Extending precolourings of circular cliques

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Abstract

Let G be a graph with circular chromatic number $\chi_c(G) = \frac{k}{q}$. Given $P \subseteq V(G)$ where the components of $G[P]$ are isomorphic to the circular clique $G_{k,q}$, suppose the vertices of P have been precoloured with a (k', q')-colouring. We examine under what conditions one can be assured the colouring extends to the entire graph. We study sufficient conditions based on $\frac{k'}{q'} - \frac{k}{q}$ as well as the distance between precoloured components of $G[P]$. In particular, we examine a conjecture of Albertson and West showing the conditions for extendibility are more complex than anticipated in their work.

1 Introduction

In [6], Thomassen asks if given a planar graph G and a 5-colouring of $P \subseteq V(G)$ with the restriction that the vertices of P are far enough apart, can the colouring be extended to the entire graph? In [1], Albertson answers the question in the affirmative provided the precoloured vertices are at distance at least four from each other. Fundamental in the result are two key components: First, one requires 5 colours for the extended colouring despite $\chi(G) \leq 4$. (In general, a precolouring of a planar graph is not extendible to a 4-colouring of the entire graph.) Second, the precoloured vertices must be sufficiently far apart. In fact, Albertson proves the following.

Theorem 1.1 (Albertson [1]). *Suppose $\chi(G) = r$ and $P \subseteq V(G)$ such that the distance between any two vertices in P is at least 4. Any $(r+1)$-colouring of P can be extended to an $(r+1)$-colouring of G.*

There are now many papers on colouring extensions. The introduction of [3] provides a nice overview. We focus on situation where the precoloured vertices induce a collection of cliques.

Given $P \subseteq V(G)$, define $d(P)$ to be the minimum distance in G between any two connected components of $G[P]$. A k-colouring of P is called a k-*precolouring*. A more general result proved by Albertson in [1] says that any $(r+1)$-precolouring of a r-colourable graph can be extended to the entire graph where the set P of precoloured vertices consists of k-cliques provided $d(P) \geq 6k-2$. This distance is improved by Kostochka to $d(P) \geq 4k$, (as cited and proved in [2]); and by Albertson and Moore to $d(P) \geq 3k$ when $r = k$, [2].

In [3], Albertson and West study extending circular colourings where the precoloured vertices form an independent set. They conjecture in the case that G is (k,q)-colourable and the precoloured components are circular cliques $G_{k,q}$, there is a distance d^* such that any (k',q')-precolouring can be extended provided $\frac{k'}{q'} - \frac{k}{q} \geq 1$ and $d(P) \geq d^*$.

We settle this conjecture in the negative by constructing an infinite family $\mathcal{F}$ with the following property. Given any ℓ and d, there is a (k,q)-colourable graph $G \in \mathcal{F}$ containing a collection $P \subseteq V(G)$ of circular cliques $G_{k,q}$, such that some (k',q')-precolouring of P is nonextendible despite having $\frac{k'}{q'} - \frac{k}{q} > \ell$ and $d(P) > d$.

We then turn our attention to positive results. Extendibility results are established for the cases when one of the q' or q equals 1, i.e. the one of the two colourings is a classical vertex colouring.

Circular colourings and the extension product

We provide some key definitions for our work here and refer the reader to [7] for standard definitions. Let k, q be positive integers such that $k \geq 2q$. A (k,q)-*colouring* of a graph G is an assignment $c : V(G) \to \{0, 1, 2, \ldots, k-1\}$ such that for $uv \in E(G)$, $q \leq |c(u) - c(v)| \leq k - q$. The *circular complete graph* or *circular clique* $G_{k,q}$ has vertices $\{0, 1, \ldots, k-1\}$ and edges $\{ij : q \leq |i-j| \leq k-q\}$. Thus $G_{k,1}$ is simply the (classical) complete graph on k vertices.

The circular complete graphs play the role in circular colourings as do the complete graphs in classical colourings. Adopting the homomorphism point of view, see [4, 5], G admits a (k,q)-colouring if, and only if, there is a homomorphism $f : G \to G_{k,q}$. Recall, a *homomorphism* $f : G \to H$ is a mapping $f : V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. We write $G \to H$ to indicate the existence of a homomorphism.

It turns out that $G_{k,q} \to G_{k',q'}$ if, and only if, $\frac{k}{q} \leq \frac{k'}{q'}$. Thus, given a graph G, if $G \to G_{k,q}$, then $G \to G_{k',q'}$ for any $\frac{k'}{q'} \geq \frac{k}{q}$. The *circular chromatic number* of a graph G is defined as

$$\chi_c(G) = \inf \left\{ \frac{k}{q} : G \to G_{k,q} \right\}$$

In [4], Bondy and Hell show the infimum may be replaced by a minimum. The proof depends on the fact that optimum colourings must be surjective. Surjective mappings play a key role in our constructions of nonextendible families.

A fundamental construction in our work comes from [3].

Definition 1.2. *Let G and H be graphs. The* extension product $G \bowtie H$ *has as its vertex set $V(G) \times V(H)$ with $(g_1, h_1)(g_2, h_2)$ an edge if $g_1g_2 \in E(G)$ and either $h_1h_2 \in E(H)$ or $h_1 = h_2$.*

Alternatively one may view $G \bowtie H$ as the categorical product of G with a reflexive copy (a loop on each vertex) of H. It is straightfoward to verify $G \bowtie H \to G$ via the projection onto the first coordinate. As a point of notation, for a fixed $i \in V(H)$, the subgraph induced by $\{(v, i)|v \in V(G)\}$ is isomorphic to G and is denoted by G^i. Given a homomorphism $f : G \bowtie H \to X$, the mapping defined by $f_i(u) := f(u, i)$ is a homomorphism $f_i : G^i \to X$. Given $\varphi : G \to X$, we may view $V(G) = V(G^i)$ and thus write $f_i = \varphi$ if $f_i(u) = \varphi(u)$ for all $u \in V(G)$.

Of particular importance for us in the product $G \bowtie P_n$. Recall P_n is the path of length $n - 1$ with vertex set $\{1, 2, \ldots, n\}$.

2 Nonextendibility for large $\frac{k'}{q'} - \frac{k}{q}$

In this section we provide a negative answer to the conjecture of Albertson and West. We examine extending circular colourings of $G_{k,q} \bowtie P_n$ where the first and last copy of the circular clique $G_{k,q}$ have been (k', q')-coloured.

We begin with our key definition.

Definition 2.1. *Let $\varphi : G \to H$ be a homomorphism and let $v \in V(G)$. The homomorphism φ is* uniquely extendible at v *if whenever $g : G \to H$*

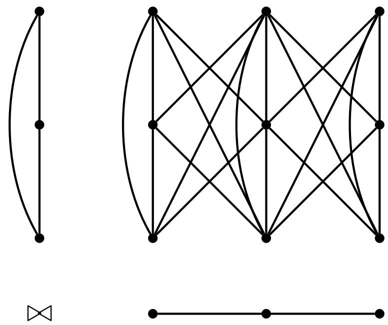

Figure 1: The product $K_3 \bowtie P_3$ together with the two factors. The projection onto the first coordinate is a homomorphism to K_3.

is a homomorphism with $g(u) = \varphi(u)$ for all $u \neq v$, then $g(v) = \varphi(v)$. If φ is uniquely extendible at v for all $v \in V(G)$, we simply say φ is uniquely extendible*.*

The following proposition provides a straightforward, but useful characterization of uniquely extendible homomorphisms.

Proposition 2.2. *Let $\varphi : G \to H$ be a homomorphism. Then φ is uniquely extendible at v if and only if*

$$\left| \bigcap_{w \in N_G(v)} N_H(\varphi(w)) \right| = 1$$

Proof. Suppose φ is uniquely extendible at v. Let $T = \bigcap_{w \in N_G(v)} N_H(\varphi(w))$. The existence of φ ensures $|T| \geq 1$. If $x, y \in T$ with $\varphi(v) \neq y$, then we define the homomorphism g by

$$g(u) = \begin{cases} \varphi(u) & \text{if } u \neq v \\ y & \text{otherwise} \end{cases}$$

contrary to the assumption that φ is uniquely extendible.

On the other hand, if φ is not uniquely extendible at v, then there is a homomorphism g such that φ and g agree on all vertices of $G - v$ but $\varphi(v) \neq g(v)$. Then $\{\varphi(v), g(v)\} \subseteq \bigcap_{w \in N_G(v)} N_H(\varphi(w))$. The result follows. $\square$

The crux of our approach is captured in the following proposition.

Proposition 2.3. *Let $\varphi : G \to H$ be uniquely extendible at v for all $v \in V(G)$. Let $f : G \bowtie P_n \to H$ be a homomorphism such that $f_1 = \varphi$. Then $f_i = \varphi$ for all i, $1 \le i \le n$.*

Proof. Consider a vertex $(v, 1)$ in $G \bowtie P_n$. The subgraph $T = G_1 - (v, 1) + (v, 2)$ is isomorphic to G. Moreover, the homomorphism f restricted to T is a homomorphism of a copy of G to H which agrees with φ on all vertices of $T - (v, 2)$. Thus $f(v, 2) = \varphi(v, 1)$. The result follows. $\square$

The following theorem demonstrates the existence of uniquely extendible homomorphisms between circular cliques.

Theorem 2.4. *Let k, q, and $t \ge 2$ be positive integers where $k \equiv \pm 1 \pmod t$, and $k \ge 3qt + 1$. Then there exists a uniquely extendible homomorphism $\varphi : G_{k,qt+1} \to G_{k,q+1}$.*

Proof. Suppose k, q, and t statisfy the hypotheses. Further suppose $k+1 = tg$ for some integer g. The case $k - 1 = tg$ is analogous. Denote the vertices of $G_{k,qt+1}$ by $\{v_i : 0 \le i \le k-1\}$ and the vertices of $G_{k,q+1}$ by $\{i : 0 \le i \le k-1\}$.

Define $\varphi : G_{k,qt+1} \to G_{k,q+1}$ by $\varphi(v_i) = gi \bmod k$. We claim φ is a homomorphism. First observe k and g are relatively prime; thus, the mapping φ is injective. Let $i \in V(G_{k,q+1})$. Consider $\varphi(v_{ti}) = gti = (k+1)i = i \bmod k$. By injectivity, v_{ti} is the unique vertex mapping to i. Consider a nonneighbour of i, say $i+j$ for some $j \in \{\pm 1, \pm 2, \ldots, \pm q\}$. By our observation, the unique pair of vertices mapping to $(i, i+j)$ is (v_{ti}, v_{ti+tj}). In particular, this is a pair of nonadjacent vertices (in $G_{k,qt+1}$) since $|tj| < qt + 1$. Thus the pre-image of any non-adjacent pair in $G_{k,q+1}$ is a unique pair of non-adjacent vertices in $G_{k,qt+1}$. Hence, φ is a homomorphism.

Next we show φ is uniquely extendible. Let v_{ti} be any arbitrary vertex in $V(G_{k,qt+1})$. Suppose to the contrary that φ is not uniquely extendible at v_{ti}. By Proposition 2.2 there is a vertex u in $\bigcap_{w \in N(v_{ti})} N(\varphi(w))$ different from i. Since $\varphi(w) \notin N(\varphi(w))$, it must be the case that $u \ne \varphi(w)$ for any $w \in N(v_{ti})$. Hence, $u = \varphi(v)$ for some $v \notin N(v_{ti})$, i.e. $v \in \{ti \pm 1, ti \pm 2, \ldots, ti \pm qt\}$. Thus assume $u = i + jg$ for some $1 \le j \le qt$. (The case $-dt \le j \le -1$ is analogous.) We now establish a neighbour of v_{ti} is mapped to a nonneighbour of u, contrary to the definition of u. Observe, $tq + 1 \le tq + j \le 2tq$. Since $k \ge 3tq+1$, v_{ti} is adjacent to $v_{ti+tq+j}$. Also, $\varphi(v_{ti+tq+j}) = i+q+jg \bmod k$. By assumption $u \in N(\varphi(v_{ti+tq+j}))$; thus, $u = i + jg$ is adjacent to $\varphi(v_{ti+tq+j}) = i + q + jg$ (in $G_{k,q+1}$), a contradiction. $\square$

From the result above we obtain an infinite family of counterexamples to the question posed by Albertson and West.

Corollary 2.5. *Given positive integers ℓ and d^*, there exists a (k,q)-colourable graph containing two circular $G_{k,q}$ cliques at distance greater than d^* such that for some $\frac{k'}{q'} \geq \frac{k}{q} + \ell$ there is a (k',q')-precolouring of the two cliques which does not extend to the entire graph.*

The reader will undoubtably notice that in the corollary above, $k' = k$. We now turn our attention to constructing (k,q)-colourable graphs that have nonextendible (k',q')-precolourings where $k' \neq k$. Continuing with the idea from the proof above, we study homomorphisms of the form $\varphi(v_i) = gi \bmod k$.

Definition 2.6. *Let $\varphi : G_{k,q} \to G_{k',q'}$ be a homomorphism. Label $V(G_{k,q}) = \{v_0, v_1, \dots, v_{k-1}\}$. If $\varphi(v_i) = gi \bmod k'$ for all $v_i \in V(G_{k,q})$, then we say g* generates *φ.*

Note that g acts on the vertices in $G_{k',q'}$. Hence, it makes sense to study mappings of this form for a fixed g and (k',q'), but differing input graphs, i.e. differing (k,q).

Theorem 2.7. *Suppose g generates a uniquely extendible homomorphism $\varphi : G_{k,q} \to G_{k',q'}$. Then g generates a uniquely extendible homomorphism $\varphi' : G_{k+2tk',q+tk'} \to G_{k',q'}$ for any integer $t \geq 1$.*

Proof. Again, let $V(G_{k,q}) = \{v_0, v_1, \dots, v_{k-1}\}$, $V(G_{k+2tk',q+tk'}) = \{v_0, v_1, \dots, v_{k+2tk'-1}\}$, and $V(G_{k',q'}) = \{0, 1, \dots, k'-1\}$.

Let $v_i \in V(G_{k,q})$. Since φ is uniquely extendible we have

$$\left| \bigcap_{w \in N_{G_{k,q}}(v_i)} N_{G_{k',q'}}(\varphi(w)) \right| = \left| \bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(\varphi(v_j)) \right| = \left| \bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j) \right| = 1$$

where the product $g \cdot j$ is reduced modulo k'. Since φ is a homomorphism, $\varphi(v_i) = gi$ is the unique vertex in $\cap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j)$, a fact we use below.

Now consider $v_i \in V(G_{k+2tk',q+tk'})$. Under φ'

$$\left|\bigcap_{w \in N_{G_{k+2tk',q+tk'}}(v_i)} N_{G_{k',q'}}(\varphi(w))\right| = \left|\bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(\varphi(v_j))\right|$$
$$= \left|\bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(g \cdot j)\right|$$

Again $g \cdot j$ is reduced modulo k' from which we obtain,

$$\left|\bigcap_{j=i+q+tk'}^{i+k+2tk'-q-tk'} N_{G_{k',q'}}(g \cdot j)\right| = \left|\bigcap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j)\right| = 1$$

By our remark above, $\varphi'(v_i) = \varphi(v_i) = gi$ is the unique vertex belonging to $\cap_{j=i+q}^{i+k-q} N_{G_{k',q'}}(g \cdot j)$. Hence φ' is a homomorphism and is uniquely extendible by Proposition 2.2 $\square$

Corollary 2.8. *Given k', q' with $\frac{k'}{q'} \geq 2$, for any $\varepsilon > 0$ there exists k, q with $\varepsilon > \frac{k}{q} - 2 \geq 0$ and a uniquely extendible homomorphism $\varphi : G_{k,q} \to G_{k',q'}$.*

Proof. Clearly, the identity map $G_{k',q'} \to G_{k',q'}$ is uniquely extendible and generated by 1. By Proposition 2.7 we may choose $k = k' + 2tk'$ and $q = q' + tk'$ for sufficiently large t. $\square$

Thus, Corollary 2.8 shows that for any $\frac{k'}{q'} \geq 2$, there are integers k and q such that:

- $3 > \frac{k}{q} \geq 2$;
- there is a (k', q')-precolouring of $P = G_{k,d}^1 \cup G_{k,d}^n$ in $G_{k,d} \bowtie P_n$; and,
- this precolouring is not extendible (for any n).

Hence, we obtain an infinite family of counterexamples to the Albertson and West conjecture where $k' \neq k$. However, in our examples $k' \leq k$. Given the positive results published for (ordinary) precolourings of (ordinary) cliques and the use of Kempe chains, it does seem reasonable to study the case when $k' > k$.

In the next section we present positive extension results; however, either the cliques or the colourings are classical cliques or colourings. We conclude this section with a small result highlighting the necessity of the hypothesis $\frac{k+3}{q} > 3$ appearing in the forthcoming theorems.

Corollary 2.9. *If $k - 2q + 2 \mid q - 1$, then there exists a uniquely extendible homomorphism $\varphi : G_{k,q} \to G_{k-2q+2,1}$.*

Proof. Let $q - 1 = t(k - 2q + 2)$. The identity map $G_{k-2q+2,1} \to G_{k-2q+2,1}$ is uniquely extendible and generate by 1. By Proposition 2.7 there is a uniquely extendible homomorphism $\varphi : G_{(k-2q+2)+2t(k-2q+1),1+t(k-2q+2)} \to G_{k-2q+1,1}$ or $\varphi : G_{k,q} \to G_{k-2q+1,1}$ □

3 Positive Results

In [2] Albertson and Moore study the problem of extending a $(k+1)$-colouring of a k-colourable graph where the precoloured components are k-cliques. They also study the problem when the precoloured components are general subgraphs. In the latter case the penalty for having general subgraphs is a larger number of colours may be required for the extension. In this spirit we now turn attention to extending a (k', q')-colouring of a (k, q)-colourable graph where the precoloured components are circular cliques. As the results in the previous section indicate, this general situation is complex. Thus we restrict our attention to two special cases: either $q = 1$ as in Theorem 3.2 or $q' = 1$ as in Theorem 3.4. In an approach similar to Albertson and Moore, we reduce to the case where the precoloured components are essentially cliques. Thus the following theorem will be of particular use.

Theorem 3.1 (Albertson and Moore [2])**.** *Let G be a graph with $\chi(G) = k$ and $P \subseteq V(G)$ such that the components of $G[P]$ are isomorphic to K_k. If $d(P) \geq 2k + 2\lfloor \frac{k}{2} \rfloor$, then any $(k+1)$-colouring of P extends to a $(k+1)$-colouring of G.*

Our first result is on circular colourings of k-cliques. Notice for $\frac{k'}{q'} \geq k+1$, the vertices $0, q', 2q', \ldots, kq'$ induce a copy of K_{k+1} in $G_{k',q'}$. Our approach in the next theorem is to reduce an arbitrary (k', d')-precolouring to a $(k+1)$-colouring using multiples of q'. At this point we use Theorem 3.1 to extend.

Theorem 3.2. *Let G be a k-colourable graph and $P \subseteq V(G)$ such that P induces k-cliques. If $\frac{k'}{q'} \geq k+1$ and $d(P) \geq 4k + 2\lfloor \frac{k}{2} \rfloor$, then any (k', q')-colouring of P extends to a (k', q')-colouring of G.*

Proof. Let φ be the (k', q')-colouring we wish to extend and let c be a k-colouring of G. Further suppose K is a clique from P. We shall extend φ to a ball of radius k around K so that all vertices on the boundary of the ball receive a colour that is a multiple of q'.

Given the symmetry of $G_{k',q'}$ we may assume for ease of demonstration one vertex receives colour from $\{0, 1, \ldots, q'-1\}$ and no vertex receives a colour from $\{k'-q', k'-q'+1, \ldots, k'-1\}$ under φ. We label the vertices of K as $v_0, v_1, \ldots, v_{k-1}$ so that $\varphi(v_0) < \varphi(v_1) < \cdots < \varphi(v_{k-1})$ and we designate the colour of v_i under c to be i, i.e. $c(v_i) = i$. Observe $c(v_i)q' \leq \varphi(v_i)$

We now extend φ to a ball of radius k around K as follows. Let u be a vertex at distance i from K, $1 \leq i \leq k$. Let v be the vertex in K such that $c(u) = c(v)$. Assign

$$\varphi(u) = \begin{cases} jq' & \text{if } c(v) = j \text{ and } j < i \\ \varphi(v) & \text{otherwise.} \end{cases}$$

At this point φ colours the vertices at distance k from K with the colours $0, q', \ldots, (k-1)q'$.

To see this extension of φ is a homomorphism, consider a vertex u at distance i from K. Let w be a neighbour of u at distance at most i from K. There are vertices u', w' in K such that $c(u') = c(u)$ and $c(w') = c(w)$. First suppose $c(u) \geq i$. If $c(w) \geq i$ as well, we have $\varphi(w) = \varphi(w')$ and $\varphi(u)\varphi(w) = \varphi(u')\varphi(w') \in E(G_{k',q'})$. If $c(w) < i$, then

$$\varphi(w) = c(w)q' \leq (i-1)q' \leq (c(u)-1)q' \leq \varphi(u) - q'$$

and again $\varphi(u)\varphi(w)$ is an edge of $G_{k',q'}$. Next suppose $c(u) < i$ and thus $\varphi(u) = c(u)q'$. If $\varphi(w) = c(w)q'$, then $1 \leq |c(u) - c(w)| \leq k-1$ implies $\varphi(u)\varphi(w)$ is an edge of $G_{k',q'}$. Finally if $c(w) \neq c(w)q'$, then $c(w) \geq i-1$. Since $c(u) < i$ and $c(u) \neq c(w)$ we conclude

$$\varphi(u) = c(u)q' \leq (c(w)-1)q' \leq \varphi(w) - q'.$$

Repeat this process for each clique in P obtaining a collection of pre-coloured components whose boundaries are coloured with with $0, q', \ldots, (k'-1)q'$. The components are distance at least $2k + 2\lfloor \frac{k}{2} \rfloor$ apart. By Theorem 3.1 this colouring may be extended to the entire graph. □

We now consider extending (classical) k'-colourings where the precoloured components are $G_{k,q}$. The general problem of extending colourings where the precoloured components are not cliques is considered in [2]. In our work the assumption that the precoloured components are circular cliques allows us to use fewer colours than is required in [2].

In the following we will consider the vertices of $G_{k,q}$ to be $\{0, 1, 2, \ldots, k-1\}$ and assume that all arithmetic is done modulo k. We think of the vertices $i, i+1, i+2, \ldots, i+t$ as being *consecutive* and forming the interval $[i, i+t]$. Given $G_{k,q}$ with $\frac{k+3}{q} > 3$ we note that any independent set must be contained in an interval of q vertices (and thus of length $q-1$). We also remind the reader that $c : G_{k,q} \to K_{\lceil k/q \rceil}$, where $c(i) = \lfloor i/q \rfloor$, is a homomorphism. The fibres of c are the intervals $[0, q-1], [q, 2q-1], [2q, 3q-1], \ldots, [(\lceil k/q \rceil - 2)q, (\lceil k/q \rceil - 2)q][(\lceil k/q \rceil - 1)q, k-1]$. We label these intervals as $I_0, I_1, I_2, \ldots, I_{\lceil k/q \rceil - 1}$. Each interval has q vertices with the possible exception of the last interval which has at most q vertices.

Any homomorphism $c' : G_{k,q} \to K_{k'}$ where the fibres of c' are the same as the fibres listed above for c is called a *canonical colouring* of $K_{k'}$. Our strategy for extending a k'-colouring to all of G is to extend the precolouring of each copy of $G_{k,d}$ to a canonical colouring on the boundary vertices of a ball around the circular clique. Once this has been done we apply Albertson's method to extend the colouring to the remainder of the graph. The ideas in the proof are straightforward but rather technical. The reader may wish to consult Figure 2 as a guide to the proof.

Lemma 3.3. *Suppose $k, q \geq 2$ are integers such that $\frac{k+3}{q} > 3$. Further suppose $k' \geq \lceil \frac{k}{q} \rceil + 1$ and $n = \lceil \frac{k}{q} \rceil + 2$. Then any k'-colouring of $G^1_{k,q}$ in $G = G_{k,q} \bowtie P_n$ can be extended to all of G in such a way that $G^n_{k,q}$ receives a canonical colouring.*

Proof. As in the previous section, we let the copies of $G_{k,q}$ in the product be denoted $G^i_{k,q}$ for $1 \leq i \leq n$. Also, $V(G^i_{k,q}) = \{(0, i), (1, i), \ldots, (k-1, i)\}$. Let c_1 be the k'-colouring of $G^1_{k,q}$. We shall define for each $i = 2, 3, \ldots n$, $c_i : G^i_{k,q} \to K_{k'}$ so that c_{i-1} and c_i are compatible homomorphisms (in the sense of the collection defining a homomorphism $G_{k,q} \bowtie P_n \to K_{k'}$).

We shall also denote the colours of $K_{k'}$ as $0, 1, 2, \ldots k'-1$ where there is no risk of confusion based on the context.

Denote the colour of $(0, 1)$ under c_1 as 0, i.e. let $c_1(0, 1) = 0$. All vertices in $G^1_{k,q}$ coloured 0 belong to an interval say $[(\ell_0, 1), (r_0, 1)]$ of length at most $q-1$.

	I_0			I_1			I_2			I_3			I_4	
$V(G_{k,q})$	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$i = 1$	0	0	1	1	2	2	2	3	3	3	4	4	4	0
$i = 2$	0	0	1	1	2	2	**5**	3	3	3	4	4	4	0
$i = 3$	0	0	1	**2**	**2**	**2**	5	3	3	3	4	4	4	0
$i = 4$	**1**	**1**	**1**	2	2	2	**5**	**5**	**5**	3	4	4	4	0
$i = 5$	1	1	1	2	2	2	5	5	5	**3**	**3**	**3**	4	0
$i = 6$	1	1	1	2	2	2	5	5	5	3	3	3	**4**	**4**

Figure 2: A 6-colouring of $G_{14,3} \bowtie P_6$ modified from $i = 1$ to $i = 6$ into a canonical colouring. Changes at each round are marked in bold. For $i = 2$ the interval $[\ell_2, r_2]$ is partitioned to avoid a multiple of $q = 3$. For $i = 3$, the vertices in I_1 are coloured 2. For $i = 4$, the intervals I_0, I_2 are made monochromatic. For $i = 5, i = 6$, the intervals I_3, I_4 are made respectively monochromatic.

Let $\ell_1 = r_0+1$ and denote by 1 the colour of $(\ell_1, 1)$, i.e. $c_1(\ell_1, 1) = 1$. Let r_1 be the last vertex in $[\ell_1, \ell_1+q-1]$ coloured 1 by c_1. Continuing in this manner, we partition $G^1_{k,q}$ into intervals $[(\ell_0, 1), (r_0, 1)], [(\ell_1, 1), (r_1, 1)], [(\ell_2, 1), (r_2, 1)], \ldots,$ $[(\ell_m, 1), (r_m, 1)]$. This partition has the property that $c_1(\ell_i, 1) = i$ and all vertices of colour i lie in the interval $[(r_i - q + 1, 1), (r_i, 1)]$.

We now define c_2 with two goals in mind. First, the intervals identified above, $[\ell_j, r_j]$, will be monochromatic under c_2. Second, we will ensure that one of these intervals does not contain a vertex of the form $(t \cdot q, 2)$.

If each interval $[(\ell_j, 1), (r_j, 1)]$ contains a vertex of the form $(t \cdot q, 1)$, then there are precisely $\lceil k/q \rceil$ such intervals. Since $k' \geq \lceil k/q \rceil + 1$, the colour $m + 1 \leq k'$ is not mapped to by c_1. In this case, we find an interval $[(\ell_p, 1), (r_p, 1)]$ of length at least two, and partition the interval into two intervals $[(\ell_p, 1), (r'_p, 1)]$, $[(r'_p + 1, 1), (r_p, 1)] = [\ell_{m+1}, r_{m+1}]$ where only one of the two intervals contains a vertex of the form $(i \cdot q, 1)$. Finally, relabel r'_p as r_p.

Define

$$c_2(i, 2) = \quad j \quad \text{if } (i, 1) \in [(\ell_j, 1), (r_j, 1)], \quad 1 \leq j \leq m + 1$$

On the other hand, if some interval $[(\ell_j, 1), (r_j, 1)]$ does not contain a vertex of the form $(t \cdot q, 1)$, then we define

$$c_2(i,2) = \quad j \quad \text{if } (i,1) \in [(\ell_j, 1), (r_j, 1)], \quad 1 \le j \le m$$

Hence at this point we have a colouring where all the vertices of colour j under c_2 are precisely the vertices in the interval $[(\ell_j, 2), (r_j, 2)]$. Also by construction there is an interval $[(\ell_s, 2), (r_s, 2)]$ not containing a vertex of the form $(t \cdot q, 2)$.

Let $t \cdot q$ be the (unique) multiple of q in $[(\ell_s - q + 1, 2), (\ell_s - 1, 2)]$.

Define

$$c_3(i,3) = \begin{cases} s & \text{if } (i,2) \in [(tq, 2), ((t+1)q - 1, 2)] \\ c_2(i,2) & \text{otherwise.} \end{cases}$$

It is straightforward to verify c_3 is a proper colouring and compatible with c_2. Recall $I_t = [t \cdot q, \min\{(t+1)q, k-1\}]$. In particular, a vertex belongs to interval I_t (in $G^3_{k,q}$) if and only if the vertex receives colour s under c_3. In other words, we have stretched the interval of vertices receiving colour s to be precisely one of the canonical intervals.

For the moment assume both I_{t-1} and I_{t+1} have length $q-1$. Let $c_3(tq - 1, 3) = j_1$ and $c_3((t+1)q, 3) = j_2$. Observe the vertices coloured j_1 by c_3 are all contained in the interval $[((t-1)q, 3), (tq-1, 3)]$ and the vertices coloured j_2 are contained in the interval $[((t+1)q, 3), ((t+1)q - 1, 3)]$. From this observation we can define

$$c_4(i,4) = \begin{cases} j_1 & \text{if } (i,3) \in [((t-1)q, 3), (tq - 1, 3)] \\ j_2 & \text{if } (i,3) \in [((t+1)q, 3), ((t+1)q - 1, 3)] \\ c_3(i,3) & \text{otherwise.} \end{cases}$$

At this point canonical intervals I_{t-1}, I_t, I_{t+1} in $G^4_{k,q}$ are monochromatic.

The correctness of this extension depends on the fact that I_{t-1} has length $q-1$ and thus all vertices of colour j_1 in $G^3_{k,q}$ belong to I_{t-1}. This argument may fail when working with $I_{\lceil k/q \rceil}$. Hence we can continue extending outward from I_t either in two directions (as described above) or in one direction so that the final interval considered is $I_{\lceil k/q \rceil}$. In the worse case ($I_t = I_0$ or $I_t = I_{\lceil k/q \rceil - 1}$), we require $n = \lceil k/q \rceil + 2$ steps to reach a canonical colouring. Should we reach a canonical colouring sooner, we simply fix that colouring for the remaining columns in the product. The result follows. □

We finish the paper with an extension result for $(\lceil k/q \rceil + 1)$-colourings of $G_{k,q}$ cliques in (k,q)-colourable graphs.

Theorem 3.4. *Suppose $\chi_c(G) = \frac{k}{q}$ and $P \subseteq V(G)$ induces of a subgraph of G consisting of circular cliques $G_{k,q}$. Further suppose $\frac{k+3}{q} > 3$ and $\gcd(k,q) = 1$. If the distance between any two cliques is at least $5\lceil k/q \rceil + 4$, then any $(\lceil k/q \rceil + 1)$-colouring of P can be extended to a $(\lceil k/q \rceil + 1)$-colouring of G.*

Proof. Let K be a circular clique in P. As above let $N_i(K)$ be the vertices at distance i from K. Let h be a (k,q)-colouring of G and let c be the $k' = (\lceil k/q \rceil + 1)$-colouring of P.

Define $c_1 = c(K)$ and $h_1 = h(K)$. Since $\gcd(k,q) = 1$, h_1 must be a bijection, [5]; hence, $c' = c_1 \circ h_1^{-1}$ is a k'-colouring of $G_{k,q}$. By Lemma 3.3 there is an extension of c' to a canonical colouring of $G_{k,q}^n$ in the product $G_{k,q} \bowtie P_n$ where $n = \lceil k/q \rceil + 2$. Consider the subgraph $N_i(K)$. The (k,q)-colouring h restricted to $N_i(K)$ is a mapping to $G_{k,q}$. Composing this with c_i gives a k'-colouring of $N_i(K)$. In particular we obtain an extension of c from K to $N_1(K) \cup N_2(K) \cup N_3(K) \cup \cdots \cup N_n(K)$ where the k'-colouring of $N_n(K)$ is a canonical colouring. We can replace $K \cup N_1(K) \cup N_2(K) \cup N_3(K) \cup \cdots \cup N_n(K)$ with a copy of $K_{k'}$ using the canonical colouring to colour the clique. Repeat this for each component of P.

We can now apply Albertson and Moore's result (Theorem 3.1) to extend the k'-colouring to the remainder of the graph. $\square$

We remark the condition $\gcd(k,q) = 1$ can be relaxed. One only needs that the colouring c_1 factors through h to obtain c' for each copy of $K \in P$. The necessity of this factoring can be seen from the following example. Consider $G_{6,2}$ on vertices $\{0, 1, \ldots, 5\}$ union K_3 on vertices $\{a, b, c\}$. Join a to $\{2, 3, 4, 5\}$, b to $\{0, 1, 4, 5\}$ and c to $\{0, 1, 2, 3\}$. The resulting graph is $(6,2)$-colourable: map $\{0, 1, a\}$ to 0; $\{2, 3, b\}$ to 2; and $\{4, 5, c\}$ to 4. The precolouring $c_1(0) = 0; c_1(1) = c_1(2) = 1; c_1(3) = c_1(4) = 2; c_1(5) = 3$ is a 4-colouring of $G_{6,2}$ that does not extend the rest of the graph, i.e. to the K_3 on $\{a, b, c\}$.

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