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# Approximate Solutions for Expanding Search Games on General Networks

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## Abstract

We study the classical problem introduced by R. Isaacs and S. Gal of minimizing the time to find a hidden point  $H$  on a network  $Q$  moving from a known starting point. Rather than adopting the traditional continuous unit speed path paradigm, we use the dynamic “expanding search” paradigm recently introduced by the authors. Here the regions  $S(t)$  that have been searched by time  $t$  are increasing from the starting point and have total length  $t$ . Roughly speaking the search follows a sequence of arcs  $a_i$  such that each one starts at some point of an earlier one. This type of search is often carried out by real life search teams in the hunt for missing persons, escaped convicts, terrorists or lost airplanes. The paper which introduced this type of search solved the adversarial problem (where  $H$  is hidden to maximize the time to be found) for the cases where  $Q$  is a tree or is 2-arc-connected. This paper’s main contribution is to give two strategy classes which can be used on any network and have expected search times which are within a factor close to 1 of the value of the game (minimax search time). These strategies classes are respectively optimal for trees and 2-arc-connected networks. We also solve the game for circle-and-spike networks, which can be considered as the simplest class of networks for which a solution was previously unknown.

**Keywords.** search games, zero-sum games, networks, defense and security

## 1 Introduction

The problem of optimal search for a stationary hidden object  $H$  (a *Hider*) on a network with given arc lengths goes back to the early work of Isaacs (1965) and Gal (1979). In the traditional version, the Searcher proceeds continuously and at unit speed from a known starting point  $O$  on the network  $Q$  until reaching  $H$  at some time  $T$ . The aim is find a randomized search that minimizes this time  $T$ , the *search time*, in the worst case. An equivalent approach, which we prefer to adopt, is to study a zero-sum dynamic game with payoff  $T$  between a minimizing Searcher and a maximizing Hider. Mixed strategies are required by both players. This article adopts an alternative search paradigm recently introduced by the authors in

Alpern and Lidbetter (2013) as “expanding search”, because the regions  $S(t)$  that have been searched by time  $t$  “expand” continuously over time. This models the actual searches that are often carried out to search for missing persons, escaped convicts, terrorists or lost airplanes.

More formally, the sets  $S(t)$ , which form a pure strategy for the searcher, start with  $S(0) = O$  and expand at unit rate of total arc length  $\lambda$ , so that  $\lambda(S(t)) = t$ . The search plan finishes at time  $\mu = \lambda(Q)$  (thus  $\mu$  is the total length of the network) because no backtracking is required, unlike the case for pathwise search. The actual search of course finishes at the capture time  $T$ , when the search region  $S(T)$  first contains  $H$ . As with pathwise search, this produces a zero-sum game with payoff  $T$ , which we call the *expanding search game*. We concentrate on expanding searches that have a simpler structure, namely, they consist of a sequence of arcs  $a_i$ ,  $i = 1, \dots, k$ , which cover  $Q$  and where the back of each arc  $a_i$  lies in one of the earlier arcs. (The direction of each arc is not prescribed in  $Q$ , which is an undirected network, but simply denotes the chosen direction of search.) Depth-first expanding searches of trees are closely related to pathwise searches of *windy* networks, as introduced by Alpern (2010) and Alpern and Lidbetter (2014), by taking the time to traverse an arc in the direction of the start node as 0.

This paper is concerned with searching for a single object. However, the expanding search game has also been studied by Lidbetter (2013a) for the case that an adversary hides *multiple* objects. In particular, the game was solved for the cases of star networks and Eulerian networks. Other approaches to the search for multiple hidings are given by Wenk (2015) (identity search) and Chapman *et al.* (2014) (agent based model).

In Alpern and Lidbetter (2013) it was explained how the notion of expanding search is also applicable to non-search problems such as mining in which the cost of moving excavation equipment through already mined trails is negligible compared to the cost of moving the equipment by excavating new areas. Minimizing the cost of reaching (finding) a randomly chosen piece of unmined coal is equivalent to minimizing the mean time that kilograms of coal are mined and ready for sale.

## 1.1 Main results

The expanding search game on a network was introduced by Alpern and Lidbetter (2013), but that paper was primarily concerned with the Bayesian problem of minimizing the expected capture time using expanding search against a *known* hider distribution. The game theoretic version only considered two special types of networks: trees and 2-arc-connected networks. Here we consider general networks.

An important graph theoretic tool that we will use is the so called *bridge-block decomposition*. The arcs of a network  $Q$  can be partitioned into *bridges* (arcs whose removal disconnects the network) and *blocks* (the connected components of what remains after the bridges are removed). When the blocks are each shrunk to a point, the bridges form a tree  $Q^t$ . We use the parameter  $r$  to denote the fraction of the total length of  $Q$  that is taken up by bridges, so that when  $r = 0$ , the network is 2-arc-connected and when  $r = 1$  the network is a tree. We call  $r$  the *bridge ratio*.

We present two new classes of mixed search strategies. In Section 6 we present the *block-optimal search strategy*, denoted  $\beta$ , which is optimal when there are no bridges. We show in Theorem 13 that for any network,  $T(\beta) \leq ((1 + \sqrt{2})/2)V(Q)$ , where the value  $V(Q)$  is the minimax search time. Note that  $(1 + \sqrt{2})/2 \simeq 1.2$ . We also show that for a network with bridge ratio  $r$ , the expected search time  $T(\beta)$  of the block-optimal strategy satisfies the inequality  $T(\beta) \leq (1 + r)V(Q)$ , so that it performs well for “block-like” networks.

In Section 7 we present the *bridge-optimal search strategy*, denoted  $\gamma$ , which is optimal when there

are no blocks (so  $Q$  is a tree), and is based on depth-first pure searches. Theorem 14 shows that for any network  $Q$  we have  $T(\gamma) \leq (2/(1+r^2))V(Q)$ . This estimate is useful when  $r$  is close to 1, so that  $Q$  is “tree-like”.

Section 2 gives a formal definition of the expanding search game. Section 3 presents results needed later on the optimal expanding search against a *known* hider distribution and begins an analysis of the *circle-with-spike network*. Section 4 reviews earlier results from Alpern and Lidbetter (2013) on the expanding search game when the network is a tree or 2-arc-connected. Section 5 completes the analysis of the *circle-with-spike network* which we began in Section 3. Sections 6 and 7 give our main results which we have already discussed. Section 8 contains our conclusions and suggestions for future work.

## 1.2 Related literature on network search games

The use of pathwise search to minimize the time to find a hidden object on a network was first proposed by Isaacs (1965). Subsequently Gal (1979) analyzed such game on general networks and gave a complete solution for trees. For trees and Eulerian networks the so-called Random Chinese Postman Tour (RCPT), consisting of an equiprobable mixture of a minimal length (Chinese Postman) tour and its time-reversed tour, is an optimal mixed strategy. Reijnierse and Potters (1993) identified a larger class (including trees) of networks where the RCPT is optimal. Gal (2000) then showed that the largest class of networks where RCPT is optimal are the weakly Eulerian networks. The difficult (and non-weakly Eulerian) network consisting of two nodes (one the start node) connected by an odd number of equal length arcs was solved by Pavlovic (1993).

In addition to generalizing the classes of networks which could be solved, other variations on the basic model have been proposed. An algorithmic approach to the problem was given by Anderson and Aramendia (1990). Search games on windy networks, where the times to traverse arcs from either direction need not be the same, were introduced by Alpern (2010) and further studied by Alpern and Lidbetter (2014). The assumption that the game ends not when the hider is found, but when he is brought back to the start node, is studied in Alpern (2011a). Search games on lattices were studied by Zoroa *et al* (2013). The possibility that the Searcher might have to choose between two or more speeds of search was considered by Alpern and Lidbetter (2015). The case where simply reaching the hidden object is not enough to find it is considered by Baston and Kikuta (2015), who posit a cost for searching, in addition to traveling. Surveys of search games on networks can be found in Garnae (2000), Gal (2006), Alpern (2011b), Lidbetter (2013b) and Hohzaki (2016). Network search is also related to patrolling a network to guard against an attack, as in Alpern *et al.* (2011) and Lin *et al.* (2013), for example.

The expanding search paradigm, as introduced by the authors in Alpern and Lidbetter (2013), has already received considerable attention in the literature. Shechter *et al.* (2015) adopt the expanding search paradigm in the constrained version of their discrete search model. Fokkink *et al.* (2016) generalize the concept of expanding search for a Hider located on one of the nodes of a tree by introducing a search model with a submodular cost function. Expanding search has provoked interest in several other areas: the search for beacons from lost airplanes in Eckman *et al.* (2016), evolutionary game theory in Kolokoltsov (2014) and Liu *et al.* (2015), contract scheduling in Angelopoulos (2015) and predator search for prey in Morice *et al.* (2013).

## 2 Formal Definition of Expanding Search

We start by repeating the formal definition of expanding search of a network given in Alpern and Lidbetter (2013). Suppose  $Q$  is a network with distinguished starting node  $O$ , which we call the *root*. Each arc  $a$  has a given length  $\lambda(a)$ , and moreover we write  $\lambda(A)$  for the measure (total length) of any subset  $A$  of  $Q$ , with the total measure of  $Q$  denoted  $\mu = \lambda(Q)$ . An expanding search of  $Q, O$  is a nested family of connected subsets of  $Q$  that starts from  $O$  and increases in measure at unit speed until filling the whole of  $Q$ . More formally, we use the definition from Alpern and Lidbetter (2013):

**Definition 1.** An expanding search  $S$  on a network  $Q$  with root  $O$  is a nested family of connected closed sets  $S(t)$  for  $0 \leq t \leq \mu$ , which satisfy

$$(i) \ S(0) = \{O\} \text{ and } S(\mu) = Q,$$

$$(ii) \ S(t') \subset S(t) \text{ for } t' < t, \text{ and}$$

$$(iii) \ \lambda(S(t)) = t \text{ for all } t.$$

We denote the set of all expanding searches of  $Q$  by  $\mathcal{S} = \mathcal{S}(Q)$ .

The first condition says that the search starts at the root and is exhaustive, the second condition says that the sets are nested and the third says that they increase at a unit rate.

This is the most general definition of an expanding search, but in practice we will mostly be concerned with a class of expanding searches called *pointwise expanding searches*. A pointwise expanding search can be thought of as a sequence of unit speed paths, each one of them beginning at a point that has already been discovered. More formally:

**Definition 2.** A pointwise expanding search of a rooted network  $Q, O$  is a piecewise continuous function  $P : [0, \mu] \rightarrow Q$  made up of a sequence  $P_1, \dots, P_k$  of unit speed paths  $P_i : [t_{i-1}, t_i] \rightarrow Q$  with  $0 = t_0 < t_1 < \dots < t_k = \mu$  such that

$$(i) \ P_1(t_0) = O,$$

$$(ii) \ \cup_{i=1}^k P_i([t_{i-1}, t_i]) = Q, \text{ and}$$

$$(iii) \ P_j(t_{j-1}) \in \cup_{i=1}^{j-1} P_i([t_{i-1}, t_i]) \text{ for all } j = 2, \dots, k.$$

Note that any such sequence  $\{P_i\}$  induces an expanding search  $S$  where  $S(t) = P([0, t])$ . The first condition says that the first path starts at the root, the second condition says the paths are an exhaustive search of  $Q$  and the third says that each path starts from a point already covered by a previous path. Not all expanding searches are not pointwise expanding searches, in particular any expanding search that moves along more than one arc at the same time. However, in Alpern and Lidbetter (2013) the authors showed that the set of pointwise expanding searches is dense in  $\mathcal{S}$ , so that every expanding search can be approximated to an arbitrary degree of accuracy by a pointwise search. We will therefore use this definition of expanding search for the rest of the paper, and will use the terms “expanding search” and “pointwise search” interchangeably.

An important concept we will use is that of *search density*, which occurs in much previous research in search games. Suppose a Hider is located on a network  $Q$  according to some fixed probability distribution  $\nu$ . Then the *search density*, or simply *density*  $\rho = \rho(A)$  of a subset  $A \subset Q$  is given by  $\rho(A) = \nu(A) / \lambda(A)$ .

When disjoint regions can be searched in either order, it is better to search the region of highest density first, as shown by the following simple lemma. The proof of a more general version of the lemma can be found in Alpern and Lidbetter (2014).

**Lemma 3.** *Suppose the Hider is located on a rooted network  $Q$  according to some given distribution  $\nu$  and let  $A$  and  $B$  be connected regions of  $Q$  that meet at a single point  $x$ . Let  $S_{AB}$  and  $S_{BA}$  be two expanding searches of  $Q$  which are the same except that, on reaching  $x$ , the search  $S_{AB}$  follows the sequence  $AB$  while  $S_{BA}$  follows  $BA$ . Then if  $\rho(A) \geq \rho(B)$ ,*

$$T(S_{AB}, \nu) \leq T(S_{BA}, \nu).$$

*If  $A$  and  $B$  have the same search density, then the regions  $A$  and  $B$  can be searched in either order.*

### 3 Searching Networks with a Known Hider Distribution

Before discussing the expanding search *game*, we give some simple results on searching for a Hider located on a network according to a *known* probability distribution. These results will be useful in our later analysis.

For a given network  $Q$  with root  $O$ , suppose the Hider is located at some point  $H$  on the network. For a given expanding search  $P$  we will write  $T(P, H)$  to denote the first time that  $H$  is “discovered” by  $P$ . That is,  $T(P, H) = \min\{t \geq 0 : P(t) = H\}$ . This was shown to be well defined in Alpern and Lidbetter (2013). We call  $T(P, H)$  the *search time*. If the Hider is located on  $Q$  according to some probability distribution  $\nu$ , we denote the *expected* search time of a search  $P$  by  $T(P, \nu)$  (or later, in the expanding search game, by  $T(p, \nu)$  if the Searcher is adopting a mixed search strategy  $p$ ).

For a given Hider distribution  $\nu$  we are interested in the problem of finding the expanding search that minimizes the expected search time. We call such a search *optimal*. In Alpern and Lidbetter (2013) the authors showed that for a rooted tree with a Hider located on it according to a known distribution, the optimal search begins with the rooted subtree of maximum density (assuming it exists). More formally the theorem is:

**Theorem 4** (Theorem 14 of Alpern and Lidbetter (2013)). *Let the Hider  $H$  be hidden according to a known distribution  $\nu$  on a rooted tree  $Q$  and suppose there is a unique rooted subtree  $A$  of maximum density. Then there is an optimal expanding search  $S$  which begins by searching  $A$ . That is,  $S(\lambda(A)) = A$ .*

When the Hider distribution  $\nu$  is concentrated on the nodes of  $Q$ , this theorem gives a simple algorithm for computing an optimal expanding search of  $Q$ . The theorem is also true when  $\nu$  is a continuous distribution.

We will later use a result that says if  $A$  is a component of  $Q$  on which the Hider is hidden uniformly, and  $A$  is connected to  $Q$  at only one point, then there is an optimal search that searches the whole of  $A$  at once. (Formally, when we say the Hider is hidden uniformly on  $A$ , we mean that given he is on  $A$ , the probability he is located within some measurable subset  $X$  of  $A$  is proportional to the measure of  $X$ .)

**Lemma 5.** *Suppose  $Q$  is composed of two subnetworks  $A$  and  $B$  which meet at a single point. Suppose a Hider is located on  $Q$  according to some distribution which is uniform on  $A$ . Then there is an optimal expanding search that searches  $A$  without interruption.*

*Proof.* Proof. Any expanding search of  $Q$  must search  $A$  in a finite number of (closed) disjoint time intervals. Let  $P$  be an optimal expanding search of  $Q$  that searches  $A$  in a minimal number  $m$  of disjoint time intervals. If  $m = 1$  then the lemma is true, so suppose  $m \geq 2$  and we will derive a contradiction.

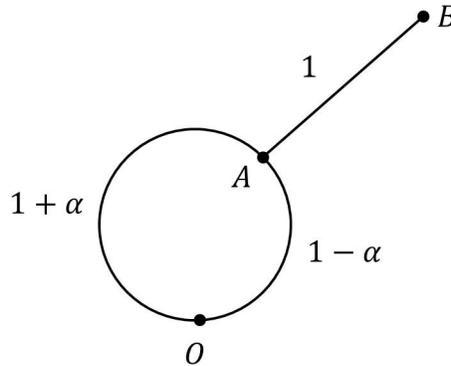
Let  $A_1$  and  $A_2$  be the subsets of  $A$  searched in the first two disjoint time intervals that  $A$  is searched, and let  $C$  be the subset of  $B$  that is searched between  $A_1$  and  $A_2$ . Note that since  $A$  and  $B$  meet at a single point, it is possible to search  $C$  at any point in time after  $P$  starts searching  $A$ . By Lemma 3, the search density of  $C$  is no more than that of  $A_1$ , otherwise the expected search time could be reduced by searching  $C$  before  $A_1$ . By a similar argument, the search density of  $A_2$  is no more than that of  $C$ . In other words,

$$\rho(A_1) \geq \rho(C) \geq \rho(A_2).$$

But since the Hider is hidden uniformly on  $A$ , the search density of  $A_1$  and  $A_2$  are the same so both of the inequalities above hold with equality. Hence, by Lemma 3, the search  $P'$  that is the same as  $P$  except that  $A_2$  and  $C$  are searched the other way around is also optimal. But  $P'$  searches  $A$  in  $m - 1$  disjoint time intervals, a contradiction.  $\square$

Every expanding search  $P$  leads to a *search tree*  $Q_P$  obtained from  $Q$  by cutting it at certain points  $x \in Q$  whose removal leaves a tree. Roughly speaking, these points  $x$  are those which are reached from more than one direction by the search  $P$ . If a search  $P$  is optimal, it has to be optimal for the corresponding distribution on the tree  $Q_P$ , so  $P$  can be found by applying Theorem 4 to all such trees.

Consider the circle-with-spike network  $Q^{CS}$  drawn in Figure 1. It consists of a circle of length 2, to which a unit length line segment is attached at a clockwise distance  $1 + \alpha$  from the root  $O$ ,  $0 \leq \alpha < 1$ . Consider Hider distributions  $\nu_p$  having atoms of weight  $\nu_p(B) = 1 - p$  at leaf node  $B$  and a uniform distribution of total weight  $p$  on the clockwise arc from  $O$  to  $A$ . For fixed  $\alpha$  and  $p$ , we determine the optimal expanding search. Every expanding search produces a subtree of  $Q^{CS}$ , so our method is first to determine these subtrees and then to find the optimal search corresponding to that subtree using Theorem 4.



**Figure 1:** The circle-with-spike network  $Q^{CS}$ .

Since the Hider distribution on the two arcs of the circle is uniform, by Lemma 5, we need only consider making cuts at the nodes. Hence there are only two trees we need consider, determined by the initial arc of the search. For the search beginning by going clockwise,  $P_+$ , the tree obtained by disconnecting the counterclockwise arc at  $O$  is optimally searched by going to  $B$  after reaching  $A$ , as the Hider is certain to

be there if not already found. So we have

$$T(P_+, \nu_p) = p \left( \frac{1 + \alpha}{2} \right) + (1 - p)(2 + \alpha).$$

For the search that begins by going counterclockwise,  $P_-$ , the tree is obtained by disconnecting the clockwise arc from  $O$  (or equivalently, disconnecting it from  $A$ ). If this arc has higher density than the one going to  $B$ , it would have been better to use  $P_+$ , so we may assume that going to  $B$  has higher density, and that

$$T(P_-, \nu_p) = p \left( 2 - \alpha + \frac{1 + \alpha}{2} \right) + (1 - p)(2 - \alpha).$$

Equating these times, we see that the search  $P_+$  is better for  $p < 2\alpha/(\alpha + 2)$ , the search  $P_-$  is better for  $p > 2\alpha/(\alpha + 2)$  and they give the same times for  $\bar{p} = 2\alpha/(\alpha + 2)$ . So in particular the least search time against  $\nu_{\bar{p}}$  is

$$\min_P T(P, \nu_{\bar{p}}) = T(P_+, \nu_{\bar{p}}) = \frac{\alpha + 4}{\alpha + 2}.$$

This analysis proves only that  $\nu_{\bar{p}}$  is the hardest Hider distribution to find within the family  $\nu_p$ , but later in Section 5 when we consider a game theoretic analysis, we will establish that in fact it is the hardest Hider distribution to search, without any restrictions. We note that if  $\alpha \leq 2/3$  then putting the uniform distribution over the long arc onto its center does not reduce its minimum search time. However if this is done when  $\alpha > 2/3$ , the Searcher can replace the strategy that begins on the long arc by one which goes to its center and then traces out the short arc from the root, saving some time as this gets to  $A$  faster.

## 4 Previous Results on the Expanding Search Game

We now assume that the Hider distribution  $\nu$  is not known to the Searcher. In this case we consider the problem of finding the mixed Searcher strategy (probability measure over expanding searches) which minimizes the expected search time in the worst case. An equivalent problem, which we prefer to adopt, is the zero-sum Expanding Search Game  $\Gamma(Q)$ . Here the maximizing Hider picks a location  $H$  in  $Q$ , the minimizing Searcher picks an expanding search  $P$  and the payoff is the search (capture) time  $T(P, H)$ . The analogous game,  $\Gamma^p(Q)$  where the Searcher picks a unit speed path on  $Q$  has been well studied, and we call this model of search *pathwise search*, denoting the value of the analogous *pathwise search game* by  $V^p(Q)$ . We note that  $V \leq V^p$ .

In Alpern and Lidbetter (2013), the authors showed that for any network  $Q$  the expanding search game has a value,  $V = V(Q)$ , and they solved the game in the cases that  $Q$  is a tree and that  $Q$  is 2-arc-connected (that is  $Q$  cannot be disconnected by the removal of fewer than 2 arcs). In general the Searcher is not restricted to using a pointwise search, but in fact in the solutions for these classes of networks the Searcher always randomizes between pointwise searches.

We first present the solution of the game for trees. In this case, the Hider's optimal distribution is concentrated on the leaf nodes of the network, since all others points are dominated. We will only consider binary trees (that is trees with maximum degree at most 3), since any tree can be transformed into a binary tree by adding arcs of arbitrarily small length. For a branch node  $x$  of a rooted tree,  $Q$ ,  $O$  with outward arcs  $a$  and  $b$ , we denote the branches starting with  $a$  and  $b$  by  $Q_a$  and  $Q_b$ , respectively, and their union by  $Q_x$ . We also denote the length of  $Q_a$  and  $Q_b$  by  $\mu_a$  and  $\mu_b$ , and write  $\mu_x = \mu_a + \mu_b$  for the length of  $Q_x = Q_a \cup Q_b$ .

**Definition 6.** Let  $Q, O$  be a rooted, variable speed tree. Let the Equal Branch Density (EBD) distribution,  $e$  be the unique probability distribution on the leaf nodes  $\mathcal{L}(Q)$  of  $Q$  such that at any branch node  $x$  of  $Q$ , all the branches rooted at  $x$  have the same search density.

For a branch node  $x$  we denote the EBD distribution on the subnetwork  $Q_x$  by  $e_x$  and similarly, for an outward arc  $a$  of  $x$  we denote the EBD distribution on  $Q_a$  by  $e_a$ .

In order to describe the optimal Searcher strategy we need to define a quantity  $D(Q)$ , which is the average distance from the root of  $Q$  to its leaf nodes, weighted with respect to the EBD distribution.

**Definition 7.** For a rooted tree,  $Q, O$ , the quantity  $D = D(Q)$  is defined by

$$D(Q) = \sum_{i \in \mathcal{L}(Q)} e(i) d(O, i),$$

where  $d(O, i)$  is the length of the path from  $O$  to  $i$  in  $Q$ .

**Definition 8.** We define the biased depth-first strategy for the Searcher as the mix of depth-first searches such that at any given branch node  $x$  with branches  $Q_a$  and  $Q_b$ , on encountering  $x$  for the first time, the Searcher searches the whole of  $Q_a$  first with probability  $p(a)$  given by

$$p(a) = \frac{1}{2} + \frac{1}{2\mu_x} (D(Q_a) - D(Q_b)).$$

In Alpern and Lidbetter (2013), the authors prove that the solution of the expanding search game on a tree is as follows.

**Theorem 9** (Theorems 18 and 19 of Alpern and Lidbetter (2013)). *The value  $V$  of the expanding search game played on a rooted tree  $Q, O$  is given by*

$$V = \frac{1}{2}(\mu + D).$$

*The EBD distribution is optimal for the Hider and the biased depth-first strategy is optimal for the Searcher.*

We also state the solution of the expanding search game for 2-arc-connected networks. We first define a *reversible expanding search* as an expanding search whose time-reverse is also an expanding search. Not all expanding searches are reversible, but it was shown in Alpern and Lidbetter (2013) that a network admits a reversible expanding search if and only if it is 2-arc-connected. This gives rise to the following theorem from the same paper.

**Theorem 10** (Theorem 20 and Corollary 22 of Alpern and Lidbetter (2013)). *A rooted tree  $Q, O$  is 2-arc-connected if and only if it admits a reversible expanding search. If  $Q$  is 2-arc-connected, then the value  $V$  of the expanding search game is  $\mu/2$ . An equiprobable choice of some reversible expanding search and its reverse is optimal for the Searcher; the uniform distribution is optimal for the Hider.*

## 5 Search Game on Circle-with-Spike Network

We continue our analysis of expanding search games by considering the circle-with-spike networks  $Q^{CS}$  of Section 3. The analysis given there of the Hider distributions  $v_{\bar{p}}$  shows that  $V = V(Q^{CS}) \geq (\alpha + 4) / (\alpha + 2)$ . We now give a mixed Searcher strategy which shows that equality holds.

The Searcher can find the Hider in time at most  $(4 + \alpha) / (2 + \alpha)$  by using the mixed strategy  $\sigma_\alpha$  of choosing searches  $(P_1, P_2, P_3)$  with probabilities  $\left(\frac{1}{2}, \frac{1}{2(2+\alpha)}, \frac{1+\alpha}{2(2+\alpha)}\right)$ , where the  $P_i$  are given as follows:  $P_1$  travels anticlockwise along the circle to  $A$ , then goes to  $B$ , and finally traverses the remaining arc from  $A$  to  $O$ ;  $P_2$  is the same as  $S_1$ , but traverses the last arc from  $O$  to  $A$ ;  $P_3$  travels clockwise along around the circle to  $A$ , then goes to  $B$ , and finally traverses the remaining arc from  $A$  to  $O$ . Note that hiding anywhere in the arc  $AB$  is dominated by hiding at  $B$  and that against  $\sigma_\alpha$  hiding on the short circular arc is dominated by  $B$ .

If the Hider is a clockwise distance  $x < 1 + \alpha$  from  $O$ , the expected search time is

$$\frac{1}{2} ((2 - \alpha) + (1 + \alpha - x)) + \frac{1}{2(2 + \alpha)} (2 - \alpha + x) + \frac{1 + \alpha}{2(2 + \alpha)} (x) = \frac{4 + \alpha}{2 + \alpha}.$$

If the Hider at  $B$ , the expected search time is

$$\left(\frac{1}{2} + \frac{1}{2(2 + \alpha)}\right) (2 - \alpha) + \left(\frac{1 + \alpha}{2(2 + \alpha)}\right) (2 + \alpha) = \frac{4 + \alpha}{2 + \alpha}.$$

So we have shown the following.

**Theorem 11.** *For the circle-with-spike networks  $Q^{CS}$  the value of the expanding search game is given by  $V = (4 + \alpha) / (2 + \alpha)$ . The optimal Hider strategy is  $\nu_{\bar{p}}$ ,  $\bar{p} = 2\alpha / (\alpha + 2)$ ; the optimal Searcher strategy is  $\sigma_\alpha$ .*

It is useful to compare this result with the solution of the pathwise search game on the same network  $Q^{CS}$ . These networks are weakly Eulerian. Roughly this means they consist of disjoint Eulerian networks which when shrunk to a point leave a tree. (In particular here such a shrinkage leads to the tree  $AB$ , with the circle shrunk to root  $A$ .) The work of Reijnierse and Potters (1993) - because these networks are also weakly cyclic - and Gal (2000) show that the pathwise search value is  $V^p = \bar{\mu}/2 = 2$ , where  $\bar{\mu}$  denotes the shortest tour time, which here is the length of the circle plus twice the length of the spike. This is the same as the expanding search value when  $\alpha = 0$ , that is, when the point  $A$  is antipodal to the root. Otherwise, the expanding search value is strictly smaller. In pathwise search, the Hider optimally hides uniformly over the circle regardless of the location of the root on the circle.

For weakly Eulerian networks, the pathwise search value  $V^p$  is independent of the location of the root. This is no longer true for such networks in expanding search. In this example, where the location of the root depends on the parameter  $\alpha$ , we have shown that the value  $V_\alpha = (4 + \alpha) / (2 + \alpha)$  is also dependent on  $\alpha$ .

## 6 The Block-Optimal Search Strategy

In Alpern and Lidbetter (2013), optimal expanding search strategies were found for networks which are 2-arc-connected or trees. In the present paper optimal expanding search strategies have also been found for the circle-with-spike networks. Since optimal strategies for searching other networks are not known, it is of use to find a class of strategies which can be calculated for any rooted network, and are ‘‘approximately optimal’’. We define a version of approximate optimality for the Searcher which is multiplicative in nature. We say that a family  $s_Q$  of mixed expanding search strategies for rooted networks  $Q$  is  $\alpha$ -factor optimal if for every  $Q$ , we have

$$T(s_Q, H) \leq \alpha V(Q), \text{ for all } H \in Q,$$

where  $\alpha \geq 1$  is a constant independent of  $Q$ . Writing  $T(s_Q)$  for the maximum value  $T(s_Q, H)$  takes over all  $H$ , we can equivalently write that the family  $s_Q$  is  $\alpha$ -factor optimal if  $T(s_Q) \leq \alpha V(Q)$ . The closer  $\alpha$  is to 1, the better the approximation given by the search family  $s_Q$ . In the language of approximation algorithms, the notion of an  $\alpha$ -factor optimal strategy is akin to the notion of an  $\alpha$ -approximate algorithm for computing the value of the game.

In this section we present such a family of search strategies, called *bridge-optimal strategies*  $\beta = \beta_Q$ , which are 1.2-factor optimal. The name comes from the fact that these strategies use the well-known “bridge-block” decomposition of an arbitrary network, as defined in Subsection 6.1. The multiplicative constant  $\alpha$  is actually  $(1 + \sqrt{2}) / 2$  which is approximately equal to 1.207.

It turns out that the range of factor constants  $\alpha$  that we need to consider goes from 1 to 2. To see that a constant of  $\alpha = 2$  is of no value consider any class  $s_Q$  of expanding search strategies. The definition of an expanding search shows that for any point  $H \in Q$  we have  $T(s_Q, H) \leq \mu$ , the total length of  $Q$ . Since the uniform hiding strategy ensures an expected search time of at least  $\mu/2$ , we know that  $V(Q) \geq \mu/2$ . Consequently, for any family  $s_Q$ , we have

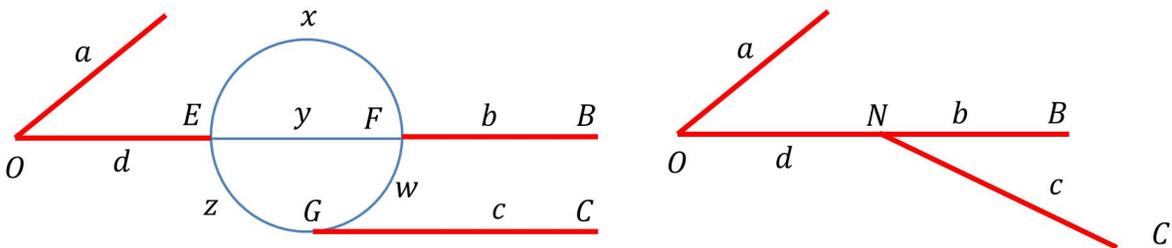
$$T(s_Q, H) \leq \mu \leq 2(\mu/2) \leq 2V(Q), \text{ for all } H \in Q,$$

and thus *any* expanding search family  $s_Q$  is 2-factor optimal.

## 6.1 The bridge-block decomposition

Let  $Q$  be a connected network. An arc of  $Q$  is called a *bridge* if removing it (but not its end nodes) disconnects the graph, or equivalently, if it is not contained in any cycle. The components of  $Q$  after removing its bridges are called the *blocks*. Note that the blocks are 2-arc-connected. We denote the set of bridge-points (points in bridges) of  $Q$  by  $Q_1$  and the set of block-points (points in blocks) by  $Q_2$ . The connected network  $Q^t$  obtained from  $Q$  by shrinking each block to a point (node) is a tree whose arcs are identical to the arcs of  $Q_1$ . We call  $Q^t$  the *bridge tree* of  $Q$ . The nodes of  $Q^t$  are of two types: the *new nodes* correspond to the blocks of  $Q$  and the *original nodes* correspond to nodes of  $Q$  which are incident only to bridges in  $Q$ .

These concepts are illustrated in Figure 2. The network  $\bar{Q}$  on the left has four bridges:  $a, b, c, d$ . It has a single block made up of arcs  $x, y, z, w$ . Its bridge tree  $\bar{Q}^t$  consists of the 4 arcs  $a, b, c, d$ . It has a new node  $N$  which corresponds to the block of  $\bar{Q}$ .



**Figure 2:** A network  $\bar{Q}$  (left) and its bridge tree  $\bar{Q}^t$ .

For any point  $H$  in  $Q$ , its height  $\pi(H)$  is defined as the distance from the root to its corresponding point in  $Q^t$ , so that all points in a given block have the same height. The notion of height is illustrated in Table 1 and Figure 3, where we assign lengths to the arcs of  $\bar{Q}$ .

We also define the *bridge ratio*  $r = r(Q)$  of a network  $Q$  as the fraction of the total length of  $Q$  that consists of bridges. So, writing  $\mu_1$  for the total length of all the bridges in  $Q$ , the bridge ratio is given by  $r = \mu_1/\mu$ .

## 6.2 Definition of the block-optimal search strategy $\beta$

We now define the block-optimal search strategy  $\beta$  as an equiprobable mixture of two expanding searches we call  $S_1$  and  $S_2$ . The second one,  $S_2$ , will be a sort of reverse path of  $S_1$  so we concentrate on describing  $S_1$ . The specification of  $S_1$  will not depend on arc lengths, so we leave that until later. To specify  $S_1$  we fix on each of its blocks a particular reversible expanding search (as defined in Section 4). For example, on the pictured network  $\bar{Q}$  we orient each arc to the right and denote a traversal to the left with a prime. So one reversible expanding search of its block is given by  $x, y', w', z'$  with reverse path  $z, w, y, x'$  (reverse order and direction). The fact that every block has a reversible expanding search follows from Theorem 10. The expanding search  $S_1$  follows the first of these two reversible expanding searches  $(x, y', w', z')$  on each block. However when reaching a cut-node such as  $F$  or  $G$  which leads out of the block, it exhaustively searches that component of the network before returning to node on which it left. When reaching a node of  $Q$  which is not in a block (in  $\bar{Q}$  the only such node is the start node  $O$ )  $S_1$  can leave via any arc; the reverse ordering will be chosen by  $S_2$ .

We describe the construction of  $S_1$  for the pictured network  $\bar{Q}$ . We begin by fixing a reversible expanding search at its only block:  $(x, y', w', z')$ . Now we start at  $O$  and can choose either of the arcs  $a$  or  $d$ . We choose  $d$  (this will mean that  $S_2$  begins with  $a$ ). We arrive at node  $E$  which is in the block so we travel according to our fixed reversible path until we reach a node which leaves the block. So we must continue with  $d, x$  which arrives at the node  $F$ . Here we leave the block by choosing  $d, x, b$ . Since the component of  $Q - F$  is now exhaustively searched we continue searching the block from  $F$ , continuing  $d, x, b, y', w'$  which brings us to another node,  $G$ , which leads out of the block. Here we continue  $d, x, b, y', w', c, z'$ . Now we have exhaustively searched the part of  $\bar{Q}$  stemming from the arc  $d$ . We now follow the same procedure on the part of  $\bar{Q}$  stemming from  $a$ , which in this case is just  $a$ , leading to the construction

$$S_1 = d, x, b, y', w', c, z', a.$$

To construct  $S_2$  we follow the reverse expanding search on each block, here  $z, w, y, x'$ . We also take the opposite ordering on the non-new branch nodes of the bridge tree, that is, we start with  $a$  rather than  $d$ . So we start with  $a, d$ . Now we follow the reverse expanding search on the block, beginning with  $a, d, z$ . Then we search the entire portion of the network that begins with arc  $c$ , which in this case is just  $c$  itself, obtaining  $a, d, z, c$ . We continue searching the block with  $a, d, z, c, w$ . Since  $b$  came before  $y$  (actually  $y'$ ) in  $S_1$ , we continue in the opposite order in  $S_2$ , with  $a, d, z, c, w, y, b$  and then finish with  $x'$ , obtaining the expanding search

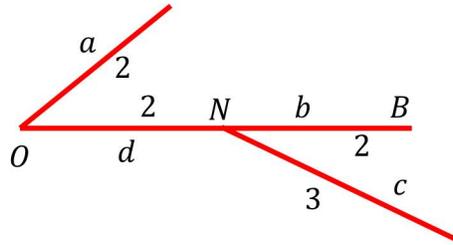
$$S_2 = a, d, z, c, w, y, b, x'.$$

We now explore the performance of  $\beta = (1/2) S_1 + (1/2) S_2$  against undominated hider strategies. A hider strategy  $H$  is undominated if it lies on an arc of a block or on one of the leaf nodes in the bridge tree. Note that arcs in blocks are traversed in opposite directions by  $S_1$  and  $S_2$ , so it does not matter where the Hider is situated in such an arc. We take the lengths of  $a, d, x, y, b$  as 2;  $c$  as 3;  $z$  and  $w$  as 1. So  $\mu = 15$ . Table 1 gives the times take for  $S_1$  and  $S_2$  to reach a Hider placed at one of the leaf nodes  $A, B, C$  or at the center of  $x, y, w, z$ .

**Table 1:** Search times for the network  $\bar{Q}$ .

	$T(S_1, H)$	$T(S_2, H)$	sum	height, $\pi(H)$	$\mu + \pi(H) = 15 + \pi(H)$
$A$	15	2	17	2	17
$B$	6	13	19	4	19
$C$	12	8	20	5	20
$x$	2.5	14.5	17	2	17
$y$	7	10	17	2	17
$z$	12.5	4.5	17	2	17
$w$	8.5	8.5	17	2	17

There is an easier way to calculate the expected time (the column labeled “sum” divided by 2) for the block-optimal strategy  $\beta$  to reach a point in  $\bar{Q}$  corresponding to any node of the bridge tree  $\bar{Q}^t$ . To understand this method, consider the bridge tree as drawn in Figure 3 with arc lengths of bridges indicated. Note that the only hiding locations that are best responses to  $\beta$  correspond to leaf nodes or new (block)



**Figure 3:** The bridge tree  $\bar{Q}^t$  with arc lengths.

nodes of  $Q^t$  (for  $\bar{Q}^t$  all nodes other than  $O$  are of one of these types). Suppose for example, that the Hider chooses  $H = C$ . Then certainly the whole lengths of arcs  $d$  and  $c$  must be covered before reaching  $H$ , so the expected search time is at least 5. Note that every other point will be reached before  $H$  by exactly one of the pure searches  $S_1$  and  $S_2$ . Since these “other points” have total length  $\mu - 5 = 15 - 5 = 10$ , the expected search time is  $5 + (1/2)10 = 10$ , as given in our table as  $20/2 = \text{sum}/2$ . This argument is quite general for any network  $Q$ . Suppose the Hider hides at a leaf node or on a non-bridge arc  $a$ , breaking it into arcs  $a'$  and  $a''$  when  $H$  is considered a node. Then as in the above argument, the bridge arcs which connect  $A$  to  $O$  in  $Q^t$  have total length  $\pi(H)$  and must definitely be traversed by  $\beta$  before it finds  $H$ . All other arcs are traversed either once by  $S_1$  or once by  $S_2$  (so on average  $1/2$  by  $\beta$ ) so the expected time for  $\beta$  to  $H$  is given by

$$\begin{aligned} T(\beta, H) &= \pi(H) + (1/2)(\mu - \pi(H)) \\ &= \frac{\mu + \pi(H)}{2}, \text{ for } H \in Q_2 \text{ or } H \text{ a leaf node.} \end{aligned}$$

Note that for other hider places  $H$  the search time may be smaller, so we have

$$T(\beta, H) \leq \frac{\mu + \pi(H)}{2}, \text{ for all } H \in Q.$$

Consequently taking  $\pi = \pi(Q) = \max_{H \in Q} \pi(H)$  to be the *height* of  $Q$ , we have

$$T(\beta) = \frac{\mu + \pi}{2}. \quad (1)$$

Note that there must be a leaf node  $H$  of  $Q_1$  such that  $\pi(H) = \pi$ .

### 6.3 Performance of $\beta$

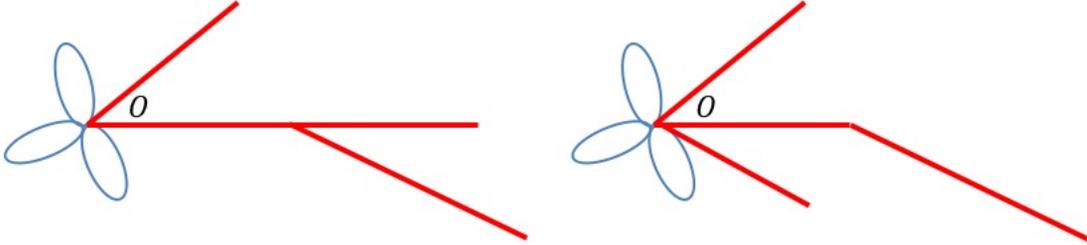
We now show that the block-optimal strategy  $\beta$  is 1.2-factor optimal. To do this, we first find a lower bound on  $V(Q)$ . This is turn is accomplished by finding the value of the game on certain networks  $Q'$  and  $Q''$  which are easier to search than  $Q$ .

**Lemma 12.** *For any rooted network  $Q$ , the value  $V(Q)$  of the game satisfies*

$$V(Q) \geq \frac{1}{2\mu}(\mu^2 + \pi^2).$$

*Proof.* Proof. We start by defining a new network  $Q'$  whose value is no greater than the value of  $Q$ . The network  $Q'$  is obtained by first identifying all the nodes in each block of  $Q$ , so that each block is now a set of loops that meet at a single node. We then remove those loops and reattach them at the root  $O$ . So  $Q'$  consists of the bridge tree  $Q^t$ , with some extra loops meeting at  $O$ . This network  $Q'$  is depicted on the left of Figure 4 for the network  $Q = \bar{Q}$  of Figure 2. The value  $V(Q')$  is lower than  $V(Q)$  since any Searcher strategy that can be used on  $Q$  can also be used on  $Q'$ .

We then define another new network  $Q''$  whose value is, again, no greater than the value of  $Q'$ . Let  $H^*$  be a leaf node of  $Q_1$  at distance  $\pi$  from  $O$ . The network  $Q''$  is obtained from  $Q'$  by “pruning” the path  $P$  from  $O$  to  $H^*$ : all subtrees rooted on this path are removed and reattached at  $O$ , so that in  $Q''$  all the vertices on  $P$  except  $O$  and the leaf nodes have degree 2. The network  $Q''$  is depicted on the right of Figure 4. It is clear that the value  $V(Q'')$  is no greater than  $V(Q')$ , since any Searcher strategy on  $Q'$  can also be executed on  $Q''$ .



**Figure 4:** The networks  $Q'$  (left) and  $Q''$  (right) depicted for the network  $Q = \bar{Q}$  of Figure 2.

We now derive a lower bound on  $V(Q'')$  (and therefore on  $V(Q)$ ) by giving an explicit Hider mixed strategy  $h$  on  $Q''$  and calculating its minimum expected search time against any Searcher strategy. The Hider strategy  $h$  hides at  $H^*$  with probability  $\pi/\mu$  and hides uniformly on the rest of  $Q''$ . This means that the search density of the path  $P$  from  $O$  to  $H^*$  is the same as the search density of the rest of the network. By Lemma 5, the two regions  $P$  and  $Q - P$  must be searched one after the other, and the fact the densities are equal means it does not matter which way around they are searched. So let  $S$  be the search that first visits  $H^*$  then searches the rest of the network (it does not matter how). The expected

search time is

$$\begin{aligned} T(S, h) &= \frac{\pi}{\mu}(\pi) + \left(1 - \frac{\pi}{\mu}\right) \left(\pi + \frac{\mu - \pi}{2}\right) \\ &= \frac{1}{2\mu} (\mu^2 + \pi^2). \end{aligned}$$

□

By comparing Equation (1) with Lemma 12, we can estimate the efficiency of the strategy  $\beta$ .

**Theorem 13.** *Let  $Q$  be a rooted network with height  $\pi$  and total length  $\mu$ . The maximum expected search time  $T(\beta)$  of the block-optimal strategy  $\beta$  on  $Q$  satisfies*

$$T(\beta) \leq (1 + r)V(Q). \quad (2)$$

Moreover,  $\beta$  is  $(1 + \sqrt{2})/2$ -factor optimal for any network.

*Proof.* Proof. It follows from Equation (1) and Lemma 12 that

$$\frac{T(\beta)}{V(Q)} \leq \frac{\frac{1}{2}(\mu + \pi)}{\frac{1}{2\mu}(\mu^2 + \pi^2)} = \frac{1 + \pi/\mu}{1 + (\pi/\mu)^2}. \quad (3)$$

Note that the definition of the height  $\pi$  ensures that it cannot exceed the total length of the bridges, so  $\pi \leq \mu_1$  and hence

$$\frac{\pi}{\mu} \leq \frac{\mu_1}{\mu} \equiv r.$$

Thus (2) follows from (3).

To show that  $\beta$  is  $(1 + \sqrt{2})/2$ -factor optimal, it is easy to check that the right-hand side of (3) is maximized when  $\pi/\mu = \sqrt{2} - 1$ , and so we get

$$\frac{T(s)}{V(Q)} \leq \frac{\sqrt{2}}{1 + (\sqrt{2} - 1)^2} = (1 + \sqrt{2})/2.$$

□

The constant  $(1 + \sqrt{2})/2$  is the best possible in general. For example, suppose  $Q$  is the star network consisting of one arc of length 1 and  $n$  arcs length  $\sqrt{2}/n$ . Then  $\mu = 1 + \sqrt{2}$  and  $\pi = 1$  so by (1), we have  $T(\beta) = (2 + \sqrt{2})/2$ . But by Theorem 9, the value  $V(Q)$  can be calculated to be

$$V(Q) = \frac{1}{2}(\mu + D) = \frac{2 + \sqrt{2} + 1/n}{1 + \sqrt{2}} \rightarrow \frac{2 + \sqrt{2}}{1 + \sqrt{2}} \text{ as } n \rightarrow \infty.$$

Hence the ratio between  $T(\beta)$  and  $V(Q)$  can be arbitrarily close to  $(1 + \sqrt{2})/2$ .

Using (3), we can say a bit more precisely how well the block-optimal strategy  $\beta$  performs for specific networks. For the circle-with-spike network  $Q^{CS}$ , the height is  $\pi = 1$ , so  $\pi/\mu = 1/3$  and by (3), we have  $T(\beta) \leq 1.2V(Q^{CS})$ . The ratio of the height to the total measure of the network  $\bar{Q}$  of Figure 2 is also  $\pi/\mu = 5/15 = 1/3$ , so we also have  $T(\beta) \leq 1.2V(\bar{Q})$ .

The inequality (2) makes it clear that the less  $Q$  is taken up with bridges, the better  $\beta$  performs. In particular, it is optimal if  $Q$  contains no bridges, and is therefore 2-arc-connected. This is also obvious from the definition of  $\beta$ , by Theorem 10.

## 7 The Bridge-Optimal Search Strategy

In Section 6, we found that the block-optimal strategy  $\beta$  approximates the optimal search strategy within a factor of 1.2 and performs particularly well on “block-like” networks. In this section we will give an alternative search strategy,  $\gamma$  that performs well in the case that the network mostly consists of bridges with a small proportion of the length taken up by blocks, so that the network is “tree-like”. We call  $\gamma$  the *bridge-optimal strategy*.

The strategy  $\gamma$  is simply an adaptation of the biased depth-first strategy, described in Section 4, which is optimal for tree networks. The strategy  $\gamma$  follows the biased depth-first strategy on the set of bridge-points  $Q_1$ , as if using the optimal strategy on the bridge tree  $Q^t$  of  $Q$ ; and whenever the Searcher encounters a block in  $Q_2$  for the first time, she performs an arbitrary search of the whole of it. Since the biased depth-first strategy is optimal for  $Q^t$ , for any point  $x$  of  $Q$  that lies on a bridge, the strategy  $\gamma$  ensures an expected search time  $T(\gamma, x)$  of

$$T(\gamma, x) \leq \mu_2 + V(Q^t) = \mu_2 + \frac{1}{2}(\mu_1 + D(Q^t)), \quad (4)$$

by Theorem 9. To see that inequality (4) also holds for any point  $x$  that lies in a block of  $Q$ , we add a leaf arc of infinitesimal length incident to some point on the block. Clearly point  $x$  is reached at an earlier time than the leaf arc, so since (4) holds for points on the leaf arc it must also hold for points in the block.

### 7.1 Performance of $\gamma$

Our main result of this section is on the performance of the search strategy  $\gamma$ .

**Theorem 14.** *Let  $Q$  be a rooted network with bridge-ratio  $r$ . The maximum expected search time  $T(\gamma)$  of the bridge-optimal strategy  $\gamma$  on  $Q$  satisfies*

$$T(\gamma) \leq \left( \frac{2}{1+r^2} \right) V(Q).$$

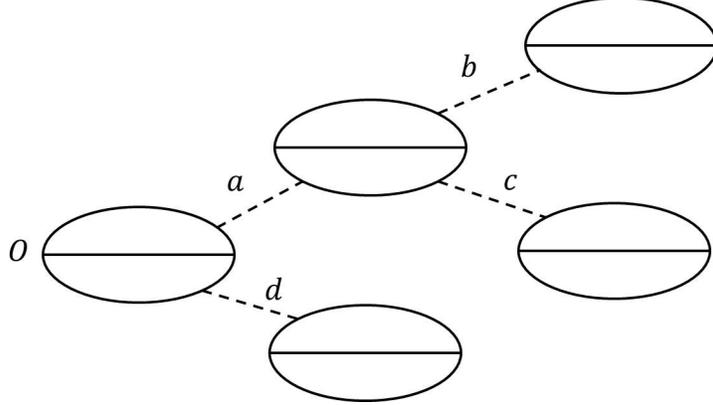
*That is, the bridge-optimal strategy  $\gamma$  is  $2/(1+r^2)$ -factor optimal.*

In order to prove the theorem, we need to give a lower bound on  $V(Q)$  that is tight in the case that  $Q$  is a tree. We will do that by describing a general Hider distribution (probability measure) on  $Q$ . We note that the lower bound on  $V$  obtained in this subsection applies as well to pathwise search games, as  $V \leq V^P$ , so the results obtained here are stronger, although sometimes the same proofs work.

We can improve on the uniform Hider distribution by placing all the uniform measure on a bridge at its forward end (away from the root). In fact we can do better than this. Moving upwards in  $Q_1$  (starting at arcs at the root), we successively remove the uniform measure on each such arc  $a$  and suitably increase the total mass of the uniform measure on the subtree of  $Q_1$  following  $a$  so that the total measure of  $Q_1$  does not change. In particular, when  $a$  is a leaf arc of  $Q_1$ , we place its measure on the corresponding leaf node. We call the resulting distribution the *pushed uniform distribution* and denote it by  $u^*$ . The measure on  $Q_1$  is thus repeatedly pushed away from the root until it is all on the leaf nodes. Note that  $u^*$  agrees with the uniform distribution  $u$  on the blocks of  $Q$  and the rest is concentrated on the leaf nodes of  $Q_1$ .

For example on any circle-with-spike network  $Q_\alpha^{CS}$ ,  $u^*(B) = 1/3$ ,  $u^*$  is uniform on its circle, and it is zero on the open arc  $AB$ . Note that if  $Q$  is already a tree, then  $u^*$  is the EBD distribution. Figure 5 depicts a network all of whose arcs have unit length and whose 2-arc-connected components are copies

of the three-arc network, given by two nodes joined by three unit length arcs. The network has 19 unit length arcs, so each has a uniform measure of density  $1/19$  under the uniform measure  $u$ . The five three-arc networks keep this measure under  $u^*$ . The measure  $1/19$  on  $a$  is split between  $b$  and  $c$ , and eventually the nodes at the forward ends of  $b$  and  $c$  each get measure  $1/19 + 1/28 = 3/28$ . The measure  $1/19$  on  $d$  gets transferred to the forward end of  $d$ . Thus the three leaf nodes of  $Q^t$  get measure  $3/28, 3/28,$  and  $2/28$  which is exactly  $\mu_1/\mu = 4/19$  times the EBD measure on  $Q_1$ . In general,  $u^*(i) = (\mu(Q_1)/\mu) \cdot e(i)$ , where  $e$  is the EBD measure on  $Q^t$ , for any leaf node  $i$  of  $Q_1$ .

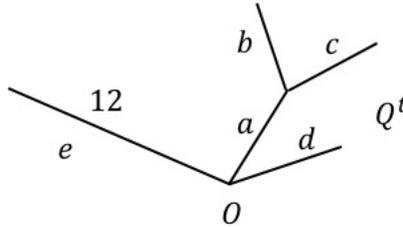


**Figure 5:** Decomposition of a particular network.

**Lemma 15.** *For any network  $Q$ , let  $\mu_1 = \mu(Q_1)$ . Then*

$$V \geq \frac{\mu + (\mu_1/\mu) D(Q^t)}{2}. \quad (5)$$

*Proof.* Proof. The Hider can ensure at least this expected capture time by adopting the pushed uniform distribution  $u^*$ . Suppose we increase the strategy space for the Searcher so that he can move freely within any given block and can thus search  $Q_1$  as if it were the tree  $Q^t$ . This is equivalent to changing the search space to the network  $\tilde{Q}$  consisting of the tree  $Q^t$  and an additional leaf arc  $e$  at the root of length  $\mu_2$ . The network  $\tilde{Q}$  corresponding to the network of Figure 5 is shown in Figure 6.



**Figure 6:** The network  $\tilde{Q}$ .

Note that any search  $S$  of  $Q$  induces a search  $\tilde{S}$  of  $\tilde{Q}$ . At time  $t$ ,  $\tilde{S}(t)$  will include a distance  $\lambda(S(t) \cap a)$  along  $a$ , the amount of  $a$  that has been searched, as well as the identical portions of  $a, b, c$  and  $d$ . Considering  $u^*$  as a distribution on  $\tilde{Q}$  (which is uniform on  $a$  and EBD on  $Q^t$ , we see that the subtrees  $a$  and  $Q^t$  both have the maximum density of  $1/\mu$ . By Lemma 5 there is an optimal search that searches  $a$  all at once, and by Lemma 3,  $Q^t$  and  $a$  can be searched in either order, so assume  $a$  is searched first and then  $Q^t$  is

searched optimally. If the Hider is located on  $a$  the expected search time is half its length,  $\mu_2/2$ . So the expected search time is

$$\begin{aligned} & \frac{\mu_2}{\mu} \left( \frac{\mu_2}{2} \right) + \frac{\mu_1}{\mu} (\mu_2 + V(Q^t)) \\ &= \frac{\mu_2}{\mu} \left( \frac{\mu_2}{2} \right) + \frac{\mu_1}{\mu} \left( \mu_2 + \frac{\mu_1 + D(Q^t)}{2} \right) \quad (\text{by Theorem 9}) \\ &= \frac{\mu + (\mu_1/\mu) D(Q^t)}{2}. \end{aligned}$$

□

To illustrate the construction in the proof of Lemma 15, we consider again the example of Figure 6, we have  $\mu = 19, \mu_1 = 4$ , and

$$D(Q^t) = 3/8 \cdot 2 + 3/8 \cdot 2 + 2/28 \cdot 1 = 14/8,$$

so the value satisfies

$$V(Q) \geq (19 + (4/19)(14/8)) / 2 = 184/19 = 9.6842.$$

We note that we have equality in (5) when  $Q$  is a tree ( $\mu_1 = \mu$ ) or if  $Q$  is 2-arc-connected ( $\mu_1 = 0$ ), by Theorem 9 and Theorem 10.

We can now prove Theorem 14.

*Proof.* Proof of Theorem 14. Combining the lower bound in (5) with our estimate (4) for the expected search time of the search strategy  $\gamma$ , we have

$$\begin{aligned} \frac{T(\gamma)}{V} &\leq \frac{2\mu_2 + (\mu_1 + D(Q^t))}{\mu + (\mu_1/\mu)D(Q^t)} \\ &= \frac{(2-r)\mu + D(Q^t)}{\mu + rD(Q^t)} \quad (\text{where } r = \mu_1/\mu) \\ &\leq \frac{(2-r)\mu + r\mu}{\mu + r^2\mu} \quad (\text{since } D \leq \mu_1) \\ &= \frac{2}{1+r^2}. \end{aligned}$$

□

We have proved that the strategy  $\gamma$  is  $2/(1+r^2)$ -factor optimal. This factor is decreasing in  $r$ : when  $r = 1$  (so that  $Q$  is a tree), then  $\gamma$  is optimal; when  $r = 0$  (so  $Q$  is 2-arc-connected), then  $T(\gamma) = 2V$  and  $\gamma$  performs very badly.

## 8 Summary and Conclusions

In many or even most searches that are carried out to find missing persons, lost airplanes or unexploded mines from earlier conflicts, the rate at which the searched area can expand is restricted by available resources. In such situations the “expanding search” model recently introduced by the authors in Alpern and Lidbetter (2013) can be applied. This model seeks the randomized strategy for expanding the search region which minimizes the expected time to find the “target” in the worst case, when the search region

has a network structure. Optimal strategies in this context were previously known only for networks that are trees or are 2-arc-connected. This paper extends the classes of solvable networks to include the new class of “circle-with-spike” networks. However our main contribution is to give two general classes of strategies which can be applied to *any* network and have expected times to find the target that are within 20% of the optimal time. One class of strategies, which was introduced and analyzed in Section 6, is the block-optimal strategy  $\beta$ , which is optimal when the network has no bridge (when the bridge-ratio  $r$  is equal to 0). This strategy is also close to optimal when the “height”  $\pi$  of the network is close to 1 (when one of the bridges is much longer than all the others combined). The other class of strategies, introduced in Section 7, is called the bridge-optimal strategy  $\gamma$ , and is optimal when when the search network is a tree. It is close to optimal when the bridge-ratio  $r$  of the network is close to 1.

The paper enables the organizer of an expanding search effort to base his search strategy simply on the bridge-ratio  $r$  of the network to be searched. This is the ratio  $r = \mu_1/\mu$  of the total length of the bridges of the network to the total length of all its arcs. Roughly speaking, when  $r$  is small the searcher can get closest to the optimal expected search time by adopting the block-optimal strategy  $\beta$  and when  $r$  is large (above 80%) by adopting the bridge-optimal strategy  $\gamma$ .

### 8.1 Factor-approximate estimates in terms of the bridge-ratio $r$

It is useful to combine the estimates on the value  $V = V(Q)$  obtained by the block-optimal strategy  $\beta$  and the bridge-optimal strategy  $\gamma$  in terms of the bridge ratio  $r = \mu_1/\mu$  of the network  $Q$ . Although we still get the factor  $(1 + \sqrt{2})/2$  for *all* networks, we can get better estimates for values of the bridge ratio  $r$  near 0 or 1, that is for block-like or bridge-like networks. (Of course even better estimates can be obtained if we make the more difficult calculation of the height  $\pi$  of the network.)

First note that, as explained in the proof of Theorem 13,  $\pi/\mu \leq r$ . Since the function  $f(r) = (1 + r)/(1 + r^2)$  is increasing on the interval  $[0, \sqrt{2} - 1] \simeq [0, 0.41421]$  with a maximum on  $[0, 1]$  of  $(1 + \sqrt{2})/2 \simeq 1.2071$  obtained at  $r = \sqrt{2} - 1$ , it follows that

$$\begin{aligned} f\left(\frac{\pi}{\mu}\right) &\leq f(r) \text{ for } r \in [0, \sqrt{2} - 1] \text{ and} \\ f\left(\frac{\pi}{\mu}\right) &\leq (1 + \sqrt{2})/2 \text{ for all } r \in [0, 1]. \end{aligned}$$

By Theorem 13, we know that by adopting the block-optimal strategy  $\beta$  the Searcher ensures that the ratio of actual expected time to optimal expected time is bounded by

$$\frac{T}{V} = \frac{T(\beta)}{V} \leq f\left(\frac{\pi}{\mu}\right).$$

Similarly from Theorem 14, we know that by adopting the bridge-optimal strategy  $\gamma$  the Searcher ensures that

$$\frac{T}{V} = \frac{T(\gamma)}{V} \leq \frac{2}{1 + r^2}.$$

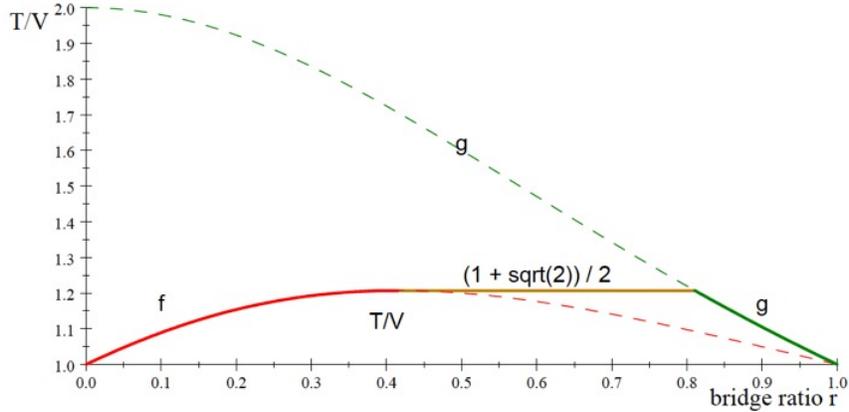
Observe that  $g(r) = 2/(1 + r^2)$  is decreasing in  $r$  and is equal to  $(1 + \sqrt{2})/2 \simeq 1.207$  for  $r = \sqrt{\frac{3-\sqrt{2}}{1+\sqrt{2}}} \simeq 0.81047$ .

So it follows that by taking strategy  $\alpha$  to be the better of the block-optimal and bridge-optimal

strategies  $\beta$  and  $\gamma$ , we can ensure that

$$\frac{T(\alpha)}{V} \leq \begin{cases} f(r) = \frac{1+r}{1+r^2} & \text{for } 0 \leq r \leq \sqrt{2} - 1 \simeq 0.41421, \\ f(\sqrt{2} - 1) = \frac{1+\sqrt{2}}{2} \simeq 1.2071 & \text{for } \sqrt{2} - 1 \leq r \leq \sqrt{\frac{3-\sqrt{2}}{1+\sqrt{2}}} \simeq 0.8104, \\ g(r) = \frac{2}{1+r^2} & \text{for } \sqrt{\frac{3-\sqrt{2}}{1+\sqrt{2}}} \leq r \leq 1. \end{cases}$$

The first two estimates are obtained by adopting  $\alpha = \beta$  and the last one by adopting  $\alpha = \gamma$ . The upper bounds on  $T/V$ , as a function of the bridge ratio  $r$ , are shown in Figure 5.



**Figure 7:**  $T/V$  bounded above by thick line, as a function of the bridge ratio  $r$ .

We can see from Figure 7 that by adopting one of only the two expanding search strategies  $\beta$  and  $\gamma$  introduced in this paper, a searcher with an expanding search to plan on a known network can be sure of getting within 20% of the optimal expected time when the network has a bridge-ratio  $r$  within  $0.41 \leq r \leq 0.81$  and can be even closer to optimal for  $r$  outside that interval. By considering the height  $\pi$  of the network, he might be able to ensure being even closer to optimal, but the calculation of  $r$  is very easy. It is worth noting that prior to this paper there were no known general strategies for expanding search on arbitrary networks.

## 8.2 Goals for future work

From the results of this paper, it seems that there two directions for further investigations in the study of expanding search on networks. One is to identify more classes of networks where optimal strategies can be found. The other direction is to see if some strategy classes can be found which reduce the factor-approximate constant of 1.2071 for the ratio of  $T/V$ . Another aspect of real-life search that might be added to the analysis is a “give-up” option, that is, when should the search be stopped. This is particularly relevant to the Bayesian problem where there is an *a priori* distribution for the target, as in the search for the Malaysian Airlines plane.

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