ASPECTS OF THE ZERO DIVISOR PROBLEM

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TABLE OF CONTENTS

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SUMMARY

The Zero Divisor Problem is the following:- If G is a torsion-free group and R is a commutative domain, is RG a domain? This thesis is concerned with three aspects of this problem.

After stating various background results in Chapter 1, we prove in Chapter 2 that RG is a domain if R is a commutative domain of characteristic zero, and G is a torsion-free group which is in one of various classes of groups, of which the most important is the class of abelian-by-finite groups.

If R is a commutative ring and G is a soluble group such that RG is a domain, then RG is an Ore domain, and so has a division ring of quotients. We are thus led in Chapter 3 to investigate under what circumstances group rings of generalised soluble groups have Artinian quotient rings. We obtain results for a class of groups which includes many (but not all) torsion-free soluble groups, and we show that, with appropriate assumptions on the coefficient rings, the quotient rings in question are QF-rings. Chapter 3 also includes several applications and examples.

In Chapter 4 we study the zero divisors of group rings by investigating the structure of the singular ideals. We explicitly describe the singular ideals of the group algebras of various classes of groups, (including soluble groups). The results obtained are reminiscent of results of Passman, Zalesskii et al on the structure of the Jacobson radical. We include various examples and applications.

Each chapter begins with a detailed introduction.

NOTATION

The following notation will be used throughout.

p, q	will denote prime numbers.
$\mathbb{Q}$	the rational field.
$\mathbb{Z}$	the ring of integers.
$\mathbb{Z}_p$	the field of p elements.
$\mathbb{N}$	the natural numbers.
$C(R)$	the centre of a ring R.
C_∞	the infinite cyclic group.
C_{p^∞}	the Prufer p-group.
C_p	the cyclic group of order p.
$\Delta(G)$	the FC-subgroup of G, (see Defn.1.1.8).
$\Delta^+(G)$	the torsion subgroup of $\Delta(G)$.
$\Delta^p(G)$	the group generated by the p-elements of $\Delta(G)$.
$\mathfrak{Z}(G)$	the Zalesskii subgroup of a soluble group G, (see Defn.4.3.8).
$\mathfrak{Z}^+(G)$	the torsion subgroup of $\mathfrak{Z}(G)$.

Results and definitions within a chapter are denoted by two natural numbers, separated by a decimal point. The first refers to the section of the chapter in which the result or definition is located, and the second refers to the numbering within that section. References to results and definitions in earlier (or, exceptionally, later) chapters incorporate three numbers, the first of which is the chapter number.

Equations, identities, and so on, are numbered consecutively within each chapter.

CHAPTER I

PRELIMINARIES

0. INTRODUCTION

In this chapter we give an account of some basic results which we shall need. For the most part, proofs are omitted.

1. GROUP THEORY

Our main reference for group theoretic results will be [40], and most of the material in this section can be found there.

DEFINITION 1.1 A *group theoretical class* $\mathfrak{X}$ is a class of groups satisfying

(a) if G is isomorphic to H, and H $\in \mathfrak{X}$, then G $\in \mathfrak{X}$;

(b) $\mathfrak{X}$ contains a unit group.

We shall use a fixed alphabet to denote certain group classes which feature prominently in this thesis; for example we write

$\mathcal{F}$ for the class of finite groups:

$\mathcal{C}$ for the class of cyclic groups;

$\mathcal{A}$ for the class of abelian groups;

$\mathcal{A}_0$ for the class of torsion-free abelian groups;

$\mathcal{G}$ for the class of finitely generated groups;

$\mathcal{D}$ for the class of unit groups.

Every group theoretical class is contained in the class of all groups, and the group theoretical classes are partially ordered by inclusion. We use the notation

$$\mathfrak{X} \leqslant \mathcal{Y}$$

to denote the fact that $\mathfrak{X}$ is a group theoretical subclass of the group theoretical class $\mathcal{Y}$.

DEFINITION 1.2 By an operation on the class of all group theoretical classes we mean a function A assigning to each class of groups $\mathcal{X}$ a class of groups $A\mathcal{X}$ such that

$$A\mathcal{D} = \mathcal{D}$$

and
$$\mathcal{X} \leq A\mathcal{X} \leq A\mathcal{Y} \qquad (1)$$

whenever $\mathcal{X} \leq \mathcal{Y}$. If $\mathcal{X} = A\mathcal{X}$, the class $\mathcal{X}$ is said to be A-closed.

We define a partial ordering on operations as follows: $A \leq B$ means that

$$A\mathcal{X} \leq B\mathcal{X}$$

for every class of groups $\mathcal{X}$.

DEFINITION 1.3 An operation A is called a closure operation if $A = A^2$.

It follows from (1) that if A is a closure operation the class $A\mathcal{X}$ is the uniquely determined smallest A-closed class of groups that contains $\mathcal{X}$. Clearly one may define a closure operation A by specifying the classes of groups which are A-closed.

The product of two operations A and B is formed according to the rule

$$(AB)\mathcal{X} = A(B\mathcal{X}).$$

Notice that the product of two closure operations need not be a closure operation. This leads to the following definition.

DEFINITION 1.4 Let $\{A_\lambda : \lambda \in \Lambda\}$ be a set of operations. We define

$$C = \langle A_\lambda : \lambda \in \Lambda \rangle,$$

the closure operation generated by the A_λ, to be that closure operation whose closed classes are the classes of groups which are A_λ-closed for every $\lambda \in \Lambda$.

One may easily check that for a class of groups $\mathfrak{X}$, $c\mathfrak{X}$ is simply the uniquely determined smallest class of groups $\mathfrak{Y}$ such that $\mathfrak{X} \leq \mathfrak{Y}$ and $\mathfrak{Y} = A_\lambda \mathfrak{Y}$ for all $\lambda \in \Lambda$.

The following are some examples of closure operations.

DEFINITION 1.5 If $\mathfrak{X}$ is a class of groups, the class of all groups G containing a finite chain

$$1 = H_0 \subset H_1 \subset \ldots \subset H_n = G,$$

where H_i is a subgroup of G, $H_i \lhd H_{i+1}$ and $H_{i+1}/H_i \in \mathfrak{X}$, $0 \leq i < n$, is called the class of poly-$\mathfrak{X}$ groups, and is denoted by $P\mathfrak{X}$.

Thus $P\mathfrak{A}$ is the class of soluble groups. Notice that if G is a soluble group, the derived series of G terminates in $\{1\}$, so G has a finite chain of characteristic subgroups with successive factors abelian. Since the torsion subgroup of an abelian group is characteristic, we deduce

LEMMA 1.6 Let $G \in P\mathfrak{A}$. Then there exists a chain

$$1 = H_0 \subset H_1 \subset \ldots \subset H_n = G$$

of characteristic subgroups H_i of G such that for each i, $0 \leq i < n$, $H_{i+1}/H_i \in \mathfrak{A}_0$ or H_{i+1}/H_i is periodic abelian.

DEFINITION 1.7 For a given class of groups $\mathfrak{X}$, $\acute{P}\mathfrak{X}$ will denote the class of all groups G for which there exists an ordinal ρ and a series

$$1 = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G \qquad (2)$$

such that $H_\alpha \lhd H_{\alpha+1}$, $H_{\alpha+1}/H_\alpha \in \mathfrak{X}$ for all α, $0 \leq \alpha < \rho$, and $H_{\lambda} = \bigcup_{\alpha<\lambda} H_{\alpha}$ if λ is a limit ordinal, $0 < \lambda \leqslant \rho$.

Note that $P \leqslant \rho$. A series of the form (2) is called <u>an ascending $\mathfrak{X}$-series</u> for G.

<u>DEFINITION 1.8</u> If $\mathfrak{X}$ is a class of groups, $L\mathfrak{X}$, the class of <u>locally $\mathfrak{X}$-groups</u>, consists of all groups G such that every finite subset of G is contained in an $\mathfrak{X}$-subgroup of G.

For example, $L\mathfrak{F}$ denotes the class of <u>locally finite</u> groups.

<u>DEFINITION 1.9</u>. If $\mathfrak{X}$ is a class of groups, $R\mathfrak{X}$, the class of <u>residually $\mathfrak{X}$-groups</u>, is the class of all groups G such that for some index set Λ, there exists a set $\{N_{\lambda} : \lambda \in \Lambda\}$ of normal subgroups of G such that $G/N_{\lambda} \in \mathfrak{X}$ for all $\lambda \in \Lambda$, and $\bigcap_{\lambda\in\Lambda} N_{\lambda} = 1$.

For example, $R\mathfrak{F}$ denotes the class of <u>residually finite</u> groups.

<u>DEFINITION 1.10</u>. Let H be a subgroup of a group G, and let Ω be a linearly ordered set. A <u>series</u> between H and G with order type Ω is a set of pairs of subgroups of G,

$$\mathcal{S} = \{\Lambda_{\sigma}, V_{\sigma} : \sigma \in \Omega\},$$

such that (i) $H \subseteq V_{\sigma} \lhd \Lambda_{\sigma}$ for all $\sigma \in \Omega$;

(ii) $G \setminus H = \bigcup_{\sigma\in\Omega}(\Lambda_{\sigma} \setminus V_{\sigma})$:

(iii) $\Lambda_{\tau} \subseteq V_{\sigma}$ if $\tau < \sigma$.

If $\mathfrak{X}$ is a class of groups, <u>a series with $\mathfrak{X}$-factors</u> between H and G is a series as defined above such that $\Lambda_{\sigma}/V_{\sigma} \in \mathfrak{X}$ for all $\sigma \in \Omega$.

The concepts we have so far introduced are discussed in detail in [40, Chapter 1], and we have for the most part used the notation given there.

We now state some basic results on particular classes of groups. It should be noted that we make no attempt here to present a unified picture of any aspect of infinite group theory: rather we shall include only those results which are fundamental to an understanding of subsequent chapters.

LEMMA 1.11. (O.J.Schmidt; [40, Thm.1.45]) The class of locally finite groups is closed with respect to forming extensions; that is, $P L \mathfrak{F} \leq L \mathfrak{F}$.

An important consequence of the above result is that the join of all the normal, locally finite subgroups of a group G is normal and locally finite. We shall denote this subgroup, the locally finite radical of G, by L(G). Note that $L(G/L(G)) = 1$.

THEOREM 1.12. Let G be a soluble group. Then the following are equivalent:

(i) G has the ascending chain condition on subgroups;

(ii) G and all its subgroups are finitely generated;

(iii) $G \in P\mathfrak{C}$.

PROOF:

(i) $\Longrightarrow$ (ii): See [40, Lemma 1.47].

(ii) $\Longrightarrow$ (iii): This follows by an induction on the derived length of G, using the structure theorem for finitely generated abelian groups.

(iii) $\Longrightarrow$ (i): Clearly cyclic groups satisfy the ascending chain condition on subgroups, and so therefore do polycyclic groups, by [40, Cor.1.48].

We denote the class of groups satisfying the ascending chain condition on subgroups by Max. At present all known groups in Max lie in the class $P(\mathcal{C} \cup \mathcal{F})$. Conversely, every group in $P(\mathcal{C} \cup \mathcal{F})$ is contained in Max, since by [40, Cor.1.48], P Max = Max. Further information about this class of groups is provided by the following result, in which $(\mathcal{C}_\infty)$ denotes the class of infinite cyclic groups.

THEOREM 1.13. Let $G \in P(\mathcal{C} \cup \mathcal{F})$. Then

(i) there exists a characteristic subgroup H of G such that $|G : H| < \infty$ and $H \in P(\mathcal{C}_\infty)$;

(ii) periodic subgroups of G are finite.

PROOF:

(i) See [40, Vol.I, p.65].

(ii) Since subgroups of G are in $P(\mathcal{C} \cup \mathcal{F})$, this is clear.

REMARKS: (i) In view of (i) above, the class $P(\mathcal{C} \cup \mathcal{F})$ is frequently referred to as the class of polycyclic-by-finite groups.

(ii) Using the Schreier Refinement Theorem, one may easily show that if $G \in P(\mathcal{C} \cup \mathcal{F})$, the number of infinite cyclic factors occurring in a series for G with finite or cyclic factors is an invariant, which is called the Hirsch Number of G, and is denoted by h(G).

We shall make frequent use of the fact that the class $(\mathcal{G} \cap \mathfrak{A}\mathcal{F})$ of finitely generated abelian-by-finite groups is polycyclic-by-finite. This follows from the structure theorem for finitely generated abelian groups, together with

LEMMA 1.14. (Schreier [40, Thm. 1.41]) If H is a subgroup of finite index in the finitely generated group G, then H is finitely generated.

THEOREM 1.15. Let G be a nilpotent group. Then

(i) the set of periodic elements of G forms a characteristic subgroup T, which is locally finite. Further, G/T is torsion-free nilpotent;

(ii) if G is finitely generated, it is polycyclic; if G is finitely generated and torsion-free, $G \in P(\mathcal{C}_\infty)$;

(iii) if G is finite, it is the direct product of its Sylow subgroups.

PROOF:

For the second part of (ii), see [40, Thm. 2.25]. For the remaining assertions, see [40, Vol. I, p. 55].

We shall use the symbol $\mathcal{N}$ to denote the class of nilpotent groups. Between the classes $(\mathcal{G} \cap \mathcal{N})$ and $P\mathcal{C}$ lies the class of supersoluble groups.

DEFINITION 1.16. (i) A group G is said to be supersoluble if it has a finite series

$$1 = H_0 \subset \ldots \subset H_i \subset H_{i+1} \subset \ldots \subset H_n = G,$$

where $H_i \lhd G$ and $H_{i+1}/H_i \in \mathcal{C}$, $0 \leqslant i < n$.

(ii) The group G is said to be an infinite dihedral group if $G = \langle a, b : b^{-1}ab = a^{-1}, b^2 = 1 \rangle$.

Note that the infinite dihedral group is supersoluble, being an extension of an infinite cyclic group by an element of order two.

THEOREM 1.17. (i) Let G be supersoluble. There exist characteristic subgroups $L \subseteq M \subseteq G$ such that L is finite of odd order, M/L is finitely generated torsion-free nilpotent, and $|G/M| = 2^{\alpha}$, where $0 \leq \alpha < \infty$.

(ii) Let G be an infinite supersoluble group. Then G has a normal subgroup N with G/N either infinite cyclic or infinite dihedral.

PROOF:

(i) See [40, Vol.I, p.67].

(ii) See [37, Lemma 13.3.8].

NOTATION: If H is a subgroup of a group G, and $x,y \in G$, we shall write x^y for $y^{-1}xy$, and define $H^y = \{ h^y : h \in H\}$.

DEFINITION 1.18.(i) If G is a group and $x \in G$, the set

$$C_G(x) = \{ y \in G : x^y = x \}$$

is called the centralizer of x in G; $C_G(x)$ is clearly a subgroup of G.

(ii) If G is a group, the set

$$\Delta(G) = \{ x \in G : |G : C_G(x)| < \infty \}$$

is a characteristic subgroup of G, called the FC-subgroup.

(iii) A group G is called an FC-group if $G = \Delta(G)$.

(iv) For a group G and a prime p, we define

$$\Delta^+(G) = \{ x \in \Delta(G) : x \text{ has finite order}\},$$

and $$\Delta^p(G) = \langle x \in \Delta^+(G) : |x| = p^{\alpha}, 0 \leq \alpha < \infty \rangle.$$

LEMMA 1.19. Let G be a group.

(i) $\Delta^+(G)$ is a characteristic locally finite subgroup of G, and $\Delta(G)/\Delta^+(G)$ is torsion-free abelian.

(ii) If S is a finite subset of $\Delta^+(G)$, there exists a finite normal subgroup H of G, such that $S \leq H$. Conversely, if H is a finite normal subgroup of G, $H \leq \Delta^+(G)$.

(iii) For any prime p, $\Delta^p(G)$ is a characteristic subgroup of G, and $\Delta^+(G)/\Delta^p(G)$ has no elements of order p.

(iv) If H is a finite normal subgroup of G, then $\Delta^+(G/H) = \Delta^+(G)/H$ and $\Delta^p(G/H) = \Delta^p(G)H/H$.

(v) If $W \leq \Delta^p(G)$ with $|\Delta^p(G) : W| < \infty$, then $|W : \Delta^p(W)| < \infty$.

(vi) If H is a finitely generated FC-group, then $\zeta(H)$, the centre of H, has finite index in H.

PROOF:

(i) - (v): See [36, Lemma 19.3].

(vi): Let $H = \langle a_1, \ldots, a_n \rangle$. Since the intersection of finitely many subgroups of finite index has finite index,

$$|H : \bigcap_{i=1}^{n} C_H(a_i)| < \infty,$$

and clearly $\zeta(H) = \bigcap_{i=1}^{n} C_H(a_i)$.

LEMMA 1.20. Let G be a finitely generated group, and suppose $|G'| < \infty$. Then $|G : \zeta(G)| < \infty$, where $\zeta(G)$ denotes the centre of G.

PROOF:

It is easy to show that since $|G'| < \infty$, G is an FC-group, so that Lemma 1.19 (vi) applies.

DEFINITION 1.21. If G is a periodic group and p is a prime, we shall say that G is (i) a p'-group if G has no non-trivial elements of order a power of p, or (ii) a p-group if every element of G has order a power of p. We shall sometimes write $\mathcal{F}_p$ for the class of finite p-groups, and $\mathcal{F}_{p'}$ for the class of finite p'-groups.

We now outline two group-theoretic constructions which we shall need.

THEOREM 1.22. Let $\{A_\alpha : \alpha \in I\}$ be a family of groups. Then there exists a group $G = \prod_\alpha^* A_\alpha$ which is uniquely determined by the following property. For each $\alpha \in I$, G contains an isomorphic copy of A_α and each element g of G is uniquely expressible as a finite product of the form $g = a_1 a_2 \ldots a_n$ for some n and some $a_i \in A_{\alpha_i}$, $a_i \neq 1$, $1 \leqslant i \leqslant n$, such that $\alpha_i \neq \alpha_{i+1}$, $1 \leqslant i < n$.

PROOF:

See [37, Thm.9.2.9].

DEFINITION 1.23. In the notation of the above theorem, G is called the free product of the A_α's. If $|I| = t < \infty$, we write $G = A_1 * \ldots * A_t$.

LEMMA 1.24. Let G be an infinite dihedral group. Then $G = C_2 * C_2$, where C_2 denotes the group of two elements.

PROOF:

See [37, Lemma 13.3.2].

DEFINITION 1.25. Let G and H be groups. The wreath product $G \wr H$ of G by H is defined as follows. For each $x \in H$, let G_x be the set of ordered pairs

$$G_x = \{ [g, x] : g \in G \},$$

with multiplication defined by

$$[g_1, x][g_2, x] = [g_1 g_2, x].$$

Thus G_x is a group, and G_x is isomorphic to G, for all $g \in G$. If $y \in H$, then y induces an isomorphism $G_x \rightarrow G_{xy}$, mapping the element $[g, x]$ to the element $[g, xy]$. In this way y induces an automorphism of

$$A = \prod_{x \in H} G_x .$$

This yields an action of H on A, and we may form the semidirect product $A \rtimes H$, which we call the wreath product $G \wr H$ of G by H. The normal subgroup A is called the base group of $G \wr H$.

We note the following useful fact.

LEMMA 1.26. Let A and B be non-trivial groups, and suppose that B is infinite. Then $\Delta(A \wr B) = 1$.

2. RING THEORY

Throughout this thesis, we shall use the term Noetherian [respectively Artinian] to mean right and left Noetherian [resp. right and left Artinian]. The term ''ideal'' will always mean ''two-sided ideal''. All rings will be assumed to contain an identity element, and all modules will be unital; we shall normally consider right modules, and write homomorphisms on the left. If R is a ring, J(R) will denote the Jacobson radical of R, and N(R) the nilpotent radical of R; that is, N(R) is the sum of the nilpotent ideals of R.

We shall need the following well-known criterion for a ring to be right Artinian.

THEOREM 2.1. The ring R is right Artinian if and only if R/N(R) is Artinian, N(R) is finitely generated as a right ideal, and N(R) is nilpotent.

DEFINITION 2.2.(i) If S is a subset of a ring R, we shall call the right ideal $\{x \in R : Sx = 0\}$ the right annihilator of S in R, and denote it by $r_R(S)$, or simply by $r(S)$ where no confusion is likely. The set $\{y \in R : yS = 0\}$ is called the <u>left annihilator</u> of S, and will be denoted by $l_R(S)$.

(ii) An element x of a ring R is said to be <u>right regular</u> if $r_R(x) = 0$; left regular elements are defined analogously, and an element is said to be regular if it is right and left regular. The set of regular elements of R will be denoted by $\mathscr{C}_R(0)$, or by $\mathscr{C}(0)$ when no confusion is likely.

(iii) An element x of a ring R is said to be a unit if there exists an element x^{-1} such that

$$x\,x^{-1} = x^{-1}x = 1.$$

The element x^{-1} is clearly unique if it exists.

<u>LEMMA 2.4</u>. If x is a right regular element of the right Artinian ring R, then x is a unit. Thus, in particular, x is regular.

<u>PROOF</u>:

There exists $n \geqslant 1$ such that $x^nR = x^{n+1}R$, and so $x^n = x^{n+1}d$ for some $d \in R$. Hence $1 = xd$ and d is right regular, so there exists $a \in R$ such that $1 = da$. Thus $x = a$, as required.

We shall have occasion to consider the following subclass of the class of Artinian rings.

<u>DEFINITION 2.4</u>. An Artinian ring R will be called a <u>quasi-Frobenius ring</u> (QF-<u>ring</u>) if

$$r\big(l(I)\big) = I, \qquad l\big(r(T)\big) = T,$$

for all right ideals I and left ideals T of R.

<u>DEFINITION 2.5</u>.(i) An R-module M is said to be <u>injective</u> if every diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \\ & M & \end{tikzcd}$$

of R-modules and R-homomorphisms can be embedded in a diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\psi"'] & B \arrow[dl, "\mu"] \\ & M & \end{tikzcd}$$

such that $\mu\varphi = \psi$.

(ii) A ring R is said to be <u>right self-injective</u> if R_R is an injective module.

<u>THEOREM 2.6</u>. (Eilenberg and Nakayama; [51, Prop.3.4, p.277]) A ring R is a QF-ring if and only if R is a right or left Noetherian ring and R is right or left self-injective. A QF-ring is right and left self-injective.

<u>PROPOSITION 2.7</u>. If I is a nil ideal of a ring R, (i.e. for all $x \in I$ there exists $n \geqslant 1$ such that $x^n = 0$), then any countable set of orthogonal idempotents of R/I can be lifted to a set of orthogonal idempotents of R.

<u>PROOF</u>:

This follows from [51, Propositions 4.1 and 4.2, p.186].

<u>DEFINITION 2.8</u>. A ring R is said to be (<u>von Neumann) regular</u> if for each $a \in R$ there exists $x \in R$ such that

$$a = axa.$$

Note that semisimple Artinian rings are regular, and it is easy to see that a ring R is regular if and only if every finitely generated right ideal of R is idempotently generated, [51, Prop.12.1, p.39]. It follows that regular rings are semisimple. Concerning the relationship between regular and Artinian rings, we have

<u>THEOREM 2.9</u>. (Kaplansky, [27, Thm.2.1]) A regular ring R is

either Artinian or contains an infinite set of orthogonal idempotents.

In the study of rings it is frequently useful to embed a given ring in a simpler ring in such a way that we may hope to deduce properties of the original ring from the structure of the over-ring. We now consider various ways of adopting this approach.

DEFINITION 2.10. Let R be a ring, T a multiplicatively closed set of regular elements of R. A partial right quotient ring of R with respect to T is a ring Q containing R such that elements of T are units in Q and every element of Q can be written in the form ac^{-1}, where $a \in R$, $c \in T$. Left quotient rings are defined analogously.

It may easily be shown that any two right quotient rings of R with respect to T are isomorphic.

If $T = \mathcal{C}(0)$, the corresponding right quotient ring Q is called the classical (right) quotient ring of R. (Note that Q need not exist.) We shall denote the classical right quotient ring of a ring R by Q(R), or sometimes by $Q_{cl}(R)$. When Q(R) exists, we say that R is a right order in Q(R).

DEFINITION 2.11. A ring R is said to satisfy the right Ore condition with respect to the multiplicatively closed set T of regular elements of R if and only if for all $a \in R$, $c \in T$, there exist $a_1 \in R$, $c_1 \in T$, such that

$$ac_1 = ca_1.$$

The left Ore condition is defined analogously.

THEOREM 2.12. (Ore) Let R be a ring containing a multiplicatively closed set of regular elements T. Then R has a partial right quotient ring with respect to T if and only if R satisfies the right Ore condition with respect to T.

For a proof of the above result, see [30, Prop.1, p.109], or for a proof which avoids the use of the maximal ring of quotients, see [37, Thm.4.4.1]. When we take $T = \mathcal{C}(0)$, Theorem 2.12 and its left-handed version provide conditions for the existence of classical right and left quotient rings of R. When both these rings exist, we have, by an elementary argument

PROPOSITION 2.13. Let R be a ring with right and left classical quotient rings Q_1 and Q_2. Then, as rings,

$$Q_1 \cong Q_2.$$

DEFINITION 2.14. In the circumstances of Prop.2.13, we shall refer to Q_1 the (classical) quotient ring of R, and we shall say that R is an order in $Q_1 = Q(R)$.

In 1958 Goldie obtained necessary and sufficient conditions for a ring to have a semisimple Artinian right quotient ring. To state these conditions, we need some more definitions. In 2.15-2.20, R denotes an arbitrary ring.

DEFINITION 2.15.(i) An R-module M is said to be an essential extension of a submodule N if, for all R-submodules X of M, $X \cap N = 0$ implies $X = 0$. Alternatively, we say that N is essential in M.

(ii) An R-module U is said to be uniform if $U \neq 0$

and every non-zero submodule of U is essential in U.

(iii) A module M has finite (uniform or Goldie) dimension if it contains no infinite direct sum of non-zero submodules.

THEOREM 2.16. [14, Thm.1.0.7] Let M be a finite dimensional R-module. There exists an integer $n \geqslant 0$ such that

(i) a direct sum of uniform submodules which is essential in M has n terms;

(ii) a direct sum of non-zero submodules in M always has at most n terms;

(iii) a submodule is essential in M if and only if it contains a direct sum of n uniform submodules.

DEFINITION 2.17. (i) The Goldie dimension of an R-module M, written $\dim_R M$, is the integer n of the above theorem, if n exists, and otherwise $\dim_R M = \infty$.

(ii) The right Goldie dimension of R is $\dim_R(R_R)$.

Clearly if $N \subseteq M$ are R-modules, $\dim_R N \leqslant \dim_R M$, with equality if and only if N is essential in M. If $\dim_R M = 0$, $M = 0$, while if $\dim_R M = 1$, M is uniform.

DEFINITION 2.18. A ring R is said to have the maximum condition on right annihilators, written max-ra, if every chain

$$r(S_1) \subset r(S_2) \subset \ldots \subset r(S_i) \subset r(S_{i+1}) \subset \ldots$$

where S_i is a subset of R, $1 \leqslant i < \infty$, terminates after finitely many terms.

We define similarly the properties max-la, min-ra and min-la.

<u>DEFINITION 2.19</u>.(i) A right ideal I of R is said to be <u>essential</u> if I_R is an essential submodule of R_R.

(ii) The <u>right singular ideal</u> Z(R) of R is defined as follows:

$$Z(R) = \{x \in R : xE = 0, \text{ for some essential right ideal } E \text{ of } R\}.$$

(iii) A ring R is said to be <u>right non-singular</u> if $Z(R) = 0$.

There are, of course, analogous definitions on the left; the left singular ideal will be denoted by Z'(R). The singular ideals were introduced by Johnson; as we shall see, they have proved useful in various aspects of ring theory.

<u>LEMMA 2.20</u>. Z(R) and Z'(R) are two-sided ideals of R.

<u>PROOF</u>:

It is clear that Z(R) is a left ideal, since if E_1 and E_2 are essential right ideals of R, then $E_1 \cap E_2$ is essential. Furthermore, if E is an essential right ideal,

$$a^{-1}E = \{ r \in R : ar \in E \}$$

is easily seen to be an essential right ideal, for all $a \in R$. It follows that Z(R) is an ideal of R

Similar remarks apply to Z'(R).

<u>LEMMA 2.21</u>.(i) Let R be a ring with max-ra. Then Z(R) is nilpotent.

(ii) If R is a commutative ring, $N(R) \subseteq Z(R)$.

<u>PROOF</u>:

(i) See [51, Lemma 2.5, p.56].

(ii) See [14, p.225].

It should be noted that both parts of the above lemma are false if we drop the appropriate hypotheses on R. For (ii), consider the ring S of all $n \times n$ lower triangular matrices with integer coefficients, for $n > 1$; $Z(S) = 0$, but $N(S) \neq 0$. For (i), see Ex.4.4.3.

It is convenient at this point to note the following basic properties of quotient rings.

LEMMA 2.22. Let R be a ring with a partial right quotient ring Q with respect to a multiplicatively closed set $T \subseteq \mathcal{C}(0)$.

(i) Let $c_1, \ldots, c_k \in T$; then there exist elements $c \in T$, $d_1, \ldots, d_k$ of R such that

$$c_i^{-1} = d_i c^{-1}, \qquad i = 1, \ldots, k.$$

(ii) Let $I \trianglelefteq_r R$. Then IQ is a right ideal of Q, and

$$IQ = \{ ac^{-1} : a \in I, c \in T \} .$$

(iii) If $P \trianglelefteq_r Q$, $P = (P \cap R)Q$.

(iv) A right ideal E of R is essential in R if and only if EQ is essential in Q.

(v) $\dim_R(Q_R) = \dim_Q(Q_Q) = \dim_R(R_R)$.

PROOF:

(i) - (iv): See [14, Lemma 1.36].

(v) This follows easily from (i), (ii) and (iii).

THEOREM 2.23.(Goldie) Let R be a ring. Then the following are equivalent:

(a) R has a semisimple Artinian classical right quotient ring Q(R).

(b) (i) R is semiprime;

(ii) R has finite right Goldie dimension;

(iii) R has max-ra.

(c) (i) R is semiprime;

(ii) R has finite right Goldie dimension;

(iii) $Z(R) = 0$.

PROOF:

(a) $\Longleftrightarrow$ (b): This is [14, Thm.1.37].

(b) $\Longrightarrow$ (c): This follows from Lemma 2.21(i).

(c) $\Longrightarrow$ (b): See [9, Cor.1.4].

DEFINITION 2.24. A ring R satisfying properties (b)(ii) and (b)(iii) above is called a right Goldie ring.

REMARK: The proofs of (a) $\Longrightarrow$ (b) and (a) $\Longrightarrow$ (c) in the above result are of course trivial applications of the previous lemma. It should be noted in particular that if R is any subring of a ring Q satisfying max-ra or min-ra, then R must also satisfy max-ra or min-ra, respectively, since if S is any subset of R,

$$r_R(S) = r_Q(S) \cap R.$$

Secondly, it is apparent from Lemma 2.22(ii) that if Q is a partial right quotient ring of the semiprime ring R, then Q is also semiprime.

If we remove the restriction to semisimple quotient rings Q from Theorem 2.23, the situation is described by the next result, originally obtained by Small, [44], and due in the form we state it to Hajarnavis, [19].

DEFINITION 2.25. If R is a ring and I is an ideal of R, we denote by $\mathcal{C}(I)$ the set of elements c of R such that $(c + I)$ is a regular element of R/I.

THEOREM 2.26. [19] A ring R has a right Artinian right quotient ring if and only if R satisfies the following conditions.

(i) R and $R/N(R)$ are right Goldie rings.

(ii) The ring $R/l_R\left(N(R)^k\right)$ has finite right Goldie dimension for $k = 1, \ldots, t-1$, where t is an integer such that $N(R)^t = 0$, but $N(R)^{(t-1)} \neq 0$.

(iii) $\mathcal{C}\left(N(R)\right) \subseteq \mathcal{C}(0)$.

It will be noted from the above result that a necessary condition for a ring R to have a right Artinian quotient ring is that $N(R)$ is nilpotent. This is always the case if R is right Goldie, as may be seen from

PROPOSITION 2.27. (Lanski) If S is a nil subring of the right Goldie ring R, then S is nilpotent.

A proof of the above result may be found in [9].

When in Chapter 3 we wish to show that certain rings have Artinian quotient rings, we shall not in fact make use of Theorem 2.26, except to deduce information about the rings in question once we know that they have Artinian quotient rings. Instead, we shall use the following result. We denote the group of ring automorphisms of a ring R by $\mathrm{Aut}(R)$.

THEOREM 2.28. (Jategaonkar, [24, Thm.3.1]) Let R be a ring with a right Artinian right quotient ring, and let $\sigma \in \mathrm{Aut}(R)$. Let $R[X;\sigma]$ denote the ring of polynomials with coefficients in R,

$$R[X;\sigma] = \{f(X) = \sum_{i=0}^{n} r_i X^i : n \geqslant 0,\ r_i \in R\},$$

where addition and multiplication are defined in the obvious manner, subject to

$$rX = X\sigma(r),$$

for all $r \in R$. Then $R[X;\sigma]$ has a right Artinian right quotient ring.

REMARK: Since X is a regular element of the ring $R[X;\sigma]$, the ring

$$R[X, X^{-1}; \sigma] = \{\sum_{i=-m}^{n} r_i X^i : m,n \geqslant 0; r_i \in R\},$$

with multiplication defined as above, is an order in $Q(R[X;\sigma])$, if this latter ring exists. Thus the above result may be applied equally well to the ring $R[X, X^{-1}; \sigma]$.

DEFINITION 2.29. The rings $R[X;\sigma]$ and $R[X,X^{-1};\sigma]$ introduced above will be referred to as twisted polynomial rings over R.

We need one more result on classical quotient rings.

LEMMA 2.30. Let R be a ring with a right Artinian right quotient ring Q. Then (i) $N(R) = N(Q) \cap R$; (ii) $N(Q) = N(R)Q = QN(R)Q$; (iii) for all positive integers k, $N(Q)^k = N(R)^kQ = Q(N(R))^kQ$.

PROOF:

This follows from Lemma 2.1 of [Talintyre, J.London Math. Soc. 41 (1966), 141-144], noting that the assumption there that R is right Noetherian may be replaced by the assumption that N(R) is nilpotent, which is the case, by Lemma 2.27.

We now turn to a related technique, that of ''localization.'' We shall only need to consider localization in circumstances very similar to the classical, commutative case; for a more detailed treatment, the reader is advised to consult [14], for example.

DEFINITION 2.31. Let R be a right Noetherian ring, I an

ideal of R. We say that I has the <u>right AR property</u> if, given a finitely generated right R-module M and a submodule N of M, there exists a positive integer n such that
$$MI^n \cap N \subseteq NI.$$

<u>PROPOSITION 2.32</u>. (Hartley) An ideal I of a right Noetherian ring R has the right AR property if and only if, given any right ideal E of R, there exists an integer $n \geqslant 1$ such that
$$E \cap I^n \subseteq EI.$$

<u>PROOF</u>:

Clearly, if I has the right AR property and E is a right ideal of R, there exists $n \geqslant 1$ such that
$$E \cap I^n \subseteq EI.$$

For the converse, let M be a finitely generated right R-module with submodule N, and suppose I satisfies the right AR property on right ideals. If M is cyclic, it is easy to show that there exists $n \geqslant 1$ such that
$$MI^n \cap N \subseteq NI.$$
If M is not cyclic, choose a submodule Q of M maximal such that
$$Q \cap N = NI,$$
and put $\bar{M} = M/Q$, $\bar{N} = (N + Q)/Q$, so that $\bar{N}$ is essential in $\bar{M}$, and $\bar{N}I = 0$. It is enough to prove that there exists $n \geqslant 1$ such that
$$\bar{M}I^n = 0.$$

Now $\bar{M} = \sum_{i=1}^{t} M_i$, say, where M_i is cyclic, for $i = 1, \ldots, t$, and $\bar{N} \cap M_i$ is essential in M_i, for $i = 1, \ldots, t$. However,
$$(\bar{N} \cap M_i)I \subseteq \bar{N}I = 0,$$
and the result follows since I has the right AR property on cyclic modules.

<u>DEFINITION 2.33</u>. Let P be a prime ideal of the Noetherian

ring R. We say that P is <u>localizable</u> if, given $c \in \mathcal{C}(P)$, $r \in R$, there exist $c_1 \in \mathcal{C}(P)$, $r_1 \in R$ such that

$$rc_1 = cr_1.$$

The next lemma is clear from the definition.

<u>LEMMA 2.34</u>. Let P be a localizable ideal of a ring R, and I an ideal of R such that $I \subseteq P$. Then P/I is a localizable ideal of R/I.

<u>LEMMA 2.35</u>. [15, Chapter 5] If P is a localizable prime ideal of a Noetherian ring R, and

$$K = \{ r \in R : rc = 0, \text{ for some } c \in \mathcal{C}(P)\},$$

then K is an ideal of R, and there exists $c \in \mathcal{C}(P)$ such that $Kc = 0$. If we put $\bar{R} = R/K$, then the set

$$\mathcal{C}(\bar{P}) = \{(c + K) : c \in \mathcal{C}(P)\}$$

consists of regular elements of $\bar{R}$. Thus if R is prime, $K = 0$ and $\mathcal{C}(P)$ consists of regular elements of R.

It follows from Lemma 2.35 that if P is a localizable prime ideal of a Noetherian ring R, we may form a partial right quotient ring of $\bar{R} = R/K$, by inverting the elements of $\mathcal{C}(\bar{P})$.

<u>DEFINITION 2.36</u>. If P is a localizable prime ideal of the Noetherian ring R, we shall denote the partial quotient ring of $\bar{R}$ described above by R_P, and call it the <u>localization</u> of R at P.

Note that if R is prime, R is a subring of R_P, which is in turn a subring of $Q_{cl}(R)$.

<u>DEFINITION 2.37</u>. A ring R is said to be <u>local</u> if

(i) R has a unique maximal ideal M, which is the

Jacobson radical of R;

(ii) R/M is Artinian.

If R/M is actually a division ring, R is said to be <u>scalar local</u>.

<u>PROPOSITION 2.38</u>. If P is a localizable ideal of the Noetherian ring R, then R_P is a right Noetherian local ring, whose elements can be written in the form $\bar{a}\bar{c}^{-1}$, where $a \in R$, $c \in \mathcal{C}(P)$, and $\bar{x}$ denotes the image in R/K of $x \in R$. If R/P is a domain, R_P is scalar local.

<u>PROOF</u>:

Retaining the notation of Lemma 2.35, $\bar{R}$ is a right Noetherian ring, and so by Lemma 2.22(iii) R_P is right Noetherian, since it is a partial quotient ring of $\bar{R}$.

Now, writing $M = PR_P \lhd R_P$, $R_P/M \cong Q_{cl}(R/P)$, which is simple Artinian by Theorem 2.23 and Lemma 2.22(iii), so that M is a maximal ideal, and R_P/M is a division ring if R/P is a domain. Finally, elements of R_P can clearly be written in the prescribed form, since $\mathcal{C}(\bar{P})$ satisfies the right Ore condition in R/K, and $M = J(R_P)$, since if $\bar{p}\bar{c}^{-1} \in M$, where $\bar{p} \in \bar{P}$, $\bar{c} \in \mathcal{C}(\bar{P})$, then

$$(1 - \bar{p}\bar{c}^{-1}) = (\bar{c} - \bar{p})\bar{c}^{-1},$$

which is a unit in R_P.

The next result reveals some of the connections between localization and the AR property.

<u>PROPOSITION 2.39</u>. [48, Prop.2.1] Let P be a prime ideal of the Noetherian ring R. If P has the right AR property, then P is localizable if and only if P/P^n is a localizable ideal of R/P^n for all positive integers n.

<u>LEMMA 2.40</u>. If M is a maximal ideal of the Noetherian ring

R, then $\mathcal{C}(M) = \mathcal{C}(M^n)$ for all positive integers n.

PROOF:

By [14, Thm.2.5], $\mathcal{C}(M^n) \subseteq \mathcal{C}(M)$. For the converse, we argue by induction on n. The result is trivial for n = 1; suppose that n is greater than 1, and that we have proved the result for (n - 1). We may suppose that $M^n = 0$, without loss of generality. We define

$$S = \{ x \in R : xc = 0, \text{ some } c \in \mathcal{C}(M)\}.$$

Since $\mathcal{C}(M) = \mathcal{C}(M^{n-1})$ by induction, Theorem 2.26 shows that $\mathcal{C}(M)$ is a right divisor set modulo M^{n-1}. Since $S \subseteq M$, it follows that S is an ideal of R. Now S is finitely generated as a left ideal of R, and so there exists $t \in \mathcal{C}(M)$ such that $St = 0$, again using our induction assumption and an adaptation of Lemma 2.22(i). If $S \neq 0$, $RtR \neq R$, and so, noting that M is the unique maximal ideal of R, we must have $t \in M$, a contradiction. Thus $S = 0$, as required.

Finally, we obtain the result for which we have been aiming.

PROPOSITION 2.41. If M is a maximal ideal of the Noetherian ring R, and M has the right AR property, then M is localizable.

PROOF:

Lemma 2.40 and Theorem 2.26 show that M/M^n is a localizable ideal of R/M^n, for all $n \geqslant 1$. The result follows from Proposition 2.39.

In view of the above result, it is necessary to have some information on when ideals of a Noetherian ring R have the AR property. If R is commutative, then every ideal has the AR property - this is the content of the Artin-Rees

Lemma, [2, Cor.10.10]. In a non-commutative Noetherian ring, however, this is not the case. By analogy with the commutative case, we make the following definition.

<u>DEFINITION 2.42</u>. An ideal I of a ring R is said to have the <u>strong right AR property</u> if and only if the ring

$$R^*(I) = \{f(X) = \sum_{i=0}^{n} a_i X^i : a_0 \in R,\ a_i \in I^i,\ 1 \leqslant i \leqslant n\}$$

is right Noetherian. (In the above, X denotes a commuting indeterminate.)

The following result, which justifies the terminology introduced above, is proved exactly as in the commutative case; see, for example, [2, §10].

<u>LEMMA 2.43</u>. If the ideal I of the right Noetherian ring R has the strong right AR property, then I has the right AR property.

As we have seen in Theorem 2.12, the classical right quotient ring of a non-commutative ring R does not in general exist. In an effort to surmount this problem, an extensive theory of 'maximal quotient rings' has been developed over the past twenty years, the maximal quotient ring of a ring having the advantage (or perhaps the disadvantage) that it always exists. Our main reference for this material is [30]; see also [51] for a more categorical treatment, and note that a very detailed discussion of the representation of Theorem 2.48 is given in [37].

<u>DEFINITION 2.44</u>. Let R be a ring, and let I_R denote the injective hull of R_R; see [30, p.92]. Let $H = \mathrm{Hom}_R(I_R, I_R)$, and finally put

$$Q_{max}(R) = \mathrm{Hom}_H({}_H I, {}_H I).$$

The ring $Q_{max}(R)$ is called the <u>maximal right quotient ring of R</u>. There is of course an analogous definition of the maximal left quotient ring of R.

We assemble some elementary properties of the maximal right quotient ring.

<u>LEMMA 2.45</u>. Let R be a ring.

(i) The map $\varphi : R \longrightarrow Q_{max}(R) : r \longmapsto (\pi_r : i \longrightarrow ir)$ is a ring monomorphism.

(ii) The map $\psi : Q_{max}(R)_R \longrightarrow I_R : q \longmapsto 1q$ is a module monomorphism.

(iii) If S is a subring of $Q_{max}(R)$, and $R \subseteq S$, then

$$Q_{max}(R) = Q_{max}(S).$$

(iv) If $Z(R) = 0$, then $Z\big(Q_{max}(R)\big) = 0$, $Q_{max}(R)$ is self-injective, and $Q_{max}(R)_R \cong I_R$.

(v) $Z(R) = 0$ if and only if $Q_{max}(R)$ is regular.

(vi) The following conditions are equivalent:

(a) The map ψ of (ii) is an epimorphism;

(b) $Q_{max}(R)_R$ is injective;

(c) $Q_{max}(R)$ is self-injective;

(d) $H \cong Q_{max}(R)$ canonically, as rings.

<u>PROOF</u>:

(i) and (ii): [30, § 4.3, Lemma 1].

(iii): Adapt [30, §4.3, Corollary to Prop.2].

(iv): [30, §4.5, and §4.3, Prop.3].

(v): [30, §4.5, Prop.3].

(vi): [30, §4.3, Prop.4.3].

It is frequently useful to have a more concrete description of the maximal quotient ring, which we now outline. Full details may be found in [30, §4.3] or [37, §4.5].

DEFINITION 2.46. A right ideal I of a ring R is said to be dense if, for all $x \in R$, $l(I:x) = 0$, where

$$(I:x) = \{r \in R : xr \in I\}.$$

Thus a dense right ideal is essential, and if I is a two-sided ideal, I is dense as a right ideal if and only if $l(I) = 0$.

LEMMA 2.47. [37, Lemma 4.5.2] (i) If D_1 and D_2 are dense right ideals of a ring R, and $f \in \mathrm{Hom}_R(D_1, R)$, then $(D_1 \cap D_2)$ and $f^{-1}D_2$ are also dense right ideals.

THEOREM 2.48. [30, §4.3, Prop.6] Let D range over the set $\mathcal{D}$ of all dense right ideals of a ring R, and consider the set

$$\bigcup_{D \in \mathcal{D}} \mathrm{Hom}_R(D, R)/\theta ,$$

where θ is the equivalence relation between two homomorphisms which agree on a dense right ideal contained in the intersection of their domains. This set may be made into a ring, isomorphic to $Q_{max}(R)$, with operations defined as follows. Let $f_i \in \mathrm{Hom}_R(D_i, R)$, $i = 1,2$, and let

$$(f_1 + f_2) \in \mathrm{Hom}_R(D_1 \cap D_2, R), \qquad f_1f_2 \in \mathrm{Hom}_R(f_2^{-1}D_1, R)$$

be defined by

$$(f_1 + f_2)d = (f_1d + f_2d), \qquad (f_1f_2)d = f_1(f_2d);$$

then we define

$$\theta f_1 + \theta f_2 = \theta(f_1 + f_2), \text{ and } \theta f_1\, \theta f_2 = \theta(f_1f_2).$$

REMARK: Notice that the ring R is represented in the above manner by $r \longmapsto (\gamma_r)$, where $\gamma_r : R_R \longrightarrow R_R : x \longmapsto rx$.

Using the above representation, we easily deduce

PROPOSITION 2.49. (i) Let R be a ring satisfying the right Ore condition. Then $R \subseteq Q_{cl}(R) \subseteq Q_{max}(R)$.

(ii) An element x of the maximal right quotient ring of a ring R is in the centre of $Q_{max}(R)$ if and only if

$$xr = rx$$

for all $r \in R$.

PROOF:

(i) See [30, §4.6, Prop.1].

(ii) Suppose x is represented by $f \in \mathrm{Hom}_R(D, R)$, and $xr = rx$ for all $r \in R$. Then clearly D may be assumed to be a two-sided ideal, and f a bimodule homomorphism of D; see [13]. If now y is an element of $Q_{max}(R)$ represented by $g \in \mathrm{Hom}_R(E, R)$, then ED is easily seen to be a dense right ideal on which fg, gf are not only defined but equal.

We now introduce some fundamental concepts from homological algebra.

DEFINITION 2.50. Let R be a ring.

(i) An R-module P is said to be projective if P is a direct summand of a free module.

(ii) Let M be an R-module. A projective resolution of M of length n is an exact sequence of non-zero R-modules

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ . \ . \ . \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where P_i is projective, for $0 \leq i \leq n$.

(iii) The projective dimension of M is

$$\inf\{n : M \text{ has a projective resolution of length } n\},$$

and is denoted by proj. dim. M.

(iv) We call

$$\sup\{\text{proj. dim. } M : M \text{ a right } R\text{-module}\}$$

the right global dimension of R, and denote it by rt.gl.dim.R.

Thus for any ring R, the right global dimension of R is either infinity or a non-negative integer, and we may easily deduce

LEMMA 2.51. A ring R has right global dimension 0 if and only if R is semisimple Artinian.

LEMMA 2.52. [28, Thm.2.1, p.191] If the ring R is not semisimple Artinian,

$$\text{rt.gl.dim.}R = 1 + \sup_I\{\text{proj.dim.}I : I \underset{r}{\triangleleft} R\}.$$

DEFINITION 2.53. A ring R is said to be right hereditary if every right ideal of R is projective; that is, if R has right global dimension one.

LEMMA 2.54. If Q is a partial right quotient ring of the ring R,

$$\text{rt.gl.dim.}Q \leqslant \text{rt.gl.dim.}R.$$

PROOF:

See [37, Lemma 10.3.14] or [28, Thm.11, p.181]; the assumption in the latter reference that R is commutative is clearly unnecessary.

We shall also need the following somewhat less elementary result, which we shall only in fact use in the special situation where the ring R is the union of an ascending chain of subrings

$$R_1 \subset R_2 \subset \ldots \subset R = \bigcup_i R_i.$$

PROPOSITION 2.55. (Berstein; [3]) Let the ring R be the direct limit of the rings $\{R_i : 1 \leqslant i < \infty\}$. Then

$$\text{rt.gl.dim.}R \leqslant 1 + \sup_{i \geqslant 1} (\text{rt.gl.dim.}R_i).$$

We have already introduced the concept of a ring of twisted polynomials over a ring R, (Def.2.29). We shall require some information about the global dimension of such rings.

PROPOSITION 2.56. Let $R[X;\sigma]$ be a twisted polynomial ring over the ring R. Then

$$\text{rt.gl.dim.}R \leq \text{rt.gl.dim.}R[X;\sigma] \leq 1 + \text{rt.gl.dim.}R.$$

PROOF:

The first inequality follows from [28, Thm.5, p.173], and the second may be proved by adapting the proof of [28, Thm.6, p.174]. This latter result may be viewed as the case σ = identity of our present proposition, and it is straightforward to check that the introduction of a non-trivial automorphism presents no obstacle to the proof that

$$\text{rt.gl.dim.}R[X;\sigma] \leq 1 + \text{rt.gl.dim.}R.$$

(Note, however, that it is not in general true that

$$\text{rt.gl.dim.}R[X;\sigma] = 1 + \text{rt.gl.dim.}R,$$

when σ is not the identity map on R).

Finally, we note that by Lemma 2.54 and [28, Thm.5, p.173], the above result may also be applied to polynomial rings of the form $R[X,X^{-1};\sigma]$.

3. GROUP RINGS

Almost all the results in this section can be found in [36] or [37]. As before, we make no attempt to be comprehensive.

DEFINITION 3.1. Let R be a ring and G a group. The *group ring* RG is the set of all formal sums $\sum_{g\in G} r_g g$ with $r_g \neq 0$ for only finitely many $g \in G$. RG is given a ring structure in the obvious way; thus if $\alpha = \sum_{g\in G} r_g g$ and $\beta = \sum_{h\in G} s_h h$ are elements of RG, we define

$$\alpha\beta = \sum_{g\in G}\Big(\sum_{\substack{x,y\in G\\ xy=g}} r_x s_y\Big) g, \qquad \alpha + \beta = \sum_{g\in G}(r_g + s_g) g.$$

DEFINITION 3.2.(i) The ring R is called the coefficient ring of RG.

(ii) If $\alpha = \sum_{g\in G} r_g g \in RG$, we call the set $\{g \in G : r_g \neq 0\}$ the support of α, and denote it by suppα.

Throughout this section, RG will denote a fixed group ring. Let H be a subgroup of G, and let $\{g_\lambda : \lambda \in \Lambda\}$ be a left transversal to H in G, where Λ is a suitable index set. We may clearly view RH as a subring of RG. If $x \in G$, the conjugation action of x on G defines a ring isomorphism from RH to $R(H^x)$. If $H \lhd G$, then x affords an automorphism of RH, fixing R. We note in addition

LEMMA 3.3.(i) With the above notation, RG is a free right RH-module, with basis $\{g_\lambda : \lambda \in \Lambda\}$.

(ii) The map π_H: $RG \longrightarrow RH : \sum_{g\in G} r_g g \longrightarrow \sum_{g\in H} r_g g$ is a left and right RH-homomorphism, called the canonical projection of RG on RH.

(iii) If $I \underset{l}{\lhd} RH$, then the left ideal of RG generated by I is simply

$$\{\sum_{\lambda\in\Lambda} g_\lambda \alpha_\lambda : \alpha_\lambda \in I, \text{ and } \alpha_\lambda = 0 \text{ for all but finitely many } \lambda \in \Lambda\}.$$

PROOF:

(i) and (ii): See [37, Lemmas 1.1.2 and 1.1.3].

(iii) This is clear from (i).

LEMMA 3.4.(i) Let H be a subgroup of G, and let c be a regular element of RH. Then c is a regular element of RG.

(ii) If RH is a domain for every finitely generated subgroup H of G, then RG is a domain.

PROOF:

(i) This is a simple application of Lemma 3.3(i).

(ii) Since every finite set of elements of RG is contained in RH for some finitely generated subgroup H of G, this follows from (i).

Not only are regular elements of RH regular in RG, when H is a subgroup of G, but we also have

LEMMA 3.5. (Smith, [47, Lemma 2.6]) Let H be a normal subgroup of G, and suppose RH has a classical right quotient ring. Then RG satisfies the right Ore condition with respect to the multiplicatively closed set $\mathcal{C}_{RH}(0)$.

PROOF:

First note that if $g \in G$ and $d \in \mathcal{C}_{RH}(0)$, then $d^g = g^{-1}dg \in \mathcal{C}_{RH}(0)$. For certainly $d^g \in RH$, since $H \lhd G$, and if $\alpha \in RH$ is such that $d^g\alpha = 0$, we have

$$(g^{-1}dg)\alpha = 0 \Rightarrow dg\alpha = 0 \Rightarrow g\alpha = 0 \Rightarrow \alpha = 0.$$

Now suppose $c \in \mathcal{C}_{RH}(0)$, and $r = \sum_{i=1}^{n} r_i g_i \in RG$. For $i = 1, \ldots, n$, we deduce, since $r_i \in R \subset RH$, that there exists $b_i \in \mathcal{C}_{RH}(0)$, $t_i \in RH$, such that

$$r_i b_i = ct_i.$$

Putting $b_i' = b_i^{g_i}$, we have $b_i' \in \mathcal{C}_{RH}(0)$ by the remark above, and

$$(r_i g_i)b_i' = (r_i b_i)g_i = c(t_i g_i).$$

Now by Lemma 2.22(i), there exist elements $b \in \mathcal{C}_{RH}(0)$,

$$e_1, \ldots, e_n \in RG,$$

such that
$$b_i' e_i = b,$$

for $i = 1, \ldots, n$. Thus

$$\begin{aligned} rb = \left(\sum_{i=1}^{n} r_i g_i\right)b &= \sum_{i=1}^{n} r_i g_i b_i' e_i \\ &= \sum_{i=1}^{n} ct_i g_i e_i \\ &= c\gamma, \end{aligned}$$

where $\gamma \in RG$, and the lemma is proved.

We state one further corollary to Lemma 3.4.

LEMMA 3.6. Let the group G have a set of subgroups $\{H_\lambda : \lambda \in \Lambda\}$ for some index set Λ, such that every finite set of elements of G lies in some H_λ, and suppose that for each $\lambda \in \Lambda$, RH_λ satisfies the right Ore condition. Then RG satisfies the right Ore condition.

PROOF:

If $\alpha \in RG$, $\beta \in \mathcal{C}_{RG}(0)$, there exists some $\lambda \in \Lambda$ such that α and β are members of RH_λ. Since RH_λ satisfies the right Ore condition, there exist $\gamma \in RH$, $\delta \in \mathcal{C}_{RH}(0)$ such that

$$\alpha\delta = \beta\gamma.$$

Since $\delta \in \mathcal{C}_{RG}(0)$ by Lemma 3.4, the proof is complete.

LEMMA 3.7. Let $G = \langle g_\lambda : \lambda \in \Lambda \rangle$, where Λ is an index set. The map

$$\gamma : RG \longrightarrow R : \sum_{g \in G} r_g g \longmapsto \sum_{g \in G} r_g$$

is an epimorphism of rings, whose kernel is generated as a right (or left) R-module by the set $\{(g - 1) : 1 \neq g \in G\}$, and as a right (or left) RG-module by $\{(g_\lambda - 1) : \lambda \in \Lambda\}$.

PROOF:

It is straightforward to check that γ is a ring epimorphism. Since $\alpha \in RG$ may be written in the form

$$\alpha = r + \sum_{1 \neq g \in G} r_g (g - 1),$$

where $r, r_g \in R$, and $r_g = 0$ for all but finitely many $g \in G$, it is clear that $\ker\gamma$ is as described. For the last part, note that the right ideal

$$I = \sum_{\lambda \in \Lambda} (g_\lambda - 1)RG$$

is certainly contained in $\ker\gamma$, while if we put

$$X = \{x \in G : (x - 1) \in I\},$$

then since, if $x, y \in G$,

$(xy - 1) = (x - 1)y + (y - 1)$, and $(x^{-1} - 1) = -(x - 1)x^{-1}$, X is a subgroup of G containing g_λ, for all $\lambda \in \Lambda$. Thus $X = G$, and $I = \ker \gamma$, as required.

DEFINITION 3.8. In the above situation, $\ker \gamma$ is called the augmentation ideal of RG, and will be denoted by $\underline{g}R$, or simply by $\underline{g}$ if no confusion seems likely. We shall occasionally write $\omega(H)$ for $\underline{h}G$ when H is a normal subgroup of G.

DEFINITION 3.9. If H is a normal subgroup of G, an ideal I of RH is said to be G-invariant if $I^g = I$ for all $g \in G$. (Here $I^g = \{g^{-1}\alpha g : \alpha \in I\}$.)

Note that $\underline{h}$, the augmentation ideal of RH, is G-invariant.

LEMMA 3.10. Let H be a normal subgroup of G, and suppose that I is a G-invariant ideal of RH. Then

$IG = \{\sum_{g\in G} \alpha_g g : \alpha_g \in I,\ \alpha_g \neq 0$ for only finitely many $g \in G\}$ is a two-sided ideal of RG, and for all $n \geq 1$, $(IG)^n = I^nG$.

PROOF:

The first part is a straightforward check, and $(IG)^n = I^nG$ is then proved by induction on n.

LEMMA 3.11. (i) If H is a normal subgroup of G, the map

$$\varphi : RG \longrightarrow R(G/H) : \sum_{g\in G} r_g g \longmapsto \sum_{g\in G} r_g \bar{g},$$

where $\bar{g}$ is the image of $g \in G$ under the canonical map $G \longrightarrow G/H$, is a ring epimorphism with kernel $\underline{h}G$. The augmentation ideal of RG is mapped by φ onto the augmentation ideal of $R(G/H)$.

(ii) If I is an ideal of R, so that by Lemma 3.10

IG is an ideal of RG, then

$$RG/IG \cong (R/I)G$$

as rings.

PROOF:

(i) Again, this is a simple check, or see [37, Ch.1].

(ii) It is clear that the kernel of the epimorphism

$$\tau : RG \longrightarrow (R/I)G : \sum_{g\in G} r_g g \longmapsto \sum_{g\in G}(r_g + I)g$$

is IG.

We shall now obtain sufficient conditions for the intersection of the powers of the augmentation ideal of a group ring to be zero.

LEMMA 3.12. If G is a finite group and K is a field, the augmentation ideal of KG is nilpotent if and only if K has characteristic $p > 0$, and G is a finite p-group.

PROOF:

If $1 \neq g \in G$ has order t, where $\mathrm{char}K \nmid t$, then

$$(1 - 1/t[\sum_{i=1}^{t} g^i]) \in \underline{\underline{g}}$$

is an idempotent. For the converse, see [36, Lemma 10.1(ii)] or, more easily, argue by induction on n, where $|G| = p^n$.

LEMMA 3.13. (Wallace: [53, Lemma 2.1]) Let $\mathcal{S} = \{H_\lambda : \lambda \in \Lambda\}$ be a set of normal subgroups of a group G, such that

(a) for all $\mu, \nu \in \Lambda$, $H_\mu \cap H_\nu \in \mathcal{S}$;

and (b) $\bigcap_{\lambda\in\Lambda} H_\lambda = 1$.

Then if R is any ring, $\bigcap_{\lambda\in\Lambda}(\underline{\underline{h}}_\lambda RG) = 0$.

LEMMA 3.14. Let K be a field of characteristic $p > 0$, and $G \in R\mathcal{F}_p$. Then in KG, $\bigcap_{n=1}^{\infty} \underline{\underline{g}}^n = 0$.

PROOF:

Since if H and L are normal subgroups of G such that

G/H and G/L are finite p-groups, then $G/(H \cap L) \in \mathcal{F}_p$, we can find a set $\mathcal{S} = \{H_\lambda : \lambda \in \Lambda\}$ of normal subgroups of G satisfying (a) and (b) of Lemma 3.13, and in addition such that $G/H_\lambda \in \mathcal{F}_p$ for all $\lambda \in \Lambda$. The result now follows from the previous three lemmas.

LEMMA 3.15. Let $\underline{\underline{g}}$ be the augmentation ideal of the group ring RG. Then $r(\underline{\underline{g}}) \neq 0$ if and only if G is finite, and in this case,

$$r(\underline{\underline{g}}) = l(\underline{\underline{g}}) = \hat{G}R,$$

where $\hat{G} = \sum_{g \in G} g$, so that $\hat{G}R$ is an ideal of RG. Further,

$$r(\hat{G}) = l(\hat{G}) = \underline{\underline{g}}.$$

PROOF:

See [37, Lemma 3.1.2], where the assumption that R is a field is unnecessary.

PROPOSITION 3.16. (Connell; [36, Thm.2.6] or [6, Thm.1]) The group ring RG is Artinian if and only if R is Artinian and G is finite.

The question of when the group ring RG is Noetherian does not admit of as straightforward an answer as that provided by the previous result.

THEOREM 3.17. (i) If RG is Noetherian, R is Noetherian and G satisfies the ascending chain condition on subgroups.

(ii) Let S be a ring, let R be a right Noetherian subring of S, with the same 1, and let G be a polycyclic-by-finite group of units in S. If $g^{-1}Rg = R$ for all $g \in G$ and $S = \langle R, G \rangle$, then S is right Noetherian.

(iii) (Hall) If R is a Noetherian ring and $G \in P(\mathcal{L} \cup \mathcal{F})$, RG is Noetherian.

PROOF:

(i) See [30, Prop.1, p.153].

(ii) See [37, Thm.10.2.7].

(iii) This is of course immediate from (ii).

Recall that the only known groups satisfying the ascending chain condition on subgroups are those in the class $P(\mathfrak{L} \cup \mathfrak{F})$, so the question of whether the converse of Theorem 3.17(i) is true is at least partly a group theoretic problem.

PROPOSITION 3.18. (Connell; [36, Thm.2.5] or [6, Thm.8]) The group ring RG is prime if and only if R is prime and G has no finite normal subgroups.

PROPOSITION 3.19. (Connell, Renault; [37, Thm.3.2.8]) The group ring RG is self-injective if and only if R is self-injective and G is finite.

Since $\Delta^{+}(G)$ is the join of the finite normal subgroups of the group G, Prop.3.18 gives one indication of the importance of FC-subgroups in the study of group rings. This can also be seen from

THEOREM 3.20. (Passman, Connell) Let K be a field and G a group. If (i) K has characteristic zero, KG is semiprime. If (ii) K has characteristic $p > 0$, then

$$N(KG) = N\big(K\Delta(G)\big)KG = J\big(K\Delta(G)\big)KG = J\big(K\Delta^{p}(G)\big)KG.$$

Moreover,

$$J\big(K\Delta^{p}(G)\big)KG = \bigcup_{W} J(KW)KG,$$

as W ranges over all finite normal subgroups of G, of order divisible by p.

PROOF:

(i) [36, Thm.3.3].

(ii) [36, Lemma 19.6 and Thm.20.2].

We give one further example which illustrates the importance of FC-subgroups in the study of group rings.

LEMMA 3.21. If RG is a group ring, the centre C(RG) of RG is a subring of $R\Delta(G)$.

PROOF:

Let $\alpha = \sum_{g\in G} r_g g \in C(RG)$, choose $x \in \operatorname{supp}\alpha$, and take $y \in G$. Then

$$\alpha^y = r_x x^y + \sum_{g\neq x} r_g g^y = \alpha,$$

so that $x^y \in \operatorname{supp}\alpha$, a finite set. Thus x has only finitely many conjugates in G; that is, $x \in \Delta(G)$.

Theorem 3.17(iii) is essentially a non-commutative version of the Hilbert Basis Theorem, [2, Thm.7.5], because of the following observation, which explains our interest in twisted polynomial rings in §2.

LEMMA 3.22. Let RG be a group ring, and H a normal subgroup of G such that $G/H \cong C_\infty$. Then RG is isomorphic to the twisted polynomial ring $RH[X,X^{-1};\sigma]$, where σ is the automorphism of RH induced by conjugation by x, where $G = \langle H, x\rangle$.

PROOF:

Elements of RG can be uniquely written in the form

$$\alpha = \sum_{i=-m}^{n} \alpha_i x^i,$$

where $m, n \geqslant 0$ and $\alpha_i \in RH$, $-m \leqslant i \leqslant n$, by Lemma 3.3(i). It is straightforward to check that the map

$$RG \longrightarrow RH[X, X^{-1};\sigma] \;:\; \sum_{i=-m}^{n}\alpha_i x^i \longmapsto \sum_{i=-m}^{n}\alpha_i X^i$$

is a ring isomorphism.

COROLLARY 3.23. Suppose that the group ring RH is a domain, and that H is a normal subgroup of a group G, such that

$G/H \in LP(\mathcal{L}_\infty)$. Then RG is a domain.

PROOF:

Suppose the result is false, and choose non-zero elements α and β of RG such that $\alpha\beta = 0$. Put

$$T = \langle H, \operatorname{supp}\alpha, \operatorname{supp}\beta \rangle,$$

so that $T/H \in P(\mathcal{L}_\infty)$. By Lemma 3.21 and a standard argument on polynomial rings, (adapted to deal with twisted rings), RT is a domain, contradicting the existence of α and β.

The proof is thus complete.

We finally consider briefly the problem which has excited most interest and research activity in the study of group rings over the past ten years - namely, that of determining the structure of J(KG), the Jacobson radical of the group algebra KG. We begin with the long-standing

CONJECTURE 3.24. If G is any group and K is a field of characteristic zero, J(KG) = 0.

The above conjecture is known to be true for large classes of groups, see for example [38, §5, §6], and it is also true for any group G if K is not algebraic over the rational field $\mathbb{Q}$. This result is due to Amitsur; see [36, §18].

Note that if G is finite, KG is Artinian, and J(KG) = N(KG). Passman, [37, 38], has conjectured that this generalises as follows:

CONJECTURE 3.25.(i) If G is finitely generated,

$$J(KG) = N(KG).$$

(ii) For any group G,

$$J(KG) = \{\alpha \in KG : \alpha \in N(KH) \text{ for all finitely generated groups } H \text{ such that } \operatorname{supp}\alpha \subseteq H \subseteq G\}.$$

Notice that Conjecture 3.25 is stronger than Conjecture 3.24, by Theorem 3.20(i). As regards (ii) of the above conjecture, clearly the right hand side of the conjectured equality is contained in $J(KG)$, and if (i) is true, we in fact have equality, by

LEMMA 3.26. Let H be a subgroup of the group G, and let K be a field. Then

$$J(KG) \cap KH \subseteq J(KH).$$

PROOF:

Take $\alpha \in J(KG) \cap KH$, and $\beta \in KH$. Now $(1 - \alpha\beta)$ is a unit in KG, and Lemma 3.3(i) may be applied to show that $(1 - \alpha\beta)$ is in fact a unit in KH, so that $\alpha \in J(KH)$, as required.

Since Conjecture 3.24 and Conjecture 3.25(i) are certainly true if G is finite, we deduce, using the above lemma, the following result.

PROPOSITION 3.27. Conjecture 3.25(ii) is true for all locally finite groups G.

COROLLARY 3.28.(i) If K is a field of characteristic zero and $G \in L\mathcal{F}$, then $J(KG) = 0$.

(ii) If K is a field of characteristic $p > 0$, and $G \in L\mathcal{F}_{p'}$, then $J(KG) = 0$.

LEMMA 3.29. Let K be a field of characteristic $p > 0$, and suppose $G \in L\mathcal{F}_p$. Then $J(KG) = \underline{\underline{g}}$.

PROOF:

By Lemma 3.12, $\underline{g}$ is a nil ideal of KG. Since $KG/\underline{g} \cong K$, by Lemma 3.7, $\underline{g}$ is maximal, and the result follows.

The set defined in Conjecture 3.25(ii), which Passman has labelled $N^*(KG)$, forms a nil ideal of KG. Using Theorem 3.20, one can prove

THEOREM 3.30. (Passman; [37, Thm.8.2.6]) If K is a field and G is a group,

$$N^*(KG) = J(K\Lambda^+(G))KG,$$

where $\Lambda^+(G) = \{x \in G : |x| < \infty, \text{ and } |H : C_H(x)| < \infty \text{ for all } H \subseteq G, H \in \mathcal{G}\}$.

Finally, as an example of the type of result which has been obtained in recent years, we have

THEOREM 3.31. (Zalesskii) Let G be a soluble group, and let K be a field. Then $N^*(KG) = J(KG)$.

For further results along these lines, see [38].

CHAPTER 2

THE ZERO DIVISOR PROBLEM

1. INTRODUCTION

The Zero Divisor Problem is one of the oldest and most difficult problems in the field of Infinite Group Rings. If G is a group and R is a ring, then clearly if RG is a domain, R must be a domain. Furthermore, if G contains a non-trivial element x of finite order n, say, then

$$(1 - x)(1 + x + \ldots + x^{n-1}) = 0,$$

so if RG is a domain, G must be torsion-free. The question at issue is whether these conditions on R and G are sufficient to ensure that RG is a domain; that is, if R is a domain and G is a torsion-free group, is RG a domain? Usually we restrict attention to commutative coefficient rings R.

There is very little evidence, other than the absence of counterexamples, to support this conjecture. The usual approach has been to try to prove the conjecture for suitable classes of groups G, and this has been successful only for fairly restricted classes. The question was first raised in the literature in a paper by G. Higman published in 1940, in which he proved that KG is a domain if K is a field and G is a group such that every finitely generated subgroup of G has an infinite cyclic image - free groups, for example, have this property. We remark in passing that one may easily deduce Higman's result from Corollary 1.2.23.

The question next appeared in Kaplansky's ''Problems in the Theory of Rings'', in 1957, but except for some work on ordered and right ordered groups, no real progress was made until 1972, when Lewin, making use of some work of Cohn on free ideal rings, proved

THEOREM 1.1. [33] Let K be afield and let G be a group. Suppose that G has subgroups A, B, and N, with $N \lhd G$, $N \leq A$, $N \leq B$, and with $G/N \cong (A/N * B/N)$. If KA and KB have no zero divisors and if KN is an Ore domain, then KG is a domain.

Finally, Formanek [11], used Theorem 1.1, Theorem 1.1.17(ii), Lemma 1.1. 24 and Cor.1.3.23 to show that if G is supersoluble and torsion-free, and K is a field, then KG is a domain. In fact, we shall make use of Formanek's argument in extending one of our results, and we shall provide another proof of Formanek's result for coefficient fields of characteristic zero, (see Prop.4.7).

In this chapter, we shall prove the Zero Divisor Conjecture for coefficient fields of characteristic zero when G is a torsion-free group which is (i) locally nilpotent-by-locally finite-p for some prime p,(Prop.4.7), or (ii) (abelian-by-locally finite)-by-locally supersoluble, (Theorem 4.11).

It will become apparent that, by making use of Thm.1.1.15(ii) and Lemma1.1.14, we may limit our attention for the most part to polycyclic-by-finite groups and their group rings. Consequently, by Theorem 1.1.17(iii), we have available the results obtained in the study of non-commutative Noetherian rings over the past twenty years.

In particular, we shall use crucially a result of Walker which states that a prime Noetherian scalar local ring of finite global dimension is a domain. We shall apply Walker's theorem by embedding suitably chosen group rings in the scalar local rings obtained by localizing at their augmentation ideals.

The results of this chapter are arranged as follows. In §2, we show how it is possible to pass to coefficient fields of positive characteristic, in which situation the augmentation ideal is more often localizable. This latter fact we establish in §3. Our approach here is via the AR property, and we shall give in this section some results which are not, strictly speaking, necessary for our immediate purpose. In particular, we shall generalise a result of Jategaonkar which shows that group rings of polycyclic-by-finite groups have ''many'' ideals with the strong AR property, (Theorem 3.5). Some of the results of §3 have been obtained independently by Roseblade, [41], using somewhat different arguments. In §4, we obtain our results on the Zero Divisor Problem, and finally we add a short discussion in §5 on the progress made towards a solution to the Zero Divisor Problem since this research was carried out.

We remark that the results of §4 can be easily extended using Cor.1.3.23; we shall not point this out on stating each particular theorem. Since a polycyclic-by-finite group G has, by Theorem 1.1.13, a normal poly-(infinite cyclic) subgroup H of finite index, and, for any field K, KH is a domain by Cor.1.3.23, we see that to prove the Zero Divisor Conjecture for this class of groups, all we must do is ''deal with'' the finite factor G/H, when G is torsion-free. It is perhaps surprising how difficult this step is, and indeed how little our methods appear to depend on the existence of the subgroup H.

2. THE COEFFICIENT RING

Suppose that G is a torsion-free group, R is a commutative domain, and K is the quotient field of R. Since $RG \subseteq KG$, if KG is a domain then so is RG. Conversely, suppose there exist non-zero elements α and β of KG such that $\alpha\beta = 0$. Clearly there exists $r \in R$ such that

$$0 \neq r\alpha, r\beta \in RG,$$

for we may take r to be a common multiple of the denominators of the coefficients of the supports of α and β, where these coefficients are expressed as fractions whose numerators and denominators are elements of R. Then

$$0 = \alpha\beta = 1/r^2 \, .r\alpha.r\beta,$$

and since, by Lemma 1.3.4, the element $1/r^2$ is regular in KG, we must have

$$0 = (r\alpha)(r\beta).$$

We have thus proved

<u>LEMMA 2.1</u>. Let G be a group, and R a commutative domain with quotient field K. Then KG is a domain if and only if RG is a domain.

<u>DEFINITION 2.2</u>. Let K be a field, and let K^* denote the set of non-zero elements of K.

(i) A <u>discrete valuation</u> on K is a surjective map $v: K^* \longrightarrow \mathbb{Z}$ satisfying (a) $v(xy) = v(x) + v(y)$;

$$\text{(b) } v(x + y) \geqslant \min\{v(x), v(y)\},$$

for all $x, y \in K^*$, where $\geqslant$ denotes the usual ordering on $\mathbb{Z}$. We make in addition the convention $v(0) = \infty$.

(ii) Let $v: K^* \longrightarrow \mathbb{Z}$ be a discrete valuation on the field K. Then

$$R = \{x \in K : v(x) \geqslant 0\}$$

is a subring of K, containing 1, called the <u>discrete valuation ring of</u> v, abbrieviated to d.v.r.

For example, if we take $K = \mathbb{Q}$, the rational field, and let p be any prime, then we may define the p<u>-adic valuation</u>

$$v_p : \mathbb{Q}^* \longrightarrow \mathbb{Z} : p^i(a/b) \longmapsto i,$$

where $a, b \in \mathbb{Z}$ and $p \nmid a$, $p \nmid b$. The corresponding d.v.r. is simply the ring obtained by localizing the integers at the prime p.

For a discussion of valuation rings, see for example [2]. We shall need the following well-known facts, which we group into a lemma.

<u>LEMMA 2.3</u>. Let R be a d.v.r., corresponding to a valuation v of a field K. Then K is the quotient field of R, and R is a principal ideal domain with unique maximal ideal M, where

$$M = \{x \in R : v(x) > 0\}.$$

Moreover M is the set of non-units of R; in fact, any non-zero element x of R can be written as

$$x = u\pi^i,$$

where $M = \pi R$, $i \geqslant 0$, and u is a unit of R. Thus M is the Jacobson radical of R, and R/M is a field.

DEFINITION 2.4. The field R/M is called the residue field of R (and of v).

We shall need one other result on valuation rings, whose proof is a straightforward adaptation of the proof of [35].

THEOREM 2.5. Let K be a field of characteristic zero, v a discrete valuation of K with corresponding d.v.r. R. Let M be the maximal ideal of R, and suppose that the residue field R/M has characteristic $p > 0$. Let $K(\tau)$ be an extension field of K. Then there exists a valuation v' of $K(\tau)$ such that $v'|_K = v$.

Now let p be any prime. Since, as we have noted, there exists a valuation of $\mathbb{Q}$ with residue field of characteristic p, the above result shows that if $K = \mathbb{Q}(a_1, \ldots, a_n)$ is any finitely generated extension field of $\mathbb{Q}$, there exists a valuation v of K with corresponding residue field of characteristic p.

We now return to consideration of the Zero Divisor Problem. Let G be a torsion-free group, K a field of characteristic zero, and p a prime. Suppose that KG is not a domain, so that there exist elements

$$0 \neq \alpha = \sum_{i=1}^{n} \lambda_i g_i, \qquad 0 \neq \beta = \sum_{j=1}^{m} \mu_j h_j$$

of KG, where $g_i, h_j \in G$ and $\lambda_i, \mu_j \in K$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, such that $\alpha\beta = 0$. Hence LG is not a domain, where $L = \mathbb{Q}(\lambda_i, \mu_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$.

As remarked above, there exists a discrete valuation v of L with corresponding d.v.r. R_v, whose maximal ideal we denote by M_v, such that R_v/M_v is a field of characteristic p. Since R_V is an order in L, it follows from Lemma 2.1 that R_VG is not a domain. Let γ, δ be non-zero elements of R_VG such that

$$\gamma\delta = 0;$$

suppose that $\gamma = \sum_{i=1}^{t} r_i g_i$, $\delta = \sum_{j=1}^{\mu} s_j h_j$, where $r_i, s_j \in R_V$, $g_i, h_j \in G$, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant \mu$. Put $M_V = \pi R_V$, and note that by Lemma 2.3 we may write

$$r_i = \pi^{a(i)} u_i, \qquad s_j = \pi^{b(j)} w_j,$$

where a(i), b(j) are non-negative integers, and u_i, w_j are units of R_V, $1 \leqslant i \leqslant t$, $1 \leqslant j \leqslant \mu$.

Let $a = \min\{a(i) : 1 \leqslant i \leqslant t\}$, $b = \min\{b(j) : 1 \leqslant j \leqslant \mu\}$, so that

$$0 = \gamma\delta = \pi^{a+b}\left(\sum_{i=1}^{t} \pi^{(a(i)-a)} u_i g_i\right)\left(\sum_{j=1}^{\mu} \pi^{(b(j)-b)} w_j h_j\right).$$

Since $\pi^{a+b} \in \mathscr{C}_{R_VG}(0)$, by Lemma 1.3.4, we must have

$$\gamma_0 \delta_0 = 0,$$

where $\gamma_0 = \sum_{i=1}^{t} \pi^{(a(i)-a)} u_i g_i$, $\delta_0 = \sum_{j=1}^{\mu} \pi^{(b(j)-b)} w_j h_j$ are non-zero elements of R_VG. Now, by choice of the non-negative integers a and b,

$$\gamma_0 \notin M_VG, \qquad \delta_0 \notin M_VG,$$

so that under the canonical homomomorphism τ of Lemma 1.3.11(ii) from R_VG onto $(R_V/M_V)G$, we have

$$0 \neq \overline{\gamma}_0 = \tau(\gamma_0), \qquad 0 \neq \overline{\delta}_0 = \tau(\delta_0),$$

but

$$\overline{\gamma}_0 \overline{\delta}_0 = \tau(\gamma_0\delta_0) = 0.$$

We have thus shown that $(R_V/M_V)G$ is not a domain, thus proving the main result of this section:

<u>LEMMA 2.6</u>. Let G be a torsion-free group, and let p be a prime. If FG is a domain for all fields F of characteristic p, then KG is a domain for all fields K of characteristic zero.

3. THE AR PROPERTY IN GROUP RINGS

As explained in §1, our approach will be to embed certain group rings in Noetherian local rings, by localizing at their augmentation ideals. Accordingly, in this section we consider the problem of ascertaining when the augmentation ideal of a group ring is localizable. In view of Prop.1.2.41, this leads us to investigate when the augmentation ideal has the AR property.

DEFINITION 3.1. Let p be a prime.

(i) A finite group G is said to be p-nilpotent if it has a normal p'-subgroup H such that G/H is a p-group.

(ii) A polycyclic-by-finite group G is said to be p-nilpotent if every finite image of G is p-nilpotent.

Note that since polycyclic-by-finite groups are residually finite, the above definition is reasonably restrictive. Indeed, since a polycyclic-by-finite group is nilpotent if and only if all of its finite images are nilpotent, we deduce easily that a polycyclic-by-finite group is nilpotent if and only if it is p-nilpotent for all primes p. Nevertheless, Roseblade has shown that if G is a polycyclic-by-finite group, G has a normal p-nilpotent subgroup of finite index, for all primes p; see [37, Lemma 11.2.16]. This last result will appear as an easy consequence of some of the results of this section, although the group theoretic proof is of course much simpler.

When G is finite and K is a field of characteristic zero, every ideal of KG has the AR property by Maschke's Theorem, [30, p.156]. If K has characteristic $p > 0$, we have the following result, due to Smith.

LEMMA 3.2. Let G be a finite group, K a field of characteristic $p > 0$. The following are equivalent:

(i) G is p-nilpotent;

(ii) $\underline{\underline{g}} \triangleleft KG$ has the right AR property;

(iii) $\underline{\underline{g}} \triangleleft KG$ is localizable.

PROOF:

By Proposition 1.3.16, KG is Artinian, so the result follows from [48, Theorem 3.4] and [49, Theorem 2.4].

In order to prove a similar result when G is polycyclic-by-finite, we shall need a strengthened version of a theorem dueto Jategaonkar. To prove this, we require a lemma which is in fact a corollary of a non-commutative form of the Hilbert Basis Theorem obtained by McConnell.

LEMMA 3.3. [25, Lemma 6] Let R be a ring, and I an ideal of R with the strong right AR property. If c is an element of R such that $(c + I^2)$ lies in the centre of R/I^2, then the ideal $(Rc + I)$ has the strong right AR property.

LEMMA 3.4. Let R be a right Noetherian ring, G a polycyclic -by-finite group with a normal subgroup H, and suppose I is a G-invariant ideal of RH with the strong right AR property. Then the ideal IG of RG has the strong right AR property.

PROOF:

In the notation of Defn.1.2.42, we must show that $RG^*(IG)$ is a right Noetherian ring. For all $s \geqslant 1$, $(IG)^s = I^sG$, by Lemma 1.3.10, so that if $\alpha = \sum_{i=0}^{n} \alpha_i X^i \in RG^*(IG)$, then

$\alpha^i \in I^iG$ for $0 \leq i \leq n$. Thus if $\{g_\lambda : \lambda \in \Lambda\}$ is a transversal to H in G, it follows that

$$\alpha = \sum_{i=0}^{n} \alpha_i X^i = \sum_{i=0}^{n}\left(\sum_{j=1}^{m(i)} \alpha_{ij} g_{\lambda(j)}\right) X^i,$$

where $\lambda(j) \in \Lambda$ and $\alpha_{ij} \in I^i$, for $0 \leq i \leq n$, $1 \leq j \leq m(i)$. Thus, putting $t = \max\{m(i) : 0 \leq i \leq n\}$, we have,

$$\alpha = \sum_{j=1}^{t}\left(\sum_{i=0}^{n} \alpha_{ij} X^i\right) g_{\lambda(j)},$$

where, for $j = 1, \ldots, t$, $\left(\sum_{i=0}^{n} \alpha_{ij} X^i\right) \in RH^*(I)$, a right Noetherian ring. Since $g_\lambda \in G$, for all $\lambda \in \Lambda$, and I is a G-invariant ideal of RH, we deduce that

$$RG^*(IG) = \langle RH^*(I), G \rangle,$$

where $g^{-1}RH^*(I)g = RH^*(I)$, for all $g \in G$, so that the result follows by Theorem 1.3.17(ii).

In the next result, the term "strong AR property" will mean "strong right and left AR property". It should be compared with [25, Theorem 2].

<u>THEOREM 3.5</u>. Let K be a field of characteristic $p > 0$, and let G be a polycyclic-by-finite group. If H is a normal subgroup of finite index in G, there exists $Q \lhd G$, $Q \leq H$, such that $|H : Q| < \infty$ and $\underline{\underline{q}}G$ has the strong AR property.

<u>PROOF</u>:

Since all subgroups of G are finitely generated, it follows from Lemma 1.1.6 that G has a series of normal subgroups

$$1 = G_0 \subset G_1 \subset \ldots \subset G_i \subset G_{i+1} \subset \ldots \subset G_d = G \qquad (1)$$

such that G_{i+1}/G_i is either finite or finitely generated torsion-free abelian, for $0 \leq i < d$. We shall prove the theorem by induction on d, the length of the series (1); specifically, we shall show that, for $0 \leq k \leq d$, if $H \lhd G$, $H \subseteq G_k$, and $|G_k : H| < \infty$, then there exists a normal subgroup Q of G, contained in H, such that $|H : Q| < \infty$ and $\underline{\underline{q}}G_k$ has the strong AR property in KG_k.

For $k = 0$, this is trivial, while the case $k = d$ is the required result. We shall assume therefore that we have shown the above statement to be true for some $k < d$, and aim to prove it for $(k+1)$.

<u>Case (i)</u>: Suppose G_{k+1}/G_k is finite. Let H be a normal subgroup of G, such that $H \subseteq G_{k+1}$ and $|G_{k+1} : H| < \infty$. Then $(H \cap G_k) \lhd G$ and $|G_k : (H \cap G_k)| < \infty$, so by our induction hypothesis there exists a normal subgroup Q of G contained in $(H \cap G_k)$, such that $(H \cap G_k)/Q$ is finite and $\underline{\underline{q}}G_k$ has the strong AR property in KG_k. Thus by Lemma 3.4, $\underline{\underline{q}}G_{k+1}$ has the strong AR property in KG_{k+1}. Since $|G_k : Q| < \infty$, $|G_{k+1} : Q| < \infty$, and the induction step is complete in this case.

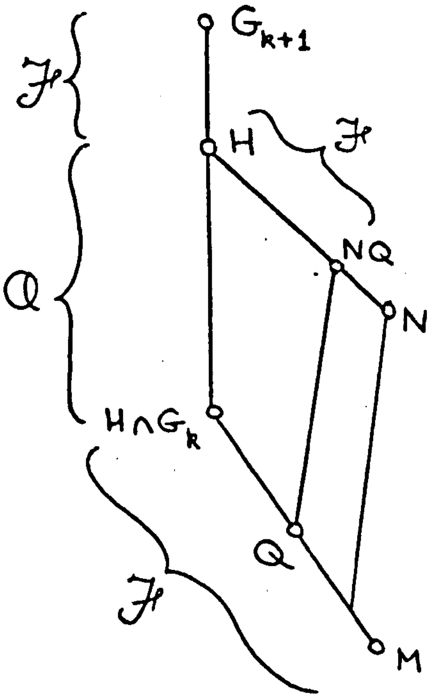
<u>Case (ii)</u>: Suppose now that G_{k+1}/G_k is finitely generated torsion-free abelian. Given a subgroup H as in Case (i), we can find a normal subgroup Q exactly as above, such that $\underline{\underline{q}}G_{k+1}$ has the strong AR property in KG_{k+1}. Let $M = Q^p$, the subgroup of G_k generated by the pth powers of the elements of Q. Note that since Q is normal in G, so is M, and Q/M, being periodic and polycyclic-by-finite, is finite by Theorem 1.1.13(ii). If $m \in M$ is a generator of M, so that $m = x^p$, say, for some $x \in Q$,

$$(m - 1) = (x - 1)^p \in (\underline{\underline{q}}G)^p \subseteq (\underline{\underline{q}}G)^2.$$

Thus $\underline{\underline{m}}G \subseteq (\underline{\underline{q}}G)^2$, by Lemma 1.3.7.

Put $T = H/M$, so that since $H/(G_k \cap H)$ is abelian and

$|(G_k \cap H) : M| < \infty$, $|T'|$ is finite. Since T is finitely generated, $|T : Z(T)| < \infty$ by Lemma 1.1.20. Let $N \subseteq H$ be such that $N/M = Z(T)$, so that since $Z(T)$ is characteristic in T, N is normal in G.



Consider now the ideal $\underline{\underline{n}}G_{k+1}$ of KG_{k+1}; if N/M is generated by the set $\{f_iM : f_i \in N,\ 1 \leqslant i \leqslant s\}$, then $\underline{\underline{n}}G_{k+1}/\underline{\underline{m}}G_{k+1}$ is generated as a right KG_{k+1}-module by $\{(f_i - 1) + \underline{\underline{m}}G_{k+1} : 1 \leqslant i \leqslant s\}$, by Lemmas 1.3.11(i) and 1.3.7. Hence $(\underline{\underline{n}}G_{k+1} + \underline{\underline{q}}G_{k+1})/\underline{\underline{q}}G_{k+1}$ is generated as a right KG_{k+1}-module by

$$\{(f_i - 1) + \underline{\underline{q}}G_{k+1} : 1 \leqslant i \leqslant s\},$$

since $\underline{\underline{q}}G_{k+1} \supseteq \underline{\underline{m}}G_{k+1}$.

Furthermore, for $i = 1, \ldots, s$ and $g \in G_{k+1}$,

$$g(f_i - 1) = ([g^{-1}, f_i^{-1}] - 1)f_ig + (f_i - 1)g.$$

Thus since $[g^{-1}, f_i^{-1}] \in M$ and $\underline{\underline{m}}G_{k+1} \subseteq (\underline{\underline{q}}G_{k+1})^2$, $(f_i - 1) + (\underline{\underline{q}}G_{k+1})^2$ is a central element of $KG_{k+1}/(\underline{\underline{q}}G_{k+1})^2$, for $i = 1, \ldots, s$.

By applying Lemma 3.3 s times we may now deduce that the ideal $(\underline{\underline{n}}G_{k+1} + \underline{\underline{q}}G_{k+1})$ of KG_{k+1} has the strong AR property. However this last ideal is simply the ideal of KG_{k+1} generated by the augmentation ideal of the group NQ, and since $|G_{k+1} : N| < \infty$, we have completed the proof of the induction step in this case also.

Note that the assumption that K is a field of positive characteristic is used in the above proof only to ensure that if P is a polycyclic-by-finite group, then the ideal $\underline{p}^2$ of KP contains the augmentation ideal of a characteristic subgroup of finite index in P. Now if I is a finitely generated ideal of a ring S, then $|S/I| < \infty$ implies $|S/I^n| < \infty$ for all integers $n \geqslant 1$, so, substituting this observation at the appropriate point in the argument, we can obtain the following variant of the above theorem.

<u>THEOREM 3.6</u>. Let R be a commutative Noetherian ring, I an ideal of R such that $|R : I| < \infty$, and suppose G is a polycyclic-by-finite group with a normal subgroup H of finite index. Then there exists a normal subgroup N of finite index in G such that $N \subseteq H$ and

$$IG + \underline{n}G$$

has the strong AR property.

We now apply Theorem 3.5 to obtain the result we shall need in the remainder of this chapter.

<u>THEOREM 3.7</u>. Let K be a field of characteristic $p > 0$, and let G be a polycyclic-by-finite group. Then the following are equivalent:

(i) G is p-nilpotent;

(ii) The augmentation ideal $\underline{g}$ of KG has the right AR property;

(iii) $\underline{g}$ is localizable.

<u>PROOF</u>:

Note first that KG is certainly Noetherian, by Theorem 1.3.17(iii).

(i) $\Longrightarrow$ (ii): We begin by showing that it is enough to prove that if a finitely generated KG-module M is an essential extension of a module N, and $N\underline{\underline{g}} = 0$, then there exists an integer $n \geqslant 1$ such that $M\underline{\underline{g}}^n = 0$. For if U is any finitely generated KG-module with a submodule V, choose a submodule Q of U maximal with respect to

$$(Q \wedge V) = V\underline{\underline{g}} .$$

Put $M = U/Q$, $N = (V + Q)/Q$, so that N is essential in M, and $N\underline{\underline{g}} = 0$. Suppose we have shown that $M\underline{\underline{g}}^n = 0$ for some $n \geqslant 1$. Then $U\underline{\underline{g}}^n \subseteq Q$, so

$$(U\underline{\underline{g}}^n \wedge V) \subseteq V\underline{\underline{g}} ,$$

as required.

Suppose therefore that M and N are as above. There exists a normal subgroup H of finite index in G such that $\underline{\underline{h}}G$ has the strong AR property, by Theorem 3.5. Since $\underline{\underline{h}}G \subseteq \underline{\underline{g}}$, it follows that $N(\underline{\underline{h}}G) = 0$, and so there exists $m \geqslant 1$ such that

$$M(\underline{\underline{h}}G)^m \wedge N \subseteq N(\underline{\underline{h}}G) = 0.$$

By Lemma 1.3.10, $(\underline{\underline{h}}G)^m = \underline{\underline{h}}^m G = G\underline{\underline{h}}^m$. Therefore, since $\underline{\underline{h}}^m$ is a G-invariant ideal of KH, $M(\underline{\underline{h}}G)^m = M\underline{\underline{h}}^m$ is a KG-submodule of M, and so since N is essential in M, we deduce that

$$M(\underline{\underline{h}}G)^m = 0.$$

By an argument similar to that used in the proof of Theorem 3.5, there exists a normal subgroup L of finite index in G, such that $\underline{\underline{l}}G \subseteq (\underline{\underline{h}}G)^m$. Now by Lemma 3.2 and the isomorphism of Lemma 1.3.11, the ideal $(\underline{\underline{g}}/\underline{\underline{l}}G)$ of $KG/\underline{\underline{l}}G$ has the right AR property, noting that since G is p-nilpotent, so is G/L. Since M is a finitely generated $KG/\underline{\underline{l}}G$-module, it follows that there exists $s \geqslant 1$ such that

$$M(\underline{\underline{g}}^s + \underline{\underline{l}}G) \wedge N \subseteq N\underline{\underline{g}} = 0,$$

and so <u>a priori</u> $M\underline{\underline{g}}^s = 0$, as required.

(ii) $\Longrightarrow$ (iii): This is a consequence of Prop.1.2.41.

(iii) $\Longrightarrow$ (i): Let H be a normal subgroup of finite index in G. By Lemma 1.2.34, $\underline{\underline{g}}/\underline{\underline{h}}G$ is a localizable ideal of $KG/\underline{\underline{h}}G$, so by Lemma 1.3.11(i) and Lemma 3.2, G/H is p-nilpotent. By definition, therefore, G is p-nilpotent.

REMARKS: (i) Using a somewhat different approach, Roseblade has obtained (i) $\Longleftrightarrow$ (ii) of the above theorem, [41, Cor.A3].

(ii) It follows from the above result that in Thm.3.5, the subgroup of finite index in G is p-nilpotent. It is thus natural to ask whether Theorem 3.7 can be strengthened to show that if K is a field of characteristic $p > 0$ and G is a polycyclic-by-finite group, G is p-nilpotent if and only if the augmentation ideal of KG has the strong AR property. This seems to be an open question.

(iii) For coefficient fields of characteristic zero, we are unable to obtain a complete analogue of Theorem 3.7. Roseblade and Smith have shown that if K is a field of characteristic zero and G is a polycyclic-by-finite group, the augmentation ideal of KG has the right AR property if and only if G is finite-by-nilpotent; see [37, Thm.11.2.14]. It is not clear, however whether the augmentation ideal of KG is localizable only if G is finite-by-nilpotent, although it seems probable that this is the case.

4. THE MAIN RESULTS

As explained in §1, we shall use crucially the following result.

THEOREM 4.1. (Walker) Let R be a right Noetherian, scalar local ring of finite right global dimension. If R is semiprime, then it is a domain.

PROOF:

This is [52, Thm.2.9], noting that if R is semiprime, then by Lemma 1.2.21(i), $Z(R) = 0$.

In order to apply the above result, we must first determine when the (right) global dimension of a group ring of a polycyclic-by-finite group is finite. This is accomplished by means of the following deep theorem.

THEOREM 4.2. (Serre; [37, Thm.10.3.2]) Let R be a commutative ring, and N a normal subgroup of finite index in a group G. Suppose that if $g \in G$ and g is periodic, $|g|$ is a unit in R. Then

$$\text{rt.gl.dim.}RG = \text{rt.gl.dim.}RN.$$

COROLLARY 4.3. Let K be a field of characteristic $p \geqslant 0$, and let G be a polycyclic-by-finite group. Suppose in addition that if $p > 0$, G contains no elements of order p. Then $\text{rt.gl.dim.}KG \leqslant h(G) < \infty$.

PROOF:

By Theorem 1.1.13, G has a normal poly-(infinite cyclic) subgroup N of finite index. By Prop.1.2.56, Cor.1.3.23, and a simple induction on $h(N)$, it follows that the right global dimension of KN is at most $h(N)$, so that the result is an immediate consequence of Theorem 4.2.

REMARK: In the situation of the above corollary, we in fact have $\text{rt.gl.dim.}KG = h(G)$; see [17, Chapter 8, Lemma 8].

We are now in a position to prove the main result of this chapter.

<u>THEOREM 4.4</u>. Let p be a prime, let K be a field of characteristic zero, and let F be a field of characteristic p. If G is a torsion-free p-nilpotent group, KG and FG are domains.

<u>PROOF</u>:

By Lemma 2.6, it is enough to prove that FG is a domain for all fields F of characteristic p. Now FG is Noetherian, by Theorem 1.3.17(iii), and by Theorem 3.7, $\underline{\underline{g}}$, the augmentation ideal of FG, is localizable. Since G has no periodic elements, Proposition 1.3.8 implies that FG is prime, so that by Lemma 1.2.35 elements of $\mathscr{C}(\underline{\underline{g}})$ are regular in FG. Hence we can embed FG in the partial right quotient ring R of FG obtained by inverting the elements of $\mathscr{C}(\underline{\underline{g}})$.

We shall show that FG is a domain by demonstrating that R satisfies the hypotheses of Theorem 4.1. As in Proposition 1.2.38, R is a right Noetherian scalar local ring, noting that $FG/\underline{\underline{g}}$ is isomorphic to F, so that $FG/\underline{\underline{g}}$ is a domain. Since FG is prime, R is prime, by Lemma 1.2.23(iii). Now G is torsion-free, and so by Theorem 4.3, FG has finite (right) global dimension. Hence by Lemma 1.2.54, R has finite right global dimension.

We have thus shown that R is a prime, right Noetherian, scalar local ring of finite right global dimension. It follows from Theorem 4.1 that R is a domain. Therefore FG is a domain, and the proof is complete.

It is perhaps worth noting how we used the assumption that G is torsion-free in the above proof. We used firstly the fact that G has no finite normal subgroups, in order to deduce that FG is prime; secondly, in applying Theorem 4.3 we used the fact that G has no elements of order p. The following easily proved lemma confirms that any p-nilpotent group with these properties must be torsion-free.

LEMMA 4.5. If G is a p-nilpotent group, G is torsion-free if and only if G has no non-trivial finite normal subgroups and no elements of order p.

Examples of p-nilpotent groups are provided by

LEMMA 4.6. Let p be a prime, and suppose the finitely generated group G has a normal nilpotent subgroup H such that G/H is a finite p-group. Then G is p-nilpotent.

PROOF:

Note first that by Lemma 1.1.14, H is finitely generated, so that by Theorem 1.1.15(ii), G is polycyclic. Let N be a normal subgroup of finite index in G. Since HN/N, being isomorphic to $H/(H \cap N)$, is a finite nilpotent group, it follows from Theorem 1.1.15(iii) that HN/N has a characteristic p'-subgroup T/N such that HN/T is a p-group. Now T is normal in G, and since $HN \supseteq H$, G/T is a p-group, so that G/N is p-nilpotent, as claimed.

We can now deduce

PROPOSITION 4.7. Let p be a prime, K a field of characteristic zero, and F a field of characteristic p. If G is a torsion-free group with a normal locally nilpotent subgroup H such that G/H is a locally finite p-group, then KG and FG are domains.

PROOF:

By Lemma 2.6, it is enough to prove that FG is a domain. If T is any finitely generated subgroup of G, then since $T/(T \cap H)$ is isomorphic to TH/H, $T/(T \cap H)$ is a finite p-group. By Lemma 1.1.14, $(T \cap H)$ is finitely generated, and so nilpotent, so it follows from Theorem 4.4 that FT is a domain. Hence, by Lemma 1.3.4(ii), FG is a domain.

REMARK: Since, by Theorem 1.1.17(i), a torsion-free supersoluble group G is an extension of a finitely generated nilpotent group by a finite 2-group, Proposition 4.7 provides another proof that KG is a domain if K is a field of characteristic zero and is torsion-free supersoluble. Of course, Formanek's original proof [11] applies equally well to coefficient fields of positive characteristic.

To obtain further corollaries to Theorem 4.4, we need the following result, due to Farkas.

THEOREM 4.8. [7] Let K be a field, and let G be a torsion-free group with a normal abelian subgroup A of finite index. Let $H_1, \ldots, H_n$ be the finitely many subgroups of G, containing A, whose images under the map $G \longrightarrow G/A$ are the Sylow subgroups of G/A. Then KG is a domain if and only if KH_i is a domain for $i = 1, \ldots, n$.

We can now deduce

THEOREM 4.9. Let K be a field of characteristic zero, and let G be a torsion-free abelian-by-finite group. Then KG is a domain.

PROOF:

This follows immediately from Proposition 4.7 and Theorem 4.8.

It is not difficult to extend Theorem 4.9 to cover torsion-free abelian-by-locally finite-by- locally nilpotent groups, using Corollary 1.3.23. However, by using an idea of Formanek, [11], we can do somewhat better. First, we must show that when the group rings with which we are concerned are domains, they are in fact Ore domains.

LEMMA 4.10. Let R be a commutative domain, and suppose G is a torsion-free group with a finite series

$$1 = G_0 \subset G_1 \subset \ldots \subset G_n = G \tag{1}$$

such that for $i = 0, \ldots, n-1$, G_i is normal in G_{i+1} and G_{i+1}/G_i is either locally finite or torsion-free abelian. If RG is a domain, then it is an Ore domain.

PROOF:

We argue by induction on n, the result being vacuous for n = 0. Suppose therefore that we have proved the lemma for all groups H having a series of the form (1) of length at most (n-1), such that RH is a domain. In particular, therefore, RG_{n-1} is an Ore domain, so $Q(RG_{n-1})$ exists and is a division ring. Furthermore, in proving that RG satisfies the right Ore condition, it is clearly sufficient, by Lemma 1.3.6, to consider the group algebras of finitely generated subgroups of G, and since all subgroups of G have a series of the form (1) of length at most n, we may without loss of generality assume that G is finitely generated. Thus G/G_{n-1} is either finitely generated torsion-free abelian or finite. In the former case, there is a finite series from G_{n-1} to G with infinite cyclic factors, and the result follows from Lemma 1.3.22 and Theorem 1.2.28, by an induction on the length of this series.

Suppose finally that G/G_{n-1} is finite. By Lemma 1.3.5, the non-zero elements of RG_{n-1} form a right divisor set of regular elements in RG, so we can obtain a partial right quotient ring Q of RG by inverting these elements. Now if $\{g_1, \ldots, g_m\}$ is a transversal to G_{n-1} in G, then since RG is generated as a right RG_{n-1}-module by $\{g_1, \ldots, g_n\}$, by Lemma 1.3.3(i), it follows that Q is generated as a right module over the quotient division ring of RG_{n-1} by $\{g_1, \ldots, g_n\}$. Since this division ring is a subring of Q, Q is right Artinian, so that regular elements of Q are units, by Lemma 1.2.3. Hence Q is the classical right quotient ring of RG, and we conclude that RG is an Ore domain.

The above lemma is essentially [33, Proposition]. It is not hard to see that it can be considerably generalised, and we shall pursue this in Chapter 3. However, Lemma 4.10 is sufficient for our present needs.

<u>THEOREM 4.11</u>. Let K be a field of characteristic zero, and let G be a torsion-free group with subgroups

$$1 \subset A \lhd T \lhd G \tag{2}$$

such that A is abelian, T/A is locally finite, and G/T is locally supersoluble. Then KG is a domain.

<u>PROOF</u>:

Since finitely generated subgroups of G are easily seen to possess a series of the form (2), we shall without loss of generality assume that G is finitely generated, so that G/T is supersoluble. We argue by induction on the Hirsch number, h(G/T), of G/T.

If $h(G/T) = 0$, G/T is finite, and T is thus finitely generated, by Lemma 1.1.14. Hence $|T/A| < \infty$, so that $|G : A| < \infty$. There therefore exists a normal subgroup B of G such that $|G/B| < \infty$ and $B \subset A$, so that in this case G is abelian-by-finite and KG is a domain by Theorem 4.9.

Now suppose that the result is known for all finitely generated torsion-free groups H with a series of the form (2) such that H/T is supersoluble with $h(H/T) \leqslant n$, for some integer $n \geqslant 0$, where T is an abelian-by-locally finite normal subgroup of H, and suppose that G satisfies $h(G/T) = n+1$. By Theorem 1.1.17(ii), there exists a normal subgroup N of G such that $T \subset N$ and G/N is infinite cyclic or infinite dihedral. By Lemma 4.10 and our induction assumption, KN is an Ore domain.

If G/N is infinite cyclic, KG is a domain by Corollary 1.3.23. On the other hand, if G/N is infinite dihedral, by Theorem 1.1.24 we have

$$G/N \cong (X/N * Y/N),$$

where X and Y are subgroups of G and $|X/N| = |Y/N| = 2$. Hence $h(X/T) = h(Y/T) = h(N/T) = n$, and so KX and KY are domains, by induction. We may thus apply Theorem 1.1 to deduce that KG is a domain.

5. FUTURE PROSPECTS

We end this chapter with a short discussion of the current position of the Zero Divisor Conjecture for group rings of torsion-free soluble groups. Building on our use of Walker's result (Thm.4.1), Farkas and Snider [8] have extended the results of §4 by showing that if K is a field of characteristic zero and G is any torsion-free polycyclic-by-finite group, KG is a domain. If on the other hand G is torsion-free polycyclic-by-finite and K is a field of positive characteristic, it is still not known in general whether KG is a domain.

Note that Theorem 4.11 shows that if K is a field of characteristic zero and G is a torsion-free metabelian group, (i.e. G is soluble of derived length 2), KG is a domain. However we cannot at present extend this result to groups of derived length 3 - perhaps the simplest example of a finitely generated torsion-free soluble group for which the Zero Divisor Conjecture in characteristic zero has not been solved is

$$G = (C_{\infty} \wr H),$$

where C_{∞} denotes an infinite cyclic group and H is any finitely generated torsion-free soluble group satisfying the following conditions:

(i) H has a normal abelian subgroup A such that H/A is finite and abelian of type (p, p), for an odd prime p;

(ii) H is not supersoluble;

(iii) H is not right orderable.

Groups satisfying all of the above conditions are fairly easy to construct, but we shall not consider the details here - see [37, Chapter 13, Exercises 15-18].

CHAPTER 3

ARTINIAN QUOTIENT RINGS

1. INTRODUCTION

Since a soluble group H has a finite series of normal subgroups with successive factors either torsion-free abelian or locally finite, by Lemma 1.1.6, it follows that H satisfies the hypothesis of Lemma 2.4.10, so that if KH is a domain for some field K, then KH is an Ore domain. Thus, if the Zero Divisor Conjecture is true for group rings of soluble groups, one might hope to at least be able to prove that if G is a torsion-free soluble group and K is a field, KG has a simple Artinian quotient ring. Such a result would, from the point of view of the Zero Divisor Conjecture, be important for two reasons. First, for K and G as above, KG would have the maximum and minimum conditions on right annihilators, by Theorem 1.2.23 and the remark following it, thus imposing at least some finiteness conditions on the zero divisors in the ring; and second, we would then have available the extensive theory of prime Goldie rings in subsequent attempts to prove the Zero Divisor Conjecture for this class of group rings.

We are thus led to pose the following question:- Which group rings of soluble groups possess right Artinian right quotient rings? We shall in fact prove our results for a class of groups $\mathcal{U}$ which includes many insoluble groups, but does not include all torsion-free soluble groups, so the question raised in the previous sentence remains open. In §4, we discuss to what extent it can be answered by the methods of this chapter.

The main theorem of this chapter will incorporate a result of Hughes, [23], who, extending work of Smith, [47], has shown that if G is a group with an ascending series

$$1 = G_0 \subset G_1 \subset \ldots \subset G_\alpha \subset G_{\alpha+1} \subset \ldots \subset G_\rho = G,$$

where ρ is an ordinal, G_α is normal in $G_{\alpha+1}$, $G_{\alpha+1}/G_\alpha$ is either finite or infinite cyclic, and $G_{\alpha+1}/G_\alpha$ is finite for only finitely many ordinals α, $0 \leqslant \alpha < \rho$, and if R is a ring with a right Artinian right quotient ring, then RG has a right Artinian right quotient ring.

To state our results, we need some notation, which will remain fixed throughout this chapter. In the notation of Chapter 1, §1, we write $\mathfrak{B} = \langle \acute{P}, L \rangle$; that is, $\mathfrak{B}$ denotes the closure operation on group classes generated by $\acute{P}$ and L. It should be noted that the B-closed classes of groups are precisely those classes which are $\acute{P}$- and L-closed, [40, Chapter 1, p.5]. We shall denote the class of torsion-free abelian groups by $\mathfrak{A}_0$, and we define a new group class, $\mathfrak{V}$, as follows. A group G is a member of the class $\mathfrak{V}$ if and only if G has an ascending series

$$1 = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G \qquad (1)$$

where ρ is an ordinal, H_α is normal in $H_{\alpha+1}$, $H_{\alpha+1}/H_\alpha$ is either finite or in the class $\mathfrak{B}\mathfrak{A}_0$, and $H_{\alpha+1}/H_\alpha$ is finite for only finitely many ordinals α, $0 \leqslant \alpha < \rho$.

If $G \in \mathfrak{V}$, we shall write m_G for the least integer occurring as the product of the orders of the finitely many finite factors appearing in a series of type (1) for G.

Given a ring R which is a right order in a right Artinian ring of right Goldie dimension n, we shall write, for each integer $r \geqslant n$, $\mathcal{P}_r(R)$ for the class of groups N such that RN has a right Artinian right quotient ring of right Goldie dimension at most r.

We can now state

<u>THEOREM 2.12</u>. Let R be a ring with a right Artinian right quotient ring of right Goldie dimension n, let $r \geqslant n$, and let H be a normal subgroup of a group G such that $H \in \mathcal{P}_r(R)$ and $G/H \in \mathcal{U}$. Then $G \in \mathcal{P}_{r\,m_{G/H}}(R)$.

Of course, there is a left-handed version of Theorem 2.12, and so by Proposition 1.2.13, if, in the above notation, RH is a (two-sided) order in an Artinian ring, then RG has a (two-sided) Artinian quotient ring. Note also that it follows from Lemma 1.2.22(iii) that if RG is semiprime, then its quotient ring Q(RG) is semisimple Artinian, while if RG is prime, it follows similarly that Q(RG) is prime, and so simple.

Horn [21] showed that if R is a Noetherian order in a QF-ring, and G is a polycyclic-by-finite group, then RG is an order in a QF-ring. Here we extend this result to

<u>THEOREM 3.8</u>. Let R, H, and G be as in Theorem 2.12, and suppose that RH is an order in a QF-ring. Then RG is an order in a QF-ring.

In particular, therefore, if R is a commutative domain, $H = \{1\}$, (and so $G \in \mathcal{U}$), then Q(RG) exists and is a QF-ring.

Note that by Lemma 1.2.45(iii) and (vi), if a ring S has a quasi-Frobenius classical quotient Q(S), then $Q(S)_{|S}$ is the injective hull of $S_{|S}$, and Q(S) is the maximal right and left quotient ring of S. We shall discuss this in more detail in §3.

The contents of this chapter are arranged as follows. We prove Theorem 2.12 in §2, and Theorem 3.8 in §3. In §4, we discuss how our results apply to group rings of soluble groups, (Theorem 4.2), and provide two examples which illustrate the limitations of our methods. We use one of these examples, (Ex.4.6), to answer a question of Lazard, [32] on infinitely generated projective modules. In §5, we include several applications of Theorems 2.12 and 3.8. For example, we show in Theorem 5.2 that if $G \in \mathcal{V}$ and R is a commutative domain, RG is a subring of a simple Artinian ring; it follows from this that every group in $\mathcal{V}$ is linear over a division ring of arbitrary characteristic. We also examine two applications to the study of group rings:- We extend some results of Jordan [26] on the Jacobson radical of the group rings of certain free products with amalgation, (Theorem 5.3), and we apply Theorem 3.8 to the study, initiated by Formanek, [13], of the centre of the maximal quotient ring of a group ring.

Finally, we remark that it has recently been brought to our notice that A.Horn has independently extended Hughes' result [23] to obtain special cases of Theorems 2.12 and 3.8; full details of Horn's work may be found in [22].

2. PROOF OF THEOREM 2.12

In common with the result of Hughes mentioned in §1, Theorem 2.12 depends on a result of Jategaonkar, namely Theorem 1.2.28, which enables us to deduce immediately

LEMMA 2.1. Let R be a ring, H a normal subgroup of a group G, such that $G/H \in (\mathcal{L} \cap \mathcal{A}_o)$. If RH is a right order in a right Artinian ring, then so is RG.

PROOF:

Since $G/H \in P(\mathcal{L}_\infty)$, we may, arguing by induction on the number of generators of G/H, suppose that G/H is infinite cyclic. The result is now an immediate consequence of Lemma 1.3.22 and Theorem 1.2.28.

Our main task in proving Theorem 2.12 is to replace the class $(\mathcal{L} \cap \mathcal{A}_o)$ in the above lemma by the class $B\mathcal{A}_o$. When we have done this, Theorem 2.12 becomes an elementary deduction. We begin with a lemma which may seem self-evident, but nevertheless we state it explicitly as it illustrates the type of argument we shall use several times in this section.

LEMMA 2.2. Let $\mathcal{P}$ and $\mathcal{X}$ be group classes which satisfy the following conditions:

(i) $\mathcal{P}\mathcal{X} = \mathcal{P}$.

(ii) For all group classes $\mathcal{Y}$ contained in $B\mathcal{X}$,

$$\mathcal{P}\mathcal{Y} = \mathcal{P} \Rightarrow \mathcal{P}(L\mathcal{Y}) = \mathcal{P}.$$

(iii) For allgroup classes $\mathcal{Y}$ contained in $B\mathcal{X}$,

$$\mathcal{P}\mathcal{Y} = \mathcal{P} \Rightarrow \mathcal{P}(\acute{P}\mathcal{Y}) = \mathcal{P}.$$

Then $\mathcal{P}(B\mathcal{X}) = \mathcal{P}$.

PROOF:

Let $\mathcal{S}$ be the class of all groups G such that $\mathcal{P}\{G\} = \mathcal{P}$, so that $\mathfrak{X} \subset \mathcal{S}$. Put $\mathcal{J} = (\mathcal{S} \cap B\mathfrak{X})$, so that $\mathcal{J}$ is $\acute{P}$- and L-closed by conditions (ii) and (iii). Since $B\mathfrak{X}$ is by definition the smallest $\acute{P}$- and L-closed class containing $\mathfrak{X}$, we must have $B\mathfrak{X} \subset \mathcal{S}$, as required.

Recall that N(S) denotes the nilpotent radical of a ring S.

LEMMA 2.3. Let R be a ring, and suppose $\mathcal{Y}$ is a class of groups satisfying the hypothesis that if L is a normal subgroup of a group M such that $M/L \in \mathcal{Y}$, and N(RL) is nilpotent, then $N(RM) = N(RL)RM$.

Then (i) if H is a subgroup of a group G such that (a) there exists an ascending series

$$H = H_0 \subset H_1 \subset \dots \subset H_\alpha \subset H_{\alpha+1} \subset \dots \subset H_\rho = G,$$

where ρ is an ordinal, $H_\alpha \lhd H_{\alpha+1}$ and $H_{\alpha+1}/H_\alpha \in \mathcal{Y}$, $0 \leqslant \alpha < \rho$, and (b) $N(RH)^n = 0$, for some integer $n \geqslant 1$, it follows that

$$N(RG) = N(RH)RG \quad \text{and} \quad N(RG)^n = 0;$$

(ii) if H is a normal subgroup of a group G, such that $G/H \in L\mathcal{Y}$, and N(RH) is nilpotent,

$$N(RG) = N(RH)RG.$$

PROOF:

(i) We prove the result by induction on ρ. Suppose that $N(RG) \neq N(RH)RG$, or $N(RG)^n \neq 0$, and let α be minimal, $0 \leqslant \alpha \leqslant \rho$, such that either $N(RH_\alpha) \neq N(RH)RH_\alpha$ or $N(RH_\alpha)^n \neq 0$. Clearly α must be a limit ordinal; but in this case, if $\beta_i \in N(RH_\alpha)$, $i = 1, \dots, n$, then there some ordinal $\gamma < \alpha$ such that for $i = 1, \ldots, n$, $\beta_i \in N(RH_\gamma)$, so that $\prod_{i=1}^{n} \beta_i = 0$, and

$$\beta_i \in N(RH)RH_\gamma \subseteq N(RH)RH_\alpha .$$

Thus $N(RH_\alpha)^n = 0$, and $N(RH_\alpha) = N(RH)RH_\alpha$. From this contradiction, the result follows.

(ii) Since $N(RG) = \sum_I \{I \lhd RG : I \text{ nilpotent}\}$, we deduce that if $T \subseteq G$,

$$N(RG) \cap RT \subseteq N(RT).$$

The result is now clear.

Next, we need an easy result on the Goldie dimension of group rings.

LEMMA 2.4. Let R be a ring , and H a subgroup of a group G. Then (i) right Goldie dimension of RG $=$ $\sup_T$\{right Goldie dimension of RT\}, as T ranges over all subgroups of G such that $T = \langle H, x_0, \ldots, x_n \rangle$, where $n \geqslant 0$ and $x_i \in G$, $0 \leqslant i \leqslant n$;

(ii) if there exists a series

$$H = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G,$$

where ρ is an ordinal, and for all ordinals α, $0 \leqslant \alpha < \rho$, rt. Goldie dimension RH_α = rt. Goldie dimension $RH_{\alpha+1}$, it follows that rt. Goldie dimension RG = rt. Goldie dimension RH.

Note that (i) incorporates

(iii) rt. Goldie dimension of RH $\leqslant$ rt. Goldie dimension of RG.

PROOF:

Note first that (iii) follows from the fact that if $I_1, \ldots I_m$ are right ideals of RH whose sum is direct, then Lemma 1.3.3(iii) shows that $\sum_{i=1}^{m} I_iG$ is a direct sum of right ideals of RG. This observation, together with a straightforward local argument, also proves (i) and finally, (ii) follows by transfinite induction, using (i) to deal with limit ordinals.

The next lemma will be crucial in our application of Lemma 2.2 to extend Lemma 2.1.

LEMMA 2.5. Let R be a ring with a right Artinian right quotient ring of right Goldie dimension n. Let m be an integer, $m \geqslant n$. Let $\mathcal{Y}$ be a class of groups satisfying

(a) $\mathcal{P}_m(R)\mathcal{Y} = \mathcal{P}_m(R)$;

and (b) if V is a normal subgroup of a group U, $U/V \in \mathcal{Y}$, and N(RV) is nilpotent, then N(RU) = N(RV)RU.

Then (i) $\mathcal{P}_m(R)(\mathrm{L}\mathcal{Y}) = \mathcal{P}_m(R)$;

and (ii) $\mathcal{P}_m(R)(\acute{\mathrm{P}}\mathcal{Y}) = \mathcal{P}_m(R)$.

PROOF:

(i) Let H be a normal subgroup of a group G such that $H \in \mathcal{P}_m(R)$ and $G/H \in \mathrm{L}\mathcal{Y}$. We must show that $G \in \mathcal{P}_m(R)$. By (a), if T is any subgroup of G such that $T \supseteq H$ and T/H is finitely generated, there exists a subgroup $T' \supseteq T$ such that T'/H is finitely generated and RT' satisfies the right Ore condition, so by Lemma 1.3.6 RG has a right quotient ring. In fact we can view Q(RG) as $\bigcup_{T'} Q(RT')$, where the union is taken over all such subgroups T'. By Lemma 2.4(i), the right Goldie dimension of RG (and hence by Lemma 1.2.22(v) of Q(RG)) is at most m. Hence to prove (i) it suffices to prove that Q(RG) is right Artinian.

Since RH has a right Artinian right quotient ring, N(RH) is nilpotent, by Proposition 1.2.27. Thus by hypothesis (b), applying Lemma 2.3(ii), N(RG) = N(RH)RG. By Lemma 1.2.30, noting that Q(RH) is a subring of Q(RG),

$N(Q(RG)) = N(RG)Q(RG) = N(RH)Q(RG) = N(Q(RH))Q(RG)$, so that in particular $N(Q(RG))$ is nilpotent, and finitely generated as a right ideal, since $Q(RH)$ is right Artinian. It thus follows from Theorem 1.2.1 that to prove $Q(RG)$ is right Artinian, it is enough to show that $Q(RG)/N(Q(RG))$ is Artinian.

We claim first that $Q(RG)/N(Q(RG))$ is von Neumann regular. Let $\alpha = ac^{-1} \in Q(RG) \setminus N(Q(RG))$, where $a, c \in RG$ and $c \in \mathcal{C}_{RG}(0)$, and put $T = \langle H, \operatorname{supp} a, \operatorname{supp} c\rangle$. Thus, by hypothesis, $Q(RT)$ exists and is right Artinian, $ac^{-1} \in Q(RT)$, and $N(Q(RG)) = N(Q(RT))Q(RG)$. Therefore, in particular, $ac^{-1} \notin N(Q(RT))$. Now $Q(RT)/N(Q(RT))$ is semisimple Artinian and thus von Neumann regular, so that there exists $\beta \in Q(RT)$ such that $(\alpha - \alpha\beta\alpha) \in N(Q(RT))$, which is contained in $N(Q(RG))$. Since $\beta \in Q(RG)$, our claim follows.

By Theorem 1.2.9, a regular ring is either Artinian or has an infinite set of orthogonal idempotents. Suppose the latter is true in this case; then in particular $Q(RG)/N(Q(RG))$ has a set $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{m+1}$ of pairwise orthogonal idempotents. Since $N(Q(RG))$ is nil, these may be lifted to a set $e_1, \ldots, e_{m+1}$ of pairwise orthogonal idempotents of $Q(RG)$, by Proposition 1.2.7. It follows that the right Goldie dimension of $Q(RG)$, and hence by Lemma 1.2.22(v) of RG, is greater than m, contradicting our earlier observation that the right Goldie dimension of RG is at most m. Hence $Q(RG)/N(Q(RG))$ must be Artinian, and so (i) is proved.

(ii) Let H be a normal subgroup of a group G such that $H \in \mathcal{P}_m(R)$ and there exists an ascending series

$$H = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G,$$

such that $H_\alpha \triangleleft H_{\alpha+1}$, and $H_{\alpha+1}/H_\alpha \in \mathcal{Y}$, $0 \leq \alpha < \rho$. Suppose $G \notin \mathcal{P}_m(R)$, and let γ be minimal, $0 < \gamma \leq \rho$, such that $H_\gamma \notin \mathcal{P}_m(R)$. Since $\mathcal{P}_m(R)\mathcal{Y} = \mathcal{P}_m(R)$, γ must be a limit ordinal. However if T/H is any finitely generated subgroup of H_γ/H, $T \subseteq H_\beta$ for some ordinal $\beta < \gamma$ and so RT has a right Artinian right quotient ring. By Lemma 2.4(ii), RH_γ has right Goldie dimension at most m, and by Lemma 2.3(i), $N(RH_\gamma) = N(RH)RH_\gamma$. By an argument exactly similar to that used in the first part of this proof, we deduce that $H_\gamma \in \mathcal{P}_m(R)$, a contradiction. Hence $G \in \mathcal{P}_m(R)$, and the proof is complete.

We now apply the general lemmas which we have obtained to a particular situation - that is, given a ring R, we shall show that the class $(\mathcal{G} \cap \mathcal{Q}_0)$ satisfies conditions (a) and (b) of the previous lemma, and that $B\mathcal{Q}_0$ satisfies (b). It is then a simple matter to apply Lemmas 2.2 and 2.5 to show that

$$\mathcal{P}_m(R)(B\mathcal{Q}_0) = \mathcal{P}_m(R),$$

for all appropriate pairs of rings of rings R and integers m, (Theorem 2.11).

Our first task is thus to show that $(\mathcal{G} \cap \mathcal{Q}_0)$ satisfies (a).

<u>LEMMA 2.6</u>. Let R be a ring, and H a normal subgroup of a group G. Then if E is an essential right ideal of RH, EG is an essential right ideal of RG.

<u>PROOF</u>:

Let $\{g_\lambda : \lambda \in \Lambda\}$ be a transversal to H in G. By Lemma 1.3.3,

$$EG_{|H} = \sum_{\lambda\in\Lambda}^{\oplus}(Eg_\lambda)_{|H} \subseteq \sum_{\lambda\in\Lambda}^{\oplus}(RHg_\lambda)_{|H} = RG_{|RH},$$

and it is thus clear that $EG_{|H}$ is an essential submodule of $RG_{|H}$. A fortiori, EG is an essential right ideal of RG.

The above lemma, which is due to Burgess, [5, Lemma 2.5], will be considerably generalised in the next Chapter. The proof of the next result is an adaptation of the proof of [43, Prop.2.5], where Shock proves a similar result for polynomial rings.

<u>LEMMA 2.7</u>. Let R be a ring, and let G be a group with a normal subgroup H such that $G/H \in \mathcal{Q}_0$. Then

rt. Goldie dimension of RH = rt. Goldie dimension RG.

<u>PROOF</u>:

By Lemma 2.4(iii), we may assume that n, the right Goldie dimension of RH, is finite, and by Lemma 2.4(i), we may assume that G/H is finitely generated, so that by induction on the number of generators of G/H, it suffices to prove the result when G/H is infinite cyclic. In this case, by Lemma 1.3.22, RG is isomorphic to $RH[X, X^{-1}; \sigma]$, where $\sigma \in \mathrm{Aut}(RH)$ is as usual induced by conjugation on H, and $rX = X\sigma(r)$ for all $r \in R$.

By Theorem 1.2.16, there exist uniform right ideals $U_1, \ldots, U_n$ of RH, such that $I = \sum_{i=1}^{n} U_i$ is a direct sum, and I is essential in RH. By Lemma 1.3.3(iii), $IG = \sum_{i=1}^{n} U_iG$ is a direct sum of right ideals of RG, and by the previous lemma, IG is essential in RG. To complete the proof, it is thus enough to show that for $i = 1, \ldots, n$, U_iG is a uniform right ideal.

Put $S = RH[X; \sigma] \subseteq RH[X, X^{-1}; \sigma]$, and fixing i,

$1 \leqslant i \leqslant n$, put $U = U_i$. It is easy to see that UG is uniform in RG if and only if US is uniform in S, so we may suppose for a contradiction that US is not uniform, and choose $0 \neq p(X), q(X) \in US$, $p(X) = \sum_{l=0}^{k} X^l a_l$, $q(X) = \sum_{j=0}^{m} X^j b_j$, (noting that $a_l \in \sigma^l(U)$), such that

$$p(X)S \cap q(X)S = 0. \tag{2}$$

Note firstly that we can ensure, by right multiplication by a suitable non-zero element of RH if necessary, that the right annihilators of the coefficients a_l of $p(X)$ are equal. Similarly, we may assume that

$$r_S(b_1) = \ldots = r_S(b_m).$$

Furthermore, we can choose $p(X)$ and $q(X)$ so that

$$(\text{degree } p(X) + \text{degree } q(X))$$

is as small as possible such that (2) and the above condition on the coefficients hold. Suppose without loss of generality that

$$\text{degree } q(X) \geqslant \text{degree } p(X).$$

Since, for all $s \geqslant 1$, $\sigma^s(U)$ is a uniform right ideal of RH, there exist $\alpha, \gamma \in RH$ such that

$$0 \neq X^k a_k X^{m-k} \alpha = X^m b_m \gamma .$$

Put

$$h(X) = p(X)X^{m-k}\alpha + q(X)(-\gamma),$$

so that $\text{degree } h(X) < \text{degree } q(X)$, and $h(X) \neq 0$ by (2).

We shall prove that

$$h(X)S \cap p(X)S \neq 0. \tag{3}$$

There exists $0 \neq \nu \in RH$ such that

$$h(X)\nu = \Big(\sum_{t=0}^{\tau} X^t \beta_t\Big)\nu \neq 0,$$

and $r_S(\beta_t \nu) = r_S(\beta_w \nu)$ whenever $\beta_t \nu \neq 0 \neq \beta_w \nu$, $0 \leqslant t, w \leqslant \tau$. Now

$$0 \leqslant \text{degree } h(X)\nu \leqslant \text{degree } h(X) < \text{degree } q(X),$$

and so (3) follows by the minimality hypothesis on $\big(\text{degree}\, p(X) + \text{degree}\, q(X)\big)$, so that

$$0 \neq h(X)f(X) = p(X)g(X), \qquad \text{say.}$$

Now the right annihilators of the coefficients of $q(X)(-\gamma)$ are all equal to

$$\gamma^{-1} r_S(b_m) = \{\delta \in S : \gamma\delta \in r_S(b_m)\}.$$

Hence if $q(X)(-\gamma)f(X) = 0$, where

$$f(X) = \sum_{u=0}^{d} \omega_u X^u,$$

then

$$\begin{aligned}
\omega_0, \ldots, \omega_d \in r_{RH}(b_m\gamma) &= r_{RH}(a_k X^{m-k}\alpha) \\
&= r_{RH}(\sigma^{m-k}(a_k)\alpha) \\
&= r_{RH}\big(\sigma^{m-k}(a_k\bar{\alpha})\big), \\
&\qquad \text{putting } \sigma^{m-k}(\bar{\alpha}) = \alpha, \\
&= \sigma^{m-k}\, r_{RH}(a_k\bar{\alpha}) \\
&= \sigma^{m-k}\big(\bar{\alpha}^{-1} r_{RH}(a_k)\big),
\end{aligned}$$

where $\bar{\alpha}^{-1} r_{RH}(a_k) = \{\eta \in RH : \bar{\alpha}\eta \in r_{RH}(a_k)\}$.

$$\begin{aligned}
\text{Also} \quad \sigma^{m-k}\big(\bar{\alpha}^{-1} r_{RH}(a_k)\big) &= \sigma^{m-k}\big(\bar{\alpha}^{-1} r_{RH}(a_l)\big), \\
&\qquad 0 \leq l \leq k, \\
&= \sigma^{m-k}\big(r_{RH}(a_l\bar{\alpha})\big) \\
&= r_{RH}\big(\sigma^{m-k}(a_l)\alpha\big) \\
&= r_{RH}(a_l X^{m-k}\alpha).
\end{aligned}$$

Thus if $\omega_0, \ldots, \omega_d \in r_{RH}(b_m\gamma)$,

$$p(X)X^{m-k}\alpha f(X) = 0,$$

and so

$$0 \neq h(X)f(X) = p(X)X^{m-k}\alpha f(X) + q(X)(-\gamma)f(X)$$
$$= 0,$$

a contradiction.

Therefore $q(X)(-\gamma)f(X) \neq 0$, and indeed

$$0 \neq q(X)(-\gamma)f(X) = \left[p(X)g(X) - p(X)X^{m-k}\alpha f(X)\right],$$

contradicting (2). Hence US is uniform, and the proof is complete.

We immediately deduce

LEMMA 2.8. Let R be a right order in a right Artinian ring of right Goldie dimension r, let m be an integer, $m \geqslant r$, and let H be a normal subgroup of a group G. If $G/H \in (\mathcal{G} \cap \mathcal{Q}_o)$ and $H \in \mathcal{P}_m(R)$, then $G \in \mathcal{P}_m(R)$.

PROOF:

By Lemma 2.1, RG has a right Artinian right quotient ring, and by Lemmas 2.7 and 1.2.22(v), this quotient ring must have right Goldie dimension at most m. Thus $G \in \mathcal{P}_m(R)$.

Next, we consider condition (b) of Lemma 2.5; first, we have

LEMMA 2.9. Let R be a ring, and let H be a normal subgroup of a group G, such that $G/H \in (\mathcal{G} \cap \mathcal{Q}_o)$. If N(RH) is nilpotent, then $N(RG) = N(RH)RG$.

PROOF:

By Lemma 1.3.10, $\left(N(RH)RG\right)^n = \left(N(RH)\right)^n RG$, for all $n \geqslant 1$, so N(RH)RG is a nilpotent ideal of RG. Hence it only remains to prove that $N(RG) \subseteq N(RH)RG$. Suppose that this is false, and choose $\beta \in N(RG) \setminus N(RH)RG$ with $|\mathrm{supp}\,\beta|$ minimal. By induction on the number of generators of G/H, we may assume G/H cyclic, say $G = \langle H, x \rangle$. By multiplying if necessary by a suitable power of x, we may assume

$\beta = \sum_{i=0}^{m} \beta_i x^i$, where $m > 0$ and $\beta_0 \neq 0$.

Since $(RG\beta RG)$ is a nilpotent ideal, say $(RG\beta RG)^t = 0$, we have for all $\alpha_j, \gamma_j \in RH$, $j = 1, \ldots, t$,

$$0 = \prod_{j=1}^{t}(\alpha_j \beta \gamma_j) = \prod_{j=1}^{t}(\alpha_j \beta_0 \gamma_j) + \eta x,$$

where $\eta = \sum_{k=0}^{r} \eta_k x^k$, r is a non-negative integer, and $\eta_k \in RH$, $0 \leq k \leq r$. Hence

$$\prod_{j=1}^{t}(\alpha_j \beta_0 \gamma_j) = 0,$$

so that $\beta_0 \in N(RH)$, and

$$(\beta - \beta_0) \in N(RG) \setminus N(RH)RG .$$

Since

$$|\mathrm{supp}(\beta - \beta_0)| < |\mathrm{supp}\beta|,$$

we have obtained a contradiction to our choice of β, and the proof is complete.

REMARK: Lemma 2.9 is false without the assumption that $N(RH)$ is nilpotent. For example, let $G = C_p \wr C_\infty$, where p is a prime, C_p denotes the cyclic group of order p, and C_∞ is an infinite cyclic group, let R be the field of p elements, and let H be the base group of G, so that H is an infinite elementary abelian p-group. Thus G/H is infinite cyclic, RG is semiprime by Theorem 1.3.20, since $\Delta(G) = 1$ by Lemma 1.1.26, and $N(RH)$ is the augmentation ideal of RH, again by Theorem 1.3.20.

LEMMA 2.10. Let R be a ring and H a subgroup of a group G. If $N(RH)$ is nilpotent and there exists an ascending series

$$H = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G,$$

where ρ is an ordinal, $H_\alpha \lhd H_{\alpha+1}$, and $H_{\alpha+1}/H_\alpha \in \mathfrak{B}\mathfrak{A}_0$, $0 \leq \alpha < \rho$, then $N(RG) = N(RH)RG$.

PROOF:

By Lemma 2.3(i), it is enough to prove the result for $\rho = 1$. Let $\mathfrak{S}$ denote the class of groups P which satisfy the property:-

$$V \lhd U,\quad U/V \cong P,\ N(RV) \text{ nilpotent} \Longrightarrow N(RV)RU = N(RU).$$

Thus $(\mathfrak{Q}_0 \wedge \mathfrak{G}) \subseteq \mathfrak{S}$ by Lemma 2.9, and Lemma 2.3 shows that $\mathfrak{S}$ is $\acute{P}$- and L-closed. The result follows as in Lemma 2.2, since $B(\mathfrak{Q}_0 \wedge \mathfrak{G}) = B\mathfrak{Q}_0$.

Now let R be a ring which has a right Artinian right quotient ring of right Goldie dimension n, and let m be an integer greater than or equal to n. We have, in this notation

<u>THEOREM 2.11</u>. $\mathcal{O}_m(R)(B\,\mathfrak{Q}_0) = \mathcal{O}_m(R)$.

<u>PROOF</u>:

We apply Lemma 2.2. Note first that $B\,\mathfrak{Q}_0 = B(\mathfrak{Q}_0 \wedge \mathfrak{G})$ and by Lemma 2.8, $\mathcal{O}_m(R) = \mathcal{O}_m(R)(\mathfrak{Q}_0 \wedge \mathfrak{G})$. Let $\mathcal{Y}$ be any class of groups contained in $B\mathfrak{Q}_0$. By Lemma 2.10, $\mathcal{Y}$ satisfies hypothesis (b) of Lemma 2.5, and hence if $\mathcal{O}_m(R)\mathcal{Y} = \mathcal{O}_m(R)$, we deduce from Lemma 2.5 that $\mathcal{O}_m(R)(L\mathcal{Y}) = \mathcal{O}_m(R)$ and $\mathcal{O}_m(R)(\acute{P}\mathcal{Y}) = \mathcal{O}_m(R)$.
It thus follows from Lemma 2.2 that

$$\mathcal{O}_m(R)\ B\,(\mathfrak{Q}_0 \wedge \mathfrak{G}) = \mathcal{O}_m(R)\ ,$$

as required.

We are now ready to prove the main result of this section. We continue with our assumption that R is a ring with a right Artinian right quotient ring of right Goldie dimension n. Suppose in addition that r is an integer greater than or equal to n. The class $\mathcal{V}$ has been defined in §1.

PROOF OF THEOREM 2.12.

Let G be a group with a normal subgroup H such that $H \in \mathcal{P}_r(R)$ and $G/H \in \mathcal{U}$. Let

$$H = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G \qquad (1)$$

be an ascending series from H to G, where ρ is an ordinal, $H_\alpha \triangleleft H_{\alpha+1}$, $H_{\alpha+1}/H_\alpha$ is either finite or a member of the class $\mathfrak{B}\mathcal{O}_0$, $0 \leq \alpha < \rho$, and the product of the orders of the finitely many finite factors in (1) is precisely $m_{G/H}$.

We shall prove first that the right Goldie dimension of RG is less than or equal to $r.m_{G/H}$, arguing by induction on ρ. That is, we shall prove that for all ordinals α, $0 \leq \alpha \leq \rho$,

$$\text{right Goldie dimension } RH_\alpha \leq r.m_{H_\alpha/H}\,. \qquad (4)$$

Note that for all such ordinals α, the product of the orders of the finite factors $H_{\gamma+1}/H_\gamma$, $\gamma < \alpha$, is precisely $m_{H_\alpha/H}$, by our choice of the series (1). Suppose (4) is false, and let α, $0 \leq \alpha \leq \rho$, be minimal such that (4) is false for RH_α. By Lemma 2.4(i), α cannot be a limit ordinal, so that $(\alpha - 1)$ exists and either $H_\alpha/H_{\alpha-1} \in \mathfrak{B}\mathcal{O}_0$, or $H_\alpha/H_{\alpha-1}$ is finite. If $H_\alpha/H_{\alpha-1} \in \mathfrak{B}\mathcal{O}_0$ we claim that

right Goldie dimension $RH_{\alpha-1}$ = right Goldie dimension RH_α.

This follows from Lemma 2.2, taking $\mathcal{P}$ to be the class of all groups T such that the right Goldie dimension of RT is less than or equal to the right Goldie dimension of $RH_{\alpha-1}$, taking $\mathfrak{X}$ to be the class $(\mathcal{Y} \cap \mathcal{O}_0)$, and applying Lemma 2.4(i), (ii) and (iii) and Lemma 2.7.

If on the other hand $H_\alpha/H_{\alpha-1}$ is finite, say $|H_\alpha/H_{\alpha-1}| = t$, then

$$r.m_{H_\alpha/H} = r.m_{H_{\alpha-1}/H}.t,$$

and RH_α is a free $RH_{\alpha-1}$-module of rank t, by Lemma 1.3.3(i). Thus

$$\text{right Goldie dimension } (RH_\alpha)|_{RH_{\alpha-1}} \leq r.m_{H_{\alpha-1}/H}.t$$

$$= r.m_{H_\alpha/H} ,$$

and so a fortiori,

$$\text{right Goldie dimension } RH_\alpha \leq r.m_{H_\alpha/H} .$$

In all cases, therefore, we have obtained a contradiction to our assumption that (4) is false, and so

$$\text{right Goldie dimension } RG \leq r.m_{G/H} .$$

We shall now prove by induction on ρ that the right quotient ring Q(RG) exists and is right Artinian. Suppose the result is false, and let τ be the least ordinal, $0 < \tau \leq \rho$, for which RH_τ fails to have a right Artinian right quotient ring.

Case (i): If τ is a limit ordinal, $Q(RH_\tau)$ exists, since we can put $Q(RH_\tau) = \bigcup_{\beta<\tau} Q(RH_\beta)$, as before. There exists an ordinal $\gamma < \tau$ such that for all ordinals β, $\gamma \leq \beta < \tau$, $H_{\beta+1}/H_\beta \in \mathfrak{b}\mathfrak{A}_0$, since there exist only finitely many finite factors in the series (1). By Lemma 2.10, $N(RH_\tau) = N(RH_\gamma)RH_\tau$, noting that $N(RH_\gamma)$ is nilpotent by Proposition 1.2.27, since RH_γ has a right Artinian right quotient ring. Since for all ordinals β, $\gamma \leq \beta < \tau$, $Q(RH_\beta)$ exists and is right Artinian, we may use an argument similar to that used in proving Lemma 2.5(i) to deduce that $Q(RH_\tau)/N(Q(RH_\tau))$ is regular, and so Artinian, since $Q(RH_\tau)$ is known by the first part of the proof to have finite right Goldie dimension. Hence $Q(RH_\tau)$ is right Artinian, by Theorem 1.2.1, since $N(Q(RH_\tau))$ is nilpotent and finitely generated as a right ideal. It follows that τ cannot be a limit ordinal.

Case (ii): If τ is not a limit ordinal, $(\tau - 1)$ exists, $RH_{\tau-1}$ has a right Artinian right quotient ring, and $H_\tau/H_{\tau-1}$ is either finite or a member of $B\mathcal{O}_0$. If $H_\tau/H_{\tau-1} \in B\mathcal{O}_0$, then $Q(RH_\tau)$ exists and is right Artinian, by Theorem 2.11. If $H_\tau/H_{\tau-1} \in \mathcal{F}$, on the other hand, $Q(RH_\tau)$ also exists and is right Artinian, since as in Lemma 2.4.10 we may put

$$Q(RH_\tau) = RH_\tau \otimes_{RH_{\tau-1}} Q(RH_{\tau-1}),$$

which is an Artinian ring since it is finitely generated as a module over the right Artinian subring $Q(RH_{\tau-1})$.

Thus the theorem follows by transfinite induction.

3. QUASI-FROBENIUS QUOTIENT RINGS

In this section we shall prove Theorem 3.8; our main tool will be the following result, extracted from [18, Theorems 3.9 and 4.2].

THEOREM 3.1. Let S be an order in an Artinian ring Q(S). Then Q(S) is quasi-Frobenius $\Longleftrightarrow$ (i) there exists a direct sum of uniform left ideals in S, containing a regular element;

(ii) there exists a direct sum of uniform right ideals in S, containing a regular element;

(iii) $l_S\big(N(S)\big) = r_S\big(N(S)\big)$.

PROOF:

From the proof of [18, Thm.3.9], it is clear that if Q(S) is known to exist and to be Artinian, then conditions (i) and (ii) imply that Q(S) is a QF-2 ring, and then [18, Thm.4.3] shows that (iii) implies that Q(S) is a QF-ring. The converse is proved exactly as in [18].

We precede the proof of Theorem 3.8 with some lemmas, in the first four of which R denotes an arbitrary ring. We extract the first from the proof of Lemma 2.7.

LEMMA 3.2. If H is a normal subgroup of a group G, $G/H \in (\mathcal{L} \wedge \mathcal{A}_0)$, and U is a uniform right ideal of RH, then UG is a uniform right ideal of RG.

The proof of the next lemma is left to the reader.

LEMMA 3.3.(i) Let H be a subgroup of G, and let U be a uniform right ideal of RH. If for all finite subsets $\{x_1, \ldots, x_n\}$ of G, UT is a uniform right ideal of RT, where $T = \langle H, x_1, \ldots, x_n \rangle$, then UG is uniform in RG.

(ii) Let $\mathcal{Y}$ be a class of groups such that if X is a normal subgroup of a group Y, $Y/X \in \mathcal{Y}$, and U is a uniform right ideal of RX, then UY is uniform in RY. Let H be a subgroup of a group G such that there exists an ascending series from H to G whose factors lie in $\mathcal{Y}$. If U is a uniform right ideal of RH, then UG is uniform in RG.

From Lemma 2.2 and the above two lemmas, we deduce

LEMMA 3.4. Let H be a normal subgroup of a group G such that $G/H \in B\mathcal{A}_0$. If U is a uniform right ideal of RH, then UG is uniform in RG.

LEMMA 3.5. Let H be a normal subgroup of G, and suppose that $N(RG) = N(RH)RG$. Then

$$l_{RG}\big(N(RG)\big) = l_{RH}\big(N(RH)\big)RG,$$

and

$$r_{RG}\big(N(RG)\big) = r_{RH}\big(N(RH)\big)RG.$$

PROOF:

We shall prove the second identity - the first is proved similarly, bearing in mind that $N(RH)$ is a characteristic ideal of RH, and so, since

$$N(RH)^g = N(RH) ,$$

for all $g \in G$, it follows that

$$l_{RH}\big(N(RH)\big)^g = l_{RH}\big(N(RH)\big),$$

for all $g \in G$. Let $\{g_\lambda : \lambda \in \Lambda\}$ be a transversal to H in G, and suppose $\beta = \sum_{\lambda\in\Lambda} \beta_\lambda g_\lambda \in r_{RH}\big(N(RH)\big)RG$, where $\beta_\lambda \in r_{RH}\big(N(RH)\big)$ for all $\lambda \in \Lambda$, and $\beta_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$. If $\gamma = \sum_{\nu\in\Lambda} g_\nu \gamma_\nu \in N(RG)$, where $\gamma_\nu \in N(RH)$ for all $\nu \in \Lambda$, and $\gamma_\nu = 0$ for all but finitely many $\nu \in \Lambda$, then clearly

$$\gamma\beta = 0,$$

and so $\beta \in r_{RH}\big(N(RH)\big)RG \subseteq r_{RG}\big(N(RG)\big)$.

The reverse inclusion is immediate from Lemma 1.3.3, since $N(RH) \subseteq N(RG)$.

LEMMA 3.6. Let H be a normal subgroup of a group G, such that $G/H \in \mathrm{B}\mathcal{Q}_0$, and suppose that RH is an order in a QF-ring. Then RG is an order in a QF-ring.

PROOF:

By Theorem 2.11 and its left-handed equivalent, RG has an Artinian quotient ring, $Q(RG)$. Applying Theorem 3.1 to the ring RH, we deduce that

$$l_{RH}\big(N(RH)\big) = r_{RH}\big(N(RH)\big) , \tag{5}$$

and that there exist elements $c, d \in \mathcal{C}_{RH}(0)$ such that

$$c \in U_1 \oplus \dots \oplus U_n$$

$$\text{and} \qquad d \quad V_1 \oplus \dots \oplus V_m ,$$

where U_i (respectively V_j) is a uniform right (resp. left) ideal of RH, $1 \leq i \leq n$, (resp. $1 \leq j \leq m$). (In fact we always have $n = m$, but we shall not need this fact here; see [18, Cor.3.8].

By Lemma 2.10, $N(RG) = N(RH)RG$, and so by (5) and Lemma 3.5,

$$l_{RG}\big(N(RG)\big) = r_{RG}\big(N(RG)\big),$$

so that RG satisfies (iii) of Theorem 3.1. Furthermore, $c \in \sum_{i=1}^{n} U_i G$, and this sum of right ideals of RG is direct, by Lemma 1.3.3(iii). Also, by Lemma 3.4, $U_i G$ is a uniform right ideal, $1 \leq i \leq n$, and by Lemma 1.3.4(i), $c \in \mathcal{C}_{RG}(0)$. We thus see that (ii) of Theorem 3.1 is satisfied; similar remarks apply to (i), and we deduce that Q(RG) is a QF-ring, as required.

The next lemma is well-known, the proof being essentially the same as that used to prove that the group algebra of a finite group is self-injective (Prop.1.3.19). In various forms, it has been proved by Burgess, [5, Lemma 2.4], Horn, [21, Lemma 3.7], and Passman, [39, Lemma 4]. Our proof will follow Passman's.

LEMMA 3.7. Let H be a normal subgroup of finite index in a group G, and let R be a ring such that RH is an order in a QF-ring. Then RG is an order in a QF-ring.

PROOF:

Let Q(RH) be the quotient ring of RH. As in the proof of Theorem 2.12, RG has an Artinian quotient ring,

$$Q(RG) = Q(RH) \otimes_{RH} RG .$$

It remains to prove that this ring is right self-injective.

Let $T = \{1 = g_1, \ldots, g_n\}$ be a transversal to H in G, so that by Lemma 1.3.3(i)

$$Q(RG)_{|Q(RH)} = \sum_{i=1}^{n}{}^{\oplus}\left(Q(RH) \otimes_{RH} g_i\right)_{|Q(RH)} .$$

Let

$$\tau : Q(RG) \longrightarrow Q(RH) : \sum_{i=1}^{n}(\alpha_i \otimes g_i) \longmapsto \alpha_1$$

be the ''trace map'', a homomorphism of right $Q(RH)$-modules. If $U \subseteq V$ are right $Q(RG)$-modules and $\sigma : U \longrightarrow Q(RG)$ is a $Q(RG)$-map, we have to extend σ to a map from V to $Q(RG)$. Now $\tau\sigma : U \longrightarrow Q(RH)$, and since $Q(RH)$ is right self-injective, $\tau\sigma$ extends to a $Q(RH)$-map $\varphi : V \longrightarrow Q(RH)$.

Now define

$$\tilde{\varphi} : V \longrightarrow Q(RG) : v \longmapsto \sum_{i=1}^{n}\left(\varphi(vg_i^{-1}) \otimes g_i\right),$$

a $Q(RH)$-map from V to $Q(RG)$, since H is normal in G. For $1 \leqslant j \leqslant n$, and $v \in V$, we have

$$\begin{aligned} \tilde{\varphi}(vg_j) &= \sum_{i=1}^{n}\left(\varphi(vg_jg_i^{-1}) \otimes g_i\right) \\ &= \left[\sum_{i=1}^{n}\{\varphi(v(g_ig_j^{-1})^{-1}) \otimes (g_ig_j^{-1})\}\right] g_j \\ &= \tilde{\varphi}(v)g_j , \end{aligned}$$

since $g_ig_j^{-1} = a_{ij}g_{k(i,j)}$, where $a_{ij} \in H$ and $1 \leqslant k(i,j) \leqslant n$. It follows that $\tilde{\varphi}: V \longrightarrow Q(RG)$ is a $Q(RG)$-map. If $u \in U$,

$$\begin{aligned} \tilde{\varphi}(u) &= \sum_{i=1}^{n}\left(\varphi(ug_i^{-1}) \otimes g_i\right) \\ &= \sum_{i=1}^{n}\left(\tau\sigma(ug_i^{-1}) \otimes g_i\right) \\ &= \sum_{i=1}^{n}\left[\tau\left(\sigma(u)g_i^{-1}\right) \otimes g_i\right] \\ &= \sigma(u), \end{aligned}$$

so that $\tilde{\varphi}$ extends σ, as required.

<u>PROOF OF THEOREM 3.8</u>.

Suppose the theorem is false, so that there exists a ring R and a group G with a normal subgroup H, such that $G/H \in \mathcal{U}$, RH is an order in a QF-ring, but RG is not an order in a QF-ring. There exists a series

$$H = H_0 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G \qquad (1)$$

where ρ is an ordinal, $H_\alpha \lhd H_{\alpha+1}$, $H_{\alpha+1}/H_\alpha$ is either finite or in $B\mathcal{O}_0$, $0 \leqslant \alpha < \rho$, and such that $H_{\alpha+1}/H_\alpha$ is finite for only finitely many ordinals α. By Theorem 2.12, $Q(RH_\beta)$ exists and is Artinian, for all β, $0 \leqslant \beta \leqslant \rho$. Let γ be the least ordinal, $0 < \gamma \leqslant \rho$, for which $Q(RH_\gamma)$ is not a QF-ring. By Lemmas 3.6 and 3.7, it is clear that γ must be a limit ordinal. Since there are only finitely many finite factors in (1), there exists an ordinal $\tau < \gamma$ such that for all ordinals β, $\tau \leqslant \beta < \gamma$, $H_{\beta+1}/H_\beta \in B\mathcal{O}_0$. Since, by minimality of γ, $Q(RH_\tau)$ exists and is a QF-ring, we deduce from Theorem 3.1 that RH_τ satisfies conditions (i), (ii) and (iii) of that theorem. By Lemma 2.10, $N(RH_\gamma) = N(RH_\tau)RH_\gamma$, so that Lemma 3.5 implies that

$$r_{RH_\gamma}\big(N(RH_\gamma)\big) = l_{RH_\gamma}\big(N(RH_\gamma)\big) .$$

Similarly, from Lemmas 3.3(ii) and 3.4 it follows that since RH_τ satisfies conditions (i) and (ii) of Theorem 3.1, so does RH_γ. We deduce from Theorem 3.1 that $Q(RH_\gamma)$ is a QF-ring, and this contradiction establishes Theorem 3.8.

It follows from Theorem 3.8 that if R and G satisfy its hypotheses, $Q(RG)_{|Q(RG)}$ is an injective module. Indeed, $Q(RG)_{|RG}$ is injective, so that $Q(RG)_{|RG}$ is the injective hull of $RG_{|RG}$, and $Q(RG)$ is the maximal right (and left) ring of quotients of RG, (see Defn.1.2.44). We assemble these and other observations in a well-known lemma.

Let S be a ring, M a right S-module, and put

$$T(M) = \{m \in M : mc = 0, \text{ some } c \in \mathcal{C}_S(0)\}.$$

It is easily seen that if S satisfies the right Ore condition $T(M)$ is a submodule of M. Clearly if $M \subseteq F$ for some free S-module F, $T(M) = 0$. If $T(M) = 0$, then $M_{|S}$ is a submodule of $(M \otimes_S Q(S))_{|S}$.

LEMMA 3.8. Let S be a ring with the right Ore condition, and let Q(S) denote its classical right quotient ring. If M is a right S-module with $T(M) = 0$, then the injective hull of M is the Q(S)-injective hull of $(M \otimes_S Q(S))$.

If Q(S) is a QF-ring, and M is a submodule of a free S-module F, the injective hull of M is $(L \otimes_S Q(S))$, where L is the maximal essential extension of M in F.

The proof is elementary.

4. SOME EXAMPLES - THE SOLUBLE CASE

In this section we discuss how Theorem 2.12 applies to group rings of soluble groups. As explained in §1, we are unable to completely determine when the group ring KG of a soluble group G over a field K has an Artinian quotient ring; the difficulty, as we shall see, lies in dealing with infinite locally finite factors T/H, where $H \lhd T \subseteq G$. Nevertheless if we impose suitable finiteness conditions either on the locally finite factors or on the torsion-free abelian factors occurring in an abelian series for G, then Theorem 2.12 is sufficient to provide an answer.

If $A \in \mathcal{Q}_0$, the rank of A is defined to be $\dim_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q})$, while if $B \in \mathcal{Q}$, the rank of B is simply the rank of $B/T(B)$ where $T(B)$ denotes the torsion subgroup of B.

We shall need the following result, due to Mal'cev. Recall from page 5 that L(G) denotes the maximal normal locally finite subgroup of G.

<u>THEOREM 4.1</u>. (Mal'cev; [34, Thm.3]) Let G be a soluble group with a finite series

$$1 = G_0 \subset \ldots \subset G_i \subset G_{i+1} \subset \ldots \subset G_n = G$$

of normal subgroups such that G_{i+1}/G_i is abelian of finite rank, $0 \leqslant i < n$. Then G/L(G) has a finite chain of characteristic subgroups, with factors which are abelian and have finite rank and finite torsion subgroups.

<u>THEOREM 4.2</u>. Let G be a soluble group and R a ring with a right Artinian right quotient ring.

(i) If G has a series

$$1 = H_0 \subset \ldots \subset H_i \subset H_{i+1} \subset \ldots \subset H_n = G \qquad (5)$$

where $H_i \lhd H_{i+1}$ and H_{i+1}/H_i is either finite or torsion-free abelian, $0 \leqslant i < n$, then RG has a right Artinian right quotient ring.

(ii) If G has a series

$$1 = H_0 \subset \ldots \subset H_i \subset H_{i+1} \subset \ldots \subset H_n = G \qquad (6)$$

where $H_i \lhd G$, and H_{i+1}/H_i is abelian of finite rank, $0 \leqslant i < n$, then RG has a right Artinian right quotient ring $\Longleftrightarrow$ L(G) is finite.

<u>PROOF</u>:

(i) This is immediate from Theorem 2.12.

(ii) If G has a series of type (6), and L(G) is finite, then by Theorem 4.1 G has a series of type (5), and the result follows from (i).

Conversely, if L(G) is infinite, then since, by

Lemma 1.3.15, if H is any finite subgroup of G,

$$r_{RG}\left(\sum_{h\in H} h\right) = \underline{\underline{h}}G,$$

it follows that RG does not have max-ra. Hence, by the remark following Theorem 1.2.23, RG cannot have a right Artinian right quotient ring.

REMARKS: (i) By using instead of Mal'cev's theorem a generalisation of it, [40, Lemma 9.34], it is possible to prove a stronger version of Theorem 4.2(ii) in which G is assumed only to be radical, rather than soluble. (A radical group is a group with an ascending series of normal subgroups with locally nilpotent factors.) However, since our main purpose in stating Theorem 4.2 is to illustrate to what extent we can answer the question raised at the beginning of §4, we shall not pursue this further.

(ii) Theorem 4.2 suggests that our question may have the following answer:-

CONJECTURE 4.3. If K is a field and G is soluble, KG has an Artinian quotient ring $\Longleftrightarrow$ G has no infinite locally finite subgroups.

We now consider two examples. The first will show that not all torsion-free soluble groups are contained in $\mathcal{V}$, a fact that we have mentioned several times previously. In this and in the following sections we shall need the following elementary facts about the classes $\mathcal{V}$ and $\mathfrak{B}\mathcal{Q}_0$.

LEMMA 4.4. (i) If $H \subseteq G$ and $G \in \mathfrak{B}\mathcal{Q}_0$, then $H \in \mathfrak{B}\mathcal{Q}_0$.

(ii) If $H \subseteq G$ and $G \in \mathcal{V}$, then $H \in \mathcal{V}$.

(iii) If $N \in \mathfrak{B}\mathcal{Q}_0$ and N is finitely generated, N has an infinite cyclic image.

(iv) If H is a finite normal subgroup of G, and $G \in \mathcal{V}$, then $G/H \in \mathcal{V}$.

PROOF:

(i) The class $\mathcal{Q}_0$ is clearly closed under taking subgroups. If we put

$$\mathcal{S} = \{\{G\} : H \leq G \Rightarrow H \in B\mathcal{Q}_0\},$$

then $\mathcal{S}$ is a subclass of $B\mathcal{Q}_0$, and $\mathcal{Q}_0 \leq \mathcal{S}$. It is trivial to check that $\mathcal{S}$ is $\acute{P}$- and L-closed, whence

$$\mathcal{S} = B\mathcal{Q}_0,$$

as required.

(ii) This follows easily from (i).

(iii) Recalling that $\mathcal{D}$ denotes the class of unit groups, let $\mathcal{P}$ be the class of groups defined by

$$\mathcal{P} = \mathcal{D} \cup \{\{G\} : 1 \neq H \leq G,\ H \in \mathcal{G} \Rightarrow H \text{ has an infinite cyclic image}\}.$$

Clearly $\mathcal{P} \geq \mathcal{Q}_0$, and one may easily show that $\mathcal{P}$ is $\acute{P}$- and L-closed. Since $\mathcal{P}$ contains $\mathcal{Q}_0$, it follows that $\mathcal{P} \geq B\mathcal{Q}_0$.

(iv) Let

$$1 = G_0 \subset \ldots \subset G_\alpha \subset G_{\alpha+1} \subset \ldots \subset G_\rho = G$$

be a $\mathcal{V}$-series for G. Consider the series

$$H = HG_0 \subset \ldots \subset HG_\alpha \subset HG_{\alpha+1} \subset \ldots \subset HG_\rho = G$$

between H and G. For $0 \leq \alpha < \rho$, we have

$$\frac{HG_{\alpha+1}}{HG_\alpha} \cong \frac{G_{\alpha+1}}{HG_\alpha \cap G_{\alpha+1}},$$

which is clearly finite if $G_{\alpha+1}/G_\alpha$ is finite. If on the other hand $G_{\alpha+1}/G_\alpha \in B\mathcal{Q}_0$, then since

$$\frac{HG_\alpha \cap G_{\alpha+1}}{G_\alpha} = \frac{(H \cap G_{\alpha+1})G_\alpha}{G_\alpha}$$

$$\cong \frac{H \cap G_{\alpha+1}}{H \cap G_\alpha},$$

and this last group is clearly finite, we must have

$$(HG_\alpha \cap G_{\alpha+1}) = G_\alpha ,$$

by (i) and (iii). Thus

$$HG_{\alpha+1}/HG_\alpha \cong G_{\alpha+1}/G_\alpha \in \mathfrak{B}\mathfrak{A}_0,$$

and so $G/H \in \mathfrak{V}$.

EXAMPLE 4.5. For each prime p, let

$$G_{(p)} = \langle b, c : [b, b^c] = c^{p^2}, [b, b^{c^i}] = 1, 2 \leq i \leq p-2; \; b^{c^{p-1} + c^{p-2} + \ldots + c + 1} = 1 \rangle .$$

The groups $G_{(p)}$ were constructed by Bowers, [4]. For each prime p, $G_{(p)}$ is polycyclic: indeed $G_{(p)}$ is a torsion-free extension of a finitely generated nilpotent group of class two by a finite abelian p-group. For our purposes, however, the important properties of $G_{(p)}$ are that it is torsion-free polycyclic and that

$$G_{(p)}/(G_{(p)})'$$

is a p-group. (Of course $(G_{(p)})'$ is the derived subgroup of $G_{(p)}$.) Proofs of all these facts can be found in [4].

Now put

$$G = \prod_p G_{(p)} ,$$

where the product is taken over all primes p. Thus G is a torsion-free soluble group of derived length 3. We claim that $G \notin \mathfrak{V}$.

Suppose that this is false, so that there exists a series

$$1 = H_0 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G \qquad (1)$$

such that $H_\alpha \lhd H_{\alpha+1}$ and $H_{\alpha+1}/H_\alpha$ is finite or in $\mathfrak{B}\mathfrak{A}_0$ for all ordinals α, $0 \leq \alpha < \rho$, and such that only finitely many of the factors are finite. Choose a prime p which does not divide the orders of these finitely many finite factors, and consider the series

$$1 = (H_0 \cap G_{(p)}) \subset \ldots \subset (H_\alpha \cap G_{(p)}) \subset \ldots \subset (H_\rho \cap G_{(p)}) = G_{(p)} . \qquad (7)$$

By (i) and (ii) of Lemma 4.4, the factors of this series are in $(\mathcal{F} \cup \mathfrak{B}\mathcal{O}_0)$, and if $(H_{\alpha+1} \cap G_{(p)})/(H_\alpha \cap G_{(p)})$ is a finite factor, then

$$|(H_{\alpha+1} \cap G_{(p)})/(H_\alpha \cap G_{(p)})| = |(H_{\alpha+1} \cap G_{(p)})H_\alpha / H_\alpha| ,$$

and so divides $|H_{\alpha+1}/H_\alpha|$. It follows that p does not divide the order of any finite factor in (7).

Now since $G_{(p)}$ is polycyclic, $G_{(p)}$ has the ascending chain condition on subgroups, by Theorem 1.1.12, and so (7) must have only finitely many distinct terms. Thus we may write (7) as

$$1 = B_0 \subset B_1 \subset \ldots \subset B_n = G_{(p)} .$$

Now either B_n/B_{n-1} is a finitely generated group in $\mathfrak{B}\mathcal{O}_0$, or B_n/B_{n-1} is a non-trivial finite p'-group. Since $G_{(p)}/(G_{(p)})'$ is finite, the first possibility cannot occur, by Lemma 4.4(iii). However, if B_n/B_{n-1} is finite, then being soluble it has a non-trivial finite abelian image, which must be a p-group since $G_{(p)}/(G_{(p)})'$ is a p-group, and since B_n/B_{n-1} is a p'-group, this too is impossible. Hence the series (7) cannot exist, and we conclude that $G \notin \mathcal{V}$.

<u>REMARK</u>: If K is a field of characteristic zero and G is the group constructed above, we know by Lemma 1.3.4(ii) and the work of Farkas and Snider [8] that KG is an Ore domain. If however K has positive characteristic, we

have no method of proving that KG is Goldie. Finally, if K is again assumed to have characteristic zero, then $K(C_\infty \wr G)$ satisfies the right Ore condition, as can be seen by using the proof of Lemma 2.4.10, since $(C_\infty \wr G)$ is an extension of a group in $\mathcal{V}$ by a locally finite group, but it is not known whether $K(C_\infty \wr G)$ is right Goldie.

In order to extend Theorem 4.2 to prove Conjecture 4.3, it will clearly be necessary to obtain a closer understanding of the relationship between the properties ''G is torsion-free'' and ''KG is right Goldie'' than that provided by Theorem 2.12. The following example will demonstrate this, and will also answer a question of Lazard, [32]. Following Bass, Lazard defines a projective module P over a ring S to be <u>uniformly big</u> if $d_S(P)$, the minimal number of generators of P, is an infinite cardinal, and $d_S(P) = d_S(P/PM)$ for all maximal ideals M of S. Clearly every infinitely generated free module is uniformly big. In [32, §V], Lazard asks the question: Are uniformly big projective modules always free? This is the case, for example, if $S/J(S)$ is Noetherian, see [32, §V] and the references given there, but as we now show, it is not true in general. Indeed, it will become clear that to construct a counter-example it is enough to find a ring which is a countable union of simple Artinian subrings, and is not itself Artinian.

<u>EXAMPLE 4.6</u>. As usual, we let C_∞ denote the infinite cyclic group, and we denote by C_{p^∞} the Prüfer group for some prime p. Let $G = (C_\infty \wr C_{p^\infty})$, and let K be any field. Let A denote the base group of G, so that A is a free

abelian group of infinite rank, and G/A is isomorphic to C_{p^∞}. There exists an ascending series of normal subgroups of G,

$$A = H_0 \subset H_1 \subset \ldots \subset H_i \subset \ldots \subset \bigcup_{i=1}^{\infty} H_i = G ,$$

where for $i \geqslant 1$, H_i/A is isomorphic to C_{p^i}, the cyclic group of order p^i.

For all $i \geqslant 0$, $H_i \in \mathcal{V}$, and since H_i has no finite normal subgroups it follows from Theorem 2.12 that KH_i is prime right Goldie, so that KH_i has a simple Artinian right quotient ring, $Q(KH_i)$. By Lemma 1.3.6, since $G = \bigcup_{i=0}^{\infty} H_i$ we can form the right quotient ring $Q = \bigcup_{i=0}^{\infty} Q(KH_i)$ of KG. Note in particular that if $\alpha \in KG$ is a regular element, α has an inverse in Q.

We shall show that Q has a uniformly big projective module which is not free. First, note that since $Q(KH_i)$ is simple for each $i \geqslant 0$, Q is also simple. Since G has infinite locally finite subgroups it follows by the same argument as that used in the proof of Theorem 4.2(ii) that KG cannot have max-ra, the ascending chain condition on right annihilators. Hence Q is not Artinian.

Now Q is right (and left) hereditary. For Q is the union of a countable chain of simple Artinian subrings, so by Proposition 1.2.55 and Lemma 1.2.51,

$$\text{rt.gl.dim.}Q \quad \leqslant \quad 1 + \sup_i\{\text{rt.gl.dim.}Q(KH_i)\} \quad = \quad 1.$$

Thus the right global dimension of Q is at most one, and since Q is not semisimple Artinian, it must be exactly one, by Lemma 1.2.51.

Since Q is simple and regular elements of Q are units, Q cannot be right Noetherian, since otherwise it would be Artinian, by Theorem 1.2.23. Let P be any right ideal of Q which is not finitely generated. (For example, we could take P to be the right ideal of Q generated by the augmentation ideal of a Prüfer subgroup of G.) Since Q is right hereditary, P is a projective Q-module, by Lemma 1.2.52, and since Q is simple it is trivial that P is uniformly big. We claim that P is not free. Suppose for a contradiction that P is free; then there exists a Q-monomorphism $\psi: Q \longrightarrow P$. Let $c = \psi(1) \in P$. Thus c is a right regular element of Q, so that c has an inverse $c^{-1} \in Q$. Hence $1 = cc^{-1} \in P$, and so $P = Q$, contradicting our choice of P.

Thus P is a uniformly big projective module which is not free.

<u>REMARKS</u>: (i) If G is the group constructed in Example 4.5, and K is any field, then since $(C_\infty \wr G)$ has a normal subgroup H such that $H \in P\mathcal{Q}_o$ and G/H is countable and locally finite, it follows by the arguments used above that $K(C_\infty \wr G)$ has a classical quotient ring Q which is simple and locally Artinian. However as the above example shows, these properties alone are not sufficient to ensure that Q is Artinian.

(ii) It seems appropriate to mention at this point that Theorem 2.11 cannot be extended to include the class $\hat{P}\mathcal{Q}_o$ (in the notation of [40]), although one can prove by the methods of [40, Chapter 8] that $B\mathcal{Q}_o \subseteq \hat{P}\mathcal{Q}_o$. Thus, for example, if F is the free group on two generators x and y, and K is any field, KF is a domain, but is not an Ore domain, as can be seen by noting that $\underline{\underline{f}}$ is freely generated as a right KF-module by $\{(x-1), (y-1)\}$; see [17, Thm.1, p.32].

5. APPLICATIONS

We include in this section various applications of Theorems 2.12 and 3.8.

First recall that by the remark following Theorem 1.2.23, it follows from Theorem 2.12 that if $G \in \mathcal{U}$ and R has a right Artinian right quotient ring, then RG has the maximum and minimum conditions on right annihilators. If we assume only that R is a ring with max-ra, it is not known whether $G \in \mathcal{U}$ implies that RG has max-ra. However, the proof of [47, Thm.5.1] can be used to obtain (i) of the result below, and we may similarly extend [47, Thms.5.2 and 5.4] to give (ii) and (iii).

THEOREM 5.1. (i) Let R be a commutative ring with the maximum condition on annihilator ideals, and suppose $G \in \mathcal{U}$. Then every nil right ideal of RG is nilpotent.

(ii) Let R be a commutative Noetherian ring, and suppose $G \in \mathcal{U}$. Then RG can be embedded in a ring which is both right and left Artinian.

(iii) Let R be a commutative Noetherian ring, and let $G \in \mathcal{U}$. If $I \triangleleft_r RG$, and I contains a regular element, then I is generated by the regular elements it contains. If I is a finitely generated right ideal, I has a finite generating set of regular elements.

There is an interesting improvement of Theorem 5.1(ii) in the case where R is assumed to be a commutative domain. Both Theorem 5.1(ii) and the result we shall prove below should be viewed in the light of the example, due to Small [45], of a right Noetherian ring which is not a subring of a right Artinian ring.

THEOREM 5.2. Let R be a commutative domain and let $G \in \mathcal{V}$. Put $m = m_G$, and let n be a positive integer. Then $M_n(RG)$ is a subring of a simple Artinian ring $M_{nm}(D)$, where D is a division ring whose centre is the quotient field of R.

PROOF:

Since $G \in \mathcal{V}$, clearly $H = (C_\infty \wr G) \in \mathcal{V}$, and $G \subseteq H$. Note that H has no non-trivial finite normal subgroups, so that RH is prime, by Proposition 1.3.18. As in the proof of Theorem 2.12, the right Goldie dimension of RH is at most m, and Q(RH) exists and is simple Artinian of right Goldie dimension at most m. It follows that $RH \subseteq M_m(D)$, where D is a division ring, so that

$$M_n(RG) \subseteq M_n(RH) \subseteq M_{mn}(D) .$$

For the last part, we may clearly assume that G is infinite, so that $\Delta(H) = 1$ by Lemma 1.1.26, and by Lemma 1.3.21 the centre of RH is R. By [46, Thm.7.4], the centre of Q(RH) is the quotient field of R, and the result follows.

REMARK: It is of course a consequence of the above result that the group of units of $M_n(RG)$, where R, G and n are as above, is linear over a division ring with centre R. In particular, therefore, if $G \in \mathcal{V}$ then G is linear over a division ring of arbitrary characteristic. Note that since a soluble group linear over a field is nilpotent-by-abelian-by-finite, we cannot in general choose the division ring D to be a field. The group $(C_\infty \wr C_\infty) \wr C_\infty$, for example, is clearly in $\mathcal{U}$, but is not linear over a field; see [40, Vol.1, p.74-79].

We now give two applications of the results of this chapter to the study of group rings. The first is a generalisation of work of Jordan, [26], on the Jacobson radical of the group rings of certain generalised free products with amalgamation. Notice first that if $G \in \mathcal{U}$ and K is a field, then by a straightforward transfinite induction, using Lemma 1.3.26 and [36, Thm.16.6], together with arguments similar to those used in Lemmas 2.3 and 2.9, one can show that

$$J(KG) = N(KG) .$$

Jordan showed [26, Thm.2] that if K is a field and G is a group with a normal subgroup H, and subgroups A and B, such that $H \subsetneq A$, $H \subsetneq B$, $G/H = (A/H * B/H)$, and either A/H or B/H is infinite, and if furthermore KH is Noetherian, then

$$J(KG) = N(KH)KG.$$

(Recall that the only known examples of Noetherian group rings are the group rings of polycyclic-by-finite groups, with Noetherian coefficient rings.)

Here we shall prove the following result:-

<u>THEOREM 5.3</u>. Let K be a field, let $H \in \mathcal{U}$, and suppose that H is a normal subgroup of a group G, with

$$G/H = (A/H * B/H) ,$$

where either A/H or B/H is infinite, and both groups are non-trivial. Then $J(KG) = N(KH)KG$.

Since our proof is similar to, and is based on, Jordan's, we shall not include all the details. Ideals of the following type are crucial to our approach.

DEFINITION 5.4. Let R be a ring and σ an automorphism of R. An element $\alpha \in R$ is said to be σ-nilpotent if, for each positive integer s, there exists a non-negative integer $r = r(s)$ such that

$$\alpha\, \sigma^{s}(\alpha) \ldots \sigma^{rs}(\alpha) = 0.$$

An ideal I of R is said to be σ-nil if every element of I is σ-nilpotent.

LEMMA 5.5. (Jordan; [26, Lemma 4]) Let K and G/H be as in Theorem 5.3. If $J(KG) \neq 0$, KH has a non-zero σ-nil ideal, where σ is an automorphism induced by conjugation by an element of G.

Jordan proves in [26, Lemma 5] that σ-nil ideals of Noetherian rings are nilpotent. Exactly the same argument may be used to prove

LEMMA 5.6. Let I be a σ-nil ideal of the right Goldie ring S. Then I is nilpotent.

We can now deduce

THEOREM 5.7. Let K be a field, and suppose the group G has a normal subgroup H such that

$$G/H = (A/H * B/H) ,$$

where either A/H or B/H is infinite, and both groups are non-trivial. Suppose further that $H \in \mathcal{U}$. Then $J(KG) \neq 0$ if and only if $N(KH) \neq 0$.

PROOF:

If $J(KG) \neq 0$, then $N(KH) \neq 0$ by Lemmas 5.5 and 5.6, since KH is right Goldie by Theorem 2.12. Conversely, if $N(KH) \neq 0$, then by Lemma 1.3.10 $N(KH)KG$ is a non-zero nilpotent ideal of KG, so that $J(KG) \neq 0$.

To deduce Theorem 5.3, we need

THEOREM 5.8. (Passman; [36, Thm.20.5]) Let K be a field of characteristic $p \geqslant 0$, and let G be a group. Then $J(KG)$ is nilpotent if and only if G has subgroups P and L such that

(i) P is a finite p-group, (where if $p = 0$ we require only that P be finite);

(ii) P is a normal subgroup of L, and $|G : L| < \infty$;

(iii) $J\big(K(L/P)\big) = 0$.

PROOF OF THEOREM 5.3.

By Theorems 5.7 and 1.3.20(i), we may assume that K has positive characteristic p. Since $H \in \mathfrak{U}$, H contains no infinite locally finite subgroups, so in the notation of Defn.1.1.18, $\Delta^p(H)$ is finite. Since $\Delta^p(H)$ is characteristic in H, $\Delta^p(H)$ is normal in G.

Let P be a Sylow p-subgroup of $\Delta^p(H)$, and put $F = N_G(P)$, the normalizer in G of P. Since each element of the finite group P has only finitely many conjugates in G, $|G : F| < \infty$, and so there exists a normal subgroup L of finite index in G such that $L \subseteq F$. We claim that

$$L \cap P = \Delta^p(L).$$

Certainly $(P \cap L) \subseteq \Delta^p(L)$. If on the other hand $x \in \Delta^p(L)$ and x is a p-element, then since x normalizes P, $\langle P \cap L, x\rangle$ is a p-subgroup of $\Delta^p(L)$. However $\Delta^p(L) \subseteq H$, since otherwise $\Delta^p(L)H/H$ is a non-trivial normal locally finite subgroup of the non-trivial free product G/H, and it is easily seen that no such subgroups exist. It follows that $\Delta^p(L) \subseteq \Delta^p(H)$, so in particular $x \in \Delta^p(H)$, and since x is a p-element normalizing P, we must have $x \in P$, P being a Sylow p-subgroup of $\Delta^p(H)$. Thus $\langle P \cap L, x\rangle \subseteq P$, and since $\langle P \cap L, x\rangle \subseteq L$, we deduce that $\langle P \cap L, x\rangle$ is precisely $(P \cap L)$, so that $x \in (P \cap L)$. Since $\Delta^p(L)$ is generated by p-elements, this proves our claim.

Now $T = (L \cap H)$ is normal in G. By a result of Karass and Solitar, [29, third Corollary to Theorem 5], since $(L/T \cap H/T) = 1$, and $H/T \lhd G/T$, L/T is a free product with factors including $(A \cap L)/T$ and $(B \cap L)/T$. Since $|A : H|$ or $|B : H|$ is infinite, and $|G : L| < \infty$, it follows that either $(A \cap L)/T$ or $(B \cap L)/T$ is infinite. Suppose without loss that $(A \cap L)/T$ is infinite, so that if L/T is not a non-trivial free product, we must have $(A \cap L) = L$; that is, $A \supseteq L$. In this case, $|G : A| < \infty$, which clearly contradicts thefact that A/H is a factor of the non-trivial free product G/H. We have thus shown that $L/T = C * D$, say, with $|C| = \infty$, is a non-trivial free product.

Since $T \leq H$, $T \in \mathfrak{U}$ by Lemma 4.4(ii), and since $P \cap L$ is a finite normal subgroup of T, $T/P \cap L \in \mathfrak{U}$, Lemma 4.4(iv). Now $\Delta^p(T) \subseteq \Delta^p(H) \cap T$, since $|H : T| < \infty$, and so we deduce that

$$\Delta^p(T) \leq \Delta^p(H) \cap L = \Delta^p(L) = P \cap L,$$

the last equality having been proved above. Since $(P \cap L)$ is finite, it follows from Lemma 1.1.19(iv) that

$$\Delta^p\bigl(T/(P \cap L)\bigr) = 1,$$

so that by Theorem 1.3.20(ii) and Theorem 5.7,

$$J\bigl(K(L/P \cap L)\bigr) = 0.$$

We have now shown that KG satisfies (i), (ii) and (iii) of Theorem 5.8, so that J(KG) is nilpotent, and the result is proved.

<u>REMARKS</u>: (i) It will be noted that the above proof is similar to that used to prove Theorem 2 of [26], although the proof given there is not completely correct, and we have made the appropriate changes above.

(ii) An immediate consequence of the above result is that if K, G and H are as given in Theorem 5.3, and KH is semiprime, then KG is semisimple. We shall show in the next chapter (Cor.4.5.8) that if KH is assumed to be prime, then under mild additional hypotheses, KG is actually primitive.

If K and G are as in Theorem 5.3, but instead of $H \in \mathcal{U}$ we assume that H is soluble, then some information can be obtained about the structure of J(KG) by applying a suitable intersection theorem due to Zalesskii, (see Thm.4.3.10), and studying σ-nil ideals of group rings of FC-groups. We shall not pursue this further here.

The second of our applications is concerned with the maximal ring of quotients of a group ring. Martha Smith showed [46] that if K is a field and G is a group such that KG is semiprime, then the classical quotient ring of the centre of KG may be identified with the centre of the classical quotient ring of KG, whenever this latter ring exists. (Note that if KG has a classical right quotient ring, then by using the anti-automorphism of KG obtained from the map $g \longmapsto g^{-1}$, $(g \in G)$, it is easy to see that KG also satisfies the left Ore condition, so that Prop.1. 2.13

applies.) In [39], Passman showed that the hypothesis that KG is semiprime could be dropped.

One might reasonably ask whether analogous results hold for the maximal right quotient ring $Q_{max}(KG)$, which of course always exists. That is, can the maximal quotient ring of the centre of KG always be ''naturally'' identified with the centre of the maximal right quotient ring of KG? (The meaning of ''naturally'' will become clear.) Formanek showed in [13] that this is the case when KG is semiprime, but the general case remains open.

We shall denote the centre of a ring R by C(R). For the remainder of this chapter, K will denote a fixed field, and G a group. We shall prove the following result.

<u>THEOREM 5.9</u>. If $\Delta^+(G)$ is finite,

$$C\big(Q_{max}(KG)\big) \cong Q_{max}\big(C(KG)\big) .$$

Recall from Theorem 1.2.48 that the maximal right quotient ring of a ring may be viewed as the set of pairs (D, f) of dense right ideals D of R and R-homomorphisms $f:D \longrightarrow R$, where two such pairs (D_1, f_1) and (D_2, f_2) are identified if

$$f_1|_{D_1 \cap D_2} = f_2|_{D_1 \cap D_2} .$$

Now Formanek showed in [13, Theorem 2] that if H is a subnormal subgroup of G, $Q_{max}(KH)$ is a subring of $Q_{max}(KG)$, where the embedding takes the element (D, f) of $Q_{max}(KH)$ to the element $(DKG, \bar{f})$ of $Q_{max}(KG)$, where if $\{g_i : i \in I\}$ denotes a right transversal to H in G,

$$\bar{f}: DKG \longrightarrow KG : \sum_{i\in I}\alpha_i g_i \longmapsto \sum_{i\in I} f(\alpha_i)g_i .$$

(Formanek shows that $\bar{f}$ is well-defined.)

The following result, which in view of the above embedding and the fact that $C(KG) \subseteq K\Delta(G)$, (Lemma 1.3.21), may not seem very surprising, will be crucial to our proof of Theorem 5.9 in that it will enable us to apply the results of this chapter to the problem.

<u>THEOREM 5.10</u>. [13, Thm.7] $C\big(Q_{max}(KG)\big)$ is a subring of $Q_{max}\big(K\Delta(G)\big)$.

In considering the above theorem it is important to bear in mind the nature of the embedding of $C\big(Q_{max}(KG)\big)$ in $Q_{max}\big(K\Delta(G)\big)$. Formanek shows that if $\alpha \in C\big(Q_{max}(KG)\big)$, then we can represent α as (D_1G, f_1), where D_1 is a G-invariant two-sided ideal of $K\Delta(G)$, (so that D_1G is an ideal of KG), $f_1(D_1) \subseteq K\Delta(G)$, and f_1 is a bimodule homomorphism. Theorem 5.10 follows from this fact via the embedding of $Q_{max}\big(K\Delta(G)\big)$ in $Q_{max}(KG)$ described above.

Suppose now that $\Delta^+(G)$ is finite. Since $\Delta(G)/\Delta^+(G)$ is torsion-free abelian, by Lemma 1.1.19(i), it follows that $\Delta(G) \in \mathcal{U}$, so that as observed in §3,

$$Q_{max}\big(K\Delta(G)\big) = Q_{cl}\big(K\Delta(G)\big), \tag{8}$$

where, as before, $Q_{cl}(R)$ denotes the classical right quotient ring of a ring R. In terms of the representation of $Q_{max}\big(K\Delta(G)\big)$ which we have been discussing, we can express this as follows. Each element (ac^{-1}) of $Q_{cl}\big(K\Delta(G)\big)$ corresponds to the element $(cK\Delta(G), [ac^{-1}])$ of $Q_{max}\big(K\Delta(G)\big)$, where

$$[ac^{-1}]: cK\Delta(G) \longrightarrow K\Delta(G) : c\beta \longmapsto a\beta;$$

the identity (8) says that every element of $Q_{max}\big(K\Delta(G)\big)$ can be so represented, so that in particular, by

Theorem 5.10, the elements of $C\big(Q_{max}(KG)\big)$ can be represented in this way. Finally, recalling the embedding of $Q_{max}\big(K\Delta(G)\big)$ in $Q_{max}(KG)$ described above, we see that each element of $Q_{max}\big(K\Delta(G)\big)$ can be represented by a pair $(cKG, \left[ac^{-1}\right])$, where

$$\left[ac^{-1}\right] : cKG \longrightarrow KG : c\beta \longmapsto a\beta \quad , \tag{9}$$

with $a, c \in K\Delta(G)$, and $c \in \mathscr{C}_{KG}(0)$.

We are now in a position to use Passman's result [39] on classical quotient rings. In fact, an examination of the proof of Passman's theorem, [39, p.224], will show that he actually proves the following.

<u>THEOREM 5.11</u>. Let KG be a group algebra containing a multiplicatively closed set T of regular elements, such that KG satisfies the right Ore condition with respect to T. Suppose that

$$\mathscr{C}_{KG}(0) \cap C(KG) \subseteq T.$$

If $\alpha \in C(Q)$, where Q denotes the partial right quotient ring of KG formed by inverting the elements of T, then there exist elements $a, c \in C(KG)$, $c \in \mathscr{C}_{KG}(0)$, such that $\alpha = ac^{-1}$.

An important ingredient of Passman's proof of Theorem 5.11 is Lemma 2 of [39], which says that an element of $C(KG)$ is regular in $C(KG)$ if and only if it is regular in KG. In fact, Passman proves slightly more than this, and it is this stronger form of his result which we shall need.

<u>LEMMA 5.12</u>. [39, Lemma 2] Let α be a central element of the group algebra KG, and put $H = \langle \operatorname{supp} \alpha \rangle$. Then α is regular in KG if and only if α is regular in the ring

$C(KG) \cap KH$.

We claim now that $Q_{cl}\big(C(KG)\big)$ is a subring of $C\big(Q_{max}(KG)\big)$.

It follows from Lemma 5.12 that if $a \in C(KG)$ and c is a regular element of $C(KG)$, cKG is a dense ideal of KG, so that $(cKG, [ac^{-1}])$ represents, in our notation, an element of $Q_{max}(KG)$. It is easy to check that this actually defines an embedding of $Q_{cl}\big(C(KG)\big)$ in $Q_{max}(KG)$. Now under our representation of $Q_{max}(KG)$ by homomorphisms defined on dense right ideals, KG embeds in $Q_{max}(KG)$ via the map

$$KG \longrightarrow \mathrm{End}(KG_{|KG}) : r \longmapsto \psi_r ,$$

where

$$\psi_r : KG \longrightarrow KG : \gamma \longmapsto r\gamma .$$

We thus see that, with a and c as above, $(cKG, \ ac^{-1})$ commutes with every element of KG, so that by Proposition 2.2.49(ii),

$$(cKG, [ac^{-1}]) \in C\big(Q_{max}(KG)\big) .$$

We have therefore shown that

$$Q_{cl}\big(C(KG)\big) \subseteq C\big(Q_{max}(KG)\big) .$$

We in fact have

<u>PROPOSITION 5.13</u>. Suppose $\Delta^+(G)$ is finite. Then

$$C\big(Q_{max}(KG)\big) = Q_{cl}\big(C(KG)\big) .$$

<u>PROOF</u>:

By Theorem 5.10 and (9), if $\alpha \in C\big(Q_{max}(KG)\big)$, α can be represented by a pair $(cKG, [ac^{-1}])$, where

$$[ac^{-1}] : cKG \longrightarrow KG : c\beta \longmapsto a\beta ,$$

with $a, c \in K\Delta(G)$ and $c \in \mathcal{C}_{KG}(0)$. Take any element r of KG, and view r as an element of $\mathrm{End}(KG_{|KG})$. Since $\alpha \in C\big(Q_{max}(KG)\big)$, we have

$$\left[ac^{-1}r\right]d = \left[rac^{-1}\right]d ,$$

for all $d \in D$, where D, the intersection of the domains of $\left[ac^{-1}r\right]$ and $\left[rac^{-1}\right]$, is a dense right ideal of KG by Lemma 1.2.47(i).

Thus $$(ac^{-1}r - rac^{-1})D = 0,$$

where the multiplication may be viewed as taking place in the partial right quotient ring Q of KG formed by inverting the elements of $\mathcal{C}_{K\Delta(G)}(0)$.

It is, however easily seen that if S is a partial right quotient ring of a ring T, and E is a dense right ideal of T, then ES is a dense right ideal of S. In particular, DQ is dense in Q, and it follows that

$$ac^{-1}r = rac^{-1}. \tag{10}$$

We can now apply Theorem 5.11, taking $T = \mathcal{C}_{K\Delta(G)}(0)$ so that the partial quotient ring Q is as defined above, to obtain $b, d \in C(KG)$, $d \in \mathcal{C}_{KG}(0)$, such that

$$bd^{-1} = ac^{-1},$$

since (10) is true for arbitrary $r \in R$.

We have thus shown that, in (8), elements of the subring $C\left(Q_{max}(KG)\right)$ of $Q_{max}\left(K\Delta(G)\right)$ can be represented by elements of $Q_{cl}\left(C(KG)\right)$ - that is,

$$C\left(Q_{max}(KG)\right) \subseteq Q_{cl}\left(C(KG)\right).$$

Since we have already obtained the reverse inclusion,

$$C\left(Q_{max}(KG)\right) = Q_{cl}\left(C(KG)\right) ,$$

completing the proof of the proposition.

Theorem 5.9 will clearly follow from the above result if we can show that

$$Q_{cl}\left(C(KG)\right) = Q_{max}\left(C(KG)\right) \tag{11}$$

under the assumption that $\Delta^{+}(G)$ is finite. Since, of course,

$$Q_{cl}\big(C(KG)\big) \subseteq Q_{max}\big(C(KG)\big),$$

by Proposition 1.2.49(i), to prove (11) it will be enough to show that

$$Q_{max}\Big(Q_{cl}\big(C(KG)\big)\Big) = Q_{cl}\big(C(KG)\big), \tag{12}$$

by Lemma 1.2.45(iii).

(12) will be proved in Lemma 5.15, but first we must show that $Q_{cl}\big(C(KG)\big)$ is ''locally Artinian''. This follows from the next lemma and Lemma 5.12.

<u>LEMMA 5.14</u>. Let H be a finitely generated normal subgroup of $\Delta(G)$. Then

$$Q_{cl}\big(C(KG) \cap KH\big)$$

is Artinian.

<u>PROOF</u>:

By Lemma 1.1.19(vi), H contains a torsion-free central subgroup Z, normal in G, such that $|H : Z| < \infty$. Since Z is torsion-free abelian, KZ is a domain by Corollary 1.3.23. Let $\alpha \in KZ - \{0\}$; then if

$$\alpha = \alpha_1, \alpha_2, \ . \ . \ . \ . \ , \alpha_n$$

denote the finitely many G-conjugates of α, which all lie in KZ as $Z \lhd G$, we have

$$0 \neq \alpha\alpha_2 \ . \ . \ . \ \alpha_n \in C(KG) \cap KZ, \tag{13}$$

since KZ is commutative. Writing $R = C(KG) \cap KZ$, so that $R^* = R \setminus \{0\}$ is a right divisor set of regular elements of KG, by Lemma 1.3.4(i), it follows from (13) that F, the partial quotient ring of KZ formed by inverting the elements of R^*, is a field, the quotient field of KZ. Furthermore, F contains the subfield

$$L = Q(R) \subseteq Q_{cl}\big(C(KG) \cap KH\big).$$

We claim that $|F : L| < \infty$. Since H is finitely generated and $|H : Z| < \infty$, Z is finitely generated, by Lemma 1.1.14, so that since $Z \subseteq \Delta(G)$ and $Z \lhd G$, it follows that $C_G(Z)$ is a normal subgroup of finite index in G, and $\bar{G} = G/C_G(Z)$ acts faithfully as a group of automorphisms of Z, and so of KZ. Notice that $\bar{G}$ fixes R, so that $\bar{G}$ acts on F as a group of field automorphisms, with fixed field L. To see this, observe that if $\beta\delta^{-1} \in F$ is fixed by all elements of G, where $\delta \in R$, then $\beta \in R$ and $\beta\delta^{-1} \in L$. Hence by Galois Theory [1, Thm.14], $|F : L| = |\bar{G}| < \infty$.

Since $|H : Z| < \infty$, $FH \otimes_{KZ} F$ is a finite dimensional algebra over F, and is therefore Artinian, so that by Lemma 1.2.3,

$$Q_{cl}(KH) = KH \otimes_{KZ} F .$$

Thus $Q_{cl}(KH)$ is a finite dimensional algebra over L, since $|F : L| < \infty$. By Lemma 5.12, we have

$$L \subseteq Q_{cl}\big(C(KG) \cap KH\big) \subseteq Q_{cl}(KH) ,$$

and we deuce that $Q_{cl}\big(C(KG) \cap KH\big)$ is a finite dimensional L-algebra, as required.

<u>REMARK</u>: The argument that $|F : L| < \infty$ in the above proof is adapted from the proof of [36, Thm.6.5].

<u>LEMMA 5.15</u>. Suppose that $\Delta^+(G)$ is finite. Then

$$Q_{cl}\big(C(KG)\big) = Q_{max}\big(C(KG)\big).$$

<u>PROOF</u>:

As observed above, it is enough to prove that $Q_{cl}\big(C(KG)\big)$ is its own maximal quotient ring. This will follow if we can show that if I is a dense ideal of $Q_{cl}\big(C(KG)\big)$ then $I = Q_{cl}\big(C(KG)\big)$. We have to prove, therefore, that every proper ideal of $Q_{cl}\big(C(KG)\big)$ has non-zero annihilator.

We prove first that:-

$$Q_{cl}\big(C(KG)\big)\Big/N\Big(Q_{cl}\big(C(KG)\big)\Big) \text{ is regular .} \tag{14}$$

We shall denote the above ring by R. Let

$$\alpha = ac^{-1} \in Q_{cl}\big(C(KG)\big) \setminus N\Big[Q_{cl}\big(C(KG)\big)\Big] ,$$

and put

$$W = \langle \operatorname{supp} a,\ \operatorname{supp} c \rangle ,$$

so that W is contained in a finitely generated normal subgroup H of $\Delta(G)$. By Lemma 5.14,

$$Q_{cl}\big(C(KG) \cap KH\big)\Big/N\Big[Q_{cl}\big(C(KG) \cap KH\big)\Big]$$

is Artinian. Note that

$$N\Big[Q_{cl}\big(C(KG)\big)\Big] \cap Q_{cl}\big(C(KG) \cap KH\big) = N\Big[Q_{cl}\big(C(KG) \cap KH\big)\Big],$$

since $Q_{cl}\big(C(KG) \cap KH\big)$ is a subring of $Q_{cl}'\big(C(KG)\big)$ by Lemma 5.12, and since both rings are commutative. Hence

$$\alpha \in Q_{cl}\big(C(KG) \cap KH\big) \setminus N\Big[Q_{cl}\big(C(KG) \cap KH\big)\Big] ,$$

so that there exists $\beta \in Q_{cl}\big(C(KG) \cap KH\big)$ such that

$$\alpha - \alpha\beta\alpha \in N\Big[Q_{cl}\big(C(KG) \cap KH\big)\Big] \subseteq N\Big[Q_{cl}\big(C(KG)\big)\Big] .$$

Thus (14) follows, since $\beta \in Q_{cl}\big(C(KG)\big)$.

By Theorem 1.2.9, R is either Artinian, or R has an infinite set of pairwise orthogonal idempotents. Suppose the latter is the case. By lifting over the nil ideal $N\Big[Q_{cl}\big(C(KG)\big)\Big]$ using Proposition 1.2.7, we deduce that $Q_{cl}\big(C(KG)\big)$ must then have an infinite set of orthogonal idempotents. However, by Lemmas 5.12 and 1.3.21, $Q_{cl}\big(C(KG)\big)$ is a subring of $Q_{cl}\big(K\Delta(G)\big)$, and since $\Delta^+(G)$ is finite, $\Delta(G) \in \mathcal{V}$, and so $Q_{cl}\big(K\Delta(G)\big)$

is Artinian, by Theorem 2.12. In particular, therefore, the subring $Q_{cl}(C(KG))$ has the minimum condition on annihilators, by the remark following Theorem 1.2.23, and so $Q_{cl}(C(KG))$ cannot have an infinite set of orthogonal idempotents. It follows that R must be Artinian, say

$$R = R\bar{e}_1 \oplus \dots \oplus R\bar{e}_n ,$$

where $\bar{e}_i^{\,2} = \bar{e}_i$, and $R\bar{e}_i$ is a field, for $i = 1, \dots, n$.

Now

$$N[Q_{cl}(C(KG))] = N(C(KG))\,Q_{cl}(C(KG)) ,$$

and

$$N(C(KG)) \subseteq N(KG).$$

By Theorem 1.3.20 and Lemma 1.3.10, $N(KG)$ is nilpotent, since $\Delta^+(G)$ is finite, and so $N[Q_{cl}(C(KG))]$ is nilpotent.

Lifting $\{\bar{e}_1, \dots, \bar{e}_n\}$ to a complete set of orthogonal primitive idempotents of $Q_{cl}(C(KG))$, we conclude that

$$Q_{cl}(C(KG)) = \sum_{i=1}^{n}{}^{\oplus} Q_{cl}(C(KG))\,e_i ,$$

where for $i = 1, \dots, n$,

$$\frac{Q_{cl}(C(KG))\,e_i}{N[Q_{cl}(C(KG))\,e_i]} \cong R\bar{e}_i ,$$

which is a field, and $N[Q_{cl}(C(KG))\,e_i]$ is nilpotent.

Suppose now that I is an ideal of $Q_{cl}(C(KG))$ such that I has zero annihilator. Then clearly, for $i = 1, \dots, n$, Ie_i is an ideal of $Q_{cl}(C(KG))e_i$, with zero annihilator in $Q_{cl}(C(KG))e_i$. It follows that for $i = 1, \dots, n$,

$$Ie_i = Q_{cl}(C(KG))\,e_i ,$$

since otherwise

$$Ie_i \subseteq N[Q_{cl}(C(KG))\,e_i]$$

and so has non-zero annihilator. Thus

$$I = \sum_{i=1}^{n}{}^{\oplus} Ie_i = Q_{cl}(C(KG)) .$$

Hence

$$Q_{cl}(C(KG)) = Q_{max}(Q_{cl}(C(KG))) ,$$

and the lemma is proved.

For convenience, we summarise the main steps in the proof of Theorem 5.9.

PROOF OF THEOREM 5.9.

We are given that $\Delta^+(G)$ is finite, and we have to show that

$$C(Q_{max}(KG)) \doteq Q_{max}(C(KG)) .$$

By Lemma 5.13,

$$C(Q_{max}(KG)) \cong Q_{cl}(C(KG)) ,$$

and by Lemma 5.15,

$$Q_{cl}(C(KG)) = Q_{max}(C(KG)) .$$

The theorem follows.

REMARKS: (i) In view of the proof of Lemma 5.15, it is natural to ask whether $Q_{cl}(C(KG))$ is actually Artinian when $\Delta^+(G)$ is finite. We are unfortunately unable to answer this question. Clearly an affirmative answer would follow via Theorem 1.2.1 if we knew that $N[Q_{cl}(C(KG))]$ was a finitely generated ideal of $Q_{cl}(C(KG))$. Now when $\Delta^+(G)$ is finite, $N(KG)$ is a finitely generated right ideal of KG, by Theorem 1.3.20, but it is not clear that we can deduce the desired result from this fact.

(ii) Since, when $\Delta^+(G)$ is finite, $Q_{cl}(C(KG))$ is its own maximal quotient ring, one might suspect that this ring is always self-injective. (This is the case if, for example, G is abelian, by Prop. 1.3.19.) However this this is certainly not the case in general. For example, if we take

$$G = \langle a, b : a^4 = b^4 = 1, a^2 = b^2, b^{-1}ab = a^{-1} \rangle,$$

the quaternion group of order eight, and K is the field of two elements, then

$$Q_{cl}\big(C(KG)\big) = C(KG),$$

by Lemma 5.12, since KG is Artinian and so regular elements of KG are units. Now C(KG) contains the ideals

$$I_1 = \{0, (a + a^3)\},$$

and

$$I_2 = \{0, \sum_{g \in G} g\}.$$

Since C(KG) has no non-trivial idempotents, (as $\underline{\underline{g}}$ is a nilpotent ideal of KG, and $KG/\underline{\underline{g}} \cong K$), it follows that if C(KG) is self-injective, it must be an essential extension of both I_1 and I_2. This is clearly impossible.

CHAPTER 4

THE SINGULAR IDEALS

1. INTRODUCTION

In this chapter, we examine the structure of the singular ideals of a group ring. In the interests of brevity we shall assume throughout that the coefficient rings of the group rings under consideration are fields, although many of our results remain valid under somewhat weaker assumptions. We shall leave such details to the reader, beyond remarking that if R is a commutative domain with quotient field K, it may very easily be shown that for any group G, $Z(RG) = Z(KG) \cap RG$.

The importance of the singular ideals should already be apparent from Theorem 1.2.23 and Lemma 1.2.45, and these ideals are prominent in many other aspects of ring theory. Since the right and left singular ideals, $Z(R)$ and $Z'(R)$, of a ring R are clearly zero when R is a domain, a first step towards proving the Zero Divisor Conjecture for group rings should be to prove $Z(KG) = Z'(KG) = 0$ when G is a torsion-free group. However the singular ideals are frequently zero even when G has periodic elements; for example if K is a field of characteristic zero and G is any group, $Z(KG) = Z'(KG) = 0$. This result is due to Snider, [50], but we include his proof here for completeness, (Lemma 2.3). More generally, one might expect that $Z(KG) = Z'(KG)$ even when K has positive characteristic, and indeed all our results lend support to this conjecture. Accordingly, in this chapter we shall work only with the right singular ideal; it will be clear throughout that the results and proofs are also valid for $Z'(KG)$.

The main results of this chapter are contained in §3, where we describe the structure of the singular ideals of KG under various assumptions about G, such as that G is locally soluble or locally FC-hypercentral, (Cor.3.14). (A group G is FC-hypercentral if every homomorphic image of G, other than $\{1\}$, has non-trivial FC-subgroup.) If G is a group, we as usual let $L(G)$ denote the unique maximal locally finite normal subgroup of G. If G is in either of the above two classes, $Z(KG) = J\big(KL(G)\big)KG$. We conjecture, (Conjecture 3.3), that this is true for an arbitrary group algebra KG, and prove several additional results which support this hypothesis. For example, if G is locally finite, Conjecture 3.3 is true, (Cor.3.5), while if G is linear then the available evidence (Theorems 3.22 and 3.25) is consistent with the conjecture.

The remainder of the chapter is organised as follows. In §2 we establish some basic results which will be needed later. In §4, we give some group theoretic applications of the results of §3, and discuss some examples; in particular, we answer a question of Gordon, [16]. This section also includes a result on the existence of non-trivial idempotents in certain group algebras, (Thm.4.2). In §5, we consider a ring-theoretic condition which implies non-singularity; namely, the property of being strongly prime. (See Definition 5.1.) We provide here some partial answers to a question of Handelman and Lawrence, [20], who ask when a group ring is strongly prime. The results we obtain enable us to construct further examples of primitive group rings, (Cor.5.8), and to give further support to Conjecture 3.3.

In §6, we review the results of this chapter, and indicate how they may be related to work of Passman, Zalesskii, and others on the structure of the Jacobson radical of a group algebra.

2. BASIC RESULTS

We begin by considering the singular ideals of group algebras of finite groups. Here the situation may be described very easily, because if K is a field and G is a finite group, KG is Artinian, (Prop.1.3.16), and KG is self-injective, (Prop.1.3.19), so that KG is a QF-ring, by Theorem 1.2.6. We can thus apply the well-known

LEMMA 2.1. If R is a quasi-Frobenius ring,

$$Z(R) = Z'(R) = N(R) = J(R) .$$

PROOF:

Since $Z(R) \subseteq N(R)$ by Lemma 1.2.21, and $N(R) = J(R)$ as R is Artinian, it is sufficient by symmetry to prove that $N(R) \subseteq Z(R)$. Let E (respectively E') denote the right (resp. left) socle of R, so that E (resp. E') is the minimal essential right (resp. left) ideal of R, and so $Z(R) = l(E)$, (resp. $Z'(R) = r(E')$). Furthermore, if I is a minimal left ideal of R, $N(R)I = 0$, and hence

$$E \leq l\big(N(R)\big), \qquad E' \leq r\big(N(R)\big).$$

Since R is a QF-ring and

$$Z(R) = l(E) \leq N(R),$$

we have $\qquad r(N(R)) \subseteq rl(E) = E,$

and so $\qquad E' \subseteq r(N(R)) \subseteq E.$

Similarly, $\qquad E \subseteq l(N(R)) \subseteq E',$

so that

$$E = E' = r(N(R)) = l(N(R)).$$

Thus $N(R) \subseteq Z(R)$ as claimed, since E is an essential right ideal of R.

In particular, it follows from the above that if K has characteristic zero and G is finite, $Z(KG) = 0$ by Maschke's Theorem. In fact, Snider has shown that this is also true if G is infinite, and we shall include his result here. First, however, we require a lemma which will also be needed later. The proof, being very similar to that of Lemma 3.2.6, is omitted.

LEMMA 2.2. If $K \subseteq F$ are fields and G is a group,

$$Z(KG) \subseteq Z(FG).$$

LEMMA 2.3. (Snider, [50]) Let K be a field of characteristic zero, and let G be a group. Then

$$Z(KG) = Z'(KG) = 0.$$

PROOF:

Let F denote a real closed field containing K; see [51a, Thm.7, p.229]. Let L be an algebraic closure of F, so that $L = F(i)$, where $i^2 = -1$. For $\lambda = a + bi \in L$, with $a, b \in F$, we write $\bar{\lambda} = a - bi$, the ''complex conjugate'' of λ. If $\alpha = \sum_{i=1}^{n} \lambda_i g_i \in LG$, we write $\alpha^* = \sum_{i=1}^{n} \bar{\lambda}_i g_i^{-1}$, so that $*$ is an involution on LG, and $\alpha\alpha^* = 0$ if and only if $\alpha = 0$, since the coefficient of 1 in $\alpha\alpha^*$ is $\sum_{i=1}^{n} \lambda_i \bar{\lambda}_i$.

Suppose that $Z(KG) \neq 0$. Then by Lemma 2.2, $Z(LG) \neq 0$, say $0 \neq \beta \in Z(LG)$. Put $E = r_{LG}(\beta)$, an essential right ideal of LG. There exists $\gamma \in LG$ such that $0 \neq \beta^*\gamma \in E$, and so

$$(\beta^*\gamma)^*(\beta^*\gamma) = (\gamma^*\beta)(\beta^*\gamma) = \gamma^*(\beta\beta^*\gamma) = 0,$$

so that $\beta^*\gamma = 0$, a contradiction. The proof is complete.

In view of the above lemma, we shall henceforth assume that all coefficient fields have positive characteristic.

We now investigate the relationship between $Z(KH)$ and $Z(KG)$, where H is a subgroup of G. One direction is easy:

<u>LEMMA 2.4</u>. [5, Lemma 2.4] Let H be a subgroup of the group G. Then

$$Z(KG) \cap KH \subseteq Z(KH).$$

<u>PROOF</u>:

Let $\alpha \in Z(KG) \cap KH$, and put $E = r_{KG}(\alpha)$. Let $\{g_i : i \in I\}$ be a right transversal to H in G, and take $\beta = \sum_{i=1}^{n} \beta_i g_i \in E$, where $0 \neq \beta_i \in KH$, $1 \leqslant i \leqslant n$. Now

$$0 = \alpha\beta = \sum_{i=1}^{n} \alpha\beta_i g_i,$$

and so, by Lemma 1.3.3(i), we deduce that $\alpha\beta_i = 0$ for all i, $1 \leqslant i \leqslant n$. If π denotes the canonical KH-homomorphism of Lemma 1.3.3(ii) from KG onto KH, it follows that $\alpha\pi(E) = 0$. If I is a non-zero right ideal of KH and

$$I \cap \pi(E) = 0,$$

then $E \subseteq \pi(E)G$ and

$$IG \cap \pi(E)G = 0,$$

contradicting our assumption that E is essential in KG. Thus $\Pi(E)$ is an essential right ideal of KH, and $\alpha \in Z(KH)$, as required.

The problem of determining when $Z(KH) \subseteq Z(KG)$ is more difficult, and we can give only a sufficient condition for this to occur. Our result is best expressed in terms of <u>subgroup theoretic classes</u>. A subgroup theoretical class is a class $\mathcal{Y}$ of pairs (H, G) of groups such that $H \subseteq G$, $\mathcal{Y}$ contains the pair (1, G) for all groups G, and $\mathcal{Y}$ contains with (H, G) all $(H\theta, G\theta)$, where θ is an isomorphism of G. For details, see [40, p.9].

Let $\mathcal{Y}$ be a subgroup theoretical class. We define $L\mathcal{Y}$ to be the class of all pairs (H, G) of groups such that H is a subgroup of G, and if $\{x_1, \ldots, x_n\}$ is any finite subset of G, there exists a subgroup T of G, containing $\langle H, x_1, \ldots, x_n\rangle$, such that $(H, T) \in \mathcal{Y}$.

We define $\acute{P}\mathcal{Y}$ to be the class of all pairs (H, G) such that H is a subgroup of G, and there exists an ascending series

$$H = H_0 \subset H_1 \subset \ldots \subset H_\alpha \subset H_{\alpha+1} \subset \ldots \subset H_\rho = G,$$

where ρ is an ordinal, $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ if λ is a limit ordinal, $0 \leq \lambda \leq \rho$, and $(H_\alpha, H_{\alpha+1}) \in \mathcal{Y}$, $0 \leq \alpha < \rho$.

Note that $L^2\mathcal{Y} = L\mathcal{Y}$, and $\acute{P}^2\mathcal{Y} = \acute{P}\mathcal{Y}$, so it is easily seen that L and $\acute{P}$ are closure operations in the sense of [40].

Let $\mathcal{X}$ denote the subgroup theoretical class defined by subnormality; thus $\mathcal{X}$ is the class of all pairs (H, G) of groups such that H is a subnormal subgroup of G. Now put $\mathcal{S} = \langle \acute{P}, L\rangle\mathcal{X}$, so that as in [40, Chapter 1],

$\mathcal{P}$ is the smallest $\acute{P}$- and L-closed class containing $\mathfrak{X}$. The importance of $\mathcal{P}$ lies in the following lemma.

LEMMA 2.5. If H is a subgroup of a group G, such that $(H, G) \in \mathcal{P}$, then $Z(KH) \subseteq Z(KG)$.

PROOF:

If H is a normal subgroup of G, and if E is an essential right ideal of KH, then by Lemma 3.2.7, EG is an essential right ideal of KG. Therefore if H is subnormal in G, $Z(KH) \subseteq Z(KG)$.

Let $\mathcal{S}$ denote the subgroup theoretical class consisting of all pairs (U, V) of groups $U \subseteq V$, such that $Z(KU) \subseteq Z(KV)$. Note that since Z(K) is trivially contained in Z(KV) for all groups V, $\mathcal{S}$ is a subgroup theoretical class in the sense of [40]. By the above, $\mathfrak{X} \subseteq \mathcal{S}$. It is not difficult to check that $\mathcal{S}$ is $\acute{P}$- and L-closed, so that since $\mathcal{P}$ is by definition the smallest $\acute{P}$- and L-closed class containing $\mathfrak{X}$, $\mathcal{P} \subseteq \mathcal{S}$ as claimed.

We isolate a special case of the above results.

COROLLARY 2.6. Let H be a subnormal subgroup of a group G. Then

$$Z(KG) \cap KH = Z(KH) .$$

Lemma 2.5 gives a sufficient but not necessary condition for Z(KH) to be contained in Z(KG) when H is a subgroup of G, even when Z(KH) is non-zero. For example, put $G = (C \times F)$, where C is a non-trivial finite p-group for some prime p, and $F = \langle x, y \rangle$ is the free group on two generators. Let K be a field of characteristic p, and put $H = (C \times \langle x \rangle)$. Then $(H, G) \notin \mathcal{P}$, but if $\underline{\underline{c}}$ denotes the augmentation ideal of KC, then

$$\underline{\underline{c}}H = Z(KH) \subseteq Z(KG) = \underline{\underline{c}}G .$$

This will follow from Theorem 2.9, for example, since a free group has a series with torsion-free abelian factors.

It will become apparent that, at least for some large classes of groups, the subgroups of G which determine the structure of $Z(KG)$ are the finite ones, and in this case Lemma 2.5 is considerably simplified by the next result.

<u>LEMMA 2.7</u>. Suppose that $(H, G) \in \mathcal{P}$, and H is locally finite. Then $H \subseteq L(G)$; moreover, if H is finite, $(H, L(G)) \in L\mathfrak{X}$.

<u>PROOF</u>:

For the first part, it will be convenient to prove the following equivalent fact:-

If $(H, G) \in \mathcal{P}$, $N \lhd H$, and $N \in L\mathfrak{F}$, then $N \leq L(G)$. (1)

Suppose first that H, and so N, is subnormal in G. Then by induction on the subnormal index of H in G, and applying Lemma 1.1.11, we may assume that $H \lhd G$. Then for all $x \in G$, N^x is a locally finite normal subgroup of H, and the result follows by Lemma 1.1.11.

For each ordinal α, we define inductively a subgroup theoretical subclass $\mathcal{P}_\alpha$ of $\mathcal{P}$, as follows:-

$$\mathcal{P}_1 = L\mathfrak{X},$$

and for non-limit ordinals α, $\mathcal{P}_\alpha = L\mathcal{P}_{\alpha-1}$ if $\mathcal{P}_{\alpha-1} = \acute{P}\mathcal{P}_{\alpha-2}$, and $\mathcal{P}_\alpha = \acute{P}\mathcal{P}_{\alpha-1}$ if $\mathcal{P}_{\alpha-1} = L\mathcal{P}_{\alpha-2}$, or if $(\alpha - 1)$ is a limit ordinal. If α is a limit ordinal, we put $\mathcal{P}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}_\beta$.

Clearly $\mathcal{P} = \bigcup_{\alpha} \mathcal{P}_{\alpha}$, where the union is taken over all ordinals α, since the right-hand-side is easily seen to be $\acute{P}$- and L-closed. We prove (1) by induction on α, where α is the least ordinal such that $(H, G) \in \mathcal{P}_{\alpha}$, so that α is not a limit ordinal.

Suppose then that (1) is known for all pairs in $\mathcal{P}_{\alpha-1}$, and that $(H, G) \in \mathcal{P}_{\alpha}$. If $\mathcal{P}_{\alpha} = L\mathcal{P}_{\alpha-1}$, the induction step is trivial. If $\mathcal{P}_{\alpha} = \acute{P}\mathcal{P}_{\alpha-1}$, let

$$H = H_0 \subset \dots \subset H_{\rho} = G$$

be an ascending series with $(H_{\beta}, H_{\beta+1}) \in \mathcal{P}_{\alpha-1}$, $0 \leq \beta < \rho$. We use induction on ρ, the case $\rho = 1$ being given by our first induction assumption. We may clearly assume that ρ is not a limit ordinal, so that by induction, $N^{H_{\rho-1}}$, the normal closure of N in $H_{\rho-1}$, is locally finite. The inductive step, and so the result, now follows by the case $\rho = 1$.

For the second part, noting that

$$(H, G) \in \mathcal{P},\ N \trianglelefteq G \implies (H \cap N, N) \in \mathcal{P},$$

we need only show that if $G = L(G)$ and $(H, G) \in \acute{P}L\mathfrak{X}$, then $(H, G) \in L\mathfrak{X}$. Suppose then that there exist an ascending series

$$H = H_0 \subset \dots \subset H_{\rho} = G,$$

such that $(H_{\alpha}, H_{\alpha-1}) \in L\mathfrak{X}$ for all ordinals $\alpha < \rho$. Arguing by induction on ρ, we may assume that $(H, H_{\alpha}) \in L\mathfrak{X}$ for all ordinals $\alpha < \rho$. We have to show that $(H, G) \in L\mathfrak{X}$. If ρ is a limit ordinal there is nothing to prove; if ρ is not a limit, we have $H \leq H_{\rho-1} \leq G$, where $(H, H_{\rho-1}) \in L\mathfrak{X}$, $(H_{\rho-1}, G) \in L\mathfrak{X}$, and it follows easily that $(H, G) \in L\mathfrak{X}$.

It is sometimes possible to completely describe

$Z(KG)$ in terms of $Z(KH)$, where H is a subgroup of G. As a simple example of this, we have

LEMMA 2.8. Let H be a normal subgroup of a group G, such that G/H is finitely generated torsion-free abelian. Then

$$Z(KG) = Z(KH)KG .$$

PROOF:

By induction on the number of generators of G/H, we may assume that G/H is infinite cyclic, say $G = \langle H, x \rangle$. By Corollary 2.6, $Z(KH)KG \subseteq Z(KG)$, so we need only consider the reverse inclusion.

Let S denote the subring of KG generated as a ring by KH and x. The following facts can easily be checked:-

(i) E is an essential right ideal of $KG \Longleftrightarrow (E \cap S)$ is an essential right ideal of S;

(ii) $Z(KG) = Z(S)KG$.

Thus we need only show that $Z(S)$ is the ideal of S generated by $Z(KH)$. Notice that S may be viewed as the twisted polynomial ring $KH[x;\sigma]$, where σ is the automorphism of KH induced by conjugation by x. Since $Z(KH)^{\sigma} = Z(KH)$, we have to show that $Z(S) = Z(KH)[x;\sigma]$. Clearly, since $Z(KH) \subseteq Z(KG)$, and using fact (i), $Z(KH)[x;\sigma] \subseteq Z(S)$; it remains to prove the converse.

Suppose $Z(S) \nsubseteq Z(KH)[x;\sigma]$, and let

$$a = \sum_{i=0}^{n} x^i a_i \in Z(S) ,$$

where $a_i \in KH$, $0 \leq i \leq n$, $a_n \neq 0$, and n is chosen minimal subject to $a \notin Z(KH)[x;\sigma]$. It follows that $a_n \in KH \setminus Z(KH)$, and hence there exists $0 \neq b \in KH$ such that $r_{KH}(a_n) \cap bKH = 0$. However, if

$$0 \neq \beta = \sum_{j=0}^{m} \beta_j x^j \in r_S(a) \cap bS ,$$

where $\beta_j \in KH$, $0 \leq j \leq m$, and $\beta_m \neq 0$, then clearly $\beta_m \in r_{KH}(a_n) \cap bKH$. Since $r_{KH}(a_n) \cap bKH = 0$, this is a contradiction.

Therefore $Z(S) = Z(KH)[x;\sigma]$, and the lemma follows.

We aim now to generalise Lemma 2.8 using the concept of a series between groups H and G: see Defn.1.1.10. Recall that $\mathcal{Q}_0$ denotes the class of torsion-free abelian groups.

THEOREM 2.9. Let H be a subgroup of a group G, and suppose there is a series between H and G whose factors are in $\mathcal{Q}_0$. Suppose further that $(H, G) \in \mathcal{P}$. Then $Z(KG) = Z(KH)KG$.

PROOF:

By Lemma 2.5, $Z(KH)KG \subseteq Z(KG)$. Suppose the reverse inclusion is false. For $\alpha \in KG$ we write

$$G(\alpha) = \langle H, \operatorname{supp}\alpha \rangle,$$

and we choose $\beta \in KG$ such that $|\operatorname{supp}\beta|$ is minimal with respect to $\beta \in Z(KG(\beta))$, $\beta \notin Z(KH)KG(\beta)$. Since for any subgroup T of G, $Z(KG) \cap KT \subseteq Z(KT)$ by Lemma 2.4, such a choice is certainly possible. Furthermore, we may clearly choose β such that $1 \in \operatorname{supp}\beta$.

By hypothesis, there exists a totally ordered set Ω and a set $\{\Lambda_\sigma, V_\sigma : \sigma \in \Omega\}$ of pairs of subgroups of G such that

(i) $H \subseteq V_\sigma \lhd \Lambda_\sigma \subseteq G$, and $\Lambda_\sigma / V_\sigma \in \mathcal{Q}_0$, for all $\sigma \in \Omega$;

(ii) $\sigma < \tau \implies \Lambda_\sigma \subseteq V_\tau$, $\sigma, \tau \in \Omega$;

(iii) $G \setminus H = \bigcup_{\sigma \in \Omega} (\Lambda_\sigma \setminus V_\sigma)$.

By choice of β, $\beta \notin KH$. For each $g \in (\operatorname{supp}\beta) \setminus H$, there exists, by (iii), $\sigma_g \in \Omega$ such that $g \in \Lambda_{\sigma_g} \setminus V_{\sigma_g}$; put

$\bar{\sigma} = \max\{\sigma_g : g \in (\operatorname{supp}\beta) \setminus H\}$. Then for all $g \in (\operatorname{supp}\beta) \setminus H$, $\sigma_g \leqslant \bar{\sigma}$, and so by (ii), $G(\beta) \subseteq \Lambda_{\bar{\sigma}}$, but $G(\beta) \nsubseteq V_{\bar{\sigma}}$. Thus $G(\beta) \cap V_{\bar{\sigma}} \lhd G(\beta)$, and $G(\beta)/G(\beta) \cap V_{\bar{\sigma}}$ is isomorphic to $G(\beta)V_{\bar{\sigma}}/V_{\bar{\sigma}}$, a non-trivial finitely generated subgroup of $\Lambda_{\bar{\sigma}}/V_{\bar{\sigma}}$.

It therefore follows from Lemma 2.8 that

$$Z(KG(\beta)) = Z\left[K\left(G(\beta) \cap V_{\bar{\sigma}}\right)\right]KG(\beta).$$

In particular, $\beta = \sum_{i=1}^{n} \beta_i g_i$, where $\{1 = g_1, g_i : i \geqslant 2\}$ is a transversal to $G(\beta) \cap V_{\bar{\sigma}}$ in $G(\beta)$, and

$$\beta_i \in Z\left[K\left(G(\beta) \cap V_{\bar{\sigma}}\right)\right], \tag{2}$$

for $i = 1, \ldots, n$. Since $1 \in \operatorname{supp}\beta$, $\beta_1 \neq 0$, and since $G(\beta) \cap V_{\bar{\sigma}} \neq G(\beta)$, there exists i, $2 \leqslant i \leqslant n$, such that $\beta_i \neq 0$. It follows that for all i, $1 \leqslant i \leqslant n$,

$$|\operatorname{supp}\beta_i| < |\operatorname{supp}\beta|.$$

Now by (2) and Lemma 2.4, $\beta_i \in Z(KG(\beta_i))$, for $i = 1, \ldots, n$. Hence by choice of β, we have

$$\beta_i \in Z(KH)KG(\beta_i) \subseteq Z(KH)KG(\beta),$$

for $i = 1, \ldots, n$, so that

$$\beta = \sum_{i=1}^{n} \beta_i g_i \in Z(KH)KG(\beta),$$

a contradiction. Thus

$$Z(KG) = Z(KH)KG,$$

and the proof is complete.

REMARKS: (i) Although one may prove by the methods of [40, §8.2] that $LP\mathfrak{A}_0 \leqslant \hat{P}\mathfrak{A}_0$, the following result is more easily proved directly, by adapting the proof of Lemma 2.8:

If H is a normal subgroup of G and $G/H \in LP\mathfrak{A}_0$, then $Z(KG) = Z(KH)KG$.

(ii) In contrast to Lemma 2.8, it is not the case that if H is normal in G and G/H is infinite cyclic, $J(KG) = J(KH)KG$. For example, if $G = C_p \wr C_\infty$, where C_p denotes the group of order p, for some prime p, and if K is a field of characteristic p, then, letting H denote the base group of G, we have $J(KH) = \underline{\underline{h}}$ by Lemma 1.3.29, while $J(KG) = 0$ by [36, Thm.21.4], although $G/H \cong C_\infty$.

3. THE STRUCTURE OF THE SINGULAR IDEALS

We begin this section by recalling the conjecture of Passman, (Conjecture 1.3.25) that $J(KG) = N^*(KG)$ for all group algebras KG, where $N^*(KG)$ is a characteristic ''locally nilpotent'' ideal of KG. Note that this conjecture is known to be true if G is soluble, linear, or locally finite; see [38] for bibliographical details.

LEMMA 3.1. For all groups G and fields K,

$$N(KG) \subseteq Z(KG).$$

PROOF:

Suppose that K has characteristic p. By Theorem 1.3.20(ii),

$$N(KG) = N\big(K\Delta^p(G)\big)KG = \bigcup_W J(KW)KG ,$$

where W ranges over all finite normal subgroups of G which are contained in $\Delta^p(G)$. The result now follows from Lemma 2.1 and Corollary 2.6.

PROPOSITION 3.2. For all groups G and fields K,

$$N^*(KG) \subseteq Z(KG) .$$

PROOF:

Let $\alpha \in KG$. By definition, $\alpha \in N^*(KG)$ if and only if $\alpha \in N(KH)$ for all finitely generated subgroups H of G containing $\operatorname{supp}\alpha$. Suppose that $\alpha \in N^*(KG)$ but that $r_{KG}(\alpha)$ is not essential in KG, so that there exists $0 \neq \beta \in KG$ such that

$$\beta KG \cap r_{KG}(\alpha) = 0.$$

Put $H = \langle \operatorname{supp}\alpha, \operatorname{supp}\beta \rangle$. Thus $\alpha \in N(KH)$, so by Lemma 3.1 $\alpha \in Z(KH)$. However, since

$$r_{KH}(\alpha) = r_{KG}(\alpha) \cap KH,$$

it follows that

$$\beta KH \cap r_{KH}(\alpha) = 0,$$

a contradiction. Hence $\alpha \in Z(KG)$.

<u>REMARK</u>: In view of Conjecture 1.3.25, one ought perhaps to try to improve Proposition 3.2 to show that J(KG) is always contained in Z(KG). However since we do not even know in general whether J(KG) consists only of zero divisors, this would seem to be very difficult. Suppose, for example, that K is a field of characteristic $p > 0$, and G is a p-group. It is not known whether, in this situation, $J(KG) = \underline{\underline{g}}$ only if G is locally finite; see [38, p.92] for a discussion of this problem. In contrast, it is not hard to prove that if F is any field and H is a non-trivial group,

$Z(FH) = \underline{\underline{h}} \iff$ F has characteristic $p > 0$ and $H \in L\mathcal{F}_p$.

The inclusion obtained in Proposition 3.2 is strict in general, (Ex.4.4), but as we shall see below, (Cor.3.5), if G is locally finite,

$$J(KG) = N^*(KG) = Z(KG)$$

for all fields K. This result may very easily be proved directly, but we prefer to obtain it as a corollary to a more general result, which is suggested by the following

CONJECTURE 3.3. For all group rings KG,

$$Z(KG) = J\big(KL(G)\big)KG .$$

Of course, we have at the moment very little evidence to support this conjecture. If G is finite, or if K has characteristic zero, then it is rather trivially true, by Lemmas 2.1 and 2.3, and Cor.1.3.28(i). That it is also true when G is locally finite is an immediate consequence of Lemma 2.1 and

PROPOSITION 3.4. Let K be a field and let $\mathcal{Y}$ be a class of groups such that if $H \in \mathcal{Y}$, $Z(KH) = J\big(KL(H)\big)KH$. Then if $G \in L\mathcal{Y}$, $Z(KG) = J\big(KL(G)\big)KG$.

PROOF:

Let $G \in L\mathcal{Y}$, and let $\{g_i : i \in I\}$ be a transversal to $L(G)$ in G. Take $0 \neq \alpha = \sum_{i=1}^{n} \alpha_i g_i \in Z(KG)$, where $0 \neq \alpha_i \in KL(G)$, $1 \leqslant i \leqslant n$. We aim to show that for $i = 1, \ldots, n$, $\alpha_i \in J\big(KL(G)\big)$, and to do this, it will be sufficient to prove that if $\beta \in KL(G)$ then $\beta\alpha_i$ is nilpotent, for $i = 1, \ldots, n$. Accordingly, choose $\beta \in KL(G)$.

We begin by noting that if $g \in G$, $g \in L(G)$ if and only if $\langle g\rangle^G$ is locally finite. Therefore if $g \notin L(G)$, there exists a finitely generated subgroup T of G, containing g, such that if N is any subgroup of G with $T \subseteq N \subseteq G$, $g \notin L(N)$. For if $\langle g^{g_1}, \ldots, g^{g_m} : g_j \in G\rangle$ $(1 \leqslant j \leqslant m)$ is infinite, we may simply put $T = \langle g, g_j : 1 \leqslant j \leqslant m\rangle$. It follows that there exists a finitely generated subgroup H of G such that $H \in \mathcal{Y}$ and

(i) $\operatorname{supp}\alpha_i \subseteq H$, for $i = 1, \ldots, n$;

(ii) $g_i \in H$, for $i = 1, \ldots, n$;

(iii) $g_i \notin L(H)$, for $i = 1, \ldots, n$;

(iv) $g_i g_j^{-1} \notin L(H)$, for all $i, j = 1, \ldots, n$, $i \neq j$;

(v) $\operatorname{supp}\beta \subseteq H$.

Clearly, for $i = 1, \ldots, n$, $\alpha_i \in KL(H)$ and $\beta \in KL(H)$. Furthermore, the set $\{g_1, \ldots, g_n\}$ can be extended to a transversal to $L(H)$ in H, by the inclusions and exclusions given above, so that

$$\beta\alpha = \sum_{i=1}^{n} \beta\alpha_i g_i \in \sum_{i=1}^{n}{}^{\oplus} KL(H)g_i \subseteq KL(H)|^{KH}.$$

Hence, as

$$\beta\alpha \in Z(KG) \cap KH \subseteq Z(KH) = J\big(KL(H)\big)KH,$$

since $H \in \mathcal{Y}$, we deduce that

$$\beta\alpha_i \in J\big(KL(H)\big),$$

for $i = 1, \ldots, n$. Thus $KL(G)\alpha_i$ is a nil left ideal of $KL(G)$, and so $\alpha_i \in J\big(KL(G)\big)$ for $i = 1, \ldots, n$, as required.

<u>COROLLARY 3.5</u>. If G is locally finite, $J(KG) = Z(KG)$.

The above corollary and Corollary 2.6 together imply that if KG is any group algebra,

$$J\big(KL(G)\big)KG \subseteq Z(KG),$$

so that to affirm Conjecture 3.3 it suffices to prove the reverse inclusion, and our main aim in the remainder of §3 will be to do this for various classes of groups G.

So far, we only know Conjecture 3.3 to be true when G is locally finite. We can, however, quickly obtain more examples by applying Theorem 2.9.

<u>LEMMA 3.6</u>. Let G be a group such that $G/L(G) \in \mathfrak{A}_0$. Then $Z(KG) = J\big(KL(G)\big)KG$. In particular, if G is an FC-group,

$Z(KG) = J\big(KL(G)\big)KG.$

PROOF:

This follows immediately from Theorem 2.9 and Corollary 3.5, noting that by Lemma 1.1.19(i), if G is an FC-group then $G/L(G) \in \mathcal{Q}_o$.

REMARK: Burgess [5, Theorem 4.11] has shown that if K is a field and G is an FC-group, $Z(KG) = 0$ if and only if $N(KG) = 0$. Bearing in mind that for FC-groups,

$$N(KG) = J\big(KL(G)\big)KG,$$

by Theorem 1.3.20, Burgess' result is clearly a consequence of Lemma 3.6. Burgess conjectures in [5] that his result remains true for arbitrary group algebras; that this is not the case may be seen from Corollary 3.5, (see, for instance, Example 4.3).

We now wish to use Lemma 3.6 to verify Conjecture 3.3 for group algebras of soluble and FC-hypercentral groups. Our main tool here will be a ''Generalised Intersection Theorem'' due to Zalesskii, which we shall quote in the form we require, (Theorem 3.10). If K is a field and H is a normal subgroup of G, an intersection theorem for KG is a result which guarantees that if I is a non-zero ideal of KG, then $I \cap KH \neq 0$. A generalised intersection theorem for KG is a result which asserts that if L is a G-invariant ideal of KH, of a particular type, and

$$LKG \subsetneqq I \lhd KG,$$

then

$$L \subsetneqq I \cap KH.$$

Such results have proved very important in the study of group rings, particularly in considering problems related to semisimplicity and primitivity.

We require some additional definitions and notation; for the most part we follow [37].

Let G be a group, H a subgroup of G. For each prime p we write $O_p(G)$ for the maximal normal locally finite p-subgroup of G. We write

$$D_G(H) = \{x \in G : |H : C_H(x)| < \infty\};$$

thus $D_G(H)$ is a subgroup of G, and if H is a normal subgroup of G, $D_G(H) \lhd G$. Notice that $D_G(G)$ is simply $\Delta(G)$. G is said to be FC-<u>hypercentral</u> if for all $N \ntrianglelefteq G$, $\Delta(G/N)$ is non-trivial.

Let KG be a group ring. An ideal I of KG is said to be <u>annihilator-free</u> if for every infinite subgroup X of G, the left annihilator of $\underline{\underline{x}}G + I/I$ in the ring KG/I is zero. Note that by Lemma 1.3.15, $\{0\}$ is always an annihilator-free ideal. In fact this condition simply ensures that the ideals we are dealing with are not "too large".

The following lemma contains all the information we require about annihilator-free ideals.

<u>LEMMA 3.7</u>. [37, Lemma 8.4.12] Let G be an FC-group, K a field of characteristic $p > 0$. If $O_p(G)$ is finite, then $J(KG)$ is annihilator-free.

Now let G be a soluble group. We define a characteristic subgroup $E(G)$ of G by induction on the derived length of G, as follows:

(i) $E(G) = G$ if G is abelian;

(ii) $E(G) = E(G')D_G\big(E(G')\big)$.

<u>DEFINITION 3.8</u>. The characteristic subgroup $\Delta\big(E(G)\big)$ of G is called the <u>Zalesskii subgroup of</u> G, (following Passman),

and is denoted by $\mathfrak{Z}(G)$. We shall write $\mathfrak{Z}^+(G)$ for $L(\mathfrak{Z}(G))$.

The crucial facts about these subgroups are contained in the following lemma, due to Zalesskii.

LEMMA 3.9. [55] (i) $E(G) \supseteq D_G(E(G)) = \mathfrak{Z}(G)$;
(ii) $E(G)$ is FC-hypercentral.

The importance of this construction is revealed by

THEOREM 3.10. [37,Thm.8.4.10] Let G be a group, H an FC-hypercentral normal subgroup of G. Let K be a field, and let L be a G-invariant, annihilator-free ideal of $KD_G(H)$. If I is an ideal of KG such that

$$I \supsetneq LKG,$$

then

$$I \cap KD_G(H) \supsetneq L .$$

Notice that since $\{0\}$ is always annihilator-free, Theorem 3.10 includes an earlier and perhaps more familiar intersection theorem, also due to Zalesskii; see [55].

THEOREM 3.11. Let G be a soluble group, K a field of characteristic $p > 0$. Let S be the normal subgroup of G given by $S/O_p(G) = \mathfrak{Z}^+(G/O_p(G))$, so that

$$O_p(G) \subseteq S \subseteq L(G).$$

Then

$$Z(KG) = J(KL(G))KG = J(KS)KG.$$

PROOF:

To illustrate the argument, we shall first prove the result under the additional hypothesis that $O_p(G) = 1$. Let T denote $\mathfrak{Z}(G)$. We have

$$Z(KG) \supseteq J\big(KL(G)\big)KG \supseteq J(KS)KG = J(KT)KG .$$

The first and second inclusions follow from Corollaries 2.6 and 3.5, noting that $S \lhd L(G)$. Finally $J(KS)KT = J(KT)$ since T is an FC-group, applying Theorem 1.3.20.

We now suppose that $Z(KG) \supsetneq J(KT)KG$, and obtain a contradiction. Since $O_p(G) = 1$, $O_p(T) = 1$, and so by Lemma 3.7 $J(KT)$ is an annihilator-free ideal of KT. Theorem 3.10 can now be applied, taking $H = E(G)$, so that H is FC-hypercentral and $D_G(H) = \mathfrak{Z}(G) = T$ by Lemma 3.9, and we conclude that

$$Z(KG) \cap KT \supsetneq J(KT).$$

Thus by Corollary 2.6, $Z(KT) \supsetneq J(KT)$. However T is an FC-group, and so this contradicts Lemma 3.6. Hence

$$Z(KG) = J(KT)KG,$$

as claimed.

We now drop the hypothesis that $O_p(G) = 1$, so that now T is the subgroup of G defined by $T/O_p(G) = \mathfrak{Z}\big(G/O_p(G)\big)$. We have

$$Z(KG) \supseteq J\big(KL(G)\big)KG \supseteq J(KS)KG \supseteq \omega\big(O_p(G)\big) .$$

The last inclusion in the above follows by, for example, Corollaries 2.6 and 3.5, and Lemma 1.3.29, since $O_p(G)$ is a normal locally finite p-subgroup of G contained in S. (Furthermore, we note, although we shall not require this fact, that it follows from the special case proved above that $J\big(KL(G)\big)KG = J(KS)KG$, since $O_p\big(G/O_p(G)\big) = 1$.)

Suppose that the required result is false, so that

$$Z(KG) \supsetneq J(KS)KG. \qquad (1)$$

Let $\psi: KG \longrightarrow K\big(G/O_p(G)\big)$ denote the canonical homomorphism of Lemma 1.3.11(i), with kernel $\omega\big(O_p(G)\big)$. If

$O_p(G) \subseteq H \subseteq G$, we shall write $\bar{H}$ for $H/O_p(G)$. Now $\psi\big(J(KS)\big) = \psi|_{KS}\big(J(KS)\big) = J(K\bar{S})$, by the radical properties of the Jacobson radical. Thus, noting that $\omega\big(O_p(G)\big)$ lies in both sides of the inequality (1), we have

$$\psi\big(Z(KG)\big) \supsetneqq J(K\bar{S})K\bar{G} \tag{2}$$

because $\psi\big(J(KS)KG\big) = \psi\big(J(KS)\big)K\bar{G}$, so that the right hand side of (2) is precisely $\psi\big(J(KS)KG\big)$.

Put $\psi\big(Z(KG)\big) = I$. Since $\bar{T}$ is an FC-group, $J(K\bar{T}) = J(K\bar{S})K\bar{T}$ by Theorem 1.3.20, and since $O_p(\bar{T}) = 1$, Lemma 3.8 shows that $J(K\bar{T})$ is annihilator-free. Applying Theorem 3.10 in the group ring $K\bar{G} = K\big(G/O_p(G)\big)$, we conclude from (2) that

$$I \cap K\bar{T} \supsetneqq J(K\bar{S})K\bar{T} = J(K\bar{T}).$$

By considering the pre-image under the map ψ of each side of the above inequality, we deduce that there exists $\alpha \in KG$ such that

$$\alpha \in \Big[Z(KG) \cap \Big(KT + \omega\big(O_p(G)\big)\Big)\Big] \setminus \Big[J(KS)KT + \omega\big(O_p(G)\big)\Big].$$

Let $\alpha = \beta + \gamma$, where $\beta \in KT$ and $\gamma \in \omega\big(O_p(G)\big)$, so that

$$\beta = (\alpha - \gamma) \in Z(KG) \setminus J(KS)KT.$$

Thus $\quad \beta \in \big(Z(KG) \cap KT\big) \setminus J(KS)KT$;

and so $\quad \beta \in Z(KT) \setminus J(KS)KT$, by Corollary 2.6. $\quad$ (3)

Since $T/O_p(G)$ is an FC-group, T/S is torsion-free abelian, and $S/O_p(G)$ is locally finite. Hence by Lemma 1.1.11, S is locally finite, and so $Z(KT) = J(KS)KT$, by Lemma 3.6. This contradicts (3) and so completes the proof of the theorem.

Clearly the proof of Theorem 3.10 could be considerably simplified if we knew that $Z(KG)$ behaved like a radical ideal when factoring by ideals like $\omega(O_p(G))$. This is of course a consequence of the theorem when G is soluble, and in fact is true in general if $O_p(G)$ is finite, (Theorem 3.23), but we have been unable to prove that this happens when $O_p(G)$ is infinite, except in special cases, as in Corollary 3.24.

The proof of the next result is similar to that of Theorem 3.11, and we leave the details to the reader. One remark is needed - the class of FC-hypercentral groups is clearly closed under taking homomorphic images.

<u>THEOREM 3.12</u>. Let G be an FC-hypercentral group, and let K be a field of characteristic $p > 0$. Let $S \lhd G$ satisfy $\Delta^+(G/O_p(G)) = S/O_p(G)$. Then

$$Z(KG) = J(KL(G))KG = J(KS)KG.$$

Part of the following corollary to the previous two theorems has been obtained independently by Snider, [13, Theorem 7].

<u>COROLLARY 3.13</u>.(i) Let G be a soluble group, K a field of characteristic $p > 0$. Then $Z(KG) \neq 0$ if and only if G has a finite two-step subnormal subgroup of order divisible by p.

(ii) Let G be an FC-hypercentral group, K a field of characteristic $p > 0$. Then $Z(KG) \neq 0$ if and only if G has a finite normal subgroup of order divisible by p.

<u>PROOF</u>:

<u>Necessity</u>. (i) Apply Theorem 3.10 to deduce that, if $Z(KG) \neq 0$, then $Z(K\mathfrak{Z}(G)) \neq 0$, (taking $L = \{0\}$). It

follows by Lemma 3.6 that $J(K\mathfrak{Z}(G)) \neq 0$, whence by Theorem 1.3.20, since $\mathfrak{Z}(G)$ is an FC-group, we deduce that $\mathfrak{Z}(G)$ has a finite normal subgroup of order divisible by p.

(ii) As in (i), $Z(K\Delta(G)) \neq 0$, and so by Lemma 3.6 and Theorem 1.3.20, $\Delta(G)$ contains an element x of order p. Thus $\langle x\rangle^G$ is a finite normal subgroup of G, of order divisible by p.

<u>Sufficiency</u>. (i) and (ii): This follows from Lemma 2.1 and Corollary 2.6.

Applying Proposition 3.4 in conjunction with Theorems 3.11 and 3.12 gives

<u>COROLLARY 3.14</u>. Let G be a locally soluble or locally FC-hypercentral group. Then $Z(KG) = J(KL(G))KG$.

Our work in Chapter 3 provides another class of groups for which we may verify Conjecture 3.3.

<u>PROPOSITION 3.15</u>. Let $G \in L\mathfrak{U}$. Then $Z(KG) = J(KL(G))KG$.

<u>PROOF</u>:

By Proposition 3.4, we may assume that $G \in \mathfrak{U}$. By Theorem 3.2.12 and the remark following Theorem 1.2.23, KG has max-ra, so that by Lemma 1.2.21(i),

$$Z(KG) \subseteq N(KG).$$

Hence by Lemma 3.1 and Theorem 1.3.20,

$$Z(KG) = N(KG) = J(K\Delta^+(G))KG.$$

However since $G \in \mathfrak{U}$, $L(G)$ is finite, so that $L(G) = \Delta^+(G)$, and the result follows.

Our aim now is to consider the validity of Conjecture 3.3 when G is a linear group over a field of characteristic $q \geqslant 0$, where $q \neq p$, p being the characteristic of the coefficient field K. Here the result we obtain is unfortunately not as powerful as some of the previous theorems of this section; nevertheless it seems very likely that the conjecture is true in this case also.

We begin with a ''going-up'' theorem similar in spirit to Theorem 2.9, to prove which we shall need the following very useful result, due to Passman.

THEOREM 3.16. [36, Thm.17.4 and Lemma 17.1(i)] Let K be an algebraically closed field of characteristic $p \geqslant 0$, and let G be a group with a normal subgroup H such that G/H is a finite abelian p'-group. If I is an ideal of KG which is invariant under all automorphisms of KG which fix K, then

$$I = (I \cap KH)KG.$$

THEOREM 3.17. Let G be a group, and let K be an algebraically closed field of characteristic $p > 0$. Let H be a subgroup of G such that $(H, G) \in \mathcal{P}$, and suppose that for all finite subsets $\{x_1, \ldots, x_n\}$ of $G \setminus H$, there exists a group S,

$$H \subseteq S \lhd \langle H, x_1, \ldots, x_n \rangle,$$

such that $\langle H, x_1, \ldots, x_n \rangle / S$ is a non-trivial finite abelian p'-group. Then $Z(KG) = Z(KH)KG$.

PROOF:

By Lemma 2.5, $Z(KH)KG \subseteq Z(KG)$. Suppose there exists $0 \neq \alpha \in Z(KG) \setminus Z(KH)KG$. Putting $G(\alpha) = \langle H, \operatorname{supp}\alpha \rangle$, we have

$$\alpha \in Z\big(KG(\alpha)\big) \setminus Z(KH)KG(\alpha), \tag{4}$$

by Lemma 2.4. Now choose $\beta \in KG$ such that (4) holds for β, and $|\operatorname{supp}\beta|$ is minimal among the set of elements of

$KG \setminus KH$ with property (4). Clearly we may choose β such that $1 \in \operatorname{supp}\beta$.

By hypothesis, there exists a normal subgroup $S(\beta)$ of $G(\beta)$ such that $H \subseteq S(\beta)$ and $G(\beta)/S(\beta)$ is a non-trivial finite abelian p'-group. Now K is algebraically closed, and so by Theorem 3.16, noting that for any ring R, Z(R) is invariant under Aut(R),

$$Z\big(KG(\beta)\big) = \Big(Z\big(KG(\beta)\big) \cap KS(\beta)\Big)KG(\beta).$$

However by Corollary 2.6 the right hand side of the above identity is simply $Z\big(KS(\beta)\big)KG(\beta)$, and so, recalling that $1 \in \operatorname{supp}\beta$, we may write β as $\beta = \sum_{i=1}^{n} \beta_i g_i$, where $0 \neq \beta_i \in Z\big(KS(\beta)\big)$, $1 \leq i \leq m$, and $\{1 = g_1, \ldots g_m\}$ forms part of a transversal to $S(\beta)$ in $G(\beta)$. By Lemma 2.4, $\beta_i \in Z\big(KG(\beta_i)\big)$ for all i, $1 \leq i \leq m$.

Since $\beta \notin KS(\beta)$, m is strictly greater than one. Since $\beta \notin Z(KH)KG(\beta)$, there exists j, $1 \leq j \leq m$, such that $\beta_j \notin Z(KH)KS(\beta)$, and so because $G(\beta_j) \subseteq S(\beta)$, $\beta_j \notin Z(KH)KG(\beta_j)$. However since $m > 1$, $|\operatorname{supp}\beta_j| < |\operatorname{supp}\beta|$. This is a contradiction to our choice of β, and so the result is proved.

We wish to remove from Theorem 3.17 the hypothesis that K is algebraically closed. To achieve this, we must strengthen our assumptions about H, so that we can apply the following well-known result.

<u>LEMMA 3.18</u>. [37, Lemma 8.2.10] Let G be a locally finite group, K a subfield of the field F. Then

$$J(FG) = F.J(KG).$$

<u>THEOREM 3.19</u>. Let K be a field of characteristic $p > 0$,

and suppose H and G are as in Theorem 3.17. If $Z(FH) = J\big(FL(H)\big)FH$, where F is an algebraic closure of K, then

$$Z(KG) = Z(KH)KG = J\big(KL(G)\big)KG.$$

PROOF:

Since $(H, G) \in \mathcal{P}$, $L(H) \subseteq L(G)$ by Lemma 2.6. Thus $\big(L(H), L(G)\big) \in \mathcal{P}$, and it follows that

$$J\big(KL(H)\big)KG \subseteq J\big(KL(G)\big)KG \subseteq Z(KG), \tag{5}$$

by Lemma 2.5 and Corollary 3.5.

Let F be an algebraic closure of K. By Lemma 2.2, $Z(KG) \subseteq Z(FG)$, and by hypothesis, applying Theorem 3.18 and Lemma 3.19,

$$Z(FG) = J\big(FL(H)\big)FG = F.J\big(KL(H)\big)FG.$$

Thus

$$J\big(KL(H)\big)FG \subseteq Z(KG) \subseteq F.J\big(KL(H)\big)KG.$$

Let $\{x_\lambda : \lambda \in \Lambda\}$ be a K-basis for F. If $\alpha \in FG$, α can be uniquely written as $\alpha = \sum_{\lambda\in\Lambda} x_\lambda \alpha_\lambda$, where $\alpha_\lambda \in KG$ for all $\lambda \in \Lambda$, and $\alpha_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$. By the uniqueness of this expression, we deduce that if $\alpha \in Z(KG)$, then $\alpha \in J\big(KL(H)\big)KG$, so that

$$Z(KG) = J\big(KL(H)\big)KG = Z(KH)KG.$$

Finally, it follows from (5) that $Z(KG) = J\big(KL(G)\big)KG$.

We shall need the following special case of the above result.

COROLLARY 3.20. Let K be a field of characteristic $p > 0$, and let H be a normal subgroup of G. Suppose that if F is an algebraic closure of K,

$$Z(FH) = J\big(FL(H)\big)FH,$$

and that for some prime q different from p, G/H is a residually finite-q group. Then

$$Z(KG) = J\big(KL(H)\big)KG = J\big(KL(G)\big)KG .$$

The relevance of the above corollary to our study of group rings of linear groups will become apparent from the following remarks.

Let G be a subgroup of $GL_n(F)$, where F is a field of characteristic $q \geqslant 0$ and n is a positive integer, and let K be a field of characteristic $p > 0$, where $q \neq p$. In the study of the linear group G, it is frequently useful to define a certain topology on G, the Zariski Topology, in such a way that for a given element x of G, the maps $y \longmapsto yx$, $y \longmapsto xy$, $y \longmapsto y^{-1}$, and $y \longmapsto y^{-1}xy$ are continuous. We shall not discuss this in detail here; see [54, Chapter 5]. For our purposes, the most important result concerning the Zariski Topology is [54, Theorem 6.4], which says that if G is a linear group and H is a normal, closed subgroup of G, (that is, the normal subgroup H is a closed subset of G under the Zariski Topology on G), then G/H is also linear over F.

We claim that $\Delta(G)$ is a closed subgroup of G. Note first that $\Delta(G)$ is the centralizer in G of G^o, the connected component of G containing $\{1\}$, by [54, Lemma 5.5], where G^o is a normal subgroup of finite index in G, by [54, Lemma 5.2]. Now by [54, Lemma 5.4], the centralizer of any subset of G is a closed set, and so our claim follows.

We therefore deduce that $G/\Delta(G) \subseteq GL_m(F)$, for some positive integer m, by [54, Theorem 6.4]. If we now assume in addition that G is finitely generated, then so is $G/\Delta(G)$

and so by [54, Theorem 4.7], $G/\Delta(G)$ is a finite extension of a residually finite-q group if $q > 0$, while if $q = 0$ there exists a finite set of primes Π such that for all primes $r \notin \Pi$, $G/\Delta(G)$ is a finite extension of a residually finite-r group.

Of course, since G itself is linear, [54, Theorem 4.7] can also be applied to G. Hence if $G \subseteq GL_n(F)$ and G is finitely generated, there exists a prime r, different from p, such that G is a finite extension of a residually finite-r group. It is trivial that a locally finite, residually finite-r group is a locally finite-r group, so that under our present hypotheses $L(G)$ has a normal r-subgroup of finite index, for some prime $r \neq p$.

Thus if $G \subseteq GL_n(F)$, where F has characteristic $q \geqslant 0$, and K is a field of characteristic $p \neq q$, it follows from Corollary 1.3.28 and Theorem 2.5.8 that if G is finitely generated, $J(KL(G))$ is nilpotent. By Lemma 1.3.10, $J(KL(G))KG$ is a nilpotent ideal of KG. Hence if Conjecture 3.3 were true, we would have $Z(KG) = N(KG)$ when G is a finitely generated subgroup of $GL_n(F)$, (provided of course that $q \neq p$). Furthermore, by Proposition 3.4, a proof of such a result would be enough to confirm the conjecture for all linear groups in characteristic $q \neq p$.

Our next result provides strong support for the conjecture that $Z(KG) = N(KG)$ if G is a finitely generated subgroup of $GL_n(F)$, and indeed is sufficient to prove this in special cases. We shall need another intersection theorem, due once more to Zalesskii.

THEOREM 3.21. [37, Theorem 9.1.8] Let I be a non-zero ideal of the group ring KG, and suppose that I contains no non-zero nilpotent ideals. Let H be a normal subgroup of G such that G/H is an FC-group. Then

$$I \cap K(H\Delta(G)) \neq 0.$$

THEOREM 3.22. Let K be a field of characteristic $p > 0$, and let G be a subgroup of $GL_n(F)$, where F has characteristic $q \geqslant 0$, and $q \neq p$.

Then (i) if G has no elements of order p, $Z(KG) = 0$;

(ii) if G is finitely generated,

$$Z(KG) \neq 0 \Longleftrightarrow N(KG) \neq 0 \Longleftrightarrow \Delta^p(G) \neq 1 \Longleftrightarrow J(KL(G)) \neq 0.$$

PROOF:

First note that (i) follows easily from (ii) since if G has no elements of order p, then, by (ii), $Z(KH) = 0$ for all finitely generated subgroups H of G, and so Lemma 2.4 gives (i).

(ii) By Theorem 1.3.20(ii), $N(KG) \neq 0 \Longleftrightarrow \Delta^p(G) \neq 1$. Since G is finitely generated, $J(KL(G))$ is nilpotent by the remarks on the previous page, so that

$$J(KL(G))KG = N(KG),$$

and we deduce that

$$\Delta^p(G) \neq 1 \Longleftrightarrow J(KL(G)) \neq 0.$$

If $N(KG) \neq 0$, then $Z(KG) \neq 0$ by Lemma 3.1.

It remains to prove that if $Z(KG) \neq 0$, then $N(KG) \neq 0$. Suppose that this is false, so that $Z(KG) \neq 0$, but $N(KG) = 0$. There exists a prime r different from p such that $G/\Delta(G)$ has a normal subgroup of finite index, $H/\Delta(G)$, which is a residually finite-r group; this follows by the remarks on the previous two pages. Since $N(KG) = 0$, $Z(KG)$

contains no non-zero nilpotent ideals, and so by Theorem 3.21,

$$Z(KG) \cap K\big(H\,\Delta(G)\big) \neq 0.$$

That is, by Corollary 2.6, and since $\Delta(G) \subseteq H$, $Z(KH) \neq 0$.

However, by Lemma 3.6 and Corollary 3.20,

$$0 \neq Z(KH) = J\big(K\Delta(G)\big)KH,$$

while by Theorem 1.3.20,

$$J\big(K\Delta(G)\big)KH \subseteq J\big(K\Delta(G)\big)KG = N(KG),$$

so that $N(KG)$ is non-zero. This contradiction implies that $Z(KG) \neq 0$ only if $N(KG) \neq 0$, and the proof is complete.

We can in fact say more about the structure of $Z(KG)$ when $G \subseteq GL_n(F)$ and G is finitely generated, but to do so we must first prove a result which is of independent interest, since it provides some evidence for the universal validity of Conjecture 3.3. If H is a locally finite normal subgroup of a group M, it is clear from Lemma 1.1.11 that $L(M/H) = L(M)/H$. For this reason, for Conjecture 3.3 to be true it is necessary that $Z(KM)$ should behave like a radical ideal when factoring by ideals like $\omega(H)$, where H is as above. Provided we assume that H is actually finite, we can prove that this is indeed what happens.

We shall have frequent recourse in the proof of the next result to Lemma 1.3.15, which states that if H is a finite subgroup of G, then

$$r_{KG}(G\underline{\underline{h}}) = \hat{H}KG,$$

where $\hat{H} = \sum_{h \in H} h$, and

$$l_{KG}(\hat{H}KG) = G\underline{\underline{h}}.$$

Note also that if $H \lhd G$, then $\hat{H}$ is a central element of KG.

<u>THEOREM 3.23</u>. Let K be a field of characteristic $p > 0$, and let H be a finite normal subgroup of a group G.

Then (i) $\big(Z(KG) + \omega(H)\big)/\omega(H) \subseteq Z(KG/\omega(H))$;

(ii) if in addition H is a p-group, (so that $\omega(H) \subseteq Z(KG)$),

$$Z(KG)/\omega(H) = Z\big(KG/\omega(H)\big).$$

<u>PROOF</u>:

(i) If $Z(KG) \subseteq \omega(H)$, the result is trivially true. Accordingly, we suppose that there exists $\alpha \in Z(KG) \setminus \omega(H)$, so that $r_{KG}(\alpha) = E$ is an essential right ideal of KG.

Put $T = \{\varphi \in KG : \varphi\hat{H} \in E\}$, a right ideal of KG. Clearly $T \supseteq \omega(H) + E$, and

$$(\alpha T)\hat{H} = \alpha(T\hat{H}) = 0 .$$

It follows that $\alpha T \subseteq \omega(H)$. We claim that $T/\omega(H)$ is an essential right ideal of $KG/\omega(H)$; this will complete the proof of (i). It is clearly sufficient to prove that if $\omega(H) \subsetneqq I \underset{r}{\lhd} KG$, then $(I \cap T) \supsetneqq \omega(H)$. Now if $I \supsetneqq \omega(H)$, $I\hat{H} \neq 0$, and so $I\hat{H} \cap E \ni \beta\hat{H} \neq 0$, where $\beta \in I$, since E is essential and $I\hat{H}$ is a right ideal of KG. Thus $\beta \in T$ by definition of T, $\beta \in I$ by choice, and $\beta \notin \omega(H)$ since $\beta\hat{H} \neq 0$.

Therefore $\beta \in \big(I \setminus \omega(H)\big) \cap \big(T \setminus \omega(H)\big)$, as required.

(ii) We need only show that

$$Z(KG)/\omega(H) \supseteq Z\big(KG/\omega(H)\big),$$

since the reverse inclusion follows from (i). Suppose therefore that $0 \neq \alpha \in KG$, $\omega(H) \subseteq E \triangleleft_r KG$, with $\alpha E \subseteq \omega(H)$ and $E/\omega(H)$ essential in $KG/\omega(H)$. We must show that $\alpha \in Z(KG)$. Clearly $\alpha E\hat{H} = 0$; we claim that $E\hat{H}$ is an essential right ideal of KG.

First, note that $\hat{H}KH$ is an essential right ideal of KH; indeed it is the unique minimal right (and left) ideal of KH. Hence by Lemma 3.2.6, $\hat{H}KG$ is essential in KG, and so if I is a non-zero right ideal of KG,

$$\hat{H}KG \cap I \ni \hat{H}\gamma = \gamma\hat{H} \neq 0,$$

noting that $\hat{H}$ is in the centre of KG. Since

$$r_{KG}(\hat{H}) = l_{KG}(\hat{H}) = \omega(H),$$

it follows that $\gamma \notin \omega(H)$, and so $\big(\gamma KG + \omega(H)\big)/\omega(H)$ is a non-zero right ideal of $KG/\omega(H)$. Thus there exists $\tau \in KG \setminus \omega(H)$ such that

$$\tau \in \big(\gamma KG + \omega(H)\big) \cap E,$$

since $E/\omega(H)$ is essential in $KG/\omega(H)$.

Let $\tau = \gamma\eta + \mu$, where $\eta \in KG, \mu \in \omega(H)$, so that

$$\gamma\eta = (\tau - \mu) \neq 0,$$

since $\tau \notin \omega(H)$. Thus $\gamma\eta \in E \setminus \omega(H)$.

It follows that

$$\gamma\eta\hat{H} = \gamma\hat{H}\eta \neq 0,$$

and $\gamma\hat{H}\eta \in E\hat{H} \cap I$, as required.

If we allow H to be infinite, the following special case is all that we can obtain.

<u>COROLLARY 3.24</u>. Let K be field of characteristic $p > 0$, let H be a normal locally finite-p subgroup of a group G, and suppose that there exists an index set Λ and a set $\{H_\lambda : \lambda \in \Lambda\}$ of subgroups of H, such that (i) $\bigcap_{\lambda \in \Lambda} H_\lambda = 1$,

(ii) $H_\lambda \triangleleft G$, for all $\lambda \in \Lambda$, and (iii) H/H_λ is finite, for all $\lambda \in \Lambda$. Then

$$Z(KG)/\omega(H) \supseteq Z\big(KG/\omega(H)\big).$$

<u>PROOF</u>:

Take $\alpha \in KG$ such that $\big(\alpha + \omega(H)\big) \in Z\big(KG/\omega(H)\big)$, and put $E = r_{KG}(\alpha)$. Choose $0 \neq \beta \in KG$; we claim that $E \cap \beta KG \neq 0$. By Lemma 1.3.13, we may assume without loss of generality that $\bigcap_{\lambda\in\Lambda}\omega(H_\lambda) = 0$. Hence we can find $\mu \in \Lambda$ such that $\beta \notin \omega(H_\mu)$.

Now, by applying Theorem 3.23(ii) in the ring $K(G/H_\mu)$, it follows that there exists $\gamma \in KG$ such that $\beta\gamma \notin \omega(H_\mu)$, but $\alpha\beta\gamma \in \omega(H_\mu)$. (We are using here the fact that $KG/\omega(H_\mu) \cong K(G/H_\mu)$.) Since H is locally finite, there exists a finite subgroup L of H_μ such that

$$\alpha\beta\gamma\hat{L} = 0:$$

however $\beta\gamma\hat{L} \neq 0$, since $1_{KG}(\hat{L}) \subseteq \omega(H_\mu)$, and $\beta\gamma \notin \omega(H_\mu)$. Therefore $0 \neq \beta\gamma\hat{L} \in E$, and so E is essential. The proof is thus complete.

We now return to linear groups. If G is a finitely generated subgroup of $GL_n(F)$, where the characteristic of F is $q \geqslant 0$, and $q \neq p$, then $\Delta^p(G)$ is a finite group. For $\Delta^p(G) \subseteq L(G)$, and as was noted earlier, L(G) has a normal r-subgroup W of finite index for some prime $r \neq p$, so that $\Delta^p(G)/\big(\Delta^p(G) \cap W\big)$ is finite. It follows by Lemma 1.1.19(v) that $|\Delta^p(G) \cap W : \Delta^p\big(\Delta^p(G) \cap W\big)| < \infty$; that is, $\Delta^p(G) \cap W$ is finite, since $\Delta^p\big(\Delta^p(G) \cap W\big) = 1$. Thus $\Delta^p(G)$ is finite, and hence by Lemma 1.1.19(iv), $\Delta^p\big(G/\Delta^p(G)\big) = 1$.

<u>THEOREM 3.25</u>. Let K be a field of characteristic $p > 0$,

and suppose G is a finitely generated subgroup of $GL_n(F)$, where F is a field of characteristic $q \geqslant 0$, and $q \neq p$. Then

$$N(KG) \subseteq Z(KG) \subseteq \omega\left(\Delta^p(G)\right).$$

PROOF:

Since $\Delta^p(G)$ is finite, $\Delta\left(G/\Delta^p(G)\right)$ is simply $\Delta(G)/\Delta^p(G)$. Since $\Delta^p\left(G/\Delta^p(G)\right) = 1$, $N\left[K\left(G/\Delta^p(G)\right)\right] = 0$, by Theorem 1.3.20(ii), and as in the proof of Theorem 3.22 we may deduce that $Z\left[K\left(G/\Delta^p(G)\right)\right] = 0$. The result now follows from Lemma 3.1 and Theorem 3.23(i).

If $\Delta^p(G)$ is a p-group, then we deduce from the above result and Lemma 1.3.29 that

$$Z(KG) = N(KG) = \omega\left(\Delta^p(G)\right).$$

4. APPLICATIONS AND EXAMPLES

We begin this section by showing how purely group theoretic results can be obtained using the methods of the previous sections. The idea is to make use of the fact that if K is a field and H is a subgroup of a group G, we know that $Z(KH) \subseteq Z(KG)$ under relatively weak assumptions about the way H is contained in G, (Lemma 2.5). In this respect the singular ideal behaves very differently from the Jacobson radical, for even when H is normal in G it does not follow that $J(KH) \subseteq J(KG)$, as we remarked after Theorem 2.9.

By a straightforward application of Theorem 3.10, with $L = \{0\}$, we can deduce that if H is an FC-hypercentral normal subgroup of a group G, and K is a field such that $Z(KG) \neq 0$, then $Z(KG) \cap KD_G(H) \neq 0$. This simple fact enables us to prove the following result, which can no doubt also be proved by purely group theoretic means.

THEOREM 4.1. (i) Let p be a prime, let G be a soluble group, and suppose there exists a finite subgroup H of G such that $p \mid |H|$ and $(H, G) \in \mathcal{P}$. Then there exists a finite subgroup T of $\mathfrak{Z}(G)$ such that $p \mid |T|$ and $T \lhd \mathfrak{Z}(G) \lhd G$.

(ii) Let p be a prime, let p be an FC-hypercentral group, and suppose there exists a finite subgroup H of G such that $p \mid |H|$ and $(H, G) \in \mathcal{P}$. Then G has a finite normal subgroup T such that $p \mid |T|$.

PROOF:

(i) Let K be any field of characteristic p. Then $Z(KH) \neq 0$ by Lemma 2.1, and so by Lemma 2.5, $Z(KG) \neq 0$. Hence by the remarks above, $Z\big(K\mathfrak{Z}(G)\big) \neq 0$, and since $\mathfrak{Z}(G)$ is an FC-group it follows from Lemma 3.6 and Theorem 1.3.20 that $\mathfrak{Z}(G)$ has a finite normal subgroup of order divisible by p.

(ii) This is proved similarly, noting that if H is a finite subgroup of $\Delta(G)$, then H^G, the normal closure of H in G, is finite.

REMARKS: (i) By Lemma 2.7, the hypothesis that $(H, G) \in \mathcal{P}$ in the above result is equivalent to the assumption that $(H, L(G)) \in L\mathfrak{X}$.

(ii) A result very similar to Theorem 4.1(i) can be proved by working solely within $L(G)$, and using well-known properties of the Jacobson radical of group algebras of locally finite groups. In the conclusion, $\mathfrak{Z}\big(L(G)\big)$ replaces $\mathfrak{Z}(G)$.

(iii) Theorem 4.1(ii) is a generalisation of a result of McLain, [40, Thm.2.25], who essentially proves the analogue of Theorem 4.1(ii) for G hypercentral. Note that if G is a hypercentral group, the condition that $(H, G) \in \mathcal{P}$, where H is a finite subgroup of G, is vacuous.

Our second application is concerned with the existence of non-trivial idempotents in group algebras. Crucial here is the elementary observation that if R is a ring and e is an idempotent in R, then $r(e) = (1 - e)R$. An immediate consequence of this is that Z(R) cannot contain any non-zero idempotents.

In view of the Zero Divisor Conjecture, one would expect that if G is a torsion-free group and K is a field, KG will have no non-trivial idempotents. To date, the best result along these lines has been obtained by Formanek, [12], who showed that if G is torsion-free and satisfies the maximum condition on cyclic subgroups, and K is a field of characteristic zero, then KG has no non-trivial idempotents.

The result we shall prove here may be viewed as a generalisation of the fact that if G is a finite p-group and K is a field of characteristic p, then KG has no non-trivial idempotents, since $KG/\underline{\underline{g}} \cong K$ and $\underline{\underline{g}}$ is nilpotent.

<u>THEOREM 4.2</u>. Let K be a field of characteristic $p > 0$, and suppose G is a group with a descending series

$$G = H_0 \supset H_1 \quad . \; . \; . \quad H_\alpha \supset H_{\alpha+1} \supset . \; . \; . \supset H_\rho = 1 \; ,$$

where ρ is an ordinal, $H_\alpha \lhd G$, and $H_\alpha / H_{\alpha+1}$ is a locally finite p-group, for all α, $0 \leq \alpha < \rho$, and where $H_\lambda = \bigcap_{\beta<\lambda} H_\beta$ if λ is a limit ordinal, $0 < \lambda \leq \rho$.

Then KG has no non-trivial idempotents.

PROOF:

If $e \in KG$ is an idempotent, then clearly either e or $(1 - e)$ is contained in $\underline{\underline{g}}$, since this ideal is the kernel of the canonical homomorphism from KG onto K. We may thus assume without loss of generality that KG has an idempotent e, contained in $\underline{\underline{g}}$. We shall prove that $e = 0$ by showing that for all ordinals α, $0 \leq \alpha < \rho$, $e \in \omega(H_\alpha)$ implies $e \in \omega(H_{\alpha+1})$.

Assume therefore that $e \in \omega(H_\alpha)$, where $0 \leq \alpha < \rho$. Now $H_{\alpha+1} \lhd G$; we shall denote the canonical map from KG to $K(G/H_{\alpha+1})$ by $x \longmapsto \bar{x}$, for $x \in KG$, $T \longmapsto \bar{T}$, $H_{\alpha+1} \subseteq T \subseteq G$. Thus $\bar{e} \in \omega(\bar{H}_\alpha)$ is an idempotent in $K\bar{G}$, and $\bar{e} = 0$ if and only if $e \in \omega(H_{\alpha+1})$. However $\omega(\bar{H}_\alpha) = Z(K\bar{H}_\alpha)K\bar{G} \subseteq Z(K\bar{G})$ by Lemma 1.3.29, Corollary 3.5, and Lemma 2.5. Thus by the remark before the statement of the theorem, $\bar{e} = 0$. That is, $e \in \omega(H_{\alpha+1})$, and so by transfinite induction, the result is proved, since if λ is a limit ordinal,

$$\bigcap_{\beta<\lambda} \omega(H_\beta) = \omega(H_\lambda),$$

by Lemma 1.3.13.

REMARK: We could not have used the Jacobson radical in place of the singular ideal in the above proof, since as was pointed out in the third remark following Theorem 2.9, it is not in general the case that if $H \lhd G$, then $J(KH)KG \subseteq J(KG)$; nor is it the case that if $H \lhd G$, $\bigcap_{n=1}^{\infty} (J(KH)KG)^n = 0$, as can be seen by taking K to be the field of p elements, and $G = H = C_{p^\infty}$, for any prime p. In this case, $J(KG) = \underline{\underline{g}} = \underline{\underline{g}}^2$.

Gordon [16, p.281] has asked whether a prime uniform ring with no non-trivial idempotent ideals is necessarily a domain. Our first example shows that the answer is ''no''.

<u>EXAMPLE 4.3</u>. Let K be a field of characteristic $p > 0$, and let C_p denote the cyclic group of order p. Let A be an infinite elementary abelian p-group, and put $G = C_p \wr A$. Put $R = KG$.

(i) Right Goldie dimension of $R = 1$.

For if I and J are non-zero right ideals of R, take $0 \neq \alpha \in I$, $0 \neq \beta \in J$, and put $H = \langle \operatorname{supp}\alpha, \operatorname{supp}\beta \rangle$, a finite p-group. Since KH has a unique minimal right ideal, namely $(\sum_{h \in H} h)KH$, and since $(I \cap KH)$ and $(J \cap KH)$ are non-zero, it follows that $0 \neq \sum_{h \in H} h \in (I \cap J)$.

(ii) By Corollary 3.5 and Lemma 1.3.29,

$$J(R) = Z(R) = \underline{\underline{g}} .$$

(iii) R has no non-trivial idempotent ideals. For, letting B denote the base group of G, we have $B \lhd G$ and $G/B \cong A$, so that in particular G/B is a residually finite-p group. It follows from Lemma 1.3.14 and the isomorphism of Lemma 1.3.11 that $\bigcap_{n=1}^{\infty} \underline{\underline{g}}^n \subseteq (\underline{\underline{b}}G)$. By a similar argument $\bigcap_{m=1}^{\infty} \underline{\underline{b}}^m = 0$, and so since $(\underline{\underline{b}}G)^m = \underline{\underline{b}}^m G$ for all $m \geqslant 1$, by Lemma 1.3.10, we deduce that

$$\bigcap_{m=1}^{\infty} (\underline{\underline{b}}G)^m = 0.$$

Note that if $\gamma \in \bigcap_{m=1}^{\infty} (\underline{\underline{b}}G)^m$, then if $\{g_t : t \in T\}$ is a transversal to B in G, and $\gamma = \sum_{t=1}^{s} \gamma_t g_t$, we have, for $t = 1, \ldots, s$, $\gamma_t \in \bigcap_{m=1}^{\infty} \underline{\underline{b}}^m = 0$, and so $\gamma = 0$.

Since $\underline{\underline{g}}$ is the unique maximal ideal of R, (iii) follows.

(iv) R is prime, by Proposition 1.3.18, since G has no non-trivial finite normal subgroups.

Thus R is a prime ring with no non-trivial idempotent ideals, such that $Z(R)$ is non-zero and is the unique maximal ideal of R.

EXAMPLE 4.4. Lawrence [31a] has given an example of a primitive singular ring, and Bergman has exhibited a prime uniform singular ring. Here we give a simple example possessing all of these properties.

Let K and C_p be as in Example 4.3, and let C_∞ denote an infinite cyclic group. Put $G = C_p \wr C_\infty$, and $R = KG$. Let B denote the base group of G; thus B is an infinite elementary abelian p-group, and $G/B \cong C_\infty$. As in Example 4.3, KB is uniform.

R is primitive, by [37, Lemma 9.2.8(ii)].

By Theorem 3.2.7, R is right (and left) uniform.

Lastly, by Corollary 3.5, Lemma 1.3.29 and Lemma 2.8,

$$Z(R) = \underline{\underline{b}}G \neq 0 .$$

Thus we see that even when one considers only primitive rings, ''right uniform'' does not imply ''right non-singular''.

5. STRONGLY PRIME RINGS

DEFINITION 5.1. A ring R is said to be *right strongly prime* if, given any non-zero element α of R, there exists a finite subset $S(\alpha)$ of R, called the *right insulator* of α in R, such that $r_R(\alpha S(\alpha)) = 0$. *Left strongly prime* rings are defined similarly.

A ring is said to be *strongly prime* if it is both right and left strongly prime.

Clearly strongly prime rings are prime. Strongly prime rings arose implicitly in the study of the coefficient rings of certain primitive group rings, [31]. Elementary properties of strongly prime rings are considered in [20], and we list here some of the results of that paper. For our purposes, the most important of these is (v), for which we include a proof. Notice that a ring R is right strongly prime if and only if every non-zero two-sided ideal of R contains a finitely generated left ideal whose right annihilator is zero.

PROPOSITION 5.2. (i) There exists a right strongly prime ring which is not left strongly prime.

(ii) If R is a prime ring with the descending chain condition on right annihilator ideals, R is right strongly prime. In particular, prime right Goldie rings are right strongly prime.

(iii) Simple rings and domains are strongly prime.

(iv) A commutative ring is strongly prime if and only if it is a domain.

(v) The right singular ideal of a right strongly prime ring is zero.

(vi) The maximal right quotient ring of a right strongly prime ring is simple.

(vii) The socle of a strongly prime ring is either $\{0\}$ or the whole ring.

PROOF:

(v) Let R be a right strongly prime ring, and suppose there exists $0 \neq \alpha \in Z(R)$. Let $\{\beta_1, \ldots, \beta_n\}$ be a right insulator of α, so that

$$r_R(\alpha\beta_1, \ldots, \alpha\beta_n) = 0.$$

However, for $i = 1, \ldots, n$, $E_i = r_R(\alpha\beta_i)$ is an essential right ideal, since $\alpha \in Z(R)$. Hence,

$$r_R(\alpha\beta_1, \ldots, \alpha\beta_n) = E = \bigcap_{i=1}^{n} r_R(\alpha\beta_i),$$

and E is an essential right ideal. This contradiction shows that $Z(R) = 0$.

Most of the remaining parts of the above proposition are equally elementary consequences of the definition; for proofs, see [20]. In [20, p.222, Q.3], Handelman and Lawrence raise the problem of characterising strongly prime group rings. They prove that necessary conditions for a group ring RG to be strongly prime are that R is strongly prime and, in our notation, $L(G) = 1$, and they show that these conditions are also sufficient if G is abelian, [20, Prop.III.1]. Provided we consider only commutative coefficient rings, we can extend their result to

<u>THEOREM 5.3</u>.(i)(Handelman and Lawrence) Let RG be a strongly prime group ring. Then R is strongly prime and $L(G) = 1$.

(ii) If R is a commutative domain and G satisfies $L(G) = 1$ and is (a) soluble, or (b) FC-hypercentral, then RG is strongly prime.

<u>PROOF</u>:

(i) If RG is strongly prime, then clearly so is R. Suppose $L(G) \neq 1$; then $\omega(L(G))$ is a non-zero two-sided ideal of RG, and if $\alpha_1, \ldots, \alpha_n$ are finitely many elements of $\omega(L(G))$, there exists a finite subgroup H of $L(G)$ such that $\alpha_i \in G\underline{\underline{h}}$, for $i = 1, \ldots, n$. Hence

$\alpha_i\hat{H} = 0$, and so RG is not strongly prime. Therefore $L(G) = 1$, as required.

(ii) (a) Clearly, RG is strongly prime if and only if Q(R)G is strongly prime, where Q(R) is the quotient field of R, so we may assume without loss that R is a field. Take $0 \neq \alpha \in RG$; it will be enough to show that the ideal $(RG\alpha RG)$ of RG contains a regular element; for if

$$c = \sum_{j=1}^{m} \beta_j \alpha \gamma_j \in RG\alpha RG$$

and c is regular, $\{\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_m\}$ is a right and left insulating set for α.

We apply Theorem 3.10, with $H = E(G)$ in the notation of §3, and $L = \{0\}$, to deduce that

$$(RG\alpha RG) \cap R\mathfrak{Z}(G) \neq 0. \tag{6}$$

However, since $L(G) = 1$, $\mathfrak{Z}^+(G) = 1$, and so by Lemma 1.1.19(i), $\mathfrak{Z}(G)$ is torsion-free abelian. Thus, by Corollary 1.3.23, $R\mathfrak{Z}(G)$ is a domain, and so by (6), $(RG\alpha RG)$ contains a regular element c of $R\mathfrak{Z}(G)$. By Lemma 1.3.4(i), c is regular in RG, and the proof is complete.

(b) The proof is similar to that of (a), taking $H = G$ in applying Theorem 3.10.

<u>REMARKS</u>: (i) Note that in proving the above theorem we have actually proved the following, at first sight stronger, result:

If R is a commutative domain and G is a soluble or FC-hypercentral group, then every non-zero two-sided ideal of RG contains a regular element if and only if $L(G) = 1$.

It seems unlikely that the above property is in general equivalent to the property of being strongly prime, and certainly by Proposition 5.2(i) there exist right strongly prime rings which do not satisfy the above condition. However, it is possible that the result is true for arbitrary group algebras. Certainly, it seems likely that Theorem 5.3 should extend to group rings of many other classes of groups; see in this regard Theorem 5.5 below, and the following remarks.

(ii) If $G \in \mathcal{U}$, $L(G) = \Delta^{+}(G)$, and so by Propositions 1.3.18 and 5.2(ii), Theorem 3.2.12 implies the following result:

If $G \in \mathcal{U}$, and R is commutative, RG is strongly prime if and only if $L(G) = 1$ and R is a domain.

(iii) The results of Chapter 2 provide, of course, many more examples of strongly prime group rings.

(iv) A straightforward local argument applied with the proof of Theorem 5.3 shows that if R is a commutative domain and G is torsion-free and locally soluble or locally FC-hypercentral, then RG is strongly prime. However it is not clear how to obtain a result corresponding to Theorem 5.3 itself, for group rings of locally soluble or locally FC-hypercentral groups; a result analogous to Proposition 3.4 seems necessary here.

(v) Bearing in mind that strongly prime rings are non-singular, it is interesting to note how Theorem 5.3 and its extensions discussed above compare with Conjecture 3.3 and the results of §3.

We mentioned at the beginning of this section that strongly prime rings arose in the consideration of coefficient rings of certain primitive group rings, [31]. Formanek, [10], showed that if $G = A * B$ is a free product, with $|A| \neq 1$, $|B| \neq 1$, and either $|A|$ or $|B| > 2$, then KG is primitive for all fields K. These group rings were among the first of the now fairly numerous examples of primitive group rings to be discovered. In [31], Lawrence adapted Formanek's proof to show, inter alia, that any strongly prime ring would suffice for the coefficient ring K. Here we shall adapt Lawrence's argument to obtain yet more examples of primitive group rings. It will be seen that the proof is in fact very similar to that devised by Formanek in [10].

DEFINITION 5.4. For a group H, we write $H^{\#} = H - \{1\}$. Let $G = A * B$; we say that $g \in G$ is of type AA and has length $(2n + 1)$ if

$$g = a_1 b_1 a_2 b_2 \cdots a_n b_n a_{n+1},$$

where $a_i \in A^{\#}$, $b_i \in B^{\#}$, $i = 1, \ldots, n+1$. We define elements of type AB, BA, and BB, and their lengths, in a similar way. We denote the length of $g \in G$ by $l(g)$, and define $l(1) = 0$.

Notice that since G is a free product, the length of an element is well-defined.

THEOREM 5.5. Let K be a field, G a group with a normal subgroup N, and subgroups $A \supsetneqq N$, $B \supsetneqq N$ such that

$$G/N \cong (A/N * B/N).$$

(Thus G is in fact the free product of A and B amalgamating the normal subgroup N.) Suppose that (i) KN is strongly prime. Then KG is strongly prime.

If in addition (i)' $0 \neq I \lhd KN$ implies that I

contains a regular element;

(ii) either $|A : N| > 2$ or $|B : N| > 2$;

and (iii) $|G/N| \geqslant |KN|$,

then KG is primitive.

PROOF:

Let $A_o = \{a_\lambda : \lambda \in \Lambda\}$, $B_o = \{b_\mu : \mu \in \mathcal{U}\}$ be transversals to N in A and B respectively. We may assume that $1 \in A_o$, $1 \in B_o$. Since $G/N \cong (A/N * B/N)$, the set G_o of words in $\{A_o \cup B_o\}$ forms a transversal to N in G. Of course, A_o, B_o and G_o are subsets of G. Note that elements of G can be uniquely written in the form

$$na_1b_1a_2b_2 \ldots a_mb_ma_{m+1},$$

where $a_i \in A_o$, $1 \leqslant i \leqslant m+1$, $b_i \in B_o - \{1\}$, $1 \leqslant i \leqslant m$, $a_j \neq 1$ for $2 \leqslant j \leqslant m$, and $n \in N$. Thus we may define the length of an element of G to be the length of its image in G/N.

We show first that (i) implies that KG is strongly prime. Take $0 \neq \alpha \in KG$, say $\alpha = \sum_{i=1}^{n} \alpha_i g_i$, where $g_i \in G_o$ and $0 \neq \alpha_i \in KN$, $1 \leqslant i \leqslant n$. Suppose g_1 has maximal length in $\{g_i : 1 \leqslant i \leqslant n\}$, and let $\{\beta_1, \ldots, \beta_m\}$ be a right and left insulating set for α_1 in KN. Take $1 \neq a \in A_o$, $1 \neq b \in B_o$, and put

$$\mathcal{S} = \{a\beta_j, b\beta_j, ab\beta_j, ba\beta_j : 1 \leqslant j \leqslant m\},$$

$$\mathcal{S}' = \{\beta_j^{g_1}a, \beta_j^{g_1}b, \beta_j^{g_1}ab, \beta_j^{g_1}ba : 1 \leqslant j \leqslant m\}.$$

It is easily checked that $\mathcal{S}$ (respectively $\mathcal{S}'$) is a left (respectively right) insulating set for α in KG, so that KG is strongly prime.

We now assume (i)', (ii) and (iii), and aim to prove that KG is primitive. Note that this will follow if we can find a proper left ideal M of KG such that $(M + I) = KG$ for every non-zero two-sided ideal I of KG.

We may assume without loss that $|A/N| \geqslant |B/N|$.

<u>Case</u> (a): $|A/N| = \infty$.

By condition (iii) and our assumption that $|A/N| \geqslant |B/N|$,

$$|G_0| = |G/N| = |A/N| \geqslant |KN| ,$$

and since $KG = \sum_{g \in G_0} KNg$, we have $|KG| = |A/N| = |A_0|$. Let $W : (KG - \{0\}) \longrightarrow A_0$ be a bijection of sets. Take $0 \neq \alpha = \sum_{i=1}^{n} \alpha_i g_i \in KG$ as above, where $g_1 \in G_0$ has maximal length in $\{g_i : 1 \leqslant i \leqslant n\}$, and $0 \neq \alpha_i \in KN$, $1 \leqslant i \leqslant n$. By hypothesis (i)', there exists a regular element $\bar{\alpha}_1$ of KN such that $\bar{\alpha}_1 \in (KN\alpha_1 KN)$, say

$$\bar{\alpha}_1 = \sum_{j=1}^{m} \gamma_j \alpha_1 \tau_j ,$$

where $0 \neq \gamma_j, \tau_j \in KN$, $1 \leqslant j \leqslant m$. Put

$$\begin{aligned} \bar{\alpha} &= \sum_{j=1}^{m} (\gamma_j \alpha (\tau_j)^{g_1}) \\ &= \Big(\sum_{j=1}^{m} \gamma_j \alpha_1 \tau_j\Big) g_1 + \sum_{i=2}^{n} \Big(\sum_{j=1}^{m} \gamma_j \alpha_i g_i (\tau_j)^{g_1} \Big) \\ &= \bar{\alpha}_1 g_1 + \sum_{i=2}^{n} \bar{\alpha}_i g_i \\ &\in (KG\alpha KG) \lhd KG, \end{aligned}$$

where $\bar{\alpha}_i \in KN$, $1 \leqslant i \leqslant n$, using the fact that N is normal in G.

Now fix any $1 \neq b \in B_0$, and define

$$T(\alpha) = \begin{cases} bW(\alpha)b\bar{\alpha}b + W(\alpha)b\bar{\alpha}b & \text{if } g_1 \text{ is of type AA;} \\ bW(\alpha)b\bar{\alpha} + W(\alpha)b\bar{\alpha}W(\alpha) & \text{if } g_1 \text{ is of type AB} \\ & \text{or } g_1 = 1 ; \\ bW(\alpha)\bar{\alpha}b + W(\alpha)\bar{\alpha}bW(\alpha)b & \text{if } g_1 \text{ is of type BA;} \\ bW(\alpha)\bar{\alpha}W(\alpha) + W(\alpha)\bar{\alpha}W(\alpha)b & \text{if } g_1 \text{ is of type BB.} \end{cases}$$

The above procedure may be followed for every non-zero element α of KG to obtain a corresponding element $T(\alpha)$ of KG. Put

$$M = \sum_{0 \neq \alpha \in KG} KG\left[T(\alpha) + 1\right],$$

a left ideal of KG. Since, if $0 \neq \alpha \in KG$, $\bar{\alpha}$ is chosen in $(KG\alpha KG)$, it follows that for any non-zero ideal I of KG,

$$(I + M) = KG.$$

Hence to prove that KG is primitive, it will be enough to show that M is a proper left ideal. Suppose that this is not the case, so that there exists $t \geqslant 1$ and

$$\alpha_1, \ldots, \alpha_t, \beta_1, \ldots, \beta_t \in KG$$

such that

$$1 = \sum_{s=1}^{t} \beta_s\left(T(\alpha_s) + 1\right). \tag{7}$$

For $s = 1, \ldots, t$, let

$$\beta_s = \sum_{l=1}^{n(s)} \beta_{sl} g_{sl},$$

where $0 \neq \beta_{sl} \in KN$ and $g_{sl} \in G_0$, $1 \leqslant l \leqslant n(s)$. Choose g' of maximal length in $\{g_{sl} : 1 \leqslant s \leqslant t,\ 1 \leqslant l \leqslant n(s)\}$, and suppose that $g' = g_{s_1 l_1}$ is of type AB or BB.

Note that if r is a regular element of a group ring SH, then r^h is regular for all $h \in H$; for if there exists $\eta \in SH$ and $h \in H$ such that

$$\eta h^{-1} r h = 0,$$

then $\eta h^{-1} r = 0$, so $\eta h^{-1} = 0$ and thus $\eta = 0$, as claimed.

It follows that if we write

$$\beta_{s_1} T(\alpha_{s_1}) = \sum_{g \in G_0} \delta_g g,$$

where $\delta_g \in KN$, then $\delta_x \neq 0$ either when

$$x = g' W(\alpha_{s_1}) \ldots g_1 \ldots W(\alpha_{s_1}) b$$

or when $x = g' W(\alpha_{s_1}) \ldots g_1 \ldots W(\alpha_{s_1})$ (8)

Notice that here we are once again using the fact that N is normal in G to enable us to move the regular element $\overline{\alpha}_1$ of KN to the left hand side of the relevant summand of $\beta_{i_1}T(\alpha_{i_1})$, to obtain a new, but still regular, element of KN.

Since the map W is a bijection, the element x of (8) cannot appear as a coset representative in the expansion of $\beta_s T(\alpha_s)$ for any $s \neq s_1$, $1 \leq s \leq t$. Hence, by (7), we must have $x \in \operatorname{supp}\beta_s$, for some s, $1 \leq s \leq t$. However,

$$l(x) \quad = \quad l(g') + l(g_1) + 4 \quad > \quad l(g'),$$

and since $l(g')$ is by choice of g' maximal among elements of $\{\operatorname{supp}\beta_s : 1 \leq s \leq t\}$, we have arrived at a contradiction.

Similar remarks apply if g' is of the form AA or BA, whence M is a proper left ideal and the proof of Case (a) is complete.

<u>Case</u> (b): $|A/N| < \infty$.

In this case, there is a bijection

$$W: (KG - \{0\}) \longrightarrow \mathbb{N},$$

where $\mathbb{N}$ denotes the natural numbers, by hypothesis (iii). Given $0 \neq \alpha \in KG$, we find $0 \neq \overline{\alpha} \in KG$ exactly as in Case (a). Since $|A_o| > 2$, we may fix elements $a, c \in A_o$, $a \neq c$, $a \neq 1 \neq c$. We also fix $1 \neq b \in B_o$. If now $\alpha = \sum \alpha_i g_i$ as before, where $\alpha_i \in KN$, $g_i \in G_o$, $1 \leq i \leq n$, and g_1 is the element of maximal length which we have chosen before obtaining $\overline{\alpha}$, we define

$$T(\alpha) = \begin{cases} (ab)^{W(\alpha)}c\overline{\alpha}b + b(ab)^{W(\alpha)}c\overline{\alpha} & \text{if } g_1 \text{ is of type BA;} \\ (ab)^{W(\alpha)}cb\overline{\alpha}a + b(ab)^{W(\alpha)}cb\overline{\alpha} & \text{if } g_1 \text{ is of type AB or } g_1 = 1; \\ (ab)^{W(\alpha)}cb\overline{\alpha}b + b(ab)^{W(\alpha)}cb\overline{\alpha} & \text{if } g_1 \text{ is of type AA;} \\ (ab)^{W(\alpha)}c\ \ a + b(ab)^{W(\alpha)}c & \text{if } g_1 \text{ is of type BB.} \end{cases}$$

We now put

$$M = \sum_{0 \neq \alpha \in KG} KG\left[T(\alpha) + 1\right],$$

and use arguments similar to those used for Case (a) to show that M is a proper left ideal of KG, and

$$(M + I) = KG$$

for all non-zero ideals I of KG.

This completes the proof of the theorem.

From the first part of the above result and Proposition 5.2(v) we immediately deduce

COROLLARY 5.6. Let K be a field, and let G be a group with a normal subgroup N such that G/N is a non-trivial free product. If KN is strongly prime, then Z(KG) = 0.

By applying a result of Passman, it is possible to weaken hypothesis (iii) of Theorem 5.5.

THEOREM 5.7. [37, Theorem 8.1.5] Let K be a field extension of F, let G be a group, and suppose FG is primitive. If K is an algebraic extension of F, or if $\Delta(G) = 1$, then KG is primitive.

COROLLARY 5.8. Let K be a field, and let G be a group with a normal subgroup N, and subgroups $A \supsetneqq N$, $B \supsetneqq N$ such that $G/N \cong (A/N * B/N)$. Suppose that

(i)' Every non-zero ideal of KN contains a regular element;

(ii) either $|A : N| > 2$ or $|B : N| > 2$;

(iii) $|G/N| \geqslant |N|$;

(iv) either $|K| \leqslant |G|$ or $\Delta(A) \cap \Delta(B) = \{1\}$.

Then KG is primitive.

PROOF:

If $|K| \leqslant |G| = |G/N|$, then $|KN| \leqslant |G/N|$ by (iii), and the result follows from Theorem 5.5.

Suppose on the other hand that $\Delta(A) \cap \Delta(B) = 1$. We claim that $\Delta(G) = \Delta(A) \cap \Delta(B)$. Now it is clear that the FC-subgroup of a non-trivial free product which satisfies hypothesis (ii) is trivial, so that $\Delta(G) \subseteq N$, and therefore $\Delta(G) \subseteq \Delta(A) \cap \Delta(B)$. The reverse inclusion is clear. Now let F denote the prime subfield of K, so that F is certainly countable, and (iii) implies that $|G/N| \geqslant |FN|$, since $|G/N|$ is infinite. Hence FG is primitive, by Theorem 5.5. The result now follows from Theorem 5.7.

REMARKS: (i) Hypothesis (iv) of the above result is certainly necessary, as by [37, Theorem 8.1.6], a primitive group ring KG satisfies either $\Delta(G) = 1$ or $|K| \leqslant |G|$. Hypothesis (ii) is also necessary, as if $N = \{1\}$ and $|A| = |B| = 2$, then $G = C_2 * C_2$, the infinite dihedral group, by Lemma 1.1.24. In this case, KG is not primitive for any field K; see [37].

(ii) It is quite possible that hypothesis (iii) is superfluous, although clearly it is necessary for our present proof. We conjecture also that hypothesis (i)' is much stronger than is necessary, and in particular we ask the following questions:

(a) If in Corollary 5.8 we require, instead of (i)', that KN be primitive, is KG primitive?

(b) Is it even sufficient to assume that KN is prime, instead of (i)', in Corollary 5.8?

6. CONCLUSION

The main purpose of this chapter has been to provide evidence in support of Conjecture 3.3. It is perhaps revealing to compare this conjecture with the conjecture of Passman and Zalesskii on the structure of the Jacobson radical of KG. Recall that they have conjectured that $J(KG) = N^*(KG)$ for an arbitrary group algebra KG. Now by Theorem 1.3.30,

$$N^*(KG) = J\big(K\Lambda^+(G)\big)KG,$$

where $\Lambda^+(G)$ is a certain characteristic locally finite subgroup of G. For a given group algebra KG, we thus have the following chains of characteristic locally finite subgroups, and ideals:-

$$\begin{array}{ccccccc}
\Delta^+(G) & \subseteq & \Lambda^+(G) & \subseteq & L(G) & \subseteq & G \\
J\big(K\Delta^+(G)\big)KG & \subseteq & J\big(K\Lambda^+(G)\big)KG & \subseteq & J\big(KL(G)\big)KG & \subseteq & KG \\
\| & & \cap\!\text{I} \;\; (a) & & \cap\!\text{I} \;\; (b) & & \\
N(KG) & \subseteq & J(KG) & & Z(KG) & \subseteq & KG \\
 & & & (c) & & &
\end{array}$$

It is well-known that we have equality at (a) for many classes of groups when K has characteristic zero, (see [38] for details) and for arbitrary K when G is, for example, locally soluble, locally FC-hypercentral, linear, or, (trivially), locally finite, [38]. The inclusion (b) is, as we have shown, an equality for all fields of characteristic zero, and for an arbitrary field K when G is locally soluble, locally FC-hypercentral, or locally finite. As we have previously remarked, it is not known whether (c) $J(KG) \subseteq Z(KG)$. Note that a proof of (c) would imply that $J(KG) = 0$ for all groups G and all fields K of characteristic zero.

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