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On Saturated $k$-Sperner Systems

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Abstract

Given a set $X$, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be $k$-Sperner if it does not contain a chain of length $k+1$ under set inclusion and it is saturated if it is maximal with respect to this property. Gerbner et al. [11] conjectured that, if $|X|$ is sufficiently large with respect to $k$, then the minimum size of a saturated $k$-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ is $2^{k-1}$. We disprove this conjecture by showing that there exists $\varepsilon > 0$ such that for every $k$ and $|X| \geq n_0(k)$ there exists a saturated $k$-Sperner system $\mathcal{F} \subseteq \mathcal{P}(X)$ with cardinality at most $2^{(1-\varepsilon)k}$.

A collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is said to be an oversaturated $k$-Sperner system if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, $\mathcal{F} \cup \{S\}$ contains more chains of length $k+1$ than $\mathcal{F}$. Gerbner et al. [11] proved that, if $|X| \geq k$, then the smallest such collection contains between $2^{k/2-1}$ and $O\left(\frac{\log k}{k}2^k\right)$ elements. We show that if $|X| \geq k^2 + k$, then the lower bound is best possible, up to a polynomial factor.

Keywords: minimum saturation; set systems; antichains

1 Introduction

Given a set $X$, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a Sperner system or an antichain if there do not exist $A, B \in \mathcal{F}$ such that $A \subsetneq B$. More generally, a collection $\mathcal{F} \subseteq \mathcal{P}(X)$ is a $k$-Sperner system if there does not exist a subcollection $\{A_1, \ldots, A_{k+1}\} \subseteq \mathcal{F}$ such that $A_1 \subsetneq \cdots \subsetneq A_{k+1}$. Such a subcollection $\{A_1, \ldots, A_{k+1}\}$ is called a $(k+1)$-chain. We say that a $k$-Sperner system is saturated if, for every $S \in \mathcal{P}(X) \setminus \mathcal{F}$, we have that $\mathcal{F} \cup \{S\}$
contains a \((k+1)\)-chain. A collection \(\mathcal{F} \subseteq \mathcal{P}(X)\) is an \textit{oversaturated \(k\)-Sperner system}\(^1\) if, for every \(S \in \mathcal{P}(X) \setminus \mathcal{F}\), we have that the number of \((k+1)\)-chains in \(\mathcal{F} \cup \{S\}\) is greater than the number of \((k+1)\)-chains in \(\mathcal{F}\). Thus, \(\mathcal{F} \subseteq \mathcal{P}(X)\) is a \(k\)-Sperner system if and only if it is an oversaturated \(k\)-Sperner system that does not contain a \((k+1)\)-chain.

For a set \(X\) of cardinality \(n\), the problem of determining the maximum size of a saturated \(k\)-Sperner system in \(\mathcal{P}(X)\) is well understood. In the case \(k = 1\), Sperner’s Theorem [17] (see also [4]), says that every antichain in \(\mathcal{P}(X)\) contains at most \(\binom{n}{\lfloor n/2 \rfloor}\) elements, and this bound is attained by the collection consisting of all subsets of \(X\) with cardinality \(\lfloor n/2 \rfloor\). Erdős [6] generalised Sperner’s Theorem by proving that the largest size of a \(k\)-Sperner system in \(\mathcal{P}(X)\) is the sum of the \(k\) largest binomial coefficients \(\binom{n}{i}\).

In this paper, we are interested in determining the minimum size of a saturated \(k\)-Sperner system or an oversaturated \(k\)-Sperner system in \(\mathcal{P}(X)\). These problems were first studied by Gerbner, Keszegh, Lemons, Palmer, Pálvölgyi and Patkós [11].

Given integers \(n\) and \(k\), let \(\text{sat}(n, k)\) denote the minimum size of a saturated \(k\)-Sperner system in \(\mathcal{P}(X)\) where \(|X| = n\). It was shown in [11] that \(\text{sat}(n, k) = \text{sat}(m, k)\) if \(n\) and \(m\) are sufficiently large with respect to \(k\). We can therefore define

\[
\text{sat}(k) := \lim_{n \to \infty} \text{sat}(n, k).
\]

We are motivated by the following conjecture of [11].

**Conjecture 1** (Gerbner et al. [11]). For all \(k\), \(\text{sat}(k) = 2^{k-1}\).

Gerbner et al. [11] observed that their conjecture is true for \(k = 1, 2, 3\). They also proved that \(2^{k/2} - 1 \leq \text{sat}(k) \leq 2^{k-1}\) for all \(k\), where the upper bound is implied by the following construction.

**Construction 2** (Gerbner et al. [11]). Let \(Y\) be a set such that \(|Y| = k - 2\) and let \(H\) be a non-empty set disjoint from \(Y\). Let \(X = Y \cup H\) and define

\[
\mathcal{G} := \mathcal{P}(Y) \cup \{S \cup H : S \in \mathcal{P}(Y)\}.
\]

It is easily verified that \(\mathcal{G} \subseteq \mathcal{P}(X)\) is a saturated \(k\)-Sperner system of cardinality \(2^{k-1}\).

In this paper, we disprove Conjecture 1 by establishing the following:

**Theorem 3.** There exists \(\varepsilon > 0\) such that, for all \(k\), \(\text{sat}(k) \leq 2^{(1-\varepsilon)k}\).

We remark that the value of \(\varepsilon\) that can be deduced from our proof is approximately \((1 - \frac{\log_2(15)}{4}) \approx 0.023277\). The proof of Theorem 3 comes in two parts. First, we give an infinite family of saturated 6-Sperner systems of cardinality 30 which shows that \(\text{sat}(6) \leq 30 < 2^5\). We then provide a method which, under certain conditions, allows us to combine

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\(^1\)In [11], this is called a \textit{weakly saturated \(k\)-Sperner system}. Since there is another notion of weak saturation in the literature (see, for instance, Bollobás [3]), we have chosen to use a different term to avoid possible confusion.
a saturated \( k_1 \)-Sperner system of small order and a saturated \( k_2 \)-Sperner system of small order to obtain a saturated \((k_1 + k_2 - 2)\)-Sperner system of small order. By repeatedly applying this method, we are able to prove Theorem 3 for general \( k \). As it turns out, our method yields the bound \( \text{sat}(k) < 2^{k-1} \) for every \( k \geq 6 \). For completeness, we will prove that \( \text{sat}(k) = 2^{k-1} \) for \( k \leq 5 \), and so \( k = 6 \) is the first value of \( k \) for which Conjecture 1 is false.

Similar techniques show that \( \text{sat}(k) \) satisfies a submultiplicativity condition, which leads to the following result.

**Theorem 4.** For \( \varepsilon \) as in Theorem 3, there exists \( c \in [1/2, 1 - \varepsilon] \) such that \( \text{sat}(k) = 2^{(1+o(1))ck} \).

Naturally, we wonder about the correct value of \( c \) in Theorem 4.

**Problem 5.** Determine the constant \( c \) for which \( \text{sat}(k) = 2^{(1+o(1))ck} \).

We are also interested in oversaturated \( k \)-Sperner systems. Given integers \( n \) and \( k \), let \( \text{osat}(n, k) \) denote the minimum size of an oversaturated \( k \)-Sperner system in \( \mathcal{P}(X) \) where \( |X| = n \). As we will prove in Lemma 7, \( \text{osat}(n, k) = \text{osat}(m, k) \) provided that \( n \) and \( m \) are sufficiently large with respect to \( k \). Similarly to \( \text{sat}(k) \), we define \( \text{osat}(k) := \lim_{n \to \infty} \text{osat}(n, k) \). Gerbner et al. [11] proved that if \( |X| \geq k \), then an oversaturated \( k \)-Sperner system in \( \mathcal{P}(X) \) of minimum size has between \( 2^{k/2 - 1} \) and \( O \left( \frac{\log(k)}{k} 2^k \right) \) elements. Together with Lemma 7, this implies

\[
2^{k/2 - 1} \leq \text{osat}(k) \leq O \left( \frac{\log(k)}{k} 2^k \right).
\]

We show that the lower bound gives the correct asymptotic behaviour, up to a polynomial factor.

**Theorem 6.** For every integer \( k \) and set \( X \) with \( |X| \geq k^2 + k \) there exists an oversaturated \( k \)-Sperner system \( \mathcal{F} \subseteq \mathcal{P}(X) \) such that \( |\mathcal{F}| = O \left( k^5 2^{k/2} \right) \). In particular,

\[
\text{osat}(k) = 2^{(1/2+o(1))k}.
\]

In Section 2, we prove some preliminary results which will be used throughout the paper. In particular, we provide conditions under which a saturated \( k \)-Sperner system can be decomposed into or constructed from a sequence of \( k \) disjoint saturated antichains. In Section 3 we show that certain types of saturated \( k_1 \)-Sperner and \( k_2 \)-Sperner systems can be combined to produce a saturated \((k_1 + k_2 - 2)\)-Sperner system, and use this to prove Theorems 3 and 4. Finally, in Section 4, we give a probabilistic construction of oversaturated \( k \)-Sperner systems of small cardinality, thereby proving Theorem 6.

Minimum saturation has been studied extensively in the context of graphs [1, 2, 5, 10, 12, 13, 18, 19, 20] and hypergraphs [7, 14, 15, 16]. Such problems are typically of the following form: for a fixed (hyper)graph \( H \), determine the minimum size of a (hyper)graph \( G \) on \( n \) vertices which does not contain a copy of \( H \) and for which adding any edge \( e \notin G \),
yields a (hyper)graph which contains a copy of $H$. This line of research was first initiated by Zykov \cite{21} and Erdős, Hajnal and Moon \cite{8}. For more background on minimum saturation problems for graphs, we refer the reader to the survey of Faudree, Faudree and Schmitt \cite{9}.

2 Preliminaries

Given a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, we say that a set $A \subseteq X$ is an atom for $\mathcal{F}$ if $A$ is maximal with respect to the property that for every set $S \in \mathcal{F}$,

\[ S \cap A \in \{\emptyset, A\}. \tag{1} \]

We say that an atom $A$ with $|A| \geq 2$ is homogeneous for $\mathcal{F}$. Gerbner et al. \cite{11} proved that if $n, m$ are sufficiently large with respect to $k$, then $\text{sat}(n, k) = \text{sat}(m, k)$. Using a similar approach, we extend this result to $\text{osat}(n, k)$.

Lemma 7. Fix $k$. If $n, m > 2^{2k-1}$, then $\text{sat}(n, k) = \text{sat}(m, k)$ and $\text{osat}(n, k) = \text{osat}(m, k)$.

Proof. Fix $n > 2^{2k-1}$ and let $X$ be a set of cardinality $n$. Suppose that $\mathcal{F} \subseteq \mathcal{P}(X)$ is an oversaturated $k$-Sperner system of cardinality at most $2^{k-1}$. We know that such a family exists by Construction 2. We will show that, for sets $X_1$ and $X_2$ such that $|X_1| = n - 1$ and $|X_2| = n + 1$, there exists $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ such that

(a) $|\mathcal{F}_1| = |\mathcal{F}_2| = |\mathcal{F}|$;
(b) $\mathcal{F}_1$ and $\mathcal{F}_2$ have the same number of $(k+1)$-chains as $\mathcal{F}$,
(c) $\mathcal{F}_1$ and $\mathcal{F}_2$ are oversaturated $k$-Sperner systems.

We observe that this is enough to prove the lemma. Indeed, by taking $\mathcal{F}$ to be a saturated $k$-Sperner system or an oversaturated $k$-Sperner system in $\mathcal{P}(X)$ of minimum order, we will have that

\[ \max\{\text{sat}(n-1, k), \text{sat}(n+1, k)\} \leq \text{sat}(n, k) \]

and

\[ \max\{\text{osat}(n-1, k), \text{osat}(n+1, k)\} \leq \text{osat}(n, k). \]

Since $n$ was an arbitrary integer greater than $2^{2k-1}$, the result will follow by induction.

We prove the following claim.

Claim 8. Given a set $X$ and a collection $\mathcal{F} \subseteq \mathcal{P}(X)$, if $|X| > 2^{|\mathcal{F}|}$, then there is a homogeneous set for $\mathcal{F}$.

Proof. We observe that every atom $A$ for $\mathcal{F}$ corresponds to a subcollection $\mathcal{F}_A := \{S \in \mathcal{F} : A \subseteq S\}$ of $\mathcal{F}$ such that $\mathcal{F}_A \neq \mathcal{F}_{A'}$ whenever $A \neq A'$. This implies that the number of atoms for $\mathcal{F}$ is at most $2^{|\mathcal{F}|}$. Therefore, since $|X| > 2^{|\mathcal{F}|}$, there must be a homogeneous set $H$ for $\mathcal{F}$. \qed
By Claim 8 and the fact that \(|X| > 2^k - 1 \geq 2^{\mid T \mid}\), there exists a homogeneous set \(H\) for \(F\). Let \(x_1 \in H\) and \(x_2 \notin X\) and define \(X_1 := X \setminus \{x_1\}\) and \(X_2 := X \cup \{x_2\}\). Let 
\[
F_1 := \{S \in F : S \cap H = \emptyset \} \cup \{S \setminus \{x_1\} : S \in F_H\}, \text{ and}
\[
F_2 := \{S \in F : S \cap H = \emptyset \} \cup \{S \cup \{x_2\} : S \in F_H\}.
\]
Since \(H\) is homogeneous for \(F\), there does not exist a pair of sets in \(F\) which differ only on \(x_1\). Thus, for \(i \in \{1, 2\}\) there is a natural bijection from \(F_i\) to \(F\) which preserves set inclusion. Hence, (a) and (b) hold. Now, let \(i \in \{1, 2\}\) and \(T_i \in \mathcal{P}(X_i) \setminus F_i\) and define 
\[
T := (T_i \setminus (H \cup \{x_2\})) \cup \{x_1\}.
\]
Then \(T \in \mathcal{P}(X) \setminus F\) since \(H\) is a non-singleton atom and \(T \cap H = \{x_1\}\), and so there exists \(A_1, \ldots, A_k \in F\) and \(t \in \{0, \ldots, k\}\) such that 
\[
A_1 \subset \cdots \subset A_t \subset T \subset A_{t+1} \subset \cdots \subset A_k.
\]
Since \(T \cap H \neq H\), we must have \(A_j \cap H = \emptyset\) for \(j \leq t\) and so \(A_1, \ldots, A_t \in F_i\) and \(A_1 \subset \cdots \subset A_t \subset T_i\). Also, since \(T \cap H \neq \emptyset\), we have \(A_j \cap H = H\) for \(j \geq t + 1\). Setting 
\[
A'_j := (A_j \cup \{x_2\}) \cap X_i,
\]
we see that \(A'_j \in F_i\) for \(j \geq t + 1\) and that \(T_i \subset A'_{t+1} \subset \cdots \subset A'_{k}\). Thus, (c) holds. \(\square\)

The rest of the results of this section are concerned with the structure of saturated \(k\)-Sperner systems. The next lemma, which is proved in [11], implies that for any saturated \(k\)-Sperner system there can be at most one homogeneous set. We include a proof for completeness.

**Lemma 9** (Gerbner et al. [11]). If \(F \subseteq \mathcal{P}(X)\) is a saturated \(k\)-Sperner system and \(H_1\) and \(H_2\) are homogeneous for \(F\), then \(H_1 = H_2\).

**Proof.** Suppose to the contrary that \(H_1\) and \(H_2\) are homogeneous for \(F\) and that \(H_1 \neq H_2\). Then, since each of \(H_1\) and \(H_2\) are maximal with respect to (1), we have that \(H_1 \cup H_2\) is not homogeneous for \(F\). Therefore, there is a set \(S \in F\) which contains some, but not all, of \(H_1 \cup H_2\). Without loss of generality, we have \(S \cap H_1 = H_1\) and \(S \cap H_2 = \emptyset\) since \(H_1\) and \(H_2\) are homogeneous for \(F\). Now, pick \(x \in H_1\) and \(y \in H_2\) arbitrarily and define 
\[
T := (S \setminus \{x\}) \cup \{y\}.
\]
Clearly \(T\) cannot be in \(F\) since \(T \cap H_1 = H_1 \setminus \{x\}\) and \(H_1\) is homogeneous for \(F\). Since \(F\) is saturated, there must exist sets \(A_1, \ldots, A_k \in F\) and \(t \in \{0, \ldots, k\}\) such that 
\[
A_1 \subset \cdots \subset A_t \subset T \subset A_{t+1} \subset \cdots \subset A_k.
\]
Since \(H_1\) and \(H_2\) are homogeneous for \(F\), and neither \(H_1\) nor \(H_2\) is contained in \(T\), we get that \(A_t \subset T \setminus (H_1 \cup H_2) \subset S\). Similarly, \(A_{t+1} \supset S\). However, this implies that \(\{A_1, \ldots, A_k\} \cup \{S\}\) is a \((k+1)\)-chain in \(F\), a contradiction. \(\square\)
By Lemma 9, if \( \mathcal{F} \) is a saturated \( k \)-Sperner system for which there exists a homogeneous set, then the homogeneous set must be unique. Throughout the paper, it will be useful to distinguish the elements of \( \mathcal{F} \) which contain the homogeneous set from those that do not.

**Definition 10.** Let \( \mathcal{F} \subseteq \mathcal{P}(X) \) be a saturated \( k \)-Sperner system and let \( H \) be homogeneous for \( \mathcal{F} \). We say that a set \( S \in \mathcal{F} \) is large if \( H \subseteq S \) or small if \( S \cap H = \emptyset \). Let \( \mathcal{F}^{\text{large}} \) and \( \mathcal{F}^{\text{small}} \) denote the collection of large and small sets of \( \mathcal{F} \), respectively. Thus, \( \mathcal{F} = \mathcal{F}^{\text{small}} \cup \mathcal{F}^{\text{large}} \).

**Lemma 11.** Let \( \mathcal{A} \subseteq \mathcal{P}(X) \) be a saturated antichain with homogeneous set \( H \). Then every set \( S \in \mathcal{P}(X) \setminus \mathcal{A} \) either contains a set in \( \mathcal{A}^{\text{small}} \) or is contained in a set of \( \mathcal{A}^{\text{large}} \).

**Proof.** Suppose, to the contrary, that \( S \in \mathcal{P}(X) \setminus \mathcal{A} \) does not contain a set of \( \mathcal{A}^{\text{small}} \) and is not contained in a set of \( \mathcal{A}^{\text{large}} \). Since \( \mathcal{A} \) is saturated, we get that either

(a) there exists \( A \in \mathcal{A}^{\text{large}} \) such that \( A \subsetneq S \), or

(b) there exists \( B \in \mathcal{A}^{\text{small}} \) such that \( S \subsetneq B \).

Suppose that (a) holds. Let \( y \in S \setminus A \) and \( x \in H \) and define \( T := (A \setminus \{x\}) \cup \{y\} \). Since \( H \) is homogeneous for \( \mathcal{A} \) and \( T \cap H = H \setminus \{x\} \), we must have \( T \notin \mathcal{A} \). Also, since \( H \) is homogeneous for \( \mathcal{A} \), any set \( T' \in \mathcal{A} \) containing \( T \) would have to contain \( T \cup \{x\} \supseteq A \) as well. Therefore, since \( \mathcal{A} \) is an antichain, no such set \( T' \) can exist. Thus, there is a set \( T'' \in \mathcal{A} \) such that \( T'' \subseteq T \subseteq S \). Since \( H \) is homogeneous for \( \mathcal{A} \) and \( T \cap H \neq H \), we get that \( T'' \in \mathcal{A}^{\text{small}} \), contradicting our assumption on \( S \).

Note that we are also done in the case that (b) holds by considering the saturated antichain \( \{X \setminus A : A \in \mathcal{A} \} \) and applying the argument of the previous paragraph. \( \square \)

### 2.1 Constructing and Decomposing Saturated \( k \)-Sperner Systems

There is a natural way to partition a \( k \)-Sperner system \( \mathcal{F} \subseteq \mathcal{P}(X) \) into a sequence of \( k \) pairwise disjoint antichains. Specifically, for \( 0 \leq i \leq k - 1 \), let \( \mathcal{A}_i \) be the collection of all minimal elements of \( \mathcal{F} \setminus \left( \bigcup_{j<i} \mathcal{A}_j \right) \) under inclusion. We say that \( (\mathcal{A}_i)_{i=0}^{k-1} \) is the **canonical decomposition** of \( \mathcal{F} \) into antichains.

In this section we provide conditions under which a sequence of \( k \) pairwise disjoint saturated antichains can be united to obtain a saturated \( k \)-Sperner system. Later we will prove a partial converse: if \( \mathcal{F} \subseteq \mathcal{P}(X) \) is a saturated \( k \)-Sperner system with a homogeneous set, then every antichain of the canonical decomposition of \( \mathcal{F} \) is saturated. We also provide an example which shows that this is not necessarily the case if we remove the condition that \( \mathcal{F} \) has a homogeneous set.

**Definition 12.** We say that a sequence \( (\mathcal{D}_i)_{i=0}^t \) of subsets of \( \mathcal{P}(X) \) is **layered** if, for \( 1 \leq i \leq t \), every \( D \in \mathcal{D}_i \) strictly contains some \( D' \in \mathcal{D}_{i-1} \) as a subset.
Note that the canonical decomposition of any set system is layered.

**Lemma 13.** If $(\mathcal{A}_i)_{i=0}^{k-1}$ is a layered sequence of pairwise disjoint saturated antichains, then every $A \in \mathcal{A}_i$ is strictly contained in some $B \in \mathcal{A}_{i+1}$

**Proof.** Let $A \in \mathcal{A}_i$. Since $\mathcal{A}_{i+1}$ is a saturated antichain disjoint from $\mathcal{A}_i$, there exists some $B \in \mathcal{A}_{i+1}$ such that either $B \not\subseteq A$ or $A \not\subseteq B$. In the latter case we are done, so suppose $B \subseteq A$. Since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, there exists some $A' \in \mathcal{A}_i$ such that $A' \not\subseteq B$. Hence we have $A' \not\subseteq B \subseteq A$, contradicting the fact that $\mathcal{A}_i$ is an antichain and completing the proof. \qed

**Lemma 14.** If $(\mathcal{A}_i)_{i=0}^{k-1}$ is a layered sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, then $F := \bigcup_{i=0}^{k-1} \mathcal{A}_i$ is a saturated $k$-Sperner system.

**Proof.** Clearly, $F$ is a $k$-Sperner system since $\mathcal{A}_0, \ldots, \mathcal{A}_{k-1}$ are antichains. Let $S \in \mathcal{P}(X) \setminus F$ be arbitrary and define $t = \max\{i : S \supseteq A_i\}$ for some $A_i \in \mathcal{A}_i$. If $t \geq 0$, then $S$ strictly contains some set $A_i \in \mathcal{A}_i$. As $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, for $0 \leq i \leq t-1$, there exist sets $A_i \in \mathcal{A}_i$ such that

$$A_0 \subsetneq \cdots \subsetneq A_t \subsetneq S.$$  

Now, if $t \geq k - 2$, then since $\mathcal{A}_{i+1}$ is a saturated antichain and $S$ does not contain a set of $\mathcal{A}_{t+1}$, there must exist $A_{t+1} \in \mathcal{A}_{t+1}$ such that $S \not\subseteq A_{t+1}$. By Lemma 13, we see that for $t + 2 \leq i \leq k - 1$ there exists $A_i \in \mathcal{A}_i$ such that

$$S \not\subseteq A_{t+1} \subsetneq \cdots \subsetneq A_{k-1}.$$  

Thus $\{A_0, \ldots, A_{k-1}\} \cup \{S\}$ is a $(k+1)$-chain, as desired. \qed

In Lemma 14, we require the sequence $(\mathcal{A}_i)_{i=0}^{k-1}$ of saturated antichains to be layered. As it turns out, if each antichain $\mathcal{A}_i$ has a homogeneous set, then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered.

**Lemma 15.** Let $(\mathcal{A}_i)_{i=0}^{k-1}$ be a sequence of pairwise disjoint saturated antichains in $\mathcal{P}(X)$, each of which has a homogeneous set. Then $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered if and only if $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered.

**Proof.** Suppose that $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered and, for some $i \geq 0$, let $A \in \mathcal{A}_i^{\text{small}}$ be arbitrary. We show that $A$ contains a set of $\mathcal{A}_i^{\text{large}}$. Otherwise, since $(\mathcal{A}_i)_{i=0}^{k-1}$ is layered, we get that there is some $B \in \mathcal{A}_i^{\text{large}}$ such that $B \not\subseteq A$. Therefore, since $\mathcal{A}_i$ is an antichain, $A$ cannot be contained in an element of $\mathcal{A}_i^{\text{large}}$. By Lemma 11 and the fact that $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ are disjoint, we get that $A$ contains a set of $\mathcal{A}_i^{\text{small}}$, as desired.

Now, suppose that $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered. Given $i \geq 0$ and $S \in \mathcal{A}_i^{\text{large}}$, we show that $S$ contains a set of $\mathcal{A}_i$, which will complete the proof. If not, then since $\mathcal{A}_i$ is saturated and disjoint from $\mathcal{A}_{i+1}$, there must exist $T \in \mathcal{A}_i$ such that $S \not\subseteq T$. Since $\mathcal{A}_{i+1}$ is an antichain, $S$ cannot be strictly contained in a set of $\mathcal{A}_{i+1}^{\text{large}}$, and so neither can $T$. Therefore, by Lemma 11, there is a set $A \in \mathcal{A}_i^{\text{small}}$ contained in $T$. However, since $(\mathcal{A}_i^{\text{small}})_{i=0}^{k-1}$ is layered, there exists $A' \in \mathcal{A}_i^{\text{small}}$ such that $A' \not\subseteq A$. But then, $A' \not\subseteq T$, which contradicts the assumption that $\mathcal{A}_i$ is an antichain. The result follows. \qed
It is natural to wonder whether a converse to Lemma 14 is true. That is: \textit{if }\mathcal{F} \textit{ is a saturated }k\text{-Sperner system, can we decompose }\mathcal{F} \textit{ into a layered sequence of }k\textit{ pairwise disjoint saturated antichains?} The following example shows that this is not always the case.

\textbf{Example 16.} Let }X := \{x_1, x_2, x_3\}, Y := \{y_1, y_2, y_3\} \textit{ and }Z := X \cup Y. \textit{ We define}

\[B_0 := \{\{x_i, x_j\} : i \neq j\} \cup \{\{x_i, y_i\} : i \in \{1, 2, 3\}\} \cup \{\{x_k, y_{j}, y_{j} : i, j, k \text{ distinct}\} \cup \{Y\},\]

\[B_1 := \{X, \{x_1, x_2, y_1\}, \{x_1, x_3, y_3\}, \{x_2, x_3, y_2\}, \{x_1, y_1, y_3\}, \{x_2, y_1, y_2\}, \{x_3, y_2, y_3\}, \{x_1, x_2, y_2, y_3\}, \{x_1, x_3, y_1, y_2\}, \{x_2, x_3, y_1, y_3\}\}.

Then }\mathcal{(B_i)}_{i=0}^{1} \textit{ is a layered sequence of disjoint antichains. In fact, }\mathcal{(B_i)}_{i=0}^{1} \textit{ is the canonical decomposition of }\mathcal{F} := B_0 \cup B_1. \textit{ Clearly }B_1 \textit{ is not saturated as }B_1 \cup \{Y\} \textit{ is an antichain. We claim that }\mathcal{F} \textit{ is a saturated }2\text{-Sperner system.}

Consider any }S \in \mathcal{P}(Z) \setminus \mathcal{F}. \textit{ We will show that }\mathcal{F} \cup \{S\} \textit{ contains a 3-chain. It is easy to check that every element of }B_0 \setminus \{Y\} \textit{ is contained in a set of }B_1. \textit{ Hence if }S \textit{ is contained in some set }B \in B_0 \setminus \{Y\}, \textit{ then }\mathcal{F} \cup \{S\} \textit{ contains a 3-chain. In particular, this completes the proof when }|S| \in \{0, 1, 2\}. \textit{ Similarly, since }\mathcal{(B_i)}_{i=0}^{1} \textit{ is layered, if }S \textit{ contains some set }B \in B_1, \textit{ then }\mathcal{F} \cup \{S\} \textit{ contains a 3-chain. Therefore, we are done if }|S| \in \{4, 5, 6\}.

It remains to consider the case that }|S| = 3. \textit{ Since }X, Y \in \mathcal{F}, \textit{ we must have }|S \cap Y| = 2, \textit{ or }|S \cap X| = 2. \textit{ If }|S \cap Y| = 2, \textit{ we have }S \in \{\{x_1, y_1, y_2\}, \{x_2, y_2, y_3\}, \{x_3, y_1, y_3\}\}. \textit{ This implies that }S \textit{ is contained in a set }B \in B_1 \textit{ and contains a set }B' \in B_0 \cap \mathcal{P}(X). \textit{ If }|S \cap X| = 2, \textit{ then }S \textit{ contains some set }\{x_i, x_j\} \in B_0. \textit{ Also, it is easily verified that }S \textit{ is contained in a set of }B_1. \textit{ Thus, }\mathcal{F} \textit{ is a saturated }2\text{-Sperner system.}

However, for saturated }k\text{-Sperner systems with a homogeneous set, the converse toLemma 14 does hold; we can partition }\mathcal{F} \textit{ into a layered sequence of }k\textit{ pairwise disjoint saturated antichains.}

\textbf{Lemma 17.} \textit{Let }\mathcal{F} \in \mathcal{P}(X) \textit{ be a saturated }k\text{-Sperner system with homogeneous set }H \textit{ and canonical decomposition }\mathcal{(A_i)}_{i=0}^{k-1}. \textit{ Then }A_i \textit{ is saturated for all }i.

\textbf{Proof.} \textit{Fix }i \textit{ and let }S \in \mathcal{P}(X) \setminus \mathcal{A}_i. \textit{ Let }x \in H \textit{ and define}

\[T := (S \setminus H) \cup \{x\}.

Then }T \notin \mathcal{F} \textit{ since }T \cap H = \{x\} \textit{ and }H \textit{ is homogeneous for }\mathcal{F}. \textit{ Therefore, there exists }\{A_0, \ldots, A_{k-1}\} \subseteq \mathcal{F} \textit{ and }t \in \{0, \ldots, k\} \textit{ such that}

\[A_0 \subseteq \cdots \subseteq A_t = T \subseteq A_{t+1} \subseteq \cdots \subseteq A_{k-1}.

By definition of the canonical decomposition, we must have }A_j \in \mathcal{A}_j \textit{ for all }j. \textit{ Also, since }H \textit{ is homogeneous for }\mathcal{F} \textit{ and }T \cap H \notin \{\emptyset, H\}, \textit{ we must have }A_{t+1} \subseteq T \setminus H \subseteq S \textit{ and }A_t \supseteq T \cup H \supseteq S. \textit{ Therefore,}

\[A_0 \subseteq \cdots \subseteq A_{t+1} \subseteq S \subseteq A_t \subseteq \cdots \subseteq A_{k-1}.

Since }S \notin \mathcal{A}_i, \textit{ we must have either }A_i \subseteq S \textit{ or }S \subseteq A_i \textit{ depending on whether or not }i < t. \textit{ Therefore, }\mathcal{A}_i \textit{ is saturated for all }i. \quad \square
3 Combining Saturated $k$-Sperner Systems

Our first goal in this section is to prove that, under certain conditions, a saturated $k_1$-Sperner system $\mathcal{F}_1 \subseteq \mathcal{P}(X_1)$ and a saturated $k_2$-Sperner system $\mathcal{F}_2 \subseteq \mathcal{P}(X_2)$ can be combined to yield a saturated $(k_1 + k_2 - 2)$-Sperner system in $\mathcal{P}(X_1 \cup X_2)$. We apply this result to prove Theorem 3. Afterwards, we prove that $\text{sat}(k) = 2^{k-1}$ for $k \leq 5$. We conclude the section with a proof of Theorem 4.

Lemma 18. Let $X_1$ and $X_2$ be disjoint sets. For $i \in \{1, 2\}$, let $\mathcal{F}_i \subseteq \mathcal{P}(X_i)$ be a saturated $k_i$-Sperner system which contains $\{\emptyset, X_i\}$ and let $H_i \subseteq X_i$ be homogeneous for $\mathcal{F}_i$. If $\mathcal{G}$ is the set system on $\mathcal{P}(X_1 \cup X_2)$ defined by

$$\mathcal{G} := \{A \cup B : A \in \mathcal{F}_1^{\text{small}}, B \in \mathcal{F}_2^{\text{small}}\} \cup \left\{S \cup T : S \in \mathcal{F}_1^{\text{large}}, T \in \mathcal{F}_2^{\text{large}}\right\},$$

then $\mathcal{G}$ is a saturated $(k_1 + k_2 - 2)$-Sperner system which contains $\{\emptyset, X_1 \cup X_2\}$ and $H_1 \cup H_2$ is homogeneous for $\mathcal{G}$.

Proof. It is clear that $\mathcal{G}$ contains $\{\emptyset, X_1 \cup X_2\}$ and that $H_1 \cup H_2$ is homogeneous for $\mathcal{G}$. We show that $\mathcal{G}$ is a saturated $(k_1 + k_2 - 2)$-Sperner system.

First, let us show that $\mathcal{G}$ does not contain a chain of length $k_1 + k_2 - 1$. Suppose that $\{A_1, \ldots, A_r\}$ is an $r$-chain in $\mathcal{G}$. We can assume that $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. Define

$$I_1 := \{i : A_i \cap X_1 \subsetneq A_{i+1} \cap X_1\}, \quad \text{and} \quad I_2 := \{i : A_i \cap X_2 \subsetneq A_{i+1} \cap X_2\}.$$

Clearly, $I_1 \cup I_2 = \{1, \ldots, r-1\}$. Also, for $i \in \{1, 2\}$, since $\mathcal{F}_i$ is a $k_i$-Sperner system, we must have $|I_i| \leq k_i - 1$. Let $t$ be the maximum index such that $A_t \cap X_1 \in \mathcal{F}_1^{\text{small}}$. Note that $t$ exists and is less than $r$ since $A_1 = \emptyset$ and $A_r = X_1 \cup X_2$. By construction of $\mathcal{G}$, $A_t \cap X_2$ is a small set for $\mathcal{F}_2$ and, for $i \in \{1, 2\}$, $A_{t+1} \cap X_i$ is a large set for $\mathcal{F}_i$. This implies that $t \in I_1 \cap I_2$ and so

$$r - 1 = |I_1 \cap I_2| = |I_t| = |I_1| + |I_2| - |I_1 \cap I_2| \leq k_1 + k_2 - 3$$

as required.

Now, let $S \subseteq \mathcal{P}(X_1 \cup X_2) \setminus \mathcal{G}$. We show that $\mathcal{G} \cup \{S\}$ contains a $(k_1 + k_2 - 1)$-chain. Fix $x_1 \in H_1$ and $x_2 \in H_2$ and define

$$T := (S \setminus (H_1 \cup H_2)) \cup \{x_1, x_2\}.$$

For $i \in \{1, 2\}$, let $T_i := T \cap X_i$. Then $T_i \notin \mathcal{F}_i$ since $T_i \cap H_i = \{x_i\}$. Therefore, there exists $A^1_{t_i}, \ldots, A^1_{k_i} \in \mathcal{F}_i$ and $t_i \in \{1, \ldots, k_i - 1\}$ such that

$$\emptyset = A^1_{t_i} \subsetneq \cdots \subsetneq A^1_{t_j} \subsetneq T_i \subsetneq A^1_{t_{i+1}} \subsetneq \cdots \subsetneq A^1_{k_i} = X_i$$

Note that $A^j_{t_i} \in \mathcal{F}_i^{\text{small}}$ for $j \leq t_i$ and $A^j_{t_i} \in \mathcal{F}_i^{\text{large}}$ for $j \geq t_i + 1$. This implies that $A^1_{t_i} \cup A^2_{t_j} \subseteq S$ and $A^1_{t_{i+1}} \cup A^2_{t_{j+1}} \supseteq S$. Therefore,

$$A^1_{t_i} \cup A^2_{t_j} \subseteq A^2_{t_{j+1}} \subseteq \cdots \subseteq A^1_{t_{i+1}} \cup A^2_{t_{j+1}} \subseteq A^2_{t_{j+2}} \subseteq \cdots \subseteq A^i_{t_{i+1}} \cup A^2_{t_{j+2}} \subseteq S.$$
Proof. Let $F \subseteq P$ (Proposition 20).

For any set $X$ exhibit an infinite family of saturated $6$-Sperner systems with cardinality $30 < 25$. We apply Lemma 18 to prove Theorem 3. The first part of the proof of Theorem 3 is to

3.1 Proof of Theorem 3

We apply Lemma 18 to prove Theorem 3. The first part of the proof of Theorem 3 is to exhibit an infinite family of saturated $6$-Sperner systems with cardinality $30 < 25$.

Proposition 20. For any set $X$ such that $|X| \geq 8$, there is a saturated $6$-Sperner system $F \subseteq P(X)$ with a homogeneous set such that $|F^{\text{small}}| = |F^{\text{large}}| = 15$.

Proof. Let $X$ be a set such that $|X| \geq 8$. Let $x_1, x_2, y_1, y_2, w$ and $z$ be distinct elements of $X$ and define $H := X \setminus \{x_1, x_2, y_1, y_2, w, z\}$. We apply Lemma 14 to construct a saturated $6$-Sperner system $F \subseteq P(X)$ of order $30$. Naturally, we define $A_0 = \{\emptyset\}$ and $A_3 := \{X\}$. Also, define

\[
A_1 := \{\{x_1\}, \{x_2\}, \{y_1\}, \{w\}, H \cup \{y_2, z\}\}, \quad \text{and} \quad A_4 := \{X \setminus A : A \in A_1\}.
\]

It is easily observed that $A_1$ and $A_4$ are saturated antichains. We define $A_2$ and $A_3$ by first specifying their small sets. Define

\[
A_2^{\text{small}} := \{\{x_i, y_j\} : 1 \leq i, j \leq 2\} \cup \{\{w, z\}\}, \quad \text{and} \quad A_3^{\text{small}} := \{\{x_1, y_1, w\}, \{x_1, y_1, z\}, \{x_2, y_2, w\}, \{x_2, y_2, z\}\}.
\]

Given any collection $B \subseteq P(X)$, a set $S \subseteq X$ is said to be stable for $B$ if $S$ does not contain an element of $B$. For $i = 2, 3$, define $A_i^{\text{large}}$ to be the collection consisting of all maximal stable sets of $A_i^{\text{small}}$ and let $A_i := A_i^{\text{small}} \cup A_i^{\text{large}}$. Note that every element of $A_i^{\text{large}}$ contains $H$. It is clear that $A_i$ is an antichain for $i = 2, 3$. Moreover, $A_i$ is saturated since every set $A \in P(X)$ either contains an element of $A_i^{\text{small}}$ or is contained in an element of $A_i^{\text{large}}$.

One can easily verify that $(A_i^{\text{small}})^5_{i=0}$ is layered. Therefore, by Lemma 15, $(A_i)^5_{i=0}$ is a layered sequence of pairwise disjoint saturated antichains. By Lemma 14, $F := \bigcup^5_{i=0} A_i$ is a saturated $6$-Sperner system. Also,

\[
|F^{\text{small}}| = \sum^5_{i=0} |A_i^{\text{small}}| = (1 + 5 + 9 + 0) = 15, \quad \text{and} \quad |F^{\text{large}}| = \sum^5_{i=0} |A_i^{\text{large}}| = (0 + 9 + 5 + 1) = 15,
\]

as desired. \(\square\)

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We remark that the construction in Proposition 20 is similar to one which was used in [11] to prove that \( \text{sat}(k, k) \leq \frac{15}{16} 2^{k-1} \) for every \( k \geq 6 \).

For the proof of Theorem 3 we require that
\[
\text{sat}(k) \leq 2 \text{sat}(k - 1). \tag{2}
\]
This was proved in [11] using the fact that if \( F \subseteq \mathcal{P}(X) \) is a saturated \((k - 1)\)-Sperner system and \( y \notin X \), then \( F \cup \{ A \cup \{ y \} : A \in F \} \) is a saturated \( k \)-Sperner system in \( \mathcal{P}(X \cup \{ y \}) \).

**Proof of Theorem 3.** First, we prove that the result holds when \( k \) is of the form \( 4j + 2 \) for some \( j \geq 1 \). In this case, we repeatedly apply Lemma 18 and Proposition 20 to obtain a saturated \( k \)-Sperner system \( F \) on an arbitrarily large ground set \( X \) such that
\[
\left| F_{\text{small}} \right| + \left| F_{\text{large}} \right| = 15^j + 15^j = 2 \cdot 15^j.
\]
Therefore, if \( k = 4j + 2 \), then \( \text{sat}(k) \leq 2 \cdot 15^j \).

For \( k \) of the form \( 4j + 2 + s \) for \( j \geq 1 \) and \( 1 \leq s \leq 3 \), apply (2) to obtain \( \text{sat}(k) \leq 2^s \text{sat}(4j + 2) \leq 2^{s+1} \cdot 15^j \). Thus, we are done by setting \( \varepsilon \) slightly smaller than \( (1 - \log_2(15) \cdot \frac{1}{4}) \).

### 3.2 Bounding \( \text{sat}(k) \) From Below

One can easily deduce from the proof of Theorem 3 that \( \text{sat}(k) < 2^{k-1} \) for all \( k \geq 6 \). For completeness, we prove that \( \text{sat}(k) = 2^{k-1} \) for \( k \leq 5 \).

**Proposition 21.** If \( k \leq 5 \), then \( \text{sat}(k) = 2^{k-1} \).

**Proof.** Fix \( k \leq 5 \). The upper bound follows from Construction 2, and so it suffices to prove that \( \text{sat}(k) \geq 2^{k-1} \). Let \( X \) be a set with \( n := |X| > 2^{k-1} \) and let \( F \subseteq \mathcal{P}(X) \) be a saturated \( k \)-Sperner system of minimum order. By Claim 8 and the fact that \( |X| > 2^{k-1} \geq |F| \), there is a homogeneous set \( H \) for \( F \).

Let \( (A_i)_{i=0}^{k-1} \) be the canonical decomposition of \( F \). By Lemma 17, we get that \( A_i \) is a saturated antichain for each \( i \). Also, since \( (A_i)_{i=0}^{k-1} \) is layered, by Lemma 13 we see that every element of \( A_i \) has cardinality between \( i \) and \( n - k + i + 1 \).

Our goal is to show that for \( k \leq 5 \), every saturated antichain \( A_i \) which satisfies (3) must contain at least \( \binom{k-1}{i} \) elements. Clearly this is enough to complete the proof of the proposition. Note that it suffices to prove this for \( i < \frac{k}{2} \) since \( \{ X \setminus A : A \in A_i \} \) is a saturated antichain in which every set has size between \( k - i - 1 \) and \( n - i \). Since \( k \leq 5 \), this means that we need only check the cases \( i = 0, 1, 2 \). In the case \( i = 0 \), we obtain \( |A_0| \geq 1 = \binom{k-1}{0} \) trivially.

Next, consider the case \( i = 1 \). Let \( A \) be the largest set in \( A_i \) such that \( H \subseteq A \). Then, by (3), we must have \( |A| \leq n - k + 2 \) and so \( |X \setminus A| \geq k - 2 \). Fix an element \( x \) of \( H \) and, for each
$y \in X \setminus A$, define $A_y := (A \setminus \{x\}) \cup \{y\}$. Since $A_i$ is saturated, $H$ is homogeneous for $F$, and $A$ is the largest set in $A_1$ containing $H$, there must be a set $B_y \in A_1$ such that $B_y \subseteq A_y$.

However, since $A_1$ is an antichain, $B_y \not\subseteq A$, and so $B_y \setminus A = \{y\}$. In particular, $B_y \not= B'_y$ for $y \not= y'$. Therefore, $|A_1| \geq |\{A \cup \{B_y : y \in X \setminus A\}| \geq 1 + |X \setminus A| \geq k - 1 = \binom{k-1}{1}$, as desired.

Thus, we are finished except for the case $i = 2$ and $k = 5$. Suppose to the contrary that $|A_2| < \binom{5}{2} = 6$. We begin by proving the following claim.

**Claim 22.** For every vertex $y \in X \setminus H$, there is a set $S_y \in \mathcal{A}_2^{\text{large}}$ containing $y$.

**Proof.** Let $x \in H$ be arbitrary and consider the set $T := \{x, y\}$. Then $T$ is not contained in $\mathcal{A}_2$ since $H$ is homogeneous for $F$. Also, no strict subset of $T$ is in $\mathcal{A}_2$ by (3). Since $\mathcal{A}_2$ is saturated, there must be some $S_y \in \mathcal{A}_2^{\text{large}}$ containing $T$, which completes the proof. □

Let us argue that $|\mathcal{A}_2^{\text{large}}| \geq 2$. By (3), each set $A \in \mathcal{A}_2^{\text{large}}$ has at most $n - 2$ elements. So, by Claim 22, if $|\mathcal{A}_2^{\text{large}}| < 3$, then it must be the case that $\mathcal{A}_2^{\text{large}} = \{A_1, A_2\}$ where $A_1 \cup A_2 = X$. Therefore, since each of $|A_1|$ and $|A_2|$ is at most $n - 2$, we can pick $\{w_1, w_2\} \subseteq A_1 \setminus A_2$ and $\{z_1, z_2\} \subseteq A_2 \setminus A_1$. Given $x \in H$ and $1 \leq i, j \leq 2$, we have that $\{x, w_i, z_j\} \not\in \mathcal{A}_2$ since $H$ is homogeneous for $F$. Note that $\{x, w_i, z_j\}$ is not contained in either $A_1$ or $A_2$, and so by Lemma 11 and (3) we must have $\{w_i, z_j\} \in \mathcal{A}_2$. However, this implies that $|A_2| \geq |\{\{w_i, z_j\} : 1 \leq i, j \leq 2\} \cup \{A_1, A_2\}| = 6$, a contradiction.

So, we get that $|\mathcal{A}_2^{\text{large}}| \geq 2$. Note that $\{X \setminus A : A \in \mathcal{A}_2\}$ is also a saturated antichain in which every set has cardinality between 2 and $n - 2$. Thus, we can apply the argument of the previous paragraph to obtain $|\mathcal{A}_2^{\text{small}}| \geq 3$. Therefore, $|\mathcal{A}_2| = |\mathcal{A}_2^{\text{small}}| + |\mathcal{A}_2^{\text{large}}| \geq 6$, which completes the proof. □

It is possible that a similar approach may prove fruitful for improving the lower bound on sat($k$) from $2^{k/2-1}$ to $2^{(1+o(1))ck}$ for some $c > 1/2$. That is, one may first decompose a saturated $k$-Sperner system $F \subseteq \mathcal{P}(X)$ of minimum size into its canonical decomposition $(\mathcal{A}_i)_{i=0}^{k-1}$ and then bound the size of $|\mathcal{A}_i|$ for each $i$ individually. Since there are only $k$ antichains in the decomposition and the bound on $|F|$ that we are aiming for is exponential in $k$, one could obtain a fairly tight lower bound on sat($k$) by focusing on a single antichain of the decomposition. Setting $i = \lceil \frac{k}{2} \rceil$ in (3), we see that it would be sufficient to prove that there exists $c > 1/2$ such that every saturated antichain $\mathcal{A}$ with a homogeneous set such that every element of $\mathcal{A}$ has cardinality between $\lfloor \frac{k}{2} \rfloor$ and $n - \lceil \frac{k}{2} \rceil + 1$ must satisfy $|\mathcal{A}| \geq 2^{(1+o(1))ck}$. The problem of determining whether such a $c$ exists is interesting in its right.

### 3.3 Asymptotic Behaviour of sat($k$)

To prove Theorem 4, we require the following fact, which is proved in [11].

**Lemma 23** (Gerbner et al. [11]). For any $n \geq k \geq 1$ and set $X$ with $|X| = n$ there is a saturated $k$-Sperner system $F \subseteq \mathcal{P}(X)$ such that $|F| = \text{sat}(n, k)$ and $\emptyset, X \subseteq F$. 

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Proof. Let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a saturated $k$-Sperner system such that $|\mathcal{F}| = \text{sat}(n, k)$. We let $(\mathcal{A}_i)_{i=0}^{k-1}$ denote the canonical decomposition of $\mathcal{F}$ and define

$$\mathcal{F}' := (\mathcal{F} \setminus (\mathcal{A}_0 \cup \mathcal{A}_{k-1})) \cup \{\emptyset, X\}.$$ 

It is clear that $\mathcal{F}' \subseteq \mathcal{P}(X)$ is a saturated $k$-Sperner system and $|\mathcal{F}'| \leq |\mathcal{F}| = \text{sat}(n, k)$, which proves the result. \hfill $\square$

Proof of Theorem 4. We show that, for all $k, \ell$,

$$\text{sat}(k + \ell) \leq 4 \text{sat}(k) \text{sat}(\ell). \quad (4)$$

Letting $f(k) := 4 \text{sat}(k)$, we see that (4) implies that $f(k + \ell) \leq f(k)f(\ell)$ for every $k, \ell$. It follows by Fekete’s Lemma that $f(k)^{1/k}$ converges, and so $\text{sat}(k)^{1/k}$ converges as well.

For $n > 2^{2k+\ell-2}$, let $X$ and $Y$ be disjoint sets of size $n$ and let $\mathcal{F}_k \subseteq \mathcal{P}(X)$ and $\mathcal{F}_\ell \subseteq \mathcal{P}(Y)$ be saturated $k$-Sperner and $\ell$-Sperner systems of cardinalities $\text{sat}(k)$ and $\text{sat}(\ell)$, respectively. By Claim 8, we can assume that $\mathcal{F}_k$ and $\mathcal{F}_\ell$ have homogeneous sets and, by Lemma 23, we can assume that $\{\emptyset, X\} \subseteq \mathcal{F}_k$ and $\{\emptyset, Y\} \subseteq \mathcal{F}_\ell$. We apply Lemma 18 and Remark 19 to obtain a saturated $(k+\ell-2)$-Sperner system $\mathcal{G} \subseteq \mathcal{P}(X \cup Y)$ of order at most $|\mathcal{F}_k||\mathcal{F}_\ell| = \text{sat}(k)\text{sat}(\ell)$. Therefore, by (2), we have

$$\text{sat}(k + \ell) \leq 4 \text{sat}(k + \ell - 2) \leq 4|\mathcal{G}| \leq 4 \text{sat}(k) \text{sat}(\ell)$$

as required. \hfill $\square$

4 Oversaturated $k$-Sperner Systems

In this section we construct oversaturated $k$-Sperner systems of small order. We first state a lemma, from which Theorem 6 follows, and then prove the lemma itself.

Lemma 24. Given $k \geq 1$, let $X$ be a set of cardinality $k^2 + k$. Then for all $t$ such that $1 \leq t \leq k^2 + k$ there exist non-empty collections $\mathcal{F}_t, \mathcal{G}_t \subseteq \mathcal{P}(X)$ that have the following properties:

(a) For every $F \in \mathcal{F}_t$ and $G \in \mathcal{G}_t$, $|F| + |G| \geq k$,

(b) $|\mathcal{F}_t| + |\mathcal{G}_t| = O(k^2 2^{k^2/2})$,

(c) For every $S \subseteq X$ such that $|S| = t$, there exists some $F \in \mathcal{F}_t$ and some $G \in \mathcal{G}_t$ such that $F \subseteq S$ and $G \cap S = \emptyset$.

We apply Lemma 24 to prove Theorem 6.

Proof of Theorem 6. First, let $X$ be a set of cardinality $k^2 + k$. For $t \in \{1, \ldots, k^2 + k\}$, let $\mathcal{F}_t$ and $\mathcal{G}_t$ be as in Lemma 24. For each $F \in \mathcal{F}_t \cup \mathcal{G}_t$, choose $F_1, \ldots, F_i \in \mathcal{P}(X)$ such that

$$F_1 \subsetneq \cdots \subsetneq F_i \subsetneq F$$
follows.

and replacing is a \((k+1)\)-chain in \(G' \cap \{S\}\) containing \(S\), which will imply that \(G\) is an oversaturated \(k\)-Sperner system. Let \(S \subseteq X\) and define \(t := |S|\). By Property (c) of Lemma 24, there exists \(F \in \mathcal{F}_t\) such that \(F \subseteq S\) and \(G \in \mathcal{G}_t\) such that \(G \cap S = \emptyset\). This implies that \(S \subseteq X \setminus G\). By Property (a) of Lemma 24 we get that

\[
\mathcal{C}_F \cup \{X \setminus T : T \in \mathcal{C}_G\} \cup \{S\}
\]

contains a \((k+1)\)-chain in \(G \cup \{S\}\) containing \(S\).

Now, suppose that \(|X| > k^2 + k\). Let \(Y \subseteq X\) such that \(|Y| = k^2 + k\) and define \(H := X \setminus Y\). As above, let \(G \subseteq \mathcal{P}(Y)\) be an oversaturated \(k\)-Sperner system of cardinality at most \(O\left(k^5 2^{k/2}\right)\). Define \(G' \subseteq \mathcal{P}(X)\) as follows:

\[
G' := \{T : T \in G\} \cup \{T \cup H : T \in G\}.
\]

Consider any set \(S \in \mathcal{P}(X) \setminus G'\). Let \(S' = S \cap Y\). We have, by definition of \(G\), that there is a \((k+1)\)-chain \(C\) in \(G \cup \{S'\}\) containing \(S'\). Adding \(H\) to every superset of \(S'\) in \(C\) and replacing \(S'\) by \(S\) in \(C\) gives us a \((k+1)\)-chain in \(G' \cup \{S\}\) containing \(S\). The result follows.

To prove Lemma 24, we use a probabilistic approach.

**Proof of Lemma 24.** Throughout the proof, we assume that \(k\) is sufficiently large and let \(X\) be a set of cardinality \(k^2 + k\). Let \(1 \leq t \leq k^2 + k\) be given. We can assume that \(t \leq \frac{k^2 + k}{2}\) since, otherwise, we can simply define \(\mathcal{F}_t := \mathcal{G}_{k^2 + k - t}\) and \(\mathcal{G}_t := \mathcal{F}_{k^2 + k - t}\). We divide the proof into two cases depending on the size of \(t\).

**Case 1:** \(t \leq \frac{k^2 + k}{8}\).

We define \(\mathcal{F}_t := \{\emptyset\}\) and let \(\mathcal{G}_t\) be a uniformly random collection of \(2^{k/2}\) subsets of \(X\), each of cardinality \(k\). Given \(S \subseteq X\) of cardinality \(t\), the probability that \(S\) is not disjoint from any set of \(\mathcal{G}_t\) is

\[
\left(1 - \prod_{i=0}^{k-1} \left(\frac{k^2 + k - t - i}{k^2 + k - i}\right)\right)^{2^{k/2}} \leq \left(1 - \left(\frac{k^2 - t}{k^2}\right)^k\right)^{2^{k/2}} \leq \left(1 - \left(\frac{7}{8} - \frac{1}{8k}\right)^k\right)^{2^{k/2}} \leq e^{-\left(\frac{7}{8} - \frac{1}{8}\right)^{k} 2^{k/2}} < e^{-(1.1)^k}.
\]
Therefore, the expected number of subsets of $X$ of cardinality $t$ which are not disjoint from any set of $G_t$ is at most $(k^2 + k) e^{-(1.1)^k}$, which is less than 1. Thus, with non-zero probability, every $S \subseteq X$ of cardinality $t$ is disjoint from some set in $G_t$.

Case 2: $k^2 + k < t \leq \frac{k^2 + k}{2}$.

Define $p := \frac{t}{k^2 + k}$ and let $a$ be the rational number such that $ak = \left\lfloor -\frac{k \log \sqrt{2}}{\log(p)} + 1 \right\rfloor$. Then, since $\frac{1}{8} \leq p \leq \frac{1}{2}$, we have

$$1/6 \leq a \leq 1/2 + 1/k < 4/7. \quad (5)$$

Now, let $F_t$ be a collection of $\left[8e^8k^22^{k/2}\right]$ subsets of $X$, each of cardinality $ak$, chosen uniformly at random with replacement. Similarly, let $G_t$ be a collection of $\left[e^2k^22^{k/2}\right]$ subsets of $X$, each of cardinality $(1 - a)k$, chosen uniformly at random with replacement. We show that, with non-zero probability, every $S \subseteq X$ of size $t$ contains a set of $F_t$ and is disjoint from a set of $G_t$.

Given $S \subseteq X$ of size $t = p(k^2 + k)$, the probability that $S$ does not contain a set of $F_t$ is at most

$$\left(1 - \prod_{i=0}^{ak-1} \left(\frac{p(k^2 + k) - i}{k^2 + k - i}\right)^{|F_t|}\right) \leq \left(1 - \left(\frac{p(k^2 + k) - ak}{k^2}\right)^{|F_t|}\right)^{|F_t|} = \left(1 - \left(1 - \frac{1 - p}{pk}\right)^{ak}\right)^{|F_t|}. \quad (6)\right.$$  

Observe that $\left(1 - \frac{1 - p}{pk}\right) \geq e^{-\frac{2(1-p)}{pk}}$ for large enough $k$. So, $\left(1 - \frac{1 - p}{pk}\right)^{ak} \geq e^{-2a(1-p)}$ which is at least $e^{-8}$ since $a < 4/7$ and $a \geq 1/8$. Thus, the expression in (6) is at most

$$\left(1 - e^{-8}p^{ak}\right)^{|F_t|} \leq e^{-e^{-8}p^{ak}|F_t|} \leq e^{-e^{-8}p^{ak}(8e^8k^22^{k/2})} = e^{-p^{ak}8k^22^{k/2}}.$$  

Using our choice of $a$ and the fact that $p \geq 1/8$, we can bound the exponent by

$$p^{ak}8k^22^{k/2} \geq p\left(\frac{\log \sqrt{2}}{\log(1-p)} + 1\right)k8k^22^{k/2} = p8k^2 \geq k^3.$$  

Therefore, the expected number of subsets of $X$ of size $t$ which do not contain a set of $F_t$ is at most

$$\left(k^2 + k\right) t e^{-k^2} \leq 2k^2 e^{-k^2}$$  

which is less than 1. Thus, with positive probability, every subset of $X$ of cardinality $t$ contains a set of $F_t$.

The proof that, with positive probability, every set of cardinality $t$ is disjoint from a set of $G_t$ is similar; we sketch the details. First, let us note that

$$a \geq \frac{-\log \sqrt{2}}{\log(p)} \geq 1 + \frac{\log \sqrt{2}}{\log(1 - p)}. \quad (7)$$

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since \( p \leq 1/2 \). For a fixed set \( S \subseteq X \) of size \( t = p(k^2 + k) \), the probability that \( S \) is not disjoint from any set of \( G_i \) is at most

\[
\left( 1 - \prod_{i=0}^{(1-a)k-1} \left( \frac{1-p)(k^2 + k - i)}{k^2 + k - i} \right) \right)^{|G_i|} \leq \left( 1 - \left( \frac{(1-p)(k^2 + k - k)}{k^2} \right)^{(1-a)k} \right)^{|G_i|} \leq \left( 1 - \frac{p}{(1-p)k} \right)^{(1-a)k} \left( 1 - p \right)^{(1-a)k} \right)^{|G_i|} \tag{8}
\]

Now, \( \left( 1 - \frac{p}{(1-p)k} \right) \geq e^{-\frac{2}{1-p}k} \) for large enough \( k \). So, \( \left( 1 - \frac{p}{(1-p)k} \right)^{(1-a)k} \geq e^{-\frac{2}{(1-p)}k(1-a)p} \), which is at least \( e^{-2} \) since \( a \geq 1/6 \) and \( \frac{1}{8} \leq p \leq \frac{1}{2} \). Therefore, the expression in (8) is at most

\[
(1 - e^{-2}(1-p)^{(1-a)k})^{\left| G_i \right|} \leq e^{-e^{-2}(1-p)^{(1-a)k} \left| G_i \right|} \leq e^{-e^{-2}(1-p)^{(1-a)k}(e^{2k^2k/2})} = e^{-(1-p)^{(1-a)k}k^2k/2}.
\]

By (7), we can bound the exponent by

\[
(1 - p)^{(1-a)k}k^2k/2 \geq (1 - p)^{(\frac{\log a/2}{\log (1-p)})k^2k/2} \geq k^2.
\]

As with \( \mathcal{F}_t \), we get that the expected number of sets of cardinality \( t \) which are not disjoint from a set of \( G_i \) is less than one. The result follows. \( \square \)

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**References**


