty$.

Then it follows from (3.1) that, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned} \lim_{y \to +\infty} P^y_{(e,\varphi)}(A) &= \lim_{y \to +\infty} E_{(e,\varphi)}\Big(I(A) h_y(e, \varphi, t)\Big) \\ &= E_{(e,\varphi)}\Big(I(A) \frac{h_r(X_t, \varphi_t)}{h_r(e, \varphi)} I\{t < H_0\}\Big) = P^r_{(e,\varphi)}(A). \end{aligned}$$

Thus, by Lemma 3.1.2 (ii), the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y \geq 0}$ converge weakly to the measure $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$. □

3.2 Conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_0 > T\}$

Let $P^T_{(e,\varphi)}$, $T > 0$, be the law of the process $(X_t, \varphi_t)_{t \geq 0}$, starting at $(e, \varphi) \in E^+_0$, conditioned on $\{H_0 > T\}$, and let $P^T_{(e,\varphi)}|_{\mathcal{F}_t}$, $t \geq 0$, be its restriction to $\mathcal{F}_t$. We are interested in weak convergence of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to +\infty$. We start by looking at the limit $\lim_{T\to+\infty} P^T_{(e,\varphi)}(A)$ for $A \in \mathcal{F}_t$.

By Lemma 1.5.4, the events $\{H_0 > T\}$, $T > 0$, are of positive probability for any $T > 0$. Thus, for $0 < t < T$ and $A \in \mathcal{F}_t$,

$$\begin{aligned} P^T_{(e,\varphi)}(A) &= P_{(e,\varphi)}(A \mid H_0 > T) \\ &= \frac{E_{(e,\varphi)}\Big(I(A)I\{H_0 > T\}\Big)}{P_{(e,\varphi)}(H_0 > T)} \\ &= \frac{E_{(e,\varphi)}\Big(I(A)P_{(X_t,\varphi_t)}(H_0 > T-t)I\{H_0 > t\}\Big)}{P_{(e,\varphi)}(H_0 > T)}. \end{aligned} \tag{3.10}$$

For fixed $t \geq 0$ and $(e,\varphi) \in E_0^+$, suppose that $\lim_{T\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists with probability one and that the process $\{\lim_{T\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)} I\{t < H_0\}, t \geq 0\}$ is a martingale. Then, by the use of Lemma 3.1.2 we can establish weak convergence of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to +\infty$.

Therefore, we shall first show that the limit $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists. In order to do that, we shall look for the asymptotic behaviour of $P_{(e,\varphi)}(H_0 > T)$.

By the Tauberian theorem (given in Feller [14] part 2, XIII.5), under certain conditions the asymptotic behaviour of a monotone function is uniquely determined by the behaviour of its Laplace transform near zero. Before we state the exact form of the theorem that we are going to use, we introduce some more notation.

For real functions $u(x)$ and $\kappa(x)$ we say

$$u(x) \sim \kappa(x) \text{ as } x \to a \quad \text{iff} \quad \lim_{x\to a} \frac{u(x)}{\kappa(x)} = 1$$

For a real positive function $L(x)$ we say that it is slowly varying at $+\infty$ iff for all $\lambda > 0$

$$L(x) \sim L(\lambda x) \quad \text{as} \quad x \to +\infty.$$

The following theorem is stated and proved in a slightly stronger form in Feller [14] part 2, XIII.5, as Theorem 4:

Theorem 3.2.1 *Let $u(x)$ be a monotone function and*

$$w(\alpha) = \int_0^\infty e^{-\alpha x}\ u(x)\ dx$$

its Laplace transform. Let $0 < \rho < \infty$. Then

$$w(\alpha) \sim \frac{1}{\alpha^\rho}\ L(\frac{1}{\alpha}) \qquad \textit{iff} \qquad u(x) \sim \frac{1}{\Gamma(\rho)}\ x^{\rho-1}\ L(x),$$

as $\alpha \to 0$ and $x \to +\infty$, respectively, where $L(x)$ is a slowly varying function as $x \to +\infty$ and $\Gamma(\rho)$ is the Gamma function.

In our case,

$$\int_0^\infty e^{-\alpha t}\ P_{(e,\varphi)}(H_0 > t)\ dt = \frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha}, \qquad \alpha > 0.$$

Since the function $t \mapsto P_{(e,\varphi)}(H_0 > t)$ is monotone, Theorem 3.2.1 and the previous equation imply that, for some $0 < \rho < \infty$ and some slowly varying function L,

$$\frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} \sim \alpha^{-\rho}\ L(\frac{1}{\alpha}) \ \text{ iff } \ P_{(e,\varphi)}(H_0 > t) \sim \frac{1}{\Gamma(\rho)}\ t^{\rho-1}\ L(t), \tag{3.11}$$

as $\alpha \to 0$ and $t \to \infty$, respectively.

By Lemmas 1.4.1 and 1.4.2, for any $\alpha > 0$ and any $(e, \varphi) \in E_0^+$,

$$E_{(e,\varphi)}(e^{-\alpha H_0}) = \Gamma_\alpha e^{-\varphi G_\alpha} J_2 1(e) = e^{-\varphi V^{-1}(Q-\alpha I)} \Gamma_\alpha J_2 1(e).$$

Hence, because $e^{-\varphi V^{-1}Q}1 = 1$,

$$\begin{aligned}\frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} &= \frac{1 - e^{-\varphi V^{-1}(Q-\alpha I)} \Gamma_\alpha J_2 1}{\alpha}(e) \\ &= e^{-\varphi V^{-1}Q} \frac{1 - \Gamma_\alpha J_2 1}{\alpha}(e) - \frac{e^{-\varphi V^{-1}(Q-\alpha I)} - e^{-\varphi V^{-1}Q}}{\alpha} \Gamma_\alpha J_2 1(e).\end{aligned} \tag{3.12}$$

We need to find the behaviour of the left-hand side of the previous equation for small $\alpha > 0$. First, we shall find the behaviour of $\frac{1-\Gamma_\alpha J_2 1}{\alpha}$. Since

$$1 - \Gamma_\alpha J_2 1 = 1 - \begin{pmatrix} \Pi_\alpha^- 1^- \\ 1^- \end{pmatrix} = \begin{pmatrix} 1^+ - \Pi_\alpha^- 1^- \\ 0 \end{pmatrix},$$

we actually have to find the behaviour of $\frac{1^+ - \Pi_\alpha^- 1^-}{\alpha}$.

We recall that μ is the invariant measure of the process $(X_t)_{t \geq 0}$.

Lemma 3.2.1

$$\frac{1^+ - \Pi_\alpha^- 1^-}{\alpha} \sim \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, (r^+ - \Pi^- r^-), \qquad \alpha \to 0.$$

Proof: Let $\beta_{min}(\alpha)$ be the eigenvalue of $V^{-1}(Q - \alpha I)$ with minimal positive real part and let $g_{min}(\alpha)$ be its associated eigenvector. Then, from (1.5) it follows that

$$\Pi_\alpha^- g_{min}^-(\alpha) = g_{min}^+(\alpha).$$

Theorem 1.8.3 gives the representation of $g_{min}(\alpha)$ for sufficiently small $\alpha > 0$. Keeping the notation of Theorem 1.8.3 and by substituting $g_{min}(\alpha)$ into the last equation, we obtain

$$\Pi_\alpha^- \left(1^- + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}} r^- + \Xi_{min}^-(\alpha^{\frac{1}{2}})\right) = 1^+ + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}} r^+ + \Xi_{min}^+(\alpha^{\frac{1}{2}})$$
$$\Pi_\alpha^- 1^- + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, \Pi_\alpha^- r^- + \Pi_\alpha^-\, \Xi_{min}^-(\alpha^{\frac{1}{2}}) = 1^+ + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}} r^+ + \Xi_{min}^+(\alpha^{\frac{1}{2}}).$$

Therefore,

$$\begin{aligned}
\Pi_\alpha^- 1^- &= 1^+ + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, (r^+ - \Pi_\alpha^- r^-) + \Xi_{min}^+(\alpha^{\frac{1}{2}}) + \Pi_\alpha^-\, \Xi_{min}^-(\alpha^{\frac{1}{2}}) \\
&= 1^+ + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, (r^+ - \Pi^- r^-) - \frac{1}{\sqrt{-\mu V r}}\, \alpha^{\frac{1}{2}}\, (\Pi_\alpha^- - \Pi^-) r^- \\
&\quad + \Xi_{min}^+(\alpha^{\frac{1}{2}}) + \Pi_\alpha^-\, \Xi_{min}^-(\alpha^{\frac{1}{2}}),
\end{aligned}$$

which gives that

$$\begin{aligned}
\frac{1^+ - \Pi_\alpha^- 1^-}{\alpha} &= -\frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, (r^+ - \Pi^- r^-) + \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}} (\Pi_\alpha^- - \Pi^-) r^- \\
&\quad + \frac{1}{\alpha}\, \Xi_{min}^+(\alpha^{\frac{1}{2}}) + \frac{1}{\alpha}\, \Pi_\alpha^-\, \Xi_{min}^-(\alpha^{\frac{1}{2}}). \qquad (3.13)
\end{aligned}$$

We recall from Theorem 1.8.3 that there exists $\varepsilon > 0$ such that the series

$$\Xi_{min}(\alpha^{\frac{1}{2}}) = \alpha\, w_2 + \alpha^{\frac{3}{2}}\, w_3 + \alpha^2\, w_4 + \ldots,$$

converges for $|\alpha| < \varepsilon$. Then, the series

$$\frac{1}{\alpha}\, \Xi_{min}(\alpha^{\frac{1}{2}}) = w_2 + \alpha^{\frac{1}{2}}\, w_3 + \alpha\, w_4 + \ldots$$

also converges for $|\alpha| < \varepsilon$. Hence, for sufficiently small $\alpha > 0$, the third term $\frac{1}{\alpha}\,\Xi^{+}_{min}(\alpha^{\frac{1}{2}})$ on the right-hand side of (3.13) is bounded.

By Lemma (1.4.9) $\Pi^{-}_{\alpha} - \Pi^{-} \to 0$ as $\alpha \to 0$. Hence, for small $\alpha > 0$, Π^{-}_{α} is bounded, and because $\frac{1}{\alpha}\,\Xi^{-}_{min}(\alpha^{\frac{1}{2}})$ is also bounded, the fourth term on the right-hand side of (3.13) is also bounded. Therefore, for sufficiently small $\alpha > 0$, the last two terms on the right-hand side of (3.13) are bounded, and since $\Pi^{-}_{\alpha} - \Pi^{-} \to 0$ as $\alpha \to 0$, we conclude from (3.13) that

$$\frac{1^{+} - \Pi^{-}_{\alpha}1^{-}}{\alpha} \sim -\frac{1}{\sqrt{-\mu V r}}\,\alpha^{-\frac{1}{2}}\,(r^{+} - \Pi^{-}r^{-}), \qquad \alpha \to 0.$$

□

We are now able to find the asymptotic behaviour of the tail probabilities of H_0.

Lemma 3.2.2 *For any $(e, \varphi) \in E_0^{+}$,*

$$P_{(e,\varphi)}(H_0 > t) \sim -\frac{1}{\sqrt{-\mu V r}}\,\frac{1}{\Gamma(\frac{1}{2})}\,t^{-\frac{1}{2}}\,e^{-\varphi V^{-1}Q}J_1\Gamma_2 r(e), \qquad t \to +\infty,$$

where $\Gamma(\frac{1}{2})$ is the Gamma function.

Proof: The lemma will be proved by the use of (3.11).

From (3.12) and Lemma 3.2.1 we can find the behaviour of $\frac{1-E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha}$ as $\alpha \to 0$. The function $\alpha \mapsto e^{-\varphi V^{-1}(Q-\alpha I)}$ is analytic for all α which implies that the limit

$$\lim_{\alpha\to 0_+}\frac{1}{\alpha}(e^{-\varphi V^{-1}(Q-\alpha I)} - e^{-\varphi V^{-1}Q})$$

exists. In addition, by Lemma 1.4.9, $\Gamma_\alpha \to \Gamma$, as $\alpha \to 0$. Hence, the second term $\frac{e^{-\varphi V^{-1}(Q-\alpha I)} - e^{-\varphi V^{-1}Q}}{\alpha}\,\Gamma_\alpha J_2 1$ on the right-hand side of (3.12) is bounded for small $\alpha > 0$.

Let $\varphi > 0$. Then, by Lemma 3.2.1,

$$e^{-\varphi V^{-1}Q}\begin{pmatrix}\frac{1^{+}-\Pi^{-}_{\alpha}1^{-}}{\alpha}\\ 0\end{pmatrix} \sim \frac{1}{\sqrt{-\mu V r}}\,\alpha^{-\frac{1}{2}}\,e^{-\varphi V^{-1}Q}\begin{pmatrix}r^{+} - \Pi^{-}r^{-}\\ 0\end{pmatrix}, \quad \alpha \to 0. \quad (3.14)$$

Since

$$\frac{1-\Gamma_\alpha J_2 1}{\alpha} = \begin{pmatrix} \frac{1^+ - \Pi_\alpha^- 1^-}{\alpha} \\ 0 \end{pmatrix} \quad \text{and} \quad J_1\Gamma_2 r = \begin{pmatrix} r^+ - \Pi^- r^- \\ 0 \end{pmatrix},$$

equation (3.14) can be rewritten as

$$e^{-\varphi V^{-1}Q}\frac{1-\Gamma_\alpha J_2 1}{\alpha} \sim \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, e^{-\varphi V^{-1}Q} J_1\Gamma_2 r, \quad \alpha \to 0.$$

Hence, from (3.12) we get that, for any $(e,\varphi) \in E \times (0,+\infty)$,

$$\frac{1-E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} \sim \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, e^{-\varphi V^{-1}Q} J_1\Gamma_2 r(e), \qquad \alpha \to 0. \tag{3.15}$$

In addition, for any $(e,\varphi) \in E^+ \times \{0\}$,

$$\frac{1-E_{(e,0)}(e^{-\alpha H_0})}{\alpha} = \frac{1^+ - \Pi_\alpha^- 1^-}{\alpha}(e) \sim \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, (r^+ - \Pi^- r^-)(e), \qquad \alpha \to 0. \tag{3.16}$$

Hence, by (3.15) and (3.16), for any $(e,\varphi) \in E_0^+$,

$$\frac{1-E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} \sim \frac{1}{\sqrt{-\mu V r}}\, \alpha^{-\frac{1}{2}}\, e^{-\varphi V^{-1}Q} J_1\Gamma_2 r(e), \qquad \alpha \to 0.$$

For fixed $(e,\varphi) \in E \times \mathbb{R}$, let $L(x)$, $x \in \mathbb{R}$, be the constant function defined by

$$L(x) = \frac{1}{\sqrt{-\mu V r}}\, e^{-\varphi V^{-1}Q} J_1\Gamma_2 r(e).$$

Then, $L(x)$ is a slowly varying function at $x = +\infty$ and

$$\frac{1-E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha} \sim \alpha^{-\frac{1}{2}}\, L(\frac{1}{\alpha}), \qquad \alpha \to 0.$$

The assertion in the lemma now follows from (3.11). □

Our aim in this section is to find the weak limit of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to \infty$.

Theorem 3.2.2 *For fixed $(e,\varphi) \in E_0^+$, let the measure $P^r_{(e,\varphi)}$ be as defined in Theorem 3.1.1, that is, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,*

$$P^r_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_r(X_t,\varphi_t) I\{t < H_0\}\Big)}{h_r(e,\varphi)},$$

where $h_r(e,\varphi) = e^{-\varphi V^{-1}Q} J_1 \Gamma_2 r(e)$.

Let $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ *be the restriction of* $P^r_{(e,\varphi)}$ *to* $\mathcal{F}_t$. *Then for any* $t \geq 0$, *the measures* $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ *converge weakly to* $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ *as* $T \to \infty$.

Proof: The theorem will be proved in the same way as Theorem 3.1.1, by the use of Lemma 3.1.2.

For fixed $(e,\varphi) \in E_0^+$ and $t, T \in [0,+\infty)$ let $h_T(e,\varphi,t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$h_T(e,\varphi,t) = \frac{P_{(X_t,\varphi_t)}(H_0 > T-t)}{P_{(e,\varphi)}(H_0 > T)}\, I\{t < H_0\}.$$

The random variables $h_T(e,\varphi,t)$, $T \geq 0$, are non-negative and by Lemma 3.2.2, for fixed $t \geq 0$,

$$\lim_{T\to+\infty} h_T(e,\varphi,t) = \frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)}\, I\{t < H_0\}, \quad a.s..$$

In addition, by (3.10),

$$E_{(e,\varphi)}\Big(h_T(e,\varphi,t)\Big) = P_{(e,\varphi)}\Big(\Omega \mid H_0 > T\Big) = 1, \quad t \geq 0,$$

and because, by Lemma 3.1.3, $\{h_r(X_t,\varphi_t) I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$,

$$E_{(e,\varphi)}\Big(\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}\Big) = 1, \quad t \geq 0.$$

Hence, the random variables $\{h_T(e,\varphi,t), T \geq 0\}$ satisfy all the conditions of Lemma 3.1.2 (i). Therefore, by Lemma 3.1.2 (i), $h_T(e,\varphi,t)$ converge to $\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $T \to +\infty$.

From (3.10) we obtain, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned}\lim_{T\to+\infty} P^T_{(e,\varphi)}(A) &= \lim_{T\to+\infty} E_{(e,\varphi)}\Big(I(A) h_T(e,\varphi,t)\Big) \\ &= E_{(e,\varphi)}\Big(I(A)\frac{h_r(X_t,\varphi_t)}{h_r(e,\varphi)} I\{t < H_0\}\Big) = P^r_{(e,\varphi)}(A),\end{aligned}$$

which, by Lemma 3.1.2 (ii), implies that the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the measure $P^r_{(e,\varphi)}|_{\mathcal{F}_t}$ as $T \to \infty$. $\square$

By comparing Theorems 3.1.1 and 3.2.2 we see that the measures $(P^y_{(e,\varphi)})_{y\geq 0}$ and $(P^T_{(e,\varphi)})_{T\geq 0}$ converge weakly to the same limit. Therefore, conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_y < H_0\}$, $y > 0$, and conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_0 > T\}$, $T > 0$, yield the same result.

Chapter 4

The negative drift case

We have already discussed conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative in the case of positive drift of the process $(\varphi_t)_{t\geq 0}$ (Section 1.10) and in the case of oscillating $(\varphi_t)_{t\geq 0}$ (Section 3). The remaining case is when the process $(\varphi_t)_{t\geq 0}$ has a negative drift. Hence, throughout this section we assume that the process $(\varphi_t)_{t\geq 0}$ has a negative drift and, as in the previous two cases, we are interested in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 = +\infty\}$.

By (1.29), the matrix G^- is conservative which, by Lemma 1.4.6 (ii), implies that the matrix Π^- is stochastic. Hence, by Lemma 1.5.2, for any $\varphi > 0$,

$$P_{(\cdot,\varphi)}(H_0 = +\infty) = \Big(I - \Gamma F(-\varphi)\Big)1 = \begin{pmatrix} 1^+ - \Pi^- \ e^{\varphi G^-} \ 1^- \\ 1^- - e^{\varphi G^-} \ 1^- \end{pmatrix} = 0,$$

and for $e \in E^+$, $\varphi = 0$,

$$P_{(e,0)}(H_0 = +\infty) = (1^+ - \Pi^- \ 1^-)(e) = 0.$$

Hence, as in the oscillating case, the probability of the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative is zero and we cannot perform conditioning of the process $(X_t, \varphi_t)_{t\geq 0}$ on this event in the standard way. But we can use the same idea as in the oscillating case, that is to condition the process $(X_t, \varphi_t)_{t\geq 0}$ on some events of positive probabilities that approximate the event $\{H_0 = +\infty\}$. In Section 4.1 we consider the approximation of the event $\{H_0 = +\infty\}$ by the events $\{H_y < H_0\}$, $y > 0$, and look at the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < H_0\}$. In Section 4.3 we consider another approximation of the event $\{H_0 = +\infty\}$, that is the approximation by the events $\{H_0 > T\}$, $T > 0$ and look at the limit as $T \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_0 > T\}$.

Conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < H_0\}$ can be seen as conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ first on the event that the process $(\varphi_t)_{t \geq 0}$ hits large levels $y > 0$, and then on the event that the process $(\varphi_t)_{t \geq 0}$ stays non-negative. Thus, instead of taking the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < H_0\}$, we can first take the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < +\infty\}$, and then we can further condition the limiting process on the event $\{H_0 = +\infty\}$. We shall see in Section 4.2 that the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < +\infty\}$ gives rise to a positive drift case of a process of the form $(X_t, \varphi_t)_{t \geq 0}$, and that further conditioning that process on the event $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_y < H_0\}$. Therefore, in order to condition the process $(\varphi_t)_{t \geq 0}$ which drifts to $-\infty$ to stay non-negative, we can condition it first to drift to $+\infty$ and then to stay non-negative.

However, in order to stay non-negative, the process $(\varphi_t)_{t \geq 0}$ does not have to go far from zero, that is to $+\infty$. It can stay around its starting point, or, in other words, it can oscillate around it. Hence, similarly to the previous case, we can first condition the process $(\varphi_t)_{t \geq 0}$ to oscillate and then we can condition the resulting process on the event $\{H_0 = +\infty\}$. We shall perform this kind of conditioning in Section 4.4 and see that, under a certain constraint, it yields the same result as the limit as $T \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t \geq 0}$ on the event $\{H_0 > T\}$.

4.1 Conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < H_0\}$

For fixed $y > 0$, let $P^y_{(e,\varphi)}$ be the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e, \varphi) \in E_0^+$, conditioned on $\{H_y < H_0\}$. Let $t \geq 0$ be fixed and let $P^y_{(e,\varphi)}|_{\mathcal{F}_t}$ denote the restriction of $P^y_{(e,\varphi)}$ to $\mathcal{F}_t$. We are interested in weak convergence of the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ as $y \to +\infty$ and we shall take the same approach as in the oscillating case.

For fixed $(e, \varphi) \in E_0^+$, by Lemma 1.5.3 the probability $P_{(e,\varphi)}(H_y < H_0)$ is positive. Hence, in the same way as in the oscillating case, we get that, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^y_{(e,\varphi)}(A) = \frac{1}{P_{(e,\varphi)}(H_y < H_0)} E_{(e,\varphi)}\Big(I(A)(I\{t < H_0 \wedge H_y\}P_{(X_t,\varphi_t)}(H_y < H_0) \\ +I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\})\Big). \tag{4.1}$$

Our next step is to show that the limit $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}$ exists and to calculate it. First we prove an auxiliary lemma.

Lemma 4.1.1 *For any vector g on E*

$$\lim_{y\to+\infty} F(y)g = 0.$$

In addition, for any non-negative vector g on E

$$\lim_{y\to+\infty} e^{-\alpha_{max} y} F(y)g = c\ J_1 f_{max}$$

for some positive constant $c \in \mathbb{R}$.

Proof: Let

$$g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \text{ and } g^+ = \sum_{j=1}^{n} a_j f_j^+,$$

for some coefficients a_j, $j = 1, \ldots, n$, where vectors f_j^+, $j = 1, \ldots, n$, form the basis $\mathcal{N}^+$ and are associated with the eigenvalues α_j, $j = 1, \ldots, n$.

Then, for $y > 0$,

$$F(y)g = \begin{pmatrix} e^{yG^+} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g^+ \\ g^- \end{pmatrix} = \begin{pmatrix} e^{yG^+} g^+ \\ 0 \end{pmatrix} = \sum_{j=1}^{n} a_j \begin{pmatrix} e^{yG^+} f_j^+ \\ 0 \end{pmatrix}. \qquad (4.2)$$

Since, by (1.10)(ii), for $Re(\alpha_j) < 0$, $j = 1, \ldots, n$, $e^{yG^+} f_j^+ \to 0$ as $y \to +\infty$, it follows from the last equation that

$$\lim_{y \to +\infty} F(y)g = 0.$$

Moreover, by Lemmas 1.4.3 and 1.4.6 (i), the matrix G^+ is an irreducible Q-matrix with the Perron-Frobenius eigenvalue α_{max} and Perron-Frobenius eigenvector f_{max}^+. Thus, by the Perron-Frobenius theorem for irreducible essentially non-negative matrices,

$$\lim_{y \to +\infty} e^{-\alpha_{max} y}\, e^{yG^+}(e, e') = c(e')\, f_{max}^+(e),$$

for some positive real constant $c(e')$. It follows that for any non-negative vector g on E^+

$$\lim_{y \to +\infty} e^{-\alpha_{max} y}\, e^{yG^+} g(e) = c\, f_{max}^+(e), \qquad (4.3)$$

for some positive constant $c \in \mathbb{R}$.

Therefore, from (4.2) and (4.3) we obtain

$$\begin{aligned} \lim_{y \to +\infty} e^{-\alpha_{max} y}\, F(y)g &= \lim_{y \to +\infty} \begin{pmatrix} e^{-\alpha_{max} y}\, e^{yG^+} g^+ \\ 0 \end{pmatrix} \\ &= c \begin{pmatrix} f_{max}^+ \\ 0 \end{pmatrix} = c\, J_1 f_{max}. \end{aligned}$$

□

Now we find the limit $\lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)}$.

Lemma 4.1.2

$$(i) \qquad h_{f_{max}}(e, \varphi) \equiv e^{-\varphi V^{-1} Q} J_1 \Gamma_2 f_{max}(e) > 0, \quad (e, \varphi) \in E_0^+,$$

$$(ii) \quad \lim_{y \to +\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{e^{-\varphi' V^{-1} Q} J_1 \Gamma_2 f_{max}(e')}{e^{-\varphi V^{-1} Q} J_1 \Gamma_2 f_{max}(e)}, \quad (e, \varphi), (e', \varphi') \in E_0^+,$$

(where "$\equiv$" means "defined to be").

Proof: (i) We recall the matrices A_y, B_y, C_y and D_y, which are, by (2.27), the components of the matrix $e^{-yV^{-1}Q}$:

$$e^{-yV^{-1}Q} = \begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix}.$$

The function $h_{f_{max}}$ can be rewritten as

$$h_{f_{max}}(\cdot,\varphi) = e^{-\varphi V^{-1}Q} J_1 \Gamma_2 f_{\max} = \begin{pmatrix} A_\varphi(I - \Pi^-\Pi^+) f^+_{max} \\ C_\varphi(I - \Pi^-\Pi^+) f^+_{max} \end{pmatrix}.$$

First we shall show that the vector $A_\varphi(I - \Pi^-\Pi^+)f^+_{max}$ is positive by showing that it is a Perron-Frobenius eigenvalue of a positive matrix. Then, because

$$C_\varphi(I - \Pi^-\Pi^+)f^+_{max} = C_\varphi A_\varphi^{-1}\, A_\varphi(I - \Pi^-\Pi^+)f^+_{max},$$

and by Lemmas 2.6.2 and 1.5.3, the matrix $C_\varphi A_\varphi^{-1}$ is positive (by Lemma 2.6.2, its entries are the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$, which are, by Lemma 1.5.3, strictly positive), we conclude that the function $h_{f_{max}}$ is positive.

Therefore, we have to prove that the vector $A_\varphi(I - \Pi^-\Pi^+)f^+_{max}$ is positive.

By (1.29), in the negative drift case the matrix G^+ is not conservative which, by Lemma 1.4.6 (ii) implies that the matrix Π^+ is strictly substochastic. Thus, by Lemma 1.4.5, the matrix $(I - \Pi^-\Pi^+)$ is invertible. Therefore, from (2.34) we deduce that

$$(A_{-y} - \Pi^- C_{-y}) = (I - \Pi^-\Pi^+)e^{yG^+}(I - \Pi^-\Pi^+)^{-1},$$

and because by (2.36) $(A_{-y} - \Pi^- C_{-y}) = (A_y - \Pi^- C_y)^{-1}$, that

$$(A_y - \Pi^- C_y)^{-1} = (I - \Pi^-\Pi^+)e^{yG^+}(I - \Pi^-\Pi^+)^{-1}.$$

By (2.30) the matrix A_φ is invertible. Hence,

$$A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1} = A_\varphi(I - \Pi^-\Pi^+)e^{yG^+}(I - \Pi^-\Pi^+)^{-1}A_\varphi^{-1}. \qquad (4.4)$$

The matrix $A_\varphi(A_y - \Pi^- C_y)^{-1}$ is positive because, by Theorem 2.3.1, it is the restriction to $E^+ \times E^+$ of the Green's function $G_0(\varphi, y)$ of the killed process $(X_t, \varphi_t)_{t\geq 0}$

which is positive. The matrix A_φ^{-1} is also positive since, by Lemma 2.6.2, its entries are the two-sided exit probabilities of the process $(X_t, \varphi_t)_{t\geq 0}$, which are, by Lemma 1.5.3, positive. Hence, the matrix $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ is positive and, by (4.4), is similar to e^{yG^+}. This means that $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ and e^{yG^+} have the same Perron-Frobenius eigenvalue and because the Perron-Frobenius eigenvector of e^{yG^+} is f_{max}^+, from (4.4) we conclude that the vector

$$A_\varphi(I - \Pi^- \Pi^+) f_{max}^+$$

is the Perron-Frobenius eigenvector of $A_\varphi(A_{-y} - \Pi^- C_{-y})A_\varphi^{-1}$ and therefore positive.

(ii) Let $X(\varphi, y)$ be an $E \times E$ matrix with entries

$$X(\varphi, y)(e, e') = P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0), \qquad e, e' \in E.$$

By decomposing the Green's function $G_0(\varphi, y)$ for the killed process $(X_t, \varphi_t)_{t\geq 0}$, we get that, for $0 < \varphi < y$,

$$G_0(\varphi, y) = X(\varphi, y)\; G_0(y, y),$$

and therefore,

$$G_0(\varphi, y)\Gamma_2 1 = X(\varphi, y)\; G_0(y, y)\Gamma_2 1. \tag{4.5}$$

For fixed $e, e' \in E$,

$$0 \leq X(\varphi, y)(e, e') \leq \sum_{e' \in E} X(\varphi, y)(e, e') = P_{(e,\varphi)}(H_y < H_0),$$

and

$$\lim_{y \to +\infty} P_{(e,\varphi)}(H_y < H_0) = P_{(e,\varphi)}(H_0 = +\infty) = 0,$$

Thus, the limit $\lim_{y\to+\infty} X(\varphi, y)(e, e')$ exists and is equal to zero.

On the other hand, by Theorem 2.4.1 and Lemma 4.1.1,

$$\lim_{y \to +\infty} G_0(y, y)\Gamma_2 1 = (I - \Gamma F(-y)\Gamma F(y))1 = 1.$$

Thus,

$$\lim_{y\to+\infty} G_0(\varphi,y)\Gamma_2 1 = \lim_{y\to+\infty} X(\varphi,y)\; G_0(y,y)\Gamma_2 1 = \lim_{y\to+\infty} X(\varphi,y)1,$$

and, by Theorem 2.4.1, $G_0(\varphi,y) = \Gamma e^{-\varphi G}\Gamma_2 F(y)\Gamma_2^{-1}$, $0<\varphi<y$.

Finally, we obtain

$$\begin{aligned}\lim_{y\to+\infty}\frac{P_{(e',\varphi')}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)} &= \lim_{y\to+\infty}\frac{X(\varphi',y)1(e')}{X(\varphi,y)1(e)}\\ &= \lim_{y\to+\infty}\frac{G_0(\varphi',y)\Gamma_2 1(e')}{G_0(\varphi,y)\Gamma_2 1(e)}\\ &= \lim_{y\to+\infty}\frac{\Gamma e^{-\varphi' G}\Gamma_2 F(y)1(e')}{\Gamma e^{-\varphi G}\Gamma_2 F(y)1(e)}\\ &= \lim_{y\to+\infty}\frac{e^{-\varphi' V^{-1}Q}\Gamma\Gamma_2 F(y)1(e')}{e^{-\varphi V^{-1}Q}\Gamma\Gamma_2 F(y)1(e)}.\end{aligned}$$

Since the vector 1 is non-negative and because $\Gamma\Gamma_2 J_1 f_{max} = J_1\Gamma_2 f_{max}$, by Lemma 4.1.1 we finally obtain

$$\lim_{y\to+\infty}\frac{e^{-\varphi' V^{-1}Q}\Gamma\Gamma_2 F(y)1}{e^{-\varphi V^{-1}Q}\Gamma\Gamma_2 F(y)1} = \frac{e^{-\varphi' V^{-1}Q}J_1\Gamma_2 f_{max}}{e^{-\varphi V^{-1}Q}J_1\Gamma_2 f_{max}}.$$

□

The function $h_{f_{max}}$ has the property that the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\}, t\geq 0\}$ is a martingale under $P_{(e,\varphi)}$. We prove this in the following lemma.

Lemma 4.1.3 *The process $\{h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\}, t\geq 0\}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function $h_{f_{max}}(e,\varphi)$ is continuously differentiable in φ. Hence, by Lemma 1.6.2, it is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq 0}$ and $\mathcal{G}h_{f_{max}} = 0$. Thus, the process $(h_{f_{max}}(X_t,\varphi_t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$ and therefore the stopped process $(h_{f_{max}}(X_{t\wedge H_0},\varphi_{t\wedge H_0}))_{t\geq 0}$ is also a local martingale under $P_{(e,\varphi)}$. By the definition of the function $h_{f_{max}}$, if the process $(X_t,\varphi_t)_{t\geq 0}$ starts in E_0^+ then $h_{f_{max}}(X_{H_0},\varphi_{H_0}) = 0$. Hence,

$$h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\} = h_{f_{max}}(X_{t\wedge H_0},\varphi_{t\wedge H_0}),\quad t\geq 0,$$

and because the process $(h_{f_{max}}(X_{t\wedge H_0}, \varphi_{t\wedge H_0}))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$, we conclude that the process $\{h_{f_{max}}(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a local martingale under $P_{(e,\varphi)}$. Since it is also bounded on every finite interval, the process $\{h_{f_{max}}(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. □

We are now prepared to prove weak convergence of the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$.

Theorem 4.1.1 *For fixed $(e,\varphi) \in E_0^+$, let a measure $P^{f_{max}}_{(e,\varphi)}$ be such that the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{f_{max}}_{(e,\varphi)}$ is the $h-$transform with the function $h_{f_{max}}$ of the killed process $(X_t, \varphi_t)_{t\geq 0}$. The function $h_{f_{max}}$ is given by*

$$h_{f_{max}}(e', y) = e^{-yV^{-1}Q} J_1 \Gamma_2 f_{max}(e'), \qquad (e', y) \in E \times \mathbb{R},$$

and the killed process $(X_t, \varphi_t)_{t\geq 0}$ is the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

More precisely, for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{f_{max}}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_{f_{max}}(X_t, \varphi_t) I\{t < H_0\}\Big)}{h_{f_{max}}(e,\varphi)}.$$

Let $P^{f_{max}}|_{\mathcal{F}_t}$ be the restriction of $P^{f_{max}}_{(e,\varphi)}$ to $\mathcal{F}_t$. Then, for any $t \geq 0$, the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the probability measure $P^{f_{max}}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y \to \infty$.

Proof: The theorem will be proved in the same way as Theorem 3.1.1, by the use of Lemma 3.1.2.

By Lemma 4.1.2 (i), the function $h_{f_{max}}$ is positive and by Lemma 4.1.3, the process $\{h_{f_{max}}(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Hence, the probability measure $P^{f_{max}}_{(e,\varphi)}$ is well-defined.

Furthermore, for fixed $(e,\varphi) \in E_0^+$ and $t, y \geq 0$, let $h_y(e,\varphi,t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$\begin{aligned} h_y(e,\varphi,t) \quad &= \quad \frac{1}{P_{(e,\varphi)}(H_y < H_0)} \Big(I\{t < H_0 \wedge H_y\} P_{(X_t,\varphi_t)}(H_y < H_0) \\ &\qquad + I\{H_y \leq t < H_0\} + I\{H_y < H_0 \leq t\} \Big). \end{aligned}$$

The random variables $h_y(e,\varphi,t)$, $y \geq 0$ are non-negative and, since $H_y \to +\infty$ as $y \to +\infty$, by Lemma 4.1.2 (ii),

$$\begin{aligned}\lim_{y\to+\infty} h_y(e,\varphi,t) &= \lim_{y\to+\infty} I\{t<H_0\wedge H_y\}\frac{P_{(X_t,\varphi_t)}(H_y<H_0)}{P_{(e,\varphi)}(H_y<H_0)}\\ &= I\{t<H_0\}\,\frac{h_{f_{max}}(X_t,\varphi_t)}{h_{f_{max}}(e,\varphi)},\quad a.s..\end{aligned}$$

In addition, by (4.1),

$$E_{(e,\varphi)}\Big(h_y(e,\varphi,t)\Big) = P_{(e,\varphi)}\Big(\Omega \mid H_y < H_0\Big) = 1, \quad t \geq 0,$$

and because, by Lemma 4.1.3, the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t<H_0\}, t\geq 0\}$ is a martingale under $P_{(e,\varphi)}$,

$$E_{(e,\varphi)}\Big(\frac{h_{f_{max}}(X_t,\varphi_t)}{h_{f_{max}}(e,\varphi)}I\{t<H_0\}\Big) = 1, \quad t \geq 0.$$

Therefore, Lemma 3.1.2 (i) applied to the random variables $\{h_y(e,\varphi,t), y\geq 0\}$ implies that $h_y(e,\varphi,t)$ converge to $\frac{h_{f_{max}}(X_t,\varphi_t)}{h_{f_{max}}(e,\varphi)}I\{t<H_0\}$ in $L^1(\Omega,\mathcal{F},P_{(e,\varphi)})$ as $y\to+\infty$.

Hence, (4.1) implies that, for fixed $t\geq 0$ and $A\in\mathcal{F}_t$,

$$\begin{aligned}\lim_{y\to+\infty} P^y_{(e,\varphi)}(A) &= \lim_{y\to+\infty} E_{(e,\varphi)}\Big(I(A)h_y(e,\varphi,t)\Big)\\ &= E_{(e,\varphi)}\Big(I(A)\frac{h_{f_{max}}(X_t,\varphi_t)}{h_{f_{max}}(e,\varphi)}I\{t<H_0\}\Big) = P^{f_{max}}_{(e,\varphi)}(A).\end{aligned}$$

Therefore, by Lemma 3.1.2 (ii), the measures $(P^y_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the measure $P^{f_{max}}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $y\to\infty$. □

4.2 Conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on the event $\{H_y < +\infty\}$

In Section 4.1 we have been looking at the limit as $y\to+\infty$ of the process $(X_t,\varphi_t)_{t\geq 0}$ conditioned on the event that the process $(\varphi_t)_{t\geq 0}$ hits level y before crossing zero. In other words, the condition was that $(\varphi_t)_{t\geq 0}$ hits large levels y and that at the same time it stays non-negative. Instead, we can condition the process $(X_t, \varphi_t)_{t\geq 0}$ first on the event that $(\varphi_t)_{t\geq 0}$ hits large levels y regardless of crossing zero (that is taking the limit as $y \to \infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < +\infty\}$), and then we can condition the resulting process on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative.

These two conditionings of the process $(X_t, \varphi_t)_{t\geq 0}$ performed in the stated order are expected to give the same result as the limit as $y \to +\infty$ of conditioning $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < H_0\}$, which was discussed in Section 4.1. In this section we shall show that this expectation is fullfilled.

Let $(e, \varphi) \in E_0^+$ and $y > \varphi$. Then,

$$\begin{aligned} P_{(e,\varphi)}(H_y < +\infty) &= P_{(e,\varphi-y)}(H_0 < +\infty) \\ &= \sum_{e' \in E^+} P_{(e,\varphi-y)}(X_{H_0} = e', H_0 < +\infty), \end{aligned}$$

By Lemma 1.5.1, all probabilities on the right hand side of the previous equation are positive. Hence, the event $\{H_y < +\infty\}$ is of positive probability and we can condition the process $(X_t, \varphi_t)_{t\geq 0}$ on $\{H_y < +\infty\}$ in the standard way.

For fixed $t \geq 0$ and any $A \in \mathcal{F}_t$, we have that

$$P_{(e,\varphi)}\left(A \mid H_y < +\infty\right) \tag{4.6}$$

$$= \frac{E_{(e,\varphi)}\Big(I(A)I\{H_y < +\infty\}\Big)}{P_{(e,\varphi)}(H_y < +\infty)}$$

$$= \frac{E_{(e,\varphi)}\Big(I(A)P_{(X_t,\varphi_t)}(H_y < +\infty)I\{t < H_y\} + I(A)I\{H_y < t\}\Big)}{P_{(e,\varphi)}(H_y < +\infty)}. \tag{4.7}$$

We shall first find the limit $\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y<+\infty)}{P_{(e,\varphi)}(H_y<+\infty)}$.

Lemma 4.2.1 *For any $(e, \varphi), (e', \varphi') \in E_0^+$,*

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < +\infty)}{P_{(e,\varphi)}(H_y < +\infty)} = \frac{e^{-\alpha_{max}\varphi'} f_{max}(e')}{e^{-\alpha_{max}\varphi} f_{max}(e)}.$$

Proof: By Lemma 1.5.2 we have that, for $0 \leq \varphi < y$,

$$P_{(e,\varphi)}(H_y < +\infty) = P_{(e,\varphi-y)}(H_0 < +\infty) = \Gamma F(y-\varphi)1.$$

The vector 1 is non-negative. Hence, by Lemma 4.1.1 and because $\Gamma J_1 f_{max} = f_{max}$,

$$\begin{aligned}
\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < +\infty)}{P_{(e,\varphi)}(H_y < +\infty)} &= \lim_{y\to+\infty} \frac{e^{-\alpha_{max}\varphi'}\Gamma e^{-\alpha_{max}(y-\varphi')}F(y-\varphi)1(e')}{e^{-\alpha_{max}\varphi}\Gamma e^{-\alpha_{max}(y-\varphi)}F(y-\varphi)1(e)} \\
&= \frac{e^{-\alpha_{max}\varphi'}\Gamma J_1 f_{max}(e')}{e^{-\alpha_{max}\varphi}\Gamma J_1 f_{max}(e)} \\
&= \frac{e^{-\alpha_{max}\varphi'} f_{max}(e')}{e^{-\alpha_{max}\varphi} f_{max}(e)}.
\end{aligned}$$

□

Let $h_{max}(e,\varphi)$ be a function on $E \times \mathbb{R}$ defined by

$$h_{max}(e,\varphi) = e^{-\alpha_{max}\varphi} f_{max}(e).$$

Lemma 4.2.2 *The process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$.*

Proof: The function $h_{max}(e,\varphi)$ is continuously differentiable in φ. Thus, by Lemma 1.6.2, h_{max} is in the domain of the infinitesimal generator $\mathcal{G}$ of the process $(X_t,\varphi_t)_{t\geq 0}$ and $\mathcal{G}h_{max} = 0$. It follows that the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$ and, because it is bounded on every finite interval, the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$. □

Finally, by Lemmas 4.2.1, 4.2.2 and 3.1.2, we prove

Theorem 4.2.1 *For fixed $(e,\varphi) \in E_0^+$, let $P_{(e,\varphi)}^{h_{max}}$ be a measure such that the process $(X_t,\varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^{h_{max}}$ is the h-transform of the process $(X_t,\varphi_t)_{t\geq 0}$ with the function $h_{max}(e',y) = e^{-\alpha_{max}y} f_{max}(e')$. More precisely, for $t \geq 0$ and $A \in \mathcal{F}_t$,*

$$P_{(e,\varphi)}^{h_{max}}(A) = \frac{E_{(e,\varphi)}\Big(I(A)\ h_{max}(X_t,\varphi_t)\Big)}{h_{max}(e,\varphi)}.$$

Then, $P^{h_{max}}_{(e,\varphi)}$ is a probability measure and, for fixed $t \geq 0$,

$$\lim_{y\to+\infty} P_{(e,\varphi)}(A \mid H_y < +\infty) = P^{h_{max}}_{(e,\varphi)}(A), \quad A \in \mathcal{F}_t.$$

Proof: First we prove that $P^{h_{max}}_{(e,\varphi)}$ is a probability measure. By the definition, the function h_{max} is positive. Hence $P^{h_{max}}_{(e,\varphi)}$ is a measure. In addition, by Lemma 4.2.2, the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$. Hence, $P^{h_{max}}_{(e,\varphi)}$ is a probability measure.

For the second part of the theorem, we shall use Lemma 3.1.2.

For fixed $(e,\varphi) \in E_0^+$ and $t, y \geq 0$, let $h_y(e,\varphi,t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$h_y(e,\varphi,t) = \frac{P_{(X_t,\varphi_t)}(H_y < +\infty)I\{t < H_y\} + I(A)I\{H_y < t\}}{P_{(e,\varphi)}(H_y < +\infty)}.$$

The random variables $h_y(e,\varphi,t)$, $y \geq 0$, are non-negative and, by Lemma 4.2.1,

$$\lim_{y\to+\infty} h_y(e,\varphi,t) = \frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}, \quad a.s..$$

In addition, by (4.7),

$$E_{(e,\varphi)}\Big(h_y(e,\varphi,t)\Big) = P_{(e,\varphi)}\Big(\Omega \mid H_y < +\infty\Big) = 1, \quad t \geq 0,$$

and because, by Lemma 4.2.2, the process $(h_{max}(X_t,\varphi_t))_{t\geq 0}$ is a martingale under $P_{(e,\varphi)}$,

$$E_{(e,\varphi)}\Big(\frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}\Big) = 1, \quad t \geq 0.$$

By applying Lemma 3.1.2 (i) to the random variables $\{h_y(e,\varphi,t), y \geq 0\}$, we get that $h_y(e,\varphi,t)$ converge to $\frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}$ in $L^1(\Omega,\mathcal{F},P_{(e,\varphi)})$ as $y \to +\infty$. Thus, (4.7) gives that, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned}\lim_{y\to+\infty} P_{(e,\varphi)}(A \mid H_y < +\infty) &= \lim_{y\to+\infty} E_{(e,\varphi)}\Big(I(A)h_y(e,\varphi,t)\Big)\\ &= E_{(e,\varphi)}\Big(I(A)\frac{h_{max}(X_t,\varphi_t)}{h_{max}(e,\varphi)}\Big) = P^{h_{max}}_{(e,\varphi)}(A).\end{aligned}$$

□

Following the disscussion from the beginning of the section, our second task is to condition the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on the event $\{H_0 = +\infty\}$. We shall first study the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$.

By Theorem 1.9.1, the process $(X_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is Markov with the Q-matrix $Q^{h_{max}}$ given by

$$Q^{h_{max}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}(Q - \alpha_{max}V)(e,e'), \qquad e,e' \in E,$$

and, by the same theorem, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$. Moreover,

Lemma 4.2.3 *The matrix $Q^{h_{max}}$ is an irreducible conservative Q-matrix.*

Proof: By the definition, the matrix $Q^{h_{max}}$ is essentially non-negative, and because

$$Q^{h_{max}}1(e) = \frac{(Q - \alpha_{max}V)f_{max}(e)}{f_{max}(e)} = 0,$$

it is a Q-matrix.

To show that $Q^{h_{max}}$ is irreducible, by Lemma 1.1.4 it is enough to show that the matrix $e^{tQ^{h_{max}}}$ is positive for all $t > 0$. For any $e, e' \in E$,

$$e^{tQ^{h_{max}}}(e,e') = \frac{f_{max}(e')}{f_{max}(e)}\, e^{t(Q-\alpha_{max}V)}(e,e').$$

By Lemma 1.1.6, the matrix $(Q - \alpha_{max}V)$ is irreducible. Thus, by Lemma 1.1.4 the matrix $e^{t(Q-\alpha_{max}V)}$ is positive.Since the vector f_{max} is also positive, it follows that the matrix $e^{tQ^{h_{max}}}$ is positive and therefore, again by Lemma 1.1.4, the matrix $Q^{h_{max}}$ is irreducible. □

We can also find the Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$.

Lemma 4.2.4 *The unique Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$ is given by*

$$V^{-1}Q^{h_{max}}\,\Gamma^{h_{max}} = \Gamma^{h_{max}}\, G^{h_{max}},$$

where, for any $(e, e') \in E \times E$,

$$\begin{aligned} G^{h_{max}}(e,e') &= \frac{f_{max}(e')}{f_{max}(e)}\,(G - \alpha_{max}I)(e,e'), \\ \Gamma^{h_{max}}(e,e') &= \frac{f_{max}(e')}{f_{max}(e)}\,\Gamma(e,e'). \end{aligned}$$

In addition, if

$$G^{h_{max}} = \begin{pmatrix} G^{h_{max},+} & 0 \\ 0 & -G^{h_{max},-} \end{pmatrix} \quad \textit{and} \quad \Gamma^{h_{max}} = \begin{pmatrix} I & \Pi^{h_{max},-} \\ \Pi^{h_{max},+} & I \end{pmatrix},$$

then $G^{h_{max},+}$ *is a conservative* $Q-$*matrix and* $\Pi^{h_{max},+}$ *is stochastic, and* $G^{h_{max},-}$ *is not a conservative* $Q-$*matrix and* $\Pi^{h_{max},-}$ *is strictly substochastic.*

Proof: By the definition of the matrix $G^{h_{max}}$,

$$\begin{aligned} G^{h_{max},+}(e,e') &= \frac{f^+_{max}(e')}{f^+_{max}(e)}(G^+ - \alpha_{max}I)(e,e'), \qquad (e,e') \in E^+ \times E^+, \\ G^{h_{max},-}(e,e') &= \frac{f^-_{max}(e')}{f^-_{max}(e)}(G^- + \alpha_{max}I)(e,e'), \qquad (e,e') \in E^- \times E^-, \end{aligned}$$

which implies that the matrices $G^{h_{max},+}$ and $G^{h_{max},-}$ are essentially non-negative. In addition, for any $e \in E^+$,

$$G^{h_{max},+}1(e) = \frac{(G^+ - \alpha_{max}I)f^+_{max}(e)}{f^+_{max}(e)} = 0.$$

Hence, $G^{h_{max},+}$ is a conservative Q-matrix.

To show that $G^{h_{max},-}$ is also a Q-matrix, we have to show that

$$G^{h_{max},-}1^- = (G^- + \alpha_{max}I)f^-_{max} \leq 0.$$

By Lemma 4.1.2 (i), the function $h_{f_{max}}$ is positive and

$$h_{f_{max}}(\cdot,\varphi) = \begin{pmatrix} (e^{-\varphi G^+} - \Pi^- e^{\varphi G^-}\Pi^+)f^+_{max} \\ (\Pi^+ e^{-\varphi G^+} - e^{\varphi G^-}\Pi^+)f^+_{max} \end{pmatrix}.$$

Hence, for any $e \in E^-$ and any $\varphi > 0$,

$$\begin{aligned} (\Pi^+ e^{-\varphi G^+} - e^{\varphi G^-}\Pi^+)f^+_{max} &= e^{-\alpha_{max}\varphi} f^-_{max} - e^{\varphi G^-} f^-_{max} \\ &= e^{-\alpha_{max}\varphi}(I - e^{\varphi(G^- + \alpha_{max}I)})f^-_{max} > 0. \end{aligned}$$

Since

$$\lim_{\varphi\to 0}\frac{\left(I-e^{\varphi(G^-+\alpha_{max}I)}\right)f^-_{max}}{\varphi}=-(G^-+\alpha_{max}I)f^-_{max},$$

and

$$(I-e^{\varphi(G^-+\alpha_{max}I)})f^-_{max}>0,$$

we conclude that

$$(G^-+\alpha_{max}I)f^-_{max}\le 0.$$

Thus, $G^{h_{max},-}$ is a Q-matrix.

Moreover, if $(G^-+\alpha_{max}I)f^-_{max}=0$ then $h_{f_{max}}(e,\varphi)=0$ for $e\in E^-$ which is a contradiction to Lemma 4.1.2. Therefore, the matrix $G^{h_{max},-}$ is not conservative.

The matrices $G^{h_{max}}$ and $\Gamma^{h_{max}}$ satisfy the equality

$$V^{-1}Q^{h_{max}}\,\Gamma^{h_{max}}=\Gamma^{h_{max}}\,G^{h_{max}},$$

which, by Lemma 1.4.1, gives the unique Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$.

Finally, Lemma 1.4.6 (ii) implies that $\Pi^{h_{max},+}$ is a stochastic and $\Pi^{h_{max},-}$ a strictly substochastic matrix. □

The next theorem justifies our expectation that conditioning the process $(X_t,\varphi_t)_{t\ge 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on the event $\{H_0=+\infty\}$ yields the same result as the limit of conditioning the process $(X_t,\varphi_t)_{t\ge 0}$ on the events $\{H_y<H_0\}$, $y>0$.

Theorem 4.2.2 *Let $P^{f_{max}}_{(e,\varphi)}$ be as defined in Theorem 4.1.1. Then, for any $(e,\varphi)\in E^+_0$ and any $t\ge 0$,*

$$P^{h_{max}}_{(e,\varphi)}(A\,|H_0=\infty)=P^{f_{max}}_{(e,\varphi)}(A),\qquad A\in\mathcal{F}_t.$$

Proof: By Lemma 4.2.3, the Q-matrix $Q^{h_{max}}$ of the process $(X_t,\varphi_t)_{t\ge 0}$ under $P^{h_{max}}_{(e,\varphi)}$ is irreducible and conservative, and by Theorem 1.9.1 the process $(\varphi_t)_{t\ge 0}$ under $P^{h_{max}}_{(e,\varphi)}$ drifts to $+\infty$. Since the positive drift case of conditioning the process $(X_t,\varphi_t)_{t\ge 0}$ on the event $\{H_0 = +\infty\}$ has been discussed in Section 1.10, we can apply the results of Section 1.10 to the process $(X_t, \varphi_t)_{t\geq 0}$ under the measure $P^{h_{max}}_{(e,\varphi)}$.

Hence, analogously to Theorem 1.10.1, for any $t \geq 0$ and any $A \in \mathcal{F}_t$,

$$P^{h_{max}}_{(e,\varphi)}(A \mid H_0 = \infty) = \frac{E^{h_{max}}_{(e,\varphi)}\Big(I(A)\ P^{h_{max}}_{(X_t,\varphi_t)}(H_0 = +\infty)\ I\{t < H_0\}\Big)}{P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty)}. \tag{4.8}$$

We need to find the probability $P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty)$.

By Lemmas 1.5.2 and 4.2.4, for $\varphi > 0$,

$$\begin{aligned}
P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty) &= (I - \Gamma^{h_{max}} e^{-\varphi G^{h_{max}}} J_2)\ 1(e) \\
&= 1 - \sum_{e',e'' \in E} \frac{f_{max}(e')}{f_{max}(e)}\ \Gamma(e,e')\ \frac{f_{max}(e'')}{f_{max}(e')} \\
&\qquad e^{-\varphi(G - \alpha_{max} I)}(e',e'')\ J_2 1(e'')\ f_{max}(e'') \\
&= 1 - \frac{e^{\alpha_{max}\varphi}}{f_{max}(e)} \sum_{e'' \in E} \Gamma e^{-\varphi G}(e,e'')\ J_2 1(e'')\ f_{max}(e'').
\end{aligned}$$

Since $J_2 1(e)\ f_{max}(e) = J_2 f_{max}(e)$, we have that

$$\begin{aligned}
P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty) &= 1 - \frac{e^{\alpha_{max}\varphi}}{f_{max}(e)} \Gamma e^{-\varphi G} J_2 f_{max}(e) \\
&= 1 - \frac{1}{h_{max}(e,\varphi)} \Gamma e^{-\varphi G} J_2 f_{max}(e) \\
&= \frac{1}{h_{max}(e,\varphi)} \Big(e^{-\alpha_{max}\varphi}\ f_{max} - \Gamma F(-\varphi) f_{max}\Big)(e),
\end{aligned}$$

and (because $\Gamma J_1 f_{max} = f_{max}$ and $(J_1 - J_2) f_{max} = \Gamma_2 J_1 f_{max}$), that

$$\begin{aligned}
P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty) &= \frac{1}{h_{max}(e,\varphi)}\ \Big(e^{-\varphi V^{-1}Q} f_{max} - \Gamma e^{-\varphi G} J_2 f_{max}\Big)(e) \\
&= \frac{1}{h_{max}(e,\varphi)}\ \Big(e^{-\varphi V^{-1}Q} \Gamma J_1 f_{max} - \Gamma e^{-\varphi G} J_2 f_{max}\Big)(e) \\
&= \frac{1}{h_{max}(e,\varphi)}\ \Big(\Gamma e^{-\varphi G} J_1 f_{max} - \Gamma e^{-\varphi G} J_2 f_{max}\Big)(e) \\
&= \frac{1}{h_{max}(e,\varphi)}\ \Gamma e^{-\varphi G} \Gamma_2 J_1 f_{max}(e) \\
&= \frac{h_{f_{max}}(e,\varphi)}{h_{max}(e,\varphi)},
\end{aligned} \tag{4.9}$$

where $h_{f_{max}}$ is as defined in Lemma 4.1.2, that is $h_{f_{max}}(e,\varphi) = \Gamma e^{-\varphi G}\Gamma_2 J_1 f_{max}(e)$.

Similarly, for $e \in E^+$,

$$
\begin{aligned}
P^{h_{max}}_{(e,0)}(H_0 = +\infty) &= (1^+ - \Pi^{h_{max}} 1^-)(e) \\
&= 1 - \frac{\Pi^- f^-_{max}(e)}{f^+_{max}(e)} \\
&= \frac{f^+_{max} - \Pi^- f^-_{max})(e)}{f^+_{max}(e)} \\
&= \frac{(I - \Pi^- \Pi^+) f^+_{max}(e)}{f^+_{max}(e)} = \frac{h_{f_{max}}(e,0)}{h_{max}(e,0)}.
\end{aligned}
$$

Therefore, from Theorem 4.2.1 and (4.8) we get that, for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$
\begin{aligned}
&P^{h_{\max}}_{(e,\varphi)}(A \mid H_0 = \infty) \\
&= \frac{E_{(e,\varphi)}\Big(I(A)\ P^{h_{max}}_{(X_t,\varphi_t)}(H_0 = +\infty)\ h_{max}(X_t,\varphi_t)\ I\{t < H_0\}\Big)}{P^{h_{max}}_{(e,\varphi)}(H_0 = +\infty)\ h_{max}(e,\varphi)} \\
&= \frac{E_{(e,\varphi)}\Big(I(A)\ h_{f_{max}}(X_t,\varphi_t)\ I\{t < H_0\}\Big)}{h_{f_{max}}(e,\varphi)} \\
&= P^{f_{\max}}_{(e,\varphi)}(A).
\end{aligned}
$$

□

Remark 1: In Theorem 1.9.1, the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{\max}}_{(e,\varphi)}$ was detemined by the sign of $\mu^{h_{\max}} V1$, where $\mu^{h_{\max}}$ is the invariant measure of the process $(X_t)_{t\geq 0}$ under $P^{h_{\max}}_{(e,\varphi)}$. However, the behaviour of the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{\max}}_{(e,\varphi)}$ directly follows from the Wiener-Hopf factorization of the matrix $V^{-1}Q^{h_{max}}$. By Lemma 4.2.4, $G^{h_{max},+}$ is a conservative Q-matrix and $G^{h_{max},-}$ is not a conservative Q-matrix. Thus, by (1.26) and Theorem 1.7.1, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_{\max}}_{(e,\varphi)}$ drifts to $+\infty$.

Remark 2: Here we give an alternative proof of Lemma 4.1.3 which states that the process $\{h_{f_{\max}}(X_t,\varphi_t) I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.

By the definition of $P^{h_{\max}}_{(e,\varphi)}$ given in Theorem 4.2.1, from the formula for conditional expectation, for any $0 < s < t$ and any random variable Z measurable with respect to $\mathcal{F}_t$,

$$E^{h_{\max}}_{(e,\varphi)}(Z \mid \mathcal{F}_s) = \frac{E_{(e,\varphi)}(Z h_{max}(X_t, \varphi_t) \mid \mathcal{F}_s)}{h_{max}(X_s, \varphi_s)}.$$

By substituting for Z the expression $E^{h_{\max}}_{(e,\varphi)}(I\{H_0 = +\infty\} \mid \mathcal{F}_t)$ we obtain

$$\begin{aligned} E^{h_{\max}}_{(e,\varphi)}\Big(E^{h_{\max}}_{(e,\varphi)}(I\{H_0 &= +\infty\} \mid \mathcal{F}_t) \mid \mathcal{F}_s\Big) \\ &= \frac{E_{(e,\varphi)}\Big(E^{h_{\max}}_{(e,\varphi)}(I\{H_0 = +\infty\} \mid \mathcal{F}_t) h_{max}(X_t, \varphi_t) \mid \mathcal{F}_s\Big)}{h_{max}(X_s, \varphi_s)}, \end{aligned}$$

which may be rearranged into

$$\begin{aligned} E_{(e,\varphi)}\Big(P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty \mid \mathcal{F}_t)\ h_{max}(X_t, \varphi_t) \mid \mathcal{F}_s\Big) \\ = P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty \mid \mathcal{F}_s)\ h_{max}(X_s, \varphi_s). \qquad (4.10) \end{aligned}$$

Hence, the process $\{P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty \mid \mathcal{F}_t) h_{max}(X_t, \varphi_t), t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. By the Markov property of the process $(X_t, \varphi_t)_{t \geq 0}$ under $P^{h_{\max}}_{(e,\varphi)}$, for any $t \geq 0$,

$$P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty \mid \mathcal{F}_t) = I\{t < H_0\} P^{h_{\max}}_{(X_t,\varphi_t)}(H_0 = +\infty).$$

Finally, from (4.9) we get that that, for any $t \geq 0$,

$$\begin{aligned} P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty \mid \mathcal{F}_t)\ h_{max}(X_t, \varphi_t) &= I\{t < H_0\}\ P^{h_{\max}}_{(e,\varphi)}(H_0 = +\infty)\ h_{max}(X_t, \varphi_t) \\ &= I\{t < H_0\}\ h_{f_{\max}}(X_t, \varphi_t), \end{aligned}$$

and therefore, using (4.10), we conclude that the process

$$\{h_{f_{\max}}(X_t, \varphi_t) I\{t < H_0\},\ t \geq 0\}$$

is a martingale under $P_{(e,\varphi)}$.

4.3 Conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 > T\}$

Let $P^T_{(e,\varphi)}$, $T > 0$, be the law of the process $(X_t, \varphi_t)_{t\geq 0}$, starting at $(e, \varphi) \in E_0^+$, conditioned on $\{H_0 > T\}$, and let $P^T_{(e,\varphi)}|_{\mathcal{F}_t}$, $t \geq 0$, be its restriction to $\mathcal{F}_t$. We are interested in weak convergence of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to +\infty$. We shall again use the approach via the Laplace transform of H_0, as we did in the oscillating case in Section 3.2.

By Lemma 1.5.4, for any $(e, \varphi) \in E_0^+$ and any $T > 0$, the probability of the event $\{H_0 > T\}$ under $P_{(e,\varphi)}$ is positive. Thus, for $0 < t < T$ and $A \in \mathcal{F}_t$,

$$\begin{aligned} P^T_{(e,\varphi)}(A) &= P_{(e,\varphi)}(A|H_0 > T) \\ &= \frac{E_{(e,\varphi)}\Big(I(A)I\{H_0 > T\}\Big)}{P_{(e,\varphi)}(H_0 > T)} \\ &= \frac{E_{(e,\varphi)}\Big(I(A)P_{(X_t,\varphi_t)}(H_0 > T - t)I\{H_0 > t\}\Big)}{P_{(e,\varphi)}(H_0 > T)}. \end{aligned} \tag{4.11}$$

Similarly to the oscillating case in Section 3.2, if, for fixed $t \geq 0$ and $(e, \varphi) \in E_0^+$, the limit $\lim_{T\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists and satisfies certain conditions, then, by the use of Lemma 3.1.2 we can prove weak convergence of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{T\geq 0}$ as $T \to +\infty$.

In order to find the limit $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ we shall look for the asymptotic behaviour of the function $t \mapsto P_{(e,\varphi)}(H_0 > t)$. By the theory of Laplace transforms (see Doetsch [12], Section 35), the asymptotic behaviour of a function is determined by the behaviour of its Laplace transform near singular points. Therefore, we shall first look for the Laplace transform of the function $t \mapsto P_{(e,\varphi)}(H_0 > t)$.

Let $\alpha > 0$ and $(e, \varphi) \in E_0^+$. Then,

$$\int_0^\infty e^{-\alpha t}\, P_{(e,\varphi)}(H_0 > t)\, dt = \frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha}.$$

For fixed $(e,\varphi) \in E_0^+$, let $L_{(e,\varphi)}(\alpha)$ be the Laplace transform of $P_{(e,\varphi)}(H_0 > t)$, that is

$$L_{(e,\varphi)}(\alpha) = \frac{1 - E_{(e,\varphi)}(e^{-\alpha H_0})}{\alpha}, \qquad \alpha > 0.$$

Then, $L_{(e,\varphi)}(\alpha)$ is analytic at least in the complex halfplane $Re(\alpha) > 0$ and its singular points lie in the halfplane $Re(\alpha) \leq 0$.

We have seen in Section 3.2 that in the oscillating case the Laplace transform of $P_{(e,\varphi)}(H_0 > t)$ has a singular point at zero and that its behaviour near zero determines the behaviour of $P_{(e,\varphi)}(H_0 > t)$ as $t \to +\infty$. However, we shall see that in the negative drift case zero is not a singular point of the Laplace transform $L_{(e,\varphi)}(\alpha)$ and that the behaviour of $L_{(e,\varphi)}(\alpha)$ at zero does not determine the asymptotic behaviour of $P_{(e,\varphi)}(H_0 > t)$. Hence, we shall look for the singular points of $L_{(e,\varphi)}(\alpha)$ in the halfplane $Re(\alpha) < 0$. We organize our approach into three steps:

1) we show that there exists finite $\alpha_0 < 0$ such that $L_{(e,\varphi)}(\alpha)$ is analytic for $Re(\alpha) > \alpha_0$ and that α_0 is a singular point of $L_{(e,\varphi)}(\alpha)$;

2) we show that $L_{(e,\varphi)}(\alpha + \alpha_0)$ for $\alpha > 0$ is the Laplace transform of the function $t \mapsto e^{-\alpha_0 t} P_{(e,\varphi)}(H_0 > t)$;

3) from steps 1) and 2) it follows that zero is a singular point of the Laplace transform $L_{(e,\varphi)}(\alpha + \alpha_0)$. Thus, by the use of the Tauberian theorem 3.2.1 we find the limit $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T-t)}{P_{(e,\varphi)}(H_0 > T)}$, assuming that it exists.

To carry out this plan, we begin with the function $L_{(e,\varphi)}(\alpha)$ written in terms of familiar matrices Q, V, and Γ_α, $\alpha > 0$. By Lemmas 1.4.1 and 1.4.2, for $\varphi > 0$,

$$E_{(e,\varphi)}(e^{-\alpha H_0}) = \Gamma_\alpha F_\alpha(-\varphi)1(e) = e^{-\varphi V^{-1}(Q-\alpha I)}\Gamma_\alpha J_2 1(e),$$

and for $e \in E^+$,

$$E_{(e,0)}(e^{-\alpha H_0}) = \Pi_\alpha^- 1^-(e) = \Gamma_\alpha J_2 1(e).$$

Thus, for $(e,\varphi) \in E_0^+$,

$$L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\varphi V^{-1}(Q-\alpha I)}\Gamma_\alpha J_2 1}{\alpha}(e) \tag{4.12}$$

In order to extend the definition of $L_{(e,\varphi)}(\alpha)$ for negative $\alpha \in \mathbb{R}$ we introduce some more notation. Let β_0 be the point at which the Perron-Frobenius eigenvalue $\alpha(\beta)$ of the matrix $(Q - \beta V)$ attains its minimum which, by Lemma 1.8.4 exists and is unique. By Lemma 1.8.5, in the negative drift case $\beta_0 < 0$. Let $\alpha_0 = \alpha(\beta_0)$ and g_0 be the Perron-Frobenius eigenvalue and right eigenvector, respectively, of the matrix $(Q - \beta_0 V)$. Then $g_0 > 0$ and, by Lemma 1.8.5, $\alpha_0 < 0$.

Let Q^0 be the $E \times E$ matrix with entries

$$Q^0(e,e') = \frac{g_0(e')}{g_0(e)}(Q - \alpha_0 I - \beta_0 V)(e,e'), \qquad e,e' \in E. \tag{4.13}$$

Lemma 4.3.1 *The matrix Q^0 is a conservative irreducible Q-matrix.*

In addition, if μ^0 is a vector on E such that $\mu^0 Q^0 = 0$ then $\mu^0 V 1 = 0$.

Proof: Since the matrices I and V are diagonal and the vector g_0 is positive, the matrix Q^0 is, by its definition, an essentially non-negative matrix. In addition,

$$Q^0 1(e) = \frac{(Q - \alpha_0 I - \beta_0 V) g_0(e)}{g_0(e)} = 0,$$

which implies that the matrix Q^0 is a conservative Q-matrix.

In order to show that Q^0 is irreducible it is, by Lemma 1.1.4, enough to show that e^{tQ^0} is positive for all $t > 0$. By the definition of Q^0, for any $e, e' \in E$,

$$e^{tQ^0}(e,e') = \frac{g_0(e')}{g_0(e)}\, e^{t(Q-\alpha_0 I - \beta_0 V)}(e,e').$$

Since the vector g_0 is positive, all we need to show is that $e^{t(Q-\alpha_0 I-\beta_0 V)}$ is positive for all $t > 0$. But, by Lemma 1.1.6, the matrix $(Q - \alpha_0 I - \beta_0 V)$ is irreducible which by Lemma 1.1.4 implies that $e^{t(Q-\alpha_0 I-\beta_0 V)}$ is positive for all $t > 0$. Therefore, e^{tQ^0} is positive for all $t > 0$, and, by Lemma 1.1.4, the matrix Q^0 is irreducible.

For the second part of the lemma we first need to find a vector μ^0 such that $\mu^0 Q^0 = 0$. Let g_0^{left} be the left Perron-Frobenius eigenvector of the matrix $(Q - \beta_0 V)$, that is

$$g_0^{left}(Q - \beta_0 V) = \alpha_0\, g_0^{left}, \quad g_0^{left} > 0.$$

Let μ^0 be a vector on E with entries

$$\mu^0(e) = g_0^{left}(e) g_0(e).$$

Then,

$$\begin{aligned}
\mu^0 Q^0(e) &= \sum_{e' \in E} \mu^0(e')\; Q^0(e', e) \\
&= \sum_{e' \in E} g_0^{left}(e') g_0(e')\; \frac{g_0(e)}{g_0(e')} (Q - \alpha_0 I - \beta_0 V)(e'e) \\
&= g_0^{left}(Q - \alpha_0 I - \beta_0 V)(e)\; g_0(e) = 0.
\end{aligned}$$

We have to show that

$$\mu^0 V 1 = g_0^{left} V g_0 = 0. \tag{4.14}$$

Let $\alpha(\beta)$, $g^{left}(\beta)$ and $g^{right}(\beta)$ be the Perron-Frobenius eigenvalue and its associated left and right eigenvectors respectively, of the matrix $(Q - \beta V)$. By Lemma 1.8.4

$$\frac{d\alpha}{d\beta}(\beta_0) = -\frac{g^{left}(\beta_0)\; V\; g^{right}(\beta_0)}{g^{left}(\beta_0)\; g^{right}(\beta_0)}.$$

By Lemma 1.8.5, $\frac{d\alpha}{d\beta}(\beta_0) = 0$. Therefore,

$$g^{left}(\beta_0)\; V\; g^{right}(\beta_0) = 0,$$

and since $g^{left}(\beta_0) = g_0^{left}$ and in the notation previous to lemma $g^{right}(\beta_0) = g_0$, the last equation and (4.14) prove the lemma. □

To simplify the notation, let G_0 be the diagonal matrix $diag(g_0(e))$. Since the vector g_0 is positive, the matrix G_0 is invertible. Hence,

$$Q^0 = G_0^{-1}\; (Q - \alpha_0 I - \beta_0 V)\; G_0.$$

By the previous lemma, Q^0 is an irreducible conservative Q-matrix. Therefore, the matrix $V^{-1}Q^0$ admits the Wiener-Hopf factorization. Moreover, all matrix lemmas from previous chapters can be applied to the matrix $V^{-1}Q^0$ and in the rest of this section we shall refer to them and use them as if they were proved for the matrix $V^{-1}Q^0$ instead of the matrix $V^{-1}Q$. We keep all the notation from Section 1.4 for the matrices associated with the matrix $V^{-1}Q$ and vectors asociated with the eigenvalues of the matrix $V^{-1}Q$, and introduce analogous notation for the same objects associated with the matrix $V^{-1}Q^0$.

By Lemmas 1.4.1 and 1.4.3, for any $\alpha \geq 0$, there exist matrices Γ_α^0 and G_α^0 such that

$$\Gamma_\alpha^0 = \begin{pmatrix} I & \Pi_\alpha^{0,-} \\ \Pi_\alpha^{0,+} & I \end{pmatrix} \qquad \text{and} \qquad G_\alpha^0 = \begin{pmatrix} G_\alpha^{0,+} & 0 \\ 0 & -G_\alpha^{0,-} \end{pmatrix}$$

and $G_\alpha^{0,+}$ and $G_\alpha^{0,-}$ are Q-matrices, and

$$V^{-1}(Q^0 - \alpha I)\,\Gamma_\alpha^0 = \Gamma_\alpha^0\, G_\alpha^0$$

(for $\alpha = 0$ we drop the subscript, thus $\Gamma_0^0 = \Gamma^0$ and $G_0^0 = G^0$).

In the following lemma we find the relation between G_α^0 and G_α and between Γ_α^0 and Γ_α.

Lemma 4.3.2

$$\begin{aligned} G_\alpha^0 &= G_0^{-1}\,(G_\alpha - \beta_0 I)\,G_0, \qquad \alpha \geq 0, \\ \Gamma_{\alpha-\alpha_0}^0 &= G_0^{-1}\,\Gamma_\alpha\,G_0, \qquad \alpha \geq 0. \end{aligned}$$

Proof: By Lemma 1.4.1 the matrix Γ_α^0 is invertible for $\alpha > 0$. Hence, the Wiener-Hopf factorization of the matrix $V^{-1}(Q^0 - \alpha I)$ can be written as

$$V^{-1}(Q^0 - \alpha I) = \Gamma_\alpha^0\, G_\alpha^0\, (\Gamma_\alpha^0)^{-1}, \quad \alpha > 0. \tag{4.15}$$

On the other hand, by the definition of the matrix Q^0 and by the Wiener-Hopf factorization of the matrix $V^{-1}(Q-\alpha I)$, $\alpha > 0$, we obtain, for every $\alpha > 0$,

$$\begin{aligned} V^{-1}(Q^0-(\alpha-\alpha_0)I) &= G_0^{-1}\,(V^{-1}(Q-\alpha I)-\beta_0 I)\,G_0 \\ &= G_0^{-1}\,(\Gamma_\alpha\,(G_\alpha-\beta_0 I)\,\Gamma_\alpha^{-1})\,G_0 \\ &= (G_0^{-1}\Gamma_\alpha G_0)\,(G_0^{-1}(G_\alpha-\beta_0 I)G_0)\,(G_0^{-1}\Gamma_\alpha^{-1}G_0). \quad (4.16) \end{aligned}$$

Let G_0^+ be the restriction of G_0 to $E^+ \times E^+$, that is $G_0^+ = diag(g_0^+(e))$, and let G_0^- be the restriction of G_0 to $E^- \times E^-$, that is $G_0^- = diag(g_0^-(e))$. Then,

$$G_0^{-1}(G_\alpha-\beta_0 I)G_0 = \begin{pmatrix} (G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+ & 0 \\ 0 & -(G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^- \end{pmatrix}.$$

Suppose that

$$(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+ \qquad \text{and} \qquad (G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^-$$

are Q-matrices. Then, by Lemma 1.4.1, (4.16) is the the Wiener-Hopf factorization of the matrix $V^{-1}(Q^0-(\alpha-\alpha_0)I)$ for $\alpha > 0$, and by the uniqueness of the Wiener-Hopf factorization it has to be equal to (4.15) for $\alpha > \alpha_0$. By comparing (4.15) and (4.16) we obtain

$$\begin{aligned} G_\alpha^0 &= G_0^{-1}\,(G_\alpha-\beta_0 I)\,G_0, \qquad \alpha \geq 0, \\ \Gamma_\alpha^0 &= G_0^{-1}\,\Gamma_\alpha\,G_0, \qquad \alpha \geq 0. \end{aligned}$$

Therefore, all we have to prove is that $(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+$ and $(G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^-$ are Q-matrices.

The matrices $(G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+$ and $(G_0^-)^{-1}(G_\alpha-\beta_0 I)G_0^-$ are essentially non-negative. In order to show that they are Q-matrices, we have to show that

$$\begin{aligned} (G_0^+)^{-1}(G_\alpha^+-\beta_0 I)G_0^+1^+ &= (G_0^+)^{-1}(G_\alpha^+-\beta_0 I)g_0^+ \leq 0 \\ (G_0^-)^{-1}(G_\alpha^-+\beta_0 I)G_0^-1^- &= (G_0^-)^{-1}(G_\alpha^-+\beta_0 I)g_0^- \leq 0. \end{aligned} \quad (4.17)$$

Let the function h be defined by $h(e,\varphi,t) = e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, g_0(e)$. Then h is continuously differentable in φ and t, and, by Lemma 1.6.1, it is in the domain of the infinitesimal generator $\mathcal{A}$ of the process $(X_t,\varphi_t,t)_{t\geq 0}$ and $\mathcal{A}h = 0$. Thus, the process $(h(X_t,\varphi_t,t))_{t\geq 0}$ is a local martingale and because it is bounded on every finite interval, a martingale. It follows that the stopped process $(h(X_{t\wedge H_y},\varphi_{t\wedge H_y},t\wedge H_y))_{t\geq 0}$ is also a martingale and that

$$E_{(e,\varphi)}\Big(e^{-\alpha_0(t\wedge H_y)}\, e^{-\beta_0\varphi_{t\wedge H_y}}\, g_0(X_{t\wedge H_y})\Big) = e^{-\beta_0\varphi}\, g_0(e).$$

The process $(h(X_{t\wedge H_y},\varphi_{t\wedge H_y},t\wedge H_y))_{t\geq 0}$ is a positive martingale and therefore it converges a.s. as $t\to+\infty$ to $h(X_{H_y},\varphi_{H_y},H_y)$. Thus, by Fatou's lemma,

$$\begin{aligned}
&E_{(e,\varphi)}\Big(e^{-\alpha_0 H_y}\, e^{-\beta_0\varphi_{H_y}}\, g_0(X_{H_y})\Big)\\
&\quad= E_{(e,\varphi)}\Big(\lim_{t\to+\infty} e^{-\alpha_0(t\wedge H_y)}\, e^{-\beta_0\varphi_{t\wedge H_y}}\, g_0(X_{t\wedge H_y})\Big)\\
&\quad\leq \lim_{t\to+\infty} E_{(e,\varphi)}\Big(e^{-\alpha_0(t\wedge H_y)}\, e^{-\beta_0\varphi_{t\wedge H_y}}\, g_0(X_{t\wedge H_y})\Big)\\
&\quad= e^{-\beta_0\varphi}\, g_0(e).
\end{aligned}\tag{4.18}$$

On the other hand, because g_0 is positive we have that, for $\alpha > \alpha_0$,

$$E_{(e,\varphi)}\Big(e^{-\alpha H_y}\, g_0(X_{H_y})\Big) \leq E_{(e,\varphi)}\Big(e^{-\alpha_0 H_y}\, g_0(X_{H_y})\Big).\tag{4.19}$$

Hence, by (4.18) and (4.19), for any $\varphi, y\in\mathbb{R}$,

$$E_{(e,\varphi)}\Big(e^{-\alpha H_y}\, g_0(X_{H_y})\Big) \leq e^{-\beta_0(\varphi-y)}\, g_0(e).\tag{4.20}$$

By Lemma 1.4.2, for $\varphi = 0$ and $y > 0$,

$$E_{(e,0)}\Big(e^{-\alpha H_y}\, g_0(X_{H_y})\Big) = \begin{pmatrix} e^{yG_\alpha^+} g_0^+ \\ \Pi_\alpha^+ e^{yG_\alpha^+} g_0^+ \end{pmatrix}.$$

Since, by (4.20),

$$E_{(e,0)}\Big(e^{-\alpha H_y}\, g_0(X_{H_y})\Big) \leq e^{\beta_0 y}\, g_0(e),$$

we obtain

$$e^{yG_\alpha^+} g_0^+ \leq e^{\beta_0 y}\, g_0^+,$$

or, equivalently,

$$e^{y(G_\alpha^+ - \beta_0)} g_0^+ \leq g_0^+.$$

From the last inequality and the limit

$$\lim_{y \to 0} \frac{e^{y(G_\alpha^+ - \beta_0)} g_0^+ - g_0^+}{y} = (G_\alpha^+ - \beta_0) g_0^+,$$

we conclude that $(G_\alpha^+ - \beta_0) g_0^+ \leq 0$ and therefore, because $(G_0^+)^{-1}$ is positive, the first inequality in (4.17) is valid.

Similarly, by Lemma 1.4.2, for $\varphi > 0$,

$$E_{(e,\varphi)}\Big(e^{-\alpha H_0}\ g_0(X_{H_0})\Big) = \begin{pmatrix} \Pi_\alpha^- e^{\varphi G_\alpha^-} g_0^- \\ e^{\varphi G_\alpha^-} g_0^- \end{pmatrix},$$

and also, by (4.20),

$$E_{(e,\varphi)}\Big(e^{-\alpha H_0}\ g_0(X_{H_0})\Big) \leq e^{-\beta_0 \varphi}\ g_0(e).$$

By comparing the last two equations we obtain

$$e^{y(G_\alpha^- + \beta_0)} g_0^- \leq g_0^-,$$

which together with the limit

$$\lim_{y \to 0} \frac{e^{y(G_\alpha^- + \beta_0)} g_0^- - g_0^-}{y} = (G_\alpha^- + \beta_0) g_0^-,$$

implies that $(G_\alpha^- + \beta_0) g_0^- \leq 0$. Hence, because $(G_0^-)^{-1}$ is positive, the second inequality in (4.17) is also valid. □

By substituting Q^0 from (4.13) and $\Gamma^0_{\alpha-\alpha_0}$ from the previous lemma into the equation (4.12) we obtain

$$L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\beta_0 \varphi}\ G_0\ e^{-\varphi V^{-1}(Q^0 - (\alpha-\alpha_0)I)}\ \Gamma^0_{\alpha-\alpha_0}\ G_0^{-1} J_2 1}{\alpha}(e). \tag{4.21}$$

Since the matrix $\Gamma^0_{\alpha-\alpha_0}$ is defined for $\alpha \geq \alpha_0$, the previous formula extends the definition of the function $L_{(e,\varphi)}(\alpha)$ to $\alpha \geq \alpha_0$. We now prove:

Lemma 4.3.3 *For any $(e,\varphi) \in E_0^+$, the function $L_{(e,\varphi)}(\alpha)$ is analytic for $Re(\alpha) > \alpha_0$.*

Proof: By Lemma 1.4.2, the matrices Π_α^+ and Π_α^- are analytic for $Re(\alpha) > 0$ because they are, by the definition, Laplace transforms. Hence, it follows from the definition of the matrix Γ_α given in Lemma 1.4.1 that Γ_α is analytic for $Re(\alpha) > 0$and therefore, the matrix $\Gamma^0_{\alpha-\alpha_0}$ is analytic for $Re(\alpha) > \alpha_0$. Since $e^{-\varphi V^{-1}(Q^0-\alpha I)}$ is analytic for all α, we conclude that the numerator of $L_{(e,\varphi)}(\alpha)$ in (4.21) is analytic for $Re(\alpha) > \alpha_0$.

Hence, $L_{(e,\varphi)}(\alpha)$ is analytic for $Re(\alpha) > \alpha_0$ except perhaps in some neighbourhood of zero. But since

$$e^{-\beta_0\varphi} G_0 e^{-\varphi V^{-1}(Q^0+\alpha_0 I)} \Gamma^0_{-\alpha_0} G_0^{-1} J_2 1 = e^{-\varphi V^{-1} Q} \Gamma J_2 1 = 1,$$

the numerator of $L_{(e,\varphi)}(\alpha)$ is equal to zero for $\alpha = 0$, and since the numerator of $L_{(e,\varphi)}(\alpha)$ is also analytic in any neighbourhood of zero, it follows that $L_{(e,\varphi)}(\alpha)$ is analytic in some neighbourhood of zero. Therefore, $L_{(e,\varphi)}(\alpha)$ is analytic for $Re(\alpha) > \alpha_0$. □

Now we have to investigate the behaviour of $L_{(e,\varphi)}(\alpha)$ near α_0. From (4.21) we see that, because $e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0)I)}$ is a power series in $(\alpha - \alpha_0)$, we actually have to find the behaviour of $\Gamma^0_{\alpha-\alpha_0} G_0^{-1} J_2 1$ for small values of $\alpha - \alpha_0 > 0$.

By the definition of the matrices Γ^0_α and J_2,

$$\Gamma^0_{\alpha-\alpha_0} G_0^{-1} J_2 1 = \begin{pmatrix} \Pi^{0,-}_{\alpha-\alpha_0} (G_0^{-1} J_2 1)^- \\ (G_0^{-1} J_2 1)^- \end{pmatrix}.$$

Hence, the problem of finding the behaviour of $\Gamma^0_{\alpha-\alpha_0} G_0^{-1} J_2 1$ for small $\alpha-\alpha_0 > 0$ reduces to the problem of finding the behaviour of $\Pi^{0,-}_{\alpha-\alpha_0}(G_0^{-1} J_2 1)^-$ for small $\alpha - \alpha_0 > 0$.

We recall the notation " $\sim$ ": for real functions $u(x)$ and $\kappa(x)$ we say

$$u(x) \sim \kappa(x) \text{ as } x \to a \quad \text{iff} \quad \lim_{x\to a} \frac{u(x)}{\kappa(x)} = 1$$

Lemma 4.3.4 *Let all non-zero eigenvalues of the matrix $V^{-1}Q^0$ be simple. If g^- is a non-negative vector on E^- then*

$$\Pi^{0,-}_\alpha g^- - \Pi^{0,-} g^- \ \sim\ c\, \alpha^{\frac{1}{2}}\, (r^{0,+} - \Pi^{0,-} r^{0,-}), \qquad \alpha \to 0,$$

where r^0 is a vector such that $V^{-1}Q^0r^0 = 1$ and c is a non-zero constant.

Proof: Let

$$g^- = \sum_{k=1}^{m} a_k \, g_k^{0,-} \tag{4.22}$$

for some constants a_k, $k = 1, \dots m$, where $g_k^{0,-}$, $k = 1, \dots m$, are the vectors on E^- associated with the eigenvalues of the matrix $G^{0,-}$. By Lemma 1.4.11, there are no non-negative vectors on E^- which are linearly independent of the vector $g_{min}^{0,-}$. Since the vector g is non-negative, it follows that the constant a_{min} which corresponds to $g_{min}^{0,-}$ in linear combination (4.22) is not zero. Thus,

$$\Pi_\alpha^{0,-} g^- = a_{min} \, \Pi_\alpha^{0,-} 1^- + \sum_{g_k^{0,-} \neq g_{min}^{0,-}} a_k \, \Pi_\alpha^{0,-} g_k^{0,-}. \tag{4.23}$$

Lemma 3.2.1 states that in the oscillating case

$$1^+ - \Pi_\alpha^- 1^- \sim \frac{1}{\sqrt{-\mu V r}} \, \alpha^{\frac{1}{2}} \, (r^+ - \Pi^- r^-), \qquad \alpha \to 0,$$

where μ is the the invariant measure of the process $(X_t)_{t \geq 0}$ and the vector r satisfies $V^{-1}Qr = 1$. By (1.28) and Lemma 4.3.1, the matrices $G^{0,+}$ and $G^{0,-}$ are both conservative, zero is an eigenvalue of the matrix $V^{-1}Q^0$ with algebraic multiplicity two and geometric multiplicity one, and there exists a vector r^0 independent of the vector 1 such that $V^{-1}Q^0r^0 = 1$. Therefore, by Lemma 3.2.1,

$$1^+ - \Pi_\alpha^{0,-} 1^- \sim \frac{1}{\sqrt{-\mu^0 V r^0}} \, \alpha^{\frac{1}{2}} \, (r^{0,+} - \Pi^{0,-} r^{0,-}), \qquad \alpha \to 0, \tag{4.24}$$

where $\mu^0 Q^0 = 0$.

We also need the behaviour of $\Pi_\alpha^{0,-} g_k^{0,-}$, $k = 1, \dots m$, $g_k^{0,-} \neq g_{min}^{0,-}$. Since all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, all vectors g_k^0, $k = 1, \dots m$, are the eigenvectors of the matrix $V^{-1}Q^0$. Let β be a non-zero eigenvalue of $V^{-1}Q^0$ and u its associated eigenvector. By Theorem 1.8.1, there exists a simple eigenvalue $\beta(\alpha)$ of the matrix $V^{-1}(Q^0 - \alpha I)$ that tends to β as $\alpha \to 0$ and that for small $|\alpha|$ can be represented by a convergent power series in α. It can also be shown (see Wilkinson [14]) that the eigenvector $u(\alpha)$ of the matrix $V^{-1}(Q^0 - \alpha I)$ associated with the eigenvalue $\beta(\alpha)$ tends to the eigenvector u of $V^{-1}Q^0$ as $\alpha \to 0$ and that for small $|\alpha|$, $u(\alpha)$ can be represented by a convergent power series in α.

Let

$$u(\alpha) = u + \sum_{n=1}^{\infty} v_n \, \alpha^n,$$

for some vectors v_n on E, $n \in \mathbb{N}$. From equality (1.5) we have that $\Pi_\alpha^{0,-} u^-(\alpha) = u^+(\alpha)$. Hence,

$$\Pi_\alpha^{0,-}(u^- + \sum_{n=1}^{\infty} v_n^- \, \alpha^n) = u^+ + \sum_{n=1}^{\infty} v_n^+ \, \alpha^n,$$

which implies that,

$$\Pi_\alpha^{0,-} u^- = u^+ + \sum_{n=1}^{\infty} \alpha^n \, (v_n^+ - \Pi_\alpha^{0,-} v_n^-).$$

The last equation applies to all eigenvectors g_k^0, $k = 1, \dots m$, $g_k^{0,-} \neq g_{min}^{0,-}$, of the matrix $V^{-1}Q^0$. For fixed k, $k = 1, \dots m$, let $v_{k,n}$, $n \in \mathbb{N}$, be vectors on E such that

$$\Pi_\alpha^{0,-} g_k^- = g_k^+ + \sum_{n=1}^{\infty} \alpha^n \, (v_{k,n}^+ - \Pi_\alpha^{0,-} v_{k,n}^-).$$

Then, because by (1.8) $g_k^+ = \Pi^- g_k^-$, $k = 1, \dots m$,

$$\begin{aligned}
\Pi_\alpha^{0,-} g_k^- - \Pi^{0,-} g_k^+ &= \sum_{n=1}^{\infty} \alpha^n \, (v_{k,n}^+ - \Pi_\alpha^{0,-} v_{k,n}^-) \\
&= \sum_{n=1}^{\infty} \alpha^n \left((v_{k,n}^+ - \Pi^{0,-} v_{k,n}^-) + (\Pi^{0,-} - \Pi_\alpha^{0,-}) v_{k,n}^- \right). \qquad (4.25)
\end{aligned}$$

From (4.23), (4.24) and (4.25), and because Lemma 1.4.9 implies that $\Pi_\alpha^{0,-} \to \Pi^{0,-}$ as $\alpha \to 0$, we obtain

$$\begin{aligned}
\Pi_\alpha^{0,-} g^- - \Pi^{0,-} g^- &= a_{min} \, (\Pi_\alpha^{0,-} 1^- - \Pi^{0,-} 1^-) + \sum_{g_k^{0,-} \neq g_{min}^{0,-}} a_k \, (\Pi_\alpha^{0,-} - \Pi^{0,-}) g_k^- \\
&\sim -\frac{a_{min}}{\sqrt{-\mu^0 V r^0}} \, \alpha^{\frac{1}{2}} \, (r^{0,+} - \Pi^{0,-} r^{0,-}), \qquad \alpha \to 0.
\end{aligned}$$

□

The previous lemma gives the behaviour of $\Pi_{\alpha}^{0,-}g^{-}$ for small $\alpha > 0$ and any non-negative vector g^{-} but under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^{0}$ are simple. This condition arises from the behaviour for small $\alpha > 0$ of the eigenvalues of the matrix $V^{-1}(Q^{0} - \alpha I)$ and their associated vectors. If the condition does not hold then the eigenvalues of $V^{-1}(Q^{0} - \alpha I)$ and their associated vectors are not necessarily power series in α and therefore, the argument used in the proof of the previous lemma is not valid any more. In the general case of the eigenvalues of the matrix $V^{-1}Q^{0}$ we are unable to find the behaviour of $\Pi_{\alpha}^{0,-}g^{-}$ for small $\alpha > 0$ and non-negative vector g^{-}. Therefore, we restrict ourselves to the case of simple non-zero eigenvalues of the matrix $V^{-1}Q^{0}$ and, by the use of the previous lemma, proceed to find the behaviour of $L_{(e,\varphi)}(\alpha)$ near α_0.

Lemma 4.3.5 *Let all non-zero eigenvalues of the matrix $V^{-1}Q^{0}$ be simple. Then, for fixed $(e, \varphi) \in E_0^{+}$,*

$$L_{(e,\varphi)}(\alpha) - L_{(e,\varphi)}(\alpha_0) \sim c\,(\alpha - \alpha_0)^{\frac{1}{2}}\, e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^{0}} J_1 \Gamma_2^{0} r^{0}(e), \quad \alpha \to \alpha_0,$$

for some non-zero constant c and

$$L_{(e,\varphi)}(\alpha_0) \equiv \lim_{\alpha \to \alpha_0} L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^{0}} \Gamma^{0} G_0^{-1} J_2 1}{\alpha_0},$$

(where " $\equiv$ " means "defined to be").

Proof: Let $L_{(e,\varphi)}(\alpha)$ be as given in (4.21). Then,

$$L_{(e,\varphi)}(\alpha_0) = \lim_{\alpha \to \alpha_0} L_{(e,\varphi)}(\alpha) = \frac{1 - e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^{0}} \Gamma^{0} G_0^{-1} J_2 1}{\alpha_0},$$

and

$$L_{(e,\varphi)}(\alpha) - L_{(e,\varphi)}(\alpha_0)$$

$$\begin{aligned}
&= \frac{1 - e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))}\Gamma^0 G_0^{-1}J_2 1}{\alpha} \\
&\quad - \frac{e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))}(\Gamma^0_{\alpha-\alpha_0} - \Gamma^0)G_0^{-1}J_2 1}{\alpha} \\
&\quad - \frac{1 - e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}Q^0}\Gamma^0 G_0^{-1}J_2 1}{\alpha_0} \\
&= - \frac{(\alpha-\alpha_0)(1 - e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}Q^0}\Gamma^0 G_0^{-1}J_2 1)}{\alpha\, \alpha_0} \\
&\quad - \frac{\alpha_0(e^{-\beta_0\varphi}G_0\, (e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))} - e^{-\varphi V^{-1}Q^0})\Gamma^0 G_0^{-1}J_2 1)}{\alpha\, \alpha_0} \\
&\quad - \frac{e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}Q^0}(\Gamma^0_{\alpha-\alpha_0} - \Gamma^0)G_0^{-1}J_2 1}{\alpha} \\
&\quad - \frac{e^{-\beta_0\varphi}G_0\, (e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))} - e^{-\varphi V^{-1}Q^0})(\Gamma^0_{\alpha-\alpha_0} - \Gamma^0)G_0^{-1}J_2 1}{\alpha}.
\end{aligned}$$

The function $\alpha \mapsto e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))}$ is analytic for all α. Hence, the function $e^{-\varphi V^{-1}(Q^0-(\alpha-\alpha_0))} - e^{-\varphi V^{-1}Q^0}$ can be written as a power series in $(\alpha - \alpha_0)$ with zero first term.

On the other hand, the vector g_0 is positive which implies that the vector $(G_0^{-1}J_2 1)^-$ is positive. Thus, by Lemma 4.3.4,

$$\begin{aligned}
(\Gamma^0_{\alpha-\alpha_0} - \Gamma^0)\; G_0^{-1}J_2 1 &= \begin{pmatrix} (\Pi^{0,-}_{\alpha-\alpha_0} - \Pi^{0,-})\; (G_0^{-1}J_2 1)^- \\ 0 \end{pmatrix} \\
&\sim c\,(\alpha-\alpha_0)^{\frac{1}{2}} \begin{pmatrix} r^{0,+} - \Pi^{0,-} r^{0,-} \\ 0 \end{pmatrix} \\
&= c\,(\alpha-\alpha_0)^{\frac{1}{2}} J_1 \Gamma_2^0 r^0, \qquad \alpha \to \alpha_0,
\end{aligned}$$

for some non-zero constant c. By Lemma 3.1.1 (i), the vector $e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0$ is positive for any $\varphi \geq 0$.. Thus,

$$L_{(e,\varphi)}(\alpha) - L_{(e,\varphi)}(\alpha_0) \sim \frac{c}{\alpha_0}\,(\alpha-\alpha_0)^{\frac{1}{2}}\, e^{-\beta_0\varphi}G_0\, e^{-\varphi V^{-1}Q^0} J_1\Gamma_2^0 r^0(e), \qquad \alpha \to \alpha_0.$$

□

The previous lemma implies that $L_{(e,\varphi)}(\alpha)$ has a many-valued singularity at α_0 which completes step 1).

Our task in step 2) is to show that $L_{(e,\varphi)}(\alpha+\alpha_0)$, $\alpha > 0$, is the Laplace transform of $e^{-\alpha_0 t}P_{(e,\varphi)}(H_0 > t)$. First we show that $L_{(e,\varphi)}(\alpha+\alpha_0)$, $\alpha > 0$, is a Laplace transform. For that we use a theorem from Feller [14] part 2, XIII.4 which establishes a necessary and sufficient condition for a function to be a Laplace transform of some measure. Before we state the theorem that we are going to use we quote the definition of a completely monotone function given in Feller [14] part 2, XIII.4 as Definition 1.

Definition 4.3.1 *A function f on $[0, +\infty)$ is completely monotone if it posseses derivatives $f^{(n)}$ of all orders and*

$$(-1)^n f^{(n)}(\alpha) \geq 0, \quad \alpha > 0.$$

Now we quote the theorem which is stated and proved in Feller [14] part 2, XIII.4 as Theorem 1a.

Theorem 4.3.1 *A function f on $[0, +\infty)$ is completely monotone iff it is of the form*

$$f(\lambda) = \int_0^\infty e^{-\lambda x} F(dx), \qquad \lambda > 0,$$

where F is a (not necessarily finite) measure on $[0, +\infty)$

Thus, in order to show that $L_{(e,\varphi)}(\alpha+\alpha_0)$, $\alpha > 0$, is a Laplace transform of some measure, by Theorem 4.3.1 it is enough to show that $L_{(e,\varphi)}(\alpha+\alpha_0)$ is completely monotone.

Lemma 4.3.6 *For any $(e, \varphi) \in E_0^+$, the function $L_{(e,\varphi)}(\alpha+\alpha_0)$, $\alpha > 0$, is completely monotone.*

Proof: By Definition 4.3.1 we have to show that $L_{(e,\varphi)}(\alpha+\alpha_0)$ is analytic for $\alpha > 0$ and that the derivatives of $L_{(e,\varphi)}(\alpha+\alpha_0)$ for $\alpha > 0$ alternate in sign.

We know from Lemma 4.3.3 that the function $L_{(e,\varphi)}(\alpha)$ is analytic for $\alpha > \alpha_0$. Hence, $L_{(e,\varphi)}(\alpha+\alpha_0)$ is analytic for $\alpha > 0$. In addition, by the definition, the function $L_{(e,\varphi)}(\alpha)$

for $\alpha > 0$ is the Laplace transform of $P_{(e,\varphi)}(H_0 > t)$. Hence, by Theorem 4.3.1 the derivatives of $L_{(e,\varphi)}(\alpha)$ for $\alpha \geq 0$ alternate in sign which implies that the derivatives of $L_{(e,\varphi)}(\alpha + \alpha_0)$ for $\alpha \geq -\alpha_0$ alternate in sign. Therefore, all we have to prove is that the derivatives of $L_{(e,\varphi)}(\alpha + \alpha_0)$ for $0 < \alpha < -\alpha_0$ alternate in sign.

By Lemma 4.3.3, the function $L_{(e,\varphi)}(\alpha)$ is analytic for $\alpha > \alpha_0$. Thus, it admits the Taylor expansion at zero

$$L_{(e,\varphi)}(\alpha) = \sum_{n=0}^{\infty} \alpha^n \, \frac{L_{(e,\varphi)}^{(n)}(0)}{n!}$$

which converges for all $\alpha > \alpha_0$.

For any $k \in \mathbb{N}$ and $\alpha > \alpha_0$, the k^{th} derivative of $L_{(e,\varphi)}(\alpha)$ is given by

$$L_{(e,\varphi)}^{(k)}(\alpha) = \sum_{n=k}^{\infty} n(n-1)\ldots(n-k+1) \, \alpha^{n-k} \, \frac{L_{(e,\varphi)}^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \alpha^n \, \frac{L_{(e,\varphi)}^{(n+k)}(0)}{n!}.$$

Since $L_{(e,\varphi)}(\alpha)$ for $\alpha > 0$ is a Laplace transform, the derivatives $L_{(e,\varphi)}^{(n)}(0)$, $n \in \mathbb{N}$, alternate in sign. Hence, $L_{(e,\varphi)}^{(n)}(0) = (-1)^n |L_{(e,\varphi)}^{(n)}(0)|$, $n \in \mathbb{N}$, and

$$L_{(e,\varphi)}^{(k)}(\alpha) = \sum_{n=0}^{\infty} (-1)^{n+k} \alpha^n \, \frac{|L_{(e,\varphi)}^{(n+k)}(0)|}{n!}, \qquad k \in \mathbb{N}, \; \alpha > \alpha_0.$$

Let $\alpha_0 < \alpha < 0$ and $k \in \mathbb{N}$. Then

$$L_{(e,\varphi)}^{(k)}(\alpha) = (-1)^k \sum_{n=0}^{\infty} (-\alpha)^n \, \frac{|L_{(e,\varphi)}^{(n+k)}(0)|}{n!} \begin{cases} > 0, & k \text{ even} \\ < 0, & k \text{ odd.} \end{cases}$$

Therefore, the derivatives of $L_{(e,\varphi)}(\alpha)$ for $\alpha_0 < \alpha < 0$ alternate in sign which implies that the derivatives of $L_{(e,\varphi)}(\alpha + \alpha_0)$ for $0 < \alpha < -\alpha_0$ alternate in sign. Thus, the proof of the lemma is complete. □

By Theorem 4.3.1 and the previous lemma, $L_{(e,\varphi)}(\alpha + \alpha_0)$, $\alpha > 0$, is a Laplace transform of some measure ν on $[0, +\infty)$ and can be written as

$$L_{(e,\varphi)}(\alpha + \alpha_0) = \int_0^{\infty} e^{-\alpha t} \nu(dt), \qquad \alpha > 0.$$

On the other hand, by the definition of $L_{(e,\varphi)}(\alpha)$ for $\alpha > 0$ from the beginning of the section we have that

$$L_{(e,\varphi)}(\alpha+\alpha_0) = \int_0^\infty e^{-(\alpha+\alpha_0)t} P_{(e,\varphi)}(H_0 > t)\, dt, \qquad \alpha > -\alpha_0.$$

Therefore, by the uniqueness of the inverse of the Laplace transform we conclude that $\nu(dt) = e^{-\alpha_0 t} P_{(e,\varphi)}(H_0 > t)\, dt$ and that $L_{(e,\varphi)}(\alpha+\alpha_0)$ for $\alpha > 0$ is the Laplace transform of $e^{-\alpha_0 t} P_{(e,\varphi)}(H_0 > t)$. This finishes step 2).

In the last step we find the limit $\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$. Let us start with the Laplace transform

$$L_{(e,\varphi)}(\alpha+\alpha_0) = \int_0^\infty e^{-\alpha t}\, e^{-\alpha_0 t} P_{(e,\varphi)}(H_0 > t)\, dt, \qquad \alpha > 0. \tag{4.26}$$

By Lemma 4.3.5, if all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, then

$$L_{(e,\varphi)}(\alpha+\alpha_0) - L_{(e,\varphi)}(\alpha_0) \sim c\, \alpha^{\frac{1}{2}}\, e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e), \quad \alpha \to 0,$$

for some non-zero constant c, or equivalently,

$$\frac{L_{(e,\varphi)}(\alpha+\alpha_0) - L_{(e,\varphi)}(\alpha_0)}{\alpha} \sim c\, \alpha^{-\frac{1}{2}}\, e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e), \quad \alpha \to 0. \tag{4.27}$$

By the property of Laplace transforms and (4.26),

$$\frac{L_{(e,\varphi)}(\alpha+\alpha_0)}{\alpha} = \int_0^\infty e^{-\alpha t}\Big(\int_0^t e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds\Big)\, dt, \qquad \alpha > 0,$$

and also $\frac{1}{\alpha} = \int_0^\infty e^{-\alpha t}\, dt$ for $\alpha > 0$. Thus, from (4.27) we deduce

$$\begin{aligned}
\frac{L_{(e,\varphi)}(\alpha+\alpha_0)}{\alpha} - \frac{L_{(e,\varphi)}(\alpha_0)}{\alpha}& \\
&= \int_0^\infty e^{-\alpha t}\Big(\int_0^t e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds - L_{(e,\varphi)}(\alpha_0)\Big)\, dt \\
&\sim c\,(\alpha)^{-\frac{1}{2}}\, e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e), \qquad \alpha \to 0.
\end{aligned}$$

Since $e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)$ is positive, the function

$$t \mapsto \int_0^t e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds - L_{(e,\varphi)}(\alpha_0)$$

is monotone. Thus, we can apply Theorem 3.2.1 to the Laplace transform

$$\frac{L_{(e,\varphi)}(\alpha+\alpha_0)}{\alpha} - \frac{L_{(e,\varphi)}(\alpha_0)}{\alpha}, \qquad \alpha > 0,$$

to obtain

$$\begin{aligned} &\int_0^t e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds - L_{(e,\varphi)}(\alpha_0) \\ &\sim \frac{c}{\Gamma(\frac{1}{2})}\, t^{-\frac{1}{2}} e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e), \quad t \to +\infty. \end{aligned}$$

Then, for fixed $(e,\varphi), (e',\varphi') \in E_0^+$,

$$\begin{aligned} &\lim_{T\to+\infty} \frac{\int_0^{T-t} e^{-\alpha_0 s} P_{(e',\varphi')}(H_0 > s)\, ds - L_{(e',\varphi')}(\alpha_0)}{\int_0^T e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds - L_{(e,\varphi)}(\alpha_0)} \\ &= \frac{e^{-\beta_0\varphi'} G_0\, e^{-\varphi' V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e')}{e^{-\beta_0\varphi} G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e)}. \end{aligned} \tag{4.28}$$

Moreover, by L'Hôpital's rule,

$$\begin{aligned} &\lim_{T\to+\infty} \frac{\int_0^{T-t} e^{-\alpha_0 s} P_{(e',\varphi')}(H_0 > s)\, ds - L_{(e',\varphi')}(\alpha_0)}{\int_0^T e^{-\alpha_0 s} P_{(e,\varphi)}(H_0 > s)\, ds - L_{(e,\varphi)}(\alpha_0)} \\ &= e^{\alpha_0 t} \lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T-t)}{P_{(e,\varphi)}(H_0 > T)}, \end{aligned} \tag{4.29}$$

if the latter limit exists.

Let h_{r^0} be a function on $E \times \mathbb{R} \times [0,+\infty)$ defined by

$$h_{r^0}(e,\varphi,t) = e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, G_0 e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e).$$

Then, it follows from (4.28) and (4.29) that

$$\text{if } \lim_{T\to+\infty} \tfrac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)} \text{ exists then it is equal to } \tfrac{h_{r^0}(e',\varphi',t)}{h_{r^0}(e,\varphi,0)}. \tag{4.30}$$

We remind ourselves that the last statement is valid under the assumption that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple. Therefore, under the same assumption we prove weak convergence of the measures $(P_{(e,\varphi)}^T|_{\mathcal{F}_t})_{T\geq 0}$.

Theorem 4.3.2 *For fixed $(e,\varphi) \in E_0^+$, let a measure $P_{(e,\varphi)}^{r^0}$ be such that the process $(X_t,\varphi_t)_{t\geq 0}$ under the measure $P_{(e,\varphi)}^{r^0}$ is the $h-$transform with the function h_{r^0} of the killed process $(X_t,\varphi_t)_{t\geq 0}$. The function h_{r^0} is given by*

$$h_{r^0}(e',y,t) = e^{-\alpha_0 t}e^{-\beta_0 y}\ G_0\ e^{-yV^{-1}Q^0}J_1\Gamma_2^0 r^0(e'), \quad (e',y,t) \in E \times \mathbb{R} \times [0,+\infty),$$

and the killed process $(X_t,\varphi_t)_{t\geq 0}$ is the process $(X_t,\varphi_t,t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero.

More precisely, for any $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P_{(e,\varphi)}^{r^0}(A) = \frac{E_{(e,\varphi)}\Big(I(A)h_{r^0}(X_t,\varphi_t,t)I\{t < H_0\}\Big)}{h_{r^0}(e,\varphi,0)}.$$

For fixed $t \geq 0$, let $P_{(e,\varphi)}^{r^0}|_{\mathcal{F}_t}$ be the restriction of $P_{(e,\varphi)}^{r^0}$ to $\mathcal{F}_t$. If all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple and if, for $t \geq 0$ and $(e,\varphi),(e',\varphi') \in E_0^+$, the limit $\lim_{T\to+\infty}\frac{P_{(e',\varphi')}(H_0>T-t)}{P_{(e,\varphi)}(H_0>T)}$ exists, then the measures $(P_{(e,\varphi)}^{T}|_{\mathcal{F}_t})_{T\geq 0}$ converge weakly to the probability measure $P_{(e,\varphi)}^{r^0}|_{\mathcal{F}_t}$ as $T \to \infty$.

Proof: First we prove that $P_{(e,\varphi)}^{r^0}$ is a probability measure. By (1.28), there exists a vector r^0 independent of the vector 1 such that $V^{-1}Q^0 r^0 = 1$, and by Lemma 4.1.2 (i), $e^{-\varphi V^{-1}Q^0}J_1\Gamma_2^0 r^0 > 0$. In addition, G_0 is positive. Therefore, the function $h_{r^0}(e,\varphi)$ is positive which implies that $P_{(e,\varphi)}^{r^0}$ is well-defined and that it is a measure. To show that $P_{(e,\varphi)}^{r^0}$ is a probability measure we need to show that the process $\{h_{r^0}(X_t,\varphi_t,t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$.

By (4.13) the function $h_{r^0}(e,\varphi,t)$ can be rewritten as

$$h_{r^0}(e,\varphi,t) = e^{-\alpha_0 t}\ e^{-\varphi V^{-1}(Q-\alpha_0 I)}G_0 J_1\Gamma_2^0 r^0(e).$$

Thus, the function $h_{r^0}(e,\varphi,t)$ is continuously differentiable in φ and t which, by Lemma 1.6.1, implies that $h_{r^0}(e,\varphi,t)$ is in the domain of the infinitesimal generator $\mathcal{A}$ of the process $(X_t,\varphi_t,t)_{t\geq 0}$ and that $\mathcal{A}h_{r^0} = 0$. Thus, the process $(h_{r^0}(X_t,\varphi_t,t))_{t\geq 0}$ is a local martingale under $P_{(e,\varphi)}$ and therefore the stopped process $(h_{r^0}(X_{t\wedge H_0}, \varphi_{t\wedge H_0}, t\wedge H_0))_{t\geq 0}$ is also a local martingale under $P_{(e,\varphi)}$. By the definition of the function h_{r^0}, if the process $(X_t, \varphi_t)_{t\geq 0}$ starts in E_0^+ then $h_{r^0}(X_{H_0}, \varphi_{H_0}, H_0) = 0$. Hence,

$$h_{r^0}(X_t, \varphi_t, t)I\{t < H_0\} = h_{r^0}(X_{t\wedge H_0}, \varphi_{t\wedge H_0}, t\wedge H_0), \quad t \geq 0,$$

and we conclude that the process $\{h_{r^0}(X_t, \varphi_t, t)I\{t < H_0\}, t \geq 0\}$ is a local martingale under $P_{(e,\varphi)}$. Since it is also bounded on every finite interval, we get that the process $\{h_{r^0}(X_t, \varphi_t, t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Therefore, $P^{r^0}_{(e,\varphi)}$ is a probability measure.

It remains to prove weak convergence of the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$. It will be proved by the use of Lemma 3.1.2 in the same way as in the second part of Theorem 3.2.2.

For fixed $(e, \varphi) \in E_0^+$ and $t, T \geq 0$, let $h_T(e, \varphi, t)$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P_{(e,\varphi)})$ by

$$h_T(e, \varphi, t) = \frac{P_{(X_t,\varphi_t)}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)}\, I\{t < H_0\}.$$

The random variables $h_T(e, \varphi, t)$, $t \geq 0$, are non-negative and, by (4.30),

$$\lim_{T\to+\infty} h_T(e, \varphi, t) = \frac{h_{r^0}(X_t, \varphi_t, t)}{h_{r^0}(e, \varphi, 0)}\, I\{t < H_0\}, \quad a.s..$$

In addition, by (4.11),

$$E_{(e,\varphi)}\Big(h_T(e, \varphi, t)\Big) = 1, \quad t \geq 0,$$

and because the process $\{h_{r^0}(X_t, \varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$,

$$E_{(e,\varphi)}\Big(\frac{h_{r^0}(X_t, \varphi_t, t)}{h_{r^0}(e, \varphi, 0)} I\{t < H_0\}\Big) = 1, \quad t \geq 0.$$

By Lemma 3.1.2 (i) applied to the random variables $\{h_T(e, \varphi, t), T \geq 0\}$, we get that the random variables $h_T(e, \varphi, t)$ converge to $\frac{h_{r^0}(X_t,\varphi_t,t)}{h_{r^0}(e,\varphi,0)} I\{t < H_0\}$ in $L^1(\Omega, \mathcal{F}, P_{(e,\varphi)})$ as $T \to +\infty$.

Hence, from (4.11) we obtain, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\lim_{T\to+\infty} P^T_{(e,\varphi)}(A) = \lim_{T\to+\infty} E_{(e,\varphi)}\Big(I(A)h_T(e,\varphi,t)\Big)$$
$$= E_{(e,\varphi)}\Big(I(A)\frac{h_{r^0}(X_t,\varphi_t,t)}{h_{r^0}(e,\varphi,0)}I\{t < H_0\}\Big) = P^{r^0}_{(e,\varphi)}(A),$$

which, by Lemma 3.1.2 (ii), implies that the measures $(P^T_{(e,\varphi)}|_{\mathcal{F}_t})_{y\geq 0}$ converge weakly to the measure $P^{r^0}_{(e,\varphi)}|_{\mathcal{F}_t}$ as $T \to \infty$. □

4.4 Conditioning the process $(\varphi_t)_{t\geq 0}$ to oscillate

In Section 4.2 we have seen that one way of conditioning the process $(\varphi_t)_{t\geq 0}$ to stay non-negative is first to condition it to drift to $+\infty$, and then condition the resulting process to stay non-negative.

Similarly, we can condition the process $(\varphi_t)_{t\geq 0}$ with a negative drift to oscillate, and then condition the resulting oscillating process to stay non-negative. In this section we shall investigate this way of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$. We are interested in the final result of these two conditionings performed in the specified order.

First we want to condition the process $(X_t, \varphi_t)_{t\geq 0}$ in such a way that the process $(\varphi_t)_{t\geq 0}$ changes its behaviour to oscillating. By Theorem 4.2.1, in order to change the behaviour of the process $(\varphi_t)_{t\geq 0}$ from a negative to a positive drift, taking the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < +\infty\}$ results in making an h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$. Therefore, in order to obtain the process $(\varphi_t)_{t\geq 0}$ which oscillates from the process $(\varphi_t)_{t\geq 0}$ which has a negative drift, we shall again look for an h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$.

We want to find a function h such that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and that the process $(\varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ oscillates. By Theorem 1.9.1, there does not exist such function h defined on $E \times \mathbb{R}$. But, if we let the function h be defined on

$E \times \mathbb{R} \times [0, +\infty)$, then, by Theorem 1.9.2, there exists exactly one function h such that the process $(X_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is Markov and that the process $(\varphi_t)_{t\geq 0}$ under $P^h_{(e,\varphi)}$ is oscillating. That function h is given by

$$h_0(e, \varphi, t) = e^{-\alpha_0 t} e^{-\beta_0 \varphi} g_0(e),$$

and β_0 is the argmin of $\alpha(\cdot)$, where $\alpha(\beta)$ is the Perron-Frobenius eigenvalue of the matrix $(Q - \beta V)$ and α_0 and g_0 are the Perron-Frobenius eigenvalue and eigenvector, respectively, of the matrix $(Q - \beta_0 V)$.

For fixed $(e, \varphi) \in E_0^+$, let a measure $P^{h_0}_{(e,\varphi)}$ be defined by

$$P^{h_0}_{(e,\varphi)}(A) = \frac{E_{(e,\varphi)}\Big(I(A) h_0(X_t, \varphi_t, t)\Big)}{h_0(e, \varphi, 0)}, \quad A \in \mathcal{F}_t,\ t \geq 0. \tag{4.31}$$

Then, by the results in Section 1.9, the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov with the Q-matrix Q^0 given by (4.13) and, by Theorem 1.9.2, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates.

Therefore, we have obtained the process $(\varphi_t)_{t\geq 0}$ that oscillates, that is the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$. Our aim now is to condition the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative. Conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ in the oscillating case on the event that $(\varphi_t)_{t\geq 0}$ stays non-negative has been discussed in Chapter 3. Since, by Lemma 4.3.1, the Q-matrix Q^0 of the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is conservative and irreducible, and, the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates, we can apply results of Chapter 3 to the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ to prove

Theorem 4.4.1 *For fixed $(e, \varphi) \in E_0^+$, let a measure $P^{h_0, h_r^0}_{(e,\varphi)}$ be such that the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0, h_r^0}_{(e,\varphi)}$ is the $h-$transform with the function h_r^0 of the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero. The function h_r^0 given by*

$$h_r^0(e', y) = e^{-yV^{-1}Q^0} J_1 \Gamma_2 r^0(e'), \qquad (e', y) \in E \times \mathbb{R},$$

where r^0 is a vector which satisfies $V^{-1}Q^0r^0 = 1$.

More precisely, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \frac{E^{h_0}_{(e,\varphi)}\Big(I(A)h_r^0(X_t,\varphi_t)I\{t < H_0\}\Big)}{h_r^0(e,\varphi)}.$$

Then, $P^{h_0,h_r^0}_{(e,\varphi)}$ is a probability measure.

In addition, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A \mid H_0 > T),$$

and

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = P^{r^0}_{(e,\varphi)}(A),$$

where $P^{r^0}_{(e,\varphi)}$ is as defined in Theorem 4.3.2.

Proof: It has been shown in Chapter 3 that for $(e,\varphi) \in E_0^+$ the event $\{H_0 = +\infty\}$ under $P_{(e,\varphi)}$ is of zero probability and that instead of conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on it we can consider limits of conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on two different approximations of the event $\{H_0 = +\infty\}$. By Theorems 3.1.1 and 3.2.2, the two suggested approximations of the event $\{H_0 = +\infty\}$ yield the same result. Therefore, by applying these results to the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$, we get that $P^{h_0,h_r^0}_{(e,\varphi)}$ is a probability measure and that

$$P^{h_0,h_r^0}_{(e,\varphi)}(A) = \lim_{y\to\infty} P^{h_0}_{(e,\varphi)}(A|H_y < H_0) = \lim_{T\to\infty} P^{h_0}_{(e,\varphi)}(A|H_0 > T).$$

In addition, by definition (4.31) of the measure $P^{h_0}_{(e,\varphi)}$ we obtain, for $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\begin{aligned} P^{h_0,h_r^0}_{(e,\varphi)}(A) &= \frac{E^{h_0}_{(e,\varphi)}\Big(I(A)h_r^0(X_t,\varphi_t)I\{t < H_0\}\Big)}{h_r^0(e,\varphi)} \\ &= \frac{E_{(e,\varphi)}\Big(I(A)\ h_0(X_t,\varphi_t,t)\ h_r^0(X_t,\varphi_t)\ I\{t < H_0\}\Big)}{h_0(e,\varphi,0)\ h_r^0(e,\varphi)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{E_{(e,\varphi)}\Big(I(A)\; h_{r^0}(X_t,\varphi_t,t)\; I\{t < H_0\}\Big)}{h_{r^0}(e,\varphi,0)} \\
&= P^{r^0}_{(e,\varphi)}(A),
\end{aligned}
$$

since

$$
\begin{aligned}
h_0(e,\varphi,t)\; h^0_r(e,\varphi) &= e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, g_0(e)\;\; e^{-\varphi V^{-1}Q^0} J_1\Gamma^0_2 r^0(e) \\
&= e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, G_0\; e^{-\varphi V^{-1}Q^0} J_1\Gamma^0_2 r^0(e) \\
&= h_{r^0}(e,\varphi,t),
\end{aligned}
$$

where $h_{r^0}(e,\varphi,t)$ is as defined in Theorem 4.3.2. □

By Theorem 4.3.2, under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, the process $(X_t,\varphi_t)_{t\geq 0}$ under $P^{r^0}_{(e,\varphi)}$ is the limiting process as $T \to +\infty$ in conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on the event $\{H_0 > T\}$. Thus, the last theorem states that transforming the process $(X_t,\varphi_t)_{t\geq 0}$ to obtain the oscillating process $(\varphi_t)_{t\geq 0}$, and then conditioning the resulting oscillating process on any of the two approximations of the event $\{H_0 = +\infty\}$, give the same result as taking the limit as $T \to +\infty$ of conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on the event $\{H_0 > T\}$.

Chapter 5

Conclusions

In Section 1.10 and Chapters 3 and 4, depending on the behaviour of the process $(\varphi_t)_{t\geq 0}$, we have discussed conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event that the process $(\varphi_t)_{t\geq 0}$ stays non-negative, that is on the event $\{H_0 = +\infty\}$. Here is a summary of the results.

In the case of positive drift of the process $(\varphi_t)_{t\geq 0}$ (Section 1.10), the event $\{H_0 = +\infty\}$ is of positive probability and conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on it is well-defined. The resulting conditioned process is the h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$ killed when the process $(\varphi_t)_{t\geq 0}$ crosses zero with the function

$$h(e, \varphi) = P_{(e,\varphi)}(H_0 = +\infty) = e^{-\varphi V^{-1} Q} J_1 \Gamma_2 1(e).$$

In the oscillating and in the negative drift case (Chapters 3 and 4), the event $\{H_0 = +\infty\}$ is of zero probability and we cannot condition the process $(X_t, \varphi_t)_{t\geq 0}$ on it in the standard way. Instead, we consider limits of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on two approximations of the event $\{H_0 = +\infty\}$. One approximation is by the events that the process $(\varphi_t)_{t\geq 0}$ hits large levels y before it crosses zero, that is by the events $\{H_y < H_0\}$, $y > 0$, and another approximation is by the events that the process $(\varphi_t)_{t\geq 0}$

stays non-negative for long times, that is by the events $\{H_0 > T\}$, $T > 0$. If the limits

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} \quad \text{and} \quad \lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)} \tag{5.1}$$

exist, and if the processes

$$\left\{ \lim_{y\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} I\{t < H_0\},\ t \geq 0 \right\}$$

and

$$\left\{ \lim_{y\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)} I\{t < H_0\},\ t \geq 0 \right\}$$

are martingales under $P_{(e,\varphi)}$, then, for fixed $t \geq 0$ and $A \in \mathcal{F}_t$,

$$\lim_{y\to+\infty} P_{(e,\varphi)}(A|H_y < H_0) = E_{(e,\varphi)}\Big(I(A)I\{t < H_0\} \lim_{y\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} \Big)$$

and

$$\lim_{T\to+\infty} P_{(e,\varphi)}(A|H_0 > T) = E_{(e,\varphi)}\Big(I(A)I\{t < H_0\} \lim_{T\to+\infty} \frac{P_{(X_t,\varphi_t)}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)} \Big).$$

In the oscillating case, both limits in (5.1) exists and

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)} = \frac{h_r(e'\varphi')}{h_r(e,\varphi)},$$

where $h_r(e,\varphi) = e^{-\varphi V^{-1}Q} J_1\Gamma_2 r(e)$ and the process $\{h_r(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Hence, the limiting process in conditioning the process $(X_t,\varphi_t)_{t\geq 0}$ on any of the two approximations of the event $\{H_0 = +\infty\}$ is the h-transform of the killed process $(X_t,\varphi_t)_{t\geq 0}$ with the function h_r.

In the negative drift case, the first limit in (5.1) exists and

$$\lim_{y\to+\infty} \frac{P_{(e',\varphi')}(H_y < H_0)}{P_{(e,\varphi)}(H_y < H_0)} = \frac{h_{f_{max}}(e'\varphi')}{h_{f_{max}}(e,\varphi)},$$

where the function $h_{f_{max}}$ is given by $h_{f_{max}}(e,\varphi) = e^{-\varphi V^{-1}Q} J_1\Gamma_2 f_{max}(e)$ and the process $\{h_{f_{max}}(X_t,\varphi_t)I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Thus, the limiting process in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_y < H_0\}$, $y > 0$, is the h-transform of the killed process $(X_t, \varphi_t)_{t\geq 0}$ with the function $h_{f_{max}}$.

The functions h, h_r and $h_{f_{max}}$ determine, in the positive drift case, the oscillating case and the negative drift case, respectively, the h-transforms of the killed process $(X_t, \varphi_t)_{t\geq 0}$ corresponding to the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < H_0\}$. By comparing the functions h, h_r and $h_{f_{max}}$, we see that they are all of the same form. Moreover, since in the positive drift case $f_{max} = 1$, the functions h and $h_{f_{max}}$ are the same function.

For the second limit in (5.1) in the negative drift case, we are unable to show its existence, but if it exists and if all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, then

$$\lim_{T\to+\infty} \frac{P_{(e',\varphi')}(H_0 > T - t)}{P_{(e,\varphi)}(H_0 > T)} = \frac{h_r^0(e'\varphi', t)}{h_r^0(e, \varphi, 0)},$$

where the function h_r^0 is given by $h_r^0(e, \varphi, t) = e^{-\alpha_0 t}\, e^{-\beta_0\varphi}\, G_0\, e^{-\varphi V^{-1}Q^0} J_1 \Gamma_2^0 r^0(e)$, and the process $\{h_r^0(X_t, \varphi_t, t) I\{t < H_0\}, t \geq 0\}$ is a martingale under $P_{(e,\varphi)}$. Thus, the limiting process in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_0 > T\}$, $T > 0$, is the h-transform of the killed process $(X_t, \varphi_t, t)_{t\geq 0}$ with the function h_r^0.

The functions h_r and h_r^0 correspond, in the oscillating and in the negative drift case, respectively, to the limits as $T \to +\infty$ of of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_0 > T\}$. Since in the oscillating case $\alpha_0 = \beta_0 = 0$, $Q^0 = Q$, $\Gamma_2^0 = \Gamma_2$ and $r^0 = r$, by substituting these values into h_r^0 we obtain h_r. Hence, h_r and h_r^0 are the same function.

In the negative drift case, making the h-transform of the process $(X_t, \varphi_t)_{t\geq 0}$ with the function $h_{max}(e, \varphi) = e^{-\alpha_{max}\varphi} f_{max}(e)$ yields the probability measure $P_{(e,\varphi)}^{h_{max}}$ such that the process $(X_t)_{t\geq 0}$ under $P_{(e,\varphi)}^{h_{max}}$ is Markov and that the process $(\varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^{h_{max}}$ has a positive drift. The process $(X_t, \varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}^{h_{max}}$ can also be seen as the limiting process as $y \to +\infty$ in conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < +\infty\}$.

Further conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_{max}}_{(e,\varphi)}$ on the event $\{H_0 = +\infty\}$ yields the same result as the limit as $y \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the event $\{H_y < H_0\}$. In other words, the diagram in Figure 5.1. commutes.

$$
\begin{tikzcd}
\begin{array}{l}(X_t,\varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)} \\ \text{(the negative drift case)}\end{array} \arrow[rr, "{\{H_y < H_0\},\ y\to+\infty}"] \arrow[dr, "{\{H_y < +\infty\},\ y\to+\infty}"'] & & (X_t,\varphi_t)_{t\geq 0} \text{ under } P^{f_{max}}_{(e,\varphi)} \\
& \begin{array}{c}(X_t,\varphi_t)_{t\geq 0} \text{ under } P^{h_{max}}_{(e,\varphi)} \\ \text{(the positive drift case)}\end{array} \arrow[ur, "{\{H_0 = +\infty\}}"'] &
\end{tikzcd}
$$

Figure 5.1: The negative drift case of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_y < H_0\}$, $y \geq 0$.

On the other hand, again in the negative drift case, making the h-transform of the process $(X_t, \varphi_t, t)_{t\geq 0}$ with the function $h_0(e, \varphi) = e^{-\alpha_0\varphi}e^{-\beta_0\varphi}g_0(e)$ yields the probability measure $P^{h_0}_{(e,\varphi)}$ such that the process $(X_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ is Markov and that the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates. Under the condition that all non-zero eigenvalues of the matrix $V^{-1}Q^0$ are simple, further conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ on the event $\{H_0 = +\infty\}$ (which, because the process $(\varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ oscillates, is the limit of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ under $P^{h_0}_{(e,\varphi)}$ on any of the two approximations of the event $\{H_0 = +\infty\}$), gives the same result as the limit as $T \to +\infty$ of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ under $P_{(e,\varphi)}$ on the event $\{H_0 > T\}$. Hence, the diagram in Figure 5.2 commutes.

$$\begin{tikzcd}[column sep=large]
\begin{array}{l}(X_t, \varphi_t)_{t\geq 0} \text{ under } P_{(e,\varphi)} \\ \text{(the negative drift case)}\end{array} \arrow[rr, "{\{H_0 > T\},\ T \to +\infty}"] \arrow[dr] & & (X_t, \varphi_t)_{t\geq 0} \text{ under } P^{r^0}_{(e,\varphi)} \\
& \begin{array}{l}(X_t, \varphi_t)_{t\geq 0} \text{ under } P^{h_0}_{(e,\varphi)} \\ \text{(the oscillating case)}\end{array} \arrow[ur, swap, "{\begin{array}{l}\{T < H_0\},\ T \to +\infty \\ \{H_y < H_0\},\ y \to +\infty\end{array}}"] &
\end{tikzcd}$$

Figure 5.2: The negative drift case of conditioning the process $(X_t, \varphi_t)_{t\geq 0}$ on the events $\{H_0 > T\}$, $T \geq 0$.

The two described commuting diagrams had been proved by Bertoin, Doney [4] for the general random walk satisfying certain conditions. The aim of this work was to show that they are also valid for the Markov process $(X_t, \varphi_t)_{t\geq 0}$. The above results confirm that the aim has largely been achieved.

Appendix A

The Perron-Frobenius theorems

Theorem A.0.2 *(***The Perron-Frobenius Theorem for Primitive Matrices***) Suppose T is an $n \times n$ primitive matrix. Then there exists an eigenvalue r such that:*

(a) r is real and positive;

(b) with r can be associated positive left and right eigenvectors;

(c) $r > |\lambda|$ for any eigenvalue λ, $\lambda \neq r$ of T;

(d) the eigenvectors associated with r are unique to constant multiples;

(e) r is a simple root of the characteristic equation of T.

(f)

$$\lim_{k \to +\infty} \frac{T^k}{r^k} = const. \neq 0.$$

Proof: For the proof of (a) to (e) see Theorem 1.1. in Seneta [31]. For the proof of (f) see Theorem 1.2. in Seneta [31]. □

Theorem A.0.3 *(***The Perron-Frobenius Theorem for Irreducible Essentially Non-negative Matrices***) Suppose B is an $n \times n$ irreducible essentially non-negative matrix. Then there exists an eigenvalue τ such that:*

(a) τ is real;

(b) with τ are associated positive left and right eigenvectors which are unique to constant multiples;

(c) $\tau > Re(\lambda)$ for any eigenvalue λ, $\lambda \neq \tau$ of B;

(d) τ is a simple root of the characteristic equation of B;

(e) $\tau \leq 0$ if and only if there exists $y \geq 0$, $y \neq 0$, such that $By \leq 0$, in which case $y > 0$; and $\tau < 0$ if and only if there is inequality in at least one position in $By \leq 0$;

(f) $\tau < 0$ if and only if $-B^{-1} > 0$.

(g)

$$\lim_{t \to +\infty} \frac{e^{tB}}{e^{t\tau}} = wv,$$

where w and v are the positive right and left eigenvectors of B associated with τ, normed so that $vw = 1$.

Proof: For the proof of (a) to (f) see Theorem 2.6. in Seneta [31]. For the proof of (g) see Theorem 2.7. in Seneta [31]. □

Appendix B

The proofs of auxiliary lemmas

Proof of Lemma 1.8.2: Let $f(x_0)$ be a local minimum of f. Then, for some $h > 0$,

$$f(x_0) \leq f(x), \qquad x \in (x_0 - h, x_0 + h).$$

Suppose that $y > x_0 + h > x_0$. Then $x_0 + \frac{1}{2}h = tx_0 + (1-t)y$ for some $t \in (0,1)$, and

$$f(x_0 + \frac{1}{2}h) = f(tx_0 + (1-t)y) \leq tf(x_0) + (1-t)f(y). \tag{B.1}$$

Since $x_0 + \frac{1}{2}h \in (x_0 - h, x_0 + h)$, we have that

$$f(x_0) \leq f(x_0 + \frac{1}{2}h),$$

which together with (B.1) gives that $f(x_0) \leq f(y)$.

In the same way, we can show that $f(x_0) \leq f(y)$ for $y < x_0 - h < x_0$. Hence, $f(x_0)$ is a global minimum of f.

Furthermore, since $f(x_0) \leq f(x)$ for every x, we have that, for $x > x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0,$$

and for $x < x_0$,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0,$$

which implies that

$$\lim_{x\to x_{0-}} \frac{f(x)-f(x_0)}{x-x_0} \leq 0 \leq \lim_{x\to x_{0+}} \frac{f(x)-f(x_0)}{x-x_0}.$$

If the function f is differentiable at x_0, then the limits in the previous inequality are equal to $f'(x_0)$ and therefore $f'(x_0)=0$. □

Proof of Lemma 1.8.3: We want to prove that $\lim_{h\to 0} u^{right}(\beta+h) = u^{right}(\beta)$ for any $\beta \in \mathbb{R}$. Suppose that for some $\beta \in \mathbb{R}$ it is not true. Then there exists $\epsilon > 0$ and a subsequence $(u^{right}(\beta+h_n))_{n\in\mathbb{N}}$ of $(u^{right}(\beta+h))_{h\geq 0}$ such that $h_n \to 0$ when $n \to +\infty$ and that for any $N \in \mathbb{N}$, $n \geq N$ implies that $\|u^{right}(\beta+h_n) - u^{right}(\beta)\| \geq \epsilon$.

Since the vectors $u^{right}(\beta)$, $\beta \in \mathbb{R}$, are unit, they all lie on the closed unit ball in $\mathbb{R}^{|E|}$ and because the closed unit ball is compact in $\mathbb{R}^{|E|}$, any sequence of vectors $u^{right}(\beta)$, $\beta \in \mathbb{R}$, has a convergent subsequence. Let $(u^{right}(\beta+h_{n_k}))_{n_k\in\mathbb{N}}$ be a convergent subsequence of $(u^{right}(\beta+h_n))_{n\in\mathbb{N}}$.

By Lemma 1.8.1, $\alpha(\beta)$ is a continuous function . Thus, by taking the limit when $n_k \to +\infty$ of both sides in the equality

$$\begin{aligned}\Big(Q-(\beta+h_{n_k})V-\alpha(\beta+h_{n_k})I\Big)\Big(u^{right}(\beta+h_{n_k})-u^{right}(\beta)\Big)\\ = \Big(Q-\beta V-\alpha(\beta+h_{n_k})I\Big)\, u^{right}(\beta) - h_{n_k} V u^{right}(\beta),\end{aligned}$$

we get that

$$\Big(Q-\beta V-\alpha(\beta)I\Big)\Big(\lim_{n_k\to+\infty} u^{right}(\beta+h_{n_k}) - u^{right}(\beta)\Big) = 0.$$

The last equation implies that

$$\lim_{n_k\to+\infty} u^{right}(\beta+h_{n_k}) - u^{right}(\beta) = \lambda u^{right}(\beta) \tag{B.2}$$

for some $\lambda \in \mathbb{R}$, because the Perron-Frobenius eigenvalue $\alpha(\beta)$ of $(Q-\beta V)$ has algebraic multiplicity one and the associated eigenvector is $u^{right}(\beta)$. From (B.2) we have that

$$\lim_{n_k\to+\infty} u^{right}(\beta+h_{n_k}) = (\lambda+1)u^{right}(\beta), \tag{B.3}$$

and because $u^{right}(\beta)$ and the sequence $(u^{right}(\beta+h_{n_k}))_{n_k\in\mathbb{N}}$ are positive, we conclude that $\lambda+1\geq 0$.

Furthermore, (B.3) implies that

$$\lim_{n_k\to+\infty}\|u^{right}(\beta+h_{n_k})\| = |\lambda+1|\ \|u^{right}(\beta)\|.$$

Since $\|u^{right}(\beta)\|=1$ for any $\beta\in\mathbb{R}$, we get that $|\lambda+1|=1$, and because $\lambda+1\geq 0$, that $\lambda+1=1$.

Hence,

$$\lim_{n_k\to+\infty} u^{right}(\beta+h_{n_k}) = u^{right}(\beta),$$

but that contradicts our assumption that for some $\epsilon>0$ and any $N\in\mathbb{N}$, $n\geq N$ implies that $\|u^{right}(\beta+h_n)-u^{right}(\beta)\|\geq\epsilon$. Therefore, $u^{right}(\beta)$ is a continuous function of β.

$\square$

Proof of Lemma 2.3.1: Let $g(e,\varphi)$ be a function on $E\times\mathbb{R}$ defined by

$$g(e,\varphi) = h((e,\varphi),(f,y)) - G_0((e,\varphi),(f,y)).$$

Then

$$\begin{aligned}\lim_{t\to+\infty} g(X_{t\wedge H_0\wedge H_y},\varphi_{t\wedge H_0\wedge H_y}) &= \lim_{t\to+\infty}\Big(g(X_t,\varphi_t)\ I\{t<H_0\wedge H_y\}\\ &+\ g(X_{H_0},\varphi_{H_0})\ I\{H_0<t\wedge H_y\} +\ g(X_{H_y},\varphi_{H_y})\ I\{H_y<t\wedge H_0\}\Big).\end{aligned}\tag{B.4}$$

From the boundary conditions for the function h and because

$$G_0((e,0),(f,y))=0,\quad e\in E^-,\quad y>0,$$
$$G_0((e,0),(f,y))=0,\quad e\in E^+,\quad y<0,$$

we have that

$$g(X_{H_y},\varphi_{H_y}) = g(X_{H_0},\varphi_{H_0}) = 0.$$

Thus, the last two terms on the right-hand side of (B.4) are equal to zero.

Now we shall show that $I\{t < H_0 \wedge H_y\} \to 0$ as $t \to +\infty$ a.s. by showing first that $P_{(e,\varphi)}(H_0 \wedge H_y = +\infty) = 0$, $(e,\varphi) \in E \times (0,y)$. We recall the processes $Y^+ = (X_{\tau_y^+})_{y \geq 0}$ and $Y^- = (X_{\tau_{-y}^-})_{y\geq 0}$ and their Q-matrices G^+ and G^- respectively. Suppose that $H_0 = H_y = +\infty$ a.s.. Then both processes Y^+ and Y^- have finite lifetimes which implies that neither of the matrices G^+ and G^- is conservative. But, by Lemma 1.7.1 at least one of the matrices G^+ and G^- is conservative. Thus, $P_{(e,\varphi)}(H_0 \wedge H_y = +\infty) = 0$ which implies that the sequence $(I\{t < H_0 \wedge H_y\})_{t\geq 0}$ converges to 0 in L^1 as $t \to +\infty$ and therefore also in probability. Since the sequence $(I\{t < H_0 \wedge H_y\})_{t\geq 0}$ is decreasing in t and bounded, it converges a.s., and it must converge a.s. to the same limit as in probability, which is 0. Thus, $I\{t < H_0 \wedge H_y\} \to 0$ as $t \to +\infty$ a.s..

In addition,

$$G_0((e,\varphi),(f,y)) = \sum_{e' \in E} P_{(e,\varphi)}(X_{H_y} = e', H_y < H_0)\; G_0((e',y),(f,y)),$$

which implies that the function $(e,\varphi) \mapsto G_0((e,\varphi),(f,y))$ is bounded. Since the function h is also bounded on $E \times (0,y)$, we conclude that the function $g(e,\varphi)$ is bounded on $E \times (0,y)$, which gives that

$$\lim_{t\to+\infty} g_-(X_t,\varphi_t) I\{t < H_0 \wedge H_y\} = 0.$$

Thus, the limit in (B.4) is equal to zero.

Moreover, the process $(g(X_{t\wedge H_0 \wedge H_y}, \varphi_{t\wedge H_0\wedge H_y}))_{t\geq 0}$ is a uniformly integrable martingale when the process $(X_t,\varphi_t)_{t\geq 0}$ starts in $E \times (0,y)$. Therefore, by (B.4),

$$\begin{aligned} g(e,\varphi) &= \lim_{t\to+\infty} E\Big(g(X_{t\wedge H_0\wedge H_y}, \varphi_{t\wedge H_0\wedge H_y})\Big) \\ &= E\Big(\lim_{t\to+\infty} g(X_{t\wedge H_0\wedge H_y}, \varphi_{t\wedge H_0\wedge H_y})\Big) = 0. \end{aligned}$$

and thus,

$$h((e,\varphi),(f,y)) = G_0((e,\varphi),(f,y)), \quad (e,\varphi) \in E \times (0,y).$$

□

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