On Quotients of the Shift associated with Dendrite Julia sets of Quadratic Polynomials

by

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SUMMARY

Chapter one considers iterations of complex mappings of the form $f_c : z \mapsto z^2 + c$ and gives sufficient conditions for the Julia set $J(f_c)$ to be homeomorphic (and conjugate) to a quotient Σ_ξ of the shift $\Sigma = \{0,1\}^{\mathbb{N}}$, the equivalence relation being given by

$$x_1 x_2 \ldots x_k 0 \xi \sim x_1 x_2 \ldots x_k 1 \xi \qquad (k \geq 0,\ x_i \in \{0,1\})$$

where ξ is a fixed element of Σ. The partition of $J(f_c)$ used involves an external ray from the Douady-Hubbard theory which converges to the critical value c. Theorem 1.5.3 establishes the conjugacy when the associated "kneading itinerary" ξ is non-recurrent.

Subsequent chapters deal abstractly with spaces Σ_ξ for general ξ. Chapter two establishes that Σ_ξ and some slightly more general quotient spaces Σ_Γ are "dendrites". It also initiates an investigation of what combinatorial information about the space Σ_ξ can be deduced from the location of ξ in a particular space Σ_Σ (akin to the Mandelbrot set).

Chapter three developes a theory of "factorization" for sequences ξ which corresponds to "renormalisation" properties of the spaces Σ_ξ.

Chapter four is concerned with giving the topological space Σ_ξ a "geodesic" metric which is (piecewise) uniformly expanded under the shift action. Theorem M establishes that such a metric can exist if and only if ξ does not admit a factorization of either of two types : either where the kneading itinerary is of generalised Feigenbaum type or where Σ_ξ contains a number of disjoint embedded copies of another such space Σ_ψ which are permuted cyclically under the shift. In the latter case Σ_ξ is given a pseudo-metric which collapses the embedded copies of Σ_ψ to points.

Chapter five is a reformulation of the Vere-Jones "Perron-Frobenius" theory for countable nonnegative matrices to allow for a relaxation of the assumption of irreducibility (necessary for the proof of Theorem M). We restrict to matrices which act as "quasi-compact" linear operators over the space ℓ^1.

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I also acknowledge the help of the excellent computing facilities without which this thesis could not have been written , and of J. Liddamore who acquainted me with the language C and the technicalities of the Unix system and got me started on the Julia set and Mandelbrot set programs which provide the illustrations .

I am grateful to my father who got me interested in complex iteration and to Colin Sparrow for his Cambridge Part III course which linked this to the general field of Dynamical Systems .

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Declaration

The work contained in this thesis is original except where otherwise indicated .

CONTENTS

INTRODUCTION

In recent years, due to the advent of computational power and computer graphics, there has been an explosion of interest in the dynamics of iterated rational maps on the Riemann sphere, which lay dormant since the classical theory of Julia and Fatou (~ 1919). The Julia set of such a map f is defined as the complement of the domain of normality (or of equicontinuity) for the iterates $\{f^n\}$. Benoit Mandelbrot, in [Ma1], appears to have been the first to produce high-resolution graphic images of Julia sets for a complex parameter family of quadratic polynomials and give a detailed description of the parameter space. An extraordinary variety of fractals is observed.

Every complex quadratic polynomial is conjugate by an affine transformation to one of the form $f_c: z \to z^2 + c$ where $c \in \mathbb{C}$. Douady and Hubbard, in [DH1], undertake a rigorous analysis of this family with the full proofs appearing in [DH2] (where more general polynomials are considered). For a polynomial f of degree at least two the point ∞ is a super-attracting fixed point and its basin of attraction is connected. The complement $K(f)$, the set of z whose iterates remain bounded, is known as the "filled in Julia set" :- corresponding to the fact that its boundary is the Julia set $J(f)$. The dynamics of f on components of the interior of $K(f)$ is governed by the general theory of rational maps. For a discussion of rational maps (and a much more complete list of references) the reader is directed to Blanchard's survey article [B].

As is generally observed for one-dimensional dynamics, the behaviour of the iterates of the critical points of f is of crucial importance. The critical points are the branch points of f regarded as a branched cover of the Riemann sphere (points where f fails to be locally a homeomorphism) and, excepting ∞, are the points where f' vanishes. For the family $z \mapsto z^2 + c$ the only finite critical point is 0. The Mandelbrot set defined as

$$\mathcal{M} = \left\{ c \in \mathbb{C} : \ (f_c)^n(0) \not\to \infty \ \text{ as } \ n \to \infty \right\}$$

provides an important dichotomy of behaviour. For $c \in \mathbb{C} \setminus \mathcal{M}$ the set $K(f_c) = J(f_c)$ is homeomorphic to a Cantor set. For $c \in \mathcal{M}$ the set $K(f_c)$ is connected.

Computer pictures of other complex parameter families of analytic maps have been observed to contain copies of the set $\mathcal{M}$. In particular the family of rational maps corresponding to Newton's method for finding a root of the cubic polynomials $p_\lambda(z) = z^3 - 3\lambda z - 1 - \lambda^3$, $\lambda \in \mathbb{C}$, is studied in [CGS]. (The Newton's method for any cubic polynomial whose roots are not all equal is affine conjugate to one of these). The λ-plane is coloured say red, green or blue according to which root $z_j = \omega^j + \omega^{-j}\lambda$, $j \in \{0, 1, 2\}$ (where $\omega = e^{2\pi i/3}$) is obtained by iterating the map $z \mapsto z - p_\lambda(z)/p_\lambda'(z)$ from the starting point $z = 0$. The remainder of the plane is left white and, by a Theorem of Fatou, contains any λ for which there exists an attracting periodic cycle other than the roots of p_λ. In [DH3] Douady and Hubbard construct a notion of "polynomial-like"

analytic mappings whose dynamics they prove mimic that of polynomials. They go on to develop a theory for "Mandelbrot-like" families of quadratic-like mappings which enable them to prove in particular that a white copy of $\mathcal{M}$ observed in the above plane is actually homeomorphic to $\mathcal{M}$ (thus ensuring that close-ups agree with $\mathcal{M}$ down to the finest detail). With this theory they also prove that along the "filaments" of $\mathcal{M}$ there exist infinitely many homeomorphic copies of $\mathcal{M}$.

A polynomial-like map of degree d is defined as a triple (U,U',f) where U',U are two simply connected domains in $\mathbb{C}$ with the closure of U' contained in U and $f: U' \to U$ is a proper analytic mapping of degree d. (This means essentially that the boundary of U' is wrapped d times around and mapped onto the boundary of U). There is an analogous definition of the "filled-in Julia set":

$$K(f,U') := \left\{ z \in U' : \ f^n(z) \in U' \text{ for } n = 1, 2, 3, \ldots \right\}.$$

The fundamental result of polynomial-like mappings $\left([D], [DH3] \right)$ is that if $f: U' \to U$ is polynomial-like then f restricted to some neighborhood of $K(f,U')$ is conjugate by a (quasi-conformal) homeomorphism to some polynomial p of degree d restricted to a neighborhood of $K(p)$.

In the light of these results it is clear that the family $f_c : z \mapsto z^2 + c$ describes behaviour of maps far more general than just the quadratic family.

When $c \in \mathbb{C} \setminus \mathcal{M}$ the classical theory gives that (J, f_c) is topologically conjugate to the shift dynamical system (Σ, σ) where $\Sigma = \{0,1\}^{\mathbb{N}}$ and $\sigma : \Sigma \to \Sigma$ is the two-to-one map which shifts sequences one place to the left. $\left(\text{Here } \{0,1\}^{\mathbb{N}} \text{ has the product topology and is homeomorphic to the standard Cantor set in } \mathbb{R} \text{ by the map } x \mapsto \sum_{n \geq 1} (2x_n)/3^n \right)$. A result of Hedlund in [H] on automorphisms of the 2-shift shows that this conjugacy is unique up to permutation of the symbol set $\{0,1\}$.

For any $c \in \mathbb{C}$ there is an analytic conjugacy φ_c which conjugates the map f_c, on some neighborhood of ∞, to the map $f_0 : w \mapsto w^2$.

When $c \in \mathcal{M}$, Douady and Hubbard show that the conjugacy φ_c can be extended to the whole basin of attraction of ∞ for f_c thereby giving a "Riemann map" φ_c^{-1} from $\mathbb{C} \cup \{\infty\} \setminus D$, where D is the closed unit disk, onto $\mathbb{C} \cup \{\infty\} \setminus K(f_c)$. Associated to such a conformal isomorphism and to any $\theta \in \mathbb{R}/\mathbb{Z}$ is the "external ray of argument θ" defined as

$$R(f_c,\theta) := \left\{ \varphi_c^{-1}(r \cdot e^{2\pi i \theta}) : \ r > 1 \right\}.$$

A description of the topological dynamics on the Julia set then reduces to the questions:

(i) Do the rays $R(f_c,\theta)$ converge as $r \searrow 1$? Do they converge uniformly in $\theta \in \mathbb{R}/\mathbb{Z}$?

(ii) Which rays converge to the same point?

The question of uniform convergence is equivalent to whether the Julia set is locally-connected. Douady and Hubbard provide the answer yes for quite a wide range of values of c in $\mathcal{M}$: in particular when c is a *Misiurewicz* point, that is the critical iterates eventually land on an unstable periodic orbit. The last question reduces to combinatorics associated with the map $\theta \mapsto 2\theta \pmod{\mathbb{Z}}$.

Fig. 0.2 Close-up of the filament of $\mathcal{M}$ containing the point i.

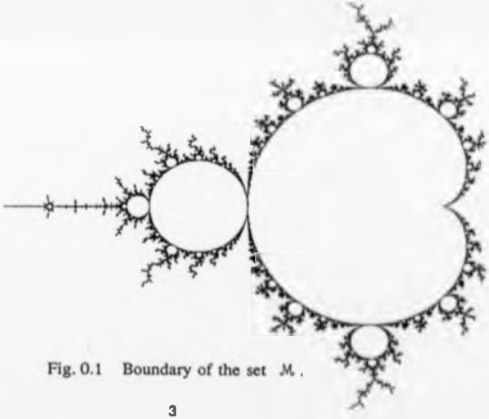


Fig. 0.1 Boundary of the set $\mathcal{M}$.

Douady and Hubbard in [DH2] give an astonishingly elegant proof that the Mandelbrot set is connected by producing a conformal isomorphism between $\mathbb{C}\setminus\mathcal{M}$ and $\mathbb{C}\setminus\mathbb{D}$ (essentially the map $c \mapsto \phi_c(c)$). They prove that the analogously defined external rays $R(\mathcal{M},\theta)$ converge whenever θ is rational : the Misiurewicz points being the limit points of those rays where θ is rational of even denominator. There is a sophisticated combinatorial description of which rays $R(\mathcal{M},\theta)$ converge to the same point of $\mathcal{M}$. However the combinatorial picture cannot be known to describe $\mathcal{M}$ exactly until the question is resolved : Is $\mathcal{M}$ locally-connected ?

The author entered the domain of complex dynamics from a slightly different perspective. Being more familiar with the dynamics of the real unimodal maps than with complex analysis it was decided to investigate the role that "kneading theory" might have to play for the complex quadratic family f_c when $c \in \mathcal{M}$. We are faced with two immediate problems which did not exist (or were easily solved) for the real quadratic map :

(i) How to partition the complex plane so as to induce a "decent" coding of the Julia set ;

(ii) How to make sense of the "ordering" of symbol sequences induced by the topological connectivity of the Julia set.

(For the real map, the natural partition to take is to divide the real line at the critical point 0 into the two halves $(-\infty,0)$ and $(0,+\infty)$, coding a point z according to the sides on which its iterates land. Milnor and Thurston [M] & [MT] construct a well-documented invertible transformation of symbol sequences to ensure that the resulting "coordinate" varies monotonically in z).

A "decent" coding in the complex case is interpreted to mean one where the set of z (in $J(f_c)$) corresponding to any given symbol sequence is a single point (or at the very least a connected set). One such coding (in the locally-connected case) is Douady and Hubbard's binary expansion of the external argument θ outlined in [DH1]. (Here θ is an external argument of z if the ray $R(f_c,\theta)$ converges to z). This can be thought of as being induced by a partition curve which intersects the filled-in Julia set $K(f_c)$ in an arc running from the more unstable fixed point β (of external argument 0) to its other inverse-image β' (of external argument $\frac{1}{2}$). If $K(f_c)$ is a "dendrite" (has no interior) then the uncountably many points of this arc (excepting β and β') will each have at least two external arguments.

The partition curve in which we shall be interested in this thesis, however, is one which intersects the filled-in Julia set in precisely one point, namely the critical point 0. This has the advantage that only countably many points (the inverse-images of 0) have more than one code associated to them. The partition concurs with the partition used for the real map $f|_{\mathbb{R}}$ in the case $c \in \mathbb{R}$, but we need the assumption that 0 lies in $J(f_c)$. Unlike for the real mapping however there does not appear to be any nice transformation to a "monotonic" coordinate.

The coding is based on the empirical observation that when c is a "generic" point on a filament of $\mathcal{M}$ and we perturb the parameter c in a certain "direction" away from the

filament then the Julia set breaks up in "continuous" fashion . Using the unique coding of $J(f_c)$ when it is a Cantor set and by unperturbing we obtain a coding for the connected Julia set . As far as visualising the coding is concerned there is really no more instructive method than to follow the coding of the Julia set $J(f_c)$ as c follows some path outside $\mathcal{M}$ starting from a standard Julia set whose dynamics are more easily visualised .

Consider the parameter $c_1 = i$ which is an endpoint on the right-hand branch of the filament shown in Figure 0.2 and is a Misiurewicz point since $0 \to i \to -1 \mapsto i \leftrightarrow -i$. The Julia set for $z \mapsto z^2 + i$ is shown in Figure 0.3 (ix) . A Cantor set perturbation is shown in Figure 0.3 (viii) . Now consider a path going outside $\mathcal{M}$ and represented by the Figs. 0.3 (i)-(ix) . It starts from the parameter $c_0 = 1/4$, the cusp of the main cardioid in Fig. 0.1 . For c equals $1/4$ or any value inside the main cardioid $\mathcal{C}$ the Julia set $J(f_c)$ is a Jordan curve and the action of f_c on this curve is the doubling map (conjugate to $z \mapsto z^2$ on the unit circle , which is the case $c = 0$) . Points z inside the curve (including the critical point 0) are all sucked towards a fixed point α . The point α is attracting except in the case $c = 1/4$ where α has merged with the other fixed point β on the Julia set to become the parabolic fixed point $z = 1/2$. As c increases in the positive-real direction beyond $1/4$ the fixed points break apart in the imaginary direction allowing the previously trapped critical iterates to squeeze through the gap created and escape to ∞ . The Julia set has broken up into a Cantor set like that in Fig. 0.3 (ii) . The dynamics of f_c on this Julia set are still quite easy to visualise . Essentially we have a doubling map on a broken up circle which we can now code by $\{0,1\}^{\mathbb{N}}$ according to taking $\mathbb{R}$ as the partition line , labelling the upper-half plane 0 and the lower-half plane 1 . $\Big($One can think of this coding as being inherited , via the bifurcation at $c = 1/4$, from Douady and Hubbard's binary expansion of the external argument for the Julia sets $J(f_c)$, $c \in (-3/4, 1/4) = \mathcal{C} \cap \mathbb{R}\Big)$.

Fig. 0.4 Standard model of a Cantor Julia set .

As c moves off the real axis into the upper-half plane the fixed point 1^∞ becomes more repelling and so less spirally than the other fixed point 0^∞ (Fig. 0.3 (iii)) . As c

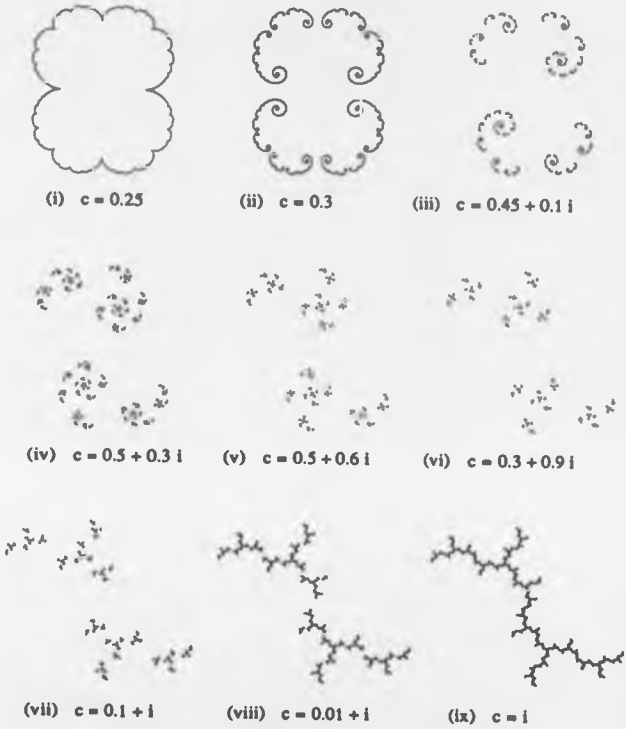


Fig. 0.3 Julia sets for the family $f_c : z \mapsto z^2 + c$.

moves further up the parameter plane the argument of the derivative of the fixed point 0^{∞} increases and so nearby bits of Julia set are wound around more and more in a whirlpool effect (Figs. 0.3 (iv)-(vii)). As c approaches its destination c_1, with the argument of the derivative of the fixed point 0^{∞} something near $2\pi/3$, the nearby bits of Julia set align themselves in a pattern which makes the point 0^{∞} tri-valent when the Julia set finally re-connects up at $c = i$.

The identifications on the Cantor set which occur can be re-traced to the standard model so that we get a picture like :

Fig. 0.5 Standard model with identifications .

In this case the two sequences identified to the critical point are $0\,0(0\,1)^{\infty}$ and $1\,0(0\,1)^{\infty}$. We write the "kneading itinerary" ξ to be the common forward image - the "critical value" - which in this case is the sequence $0(0\,1)^{\infty}$. The quotient of Σ obtained is completely determined by ξ and is written Σ_{ξ}.

In our text we shall analyse the family $z \mapsto z^{d}+c$ not because this represents any substantial generalisation to higher degree polynomials, but because of the ease with which the ideas extend . Our sequences (whether "phase-space" or "parameter-space") are indexed by $\mathbb{N}$ rather than $\mathbb{Z}_{+}$ because we will often be working with parameter space sequences and these represent the critical value as opposed to the critical point . (Such notation would not be recommended when studying polynomials with more than one criticalpoint).

Most of Chapter one is concerned with making of some the above ideas rigorous : principally the notion that $J(f_c)$ might be a quotient of Σ by such a partition . The main device for constructing the partition is to use an external ray from the Douady and Hubbard theory which converges to the image of the critical point . The main result is Theorem 1.5.3 . The essence of the proof here is the same as the proof , given in much of

the literature including [B], that the Julia set is a Cantor set in the case the critical iterates tend to infinity. However considerable intricacy is required to adapt this to our case and most of what precedes the theorem is the rather technical preparations required. The question of trying to prove the hypothesis of the convergence of an external ray to the critical value (other than in the Misiurewicz case, where the result is otherwise deducible from the Douady-Hubbard theory) is not really addressed in this thesis, although the author has been able to obtain examples for nonperiodic perturbations of Misiurewicz points. (The author has recently heard of a new and independent result of J.C.Yoccoz which may well improve on Theorem 1.5.3 and certainly seems to establish the local-connectivity of a very large class of Julia sets. However I have been unable as yet to obtain a copy of the fully written-up version of this result).

Chapters two, three and four, more abstract in nature, are concerned with the structure of the quotient spaces Σ_ξ that arise. Chapter two deals with the elementary theory of such spaces (and infact some slightly more general spaces Σ_Γ so as to apply to another space Σ_Γ which is somewhat akin to the Mandelbrot set). For technical reasons we shall often work the pre-identified "glueing spaces" (Σ,Γ). Section 2.6 discusses the special case which corresponds to the dynamics for real quadratic mappings and outlines the relationship to the Thurston-Milnor kneading theory.

Chapter three describes a "renormalisation" semi-group under a "$\circ$-product" which is an abstract version of the more commonly studied renormalisations for real unimodal mappings (but which extends to kneading itineraries ξ which are not "admissible" in the sense of Milnor [M]). It also sets up some technical results required for the next chapter.

The main result in Chapter four is Theorem M which is the attempt to metrize the topological space Σ_ξ in such a way that the shift map σ acts in a piecewise uniformly expanding fashion. The metric is required to be "geodesic" in terms of the dendritic structure of Σ_ξ. The existence of such a metric turns out to depend on decomposability properties of the kneading itinerary ξ. For "most" ξ the metric exists and the remaining cases are of two types: either where the kneading itinerary is of generalised Feigenbaum type (type (A)) or where Σ_ξ contains a number $\ell\,(>1)$ of disjoint embedded copies of another such space Σ_ψ which are permuted cyclically under the shift action (type (B)). In the latter case ψ is the sequence $\xi_\ell\,\xi_{2\ell}\,\xi_{3\ell}\ldots$ and it is still possible to give the space Σ_ξ a pseudo-metric (with the above properties) by which the embedded copies of Σ_ψ are effectively shrunk to points. Such a metric (or pseudo-metric) in the "real" case is equivalent to conjugacy (resp. semi-conjugacy) of a real unimodal map to a "tent" map (after identifying "monotone equivalent" points (see [MT])). Theorem M can thus be viewed as extending the Thurston-Milnor kneading theory for real unimodal maps to the complex quadratic case.

The essential tool in the construction of the (pseudo-)metric is an infinite-dimensional nonnegative linear operator φ_ξ^* which is the dual of an operator φ_ξ which essentially acts like iterating the map σ on the countable set of "arcs" between forward iterates of the "critical value" ξ. The (pseudo-)metric is essentially the "Perron-

Frobenius" eigenvector of φ_ξ^* and the corresponding eigenvalue is the constant of uniform expansion. The last two sections of Chapter four pursue the theory of φ_ξ further with section 4.8 emphasising the role of the space Σ_ξ in the variation of the "expansion constant" with ξ.

Chapter five is really an appendix and does not depend on the other chapters. It is needed for proof of the Theorem M and is really a pre-requisite for Chapter four. It is heavily based on the Vere-Jones "Perron-Frobenius" theory for countable nonnegative matrices (and is ultimately derived from the theory for countable-state Markov chains). The chapter is essentially an adaptation of the Vere-Jones theory to allow for the possibility that the matrix T under study is not irreducible. However we restrict to matrices which act as bounded linear operators over the space ℓ^1. We use the additional strong assumption that T is "quasi-finite" in place of the Vere-Jones assumptions of *R-recurrence* and *R-positivity* for T.

The author suspects that a deeper understanding of the meromorphic behaviour of the resolvent operator $(\lambda-T)^{-1}$ in the case T is a "quasi-compact" operator would probably result in a substantial simplification of this chapter.

CHAPTER 1

§1.1 POLYNOMIAL MAPS WITH ONE CRITICAL POINT :

We are concerned with iterations, on the complex plane, of a map f the form $f_c : z \mapsto z^d + c$ where $c \in \mathbb{C}$ and $d \geq 2$. The Julia set $J = J(f)$ is the boundary of the compact set $K(f) = \left\{ z \in \mathbb{C} : f^n(z) \not\to \infty \text{ as } n \to \infty \right\}$. Observe that f has a unique critical point at $z = 0$ and that the corresponding critical value is the complex parameter c. From the classical theory of Julia and Fatou (See Blanchard [B]), there are two possibilities for J depending on the iterates of 0:

Either the iterates of 0 escape to ∞, in which case J is homeomorphic to the Cantor set $\Sigma_d = L^{\mathbb{N}}$ where L is a set of d symbols, say $L = \{0, 1, \ldots, d-1\}$, with the action of f on J conjugate to the shift map σ $\left(\text{where } (\sigma(x))_i = x_{i+1} \text{ for all } i \in \mathbb{N} \right)$

or the iterates of 0 remain bounded, in which case J is connected. In the latter case Douady and Hubbard [DH1] prove that the local conjugacy between f_c and the map $f_0 : w \mapsto w^d$ for large $|z|$, extends to a conjugacy φ_c from $\mathbb{C} \setminus K(f)$ to $\mathbb{C} \setminus D$, where D is the closed unit disc. Here $\varphi_c(z)/z \to 1$ as $z \to \infty$ and the "potential" function $G_c : \mathbb{C} \setminus K(f) \to \mathbb{R}_+$ defined by

$$G_c(z) = \lim_{n \to \infty} \left(\log |f^n(z)| \right) / d^n$$

satisfies $\log |\varphi_c(z)| = G_c(z)$ for $z \in \mathbb{C} \setminus K(f)$.

DEFINITION 1.1.1 : The *external ray* $R(f_c, \theta)$ *of argument* θ is defined, for $\theta \in \mathbb{R}/\mathbb{Z}$, as the image-set of the map $r \mapsto \varphi_c^{-1}(r \cdot e^{2\pi i \theta})$ from $(1, +\infty)$ to $\mathbb{C}$.

Locally connected Julia Sets :

A topological space is *locally-path-connected* if every neighborhood U of a point x has a sub-neighborhood (of x), every point of which is connected to x by a path contained in U. For closed subsets of $\mathbb{C}$, *locally-connected* is the same as locally-path-connected. (See [DH2, exp. II]).

If the Julia set J is connected and also locally-connected then, by a theorem of Caratheodory, the "Riemann map" φ_c^{-1} defined on $\mathbb{C} \setminus D$ extends continuously to the boundary S^1 and thereby gives a semi-conjugacy $\gamma : S^1 \to J$. Given a point z in J, a value $\theta \in \mathbb{R}/\mathbb{Z}$ is called an *external argument* of z if $\gamma(e^{2\pi i \theta}) = z$. In [DH2, exp. III & X] Douady and Hubbard establish that the Julia set J is locally-connected if either :

(i) The iterates of 0 converge on an attracting periodic orbit ;

(ii) The iterates of 0 converge on a periodic orbit whose derivative is a root of unity ;

or (iii) The critical point is *pre-periodic* : that is , the iterates of 0 land , in finite time , on a periodic orbit not containing 0 .

CASE Critical Point 'Preperiodic' :

The proof of the local-connectivity of J in case (iii) involves constructing a "sub-hyperbolic" metric on a neighborhood U of J . This means a Riemannian metric $\| \ \|$ on $U \setminus A$ where A is the finite set $\left\{ f^n(0) : n \in \mathbb{N} \right\}$, that is given by $\|dz\| = u(z) \cdot |dz|$ where $| \ |$ is the Euclidean norm on $\mathbb{C}$ and $u : U \setminus A \to \mathbb{R}$ is a continuous positive-valued function such that $|z - a|^{1-1/d} \cdot u(z)$ tends to a finite non-zero limit as z tends to a , for every $a \in A$. $\Big($ The fact that the exponent $1-1/d$ is less than 1 guarantees that every compact $\mathbb{R}$-analytic curve γ in U has finite $\| \ \|$-length $\int_\gamma \|dz\|$ $\Big)$. The Riemannian metric has the added property that there exists $\rho > 1$ such that

$$u\big(f(z)\big) \cdot |f'(z)| \geq \rho \cdot u(z)$$

for all $z \in U \cap f^{-1}(U \setminus A)$, and hence that the map is expanding in the sense that

$$\|df(z)\| \geq \rho \cdot \|dz\| .$$

EXAMPLE : If f is the map $z \mapsto z^2 - 2$ then $0 \to -2 \to +2 \circlearrowleft$ and J is the real interval $\left[-2, +2\right]$. The sub-hyperbolic metric

$$\|dz\| = |dz| / |z^2 - 4|^{\frac{1}{2}}$$

defined on any neighborhood of J satisfies

$$\|df(z)\| = 2 \cdot \|dz\| .$$

§1.2 CODINGS OF JULIA SETS :

Thurston , in $\left[T \right]$, considers the structure and dynamics of locally-connected polynomial Julia sets , in terms of "laminations" on the disk D $\Big($ equivalence relations on S^1 , invariant under the action $z \mapsto z^d$, which are theoretically induced when extending the Riemann map conjugacy for such polynomials $\Big)$.

We shall adopt the alternative point of view of treating the Julia sets as the quotient of Σ_d with the action of f represented by the shift map σ . We shall have to restrict to cases where the Julia set turns out to be a dendrite . $\Big($ A subset of $\mathbb{R}^2$ is said to be a *dendrite* if it is compact , connected , locally-connected , simply-connected and has no interior $\Big)$.

Informally , these cases occur when the parameter c_0 lies in a certain set of boundary points of the set $\mathcal{M} = \mathcal{M}_d := \left\{ c \in \mathbb{C} : J(f_c) \text{ is connected} \right\}$ and is the limit of a curve γ in $\mathbb{C} \setminus \mathcal{M}$. The semi-conjugacy $\Sigma_d \to J(f_{c_0}) \left(\subset \mathbb{C} \right)$ can then be thought of as the limit of conjugacies $\Sigma_d \to J(f_c) \left(\subset \mathbb{C} \right)$ as c tends to c_0 along γ .

CASE $K(f)$ is 'Partitionable' :

We suppose that the critical point 0 lies in the Julia set J and furthermore that, for some $\eta \in \mathbb{R}$, the external ray $\left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i\eta}\right) : r > 1 \right\}$ converges, as $r \searrow 1$, on $c \left(= f(0)\right)$. The d inverse-image rays $\left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i(\eta+j)/d}\right) : r > 1 \right\}$ (for $j = 0, 1, \ldots, d-1$) therefore converge on 0 as $r \searrow 1$ and partition the remainder of the complex plane into d components $\mathbb{C}_0, \mathbb{C}_1, \ldots, \mathbb{C}_{d-1}$. We may adopt the convention that $\mathbb{C}_j$ contains the region

$$\varphi_c^{-1}\left\{ r \cdot e^{2\pi i\theta} : r > 1 \text{ and } (\eta+j)/d < \theta < (\eta+j+1)/d \right\}.$$

Since f restricted to $\mathbb{C} \setminus f^{-1}\left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i\eta}\right) : r > 1 \right\} \setminus \{0\}$ is a covering map onto the simply-connected domain $\mathbb{C} \setminus \left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i\eta}\right) : r > 1 \right\} \setminus \{c\}$ it follows that it admits d inverse maps g_j $(j = 0, 1, \ldots, d-1)$ defined on $\mathbb{C} \setminus \left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i\eta}\right) : r > 1 \right\} \setminus \{c\}$ with images the corresponding $\mathbb{C}_j$.

Coding of $K(f)$:

For $z \in K(f)\setminus\{0\}$ we define $\alpha(z) = j$ where $z \in \mathbb{C}_j$. Given $z \in K(f) \setminus \bigcup\left\{ f^{-n}(0) : n \in \mathbb{Z}_+ \right\}$ we can define a sequence $x(z)$ in Σ_d by $\left(x(z)\right)_n = \alpha\left(f^{n-1}(z)\right)$. Notice $c \notin \bigcup\left\{ f^{-n}(0) : n \in \mathbb{Z}_+ \right\}$ since otherwise 0 would be periodic and so would lie in a (super-)attracting cycle of f thereby contradicting the assumption $0 \in J$. We can therefore define the *kneading itinerary* of f by

$$\xi = \xi(f,\eta) := x(c).$$

If, on the other hand, $f^{m-1}(z) = 0$ for some $n \geq 1$ then there are d possible choices for corresponding sequence $x(z)$ according to the choice of value $\left(x(z)\right)_n$ in $\{0, 1, \ldots, d-1\}$ - the remaining entries being given by

$$\left(x(z)\right)_m = \begin{cases} \alpha\left(f^{m-1}(z)\right) & \text{if } m \neq n \\ = \xi_{m-n} & \text{if } m > n \end{cases}.$$

One might suspect therefore that there is induced a continuous map $\Pi : K(f) \to \Sigma_\xi$ where Σ_ξ is the space obtained from Σ_d by identifying together those groups of points $\left\{ x_1 x_2 \ldots x_{n-1} \alpha\, \xi \;:\; \alpha \in \{0, 1, \ldots, d-1\} \right\}$ where $\left(x_1, x_2, \ldots, x_{n-1}\right)$ ranges through $\{0, 1, \ldots, d-1\}^{n-1}$ and n ranges through $\mathbb{N}$. We shall see that this map Π does indeed exist and is surjective. The question of whether the map is one-to-one is left to a later section.

§1.3 CUTS AND CODING RELATIONS :

DEFINITION 1.3.1 Let f be a polynomial $z \mapsto z^d + c$. A *cut* R is the image-set of a continuous embedding $\gamma : [0, +\infty) \to \mathbb{C}$ satisfying $\gamma(t) \to \infty$ as $t \to +\infty$ and $\gamma(0) = c$. Now $\mathbb{C} \setminus R$ is a simply-connected domain. The map f restricted to $\mathbb{C} \setminus f^{-1}(R)$ is a d-to-one covering onto $\mathbb{C} \setminus R$ and so admits d inverses

$g_0, g_1, \ldots, g_{d-1}$ defined on $\mathbb{C} \setminus R$. The sets $\mathbb{C}_\alpha := g_\alpha(\mathbb{C} \setminus R)$ are the connected components of $\mathbb{C} \setminus f^{-1}(R)$. Furthermore, if U is a (bounded) connected domain then the sets $U_\alpha = g_\alpha(U \setminus R)$ will be connected provided that U is transverse to R in the following sense:

If R' is the image-set of an embedding $\gamma : [0, +\infty) \to \mathbb{C}$ satisfying $\gamma(t) \to \infty$ as $t \to +\infty$ then a bounded open set U is said to be *transverse* to R' if either $\gamma^{-1}(\partial U) = \{t_0\}$ and $\gamma^{-1}(U) = [0, t_0)$ for some $t_0 \geq 0$ or $\gamma^{-1}(\mathrm{clos}(U)) = \emptyset$.

DEFINITION 1.3.2: A bounded simply-connected domain U in $\mathbb{C}$ is called a B.I.D. ("backward invariant disk") if $\mathrm{clos}(f^{-1}(U)) \subset U$.

For any B.I.D. U we have $K(f) = \bigcap \{ f^{-n}(U) : n \geq 0 \}$.
(To see this, observe that $\mathbb{C} \setminus \mathrm{clos}(f^{-1}(U))$ is forward invariant and so is contained in the domain F of normality for $\{f^n : n \in \mathbb{N}\}$ (with respect to the spherical metric on the Riemann sphere $\mathbb{C} \cup \{\infty\}$, see (Hille, section 15.2)). Now the set $\mathbb{C} \setminus K(f) = \{ z \in \mathbb{C} : f^n(z) \to \infty \}$ lies in F, its boundary $J(f)$ is disjoint from F. Hence $\mathbb{C} \setminus K(f)$ contains the unbounded component of F and so contains $\mathbb{C} \setminus U$. Thus $K(f) \subseteq U$ and so $K(f)$ is contained in the inverse-images of U. On the other hand, since U is bounded, any z not in $K(f)$ eventually escapes from U. For a fuller discussion of these ideas see [B]).

Write $U_\alpha = g_\alpha(U \setminus R)$, for $\alpha \in \{0, 1, \ldots, d-1\}$. We can define also $U_{x_1 x_2 \ldots x_n}$ for finite words $x_1 x_2 \ldots x_n$ in $\{0, 1, \ldots, d-1\}^n$ for $n \geq 2$ recursively in n by

$$U_{x_1 x_2 \ldots x_n} = g_{x_1}(U_{x_2 \ldots x_n} \setminus R).$$

Similarly we define $\mathbb{C}_{x_1 x_2 \ldots x_n}$ recursively in n by

$$\mathbb{C}_{x_1 x_2 \ldots x_n} = g_{x_1}(\mathbb{C}_{x_2 \ldots x_n} \setminus R).$$

NOTE: Given $V \subseteq \mathbb{C}$ we have $g_\alpha(V \setminus R) = \mathbb{C}_\alpha \cap f^{-1}(V)$. In particular, $U_\alpha = \mathbb{C}_\alpha \cap f^{-1}(U)$ and $U_{x_1 \ldots x_n} = \mathbb{C}_{x_1 \ldots x_n} \cap f^{-n}(U)$ for all $\alpha \in L$ and words $x_1 \ldots x_n$.

DEFINITION 1.3.3: We define a relation S between U and Σ_d by

$$(z, x) \in S \iff f^{n-1}(z) \in U_{x_n} \text{ for all } n \in \mathbb{N}.$$

We also define, for $N \geq 1$, the relations $S_N \subseteq U \times \Sigma_d$ by

$$(z, x) \in S_N \iff f^{n-1}(z) \in U_{x_n} \text{ for all } n \leq N.$$

Clearly S is the (decreasing) intersection of the (open) sets S_N.

The actual relation we want is not S but $S' := \bigcap \{ \mathrm{clos}(S_N) : N \geq 1 \}$. Note that one can show that S' is the closure of S unless $K(f)$ is covered by the inverse images of R.

PROPOSITION 1.3.1 : *If* $z \in U$ *and* $x \in \Sigma_d$ *then*

(i) $(z,x) \in S_N \iff z \in U_{x_1 x_2 \ldots x_N}$;

(ii) $(z,x) \in \mathrm{clos}(S_N) \iff z \in \mathrm{clos}\left(U_{x_1 \ldots x_N}\right)$.

Proof . We prove (i) by induction on N . Our inductive hypothesis is

$$(z,x) \in S_{N-1} \iff z \in U_{x_1 \ldots x_{N-1}} .$$

Thus

$$\begin{aligned}(z,x) \in S_N &\iff z \in U_{x_1} \text{ and } \left(f(z),\sigma x\right) \in S_{N-1} \\ &\iff z \in g_{x_1}\left(U \setminus R\right) \text{ and } f(z) \in U_{x_2 \ldots x_N} \\ &\iff z \in g_{x_1}\left(U \setminus R\right) \cap f^{-1}\left(U_{x_2 \ldots x_N}\right) \\ &\iff z \in g_{x_1}\left(U_{x_2 \ldots x_N} \setminus R\right) = U_{x_1 x_2 \ldots x_N}\end{aligned}$$

To prove (ii) , if $(z,x) \notin \mathrm{clos}(S_N)$ then , for some neighborhoods V of z and W of x , we have $\left(V \times W\right) \cap S_N = \emptyset$ and so in particular $z' \notin U_{x_1 \ldots x_N}$ for all $z' \in V$, and so $z \notin \mathrm{clos}\left(U_{x_1 \ldots x_N}\right)$. On the other hand , if $V \cap U_{x_1 \ldots x_N} = \emptyset$ for some neighborhood V of z then $\left(V \times \left[x_1 \ldots x_N\right]\right) \cap S_N = \emptyset$ and so $(z,x) \notin \mathrm{clos}(S_N)$. □

LEMMA 1.3.2 : *If* $V \subseteq \mathbb{C}$ *and* $\alpha \in L$ *then* $\mathrm{clos}\left(g_\alpha(V \setminus R)\right) \supseteq g_\alpha\left(\mathrm{clos}(V) \setminus R\right)$. *However* $\mathrm{clos}\left(g_\alpha(V \setminus R)\right) \subseteq g_\alpha\left(\mathrm{clos}(V) \setminus R\right) \cup f^{-1}\left(R \cap \mathrm{clos}(V)\right)$.

Proof . If $z \in g_\alpha\left(\mathrm{clos}(V) \setminus R\right)$ then $f(z) \in \mathrm{clos}(V) \setminus R$ and so $f(z) \in \mathrm{clos}(V \setminus R)$ since R is closed . The continuity of g_α implies $z = g_\alpha\left(f(z)\right) \in \mathrm{clos}\left(g_\alpha(V \setminus R)\right)$.

If z adheres to $g_\alpha(V \setminus R)$ then $f(z)$ adheres to $V \setminus R$, and hence to V . Thus either $f(z) \in R \cap \mathrm{clos}(V)$ or $z \in f^{-1}\left(\mathrm{clos}(V) \setminus R\right)$. In the latter case $z \in g_\beta\left(\mathrm{clos}(V) \setminus R\right)$ for some $\beta \in L$. Now $g_\beta\left(\mathrm{clos}(V) \setminus R\right)$ is relatively open in $f^{-1}\left(\mathrm{clos}(V) \setminus R\right)$. So $\beta = \alpha$ since otherwise $g_\alpha(V \setminus R)$ is contained in $f^{-1}\left(\mathrm{clos}(V) \setminus R\right) \setminus g_\beta\left(\mathrm{clos}(V) \setminus R\right)$ which is separated from z . □

COROLLARY 1.3.2.1 : *If* $R \cap \mathrm{clos}(V) = \emptyset$ *then* $\mathrm{clos}\left(g_\alpha(V)\right) = g_\alpha\left(\mathrm{clos}(V)\right)$. □

COROLLARY 1.3.2.2 : *For all words* $x_1 \ldots x_N$ *we have*

$$\mathrm{clos}\left(U_{x_1 \ldots x_N} \cap f^{-N-1}(U)\right) = \bigcup_{\beta \in L} \mathrm{clos}\left(U_{x_1 \ldots x_N \beta}\right) .$$

Proof. First observe that $f^{-1}(U) \subseteq \mathrm{clos}\left(\bigcup_{\beta \in L} U_\beta\right)$. So "applying" g_{x_N} gives us $U_{x_n} \cap f^{-2}(U) = g_{x_N}\left(f^{-1}(U) \setminus R\right) \subseteq g_{x_N}\left(\mathrm{clos}\left(\bigcup_{\beta \in L} U_\beta\right) \setminus R\right)$ which is contained in $\mathrm{clos}\left(g_{x_N}\left(\left(\bigcup_{\beta \in L} U_\beta\right) \setminus R\right)\right) = \mathrm{clos}\left(\bigcup_{\beta \in L} g_{x_N}(U_\beta \setminus R)\right) = \mathrm{clos}\left(\bigcup_{\beta \in L} U_{x_N \beta}\right)$. Now successively "applying" $g_{x_{N-1}}, \ldots, g_{x_1}$ gives

$$U_{x_1 \ldots x_N} \cap f^{-N-1}(U) \subseteq \mathrm{clos}\left(\bigcup_{\beta \in L} U_{x_1 \ldots x_N \beta}\right)$$

and taking closures gives the inclusion one way. On the other hand $\bigcup_{\beta \in L} U_{x_1 \ldots x_N \beta}$ is contained in $U_{x_1 \ldots x_N} \cap f^{-N-1}(U)$ and so taking closures gives the inclusion the other way. □

LEMMA 1.3.3: *For all sequences* $x \in \Sigma_d$ *and all* $N \in \mathbb{N}$ *we have*

$$z \in \mathrm{clos}\left(U_{x_1 \ldots x_N}\right) \Rightarrow f^{n-1}(z) \in \mathrm{clos}\left(U_{x_n}\right) \quad \text{whenever} \quad 1 \le n \le N .$$

Proof. We prove by induction on N. Suppose $N \ge 2$.

$$\begin{aligned} z \in \mathrm{clos}\left(U_{x_1 \ldots x_N}\right) &\Leftrightarrow z \in \mathrm{clos}\left(g_{x_1}\left(U_{x_2 \ldots x_N} \setminus R\right)\right) \\ &\Leftrightarrow z \in \mathrm{clos}\left(g_{x_1}(U \setminus R) \cap f^{-1}\left(U_{x_2 \ldots x_N}\right)\right) \\ &\Rightarrow z \in \mathrm{clos}\left(g_{x_1}(U \setminus R)\right) \text{ and } z \in \mathrm{clos}\left(f^{-1}\left(U_{x_2 \ldots x_N}\right)\right) \\ &\Leftrightarrow z \in \mathrm{clos}\left(U_{x_1}\right) \text{ and } f(z) \in \mathrm{clos}\left(U_{x_2 \ldots x_N}\right). \end{aligned}$$ □

NOTE: We cannot obtain the two-sided implication without further restrictions on R. See Proposition 1.4.2.

THEOREM 1.3.4: *Observe that* S' *is much like a surjective function defined on* $K(f)$:

(i) $\forall z \in K(f)\ \exists x \in \Sigma_d$ with $(z,x) \in S'$;

(ii) $(z,x) \in S$ and $(z,y) \in S' \Rightarrow x = y$;

(iii) $\forall x \in \Sigma_d\ \exists z \in K(f)$ with $(z,x) \in S'$.

Proof of (i): If $z \in K(f)$ then $z \in \mathrm{clos}\left(U_{x_1}\right)$ for some $x_1 \in L$. Since z lies in (the interior of) $f^{-2}(U)$ it follows that $z \in \mathrm{clos}\left(U_{x_1} \cap f^{-2}(U)\right)$ and so, by Corollary 1.3.2.2, $z \in \mathrm{clos}\left(U_{x_1 x_2}\right)$ for some $x_2 \in L$. Repeating this argument gives that $z \in \mathrm{clos}\left(U_{x_1 \ldots x_N}\right)$ for all N and so for some choice of sequence $x \in \Sigma_d$. Thus $(z,x) \in S'$.

of (ii): If $(z,x), (z,y) \in S'$ and $x \ne y$ then $x_n \ne y_n$ for some $n \ge 1$. Lemma 1.3.3 then tells us $f^{n-1}(z) \in \mathrm{clos}\left(U_{x_n}\right) \cap \mathrm{clos}\left(U_{y_n}\right) \subseteq f^{-1}(R)$. So $(z,x), (z,y) \in S' \setminus S$.

of (iii): If $x \in \Sigma_d$ then for all $N \ge 1$, the sets $U_{x_1 \ldots x_N}$ are non-empty and open. Consequently $\mathrm{clos}\left(U_{x_1 \ldots x_N}\right), \left(N \ge 1\right)$ forms a decreasing

sequence of non-empty closed (compact) sets , and so their intersection is non-empty , containing z , say . Thus $(z,x) \in S'$. □

DEFINITION 1.3.4 : For $x \in \Sigma$ we define

$$K_x := \left\{ z \in U : (z,x) \in S' \right\} = \bigcap \left\{ \mathrm{clos}(U_{x_1 \ldots x_N}) : N \geq 1 \right\}.$$

By the above theorem we have that K_x is always nonempty and $K(f) = \bigcup_{x \in \Sigma} K_x$. Furthermore K_x is independent of the choice of U as $K_x = K(f) \cap \bigcap \left\{ \mathrm{clos}\left(\mathbb{C}_{x_1 \ldots x_N}\right) : N \geq 1 \right\}$. (Clearly $K_x \subseteq K(f)$ and $K_x \subseteq \mathrm{clos}\left(\mathbb{C}_{x_1 \ldots x_N}\right)$ for each N . On the other hand if $z \in K(f)$ and $z \in \mathrm{clos}\left(\mathbb{C}_{x_1 \ldots x_N}\right)$ then z lies in the interior of $f^{-N}(U)$ and so $z \in \mathrm{clos}\left(f^{-N}(U) \cap \mathbb{C}_{x_1 \ldots x_N}\right) = \mathrm{clos}\left(U_{x_1 \ldots x_N}\right)$.)

Equivalence Relations on Σ_d :

We obtain an induced relation S'_Σ on Σ_d by

$$(x,y) \in S'_\Sigma \iff \exists\, z \in K(f) \text{ with } (z,x), (z,y) \in S'$$
$$\left(\iff K_x \cap K_y \neq \emptyset \right).$$

Notice that this is a closed relation since S' is closed and $K(f)$ is compact . However it is not a priori an equivalence relation .

PROPOSITION 1.3.5 : *If $\sim$ is a closed equivalence relation on Σ_d which contains S'_Σ then the map $\pi : K(f) \to \Sigma_d/\sim$ induced by the relation S' is continuous (where $\Sigma_d/\sim$ is endowed with the quotient topology).*

NOTE : If $\sim$ is S'_Σ then the map π is given by $\pi(z) = \left\{ x \in \Sigma_d : (z,x) \in S' \right\}$. Otherwise we must write $\pi(z) = \left\{ x \in \Sigma_d : \exists\, y \text{ with } x \sim y \text{ and } (z,y) \in S' \right\}$. The map is obviously well-defined and is surjective . The insistance that $\sim$ is a closed equivalence relation guarantees that the space $\Sigma_d/\sim$ is Hausdorff . We write $\pi_\sim$ for the associated quotient map $\Sigma_d \to \Sigma_d/\sim$.

Proof. Let V be a closed subset of Σ_d which is a union of $\sim$-equivalence classes (so that $\pi_\sim(V)$ is an arbitrary closed subset of $\Sigma_d/\sim$). We see that the set $\pi^{-1}\left(\pi_\sim(V)\right)$, which is just $\left\{ z \in K(f) : \exists\, y \in V \text{ with } (z,y) \in S' \right\}$, is closed in $K(f)$. This follows because V is compact and S' is a closed subset of $K(f) \times \Sigma_d$. □

In general , it is not much use having a semi-conjugacy if the set K_x of z corresponding to a given symbol sequence x consists of many points scattered all over the place . So we restrict attention to a case where the sets K_x are at least connected .

§1.4 FOLIATORY CUTS :

DEFINITION 1.4.1 : We say that a cut R is *foliatory* if $R \cap f^n(R) = \emptyset$ for all $n \in \mathbb{N}$.

NOTE : If R is foliatory then the inverse-images $f^{-n}(R)$, $(n \geq 0)$, of R are all disjoint . Furthermore f^n restricted to R is a homeomorphism onto $f^n(R)$, for all $n \in \mathbb{N}$.

THEOREM 1.4.1 : *If R is foliatory then for every word $x_1 \ldots x_N$ the set $\mathbb{C}_{x_1 \ldots x_N}$ is a connected domain whose boundary consists of branches of $f^{-n}(R)$ $\left(\text{for } 1 \leq n \leq N\right)$. Furthermore if U is a B.I.D. which is transverse to $f^n(R)$ for all $n \geq 0$ then $U_{x_1 \ldots x_N}$ is a connected domain for every word $x_1 \ldots x_N$ and its boundary consists of initial segments of branches of $f^{-n}(R)$ $\left(\text{for } 1 \leq n \leq N\right)$ and pieces of arc of $\partial\left(f^{-N}(U)\right)$.*

Proof. We ~~prove~~ (proceed) by induction on N . By inductive hypothesis the boundary of $\mathbb{C}_{x_2 \ldots x_N}$ is contained in $\bigcup_{1 \leq n \leq N-1} f^{-n}(R)$ and so cannot intersect R . Consequently R is either contained in or disjoint from the connected domain $\mathbb{C}_{x_2 \ldots x_N}$ and so $\mathbb{C}_{x_2 \ldots x_N} \setminus R$ is also connected . Applying g_{x_1} gives that $\mathbb{C}_{x_1 x_2 \ldots x_N}$ is a connected domain whose boundary consists of branches of $f^{-n}(R)$ $\left(\text{for } 1 \leq n \leq N\right)$.

In the second case the hypothesis is that $U_{x_2 \ldots x_N}$ is connected with boundary contained in $f^{-N+1}(\partial U) \cup \bigcup_{1 \leq n \leq N-1} f^{-n}(R)$. Since U is transverse to $f^{N-1}(R)$ it follows that R intersects $f^{-N+1}(\partial U)$, and so the boundary of $U_{x_2 \ldots x_N}$, in at most one point . Hence $U_{x_2 \ldots x_N} \setminus R$ is connected and $U_{x_1 x_2 \ldots x_N}$ has the desired properties . □

COROLLARY 1.4.1 : *If R is foliatory and U is transverse to $f^n(R)$ for all $n \geq 0$ then the set $K_x = \left\{ z \in K(f) : (z,x) \in S' \right\}$ is connected for each $x \in \Sigma_d$.*

Proof. The set K_x is just $\bigcap\left\{ \operatorname{clos}\left(U_{x_1 \ldots x_N}\right) : N \geq 1 \right\}$ which , being the decreasing intersection of connected closed (compact) sets , is connected . □

PROPOSITION 1.4.2 : *If R is foliatory and U is a B.I.D. transverse to $f^n(R)$ for all $n \geq 0$ then for every cylinder $\left[x_1 \ldots x_N\right]$ we have*

$$\operatorname{clos}\left(U_{x_1 \ldots x_N}\right) = \bigcap\left\{ f^{-n+1}\left(\operatorname{clos}\left(U_{x_n}\right)\right) : 1 \leq n \leq N \right\} .$$

Proof. Refering back to the proof of Lemma 1.3.3 we see that if z is adherent to both $g_{x_1}(U\backslash R)$ and $f^{-1}\left(U_{x_2..x_N}\right)$ but not to their intersection then z must be a boundary point of both. It follows that z lies in $f^{-1}(R)$ and hence also in $\partial\left(f^{-N}(U)\right)$. But the intersection of these two arcs is "transverse" thus ensuring that z is still adherent to $g_{x_1}(U\backslash R) \cap f^{-1}\left(U_{x_2..x_N}\right)$, which is a contradiction. □

Determination of S'_Σ :

From now on, we assume that R is foliatory and that U is a B.I.D. transverse to $f^n(R)$ for all $n \geq 0$. We have, for all $N \geq 1$, that $R \cap f^{-N}(U)$ is either empty or a connected set containing the critical value c. The same is therefore true of $R \cap K(f)$.

If $c \notin K(f)$ then $R \cap K(f) = \emptyset$ and so every $z \in K(f)$ has a unique $x \in \Sigma_d$ such that $(z,x) \in S$. Thus $S\left(= S'\right)$ is a bona fide function (Theorem 1.3.4). It is well-known that in this case $K(f) = J(f)$ is a Cantor set, and so S is one-to-one and a homeomorphism (each of the sets K_x being connected).

We are interested in the case $c \in K(f)$. All the forward iterates of $R \cap K(f)$ lie in $f^{-1}\left(U\backslash R\right)$. Consequently, there is a unique *kneading itinerary* $\xi \in \Sigma_d$ such that (z,ξ) lies in $S\left(\text{or } S'\right)$, for $z \in R \cap K(f)$.

The only $z \in K(f)$ for which there is more than one sequence x with $(z,x) \in S'$ are those for which there exists (a unique) $N \geq 1$ with $f^N(z) \in R \cap K(f)$. Thus $z \in U_{x_1..x_{N-1}}$ for some unique $x_1 \ldots x_{N-1}$ and $f^{N-1}(z)$ lies in the common boundary of two U_α $\left(\text{or to all the } U_\alpha \text{ in the case } f^{N-1}(z) = 0\right)$. Clearly $f^{n-1}(z) \in U_{\xi_{n-N}}$ for $n > N$.

Observe now that the induced relation S'_Σ is determined purely by the sequence ξ. Given $x \in \Sigma_d$ the elements of Σ_d related, by S'_Σ, to x are precisely : x itself ; and the elements $x_1..x_{N-1}\alpha\,\xi$ for $\alpha \in L$ and $N \geq 1$ satisfying $x = x_1..x_{N-1}x_N\,\xi$. There can be at most one such N unless ξ is a periodic sequence.

If ξ is not periodic then $(x,y),(x,y') \in S'_\Sigma \Rightarrow (y,y') \in S'_\Sigma$ and so S'_Σ is an equivalence relation. The nontrivial equivalence classes of S'_Σ are then of the form

$$\left\{x_1..x_{N-1}\alpha\,\xi : \alpha \in L\right\} \tag{1.4}$$

for $N \geq 1$. However if ξ is periodic, that is $\sigma^p\xi = \xi$ for some $p \geq 1$, then the elements related to a sequence $x = x_1 \ldots x_{N-1}x_N\,\xi$ include the infinite set

$$\left\{x_1 \ldots x_{N-1+mp}\,\alpha\,\xi : m \geq 0,\ \alpha \in L\right\}$$

and so S'_Σ is certainly not an equivalence relation.

External Rays as Foliatory Cuts :

Under our assumption that $\varphi_c^{-1}\left(r\cdot e^{2\pi i\eta}\right) \to c$ as $r \searrow 1$, we put $\gamma(t) = \varphi_c^{-1}\left(e^t + 2\pi i\eta\right)$ for $t \in \left(0,+\infty\right)$ and $\gamma(0) = c$. Since c cannot be periodic it follows that η cannot be rational of denominator coprime to d. Writing R for the image

set of γ, we get that $R \cap f^n(R) = \emptyset$ for all $n \geq 1$. To ensure that U is transverse to $f^n(R)$ for all n, we can either choose U to be a sufficiently large disk centred at the origin or simply put

$$U = \mathbb{C} \setminus \left\{ \varphi_c^{-1}\left(r \cdot e^{2\pi i\theta}\right) : \ r \geq r_0 , \ 0 \leq \theta < 2\pi \right\}$$

for some $r_0 > 1$.

§1.5 INJECTIVITY OF $\bar{x}$:

It is clear that if ξ is periodic then $\bar{x}$ cannot be one-to-one (since otherwise S'_Σ would be an equivalence relation). One might conjecture that $\bar{x}$ is one-to-one whenever ξ is not periodic. (This will be true if we know that $K(f)$ is locally-connected, using a Thurston lamination argument). We will establish that $\bar{x}$ is one-to-one if at least ξ is *non-recurrent*, that is, there exists $M \geq 1$ such that for all $n \geq 1$ we have $\sigma^n\xi \notin \left\{\xi_1 \ldots \xi_M\right\}$. But first we need :

LEMMA 1.5.1 "Unused cut" : *If* $x_1 \ldots x_n \neq \xi_1 \ldots \xi_n$ *then* $R \cap \mathbb{C}_{x_1 \ldots x_n} = \emptyset$.

□

COROLLARY 1.5.1 : *If* $0 \leq k < n$ *and* $x_{n-i+1} \ldots x_n \neq \xi_1 \ldots \xi_i$ *for all* i *with* $n-k \leq i < n$ *then* $\mathbb{C}_{x_1 \ldots x_n} = \left(g_{x_1} \circ \cdots \circ g_{x_k}\right)\left(\mathbb{C}_{x_{k+1} \ldots x_n}\right)$.

Proof. Let n be fixed. We proceed by induction on k. The result is trivial if $k = 0$. Now suppose that the hypothesis and result are true for $k = \ell - 1$ and in addition that $\ell < n$ and $x_{\ell+1} \ldots x_n \neq \xi_1 \ldots \xi_{n-\ell}$. Then

$$\mathbb{C}_{x_1 \ldots x_n} = \left(g_{x_1} \circ \cdots \circ g_{x_{\ell-1}}\right)\left(\mathbb{C}_{x_\ell x_{\ell+1} \ldots x_n}\right).$$

Now since $\mathbb{C}_{x_\ell x_{\ell+1} \ldots x_n}$ equals $g_{x_\ell}\left(\mathbb{C}_{x_{\ell+1} \ldots x_n} \setminus R\right)$ which equals $g_{x_\ell}\left(\mathbb{C}_{x_{\ell+1} \ldots x_n}\right)$, the result is true for $k = \ell$. □

THEOREM 1.5.2 : *Suppose* $c \in K(f)$ *and* R *is a foliatory cut and* U *is a* B.I.D. *transverse to* $f^n(R)$, $n \geq 0$. *If the associated kneading itinerary* ξ *is non-recurrent then the coding map* $\bar{x} : K(f) \to \Sigma_\xi$ *is one-to-one*.

Proof. Given a sequence $x \in \Sigma_d$ we show that $\bigcap\left\{ \operatorname{clos}\left(U_{x_1 \ldots x_N}\right) : N \geq 1 \right\}$ is a single point. First we suppose that x is not a backward iterate of ξ, that is $\sigma^k x \neq \xi$ for all $k \in \mathbb{N}$. We define a sequence $\left(m_n\right)$, $n \in \mathbb{N}$ by

$$m_n = \max\left\{ i \in \mathbb{Z}_+ : \ 0 \leq i \leq n-1 \ \text{ and } \ \xi_1 \ldots \xi_i = x_{n-i+1} \ldots x_n \right\}.$$

Observe, by the above technical results, that the function $g_{x_1} \circ \cdots \circ g_{x_{n-m_n-1}}$ is defined on the domain $\mathbb{C}_{x_{n-m_n}\xi_1 \cdots \xi_{m_n}}$ and that the image-set is $\mathbb{C}_{x_1 \cdots x_n}$. We show $m_n \le M$ for infinitely many n. We compare m_{n+1} with m_n. There are two cases:

either $\xi_{m_n+1} = x_{n+1}$ in which case $m_{n+1} = m_n + 1$;

or $\xi_{m_n+1} \ne x_{n+1}$ in which case $m_{n+1} \le m_n$ and moreover $\xi_1 \cdots \xi_{m_n} x_{n+1} = x_{n+1-m_n} \cdots x_n x_{n+1} = x_{n+1-m_n} \cdots x_{n-m_{n+1}} \xi_1 \cdots \xi_{m_{n+1}}$ and so $\xi_1 \cdots \xi_{m_n}$ ends in $\xi_1 \cdots \xi_{m_{n+1}-1}$. In this case the non-recurrent property of ξ guarantees that $m_{n+1} \le M$.

If $m_n \le M$ for only finitely many n then we must have had $m_{n+1} = m_n + 1$ for all n greater than some N so that $n - m_n$ had a common value $k \ge 1$ for $n > N$. But this would imply that $\xi_1 \cdots \xi_{n-k} = x_{k+1} \cdots x_n$ for all $n > k$, and so that $\sigma^k x = \xi$ contradicting our assumption.

Therefore there exists $m \le M$ such that $m_{n_r} = m$ for an infinite subsequence $n_r \to \infty$. The sequence of functions $g_{x_1} \circ \cdots \circ g_{x_{n_r-m-1}} \circ g_{x_{n_r-m}}$ are all defined on the domain $\mathbb{C}_{\xi_1 \cdots \xi_m} \setminus R$. The symbol x_{n_r-m} must take some value α infinitely often, and we refine our subsequence to ensure $x_{n_r} = \alpha$ for all r. The functions $h_r := g_{x_1} \circ \cdots \circ g_{x_{n_r-m-1}}$ are then all defined on $\mathbb{C}_{\alpha\, \xi_1 \cdots \xi_m}$. Now suppose that we can enlarge to a domain W, which contains the closure of $U_{\alpha\, \xi_1 \cdots \xi_m}$, on which the functions h_r are still all defined. The h_r (being uniformly bounded on compact subsets of W) form a normal family (Hille, section 15.2) and so there is a subsequence h_{r_s} which converges (uniformly on compact subsets of W) to an analytic function h.

We show that h is a constant function. Choose z in $U_{\alpha\, \xi_1 \cdots \xi_m} \setminus K(f)$. So $h_r(z) \notin K(f)$ for all r. Now let V be a neighborhood of z which is relatively compact in $U_{\alpha\, \xi_1 \cdots \xi_m}$. For some sequence $\varepsilon_s \searrow 0$, we have that $h(V)$ is contained in the ε_s-neighborhood of $h_{r_s}(V)$, and so of $f^{-n_{r_s}}(U)$. It follows that $h(V)$ is contained in $K(f)$, the ω-limit set of $f^{-n_{r_s}}(U)$. Were h not constant then it would be an open map and so $h(z)$ would lie in the interior of $K(f)$. But this would contradict the fact that $h_{r_s}(z) \to h(z)$.

Thus h is constant and the diameter of $\operatorname{clos}\left(U_{x_1 \cdots x_{n_{r_s}}}\right)$, which (by Corollary 1.3.2.1) is $h_{r_s}\left(\operatorname{clos}\left(U_{\alpha\, \xi_1 \cdots \xi_m}\right)\right)$, tends to zero as s tends to infinity, and so $\bigcap\left\{\operatorname{clos}\left(U_{x_1 \cdots x_n}\right) : n \ge 1\right\}$ is a single point, the value of h.

We now deal with the construction of the enlargement W. To do this we

use the fact that R has a neighborhood Y which is disjoint from $f^n(R)$ for all $n \geq 1$. Define $Y := \mathbb{C}_{\xi_1 \ldots \xi_M}$. This is an open connected set which contains the critical value c but not its future iterates. By Theorem 1.4.1 it (and its closure) therefore contain R but are disjoint from $f^n(R)$ for $n \geq 1$. Now put $W_0 := \mathbb{C}$ and recursively define W_n, for $0 \leq n \leq m$, by $W_{n+1} := g_{\xi_{m-n}}(W_n \setminus R) \cup I$ where I is $f^{-1}(Y)$ or $\emptyset$ according as $c \in W_n$ or not (and thus according as $\xi_{m-n+1} \ldots \xi_m = \xi_1 \ldots \xi_n$ or not). Finally put $W := g_\alpha(W_m) \cup f^{-1}(Y)$.

One can verify, using Lemma 1.3.2, that at every stage W_n is a connected neighborhood of $\mathrm{clos}(U_{\xi_{m-n+1} \ldots \xi_m})$ and so that W is a connected neighborhood of $\mathrm{clos}(U_{\alpha\, \xi_1 \ldots \xi_m})$. By construction, $W \setminus \mathbb{C}_{\alpha\, \xi_1 \ldots \xi_m}$ is contained in $\bigcup_{1 \leq n \leq m+1} f^{-n}(Y)$ on which all concatenated backward iterate maps are defined. So W does indeed extend the domain of definition of the h_r.

We have therefore proved that $\bigcap \{ \mathrm{clos}(U_{x_1 \ldots x_n}) : n \geq 1 \}$ is a single point when x is not a strict inverse-image of ξ. In particular we get the point $\{c\}$ if $x = \xi$. Given $\alpha \in L$, we have $U_{\alpha\, \xi_1 \ldots \xi_n} \subseteq f^{-1}(U_{\xi_1 \ldots \xi_n})$ for all n, and so $\bigcap \{ \mathrm{clos}(U_{\alpha\, \xi_1 \ldots \xi_n}) : n \geq 1 \}$ is contained in $f^{-1}(c) = \{0\}$. Finally if $x = x_1 \ldots x_{k-1} x_k\, \xi$ then for $n \geq M$ we have $\mathrm{clos}(U_{x_k\, \xi_1 \ldots \xi_n}) \subseteq \mathrm{clos}(f^{-1}(Y))$ and so $\mathrm{clos}(U_{x_1 \ldots x_{k-1} x_k\, \xi_1 \ldots \xi_n}) = (g_{x_1} \circ \ldots \circ g_{x_{k-1}})(\mathrm{clos}(U_{x_k\, \xi_1 \ldots \xi_n}))$. We therefore get $\bigcap \{ \mathrm{clos}(U_{x_1 \ldots x_{k-1} x_k\, \xi_1 \ldots \xi_n}) : n \geq 1 \} = g_{x_1} \circ \ldots \circ g_{x_{k-1}}\{0\}$ which is a single point.

□

Since $K(f)$ is compact and Σ_ξ is Hausdorff the conclusion of the above Theorem is that $\mathbb{I}$ is a homeomorphism. We will see in Chapter 2 that the space Σ_ξ is locally-arcwise-connected and uniquely-arcwise-connected and so it follows that $K(f)$ is locally-connected with no interior and so $K(f) = J(f)$ is a dendrite.

We now re-word the Theorem in terms of external rays.

THEOREM 1.5.3: *If $f : z \mapsto z^d + c$ satisfies the following conditions:*

(i) *The critical point 0 lies in $J(f)$ $(= \partial K(f))$;*

(ii) *The critical value c is accessible, is the limit point of an external ray $\{ \varphi_c^{-1}(r \cdot e^{2\pi i \eta}) : r > 1 \}$ for some $\eta \in \mathbb{R}$;*

(iii) *The associated kneading itinerary $\xi\,(= \xi(f, \eta))$ is non-recurrent,*

then $J(f) = K(f)$ and there is a semi-conjugacy $\kappa : \Sigma_d \to J(f)$ given by $K_x = \{\kappa(x)\}$ and satisfying $\kappa(\sigma x) = f(\kappa(x))$ where σ is the shift map $\Sigma_d \circlearrowleft$. The associated equivalence relation on Σ_d is given by (1.4).

COROLLARY 1.5.3 : *If the critical point* 0 *is pre-periodic then the result holds.*

Proof. We use the sub-hyperbolic metric defined in [DH2, exp. III]. Every periodic orbit of $K(f)$ is repelling - in particular, the eventual critical orbit. This orbit and so the critical point must lie in the dis-equicontinuity set which is $J(f)$. By a theorem of Douady-Hubbard [DH2, exp. III] the Julia set is locally-connected and so the Riemann map conjugacy φ_c^{-1} extends to S^1. Hence every point of J, in particular the critical value c, is accessible by an external ray. Since the critical point is strictly pre-periodic, that is $f^{\ell+1}(0) = f^{\ell+p+1}(0)$ but $f^{\ell}(0) \neq f^{\ell+p}(0)$ for some $\ell, p \in \mathbb{N}$, it follows that the associated kneading itinerary ξ is pre-periodic or periodic. However the latter case can be ruled out since $f^{\ell}(0)$ and $f^{\ell+p}(0)$ are distinct inverse-images of the same point and hence lie in different $\mathbb{C}_\alpha$. We have $\xi_\ell \neq \xi_{\ell+p}$ whereas $\xi_{\ell+n} = \xi_{\ell+p+n}$ for $n \geq 1$. Had we that $\sigma^n\xi \in [\xi_1 \ldots \xi_{\ell+p}]$ for some $n \geq 1$ it would follow that $\xi_\ell = \xi_{\ell+n} = \xi_{\ell+p+n} = \xi_{\ell+p}$ which is a contradiction. Thus ξ is non-recurrent. □

NOTE : It is possible using a stability result, in [DH2, exp. VIII], for external rays of a repelling periodic point, to construct examples of Julia sets where the conditions of Theorem 1.5.3 hold but where the critical point is not pre-periodic : Perturbations of preperiodic cases where the critical point eventually lands on a hyperbolic "subshift of finite type" which when unperturbed contained the eventual critical cycle but not the critical value.

§1.6 POSSIBLE VALUES FOR THE ITINERARY ξ :

Given a polynomial $f : z \mapsto z^d + c$ for which the hypotheses (i) and (ii) of Theorem 1.5.3 hold it is possible that two or more external rays converge on the critical value c. One might expect therefore to obtain different corresponding values ξ. However we see that in fact the kneading itinerary ξ is essentially uniquely determined :

THEOREM 1.6.1 : *If distinct external rays* $\varphi_c^{-1}(r \cdot e^{2\pi i\eta})$, $\varphi_c^{-1}(r \cdot e^{2\pi i\eta'})$ *converge on* c *as* $r \searrow 1$ *then, up to permutation of the symbols* L, *we have* $\xi(f_c,\eta) = \xi(f_c,\eta')$.

Proof. Write W for the component of $\mathbb{C} \setminus \left(R(f_c,\eta) \cup R(f_c,\eta') \cup \{c\}\right)$ which contains the critical point 0. Write V for the other component. We show that $f_c^n(c) \notin V$ for all $n \geq 1$. Recall that neither η nor η' are periodic under the action $\theta \mapsto d \cdot \theta \pmod{\mathbb{Z}}$. We assume η, η' are the representatives $(\bmod \mathbb{Z})$ which lie in the interval $(0, 1)$. Without loss of generality $\eta' > \eta$. Hence for $r > 1$, $\varphi_c^{-1}(r \cdot e^{2\pi i\theta}) \in V \Leftrightarrow \theta \in (\eta, \eta') + \mathbb{Z}$. The subinterval of $\mathbb{R}/\mathbb{Z}$ corresponding to V must be this one since it cannot contain η/d (or any of the points $(\eta+j)/d$ or

$(\eta'+j)/d$, $(j \in \mathbb{Z})$ which correspond to the critical point 0). Suppose that for some $n \geq 1$ we have $f_c^n(c) \in V$, or equivalently , $d^n\eta \in (\eta, \eta') + \mathbb{Z}$. We suppose that n is the least such . Say $\eta+j < d^n\eta < \eta'+j$ where $j \in \mathbb{Z}$ (and $0 \leq j \leq d^n-1$). We show by induction on k ($\leq n$) that the rays $R(f_c, (\eta+j)/d^k)$ and $R(f_c, (\eta'+j)/d^k)$ converge on the same inverse-image of c and that the component V_k of the remaining part of $\mathbb{C}$ "cut off " by these rays and containing the point $f_c^{n-k}(c)$ satisfies $V_k \backslash K(f) = \varphi_c^{-1}\{ re^{2\pi i\theta} : r > 1 \text{ and } (\eta+j)/d^k < \theta < (\eta'+j)/d^k \}$ and does not contain any of the points $f_c^{\ell}(c)$ for $0 \leq \ell < n-k$. This is clearly true for $k = 0$. We establish the statement for $k+1$ given its truth for k ($\leq n-1$) :

The critical value c is not contained in the region V_k (nor is it a boundary point of V_k unless $k = 0$). Hence there is an inverse branch map g , of f_c , defined on the closure of V_k , analytic everywhere (except , in the case $k = 0$, at the branch point c where it is still continuous) , with its restriction to $\mathrm{clos}(V_k)\backslash K(f)$ satisfying $g(\varphi_c^{-1}(r\cdot e^{2\pi i\theta})) = \varphi_c^{-1}(r^{1/d}\cdot e^{2\pi i\theta/d})$ for $r > 1$, $(\eta+j)/d^k \leq \theta \leq (\eta'+j)/d^k$. Thus $V_{k+1} := g(V_k)$ is bounded by $g(\partial V_k)$ which consists of external rays $R(f_c, (\eta+j)/d^{k+1})$ and $R(f_c, (\eta'+j)/d^{k+1})$ together which their common limit point . Furthermore $(\eta+j)/d^{k+1} < d^{n-(k+1)}\eta < (\eta'+j)/d^{k+1}$ and so the ray $R(f_c, 2^{n-(k+1)}\eta)$ together with its limit point $f_c^{n-(k+1)}(c)$ lie in V_{k+1} .

Now the statement holds when $k = n$ and so V_n contains the point c (but not its inverse image 0 , since otherwise $f_c^{n-1}(c) \in f_c^n(V_n) = V$). Thus both external rays $R(f_c,\eta)$ and $R(f_c,\eta')$, which cannot cross the boundary of V_n , must be contained in V_n . But this implies that both η and η' are contained in the $\mathbb{R}/\mathbb{Z}$-interval $((\eta+j)/d^n, (\eta'+j)/d^n) + \mathbb{Z}$, and so the real interval $((\eta+j)/d^n, (\eta'+j)/d^n) \subseteq (0,1)$. This is an absurdity since $|\eta'-\eta| > (\eta'-\eta)/d^n$. So we must have had that $f_c^n(c) \in K(f)\backslash V$ for all $n \geq 1$.

Given two codings of $K(f)$ according to respective external ray cuts $R(f_c,\eta)$ and $R(f_c,\eta')$ we can , by appropriate permutation of the symbols of the second coding , ensure that the corresponding inverse branch maps g_j , g_j' of f agree on W for all $j \in L$. This means that for each (first) pre-image z of the critical point the same symbol has been assigned : $\alpha(z) = \alpha'(z)$. (Note that we need no permutation of L if $\lceil\eta\rceil = \lceil\eta'\rceil$ and we are adopting the convention in §1.2).

For all $n \geq 1$ we have $f_c^n(c) \in K(f)\backslash V \backslash \{c\} \subseteq W$, so the j for which $f^{n-1}(c) \in g_j(W)$ is also the j for which $f^{n-1}(c) \in g_j'(W)$. The result follows . □

NOTE : The main argument used here can be expressed in terms of laminations of the disk and bears a close relationship with Thurston's "centrally-enlarging " Lemma 9.1 in [T] . It is also possible to deduce this result , under the additional hypothesis that $\xi(f_c,\eta)$ is not periodic , from the corollary to Theorem 2.5.3 in the next chapter which gives a more "quantitative " statement .

If we adopt the convention in §1.2 then the sequence ξ can be obtained from η by a purely combinatorial formula $\xi_n = \lceil (d^n-1)\cdot\eta \rceil$ (mod d) $(n \in \mathbb{N})$ since

$$\begin{aligned} \xi_n = j \iff & d^{n-1}\cdot\eta \in \left((\eta+j)/d\,,\,(\eta+j+1)/d\right) \pmod 1 \\ \iff & d^n\cdot\eta \in \left(\eta+j\,,\,\eta+j+1\right) \pmod d \\ \iff & \lceil (d^n-1)\cdot\eta \rceil = j \pmod d\,. \end{aligned}$$

We therefore have a map $\xi : (\mathbb{R}\setminus\mathbb{Q}_d)/d\mathbb{Z} \to \Sigma$ where $\mathbb{Q}_d$ denotes the set of rationals of denominator coprime to d . In order to extend this to a continuous map from the whole of $\mathbb{R}/d\mathbb{Z}$ it is necessary to make some identifications on Σ . The elements of $\mathbb{Q}_d$ are of the form $a/(d^n-1)$ where $a \in \mathbb{Z}$ (and we shall assume that $n \in \mathbb{N}$ is minimal). Observe that as $\eta \nearrow a/(d^n-1)$ the corresponding kneading itinerary converges (in the product topology) on some sequence $\xi^-(a/(d^n-1)) = (x_1x_2\ldots x_{n-1}\alpha)^\infty$ of period n , and that as $\eta \searrow a/(d^n-1)$ the corresponding itinerary converges to $\xi^+(a/(d^n-1)) = (x_1x_2\ldots x_{n-1}\alpha')^\infty$ where $\alpha' = \alpha+1$ (mod d). Hence if we define Σ_Σ as the (Hausdorff) space obtained from Σ by identifying together the points $\{(x_1\ldots x_{n-1}\alpha)^\infty : \alpha \in L\}$ for each $x_1x_2\ldots x_{n-1} \in L^{n-1}$ and $n \geq 1$, then the map $\eta \to [\xi(\eta^-)] = [\xi(\eta^+)]$ is a continuous function from $\mathbb{R}/d\mathbb{Z}$ to Σ_Σ . (Here $[x]$ denotes the equivalence class of the sequence x).

In the light of Theorem 1.6.1 and various results of Douady & Hubbard [DH2] we conjecture that the map $\eta \mapsto [\xi(\eta)]$ factors through the equivalence relation given by Thurston's "quadratic minor lamination" described in [T] to give a continuous map from a d-fold cover of $\partial\mathcal{M}$ (and extendable to a d-fold cover of $\mathcal{M}$) to the space Σ_Σ . Such a map will not be onto even in the case d = 2 (see Example 2.2 in Section 4 of the next chapter) .

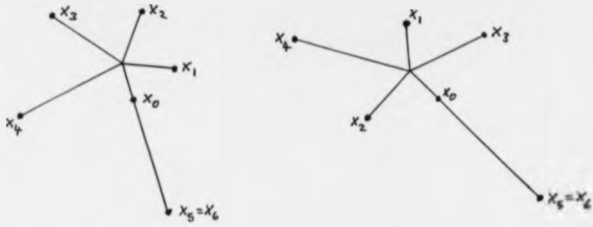
DEFINITION 1.6.1 : A sequence $\xi \in \Sigma_d$ is *complex-admissible* if it is in the closure of the set $\{\xi(\eta) : \eta \in \mathbb{R}\setminus\mathbb{Q}_d\}$.

NOTE : At least in the quadratic case (d = 2) Douady and Hubbard [DH2, exp. XIII] show that whenever η is rational of <u>even</u> denominator the ray $R(\mathcal{M},\eta)$ converges to a parameter value c , a Misiurewicz point , for which the corresponding ray $R(K(f_c),\eta)$ converges to the critical value c in $K(f_c)$. (In these cases $\xi(f_c,\eta)$ is defined and is strictly pre-periodic).

Hence the set of ξ -values obtainable from partitionable Julia sets from the quadratic family are dense in the set of complex-admissible sequences .

It is possible that a given value ξ might represent different c values . For example , consider the two distinct Misiurewicz points which are the limit points of the rays $R(\mathcal{M},\eta_1)$, $R(\mathcal{M},\eta_2)$ where $\eta_1 = 1/16$ and $\eta_2 = 5/16$ respectively . It is an easy

calculation that $\xi(\eta_1) = \xi(\eta_2) = 0\,0\,0\,0\,1^{\infty}$. The Hubbard tree (defined in [DH1] essentially as the union of the arcs in $K(f_c)$ connecting the marked forward images of the critical point) for each of these c values is drawn below .



$\eta_1 = 1/16$ $\qquad$ $\eta_2 = 5/16$

Fig. 1.1

Corollary 1.5.3 asserts that the respective Julia sets are homeomorphic (by a map which conjugates their respective maps f_c). The above diagrams show that this homeomorphism cannot extend to a homeomorphism between neighborhoods of the Julia sets .

Finally , with important consequences for Theorem 1.5.3 , we make the following combinatorial

CONJECTURE : It is fairly clear that (for $\eta \notin \mathbb{Q}_d$) if $\xi(\eta)$ is non-recurrent then η must be non-recurrent in the sense that

$$(\exists\, \varepsilon > 0)(\forall\, n \geq 1)(d^n\eta \notin (\eta-\varepsilon, \eta+\varepsilon) + \mathbb{Z}).$$

We conjecture that the converse of this is also true , namely (η non recurrent $\Rightarrow$ $\xi(\eta)$ is non-recurrent) .

CHAPTER 2

In this chapter, we consider a special type of equivalence relation $\sim_\Gamma$ on a one-sided sequence space Σ, the quotient Σ_Γ of which turns out to be a dendrite. We then proceed to the special case where Σ_Γ arises as a model for Julia sets of certain polynomials of the form $z^d + c$.

§2.1 GLUEING SPACES :

Let L be an alphabet of d (≥ 2) symbols (with the discrete topology). We define $\Sigma := L^{\mathbb{N}}$ to be the space of one-sided sequences, endowed with the product topology (and is thereby compact, totally disconnected and homeomorphic to a Cantor set).

Cylinders :

For $k \geq 0$, consider the equivalence relation $\sim_k$ on Σ given by :

$$x \sim_k y \iff x_i = y_i \text{ for all } i \leq k .$$

The equivalence class of x, denoted by $[x]_k$ or $[x_1\, x_2 \ldots x_k]$, is called the "k-cylinder" of x.

Define $C_k \left(= C_k(\Sigma) \right)$ to be the set $\Sigma/\sim_k$ of k-cylinders.

Define $C \left(= C(\Sigma) \right)$ to be the disjoint union of the sets C_k for $k \geq 0$, the set of "cylinders". Thus C is a base for the topology on Σ.

Let $\pi : C \setminus C_0 \to C$ be the d-to-one "parent" map :

$$[x_1\, x_2 \ldots x_{k-1}\, x_k] \mapsto [x_1\, x_2 \ldots x_{k-1}] .$$

The "children" of a cylinder $[x_1\, x_2 \ldots x_{k-1}]$ are naturally labelled by the elements of L. Often we will write them as $[x_1\, x_2 \ldots x_{k-1}; \alpha]$, $\alpha \in L$, to emphasise the role of the parent. Thus we have a natural identification $C \times L \to C \setminus C_0$ given by

$$\left([x_1 \ldots x_{k-1}], \alpha \right) \mapsto [x_1 \ldots x_{k-1}; \alpha] \quad \left(= [x_1 \ldots x_{k-1}\, \alpha] \right).$$

Glueing operations on Σ :

We shall see that by identifying together countably many pairs of points in a certain fashion the space Σ "connects up".

DEFINITION 2.1.1 : A <u>gluepoint map</u> is a map $\Gamma : C \setminus C_0 \to \Sigma$ satisfying the rule that

$$\Gamma\left([x_1\, x_2 \ldots x_{k-1}; x_k] \right) \in [x_1\, x_2 \ldots x_{k-1}\, x_k]$$

for all $k \geq 1$ and $[x_1\, x_2 \ldots x_{k-1}\, x_k] \in C_k$.

Define $\sim_\Gamma$ to be the minimal closed equivalence relation on Σ whose associated projection $\pi_\Gamma : \Sigma \to \Sigma/\sim_\Gamma$ has the property that $\pi_\Gamma \circ \Gamma$ factors through

π. In other words $\Sigma_\Gamma := \Sigma/\sim_\Gamma$ is the topological space obtained by identifying together the "glue" points of those cylinders which have the same "parent" cylinder.

$$\begin{tikzcd} C \times L \arrow[r, "\Gamma"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi_\Gamma"] \\ C \arrow[r, dashed] & \Sigma_\Gamma \end{tikzcd}$$

In the case Γ is one-to-one the nontrivial equivalence classes are easily verified to be of the form :

$$\left\{ \Gamma[x_1 \ldots x_{k-1};\alpha] : \alpha \in L \right\}$$

for $k \geq 1$ and $x_1, \ldots, x_{k-1} \in L$. In the case Γ is not one-to-one, some of the equivalence classes may be of infinite cardinality, so we have to ask that $\sim_\Gamma$ be a closed relation in order that these are closed subsets and furthermore that the quotient space Σ_Γ is Hausdorff. For notational convenience we may write $\bar{x}$ for the element $\pi_\Gamma(x)$ of Σ_Γ.

A pair (Σ,Γ) satisfying the above is called a *glueing space*.

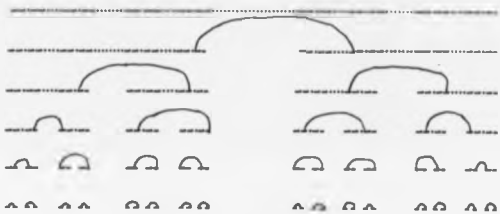


Fig. 2.1

EXAMPLE 1 : Suppose $L = \{0,1\}$ and Γ is the gluepoint map given by

$$\Gamma[x_1 \ldots x_{k-1}; 1] = x_1 \ldots x_{k-1}\, 0\, 1^\infty \quad \text{and}$$
$$\Gamma[x_1 \ldots x_{k-1}; 1] = x_1 \ldots x_{k-1}\, 1\, 0^\infty ,$$

then Γ is clearly one-to-one and so the nontrivial equivalence classes of $\sim_\Gamma$ are

$$\left\{ x_1 \ldots x_{k-1}\, 0\, 1^\infty, x_1 \ldots x_{k-1}\, 1\, 0^\infty \right\}$$

for $k \geq 1$ and $x_1, \ldots, x_{k-1} \in \{0,1\}$. The resulting space Σ_Γ is easily seen to be homeomorphic to the real interval $[0,1]$ via the map $x \mapsto \sum_{i \geq 1} x_i/2^i$.

EXAMPLE 2 : Given an element $\xi \in \Sigma$, let $\Gamma = \Gamma_\xi$ be the gluepoint map defined by

$$\Gamma[x_1 \ldots x_{k-1}; x_k] = x_1 \ldots x_{k-1}\, x_k\, \xi \,.$$

If ξ is not periodic then Γ is one-to-one and the nontrivial equivalence classes of $\sim_\Gamma$ are

$$\left\{ x_1 \ldots x_{k-1}\, \alpha\, \xi \;:\; \alpha \in L \right\}$$

for $k \geq 1$ and $x_1, \ldots, x_{k-1} \in L$. The quotient space will be written Σ_ξ and associated projection π_ξ.

(Note that if ξ is periodic and of type (A) then the necessity to make Σ_ξ Hausdorff reduces the space to a single point. See Chapter 3 for definition of type (A)).

Observe that the shift map $\sigma : \Sigma \to \Sigma$, defined by $(\sigma(x))_i = x_{i+1}$, for $i \in \mathbb{N}$, factors through the projection $\pi_\xi : \Sigma \to \Sigma_\xi$ to give a well-defined and continuous map $\bar{\sigma} : \Sigma_\xi \to \Sigma_\xi$ which is d-to-one except for being one-to-one at the "critical" point $\bar{\sigma}^{-1}(\xi)$.

EXAMPLE 3 : Consider the gluepoint map $\Gamma = \Gamma_\Sigma$ defined by

$$\Gamma[x_1 \ldots x_{k-1}; x_k] = (x_1 \ldots x_{k-1}\, x_k)^\infty \,.$$

Here Γ is not one-to-one and the nontrivial equivalence classes of $\sim_\Gamma$ are of the form

$$\left\{ \beta \circ x \mid x \text{ is a sequence in } \Sigma \text{ of type (A)} \right\}$$

where β is an "island" block. (See the Chapter 3 for definitions). The "largest" class corresponding to $\beta = 1$ consists of the sequences of type (A) and is the closure of the set of "continent" sequences obtainable from period one sequences by a finite number of Γ- "jumps". (Compare with B.B.Mandelbrot 's "continent molecule" of the Mandelbrot set [Ma1]).

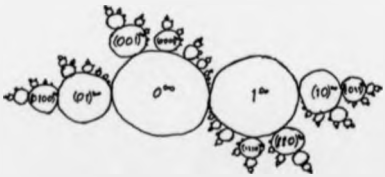


Fig. 2.2

The exact relationship between the quotient space Σ_Γ, which we shall write Σ_Γ, and the Mandelbrot set (for case $d = 2$) is not clear but it certainly demands investigation.

EXAMPLE 4 : Any cylinder c of Σ can itself be regarded as one-sided sequence space . A gluepoint map Γ for Σ, when restricted to the strict sub-cylinders of c gives a gluepoint map for c .

PROPOSITION 2.1.1 "Connected" : *If* U *is an open and closed subset of* Σ *with the property that for all* $c_1, c_2 \in C \setminus C_0$:

$$\Big((\bullet) \quad \pi(c_1) = \pi(c_2) \;\Rightarrow\; \big(\Gamma(c_1), \Gamma(c_2) \in U \;\text{ or }\; \Gamma(c_1), \Gamma(c_2) \in \Sigma \setminus U \big) \Big)$$

then $U = \Sigma$ *or* $U = \emptyset$.

COROLLARY 2.1.1 : *The quotient space* Σ_Γ *is connected* .

Proof . Let U be an open and closed subset of Σ . Thus U is compact and is therefore a finite union of cylinders . If U is neither Σ nor $\emptyset$ then there exists $k \geq 1$ such that U is a union of k-cylinders , but not a union of $(k-1)$-cylinders . It follows that there is a $(k-1)$-cylinder $[x_1 \ldots x_{k-1}]$ which intersects , but is not contained in U . The intersection set must be a union of some , but not all , cylinders $[x_1 \ldots x_{k-1}\, \alpha]$ as α runs through L , and so contains some , but not all , of the points $\Gamma[x_1 \ldots x_{k-1}; \alpha]$. Thus U does not satisfy property $(\bullet)$. □

Clearly , the above argument for proving the connectedness of Σ_Γ would fail if merely one identification correponding to a "glue" point $\Gamma[x_1 \ldots x_{k-1}; x_k]$ were removed from $\sim_\Gamma$. One might suspect that , upon removing such an identification , a component "breaks off " from the rest of Σ_Γ . Motivated by this , we make the following definition .

DEFINITION 2.1.2 : The *branch* B of a k-cylinder $\big($ for $k \geq 1 \big)$ is defined recursively as follows :

$$B[x_1] := [x_1] \; .$$

For $k = 1, 2, 3, \ldots$

$$B[x_1 \ldots x_{k-1}; x_k] := [x_1 \ldots x_{k-1}\, x_k] \cup$$
$$\bigcup \Big\{ B[x_1 \ldots x_{r-1}; \beta] \;:\; 1 \leq r < k \text{ with } \Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{k-1}\, x_k] \text{ and } \beta \neq x_r \Big\} .$$

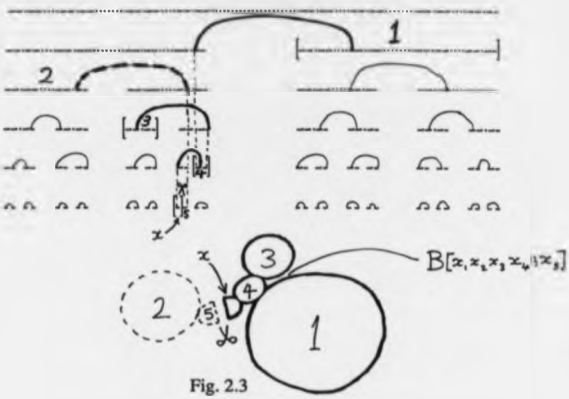


Fig. 2.3

§2.2 COMPLEMENTARY CYLINDERS :

Given a cylinder $[x_1 \ldots x_k]$, observe that the complement $\Sigma \setminus [x_1 \ldots x_k]$ is the disjoint union of "complementary " cylinders $[x_1 \ldots x_{r-1}\ \beta]$, for $1 \le r \le k$ and $\beta \ne x_r$. Similarly, given a point x, observe that $\Sigma \setminus \{x\}$ is the disjoint union of complementary cylinders $[x_1 \ldots x_{r-1}\ \beta]$, for $r \ge 1$ and $\beta \ne x_r$.

Consider the effect of removing a point x or a cylinder $[x_1 \ldots x_k]$ from the space Σ and examine the image of the resulting space under π_Γ. Clearly, the image of each complementary cylinder $[x_1 \ldots x_{r-1}\ \beta]$ will be connected (Example 4 and Proposition 2.1.1). Furthermore the images of two complementary cylinders $[x_1 \ldots x_{r-1}\ \beta]$, $[x_1 \ldots x_{s-1}\ \gamma]$ (where $r \le s$) will have connected union if either $r = s$ or $\Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{s-1}\ \gamma]$. With the desire to identify the connected components of $\pi_\Gamma(\Sigma \setminus \{x\})$ or $\pi_\Gamma(\Sigma \setminus [x_1 \ldots x_k])$ we introduce the following ideas.

DEFINITION 2.2.1 : Given a point $x \in \Sigma$ the map Γ induces a "successor " function $\nu_x : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ defined by

$$\nu_x(r) = \begin{cases} s & \text{if } \Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{s-1}\ \gamma] \text{ for some } \gamma \in L \setminus \{x_s\} \\ +\infty & \text{if } \Gamma[x_1 \ldots x_{r-1}; x_r] = x \end{cases} .$$

Clearly $\nu_x(r) \geq r+1$ for all $r \in \mathbb{N}$.

Given a cylinder $[x_1 \ldots x_{k-1}\, x_k]$ the map $\nu_{[x_1 \ldots x_k]}$ can still be defined as a map from $\{1, 2, \ldots, k\}$ to $\{1, 2, \ldots, k\} \cup \{+\infty\}$ by

$$\nu_{[x_1 \ldots x_k]}(r) = \begin{cases} s\ (\leq k) & \text{if } \Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{s-1}\, \gamma] \text{ for some } \gamma \in L \setminus \{x_s\} \\ +\infty & \text{if } \Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{k-1}\, x_k] \end{cases}.$$

Again, $\nu_{[x_1 \ldots x_k]}(r) \geq r+1$ for all $r \leq k$.

Note that if either $\nu_x(r) \leq k$ or $\nu_{[x_1 \ldots x_k]}(r) \leq k$ then $\nu_x(r) = \nu_{[x_1 \ldots x_k]}(r)$.

We state without proof the following obvious result.

LEMMA 2.2.1 "ν-transfer": *Given* $x_1, \ldots, x_k \in L$, *suppose that* $1 \leq r < s \leq k$ *then*:

(i) *If, for some* $\gamma \in L \setminus \{x_s\}$, *either of* $\nu_{[x_1 \ldots x_{s-1}\, \gamma]}(r)$ *and* $\nu_{[x_1 \ldots x_k]}(r)$ *is strictly less than* s *then they are coincide*;

(ii) $\left(\exists\, \gamma \in L \setminus \{x_s\} \text{ with } \nu_{[x_1 \ldots x_{s-1}\, \gamma]}(r) = +\infty\right) \iff \nu_{[x_1 \ldots x_k]}(r) = s$.

□

We can now give an explicit expression for $B[x_1 \ldots x_{k-1}; x_k]$.

THEOREM 2.2.2: *Whenever* $[x_1 \ldots x_{k-1}\, x_k] \in C \setminus C_0$ *we have*

$$B[x_1 \ldots x_{k-1}; x_k] = [x_1 \ldots x_{k-1}\, x_k] \cup$$
$$\bigcup \left\{ [x_1 \ldots x_{r-1}; \beta] : 1 \leq r < k \text{ with } \left(\exists\, n \text{ such that } \nu^n(r) = k\right), \text{ and } \beta \neq x_r \right\},$$

where $\nu^n(r)$ $(n \geq 0)$ *denotes* n *successive applications of the map* $\nu \left(= \nu_{[x_1 \ldots x_k]}\right)$ *on* r.

Proof. (By induction on k). When $k \geq 2$ we have, from the definition, that

$$B[x_1 \ldots x_{k-1}; x_k] = [x_1 \ldots x_{k-1}\, x_k] \cup$$
$$\bigcup \left\{ B[x_1 \ldots x_{s-1}; \gamma] : 1 \leq s < k \text{ with } \nu(s) = +\infty, \text{ and } \gamma \neq x_s \right\}.$$

By inductive hypothesis, each sub-union $\bigcup \left\{ B[x_1 \ldots x_{s-1}; \gamma] : \gamma \neq x_s \right\}$ for fixed $s\ (< k)$ consists of $[x_1 \ldots x_{s-1}\, \gamma]$ for $\gamma \neq x_s$, together with those complementary cylinders $[x_1 \ldots x_{r-1}\, \beta]$, $1 \leq r < s$ and $\beta \neq x_r$, for which there exists $n \geq 1$ such that:

$$\exists\, \gamma \in L \setminus \{x_s\} \text{ with } \nu_{[x_1 \ldots x_{s-1}\, \gamma]}{}^{n-1}(r) < s \text{ and } \nu_{[x_1 \ldots x_{s-1}\, \gamma]}{}^{n}(r) = +\infty,$$

or equivalently, by Lemma 2.2.1 "ν-transfer", such that:

$$\nu_{[x_1 \ldots x_k]}{}^{n-1}(r) < s \text{ and } \nu_{[x_1 \ldots x_k]}{}^{n}(r) = s.$$

Consequently, $B[x_1 \ldots x_{k-1};x_k]$ is given by

$$[x_1 \ldots x_{k-1}\, x_k] \cup \bigcup\Big\{ [x_1 \ldots x_{r-1};\beta] \;:\; \beta \neq x_r \text{ and there exists } s < k \text{ with } \nu(s) = +\infty \text{ such that either } r = s \text{ or } \big(1 \le r < s \text{ and } \nu^n(r) = s \text{ for some } n \ge 1\big)\Big\}$$

$$= [x_1 \ldots x_{k-1}\, x_k] \cup \bigcup\Big\{ [x_1 \ldots x_{r-1};\beta] \;:\; 1 \le r < k \text{ and there exist } s \text{ and } n \ge 0 \text{ such that } \nu^n(r) = s < k \text{ and } \nu^{n+1}(r) = +\infty \text{, and } \beta \neq x_r \Big\},$$

which is simply

$$[x_1 \ldots x_{k-1}\, x_k] \cup \bigcup\Big\{ [x_1 \ldots x_{r-1};\beta] \;:\; 1 \le r < k \text{ and } \big(\exists\, n,\ \nu^n(r) = k\big), \text{ and } \beta \neq x_r \Big\}.$$

□

COROLLARY 2.2.2 : *For each cylinder* $[x_1 \ldots x_{k-1}]$ *we have that* Σ *is the disjoint union of branches* $B[x_1 \ldots x_{k-1};\alpha]$ *as* α *runs through* L.

Proof. For each $r\ (< k)$ there exists $n \ge 0$ such that $\nu[x_1 \ldots x_{k-1}\,\alpha]^{n}(r) < k$ but $\nu[x_1 \ldots x_{k-1}\,\alpha]^{n+1}(r) \ge k$ (with n independent of the choice of $\alpha \in L$, by Lemma 2.2.1 "ν-transfer") . Furthermore $\nu[x_1 \ldots x_{k-1}\,\alpha]^{n+1}(r)$ equals $+\infty$ for precisely one $\alpha \in L$ (and equals k for all other α).
Consequently, every cylinder $[x_1 \ldots x_{r-1}\,\beta]$, $1 \le r < k$, $\beta \neq x_r$, must be part of precisely one of $B[x_1 \ldots x_{k-1};\alpha]$ as α runs through L. The remaining part, $[x_1 \ldots x_{k-1}]$, of Σ is the disjoint union of $[x_1 \ldots x_{k-1}\,\alpha]$, $\alpha \in L$, each of which is allocated to precisely one of the branches – the corresponding $B[x_1 \ldots x_{k-1};\alpha]$. □

LEMMA 2.2.3 "Branch of Complementary Cylinder" : *If* $s \le k$, $\gamma \neq x_s$ *and* $[x_1 \ldots x_{s-1}\,\gamma] \subseteq B[x_1 \ldots x_{k-1};x_k]$ *then*

$$B[x_1 \ldots x_{s-1};\gamma] \subseteq B[x_1 \ldots x_{k-1};x_k] \setminus [x_1 \ldots x_{k-1}\, x_k].$$

Proof. The hypothesis says that $\exists\, m \ge 0$ with $\nu^m(s) = k$. Thus if $[x_1 \ldots x_{r-1}\,\beta]$ is any complementary cylinder of $[x_1 \ldots x_{s-1}\,\gamma]$ which makes up $B[x_1 \ldots x_{s-1};\gamma]$ then $r < s \le k$ and $\exists\, i \ge 1$ with $\nu^i(r) = s$ (by Lemma 2.2.1 "ν-transfer") and so $\exists\, n \ge 1$ with $\nu^n(r) = k$. This says

$$B[x_1 \ldots x_{s-1};\gamma] \setminus [x_1 \ldots x_{s-1}\,\gamma] \subseteq B[x_1 \ldots x_{k-1};x_k] \setminus [x_1 \ldots x_{k-1}\, x_k]$$

and the result follows. □

NOTE that in the above, if $[x_1 \ldots x_{s-1}\,\gamma] \not\subseteq B[x_1 \ldots x_{k-1};x_k]$ then $B[x_1 \ldots x_{s-1}\,\gamma]$ is contained in $B[x_1 \ldots x_{k-1};\alpha]$ for some $\alpha \neq x_k$ and so is disjoint from $B[x_1 \ldots x_{k-1};x_k]$.

We now verify that the branch B of a cylinder $[x_1 \ldots x_{k-1}\, x_k]$ is indeed what we hoped it would be – which in precise language is – : the inverse image under $\pi_{\Gamma'}$ of the component containing $\pi_{\Gamma'}\left([x_1 \ldots x_{k-1}\, x_k]\right)$ of the space $\pi_{\Gamma'}(\Sigma)$. Here Γ' is the restriction of Γ to $C \setminus C_0 \setminus \left\{[x_1 \ldots x_{k-1}\, x_k]\right\}$ and $\pi_{\Gamma'}$ is the projection corresponding to the minimal (closed) equivalence relation on Σ satisfying the property that the map $\pi_\Gamma \to \Gamma'$ factors through (the restriction of) π.

PROPOSITION 2.2.4 "components". *Given* $[x_1 \ldots x_{k-1}\, x_k]$ *in* $C \setminus C_0$ *the only open and closed subsets* U *of* Σ *satisfying the property*

$$(\bullet) \qquad \forall\, c_1, c_2 \in C \setminus C_0 \text{ with } \pi(c_1) = \pi(c_2):$$
$$\left(\Gamma(c_1) \in U \text{ and } \Gamma(c_2) \in \Sigma \setminus U\right) \Rightarrow \left([x_1 \ldots x_{k-1}\, x_k] = c_1 \text{ or } c_2\right)$$

are $\emptyset$, Σ, B *and* $\Sigma \setminus B$ *where* $B = B[x_1 \ldots x_{k-1}; x_k]$.

Proof. We first define $W(r)$, for $r \le k$, by

$$W(r) := \bigcup\left\{[x_1 \ldots x_{r-1}\, \beta] : \beta \in L \setminus \{x_r\}\right\}$$

and we put

$$W(+\infty) := [x_1 \ldots x_k].$$

Thus Σ is the disjoint union of $W(r)$ as r runs through $\{1, 2, \ldots, k, +\infty\}$, and B is the disjoint union of those $W(r)$ for which $\exists\, n \ge 0$ with $\nu^n(r) = k$.

We first verify that $(\bullet)$ holds when $U = B$ (and hence also when $U = \Sigma \setminus B$). Suppose that $\pi(c_1) = \pi(c_2)$, $\Gamma(c_1) \in U$ and $\Gamma(c_2) \in \Sigma \setminus U$. Let $[x_1 \ldots x_{r-1}]$ be the "common ancester" of $[x_1 \ldots x_k]$ and $\pi(c_1)$, in other words – the smallest cylinder containing them both.

We must have $r \le k$ since if $r-1 = k$ then $\Gamma(c_1), \Gamma(c_2) \in \pi(c_1) \subseteq [x_1 \ldots x_k] \subseteq B$.

We must have $\pi(c_1) = [x_1 \ldots x_{r-1}]$ since otherwise $\Gamma(c_1), \Gamma(c_2) \in \pi(c_1) \subseteq W(r)$.

We must have $c_i = [x_1 \ldots x_{r-1}\, x_r]$ and $c_j \subseteq W(r)$ where $\{i,j\} = \{1,2\}$ since otherwise $c_1 = c_2$ or $\Gamma(c_1), \Gamma(c_2) \in c_1 \cup c_2 \subseteq W(r)$.

Furthermore $\Gamma(c_i) \in W(\nu(r))$. The only possible way in which exactly one of $\Gamma(c_i), \Gamma(c_j)$ (and so exactly one of $W(r)$ and $W(\nu(r))$) is contained in B is that $r = k$ and $\nu(r) = +\infty$. Thus $c_i = [x_1 \ldots x_k]$.

Suppose, now, that U is an open and closed subset satisfying $(\bullet)$. If c is either $[x_1 \ldots x_k]$ or a complementary cylinder $[x_1 \ldots x_{r-1}; \beta]$, $1 \le r \le k$, $\beta \ne x_r$ then $[x_1 \ldots x_k]$ is not a strict subcylinder of c, so applying Proposition 2.1.1 to c gives that either $c \subseteq U$ or $c \cap U = \emptyset$. Thus U is a disjoint union of complementary cylinders and possibly $[x_1 \ldots x_k]$. Furthermore if $r \le k$ then property $(\bullet)$ yields that

$$[x_1 \ldots x_{r-1}\, \beta] \subseteq U \iff [x_1 \ldots x_{r-1}\, \beta'] \subseteq U$$

whenever $\beta' \ne x_r \ne \beta$. Thus U is a disjoint union $\bigcup\left\{W(r) : r \in u\right\}$ for some subset u of $\{1, 2, \ldots, k, +\infty\}$.

We now verify that

$$r < k \Rightarrow \left(r \in u \Leftrightarrow \nu(r) \in u \right):$$

If $\nu(r) = s \le k$ then $\Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_{s-1}\ \gamma]$ for some $\gamma \neq x_s$ and so property $(*)$ applied to $[x_1 \ldots x_{r-1}\ x_r]$ and $[x_1 \ldots x_{r-1}\ \beta]$, where $\beta \neq x_r$, gives that $[x_1 \ldots x_{s-1}\ \gamma]$ and $[x_1 \ldots x_{r-1}\ \beta]$ either both lie in U or both lie in $\Sigma \setminus U$, that is, $r \in u \Leftrightarrow s \in u$. If $\nu(r) = +\infty$ then $\Gamma[x_1 \ldots x_{r-1}; x_r] \in [x_1 \ldots x_k]$ and so, provided $r \neq k$, property $(*)$ applied to $[x_1 \ldots x_{r-1}\ x_r]$ and $[x_1 \ldots x_{r-1}\ \beta]$, where $\beta \neq x_r$, gives that $r \in u \Leftrightarrow +\infty \in u$.

Hence, if the ν-iterates of r never hit k then $r \in u \Leftrightarrow +\infty \in u$. On the other hand, if $\nu^n(r) = k$ for some $n \ge 0$ then $r \in u \Leftrightarrow k \in u$.

Thus U is determined purely by whether or not it contains $W(+\infty)$ and $W(k)$. The four possibilities for U are: Σ, B, $\Sigma \setminus B$ and $\emptyset$. □

COROLLARY 2.2.4: *Writing* x *for the gluepoint* $\Gamma[x_1 \ldots x_{k-1}; x_k]$, *we see that* $\Sigma_\Gamma \setminus \{\xi\}$ *is the disjoint union of open sets* $\pi_\Gamma(B) \setminus \{\xi\}$ *and* $\pi_\Gamma(\Sigma \setminus B) \setminus \{\xi\}$.

Proof. The proof is made more tricky by the possibility Γ is not one-to-one. Let $[x]$ denote the $\sim_\Gamma$-equivalence class of x. First verify that $\left(B \times B \cup (\Sigma \setminus B) \times (\Sigma \setminus B) \cup [x] \times [x] \right) \cap \sim_\Gamma$ is a closed equivalence relation on Σ such that for every cylinder there is an equivalence class containing the gluepoints of all its "children". The minimality of $\sim_\Gamma$ then implies that $\sim_\Gamma$ is contained in $B \times B \cup (\Sigma \setminus B) \times (\Sigma \setminus B) \cup [x] \times [x]$. Hence $\pi_\Gamma^{-1}\left(\pi_\Gamma(\Sigma \setminus B)\right) = (\Sigma \setminus B) \cup [x]$ and $\pi_\Gamma^{-1}\left(\pi_\Gamma(B)\right) = B \cup [x]$. Thus $\pi_\Gamma(B)$ and $\pi_\Gamma(\Sigma \setminus B)$ are closed sets which intersect in $\{\xi\}$. □

§2.3 ARCWISE CONNECTIVITY OF Σ_Γ:

DEFINITION 2.3.1: Given points $x, z \in \Sigma$ we define the *Γ-arc from x to z* as the set

$$\Gamma\text{-arc}(x, z) := \left\{ y \in \Sigma : \ \forall k \ge 1,\ B[y_1 \ldots y_{k-1}; y_k] \cap \{x, z\} \neq \emptyset \right\}.$$

This is at first sight a somewhat obscure definition. It is easier to understand the complement $\Sigma \setminus \Gamma\text{-arc}(x, z)$. This is just the set of $y \in \Sigma$ which are "separable" from $\{x, z\}$ in the sense that $B[y_1 \ldots y_{k-1}; y_k] \cap \{x, z\} = \emptyset$ for some $k \ge 1$.

To progress further it is necessary to prove some technical results concerning the intersection of branches of a decreasing chain of cylinders.

LEMMA 2.3.1: *For all* $[y_1 \ldots y_N] \in C \setminus C_0$ *we have*

$$\bigcap_{1 \le k \le N} B[y_1 \ldots y_k] = [y_1 \ldots y_N].$$

Proof. This is clear for $N = 1$. The result follows by induction on N using the fact

$$[y_1 \ldots y_N] \cap B[y_1 \ldots y_N; y_{N+1}] = [y_1 \ldots y_N y_{N+1}] \ . \qquad \square$$

LEMMA 2.3.2: *If* $B[y_1 \ldots y_k] \not\supseteq B[y_1 \ldots y_\ell]$ *for all* ℓ *in the range* $k+1 \le \ell \le N$ *then* $\Gamma[y_1 \ldots y_k] \in [y_1 \ldots y_N]$.

Proof. Suppose, on the contrary, that $\Gamma[y_1 \ldots y_k] \in [y_1 \ldots y_{\ell-1}\beta]$ for some ℓ in the range $k+1 \le \ell \le N$ and some $\beta \ne y_\ell$. Then, for each $\alpha \ne y_k$, the cylinder $[y_1 \ldots y_{k-1}\alpha]$ is complementary to $[y_1 \ldots y_{\ell-1}\beta]$ and so, by the (recursive) definition of *branch*, we have $B[y_1 \ldots y_{k-1};\alpha] \subseteq B[y_1 \ldots y_{\ell-1};\beta]$. Hence, by Corollary 2.2.2, $B[y_1 \ldots y_{k-1};\alpha]$ is disjoint from $B[y_1 \ldots y_{\ell-1};y_\ell]$ for each $\alpha\ (\ne y_k)$, and so, again by Corollary 2.2.2, we must have that $B[y_1 \ldots y_{\ell-1};y_\ell]$ is contained in $B[y_1 \ldots y_{k-1};y_k]$. This contradicts the hypothesis. $\square$

We can now see at least that Γ-arc(x, z) is contained in the closure of the set of points which "separate" x and z.

PROPOSITION 2.3.3: *The set* Γ-arc(x, z) *is contained in*
$\{z\} \cup \mathrm{clos}\{\Gamma[y_1 \ldots y_k] : y_1 \ldots y_k,\ k \ge 1$ such that $B[y_1 \ldots y_k]$ contains x but not z$\}$.

Proof. Given $y \in \Gamma$-arc(x, z) and $N \ge 1$, we show that either $z \in [y_1 \ldots y_N]$ or $\left(\exists k \le N \text{ such that } \left(\Gamma[y_1 \ldots y_k] \in [y_1 \ldots y_N] \text{ and } B[y_1 \ldots y_k] \text{ contains } x \text{ but not } z\right)\right)$.

If $z \in B[y_1 \ldots y_k]$ for all $k \le N$ then, by Lemma 2.3.1, $z \in [y_1 \ldots y_N]$. Otherwise there exists $k\ (\le N)$ maximal such that $z \notin B[y_1 \ldots y_k]$. Hence, for ℓ in the range $k+1 \le \ell \le N$ we have $z \in B[y_1 \ldots y_\ell]$. So, by Lemma 2.3.2, $\Gamma[y_1 \ldots y_k] \in [y_1 \ldots y_N]$. Since $y \in \Gamma$-arc(x, z) we must have $x \in B[y_1 \ldots y_k]$. The result follows. $\square$

NOTE: If $y = \Gamma[y_1 \ldots y_k]$ where $B[y_1 \ldots y_k]$ contains x but not z then, by Corollary 2.2.4, $\bar{x}$ and $\bar{z}$ do not lie in the same component of $\Sigma_\Gamma \setminus \{y\}$.

Parametrization of Γ-arc :

DEFINITION 2.3.2 : Given $x, z \in \Sigma$ a continuous map $P : \{0,1\}^{\mathbb{N}} \to \Sigma$ is said to be a *pre-path from* x *to* z if $P(0^{\infty}) = x$, $P(1^{\infty}) = z$ and for each word $\varepsilon_1 \ldots \varepsilon_k$ the points $P(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty})$ and $P(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^{\infty})$ are either equal or the gluepoints of cylinders with the same parent $\big($ thus ensuring $P(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty}) \sim_{\Gamma} P(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^{\infty})\big)$. It follows that the composed map $\pi_{\Gamma} \circ P$ factors through the surjection $\{0,1\}^{\mathbb{N}} \to [0,1]\ \big(\subset \mathbb{R}\big)$, given by $\varepsilon \mapsto \sum_{i \geq 1} \varepsilon_i / 2^i$, to give a continuous map $P_{\Gamma} : [0,1] \to \Sigma_{\Gamma}$ satisfying $P_{\Gamma}(0) = \bar{x}$ and $P_{\Gamma}(1) = \bar{z}$.

PROPOSITION 2.3.4 : Γ-arc(x, z) *contains the image set of some pre-path from* x *to* z.

Proof. We define a map I on the set of words in 0s and 1s as a sequence of maps $\{0,1\}^k \to L^k$ for $k \geq 1$, defined recursively as follows :

$$I(0) := x_1 ,$$
$$I(1) := z_1 ,$$

and then for $k = 2, 3, \ldots$ by :

$$I(\varepsilon_1 \ldots \varepsilon_{k-1}\, 0) := I(\varepsilon_1 \ldots \varepsilon_{k-1})\, \alpha ,$$
$$I(\varepsilon_1 \ldots \varepsilon_{k-1}\, 1) := I(\varepsilon_1 \ldots \varepsilon_{k-1})\, \beta ,$$

where α, β are the uniquely determined elements of L such that $x \in B\big[I(\varepsilon_1 \ldots \varepsilon_{k-1}); \alpha\big]$ and $z \in B\big[I(\varepsilon_1 \ldots \varepsilon_{k-1}); \beta\big]$. Now I induces a continuous function $I^* : \{0,1\}^{\mathbb{N}} \to \Sigma$ given by $\bigcap_{k \geq 1} \big[I(\varepsilon_1 \ldots \varepsilon_k)\big] = \{I^*(\varepsilon)\}$. Clearly, by construction, the image of I^* is contained in Γ-arc(x, z). Furthermore $I^*(0^{\infty}) = x$ and $I^*(1^{\infty}) = z$.

Unfortunately the map I^* is not quite a pre-path as it stands, so we modify it by composing with another map $J : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ defined by :

$$J(\varepsilon) := \varepsilon \qquad \text{if } \big(\forall k \geq 1\big)\big(\varepsilon_k = 0 \text{ or } x \notin B\big[I(\varepsilon_1 \ldots \varepsilon_k)\big]\big),$$
$$J(\varepsilon) := \varepsilon_1 \ldots \varepsilon_k\, 1^{\infty} \quad \text{where } k \text{ is minimal with } \big(\varepsilon_k = 1 \text{ and } x \in B\big[I(\varepsilon_1 \ldots \varepsilon_k)\big]\big).$$

Now put $P := I^* \circ J$. Clearly $J(0^{\infty}) = 0^{\infty}$, so $P(0^{\infty}) = x$. Also $J(1^{\infty}) = 1^{\infty}$ and so $P(1^{\infty}) = z$. Furthermore P is continuous since both I^* and J are.

Now given $\varepsilon_1 \ldots \varepsilon_k$, $(k \geq 0)$, we examine the values $P(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty})$ and $P(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^{\infty})$:

If $\big(\exists\, i \leq k$ with $\varepsilon_i = 1$ and $x \in B\big[I(\varepsilon_1 \ldots \varepsilon_i)\big]\big)$ then $J(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty}) = J(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^{\infty})$ and consequently the P-values are equal.

Otherwise $J(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty}) = \varepsilon_1 \ldots \varepsilon_k\, 0\, 1^{\infty}$. In this case if $x \in B\big[I(\varepsilon_1 \ldots \varepsilon_k\, 1)\big]$ then $J(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^{\infty}) = \varepsilon_1 \ldots \varepsilon_k\, 1\, 1^{\infty}$ and also by the construction of I we have $I(\varepsilon_1 \ldots \varepsilon_k\, 0) = I(\varepsilon_1 \ldots \varepsilon_k\, 1)$ and moreover

$I^*(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^\infty) = I^*(\varepsilon_1 \ldots \varepsilon_k\, 1\, 1^\infty)$. Hence the P-values are again equal .

Otherwise $x \notin B\left[I(\varepsilon_1 \ldots \varepsilon_k\, 1)\right]$

and so $J(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty) = \varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty$. The construction of I now gives $x \in B\left[I(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^n)\right]$ for all $n \geq 1$ and so by Lemma 2.3.2 , $\Gamma\left[I(\varepsilon_1 \ldots \varepsilon_k\, 1)\right] = I^*(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty) = P(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty)$. We must also have $z \notin B\left[I(\varepsilon_1 \ldots \varepsilon_k\, 0)\right]$ whereas $z \in B\left[I(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^n)\right]$ for all $n \geq 1$. Hence , again by Lemma 2.3.2 , we have $\Gamma\left[I(\varepsilon_1 \ldots \varepsilon_k\, 0)\right] = I^*(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^\infty) = P(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^\infty)$.

We have therefore proved that P is a pre-path from x to z . The image of P is contained in Γ-arc(x, z) . □

PROPOSITION 2.3.5 : *The set of gluepoints of cylinders whose branch contains* x *but not* z *is contained in the image set of any pre-path from* x *to* z .

Proof . Let P be any pre-path from x to z , and let y be the gluepoint of a cylinder $\left[y_1 \ldots y_\ell\right]$ whose branch B contains x but not z . Now B is an open and closed subset of Σ and so $P^{-1}(B)$ is an open and closed subset of $\{0,1\}^{\mathbb{N}}$ containing 0^∞ but not 1^∞ . It follows from Proposition 2.1.1 "connected" (applied to the glueing space of Example 1) that for some word $\varepsilon_1 \ldots \varepsilon_k$ precisely one of $\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^\infty$ and $\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty$ is contained in $P^{-1}(B)$. Thus since P is a pre-path , the points $P(\varepsilon_1 \ldots \varepsilon_k\, 0\, 1^\infty)$ and $P(\varepsilon_1 \ldots \varepsilon_k\, 1\, 0^\infty)$, precisely one of which is contained in B , must be the gluepoints of two cylinders with the same parent . It follows from Proposition 2.2.4 "components" that one of these cylinders is $\left[y_1 \ldots y_\ell\right]$ and so the corresponding gluepoint is y . Hence we have shown that y lies in the image set of P . □

We now collect the results of Propositions 2.3.3 , 2.3.4 and 2.3.5 into a theorem .

THEOREM 2.3.6 : *For any two points* $x, z \in \Sigma$ *the set* Γ-arc(x, z) *is the image-set of a pre-path from* x *to* z *and is contained in the image-set of any other pre-path from* x *to* z . *Furthermore if* $x \neq z$ *then* Γ-arc(x, z) *is the closure of the set of gluepoints of cylinders whose branch contains precisely one of* x *and* z . □

We list a few obvious properties of Γ-arc :

(i) $\Gamma\text{-arc}(x, z) \subseteq \Gamma\text{-arc}(x, y) \cup \Gamma\text{-arc}(y, z)$;

(ii) $y \in \Gamma\text{-arc}(x, z) \Rightarrow \Gamma\text{-arc}(x, y) \subseteq \Gamma\text{-arc}(x, z)$;

(iii) $x_1 \ldots x_k = z_1 \ldots z_k \Rightarrow \Gamma\text{-arc}(x, z) \subseteq \left[x_1 \ldots x_k\right]$;

(iv) $\left(x_1 \ldots x_{k-1} = z_1 \ldots z_{k-1}\ \&\ x_k \neq z_k\right) \Rightarrow$
$\Gamma\text{-arc}(x, z) = \Gamma\text{-arc}\left(x, \Gamma\left[x_1 \ldots x_k\right]\right) \cup \Gamma\text{-arc}\left(\Gamma\left[z_1 \ldots z_k\right], z\right)$.

Unique-arc-connectivity :

DEFINITION 2.3.2 : Given points $\bar{x}, \bar{z} \in \Sigma_\Gamma$ define the Σ_Γ-*arc from* $\bar{x}$ *to* $\bar{z}$ to be the closure of set of points $\bar{y} \in \Sigma_\Gamma$ such that $\bar{x}$ and $\bar{z}$ do not lie in the same component of $\Sigma_\Gamma \setminus \{\bar{y}\}$.

Clearly, this set is contained in the image-set of any path from $\bar{x}$ to $\bar{z}$. On the other hand, for any $x \in \pi_\Gamma^{-1}(\bar{x})$ and $z \in \pi_\Gamma^{-1}(\bar{z})$, we have, dense in the set Γ-arc(x, z), points y for which $\left(\text{by Corollary 2.2.4}\right)$ $\bar{x}$ and $\bar{z}$ do not lie in the same component of $\Sigma \setminus \pi_\Gamma(y)$. Consequently the Σ_Γ-arc from $\bar{x}$ to $\bar{z}$ equals $\pi_\Gamma\left(\Gamma\text{-arc}(x, z)\right)$ and is the image of a path from $\bar{x}$ to $\bar{z}$.

Local-arc-connectivity :

PROPOSITION 2.3.7 : *The space* Σ_Γ *is locally-path-connected.*

Proof. Given a point $\bar{x} \in \Sigma_\Gamma$ and open set U containing $\bar{x}$, we have that the open subset $\pi_\Gamma^{-1}(U)$ of Σ contains all the points x of $\pi_\Gamma^{-1}(\bar{x})$. For each such x there is a cylinder c_x containing x, but contained in $\pi_\Gamma^{-1}(U)$. Now $V := \bigcup\left\{ c_x : x \in \pi_\Gamma^{-1}(\bar{x}) \right\}$ is an open subset of Σ containing $\pi_\Gamma^{-1}(\bar{x})$. So $\pi_\Gamma(\Sigma \setminus V)$, being the continuous image of a compact set, is compact and hence a closed subset of the Hausdorff space Σ_Γ. Consequently $\pi_\Gamma(V)$ is a neighborhood (albeit not necessarily open) of $\bar{x}$. For all $\bar{z} \in \pi_\Gamma(V)$ there exists z in $\pi_\Gamma^{-1}(\bar{z}) \cap V$ and so $z \in c_x$ for some $x \in \pi_\Gamma^{-1}(\bar{x})$. It is easily seen that Γ-arc(x, z) is contained in c_x. Hence the Σ_Γ-arc from $\bar{x}$ to $\bar{z}$ is contained in $\pi_\Gamma(V)$ and so in U. □

COROLLARY 2.3.7 : *For all* $\bar{y}$ *in the* Σ_Γ-*arc from* $\bar{x}$ *to* $\bar{z}$, *the points* $\bar{x}$ *and* $\bar{z}$ *do not lie in the same component of* $\Sigma_\Gamma \setminus \{\bar{y}\}$.

Proof. Since $\Sigma_\Gamma \setminus \{y\}$ is locally-path-connected its components are path-components. Hence if $\bar{x}$ and $\bar{z}$ lie in the same component of $\Sigma_\Gamma \setminus \{\bar{y}\}$ then $\{\bar{y}\}$ is disjoint from the Σ_Γ-arc from $\bar{x}$ to $\bar{z}$. □

§2.4 VALENCY :

DEFINITIONS 2.4.1 : We say a number $s \in \mathbb{N}$ is an *adjacent* of a point $x \in \Sigma$ if $\Gamma[x_1 \ldots x_{s-1}; x_s] = x$, or equivalently if $\nu_x(s) = +\infty$. Generically x will have no adjacents.

Define an equivalence relation $\sim_x$ on $\mathbb{N}$ by :

$$r \sim_x s \iff \left(\exists i, j \geq 0\right)\left(\nu_x^i(r) = \nu_x^j(s) \neq +\infty \right).$$

The equivalence classes which do not contain adjacents will be called *asymptotics*. Define the *asymptotic valency* of x to be the number of asymptotics, and put the *valency* of x to be the number of asymptotics plus the number of adjacents.

LEMMA 2.4.1 : *If* y *lies in a complementary* r*-cylinder of* x *and* z *lies in a complementary* s*-cylinder of* x *then* $r \sim_x s \iff x \notin \Gamma\text{-arc}(y, z)$.

Proof. $x \notin \Gamma\text{-arc}(y, z) \iff \left(\exists\, k \geq 1\right)\left(B[x_1 \ldots x_k] \cap \{y, z\} = \emptyset\right)$

$$\iff \left(\exists\, k \geq 1 \text{ and } \exists\, i, j \geq 0\right)\left(\nu_x^{\,i}(r) = k = \nu_x^{\,j}(s)\right)$$

$$\iff r \sim_x s$$

(using Theorem 2.2.2). □

COROLLARY 2.4.1 : *If* $\{x\}$ *is a* $\sim_\Gamma$*-equivalence class then for each* $\sim_x$*-class we have that the union of complementary* r*-cylinders*, r *varying over this class*, *is the inverse image under* π_Γ *of a component of* $\Sigma_\Gamma \backslash \{\bar{x}\}$. *Consequently the valency of* x *is the number of components of* $\Sigma_\Gamma \backslash \{\bar{x}\}$.

Proof. Pick y from a complementary r-cylinder of x . By hypothesis, $\bar{y} \neq \bar{x}$ so, by Corollary 2.3.7, the set of $\bar{z}$ for which $\bar{x}$ does not lie in the Σ_Γ-arc from $\bar{y}$ to $\bar{z}$ is a component of $\Sigma_\Gamma \backslash \{\bar{x}\}$. The inverse-image of this component is therefore the set of $z \in \Sigma$ such that $\pi_\Gamma^{-1}(\bar{x})\ (= \{x\})$ is disjoint from $\Gamma\text{-arc}(y, z)$. But this set is just the union of complementary s-cylinders, s varying over the $\sim_x$-class of r .

□

EXAMPLE 2.1 :

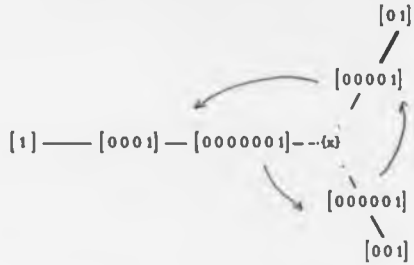
Consider the space Σ_ξ where $\Sigma = \{0,1\}^{\mathbb{N}}$ and ξ is a nonperiodic sequence lying in $[0\,0\,1]$. Let x be the period one point 0^{∞} . Now x is not an inverse-image of ξ (under the shift map σ) and so $\{x\} = \pi_\Gamma^{-1}(\bar{x})$. To compute the valency of x we calculate the action of ν_x on $\mathbb{N}$. This is really a map on the complementary cylinders:

$$[1], [0\,1], [0\,0\,1], [0\,0\,0\,1], [0\,0\,0\,0\,1], \ldots$$

Now the complementary cylinder $[1]$ is attached to the cylinder $[0]$ (containing x) at the gluepoint $0\,\xi$ which lies in the complementary cylinder $[0\,0\,1]$. This in turn is attached to the cylinder $[0\,0\,0\,0]$ at the gluepoint $0\,0\,0\,0\,\xi$ which lies in the complementary cylinder $[0\,0\,0\,0\,0\,0\,1]$, and so on . Thus $\nu_x(1) = 4$, $\nu_x(4) = 7$, . . .

Similarly, the complementary cylinder $[0\,1]$ is attached to the cylinder $[0\,0]$ at the gluepoint $0\,0\,\xi$ which lies in the complementary cylinder $[0\,0\,0\,0\,1]$, and so on . So $\nu_x(2) = 5$, $\nu_x(5) = 8$, ... Finally we get $\nu_x(3) = 6$, $\nu_x(6) = 9$, ... Hence $\sim_x$ has three asymptotic equivalence classes and x has valency 3 .

Here we also have the action of the shift map σ which, leaving x invariant, must send complementary r-cylinders to $(r-1)$-cylinders, which are complementary unless $r = 1$. Hence σ permutes the asymptotic classes in a cyclic fashion .



We can condense this picture into the following diagram .

It is a pretty clear generalisation of the above that if ξ is not periodic and lies in the cylinder $[0^{v-1}1]$, $v \geq 1$, then the fixed point 0^{∞} has (asymptotic) valency v, and that the asymptotics are permuted cyclically by σ. The other fixed point 1^{∞} has valency one unless $v = 1$.

One can see that for given ξ (not periodic) and given point x (say periodic or pre-periodic) it is not too hard to compute the valency of x. However there does not seem to be any nice formula for valency in terms of general x and ξ. We leave as unproved the following

CONJECTURE : *If Γ is as in Example 2 for some sequence $\xi \in \Sigma$ and x is a point of asymptotic valency three or more then x is periodic or pre-periodic .*

NOTE : Compare this to Theorem 9.2 "gaps-eventually-cycle" in [T].

Example of a Non complex-admissible Kneading Itinerary :

EXAMPLE 2.2 : For $\Sigma = \{0,1\}^{\mathbb{N}}$, let Γ be the gluepoint map corresponding to some (non periodic) sequence ξ in $[0\,1\,0\,0\,1\,1]$ and let x be the period three point $(0\,1\,0)^{\infty}$. To compute the valency of x we note that for $n \geq 1$, $v_x(n) = n + k$ where k is the discrepancy of $\sigma^n x$ and ξ, in other words , the first ordinate where they disagree .

Here are the ν_x-iterates of 1 :

$$x = 0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,\ldots$$
$$(\xi =)\ 0\ \ldots$$
$$(\xi =)\ 0\ 1\ \ldots$$
$$(\xi =)\ 0\ \ldots$$
$$(\xi =)\ 0\ 1\ \ldots$$
$$(\xi =)\ 0\ \ldots$$
$$(\xi =)\ 0\ 1\ \ldots$$

Now for the ν_x-iterates of 3 :

$$x = 0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,\ldots$$
$$(\xi =)\ 0\,1\,0\,0\,1\,1\ \ldots$$
$$(\xi =)\ 0\,1\,0\,0\,1\,1\ \ldots$$

Similarly the ν_x-iterates of 6 :

$$x = 0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,\ldots$$
$$(\xi =)\ 0\,1\,0\,0\,1\,1\ \ldots$$
$$(\xi =)\ 0\,1\,0\,0\,1\,1\ \ldots$$

These ν_x-iterates are all disjoint and account for all of $\mathbb{N}$. Hence x has valency three . Moreover the action of σ^3 , which fixes x , on the complementary r-cylinders ($r \geq 4$) induces a permutation on the asymptotics which is to fix the asymptotic class of 1 , but switch the other two . It follows that Σ_ξ cannot be embedded as a subset J of the complex plane in such a way that the map induced by σ^3 extends to a neighborhood of J and is locally an orientation-preserving homeomorphism about (the embedded copy of) x . In particular , since $\mathbb{R}$ is not an inverse-image of the "critical point" $\sigma^{-1}(\xi)$, the space Σ_ξ (together with the action of σ) cannot be topologically conjugate to a Julia set of any map of the form $z \mapsto z^2 + c$. (It is possible however , for instance if $\xi = 0\,1\,0\,0\,(1)^\infty$, that Σ_ξ represents the "Julia set" of some complex-conjugate polynomial map $z \mapsto \bar{z}^2 + c$.)

§2.5 THE SPACE Σ_ξ :

We now concentrate on the glueing space (Σ,Γ) given in Example 2 . Let ξ in Σ be fixed . The map $\Gamma = \Gamma_\xi$ was given by

$$\Gamma[x_1 \ldots x_{k-1};x_k] = x_1 \ldots x_{k-1}\, x_k\, \xi \ .$$

LEMMA 2.5.1 "backward-iterating Γ_ξ-branches" : *If* $\xi \notin B[x_1 \ldots x_{k-1};x_k]$ *then* $B[x_0\, x_1 \ldots x_{k-1};x_k] = x_0\, B[x_1 \ldots x_{k-1};x_k]$ *for all* $x_0 \in L$.

Proof . (By induction on k) . If $\xi \notin B[x_1 \ldots x_{k-1};x_k]$ then ξ is not contained in any "sub-branch" $B[x_1 \ldots x_{r-1}\ \beta]$ (where $1 \le r < k$ with $\Gamma[x_1 \ldots x_{r-1};x_r] \in [x_1 \ldots x_{k-1}\, x_k]$ and $\beta \in L\setminus\{x_r\}$) . So, for these r , by inductive hypothesis :

$$B[x_0\, x_1 \ldots x_{r-1}\, \beta] = x_0\, B[x_1 \ldots x_{r-1}\, \beta] .$$

Now , from the original recursive definition of "branch" we have

$$B[x_0\, x_1 \ldots x_{k-1};x_k] = [x_0\, x_1 \ldots x_{k-1}\, x_k] \cup$$
$$\bigcup\{\, B[x_0\, x_1 \ldots x_{r-1};\beta] \ :\ 0 \le r < k \text{ with } x_0\, x_1 \ldots x_{r-1}\, x_r\, \xi \in [x_0\, x_1 \ldots x_{k-1}\, x_k] \text{ and } \beta \ne x_r \,\}$$
$$= x_0\, [x_1 \ldots x_{k-1}\, x_k] \cup$$
$$\bigcup\{\, x_0\, B[x_1 \ldots x_{r-1};\beta] \ :\ 1 \le r < k \text{ with } x_1 \ldots x_{r-1}\, x_r\, \xi \in [x_1 \ldots x_{k-1}\, x_k] \text{ and } \beta \ne x_r \,\}$$

since (for $r = 0$) we have $\xi \notin [x_1 \ldots x_{k-1};x_k]$. The result follows . □

NOTE : It is not hard to amend this proof to see that if $\xi \in B[x_1 \ldots x_{k-1};x_k]$ then $B[x_0\, x_1 \ldots x_{k-1};x_k] = x_0\, B[x_1 \ldots x_{k-1};x_k] \cup \bigcup\{\, \{\beta\} \ :\ \beta \in L\setminus\{x_0\} \,\}$.

As a Corollary to this , or otherwise , we can see :

PROPOSITION 2.5.2 : $x_1 = z_1 \Rightarrow \Gamma_\xi\text{-arc}(\sigma x\, , \sigma z) = \sigma(\Gamma_\xi\text{-arc}(x, z))$. □

The Nonperiodicity Function :

Consider the parameter ξ now as a point -the 'critical value'- in the space (Σ,Γ_ξ) . The associated successor function $\nu_\xi : \mathbb{N} \to \mathbb{N}\cup\{+\infty\}$ is of special interest . The successor of a number n is the point at which ξ disagrees with the gluepoint $\Gamma[\xi_1 \ldots \xi_n] = \xi_1 \ldots \xi_n\, \xi$. Thus

$$\nu_\xi(n) = \min\{\, j > n \ :\ \xi_j \ne \xi_{j-n} \,\} \ \left(\text{or } = +\infty \text{ if } \xi = \xi_1 \ldots \xi_n\, \xi \right).$$

For each $n \in \mathbb{N}$ the number $\nu_\xi(n)$ is then also the point at which ξ fails to be periodic of period n , so we call it the *point of nonperiodicity* of n . We call the ν_ξ the *nonperiodicity function* (of ξ) . A number m ($< +\infty$) is called a *nonperiodicity*

of n if there exists $i \geq 0$ such that $\nu_\xi^i(n) = m$. A number m is called a *principal nonperiodicity* (or just *principal*) if it is a nonperiodicity of 1. (The principal nonperiodicities turn out to be of considerable importance).

The valency of the point ξ is now the maximal number of disjoint "chains" of nonperiodicities. For general n (≥ 3) there it is not always true that its nonperiodicities should eventually become principal. For instance if $\xi = 0\,1\,(1\,0\,0)^\infty$ and $n = 4$ we have

$$\xi = 0\,1\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,.\,.$$

and so the nonperiodicities of 4 are never principal (see section 2.6 for the implication of this in this example). However if n can be expressed in the form $\nu_\xi(k) - k$ for some $k \geq 1$ then we shall see that its nonperiodicities do eventually become principal.

From now on we shall write ν for ν_ξ unless there is some ambiguity. Associated to the nonperiodicity function is the "lower orbit " function $\mu : \mathbb{N} \to \mathbb{N} \cup \{+\infty\}$ (see §4.3) which is defined by

$$\mu(k) = \nu(k) - k \ .$$

For $k \geq 1$, assuming that the points ξ and $\sigma^k\xi$ are distinct, they lie in the same $(\mu(k)-1)$-cylinder, but distinct $\mu(k)$-cylinders : respectively $[\xi_1 \ldots \xi_{\mu(k)}]$, $[\xi_{k+1} \ldots \xi_{\nu(k)}]$.

THEOREM 2.5.3 "$\mu(k)$ nonperiodicities principal before $\nu(k)$" :
If $k \geq 1$ *then there exist* $i, j \geq 0$ *such that*

$$\nu^i(\mu(k)) = \nu^j(1) \leq \nu(k) \ .$$

Proof. We take it that ξ and $\sigma^k\xi$ are distinct. The cylinders $[\xi_{k+1} \ldots \xi_{\nu(k)}]$ (containing $\sigma^k\xi$) and $[\xi_1 \ldots \xi_{\mu(k)}]$ (containing ξ) are distinct with the same "parent". Thus $B[\xi_{k+1} \ldots \xi_{\nu(k)}] \cap [\xi_1 \ldots \xi_{\mu(k)}] = \emptyset$. Find p minimal such that $\sigma^p\xi \in B[\xi_{k+1} \ldots \xi_{\nu(k)}]$. So $1 \leq p \leq k$ and $\sigma^p\xi$ lies in either $[\xi_{k+1} \ldots \xi_{\nu(k)}]$ or a cylinder complementary to $[\xi_{k+1} \ldots \xi_{\nu(k)-1}]$, which is in either case a cylinder complementary to $[\xi_1 \ldots \xi_{\mu(k)}]$. This complementary cylinder, which is in fact $[\xi_{p+1} \ldots \xi_{\nu(p)}]$ (with length $\mu(p) \leq \mu(k)$), is contained in $B[\xi_{k+1} \ldots \xi_{\nu(k)}]$. Thus, by Lemma 2.2.3 "Branch of complementary cylinder",

$$B[\xi_{p+1} \ldots \xi_{\nu(p)}] \subseteq B[\xi_{k+1} \ldots \xi_{\nu(k)}] \ .$$

Now, by the minimality of p, we have, for $i = 0, 1, \ldots, p-1$, that

$$\xi \notin \xi_{p-i+1} \ldots \xi_p \, B[\xi_{k+1} \ldots \xi_{\nu(k)}] \ ,$$

and hence that

$$\xi \notin \xi_{p-i+1} \ldots \xi_p \, B[\xi_{p+1} \ldots \xi_{\nu(p)}] \ .$$

It follows, by successive applications of Lemma 2.5.1 "backward iterating . . ", that

$$B[\xi_1 \cdots \xi_p\, \xi_{p+1} \cdots \xi_{\nu(p)}] = \xi_1 \cdots \xi_p\, B[\xi_{p+1} \cdots \xi_{\nu(p)}] .$$

This set is too small to contain either a complementary 1-cylinder or $B[\xi_{\mu(k)+1} \cdots \xi_{\nu(k)}]$, and hence, by Lemma 2.2.3 "Branch of Complementary . . . ", the complementary $\mu(k)$-cylinder $[\xi_{\mu(k)+1} \cdots \xi_{\nu(k)}]$. In terms of nonperiodicities, this means that there exist $i, j \geq 0$ such that $\nu^i(\mu(k)) = \nu(p) = \nu^j(1)$ (by Theorem 2.2.2). Finally we have $(\exists l \geq 0)(\nu^l(\mu p) = \mu k)$. If $l \geq i$ then $\nu^{l-i}(\nu(p)) = \mu k$. Otherwise $l < i$ and $\nu^{i-l}(\mu k) = \nu(p) = p + \mu(p) \leq k + \mu(k) = \nu(k)$. □

COROLLARY 2.5.3.1 : *If ξ is not periodic then the points $\sigma^k(\xi)$, for $k \geq 1$, all lie in the component of $\Sigma_\xi \setminus (\xi)$ which contains $\sigma^{-1}(\xi)$.*

Proof. Fix $k \geq 1$. Since ξ is not periodic, the points $\pi_\xi(\xi)$ and $\pi_\xi(\xi_1 \cdots \xi_{\nu(p)}\, \xi)$ are distinct. Now $\Gamma_\xi\text{-arc}(\sigma^{-1}\xi, \sigma^k\xi) \subseteq \Sigma \setminus B[\xi_1 \cdots \xi_{\nu(p)}]$. Hence by Corollary 2.2.4 the Σ_ξ-arc from $\sigma^{-1}(\xi)$ to $\sigma^k(\xi)$ does not contain the point $\pi_\xi(\xi)$. □

COROLLARY 2.5.3.2 : *Up to permutation of symbols in L, the kneading itinerary ξ is an invariant of the topological dynamical system (Σ_ξ, σ).* (Here we are restricting attention to the "non-degenerate " systems arising from $\{\xi \in \Sigma : \xi \text{ not periodic}\}$).

Proof. Specifying a labelling of the inverse-images under σ of the "critical" point $\sigma^{-1}(\xi)$ by the elements of L, we can recover the kneading itinerary ξ by observing for each $n \in \mathbb{N}$ which inverse-image of $\sigma^{-1}(\xi)$ is contained in the component of $\Sigma_\xi \setminus \{\sigma^{-1}(\xi)\}$ which contains $\sigma^{n-1}(\xi)$. □

The space Σ_Σ :

In Example 3, the map $\Gamma = \Gamma_\Sigma$ was given by

$$\Gamma_\Sigma[x_1 \cdots x_{k-1}; x_k] = (x_1 \cdots x_{k-1}\, x_k)^\infty .$$

For any $\xi \in \Sigma$ the associated successor function ν_ξ sends each n in $\mathbb{N}$ to the point at which ξ disagrees with the gluepoint $\Gamma[\xi_1 \cdots \xi_n] = (\xi_1 \cdots \xi_n)^\infty$. Hence ν_ξ is actually the nonperiodicity function of ξ. Hence in particular the valency of ξ in (Σ, Γ_Σ) is equal to the valency of the "critical value" ξ in (Σ, Γ_ξ). (Compare with Theorem 2 in [DH2, exp. VIII] which equates the external arguments of a Misurewicz point c in the Mandelbrot set with the external arguments of the critical value c in the Julia set $K(f_c)$).

The space Σ_Σ (or rather the glueing space (Σ, Γ_Σ)) can be thought of as a "parameter space" for the set of Σ_ξ (or rather of spaces (Σ, Γ_ξ)) as ξ varies through Σ. Figure 2.4 shows one half, the cylinder $[0]$, of the space (Σ, Γ_Σ) in the

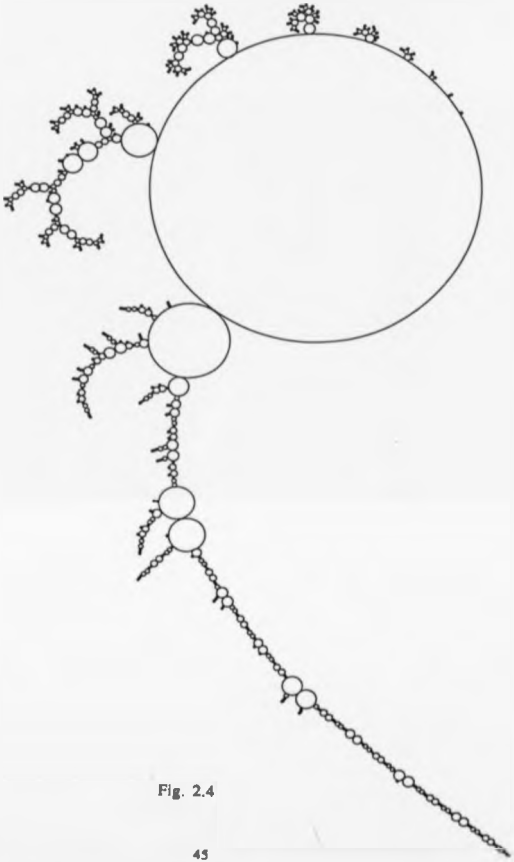


Fig. 2.4

case $\Sigma = \{0,1\}^{\mathbb{N}}$, down to "level" 10. (The radius of each 10-cylinder is drawn proportional to the inverse square of the minimal length cylinder whose gluepoint it contains, i.e. the minimal period among periodic points it contains).

We now concentrate on the "spine" of this set.

§2.6 REAL-ADMISSIBLE KNEADING ITINERARIES :

DEFINITION 2.6.1 : We say a sequence ξ in Σ is *real-admissible* if $\sigma^2\xi \in \Gamma_\xi\text{-arc}(\xi,\sigma\xi)$. Essentially, this means that the space Σ_ξ contains a forward-invariant "interval". It is not hard to check that ξ is real-admissible if and only if $\sigma^k\xi \in \Gamma_\xi\text{-arc}(\xi,\sigma\xi)$ for all $k \geq 2$. (Induction on k and separating the cases $\sigma^k\xi \in \Gamma_\xi\text{-arc}(\xi, \xi_1\xi)$ and $\sigma^k\xi \in \Gamma_\xi\text{-arc}(\xi_2\xi, \sigma\xi)$).

PROPOSITION 2.6.1 :

$$\xi \textit{ is real-admissible} \leftrightarrow \left(\exists\alpha, \beta \in L, \alpha \neq \beta\right)\left(\xi \in \Gamma_\Sigma\text{-arc}(\alpha^\infty, \alpha\,\beta^\infty)\right).$$

Proof. Let $\xi \in \Sigma$ be given. If $\xi_1 = \xi_2$ then $\Gamma_\xi\text{-arc}(\xi,\sigma\xi) \subseteq \left[\xi_1\right]$ and so ξ is real-admissible if and only if $\xi = (\xi_1)^\infty$. Furthermore for each $\alpha, \beta \in L$ the set $\Gamma_\Sigma\text{-arc}(\alpha^\infty, \alpha\,\beta^\infty)$ is contained in $\left[\alpha\right]$ and intersects $\left[\alpha\alpha\right]$ only in the point α^∞, and therefore contains ξ if and only if $\xi = \alpha^\infty$.

From now on we suppose $\xi_1 \neq \xi_2$. If $\sigma^2\xi = \sigma\xi$ then ξ equals $\xi_1(\xi_2)^\infty$ and is real-admissible. Otherwise find $k \geq 1$ minimal such that $(\sigma^2\xi)_k \neq (\sigma\xi)_k$, in other words such that $\xi_{k+2} \neq \xi_{k+1}$. We have $\xi_1 \neq \xi_2 = \xi_3 = \ldots = \xi_{k+1} \neq \xi_{k+2}$. Now, from the definition of k, we have that the point $\xi_1\,\xi_2\,\sigma\xi$ lies in a complementary $(k+2)$-cylinder of ξ whereas the point $\xi_1\,\xi_2\,\xi\ \left(= \Gamma_\xi\left[\xi_1;\xi_2\right]\right)$ lies in a complementary $\nu_\xi(2)$-cylinder of ξ (or equals ξ if $\nu_\xi(2) = +\infty$). Hence

$$\begin{aligned}
\xi \text{ is real-admissible} &\leftrightarrow \sigma^2\xi \in \Gamma_\xi\text{-arc}(\xi, \sigma\xi) \\
&\leftrightarrow \xi \in \Gamma_\xi\text{-arc}\left(\xi_1\,\xi_2\,\xi\ ,\ \xi_1\,\xi_2\,\sigma\xi\right) \\
&\leftrightarrow \text{the } \xi\text{-nonperiodicities of } \nu_\xi(2) \text{ are disjoint from those of } k+2 \\
&\leftrightarrow \text{the } \xi\text{-nonperiodicities of } 2 \text{ and of } k+2 \text{ are disjoint} \\
&\leftrightarrow \xi \in \Gamma_\Sigma\text{-arc}\left((\xi_1)^\infty,\ \xi_1\,(\xi_2)^\infty\right)
\end{aligned}$$

since $(\xi_1)^\infty$ lies in complementary 2-cylinder of ξ whereas $\xi_1(\xi_2)^\infty$ lies in a complementary $(k+2)$-cylinder of ξ. (Here we are using Lemma 2.4.1.) □

For the quadratic family $f_c : z \mapsto z^2+c$ it is well-known that when c lies in the real interval $[-2, 1/4]$ the iterates of the critical point 0 remain bounded and the set $K(f_c)$ intersects $\mathbb{R}$ in the interval $[-w, w]$ where $w = (1/2)+\sqrt{((1/4)-c)}$ is the larger of the two fixed points. If $K(f_c)$ is partitionable in the sense of §1.2 then the partition thereof chops the interval $[-w, w]$ into sets $[-w, 0)$ and $(0, w]$. However this natural partition of $[-w, w]$ can be applied to f_c for any c in $[-2, 1/4]$. Many authors have studied the dynamics of real quadratic maps (or more generally "one hump" maps) of the interval. In particular, Milnor & Thurston (in [MT] & [M]) introduce the notion of "itinerary" of a point z in $[-w, w]$ which in our set-up means the sequence $x(z) = x_c(z) = \alpha_1 \alpha_2 \alpha_3 \ldots \in \{0, 1\}^{\mathbb{N}}$, where

$$\alpha_n = 0 \quad \text{if} \quad f_c^{n-1}(z) < 0 ,$$

$$\alpha_n = 1 \quad \text{if} \quad f_c^{n-1}(z) > 0 ,$$

(or the pair $\{x(z^-), x(z^+)\}$ in the case z is the critical point 0 or one of its inverse-images). They then construct the "invariant coordinate"

$\theta(z) = \theta_1 \theta_2 \theta_3 \ldots \in \{-1, +1\}^{\mathbb{N}}$ by

$$\theta_n = \theta_n(z) := \prod_{1 \le i \le n} \left(2 \cdot \alpha_i - 1 \right)$$

(or the pair $\{\theta(z^-), \theta(z^+)\}$ in the case z is 0 or one of its inverse images). Thus θ_n equals the sign of the derivative of f_c^n at the point z. One advantage of using the invariant coordinate is that it behaves monotonically in $z \in [-w, w]$ in that: $z_1 < z_2$ implies $\theta(z_1) \le \theta(z_2)$ with respect to the lexicographic order on $\{-1, +1\}^{\mathbb{N}}$ (see [M] and [MT]). The set of "admissible" kneading coordinates $\{\theta(z^-), \theta(z^+) : z \in [-w, w]\}$ is determined purely by the "kneading invariant" $\theta(c^+)$. The strict definition of kneading invariant given in [M] and reiterated in [JR] is the formal power series $P(t) := 1 + \sum_{n \ge 1} \theta_n(c^+) \cdot t^n$ (where $\theta_n(c^+) = \lim_{z \searrow c} \theta_n(z)$). Treating $P(t)$ as an analytic function in t, it is proved in [M] that the reciprocal of the smallest positive zero of $(1-t) \cdot P(t)$ is the growth-rate of the number of real solutions z of $f_c^n(z) = 0$ (for $c < 0$) and so, equals the exponential of the topological entropy of $f_c|_{[-w,w]}$ ([MS]).

In the case 0 is not periodic under f_c we can define the *'real' kneading itinerary* $\xi = \xi(c) := x_c(c)$. Given $z_1, z_2 \in [-w, w]$ with $z_1 \le z_2$, the map $z \mapsto x(z)$ from $[z_1, z_2]$ to $\Sigma = \{0,1\}^{\mathbb{N}}$ induces a pre-path from $x(z_1^-)$ to $x(z_2^+)$ where the relevent gluepoint map is Γ_ξ. The monotonic behaviour of $z \mapsto \theta(z)$ together with the invertibility of the transformation relating itineraries and invariant coordinates insures that the image-set $\{x(z^-), x(z^+) : z_1 \le z \le z_2\}$ of this pre-path is precisely $\Gamma_\xi\text{-arc}(x(z_1^-), x(z_2^+))$.

Now if c lies in $[-2, 0)$ and is not periodic then $c \le f_c^2(c) \le f_c(c)$ and so $\sigma^2\xi \in \Gamma_\xi\text{-arc}(\xi, \sigma\xi)$, in other words ξ is real-admissible.

The map $c \mapsto \xi(c)$ on $[-2, 0)$ is defined and continuous except at a

countable number of "jumps" where 0 lies in a superstable periodic orbit. If c_0 is the parameter value of such a super-stable orbit, of (minimal) period p say, then, due to the continuity of the map $(c, x) \mapsto \left(\partial/_{\partial x}\right)\left(f_c^p(x)\right)$ at $(c_0, 0)$, the kneading itinerary $\xi(c)$ will take some constant value $(\xi_1 \ldots \xi_{p-1}\, \alpha)^\infty$ for c in some interval whose infimum is c_0, and will take a constant value $(\xi_1 \ldots \xi_{p-1}\, \beta)^\infty$ for c in some interval whose supremum is c_0. $\Big($ That $\beta \neq \alpha$ follows from the fact that $c = c_0$ is a simple root of the equation $f_c^p(0) = 0$ [DH2, exp. XIX] $\Big)$. We can therefore conclude that the set $\left\{\xi(c) : c \in [-2, 0),\ 0 \text{ not periodic under } f_c\right\}$ is the image-set of a pre-path from $\xi(0^-) = 0^\infty$ to $\xi(-2) = 0\,(1)^\infty$ $\Big($ where the relevant gluepoint map on $\Sigma = \{0,1\}^{\mathbb{N}}$ is $\Gamma_\Sigma\Big)$ and so contains $\Gamma_\Sigma\text{-arc}\left(0^\infty, 0(1)^\infty\right)$.

It therefore follows, by Proposition 2.6.1 that a sequence $\xi \in \{0,1\}^{\mathbb{N}}$ (with $\xi_1 = 0$) is real-admissible if only if ξ is the 'real' kneading itinerary of a polynomial $z \mapsto z^2 + c$ with c real. The proof of Prop. 2.6.1 gives an algorithm for deciding when a sequence ξ is real-admissible, in terms of its nonperiodicity function.

CHAPTER 3

In this Chapter we develop some of the theory of "factorization" of one-sided sequences. In section 3.3 it is shown that a factorization of a kneading itinerary ξ corresponds to a renormalization of the dynamical system $\left(\Sigma_\xi, \sigma\right)$ whereby some power of the shift map σ restricted to various subsets of Σ_ξ is conjugate to another such dynamical system $\left(\Sigma_\psi, \sigma\right)$.

NOTATION : Let L be an alphabet and $\Sigma = L^{\mathbb{N}}$ be the set of one-sided sequences $x = (x_1\, x_2\, x_3 \ldots)$ where $x_j \in L$ for all j.

We say $x\ (\in \Sigma)$ is *periodic with period* p if $x_j = x_{j+p}$ for all $j \geq 1$.

We say $x\ (\in \Sigma)$ is *factorizable with period* q if $x_j = x_{j+q}$ whenever j is not divisible by q. In this case the sequence $(x_q\, x_{2q}\, x_{3q} \ldots)$ can be an arbitrary element of Σ.

§3.1 COMMON FACTORIZABILITY :

DEFINITION 3.1.1 : Suppose $x \in \Sigma$ and $k \geq 1$. We say x has property P *as far as* k if there exists $y \in \Sigma$ with property P and $x_j = y_j$ for all $j \leq k$.

THEOREM 3.1.1 "Common factorizable-periodic"

Suppose $x \in \Sigma$ *is periodic with period* p *as far as* k, *and also factorizable with period* q *as far as* k. *Let* $h = \mathrm{HCF}(p, q)$.

If $k \geq p+q-h$ *then* x *has period* h *as far as* q-1.

Furthermore if either

(i) $q \geq p$

or (ii) $q < p$, $q \nmid p$ and $k \geq p + \lfloor p/q \rfloor q$

then x *is periodic with period* h *as far as* k.

Proof.

Put $m = p+q$. Let $S := \left\{h, 2h, \ldots, m-h\right\}$.

Since $\mathrm{HCF}(q, m) = h$, multiplication by q (modulo m) gives a bijection $\bar{q} : \{1, 2, \ldots, m/h - 1\} \to S$. One can notionally define $\bar{q}(m/h) = m$.

Consider the range $\left\{j \in \mathbb{N} : 1 \leq j \leq p+q-h\right\}$. This is a disjoint union of (mod h) congruence classes $S-r$, $(0 \leq r \leq h-1)$.

To each congruence class $(r \neq 0)$, we shall establish a common value x_j for j in $S-r$.

For the zero-class ($r=0$), we establish a common value for x_j among those j, in S, not divisible by q.

Given r (with $1 \le r \le h-1$), suppose $1 \le i < m/h - 1$. We show :

$$(*) \qquad x_{\bar{q}(i+1)-r} = x_{\bar{q}(i)-r} .$$

In the case $r = 0$ we suppose $\lceil m/q \rceil + 1 \le i < m/h - 1$ (noting that $q \mid \bar{q}(i) \Leftrightarrow i \le \lceil m/q \rceil$). So both $\bar{q}(i)-r$ and $\bar{q}(i+1)-r$ lie in the range $1 \le j \le p+q-h$ and neither are divisible by q.

If $\bar{q}(i+1) = \bar{q}(i)+q$ then $(*)$ follows by the period q factorizability of x as far as $p+q-h$.

Otherwise $\bar{q}(i+1) = \bar{q}(i)+q-m = \bar{q}(i)-p$ and so $(*)$ follows by the period p property of x as far as $p+q-h$.

Hence we have at least that x has period h as far as $q-1$.

If (i) holds with $q > p$ then x has period h as far as p, and therefore as far as k by the period p property of x.

If $q = p$ then $h = p$ and there is nothing to prove.

In case (ii) we first extend the common value x_j for $j \in S$ to include those j ($\le p$) divisible by q. For all $i \le \lfloor p/q \rfloor$ the period p property of x as far as $p+\lfloor p/q \rfloor q$ gives $x_{iq} = x_{p+iq}$ which equals x_p (our common value) by the period q factorizability of x as far as $p+\lfloor p/q \rfloor q$, using the assumption that q does not divide p. Thus x has period h as far as p, and so as far as k by the period p property of x. □

COROLLARY 3.1.1.1 "Common periodic".

If a sequence x *is periodic with period* p *and also with period* q *as far as* k *and* $k \ge p+q-1$ *then* x *is periodic with period* $\mathrm{HCF}(p, q)$ *as far as* k.

Proof. By symmetry, we may assume $p \le q$ and the result follows by case (i). □

COROLLARY 3.1.1.2 "Common factorizable".

Suppose that a sequence x *is factorizable with period* p *and with period* q *as far as* k *and that* $p \nmid q$, $q \nmid p$ *and* $k \ge 2 \cdot \max\{p, q\}-1$. *Then* x *is periodic with period* $\mathrm{HCF}(p, q)$ *as far as* $\min\{k, \mathrm{LCM}(p, q)-1\}$ *and is also factorizable with period* $\mathrm{LCM}(p, q)$ *as far as* k.

Proof. Without loss of generality $q < p$. The sequence x is in fact periodic with period p as far as $2p-1$ and so the theorem gives us that x is periodic with period h ($:= \mathrm{HCF}(p,q)$) as far as $2p-1$. The period p factorizability of x will prove the result provided that we can establish a common value x_j for those j ($\le k$) divisible by p, but not divisible by q.

If $p \mid j < \mathrm{LCM}\{p, q\}$ then $q \nmid j$ and either $j = p$ or $j > p$ in which case $x_j = x_{j-q} = x_{j-p-q} = x_{j-p}$. It follows by induction that $x_j = x_p$. If $p \mid j > \mathrm{LCM}\{p, q\}$ and $\mathrm{LCM}\{p,q\} \nmid j$ then $q \nmid j$ and so $x_j = x_{j-q} = x_{j-2q} = \ldots = x_{j-(p/b)q} = x_{j-\mathrm{LCM}(p,q)}$. It follows again by induction that $x_j = x_{j'}$ where $j' = j - \left\lfloor j / \mathrm{LCM}\{p, q\} \right\rfloor \mathrm{LCM}\{p, q\}$ which is less than $\mathrm{LCM}\{p, q\}$. Hence $x_j = x_p$. □

§3.2 FACTOR BLOCKS :

DEFINITIONS 3.2.1 : A *block* $\beta = \left[\beta_1 \beta_2 \ldots \beta_{\ell-1} \_\right]$ of *'length'* $\ell \geq 1$ (with $\beta_i \in L$) acts on the set Σ of sequences y by

$$\beta \circ y = \beta_1 \beta_2 \ldots \beta_{\ell-1} y_1 \beta_1 \ldots \beta_{\ell-1} y_2 \beta_1 \ldots \beta_{\ell-1} y_3 \ldots$$

Observe that

$$\sigma^{\ell}(\beta \circ y) = \beta \circ \sigma y .$$

The composed action , on Σ , of two blocks α and β (of lengths k and ℓ respectively) is represented by another block which we write $\alpha \circ \beta$. Thus

$$(\alpha \circ \beta) \circ y = \alpha \circ (\beta \circ y) \quad \text{for all} \quad y \in \Sigma$$

and

$$\alpha \circ \beta = \left[\alpha_1 \ldots \alpha_{k-1} \beta_1 \ \ldots \ \alpha_1 \ldots \alpha_{k-1} \beta_{\ell-1} \alpha_1 \ldots \alpha_{k-1} \_\right]$$

which has length $k \cdot \ell$. The set of blocks forms an associative semi-group under the $\circ$ "product" and has identity element $\left[\_\right]$ $\left(\text{the "trivial" block , of length one}\right)$.

Note that if $x = \beta \circ y$ then x is factorizable with period ℓ , the length of β . If x is factorizable with period q then it "decomposes" as :

$$x = \left[x_1 x_2 \ldots x_{q-1} \_\right] \circ \left(x_q x_{2q} x_{3q} \ldots\right) .$$

We say $\left[x_1 x_2 \ldots x_{q-1} \_\right]$ is the q-*block of* x .

NOTE : The $\circ$-product here corresponds closely to the similar notation used in [JR] to describe the renormalisation properties of the kneading invariant for real unimodal maps of the interval .

DEFINITIONS 3.2.2 : We say that a block β of length ℓ is *atomic* if

$$\beta_1 = \beta_2 = \ldots = \beta_{\ell-1} .$$

If $\ell > 1$ then the common entry β_1 is called the *symbol* of β. The $\circ$-product of two atomic blocks is again atomic unless they are of different symbols .

We say that β is a *continent* block if it is the $\circ$-product of finitely many atomic blocks .

On the other hand , we say that β is an *island* block if :

$$\beta = \alpha \circ \gamma \ \text{ and } \ \gamma \text{ is atomic} \ \Rightarrow \ \gamma = \left[\_\right] .$$

We say a sequence is of *type (A)* if it is the $\circ$-product of infinitely many atomic factors . We say a sequence is of *type (B)* if it can written as $\beta \circ y$ where β is a nontrivial island factor .

The island-continent split :

DEFINITION 3.2.3 : The *confluent* of a block β is defined as the minimal length of α among possible decompositions $\beta = \alpha \circ \gamma$ where γ is atomic .

Thus β is an island block if and only if its confluent is its length .

LEMMA 3.2.1 : *Let β be a block with confluent p . If $y \in \Sigma$ then $\beta \circ y$ can only be factorizable with period q if $q \mid p$ or $p \mid q$.*

Proof . Suppose , on the contrary that $x := \beta \circ y$ is factorizable with period q where $p \nmid q$ and $q \nmid p$. Then , by Corollary 3.1.1.2 "common factorizable" , x is periodic with period $HCF\{p,q\}$ as far as $LCM\{p,q\} - 1 \geq p$. Since x is periodic with period p as far as $\ell - 1$, where ℓ is the length of β , it follows that x is periodic with period $HCF\{p,q\}$ as far as $\ell - 1$. Since $HCF\{p,q\} < p$ this contradicts the minimality of p . □

LEMMA 3.2.2 : *If ξ is of type (A) and $\xi = \beta \circ \psi$ then β is a continent block .*

Proof . It suffices , by induction on the length of β , to show that β is trivial or $\beta = \beta' \circ \gamma$ where γ is nontrivial and atomic . Suppose β has length $\ell > 1$. Since ξ is of type (A) we have a strictly increasing sequence of natural numbers q_n ($n \in \mathbb{Z}_+$) where $q_0 = 1$ and for each n we have $q_n \mid q_{n+1}$ and ξ is periodic of period q_n as far as $q_{n+1} - 1$. Find $n \geq 0$ such that $q_n < \ell \leq q_{n+1}$. The confluent c of the q_{n+1}-block of ξ must satisfy $c \leq q_n$ and furthermore $c \mid \ell$ by Lemma 3.2.1 . Now ξ is periodic with period c as far as $q_{n+1}-1$ and so as far as $\ell - 1$. Hence $\beta = \beta' \circ \gamma$ where β' is the c-block of ξ and γ is nontrivial atomic . □

COROLLARY 3.2.2 : *If $\xi = \alpha \circ \psi = \alpha' \circ \psi'$ and α is an island block and ψ' is of type (A) then the length of α divides the length of α' .*

Proof . Write p, p' for the lengths of α, α' respectively . From Lemma 3.2.1 and the fact that p is the confluent of α we have either $p \mid p'$ or $p' \mid p$. If $p' \mid p$ then ψ' decomposes as $\gamma \circ \psi$ for some block γ of length p/p' . The above Lemma then gives us that γ is a continent block . Since $\alpha = \alpha' \circ \gamma$ is an island block we must have $\gamma = [\_]$, whence $p' = p$. □

THEOREM 3.2.3 : *Any block β has a unique decomposition of the form $\beta = \alpha \circ \gamma$ where α is an island block and γ is a continent block .*

Proof. Uniqueness.

If $\beta = \alpha * \gamma = \alpha' * \gamma'$ are two such decompositions then writing $\xi = \beta * \psi$ for some type (A) sequence ψ, we obtain that $\gamma * \psi$ and $\gamma' * \psi$ are also of type (A). Now Corollary 3.2.2 proves $\alpha = \alpha'$.

Existence.

Define a sequence of natural numbers q_n ($n \in \mathbb{Z}_+$) as follows. Put q_0 equal to the length of β, and for $n = 0, 1, 2, \ldots$ put q_{n+1} equal to the confluent of $[\beta_1 \ldots \beta_{q_n - 1} \_]$. Hence (q_n) is a decreasing sequence and so must eventually have $q_N = q_{N+1}$ for some $N \geq 0$. The block $[\beta_1 \ldots \beta_{q_N - 1} \_]$ is an island block. Since $q_{n+1} \mid q_n$ for all $n < N$ we can decompose the block $[\beta_{q_N} \beta_{2 \cdot q_N} \cdots \beta_{q_0 - q_N} \_]$ into a $*$-product of factors $[\beta_{q_n} \beta_{2 \cdot q_n} \cdots \beta_{q_{n-1} - q_n} \_]$, $N \geq n \geq 1$, each of which is atomic. □

NOTE: Compare this process here to Mandelbrot's notion of "confluence" for components of the interior of the Mandelbrot set [Ma2].

§3.3 RENORMALISATION FOR SPACES Σ_ξ:

To avoid technicalities we shall assume, here, that $\xi (\in \Sigma)$ is not periodic.

If the sequence ξ is "decomposable" with $\xi = \beta * \psi$ and β is of length ℓ then for each $m \in \{0, 1, \ldots, \ell - 1\}$ the map $e_{\beta,m} : \Sigma \to \Sigma$ given by

$$y \mapsto \sigma^m(\beta * y)$$

"projects" through the respective equivalence relations $\sim_{\Gamma_\psi}$ and $\sim_{\Gamma_\xi}$ to give a well-defined map $\bar{e}_{\beta,m}$ from Σ_ψ into Σ_ξ. (In other words $\pi_\xi \circ e_{\beta,m}$ factors through π_ψ and $\pi_\xi \circ e_{\beta,m} = \bar{e}_{\beta,m} \circ \pi_\psi$). We shall see (Corollary 3.3.1) that $\bar{e}_{\beta,m}$ is injective (and hence a homeomorphism onto its image since Σ_ψ is compact and Σ_ξ is Hausdorff).

The ℓ resulting image-sets $\bar{e}_{\beta,m}(\Sigma_\psi)$, $0 \leq m < \ell$, are cyclically permuted by σ. There are other embedded copies of Σ_ψ in Σ_ξ which are the inverse-images of this cycle.

Figure 3.1 shows a hyperbolic Julia set which "represents" the island block $\beta = [0\,0\,0\,1\,\_]$. Here, on collapsing the embedded circles to points one obtains the "semi-degenerate" quotient space $\Sigma_{\beta * x}$ where x is any periodic sequence of type (A). Figure 3.2 shows the Julia set for $f_1 : z \mapsto z^2 + 1$ which has kneading itinerary $0(0\,1)^\infty$ or $1(1\,0)^\infty$. Figure 3.3 shows a Julia set with kneading itinerary $[0\,0\,0\,1\,\_] * 1(1\,0)^\infty$ which contains embedded copies of $J(f_1)$. See Douady's notion of "tuning" [D], [DH3]. (Note that the sequence $[0\,0\,0\,1\,\_] * 0(0\,1)^\infty$ is not complex-admissible).

Fig. 3.1 $J(f_c)$, $c = 0.359259 + 0.642514\,i$.

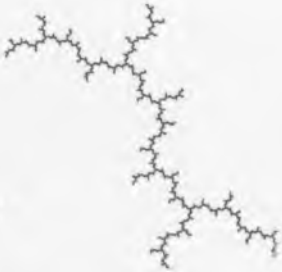


Fig. 3.2 $J(f_i)$

Fig. 3.3 $J(f_c)$, $c = 0.363042 + 0.64473\,i$

LEMMA 3.3.1 : *If* $m,n \in \{0, 1, \ldots, \ell-1\}$ *and* $m \neq n$. *Then for sequences* $y, z \in \Sigma$ *we have*

$$\sigma^m(\beta \bullet y) = \sigma^n(\beta \bullet z)$$
$$\Leftrightarrow \quad \beta = [\beta_1 \ldots \beta_{h-1} \_] \bullet [(\beta_h)^{\ell/h-1} \_] \quad \text{and} \quad y = z = (\beta_h)^\infty$$

where $h := \mathrm{HCF}\{\ell, m-n\}$.

Proof . Without loss of generality $m > n$. First suppose $\sigma^m(\beta \bullet y) = \sigma^n(\beta \bullet z)$. Comparing the jth ordinates of both sides , for j in the range $\ell-m < j < \ell-n$ gives us $\beta_1 \ldots \beta_{m-n-1} = \beta_{\ell-m+n+1} \ldots \beta_{\ell-1}$. Hence $\beta \bullet x$ (for any x) is periodic of period $\ell-m+n$ as far as $\ell - 1$. Similarly comparing the jth ordinates of both sides for j in the range $\ell-n < j < 2\cdot\ell - m$ gives us $\beta_{m-n+1} \ldots \beta_{\ell-1} = \beta_1 \ldots \beta_{\ell-m+n-1}$. Hence $\beta \bullet x$ (for any x) is periodic of period $m-n$ as far as $\ell - 1$. It follows by Corollary 3.1.1.1 "common periodic" that $\beta \bullet x$ is periodic of period h as far $\ell - 1$. In other words $\beta = [\beta_1 \ldots \beta_{h-1} \_] \bullet [(\beta_h)^{\ell/h-1} \_]$.

Now for each $i \in \mathbb{N}$, comparing the $(\ell\cdot i - m)$ ordinate of both sides gives $y_i = \beta_{\ell-m+n}$. For each $i \in \mathbb{N}$, comparing the $(\ell\cdot i - n)$ ordinate of both sides yields $\beta_{m-n} = x_i$. Hence $y = z = (\beta_h)^\infty$.

The reverse implication is obvious since $\beta \bullet y = \beta \bullet z = (\beta_1 \ldots \beta_{h-1}\, \beta_h)^\infty$ and so $\sigma^{m-n}(\beta \bullet y) = \beta \bullet z$. □

COROLLARY 3.3.1 : *If* ψ *is not periodic then for each* $m \in \{0,1, \ldots, \ell-1\}$ *the set* $\{\sigma^m(\beta \bullet y) : y \in \Sigma\}$ *is a union of* $\sim_{\Gamma_{\beta\bullet\psi}}$ *equivalence classes which are precisely the images under* $e_{\beta,m}$ *of all the* $\sim_{\Gamma_\psi}$ *classes of* Σ .

Proof . We have already observed that the image under $e_{\beta,m}$ of any $\sim_{\Gamma_\psi}$ equivalence class is contained in a $\sim_{\Gamma_{\beta\bullet\psi}}$ equivalence class . Now suppose that $x = x_1 \ldots x_{k-1}\, x_k\, (\beta\bullet\psi)$, $k \geq 1$, is an element of a nontrivial $\sim_{\Gamma_{\beta\bullet\psi}}$ class which is contained in $e_{\beta,m}(\Sigma)$. If $\ell \nmid k+m$ then applying the Lemma to $\sigma^k x \in e_{\beta,0}(\Sigma) \cap e_{\beta,n}(\Sigma)$ where n (> 0) is $k+m$ reduced modulo ℓ , we see that $\sigma^k x = \beta\bullet\psi$ is periodic contradicting the assumption that ψ is not periodic . Hence $k = p\cdot\ell - m$ for some $p \geq 1$. Now $\beta_1 \ldots \beta_m\, x = \beta \bullet y$ for some y . So in particular $y_p = x_k$ and $\sigma^p y = \psi$.

The other elements of the $\sim_{\Gamma_{\beta\bullet\psi}}$ equivalence class of x are of the form $x' = x_1 \ldots x_{k-1}\, x'_k\, (\beta\bullet\psi)$ where $x'_k \neq x_k$, and so are of the form $\sigma^m(\beta \bullet y')$ where $\sigma^p y' = \psi$ and $y'_i = y_i$ for $i < p$ but $y'_p \neq y_p$, and so where $y' \sim_{\Gamma_\psi} y$. □

THEOREM 3.3.2 : *Theimage-sets* $\bar{e}_{\beta,m}(\Sigma_\psi)$, $0 \le m < \ell$, *are mutually disjoint if and only if* β *is an island factor . On the other hand , if* β *is an atomic factor then they have a common single point of intersection .*

Proof . If $m \neq n$ and $m,n \in \{0,1,\ldots,\ell-1\}$ then writing $h = HCF(\ell, m-n)$ we have , by Lemma 3.3.1 and its Corollary :

$$\bar{e}_{\beta,m}(\Sigma_\psi) \cap \bar{e}_{\beta,n}(\Sigma_\psi) = \emptyset \iff e_{\beta,m}(\Sigma) \cap e_{\beta,n}(\Sigma) = \emptyset$$
$$\iff \beta = \left[\beta_1 \ldots \beta_{h-1}\_\right] \circ \left[(\beta_h)^{\ell/h-1}\_\right] .$$

For any fixed m we have that h ranges through the divisors of ℓ as n ranges through the elements of $\{0,1,\ldots,\ell-1\} \setminus \{m\}$. Hence the sets $\bar{e}_{\beta,m}(\Sigma_\psi)$ are mutually disjoint if and only if β is an island block .

If β is atomic then $e_{\beta,m}(\Sigma) \cap e_{\beta,n}(\Sigma) = \{(\beta_1)^\infty\}$ whenever $m \neq n$. □

Factorization and Σ_Σ :

The equivalence relation $\sim_{\Gamma_\Sigma}$ introduced in Chapter 2 , Example 3 , can be expressed as the closed equivalence relation on Σ generated by the identifications :

$$\beta \circ x \sim_{\Gamma_\Sigma} \beta \circ y$$

where β is any block and x and y are any sequences of period one .

DEFINITION 3.3.1 : *Given a block* β *we define*

$$A(\beta) := \left\{ \beta \circ x \ : \ x \text{ is a sequence of type } (A) \right\} .$$

PROPOSITION 3.3.3 : *The nontrivial* $\sim_{\Gamma_\Sigma}$ *equivalence classes are of the form* $A(\beta)$ *as* β *runs through all island blocks .*

Proof . First of all , given any block β , we show $A(\beta)$ is contained in a single $\sim_{\Gamma_\Sigma}$ equivalence class . Now $\beta \circ x \sim_{\Gamma_\Sigma} \beta \circ y$ if x and y have period one . Let x be any sequence of type (A) . So there exists a strictly increasing sequence of natural numbers (q_n) , $n \in \mathbb{Z}_+$, with $q_0 = 1$ and $q_n \mid q_{n+1}$ for all n , such that x is periodic with period q_n as far as $q_{n+1}-1$ for all n . We see

$\beta \circ (x_1 \ldots x_{q_n})^\infty \sim_{\Gamma_\Sigma} \beta \circ (x_1 \ldots x_{q_{n+1}})^\infty$ for each n . This is clear since

$\beta \circ (x_1 \ldots x_{q_n})^\infty = \beta \circ \left[x_1 \ldots x_{q_{n+1}-1}\_\right] \circ (x_{q_n})^\infty$ and

$\beta \circ (x_1 \ldots x_{q_{n+1}})^\infty = \beta \circ \left[x_1 \ldots x_{q_{n+1}-1}\_\right] \circ (x_{q_{n+1}})^\infty$. Hence $\beta \circ (x_1)^\infty$ is equivalent to all the sequences $\beta \circ (x_1 \ldots x_{q_n})^\infty$ and hence to their limit point $\beta \circ x$.

Corollary 3.2.2 shows $A(\beta) \cap A(\beta') = \emptyset$ if β and β' are distinct island blocks . It suffices therefore to show that $\left\{ (x,x) : x \in \Sigma \right\} \cup \bigcup \left\{ A(\beta) \times A(\beta) : \text{island blocks } \beta \right\}$ is a closed equivalence relation on Σ containing all pairs $(\alpha \circ x , \alpha \circ y)$ where α is a block and x , y are sequences of period one .

The fact that it is closed is clear because for any length ℓ there are only finitely many (island) blocks β of length ℓ and any such block β has $A(\beta)$ contained in an $(\ell-1)$-cylinder. The fact that it is an equivalence relation comes from the disjointness of the sets $A(\beta)$ for island blocks β. By Theorem 3.2.3 we have that any block α decomposes as $\beta \circ \gamma$ where β is an island block and γ is a continent block. Consequently if x and y have period one then $\gamma \circ x$ and $\gamma \circ y$ are of type (A) and so $(\alpha \circ x, \alpha \circ y) = (\beta \circ (\gamma \circ x), \beta \circ (\gamma \circ y)) \in A(\beta)$. □

CHAPTER 4

In this chapter we construct a metric on the topological space Σ_ξ (assuming that the kneading itinerary ξ does not admit certain types of "factorization") which we obtain essentially as the "Perron-Frobenius" eigenvector of a linear operator acting on the Banach space of bounded functions : $\Sigma \times \Sigma \to \mathbb{R}$. We use results from Chapter 5 .

A Metric on Σ_ξ :

This chapter is devoted mainly to the proof of

THEOREM M : *Let ξ be an element of Σ . If ξ is not decomposable of type (A) then there exists a pseudo-metric $d\ (\neq 0)$ on Σ_ξ and a constant $\lambda > 1$ which satisfy :*

$$\begin{pmatrix} \text{M 1} \\ \text{M 2} \end{pmatrix} \qquad \begin{array}{l} d(\sigma\bar{x},\sigma\bar{y}) = \lambda \cdot d(\bar{x},\bar{y}) \quad \textit{whenever } x_1 = y_1 \ ; \\ d(\bar{x},\bar{z}) = d(\bar{x},\bar{y}) + d(\bar{y},\bar{z}) \quad \textit{whenever } \bar{y} \textit{ lies on the } \Sigma_\xi\textit{-arc} \end{array}$$

from $\bar{x}$ to $\bar{z}$ (in other words , whenever $\bar{x}$ and $\bar{z}$ are not in the same component of $\Sigma_\xi \setminus \{\bar{y}\}$) .

The constant λ is uniquely determined and d is unique up to scalar multiples .

If furthermore ξ is not of type (B) then d is a metric . The topology induced by this metric is the same as the quotient topology on Σ_ξ inherited from the product topology on Σ .

If ξ is of type (B) then writing $\xi = \beta \circ \psi$ where β is the island factor of minimal length greater than one we have

$$d(\bar{x},\bar{y}) = 0 \iff x = y \ \text{ or } \left(x,y \in x_1 \ldots x_n\, \sigma^m(\beta \circ \Sigma) \text{ for some } m,n \geq 0 \right).$$

If ξ is of type (A) then for no $\lambda > 1$ can there exist a pseudo-metric on Σ_ξ satisfying (M1) and (M2) .

NOTE : The factorization types (A) and (B) are given in Definitions 3.2.2 .

Recall that $d\ (\neq 0)$ is a pseudo-metric on a space Σ_ξ if for all $\bar{x}, \bar{y}, \bar{z} \in \Sigma_\xi$:

(i) $d(\bar{x},\bar{y}) \geq 0$;

(ii) $d(\bar{x},\bar{y}) = d(\bar{y},\bar{x})$;

(iii) $d(\bar{x},\bar{z}) \leq d(\bar{x},\bar{y}) + d(\bar{y},\bar{z})$,

(the condition $\bar{x} \neq \bar{y} \Rightarrow d(\bar{x},\bar{y}) > 0$ not being required) .

§4.1 THE OPERATOR φ_ξ^* :

Write $\mathfrak{D} = \left\{ d: \Sigma\times\Sigma \to \mathbb{R} \mid \sup |d(x,y)| < \infty \right\}$. This is a Banach space under the norm $\|d\| = \sup |d(x,y)|$.

The operator $\varphi_\xi^* : \mathfrak{D} \to \mathfrak{D}$ is defined by the rule :

$$\left(\varphi_\xi^*(d)\right)(x,y) = \begin{cases} d(\sigma x,\sigma y) & \text{if } x_1 = y_1 \\ d(\sigma x,\xi) + d(\xi,\sigma y) & \text{if } x_1 \neq y_1 \end{cases} .$$

First observe that $\mathfrak{D}$ is in fact the (continuous) dual of the Banach space $\ell^1(\Sigma\times\Sigma)$, the set of formal linear combinations $v = \sum v_{x,y} \cdot (x,y)$ of elements $(x,y) \in \Sigma\times\Sigma$, whose norm $\|v\|_1 = \sum |v_{x,y}|$ is finite. This duality is obtained by extending the domain of definition of d ($\in \mathfrak{D}$) from $\Sigma\times\Sigma$ to $\ell^1(\Sigma\times\Sigma)$ via the rule :

$$d\left(\sum v_{x,y} \cdot (x,y)\right) = \sum v_{x,y} \cdot d(x,y) .$$

The operator φ_ξ^* is thus the dual of a linear operator φ_ξ on $\ell^1(\Sigma\times\Sigma)$ defined by the action on basis elements :

$$\varphi_\xi(x,y) = \begin{cases} (\sigma x,\sigma y) & \text{if } x_1 = y_1 \\ (\sigma x,\xi) + (\xi,\sigma y) & \text{if } x_1 \neq y_1 \end{cases} .$$

Note that φ_ξ commutes with the coordinate flip which is generated linearly by

$$(x,y) \mapsto (y,x) .$$

A vector will be called *symmetric* if it is invariant under the flip .

Eigenvectors of φ_ξ^* :

We now show that pseudo-metrics on Σ_ξ satisfying $(M1)$ and $(M2)$ are equivalent to nonnegative eigenvectors of φ_ξ^* corresponding to eigenvalue $\lambda > 1$.

THEOREM 4.1.1 : *If $\bar{d} : \Sigma_\xi\times\Sigma_\xi \to \mathbb{R}$ satisfies $(M1)$ and $(M2)$ then the lift $d := \bar{d}\circ(\pi_\xi\times\pi_\xi)$ defined on $\Sigma\times\Sigma$ is an eigenvector of φ_ξ^* corresponding to eigenvalue λ . Conversely if d is a nonnegative eigenvector of φ_ξ^* corresponding to $\lambda > 1$ then d is a continuous function on $\Sigma\times\Sigma$ which factors through $\pi_\xi\times\pi_\xi$ to give a pseudo-metric on Σ_ξ satisfying $(M1)$ and $(M2)$.*

Proof . If $\bar{d}$ is a function on $\Sigma_\xi\times\Sigma_\xi$ which satisfies $(M1)$ and $(M2)$ then writing d as its lift to a function on $\Sigma\times\Sigma$, we have $d(\sigma x,\sigma y) = \lambda \cdot d(x,y)$ whenever $x_1 = y_1$.

If $x_1 \neq y_1$ then

$$d(\sigma x,\xi) + d(\xi,\sigma y) = \lambda \cdot d(x,x_1\xi) + \lambda \cdot d(y_1\xi,y)$$
$$= \lambda \cdot d(x,y)$$

since the "critical point" $\overline{x_1\xi} = \overline{y_1\xi}$ lies on the Σ_ξ-arc from $\bar{x}$ to $\bar{y}$.

Hence d is an eigenvector of $\varphi_\xi{}^*$ corresponding to eigenvalue λ.

Conversely, given λ with $|\lambda| > 1$, consider the action of the re-scaled operator $\varphi_\xi{}^*/\lambda$ on functionals $d \in \mathfrak{D}$. We are particular interested, in the case $\lambda > 1$, on the action of the re-scaled operator on nonnegative $d \in \mathfrak{D}$ and we see that the amount by which such d fail to be a pseudo-metric shrinks to zero under iteration :

(1) For any $x \in \Sigma$ we have

$$\left((\varphi_\xi{}^*)^n(d)\right)(x,x) = d\left((\varphi_\xi)^n(x,x)\right) = d(\sigma^n x,\sigma^n x)$$

so

$$\left|\left((\varphi_\xi{}^*/\lambda)^n(d)\right)(x,x)\right| \leq \|d\|/|\lambda|^n \qquad \text{for all } n.$$

Thus if d is a fixed point of $\varphi_\xi{}^*/\lambda$ then $d(x,x) = 0$ for all $x \in \Sigma$.

(2) Now suppose that $\bar{x} = \bar{y}$ but $x \neq y$, specifically that $x_i = y_i$ for i less than some k but that $x_k \neq y_k$ and $\sigma^k x = \sigma^k y = \xi$. Then $\varphi_\xi{}^k(x,y) = \varphi_\xi{}^i(x_k\xi,y_k\xi) = 2 \cdot (\xi,\xi)$. Thus applying the functional d we obtain

$$\left|\left((\varphi_\xi{}^*/\lambda)^n(d)\right)(x,y)\right| \leq 2 \cdot \|d\|/|\lambda|^n \qquad \text{for all } n$$

and so $d(x,y) = 0$ if d is again a fixed point of $\varphi_\xi{}^*/\lambda$.

(3) In order to establish the "convergence to triangle inequality" for iterates of nonnegative $d \in \mathfrak{D}$ we introduce *triangle vectors* $\Delta(x,y,z) = (x,y) + (y,z) - (x,z)$. Direct computation yields the following

LEMMA 4.1.2 :

$$\varphi_\xi\left(\Delta(x,y,z)\right) = \Delta(\sigma x,\sigma y,\sigma z) \qquad \textit{if } x_1 = y_1 = z_1,$$
$$\varphi_\xi\left(\Delta(x,y,z)\right) = \Delta(\sigma x,\sigma y,\xi) \qquad \textit{if } x_1 = y_1 \neq z_1,$$
$$\varphi_\xi\left(\Delta(x,y,z)\right) = \Delta(\xi,\sigma y,\sigma z) \qquad \textit{if } x_1 \neq y_1 = z_1,$$
$$\varphi_\xi\left(\Delta(x,y,z)\right) = \Delta(\sigma x,\xi,\sigma z) + (\sigma y,\xi) + (\xi,\sigma y) \qquad \textit{if } x_1 = z_1 \neq y_1$$

and

$$\varphi_\xi\left(\Delta(x,y,z)\right) = (\sigma y,\xi) + (\xi,\sigma y) \qquad \textit{if } x_1, y_1 \textit{ and } z_1$$

are distinct (not possible in the quadratic case). □

It follows that $\varphi_\xi{}^n\left(\Delta(x,y,z)\right)$ equals $\Delta(x',y',z') + s$ (or possibly just s if $|\lambda| > 2$) for some $x',y',z' \in \Sigma$ and nonnegative symmetric $s \in \ell^1(\Sigma \times \Sigma)$. Applying a nonnegative functional $d \in \mathfrak{D}$ we obtain

$$d\left(\varphi_\xi{}^n\left(\Delta(x,y,z)\right)\right) \geq -d(x',z') \qquad (\text{ or } \geq 0),$$

so in any case

$$\left((\varphi_\xi{}^*/\lambda)^n(d)\right)\left(\Delta(x,y,z)\right) \geq -\|d\|/\lambda^n \qquad \text{for all } n.$$

Hence if d is a nonnegative fixed point of $\varphi_\xi{}^*/\lambda$ then

$$d(x,y) + d(y,z) - d(x,z) = d\big(\Delta(x,y,z)\big) \geq 0$$

and d satisfies the triangle inequality.

(4) To establish that equality occurs whenever $\bar{y}$ lies on the $\Sigma\xi$-arc from $\bar{x}$ to $\bar{z}$ we define $A = \big\{(x,y,z) \in \Sigma^3 : \ y \text{ lies on the } \Gamma\xi\text{-arc from } x \text{ to } z\big\}$. We show

$$(x,y,z) \in A \quad \Rightarrow \quad \varphi_\xi\big(\Delta(x,y,z)\big) = \Delta(x',y',z') \text{ for some } (x',y',z') \in A.$$

Given $(x,y,z) \in A$, if $x_1 = y_1 = z_1$ then $(\sigma x,\sigma y,\sigma z) \in A$ (by Proposition 2.5.2). If $x_1 = y_1 \neq z_1$ then $(x,y,x_1\xi) \in A$ and so $(\sigma x,\sigma y,\xi) \in A$. Similarly if $x_1 \neq y_1 = z_1$ then $(z_1\xi,y,z) \in A$ and so $(\xi,\sigma y,\sigma z) \in A$. The implication therefore follows by Lemma 4.1.2 as the other two cases cannot occur since $\Gamma\xi\text{-arc}(x,z) \subseteq \{x_1\} \cup \{z_1\}$. Iterating this and applying any $d \in \mathfrak{D}$ gives

$$\big|\big((\varphi_\xi{}^*/\lambda)^n(d)\big)\big(\Delta(x,y,z)\big)\big| \leq 3\cdot\|d\|/|\lambda|^n \qquad \text{for all } n.$$

Therefore $d(x,y) + d(y,z) - d(x,z) = d\big(\Delta(x,y,z)\big) = 0$ whenever $(x,y,z) \in A$ and d is a fixed point of $\varphi_\xi{}^*/\lambda$.

(5) In order to prove that a fixed functional d (of $\varphi_\xi{}^*/\lambda$) is *symmetric*, in other words that $d(x,y) = d(y,x)$ for all $x,y \in \Sigma$, we observe that the action of φ_ξ on basis elements can be rewritten :

$$\varphi_\xi(x,y) = (\sigma x,\sigma y) + t$$

where t is 0 or $\Delta(\sigma x,\xi,\sigma y)$ according as $x_1 = y_1$ or not. It follows by Lemma 4.1.2 that :

$$\varphi_\xi{}^n(x,y) = (\sigma^n x,\sigma^n y) + t + s$$

where t is a sum of at most n triangle vectors and s is a symmetric vector. Hence :

$$\varphi_\xi{}^n\big((x,y) - (y,x)\big) = (\sigma^n x,\sigma^n y) - (\sigma^n y,\sigma^n x) + t - t'$$

where t' is the image of t under the coordinate flip. Applying a functional $d \in \mathfrak{D}$ we obtain :

$$\big|\big((\varphi_\xi{}^*/\lambda)^n(d)\big)\big((x,y) - (y,x)\big)\big| \leq 2\cdot\|d\|\cdot\big(1 + 3n\big)/|\lambda|^n.$$

If d is a fixed point of $\varphi_\xi{}^*/\lambda$, letting n tend to infinity gives $d\big((x,y) - (y,x)\big) = 0$.

(6) Now suppose that d is a fixed point of $\varphi_\xi{}^*/\lambda$ where $|\lambda| > 1$. We show that d is a continuous function on $\Sigma\times\Sigma$. Suppose basis vectors $(x,y), (x',y') \in \Sigma\times\Sigma$ satisfy $x_k = x'_k$ and $y_k = y'_k$ for all k less than some $n \in \mathbb{N}$. It is an easy induction to verify that for each $k \leq n$ either

$$\varphi_\xi{}^k\big((x',y') - (x,y)\big) = (\sigma^k x',\sigma^k y') - (\sigma^k x,\sigma^k y)$$

$\big($in the case $x_l = y_l$ for all $l \leq k\big)$ or there exist $l, j < k$ such that

$$\varphi_\xi{}^k\big((x',y') - (x,y)\big) = (\sigma^k x',\sigma^l\xi) - (\sigma^k x,\sigma^l\xi) + (\sigma^l\xi,\sigma^k y') - (\sigma^l\xi,\sigma^k y).$$

It follows that

$$
\begin{aligned}
|d(x',y') - d(x,y)| &= \left|\left((\varphi_\xi^*/\lambda)^n(d)\right)\left((x',y') - (x,y)\right)\right| \\
&= \left|d\left(\varphi_\xi^n\left((x',y') - (x,y)\right)\right)\right| / \lambda^n \\
&= 4 \cdot \|d\| / \lambda^n .
\end{aligned}
$$

In the case ξ is not periodic it follows from (1), (2) and (4) that d factors through $\pi_\xi \times \pi_\xi$. To deal with the case ξ is periodic, consider the subset $d^{-1}\{0\}$ of $\Sigma \times \Sigma$. Assuming that d is nonnegative we have (3) and so this gives via (1), (2), (5) and (6) that $d^{-1}\{0\}$ is a closed equivalence relation containing all pairs $\left(x_1 \ldots x_{k-1}\alpha\,\xi, x_1 \ldots x_{k-1}\beta\,\xi\right)$, $k \geq 1$, $\alpha \neq \beta$. The minimality of $\sim_{\Gamma_\xi}$ then insures that $d(x,y) = 0$ whenever $x \sim_{\Gamma_\xi} y$. It follows by (3) that d factors through $\pi_\xi \times \pi_\xi$.

The induced function $\bar{d} : \Sigma \times \Sigma \to \mathbb{R}$ is continuous and, by (4), satisfies $(M1)$ and $(M2)$. In the case d is nonnegative, properties (3) and (5) say that $\bar{d}$ is a pseudo-metric. □

PROPOSITION 4.1.3 : *If* d *is a nonnegative eigenvector of* φ_ξ^* *corresponding to* $\lambda > 1$ *then the pseudo-metric* $\bar{d}$ *on* Σ_ξ *is actually a metric if* $d(\xi,\sigma^n\xi) > 0$ *for all* $n \geq 1$. *Futhermore the topology generated by* $\bar{d}$ *is the quotient topology on* Σ_ξ.

Proof. Given $x,y \in \Sigma$ with $\bar{x} \neq \bar{y}$. Suppose $x_i = y_i$ for $i < k$ but $x_k \neq y_k$. Since not both $\sigma^k x$ and $\sigma^k y$ can be equal to ξ we may assume for some $n \geq 1$ that $x_{k+n} \neq \xi_n$ but that $x_{k+i} = \xi_i$ for $i < n$. So $\varphi_\xi^k(x,y) = (\sigma^k x,\xi) + (\xi,\sigma^k y)$ and

$$\varphi_\xi^n(\sigma^k x,\xi) = (\sigma^{k+n}x,\xi) + (\xi,\sigma^n\xi) .$$

Thus applying d we obtain

$$
\begin{aligned}
&\lambda^{k+n} \cdot d(x,y) = \\
&\left((\varphi_\xi^*)^{k+n}(d)\right)(x,y) = d\left((\sigma^{k+n}x,\xi) + (\xi,\sigma^n\xi) + \varphi_\xi^n(\xi,\sigma^k\xi)\right) \\
&\qquad \geq d(\xi,\sigma^n\xi) .
\end{aligned}
$$

We therefore have $d(x,y) > 0$ whenever $\bar{x} \neq \bar{y}$.

Let U be any open subset of Σ containing the equivalence class of x. Since d is continuous and $\Sigma \setminus U$ is compact the function $y \mapsto d(x,y)$ defined on $\Sigma \setminus U$ must attain its infimum r and so $r > 0$. Consequently $x \in \left\{y \in \Sigma : d(x,y) < r\right\} \subseteq U$.

This, together with the continuity of d, proves that the metric topology on Σ_ξ agrees with the quotient topology. □

For the eigenvectors d of φ_ξ^* that we will obtain it will not always be the case that $d(\xi,\sigma^n\xi) > 0$ for all $n \geq 1$. However, we do have the following result.

PROPOSITION 4.1.4 : *If* $d \neq 0$ *is a nonnegative eigenvector of* φ_ξ^* *corresponding to eigenvalue* $\lambda > 1$ *then* $d(\xi,\sigma\xi) > 0$ *and*

$$\|d\| = 2 \cdot d(\xi,\sigma\xi)/\big(\lambda(\lambda-1)\big) .$$

Proof. Consider an element (ξ,x) of $\Sigma\times\Sigma$. We have

$$\varphi_\xi(\xi,x) = (\sigma\xi,\xi) + (\xi,\sigma x) - t$$

where t equals $\Delta(\sigma\xi,\xi,\sigma x)$ or 0 according as $\xi_1 = x_1$ or not. Thus, by induction,

$$(4.1) \qquad \varphi_\xi^k(\xi,x) = \Big[\varphi_\xi^{k-1}(\sigma\xi,\xi) + \ldots + \varphi_\xi(\sigma^k\xi,\xi) + (\sigma\xi,\xi)\Big] + (\xi,\sigma^k x) - t - s$$

where t is a sum of at most k triangle vectors and s is a nonnegative (symmetric) vector. Notice that $s = t = 0$ if $\xi_1 \neq x_i$ for all $i \leq k$.

Applying a nonnegative fixed point d of φ_ξ^*/λ (and dividing by λ^k) we obtain

$$d(\xi,x) = d\big(\varphi_\xi^k(\xi,x)\big)/\lambda^k \leq \big[1/\lambda + \ldots + 1/\lambda^{k-1} + 1/\lambda^k\big] \cdot d(\sigma\xi,\xi) + d(\xi,\sigma^k x)/\lambda^k .$$

Letting k tend to infinity and noting that $d(\xi,\sigma^k x) \leq \|d\| < \infty$ gives that

$$d(\xi,x) \leq d(\sigma\xi,\xi)/(\lambda-1)$$

and equality occurs if $\xi_i \neq x_i$ for all i. Now, given an arbitrary element (x,y) where $x \neq y$ there exists $k \geq 1$ with $x_i = y_i$ for $i < k$ but $x_k \neq y_k$. Thus we have

$$\begin{aligned} d(x,y) = d\big(\varphi_\xi^k(x,y)\big)/\lambda^k &= \big(d(\sigma^k x,\xi) + d(\xi,\sigma^k y)\big)/\lambda^k \\ &\leq 2 \cdot d(\sigma\xi,\xi)/\big(\lambda^k(\lambda-1)\big) \\ &\leq 2 \cdot d(\sigma\xi,\xi)/\big(\lambda(\lambda-1)\big) \end{aligned}$$

and equality occurs whenever $x_1 \neq y_1$ and $x_i \neq \xi_1 \neq y_i$ for all $i > 1$. □

The invariant subspace $V := \ell^1\big\{(\sigma^m\xi,\sigma^n\xi) : m,n \geq 0,\ m \neq n\big\}$.

We now see that the important part of the dynamics of φ_ξ is contained in V and that an eigenvector (in V^*) of $\big(\varphi_\xi|_V\big)^*$ corresponding to eigenvalue λ with $|\lambda| > 1$ extends uniquely to an eigenvector of φ_ξ^*.

Recallthat

$$\varphi_\xi(x,y) = \begin{cases} (\sigma x,\sigma y) & \text{if } x_1 = y_1 \\ (\sigma x,\xi) + (\xi,\sigma y) & \text{if } x_1 \neq y_1 \end{cases} .$$

Hence it follows by induction that for $n \geq 1$:

$$\varphi_\xi^n(x,y) = \begin{cases} (\sigma^n x,\sigma^n y) & \text{if } x_1 \ldots x_n = y_1 \ldots y_n , \\ (\sigma^n x,\sigma^k\xi) + v + (\sigma^\ell\xi,\sigma^n y) & \text{otherwise} , \end{cases}$$

where $v \in V$ and $k, \ell < n$.

Thus, modulo V, there is no growth in the norm of ℓ^1 vectors beyond one doubling.

Now suppose that f in V^* (in other words , a bounded function on $\{(\sigma^m\xi,\sigma^n\xi) : m,n \geq 0 , m \neq n\}$) is an eigenvector of $(\varphi_\xi|_V)^*$ corresponding to eigenvalue λ and that $|\lambda| > 1$. Let d_0 be any bounded extension of f to the whole of $\Sigma\times\Sigma$ and put $d_n := (\varphi_\xi^*/\lambda)^n d_0$ for $n \geq 1$. First observe that $d_n|_V = f$ for all $n \geq 0$, for if $v \in V$ then

$$d_n(v) = \left((\varphi_\xi^*/\lambda)^n d_0\right)(v) = d_0\left(\varphi_\xi^n(v)\right)/\lambda^n = f\left(\varphi_\xi^n(v)\right)/\lambda^n = \left((\varphi_\xi|_V)^{*n} f\right)(v)/\lambda^n$$
$$= f(v) .$$

Now suppose that d and d' , in $\ell^\infty(\Sigma\times\Sigma)$, satisfy $d|_V = d'|_V$. We have

$$\left((\varphi_\xi^*/\lambda)^n d' - (\varphi_\xi^*/\lambda)^n d\right)(x,y) = (d'-d)\left(\varphi_\xi^n(x,y)\right)/\lambda^n$$
$$= \begin{cases} (d'-d)(\sigma^n x,\sigma^n y)/\lambda^n & \text{if } x_1\ldots x_n = y_1\ldots y_n \\ (d'-d)\left((\sigma^n x,\sigma^k\xi) + (\sigma^\ell\xi,\sigma^n y)\right)/\lambda^n & \text{, with } k,\ell < n \text{, otherwise .} \end{cases}$$

Thus

$$\|(\varphi_\xi^*/\lambda)^n d' - (\varphi_\xi^*/\lambda)^n d\| \leq 2\cdot\|d'-d\|/|\lambda|^n \quad \left(\to 0 \text{ as } n\to\infty\right).$$

In particular

$$\|d_{n+1} - d_n\| \leq 2\cdot\|d_1 - d_0\|/|\lambda|^n ,$$

and so $\{d_n \mid n \geq 0\}$ forms a Cauchy sequence. The limit d_∞ satisfies $d_\infty|_V = f$, is independent of the choice of d_0 , and is the unique eigenvector of φ_ξ^* whose restriction to V is f . The nonnegativity of φ_ξ guarantees that if f is nonnegative then so is d_∞ .

NOTE : If f gives the correct "distances" between forward-iterates of ξ , in other words $(\varphi_\xi|_V)^*(f) = \lambda\cdot f$, then the sequence $(d_n)_{n\geq 0}$ gives the correct "distance" between any two (given) backward-iterates of ξ in finite time :

$$d_n(x,y) = d_\infty(x,y) \quad \text{whenever} \quad x, y \in \sigma^{-n}\{\sigma^m\xi : m \geq 0\} .$$

We can now see the existence of the required metric in certain cases . For the polynomial $f_i : z \mapsto z^2 + i$, the critical iterates are : $0 \to i \to -1+i \leftrightarrow -i$. Since 0 is preperiodic it follows , by Corollary 1.5.3 , that for some argument η the external ray $R(f_i,\eta)$ converges on i , and that $K(f_i)$ is topologically conjugate to Σ_ξ where ξ is the associated kneading itinerary . Now $\sigma\xi$ is periodic of strict period 2 and $\xi_1 \neq \xi_3$. Hence (by suitable assignment of the symbol set) $\xi = 0\,(0\,1)^\infty$. (The only solutions η to $\xi(\eta) = 0\,(0\,1)^\infty$ (or $1\,(1\,0)^\infty$) are $\eta = \pm 1/6$ (mod $\mathbb{Z}$) . In fact $\eta = 1/6$). The action of φ_ξ on basis elements of V is :

$$\varphi_\xi\left(0\,(0\,1)^\infty, (0\,1)^\infty\right) = \left((0\,1)^\infty, (1\,0)^\infty\right);$$
$$\varphi_\xi\left((0\,1)^\infty, (1\,0)^\infty\right) = \left((1\,0)^\infty, 0\,(0\,1)^\infty\right) + \left(0\,(0\,1)^\infty, (0\,1)^\infty\right);$$
$$\varphi_\xi\left((1\,0)^\infty, 0\,(0\,1)^\infty\right) = \left((0\,1)^\infty, 0\,(0\,1)^\infty\right) + \left(0\,(0\,1)^\infty, (0\,1)^\infty\right)$$

with the rest of the action being determined by the symmetry of φ_ξ. Hence, with respect to the basis $\left\{ \left(0(01)^\infty, (01)^\infty\right), \left((01)^\infty, (10)^\infty\right), \left((10)^\infty, 0(01)^\infty\right), \left((01)^\infty, 0(01)^\infty\right), \left((10)^\infty, (01)^\infty\right), \left(0(01)^\infty, (10)^\infty\right) \right\}$, $\varphi_\xi|_V$ has the matrix:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

As far as eigenvalues of absolute value greater than one (and their associated eigenvectors) are concerned, we need only consider the action of φ_ξ on the unordered pairs $\left\{ \{ 0(01)^\infty, (01)^\infty \}, \{ (01)^\infty, (10)^\infty \}, \{ (10)^\infty, 0(01)^\infty \} \right\}$ with the corresponding matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$$

The characteristic equation $\lambda^3 - \lambda - 2 = 0$ has one real root $\lambda = 1 \cdot 52 \ldots$ which, by the standard Perron-Frobenius theorem $\left(\text{Seneta } 1.1\right)$ (observing that the matrix is irreducible and aperiodic), must be strictly greater than the absolute value of any other eigenvalue, and furthermore must correspond to unique row and column eigenvectors, each having positive entries. The column eigenvector

$$\begin{pmatrix} d\left(0(01)^\infty, (01)^\infty\right) \\ d\left((01)^\infty, (10)^\infty\right) \\ d\left((10)^\infty, 0(01)^\infty\right) \end{pmatrix}$$

extends uniquely to a nonnegative eigenvector d of φ_ξ^*. By Theorem 4.1.1 and Proposition 4.1.3, d factors through $\pi_\xi \times \pi_\xi$ to give the required metric on $\Sigma_{0(01)^\infty}$ $\left(\text{and hence on } J(f_1)\right)$.

Observe that the fixed point $\pi_\xi(0^\infty)$ of $\Sigma_{0(01)^\infty}$ is tri-valent and that the points $\xi, \sigma\xi$ and $\sigma^2\xi$ lie in distinct components of $\Sigma_{0(01)^\infty} \setminus \pi_\xi(0^\infty)$. To construct the metric in this case it would seem more natural to first compute the distances $d(0^\infty,\xi)$, $d(0^\infty,\sigma\xi)$ and $d(0^\infty,\sigma^2\xi)$ via the equations :

$$\lambda \cdot d(0^\infty,\xi) = d(0^\infty,\sigma\xi),$$
$$\lambda \cdot d(0^\infty,\sigma\xi) = d(0^\infty,\sigma^2\xi),$$
$$\lambda \cdot d(0^\infty,\sigma^2\xi) = 2 \cdot d(0^\infty,\xi) + d(0^\infty,\sigma\xi)$$

(observing that the Γ_ξ-arc from 0^∞ to $\sigma^2\xi$ contains the "fold point" $\sigma^{-1}(\xi)$). However to generalise this approach requires a knowledge of multi-valent points in Σ_ξ. For general ξ this seems a hard problem.

In order to make some headway for general ξ it is advisable to understand the relationship between the dynamics of φ_ξ and the $*$-factorization of ξ.

§4.2 ACTION OF φ_ξ FOR DECOMPOSABLE ξ :

Suppose that $\xi = \beta * \psi$ where β is block of length ℓ. We will examine the action of φ_ξ restricted to the invariant subspace $\Lambda = \Lambda_\beta$ generated by the basis elements $\left(\sigma^m(\beta*x),\sigma^n(\beta*y)\right)$ where $x, y \in \Sigma$ and $0 \le m, n \le \ell - 1$.
Notice that Λ includes all pairs of forward iterates of ξ and therefore the essential part of the dynamics of φ_ξ.

We first restrict to the action of φ_ξ on the "quasi-diagonal" subspace $Q = Q_\beta$ generated by the elements $\left(\sigma^n(\beta*x),\sigma^n(\beta*y)\right)$ $(x, y \in \Sigma)$.

If $1 \le n < \ell$ then $\varphi_\xi\left(\sigma^{n-1}(\beta*x),\sigma^{n-1}(\beta*y)\right) = \left(\sigma^n(\beta*x),\sigma^n(\beta*y)\right)$

and
$$\varphi_\xi\left(\sigma^{\ell-1}(\beta*x),\sigma^{\ell-1}(\beta*y)\right) = \begin{cases} (\beta*\sigma x, \beta*\sigma y) & \text{if } x_1 = y_1 \\ (\beta*\sigma x, \beta*\psi) + (\beta*\psi, \beta*\sigma y) & \text{if } x_1 \ne y_1 \end{cases}$$

We can thus see that this restricted action of φ_ξ is intimately related to the action of φ_ψ via :

$$\varphi_{\beta*\psi}^{\ell}\left(\beta * (x,y)\right) = \beta * \varphi_\psi(x,y)$$

(extending the action of β to (linear combinations of) pairs (x,y) via
$$\beta * (x,y) = (\beta * x, \beta * y) \quad).$$

In particular, the spectral radius of $\varphi_{\beta*\psi}|_Q$ is the ℓth root of the spectral radius of φ_ψ.

We now return to the action of $\Phi_{\beta\circ\psi}$ on "off-diagonal" pairs and we see that, modulo Q, this action is determined essentially only by β and is independent of ψ.

If $1 \le m,n < \ell$ then

$$\Phi_\zeta\left(\sigma^{m-1}(\beta\circ x),\sigma^{n-1}(\beta\circ y)\right) = \begin{cases} \left(\sigma^m(\beta\circ x),\sigma^n(\beta\circ y)\right) & \text{if } \beta_m = \beta_n \\ \left(\sigma^m(\beta\circ x),\ \beta\circ\psi\right) + \left(\beta\circ\psi,\sigma^n(\beta\circ y)\right) & \text{if } \beta_m \ne \beta_n . \end{cases}$$

If $1 \le m < \ell$ and $n = \ell$ then

$$\Phi_\zeta\left(\sigma^{m-1}(\beta\circ x),\sigma^{\ell-1}(\beta\circ y)\right) = \begin{cases} \left(\sigma^m(\beta\circ x),\ \beta\circ\sigma y\right) & \text{if } \beta_m = y_1 \\ \left(\sigma^m(\beta\circ x),\ \beta\circ\psi\right) + \left(\beta\circ\psi,\ \beta\circ\sigma y\right) & \text{if } \beta_m \ne y_1 . \end{cases}$$

There is a similar expression if $m = \ell$ and $1 \le n < \ell$.

Taking π to be the (essentially well-defined) map

$$\sigma^m(\beta\circ x) \mapsto m ,$$

we see that the above action, modulo Q, factors through $\pi\times\pi$ to give a finite-dimensional operator Φ_β acting on the set (of linear combinations of) $\left\{(m,n):\ 0 \le m,n < \ell \text{ with } m \ne n\right\}$. Thus if $1 \le m,n < \ell$ then

$$\Phi_\beta(m-1,\ n-1) = \begin{cases} (m,\ n) & \text{if } \beta_m = \beta_n \\ (m,\ 0) + (0,\ n) & \text{if } \beta_m \ne \beta_n \end{cases} ,$$

$$\Phi_\beta(m-1,\ \ell-1) = (m,\ 0) ,$$

and

$$\Phi_\beta(\ell-1,\ n-1) = (0,\ n) .$$

PROPOSITION 4.2.1 : *Let β be a block of length $\ell > 1$. If β is atomic then $\rho(\Phi_\beta) = 1$. If β is an island block then $\rho(\Phi_\beta) \ge 2^{1/(\ell-1)}$.*

NOTE : Here ρ denotes spectral radius (see (Dunford & Schwartz) page 567).

Proof. Observe that after one iterate of Φ_β the vector (m,n) either "splits" :-

$$\Phi_\beta(m,n) = (m+1,0) + (0,n+1) ,$$

or the quantity $m - n$ is preserved modulo ℓ.

If β is an atomic block then the finite graph corresponding to the action of Φ_β on the basis elements $(m,\ n)$ consists of disjoint cycles of length ℓ. These cycles are the (nonzero) congruence classes of $m - n$ (mod ℓ). As there are no splits the spectral radius of Φ_β is 1.

If β is an island block then we observe that every vector $(0,n)$ must split within $\ell - 1$ iterates. Suppose conversely that $\Phi_\beta^{\ell-1}(0,n) = (\ell-1,n-1)$. Then ξ must be periodic with period n as far as $\ell-1$ and since $\Phi_\beta^{n-1}(\ell-n,0) = \Phi_\beta^{n-1}\left(\Phi_\beta^{\ell-n}(0,n)\right) = (\ell-1,n-1)$ then ξ is also periodic with period $\ell-n$ as far as $\ell-1$. It would follow by Corollary 3.1.1.1 "common periodic" that ξ is periodic as far as $\ell-1$ with period $h := \mathrm{HCF}(\ell,n)$. This would mean that

$$\beta = \left[\beta_1 \ldots \beta_{h-1}\_\right] * \left[\beta_h^{\ell/h-1}\_\right]$$

contradicting the assumption that β is an island factor. Similarly, every vector $(m,0)$ must split within $\ell-1$ iterates. Therefore, for each n, we have that

$$\|\Phi_\beta^{(\ell-1)k}(0,n)\|_1 \geq 2^k \quad \text{for all } k \geq 0,$$

and hence that

$$\rho(\Phi_\beta) \geq 2^{1/(\ell-1)}. \qquad \square$$

COROLLARY 4.2.1.1 : *If* $\xi = \beta * \psi$ *where* β *is of length* ℓ *then* $\rho(\varphi_\xi)$ *equals* $\left(\rho(\varphi_\psi)\right)^{1/\ell}$ *or* $\rho(\Phi_\beta)$ *according as* β *is a continent block or not.*

Proof. From the fact $\rho\left(\varphi_{\beta*\psi}|_\Lambda\right) = \max\left\{\rho(\Phi_\beta), \rho\left(\varphi_{\beta*\psi}|_Q\right)\right\}$ we can deduce

$$\rho(\varphi_{\beta*\psi}) = \max\left\{\rho(\Phi_\beta), \left(\rho(\varphi_\psi)\right)^{1/\ell}\right\}. \tag{4.2}$$

It follows by induction that if $\beta = \beta^1 * \beta^2 * \ldots * \beta^k$ where β^i are (nontrivial) atomic factors then

$$\rho(\varphi_\xi) = \left(\rho(\varphi_\psi)\right)^{1/\ell}.$$

If, on the other hand, $\beta = \alpha * \gamma$ where α is an island factor of length $k > 1$ then

$$\rho(\varphi_\xi) = \max\left\{\rho(\Phi_\alpha), \rho(\varphi_{\gamma*\psi})^{1/k}\right\}$$
$$= \rho(\Phi_\alpha)$$

since $\rho(\Phi_\alpha) \geq 2^{1/(k-1)} > 2^{1/k}$. The result follows since $\rho(\varphi_\xi) \geq \rho(\Phi_\beta) \geq \rho(\Phi_\alpha)$. $\square$

COROLLARY 4.2.1.2 : *If* β *is a continent block of length* $\ell > 1$ *then* $\rho(\Phi_\beta) = 1$.

Proof. Substituting ψ to have period one in the previous Corollary gives

$$\rho(\Phi_\beta) \leq \rho(\varphi_{\beta*\psi}) = \left(\rho(\varphi_\psi)\right)^{1/\ell} = 1. \qquad \square$$

We re-state some definitions from Chapter 3.

DEFINITION 4.2.1 : We say that a sequence ξ is decomposable of :

type (A) if ξ is an infinite $*$-product of atomic blocks ;

type (B) if ξ can be written as $\beta * \psi$ where β is a nontrivial island block (or equivalently if there is such a decomposition where β is not a continent factor).

If ξ is of type (A) then, by Corollary 4.2.1.1, we have $\rho(\varphi_\xi) \le 2^{1/\ell}$ for ℓ arbitrarily large, and so $\rho(\varphi_\xi) = 1$. Since $|\lambda| \le \rho(\varphi_\xi)$ for any eigenvalue λ there is no chance of a metric or pseudo-metric on Σ_ξ satisfying $(M1)$ and $(M2)$.

If $\xi = \beta * \psi$ where β is an island factor of length $\ell > 1$ then $\rho(\Phi_\beta) > \left(\rho(\varphi_\psi)^{1/\ell}\right)$. Any eigenvector d of $\varphi_\xi{}^*$ corresponding to an eigenvalue λ, where $|\lambda| = \rho = \rho(\varphi_\xi)$, must take value zero on Q, since if $(x,y) \in Q$:

$$\left|\left((\varphi_\xi{}^*/\lambda)^n d\right)(x,y)\right| = \left|d\left(\varphi_\xi{}^n(x,y)\right)/\lambda^n\right| \le \|d\| \, \|\left(\varphi_\xi|_Q\right)^n(x,y)\|/\rho^n \to 0 \quad \text{as} \quad n \to \infty .$$

We will see later that if $\lambda > 1$ is the eigenvalue of a nonnegative eigenvector of $\varphi_\xi{}^*$ then $\lambda = \rho$. Consequently, if ξ is of type (B) (and not periodic) then there can be no metric on Σ_ξ satisfying properties $(M1)$ and $(M2)$. However one can categorise those subsets of Σ_ξ which are "collapsed to points" by any pseudo-metric eigenvector, using the following analogue of Proposition 4.1.3.

PROPOSITION 4.2.2: *If* d *is a nonnegative eigenvector of* $\varphi_\xi{}^*$ *corresponding to* $\lambda > 1$ *and satisfying* $d(\xi, \sigma^k\xi) > 0$ *whenever* k *is not a multiple of the length* ℓ *of some fixed* $*$*-factor* β *of* ξ *then*

$$d(x,y) = 0 \iff x = y \quad \text{or} \quad \left(x, y \in x_1 \ldots x_n \, \sigma^m(\beta * \Sigma) \text{ for some } m,n \ge 0\right).$$

Proof. Suppose $x \ne y$. Let k be minimal such that $x_k \ne y_k$. If both $\sigma^k x$ and $\sigma^k y$ lie in $\beta * \Sigma$ then $x, y \in x_1 \ldots x_{k-1} \, \sigma^{\ell-1}(\beta * \Sigma)$. Otherwise we may assume $\sigma^k x \notin \beta * \Sigma$ so that there exists some (least) n, not divisible by ℓ, with $x_{k+n} \ne \beta_{n-\ell[n/\ell]}$.

Now $\varphi_\xi{}^k(x,y) = (\sigma^k x, \xi) + (\xi, \sigma^k y)$ and, for some sequence (r_i) of integers with $0 = r_0 < r_1 < \ldots < r_s \le [n/\ell]$, we have

$$\varphi_\xi{}^{(r_{i+1}-r_i)\ell}\left(\sigma^{k+r_i\ell} x, \xi\right) = \left(\sigma^{k+r_{i+1}\ell} x, \xi\right) + \left(\xi, \sigma^{(r_{i+1}-r_i)\ell} \xi\right) \qquad (i < s)$$

and

$$\varphi_\xi{}^{n-r_s\ell}\left(\sigma^{k+r_s\ell} x, \xi\right) = \left(\sigma^{k+n} x, \xi\right) + \left(\xi, \sigma^{n-r_s\ell} \xi\right) .$$

It follows that $d(x,y) = \left((\varphi_\xi{}^*/\lambda)^{k+n}(d)\right)(x,y) \ge d\left(\xi, \sigma^{n-r_s\ell}\xi\right) / \lambda^{k+n} > 0$. □

§4.3 ACTION OF φ_ξ ON FORWARD ORBIT OF ξ:

Consider the linear map Φ_ξ on $\ell^1\left(\mathbb{Z}_+ \times \mathbb{Z}_+\right)$ generated by the action:

$$\Phi_\xi(m-1, n-1) = \begin{cases} (m, n) & \text{if } \xi_m = \xi_n \\ (m, 0) + (0, n) & \text{if } \xi_m \ne \xi_n \end{cases} .$$

The substitution $(m,n) \mapsto (\sigma^m\xi, \sigma^n\xi)$ is a map from $\mathbb{Z}_+ \times \mathbb{Z}_+$ into $\Sigma \times \Sigma$ which is one-to-one unless ξ is eventually periodic. Thus Φ_ξ represents the action of φ_ξ restricted to V. It will be convenient to study the operator Φ_ξ rather than φ_ξ. The dual operator Φ_ξ^* acts on the space $\ell^\infty(\mathbb{Z}_+ \times \mathbb{Z}_+)$. It is a technical formality to check, in the case ξ satisfies $\sigma^{\ell+p}\xi = \sigma^\ell\xi$ (with $\ell \geq 0$ and $p \geq 1$ minimal), that any eigenvector F of Φ_ξ^*, corresponding to eigenvalue λ with $|\lambda| > 1$, satisfies $F(m+p,n) = F(m,n)$ $\forall\, m \geq \ell,\ n \geq 0$ and $F(m,n+p) = F(m,n)$ $\forall\, n \geq \ell,\ m \geq 0$ and so gives a well-defined eigenvector of $(\varphi_\xi|_V)^*$.

NOTATION : We will often refer to the partial order on $\ell^1(\mathbb{Z}_+ \times \mathbb{Z}_+)$ given by $u \geq v$ if $u - v$ is nonnegative (all coefficients nonnegative).

In what follows, we simultaneously consider three interpretations of the operator Φ_ξ :

(i) Its action on $\ell^1(\mathbb{Z}_+ \times \mathbb{Z}_+)$ and in particular on the "basis" vectors (m,n) ;

(ii) The countable 0,1-matrix $T = (t_{ij})$ associated to the basis $\{(m,n) : m,n \in \mathbb{Z}_+\}$;

(iii) The directed graph on $\{(m,n) : m,n \in \mathbb{Z}_+\}$ with an edge running from vertex (m,n) to vertex (m',n') whenever $\Phi_\xi(m,n) \geq (m',n')$.

The statement $\Phi_\xi^k(m,n) \geq (m',n')$ is then equivalent to the statement $t_{ij}^{(k)} > 0$ (where $i = (m,n)$ and $j = (m',n')$) or to saying that there is path of length k from (m,n) to (m',n'). We say that (m,n) *leads to* (m',n') if there exists $k \geq 1$ such that $\Phi_\xi^k(m,n) \geq (m',n')$.

It will often be convenient to deal with the "symmetrized" action of Φ_ξ on (linear combinations of) unordered pairs $\{m,n\}$ ($m,n \geq 0$, $m \neq n$). One can think of the new basis $\{m,n\}$ either as the pair $\{(m,n),(n,m)\}$ or as the vector $\frac{1}{2}(m,n) + \frac{1}{2}(n,m)$.

The Thurston-Milnor simplification in the case of ξ real-admissible :

If ξ is such that the future iterates of ξ all lie on the Γ_ξ-arc from ξ to $\sigma\xi$ (which occurs, in the quadratic case, when ξ is the kneading itinerary corresponding to a real polynomial) then identifying to zero all triangle vectors $\Delta(x,y,z)$ where y lies on the Γ_ξ-arc from x to z gives a simplification of Φ_ξ since $(n,1)$ becomes $(0,1) - (0,n)$. The symmetrized action of Φ_ξ thus becomes :

$$\{0,n-1\} \mapsto \begin{cases} \{0,1\} - \{0,n\} & \text{if } \xi_1 = \xi_n \\ \{0,1\} + \{0,n\} & \text{if } \xi_1 \neq \xi_n \end{cases}.$$

The dynamics has thus been considerably simplified from a two-dimensional array of basis vectors to the one-dimensional array $\{\{0,n\} : n \in \mathbb{N}\}$. Furthermore, the spectral radius of Φ_ξ can be determined as the reciprocal of the positive first zero (or the radius of convergence, whichever is smaller) of $1 - F(z)$ where $F(z)$ is the 'first-return' power series of the index $\{0,1\}$:

$$F(z) = z + \varepsilon_2 \cdot z^2 + \varepsilon_2\varepsilon_3 \cdot z^3 + \varepsilon_2\varepsilon_3\varepsilon_4 \cdot z^4 + \dots, \text{ where } \varepsilon_n = +1 \text{ if } \xi_1 \neq \xi_n$$
$$\text{and } \varepsilon_n = -1 \text{ if } \xi_1 = \xi_n.$$

In the quadratic case, the power series $1-F(z)$ is none other than the Thurston-Milnor kneading invariant [MT] (or see §2.6) for a (real) unimodal map whose kneading itinerary is ξ.

It does not seem that such a simplification can be performed for general ξ or even ξ corresponding to polynomials z^2+c for c nonreal.

Axis-return maps :

The first step to understanding the dynamics of Φ_ξ is to observe that whenever a vector (m,n) grows in size (under iteration) it must do so by passing through the graph's "axes"

$$\{(m,0) \mid m\geq 1\} \cup \{(0,n) \mid n\geq 1\}.$$

Of crucial importance therefore is the "axis-return" equation :- for $n \geq 1$,

$$\Phi_\xi^{\mu(n)}(0,n) = (\mu(n),0) + (0,\nu(n)) \tag{4.3}$$

where μ and ν are maps from $\mathbb{N}$ to $\mathbb{N}\cup\{+\infty\}$ defined by:

$$\mu(n) = \min\{ j \geq 1 \ : \ \xi_j \neq \xi_{n+j} \}$$

and

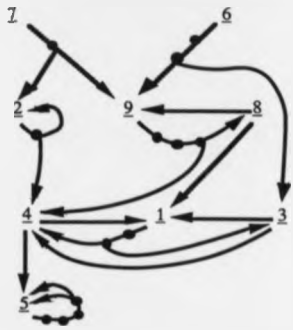
$$\nu(n) = \min\{ j \geq n+1 \ : \ \xi_{j-n} \neq \xi_j \} = n + \mu(n).$$

(In the case $\xi = \sigma^n\xi$, the Φ_ξ-iterates of $(0,n)$ never return to the axes and we write $\nu(n) = \mu(n) = +\infty$). Observe that ν is none other than the nonperiodicity function for the sequence ξ introduced in Chapter 2.

The following example shows that the dynamics of Φ_ξ can get quite complicated.

EXAMPLE : Let ξ be the sequence $\xi = 0\,0\,0\,1\,1(0\,0\,0\,1\,0)^\infty$.

The graph below shows the symmetrized action of Φ_ξ on the vertices $\{m,n\}$, $0 \leq m < n < 10$ (taking the identifications $(m+5,n) = (m,n)$ $\forall\ m \geq 5$ and $(m,n+5) = (m,n)$ $\forall\ n \geq 5$). We have emphasised the axis vertices $\underline{n} = \{0,n\}$. (Vertices not reachable from an axis vertex are omitted). The layout is that suggested in (Seneta 1.2) to clarify the cyclic structure of the graph.



Observe here that the graph is very far from irreducible , with many self-communicating classes . However there are only two classes whose growth-rate is strictly greater than one , and one of these - the class of $\underline{5}$ - corresponds to the "quasi-diagonal" subspace Q associated to the decomposition of ξ as $\left[0\,0\,0\,1\,\_\right] * 1\,(0)^{\infty}$, and consequently has the smaller growth-rate $\left(\text{Proposition } 4.2.1\right)$.

We shall see that for general ξ , not of type (A) , there will be one self-communicating class whose growth-rate exceeds the growth-rate of all other such classes , and that the only other classes of growth-rate greater than 1 must be contained in the "quasi-diagonal" subspace associated to some (nontrivial) island factor of ξ .

"Oscillatory" numbers :

We are concerned with successive applications of the maps μ and ν . First let us concentrate on the "lower orbit" $\left\{\mu^k(n)\right\}$ of a number $n \in \mathbb{N}$.

DEFINITION 4.3.1 : A natural number n is said to be *oscillatory* if $\mu\left(\mu(n)\right) = n$. It follows that

$$\mu^k(n) = \begin{cases} \nu(n) - n & \text{for } k \text{ odd} \\ n & \text{for } k \text{ even} \end{cases} ,$$

and for all k :

$$\nu\left(\mu^k(n)\right) = \nu(n) .$$

We now see that n is oscillatory if at least n divides $\nu(n)$.

LEMMA 4.3.1: *Let* m *be any positive multiple of* $h\ (\geq 1)$ *which is less than* $\nu(h)$. *Then* $\nu(m) = \nu(h)$.

Proof. For all j with $m < j < \nu(h)$ we have

$$\xi_j = \xi_{j-h} = \cdots = \xi_{j-m}.$$

But

$$\xi_{\nu(h)} \neq \xi_{\nu(h)-h} = \cdots = \xi_{\nu(h)-m}. \qquad \square$$

COROLLARY 4.3.1 : *If* $h := \mathrm{HCF}(n, \nu(n))$ *satisfies* $\nu(h) > n$ *then* $\nu(h) = \nu(n) = \nu(\mu(n))$ *and consequently* $\mu(\mu(n)) = n$.

Proof. Substituting $m = n$ and then $m = \mu(n)$ into the Lemma gives the result. $\square$

NOTE : The condition $\nu(h) > n$ is necessary and sufficient for $\mu(\mu(n)) = n$. If $\nu(h) \leq n$ then $\mu(\mu(n)) < n$. (This must be true for if $\nu(\mu(n)) \geq n + \mu(n) = \nu(n)$ then by Corollary 3.1.1.1 "common periodic" $\nu(h) \geq \nu(n) > n$).

Factorization and nonperiodicities :

We now turn to the "upper orbit" and its relationship to the $*$-factorization of ξ.

LEMMA 4.3.2 : *The sequence* ξ *has a left* $*$*-factor of length* ℓ *if and only if* $\ell \mid \nu^k(\ell)$ *for all* k.

Proof. Factorizability of ξ tells us that $\xi_n = \xi_{n+\ell}$ whenever $\ell \nmid n$. So if $\ell \mid m$ then whenever $\ell \nmid j \geq 1$ we must have $\xi_j = \xi_{m+j}$. Thus the nonperiodicity of m must occur at a multiple of ℓ. Since $\ell \mid \ell$ it follows by induction that $\ell \mid \nu^k(\ell)$ for all $k \geq 0$.

Conversely suppose that ξ is factorizable of period ℓ as far as m and that $\ell \mid m$ then ξ must also be period ℓ factorizable as far as $\nu(m)-1$. This follows since whenever $m < j < \nu(m)$ we have $\xi_j = \xi_{j-m}$ and so there is a single value of ξ_j to each of the sets

$$\{ j : 1 \leq j < \nu(m) \text{ with } j = n \pmod{\ell} \}$$

except for the set $n = 0 \pmod{\ell}$. If, further, $\ell \mid \nu(m)$ then ξ is *de facto* factorizable (of length ℓ) as far as $\nu(m)$. The assumption that $\ell \mid \nu^k(\ell)$ for all $k \geq 0$ then yields inductively that ξ is factorizable of length ℓ as far as $\nu^{k+1}(\ell)-1$ for all k and so in its entirety. $\square$

We now consider the *principal* nonperiodicities $\nu_k := \nu^k(1)$, $k \geq 0$.

PROPOSITION 4.3.3 : *A sequence* ξ *is of type* (A) *if and only if* $\nu^k(1) \mid \nu^{k+1}(1)$ *for all* $k \geq 1$.

Proof. If ξ is decomposable of type (A) then we can write ξ as an infinite product

$$\xi = [\xi_1^{\ell_1 - 1}\_] * [\xi_{\ell_1}^{\ell_2 - 1}\_] * [\xi_{\ell_1\ell_2}^{\ell_3 - 1}\_] * \ldots .$$

We can assume that $\xi_1 \neq \xi_{\ell_1} \neq \xi_{\ell_1\ell_2} \neq \ldots$ since two successive atomic blocks of the same symbol can be coalesced into one . Clearly then $\nu^k(1) = \ell_1\ell_2 \cdot \ldots \cdot \ell_k$ for all $k \geq 1$.

Conversely if $\nu_k \mid \nu_{k+1}$ for all k then by Lemma 4.3.2 , ξ has a left factor β_k of length ν_k for each k . It is easy to verify that $\beta_k = \beta_{k-1} * [\xi_{\nu_{k-1}}^{\ell_k - 1}\_]$ where $\ell_k := \nu_k / \nu_{k-1}$ and so we obtain the above expression for ξ . □

§4.4 SEQUENCES NOT FACTORIZABLE OF TYPE (A) :

We have already seen that if ξ is of type (A) then the operator Φ_ξ has spectral radius $\rho = 1$. In order to see that $\rho > 1$ otherwise , we find a vertex a $\big(= (m,n) \big)$ such that

$$\Phi_\xi^{\,p}\, a \geq 2 \cdot a$$

for some $p \geq 1$. From this it would follow that

$$\Phi_\xi^{\,kp}\, a \geq 2^k \cdot a$$

for all k . Hence

$$\rho(\Phi_\xi) = \rho(\Phi_\xi) = \lim_{r\to\infty} \sup_{m,n\in\mathbb{N}} \|\Phi_\xi^{\,r}(m,n)\|_1^{\,1/r} \geq \limsup_{r\to\infty} \|\Phi_\xi^{\,r}\, a\|_1^{\,1/r}$$
$$\geq 2^{1/p} .$$

The "apex" of the graph :

It follows from Proposition 4.3.3 that if ξ fails to be of type (A) then there exists t such that $\nu^t(1) \nmid \nu^{t+1}(1)$. We take it that $\nu^i(1) \mid \nu^{i+1}(1)$ for all $i \leq t-1$. Thus we call $\nu_t = \nu^t(1)$ the *first nondividing principal nonperiodicity* of ξ . The vertex $(0,\nu_t)$ will be called the *apex* .

Observe that all natural numbers n ($\leq \nu_{t+1}$) have a unique "ν-ary" representation:

$$n = c_t \cdot \nu_t - \sum_{0\leq i<t} c_i \cdot \nu_i \ .$$

where $c_t \in \mathbb{N}$ and $0 \leq c_i < \nu_{i+1}/\nu_i$ for $i = 0, 1, \ldots, t-1$. The reason for doing this is that the value ξ_n (for $n < \nu_{t+1}$) can now be read off as ξ_{ν_i} where i is least such that $c_i > 0$. Note

$$\xi_1 \neq \xi_{\nu_1} \neq \xi_{\nu_2} \neq \ldots \neq \xi_{\nu_t} .$$

We record the ν-ary expansion of ν_{t+1} :

$$\nu_{t+1} = (b_t+1)\nu_t - \sum_{0\le i<t} b_i \nu_i \ .$$

Here $b_t \ge 1$ since $\nu_{t+1} \ge \nu_t + 1$. Furthermore there exists $i < t$ such that $b_i > 0$. We take r to be the least such i and s ($\le t$) to be the next least such i . If $s = r+1$ then $\xi_{\nu_s} \ne \xi_{\nu_r}$.

PROPOSITION 4.4.1 : *The number* ν_t *satisfies* $\mu\big(\mu(\nu_t)\big) < \nu_t$ *and the vertex* $(0,\nu_t)$ *satisfies*

$$\Phi_\xi^{\nu_{t+1}+\mu\mu(\nu_t)}(0,\nu_t) \ \ge \ (0,\nu_t) + (\nu_t, 0) \ .$$

COROLLARY 4.4.1 : $\rho(\Phi_\xi) \ge 2^{1/(\nu_{t+1} + \mu\mu(\nu_t))}$.

Proof . We examine the Φ_ξ-iterates of the vector $(0,\nu_t)$ using the axis-return equation $\big(4.3\big)$. We obtain

$$\Phi_\xi^{\mu(\nu_t)}(0,\nu_t) \ = \ \big(\mu(\nu_t), 0\big) + (0, \nu_{t+1}) \ .$$

Since we have no information on ξ beyond ν_{t+1} we concentrate on the iterates of $\big(\mu(\nu_t),0\big)$. Now

$$\mu(\nu_t) = \nu_{t+1} - \nu_t = b_r\nu_t - \sum_{0\le i<t} b_i\nu_i \ .$$

We see how far $\sigma^{\mu(\nu_t)}\xi$ agrees with ξ . The first place where discrepancy might occur is at $n = \mu(\nu_t) + b_r\nu_r$. Here $\xi_n = \xi_{\nu_s}$ whereas $\xi_{b_r\nu_r} = \xi_{\nu_r}$. In the less usual case that $\xi_{\nu_s} = \xi_{\nu_r}$ (where we must have had that $s \ge r+2$) the first discrepancy certainly does occur at $n = \mu(\nu_t) + \nu_{r+1}$. Here $\xi_n = \xi_{\nu_r} \ne \xi_{\nu_{r+1}}$.

Thus there are two cases :

CASE I $\xi_{\nu_s} \ne \xi_{\nu_r}$, $\mu\big(\mu(\nu_t)\big) = b_r\nu_r$, $\nu\big(\mu(\nu_t)\big) = b_r\nu_t - \sum_{s\le i<t} b_i\nu_i$;

CASE II $\xi_{\nu_s} = \xi_{\nu_r}$, $\mu\big(\mu(\nu_t)\big) = \nu_{r+1}$, $\nu\big(\mu(\nu_t)\big) = b_r\nu_t - \sum_{s\le i<t} b_i\nu_i + \nu_{r+1} - b_r\nu_r$.

In either case $\nu\big(\mu(\nu_t)\big) < \nu_{t+1}$ and $\mu\big(\mu(\nu_t)\big) < \nu_t$ and we can see that the "upper orbit" (i.e. nonperiodicities) of $\mu\big(\mu(\nu_t)\big)$ makes its way back to ν_t :

Just before we examine the effect of subsequent μ , ν applications on $\mu\big(\mu(\nu_t)\big)$ we consider their effect on more general $c_i\cdot\nu_i$ (where $0 \le i < t$ and $0 < c_i < \nu_{i+1}/\nu_i$). By Corollary 4.3.1 and the "axis-return" equation $\big(4.3\big)$, we have

$$\begin{aligned}\Phi_\xi^{\nu_{i+1}-c_i\nu_i}(0, c_i\nu_i) \ &= \ (\nu_{i+1}-c_i\nu_i ,0) + (0,\nu_{i+1}) \\ &\ge \ (0,\nu_{i+1}) \ .\end{aligned}$$

It follows therefore , by a succession of axis-returns , that

$$\Phi_\xi^{\nu_t - c_i\nu_i}(0, c_i\nu_i) \ \ge \ (0,\nu_t) \ .$$

Substituting $c_1 v_1 = \mu(\mu(v_t))$ gives us a first-return to the vertex $(0, v_t)$ in a total time of

$$\mu(v_t) + \mu(\mu(v_t)) + (v_t - \mu(\mu(v_t))) = v_{t+1} \quad \text{iterates}.$$

There is a slightly later 'arrival' at the vertex $(v_t, 0)$ given by

$$\Phi_\xi^{v_t}(0, \mu(\mu(v_t))) \geq \Phi_\xi^{v_t - \mu\mu\mu(v_t)}(\mu\mu\mu(v_t), 0)$$
$$\geq (v_t, 0).$$

We now verify that simultaneous with this arrival is another return of the vertex $(0, v_t)$ created via the iterates of $\nu(\mu(v_t))$. We compute $\mu\nu\mu(v_t)$:

In CASE II this is easy since the first discrepancy between $\sigma^{\nu\mu(v_t)}\xi$ and ξ must occur at $n = \nu\mu(v_t) + b_r v_r$ $(< v_{t+1})$. Here $\xi_n = \xi_{v_{r+1}} \neq \xi_{v_r} = \xi_{b_r v_r}$. Thus

CASE II $\qquad \mu\nu\mu(v_t) = b_r v_r, \quad \nu\nu\mu(v_t) = b_r v_t - \sum_{s \leq i < t} b_i v_i + v_{r+1};$

In CASE I there are several possibilities. If $s < t$ put $s' = \min\{i > s : b_i > 0\}$.

CASE I(i) $\quad s < t$ and $\xi_{v_s} \neq \xi_{v_{s'}}$, $\mu\nu\mu(v_t) = b_s v_s$, $\nu\nu\mu(v_t) = b_r v_t - \sum_{s' \leq i < t} b_i v_i$;

CASE I(ii) $\quad s < t-1$ and $\xi_{v_s} = \xi_{v_{s'}}$, $\mu\nu\mu(v_t) = v_{s+1}$,

$$\nu\nu\mu(v_t) = b_r v_t - \sum_{s \leq i < t} b_i v_i + v_{s+1};$$

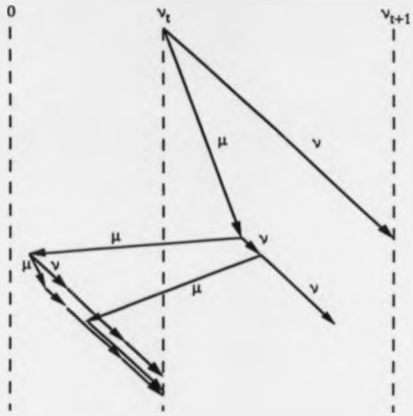
CASE I(iii) $\quad s = t \ (\Leftrightarrow v_t \mid \nu\mu(v_t))$, $\mu\nu\mu(v_t) = v_t - b_r v_r$, $\nu\nu\mu(v_t) = v_{t+1}$.

In all cases $\mu\nu\mu(v_t)$ is less than v_t and its upper orbit leads back to v_t. In all cases except the last $\nu\nu\mu(v_t) < v_{t+1}$. We obtain

$$\Phi_\xi^{v_t}(\nu\mu(v_t), 0) \geq \Phi_\xi^{v_t - \mu\nu\mu(v_t)}(0, \mu\nu\mu(v_t))$$
$$\geq (0, v_t).$$

Hence

$$\Phi_\xi^{v_{t+1} + \mu\mu(v_t)}(0, v_t) \geq \Phi_\xi^{v_t + \mu\mu(v_t)}(\mu(v_t), 0)$$
$$= \Phi_\xi^{v_t}((\nu\mu(v_t), 0) + (0, \mu\mu(v_t)))$$
$$\geq (0, v_t) + (v_t, 0). \qquad \square$$



Polynomial "first entrance" to the set $\{(\nu_t\ ,0),(0,\nu_t\ )\}$:

To obtain more accurate estimates (lower bounds) for $\rho(\Phi_\xi)$ it is convenient to deal with the "symmetrized" action of Φ_ξ . The apex a is then the vertex $\{0,\nu_t\}$ or the set $\{(\nu_t,0),(0,\nu_t)\}$. We shall often write n for the "axis" vertex $\{0,n\}$.

We appeal to renewal type arguments and consider , in particular , the number $f_{ij}^{(k)}$ of paths of length k from a vertex i (= $\{m,n\}$) which end with a "first-entrance" to the vertex j (= $\{m',n'\}$) . The "first entrance" power series $F_{ij}(z)$ are then defined by :

$$F_{ij}(z) := \sum_{k\geq 1} f_{ij}^{(k)}.\ z^k\ .$$

In particular , we have the "first return" power series $F_{a\,a}(z)$ of the vertex a , from which the "cumulative return" power series $T_{a\,a}(z) = \sum_{k\geq 0} t_{a\,a}^{(k)}.\ z^k$ can be obtained via the equation

$$T_{a\,a}(z) = \left(1 - F_{a\,a}(z)\right)^{-1}\ .$$

Here $t_{a\,a}^{(k)}$ is the coefficient of a in $\Phi_\xi^k(a)$. The growth-rate $g_a = \limsup_k \left((t_{a\,a}^{(k)})^{1/k}\right) = \sup_k \left((t_{a\,a}^{(k)})^{1/k}\right)$ is then the reciprocal of the radius of convergence of $T_{a\,a}(z)$ which is either the radius of convergence of $F_{a\,a}(z)$ or the smallest positive solution z of the equation $F_{a\,a}(z) = 1$ whichever is smaller . It is clear that $\rho(\Phi_\xi)$ is greater than or equal to g_a . We will see later that in fact $\rho(\Phi_\xi) = g_a$.

We define the "restricted first-entrance" power series ${}_hF_{ij}(z) = \sum_{k\geq 1} {}_hf_{ij}^{(k)}$ for $i \neq h \neq j$, where ${}_hf_{ij}^{(k)}$ denotes the number of paths of length k from i which end with a first entrance to the vertex j and which are disjoint from h.

LEMMA 4.4.2 : *If $i \neq h \neq j$ are vertices then*

$$F_{ij}(z) = {}_hF_{ij}(z) + {}_jF_{ih}(z) \cdot F_{hj}(z).$$

Proof. Any path with of length k starting from i and ending with a first-entrance to j is either disjoint from h or there is a unique ℓ (with $0 < \ell < k$) such that the initial subpath of length ℓ ends with a first-entrance to the vertex h (and is disjoint from j), the remainder of the path ending with a first-entrance to j. (Compare Vere-Jones [VJ3]). □

COROLLARY 4.4.2.1 :

$$F_{\underline{n}\,\underline{\nu(n)}}(z) = z^{\mu(n)} \cdot \left(1 + F_{\underline{\mu(n)}\,\underline{\nu(n)}}(z)\right).$$

Proof. Substitute $i = \underline{n}$, $h = \underline{\mu(n)}$ and $j = \underline{\nu(n)}$ and use the "axis-return" equation (4.3). □

COROLLARY 4.4.2.2 : *If $\mu(\mu(n)) = n$ then*

$$F_{\underline{n}\,\underline{\nu(n)}}(z) = \left(z^{\mu(n)} + z^{\nu(n)}\right)/\left(1 - z^{\nu(n)}\right).$$ □

COROLLARY 4.4.2.3 : *If n is of the form $c_i\nu_i$ (with $i < t$ and $1 \leq c_i < \nu_{i+1}/\nu_i$) then* $F_{\underline{n}\,\underline{a}}(z) =$

$$z^{\nu_t - n} \cdot \left((1 + z^{\nu_i})/(1 - z^{\nu_{i+1}})\right) \cdot \left((1 + z^{\nu_{i+1}})/(1 - z^{\nu_{i+2}})\right) \cdots \left((1 + z^{\nu_t - 1})/(1 - z^{\nu_t})\right).$$

Proof. If n is of the above form then $\nu(n) = \nu_{i+1}$ which is also of the same form unless $i+1 = t$. The result then follows by induction on i using the Lemma and Corollary 4.4.2.2 and the fact

$$z^{\mu(n)} + z^{\nu(n)} = z^{\nu(n) - n} \cdot (1 + z^n).$$ □

Now $\mu\mu(\nu_t)$ is of the above form $c_i\nu_i$. Also $\mu\nu\mu(\nu_t)$ is of the above form (except in CASE I(iii) where $\nu\mu\nu\mu(\nu_t) = \nu_t$ and $\mu\mu\nu\mu(\nu_t)$ is of the above form). Hence one can obtain a much better estimate of $\rho(\Phi_{\underline{a}})$ using the fact that $F_{\underline{a}\,\underline{a}}(z)$ is coefficientwise greater than or equal to

$$z^{\mu(\nu_t)} + \mu\mu(\nu_t) \cdot \left(F_{\underline{\mu\mu(\nu_t)}\,\underline{a}}(z) + z^{\mu\nu\mu(\nu_t)} \cdot F_{\underline{\mu\nu\mu(\nu_t)}\,\underline{a}}(z)\right)$$

for instance. From this, with some calculation, one can deduce

$$\beta_{\underline{a}} > 3^{1/\nu_{t+1}}.$$

Type (B) factorizability :

LEMMA 4.4.3 : *If* $\xi = \beta \circ \psi$ *where* β *is an island factor of length* $\ell > 1$ *then*

$$\nu_{t+1} < 2 \cdot \ell \ .$$

Proof. If $\nu_{t+1} \le \ell$ then we are done. Otherwise let $i\ (\le t)$ be the largest such that $\nu_i \mid \ell$. Clearly $i \ne t$ since otherwise $\nu_t \mid \ell \mid \nu(\ell) = \nu(\nu_t)$ by Lemmas 4.3.1 and 4.3.2 thereby contradicting the definition of t. Were it true that $\ell \le \nu_{i+1}$ then we would have

$$\left[\xi_1 \cdots \xi_{\ell-1}\ \_\right] = \left[\xi_1 \cdots \xi_{\nu_i - 1}\ \_\right] \circ \left[\xi_{\nu_i}{}^{\ell/\nu_i - 1}\ \_\right]$$

and so $\left[\xi_1 \cdots \xi_{\ell-1}\ \_\right]$ would be a $\circ$-product of atomic factors. Thus we must have $\nu_{i+1} < \ell$. $\nu_{i+1} \nmid \ell$ and we know ξ is periodic with period ℓ as far as $2\ell - 1$. We know also that ξ is factorizable with period ν_{i+1} as far as $\nu_{i+1} - 1$. If it were true that $\nu_{i+1} - 1 \ge 2\ell - 1$ then by Theorem 3.1.1 "common factorizable-periodic" we would have that

$$\left[\xi_1 \cdots \xi_{\ell-1}\ \_\right] = \left[\xi_1 \cdots \xi_{h-1}\ \_\right] \circ \left[\xi_h{}^{\ell/h - 1}\ \_\right]$$

where $h = \mathrm{HCF}(\nu_{i+1}, \ell) < \ell$ contradicting the assumption that $\left[\xi_1 \cdots \xi_{\ell-1}\ \_\right]$ is an island factor. Thus $\nu_{i+1} - 1 > 2\ell - 1$. □

It follows therefore that ν_t is not a multiple of ℓ and so vectors (m,n) with $m \equiv n \pmod{\ell}$ $\left(\text{contained in the invariant subspace } Q \text{ discussed before}\right)$ can never lead to the vertex $(0,\nu_t)$.

We now prove a converse to this observation regarding the structure of the graph.

THEOREM 4.4.4 : *If the set* $S := \left\{ n \in \mathbb{N} : \Phi_\xi^j(0,n) \ge (0,\nu_t) \text{ for all } j \ge 1 \right\}$ *is nonempty then it consists of multiples of some least element* ℓ. *The sequence* ξ *is island-factorizable of length* ℓ.

Proof. Let ℓ be the least element of the set S, which we assume nonempty, and we claim that $\ell \mid \nu^i(\ell)$ for all $i \ge 1$ and hence, by Lemma 4.3.2 that ξ is factorizable of length ℓ. Suppose, on the contrary, that there exists $k \ge 0$ such that $\ell \mid \nu^i(\ell)$ for all $i \le k$ but that $\ell \nmid \nu^{k+1}(\ell)$. So we have that ξ is factorizable of length ℓ as far as $\nu^{k+1}(\ell) - 1 \ge \ell + \mu\left(\nu^k(\ell)\right) - 1$.

Now consider $\mu\mu\left(\nu^k(\ell)\right)$. Were this to be less than ℓ then, by the minimality of ℓ, for some $j \ge 1$ we would have

$$\Phi_\xi^j\left(0, \mu\mu(\nu^k(\ell))\right) \ge (0,\nu_t) \ .$$

But this would imply

$$\Phi_\xi^{j + \mu\mu\left(\nu^k(\ell)\right) + \mu\left(\nu^k(\ell)\right) + \nu^k(\ell) - \ell}(0,\ell) \ \ge\ (0,\nu_t)$$

contradicting the fact that $\ell \in S$.

Thus $\mu\mu\left(\nu^k(\ell)\right) \geq \ell$ and so ξ is periodic with period $\mu\left(\nu^k(\ell)\right)$ as far as $\ell+\mu\left(\nu^k(\ell)\right)-1$.

Hence, applying Theorem 3.1.1 "common factorizable-periodic", we have that ξ is periodic with period $h := \mathrm{HCF}\left(\ell, \mu(\nu^k(\ell))\right)$ as far as $\ell-1$.

Since $h \mid \ell$ and $\nu(h) \geq \ell$ we have, by Lemma 4.3.1, that $\nu(h)$ equals either ℓ or $\nu(\ell)$ and is hence a multiple of h $\left(\text{even if } k = 0\right)$. It follows that $\mu\mu(h) = h$ and so neither h nor $\mu(h)$ can be ν_t $\left(\text{which satisfies } \mu\mu(\nu_t) < \nu_t\right)$. Since $\nu\left(\mu(h)\right) = \nu(h)$ lies in the ν-orbit of ℓ, the only "axis" vertices of the Φ_ξ-orbit of $(0,h)$ not reached by the Φ_ξ-orbit of $(0,\ell)$ $\left(\text{or } (\ell,0)\right)$ are possibly the vertices $(0,h)$ and $\left(\mu(h),0\right)$ neither of which is the vertex $(0,\nu_t)$. Thus $h \in S$ and so the hypothesis $h < \ell$ contradicts the minimality of ℓ.

Thus we have established that ξ is factorizable of length ℓ. To see that $\left[\xi_1 \ldots \xi_{\ell-1}\_\right]$ is an island factor we repeat the argument of the last paragraph with h defined by the hypothesis that

$$\left[\xi_1 \ldots \xi_{\ell-1}\_\right] = \left[\xi_1 \ldots \xi_{h-1}\_\right] * \left[\xi_h^{\ell/h-1}\_\right]$$

and obtain a contradiction if $h < \ell$.

We now check that S consists precisely of the multiples of ℓ:
If $\ell \mid n$ then clearly $\ell \mid \nu(n)$ and $\ell \mid \mu(n)$ so the Φ_ξ-iterates of $(0,n)$ never hit $(0,\nu_t)$, and so $n \in S$. On the other hand, if $n \in S$ then the vertex $\left(\mu(n),0\right)$ cannot lead to $(0,\nu_t)$ and so nor to $(\nu_t, 0)$. The symmetry of Φ_ξ implies that $\mu(n) \in S$ and so $\mu(n) \geq \ell$. Thus ξ is periodic with period n as far as $n+\ell-1$ and so Theorem 3.1.1 "common factorizable-periodic" tells us that ξ is periodic with period h as far as $\ell-1$. Since $\left[\xi_1 \ldots \xi_{\ell-1}\_\right]$ is an island block we have that $h = \ell$ and therefore n must have been a multiple of ℓ. □

§4.5 QUASI-COMPACTNESS OF Φ_ξ:

In this section we always assume that ξ is not of type (A) and hence that $\rho(\Phi_\xi) > 1$.
We shall establish the important "quasi-compact" property of Φ_ξ — namely that Φ_ξ can be written as the sum of two bounded operators: one of which has spectral radius strictly less than that of Φ_ξ and the other which is compact. More specifically, we show that the nonnegative matrix associated to Φ_ξ is "quasi-finite" in the sense of §5.5.

PROPOSITION 4.5.1: *Given* $N \in \mathbb{N}$, *let* $H := \left\{(n,0),(0,n) : 1 \leq n < N\right\}$. *Let* ${}_H\Phi_\xi$ *denote the operator obtained from* Φ_ξ *by eradicating the edges incident to the vertices* H *from the associated graph. Then for all* $m,n \in \mathbb{N}$ *we have* $\|({}_H\Phi_\xi)^k(m,n)\|_1 \leq 2^{1+\left\lceil (k-1)/N\right\rceil}$. *Hence*

$$\rho({}_H\Phi_\xi) \leq 2^{1/N}.$$

Proof. We write the action of ${}_H\Phi_\xi$:

$$\left({}_H\Phi_\xi\right)(m-1,n-1) = \begin{cases} (m,n) & \text{if } \xi_m = \xi_n \\ (m,0)+(0,n) & \text{if } \xi_m \neq \xi_n \text{ and } N \le m,n \\ (m,0) & \text{if } \xi_m \neq \xi_n \text{ and } n < N \le m \\ (0,n) & \text{if } \xi_m \neq \xi_n \text{ and } m < N \le n \\ 0 & \text{if } \xi_m \neq \xi_n \text{ and } m,n < N \end{cases}$$

A basis vector $(0,n)$ $\left(\text{and similarly } (m,0)\right)$ can only split into a sum of two basis vectors after N iterates of ${}_H\Phi_\xi$ since, by induction on $k\ (< N)$, we have that $\left({}_H\Phi_\xi\right)^k(0,n)$ equals either $(k',n+k)$, where $0 \le k' \le k < N$, or 0. It follows that $\left\|\left({}_H\Phi_\xi\right)^k(0,n)\right\|_1 \le 2^{\lceil k/N \rceil}$ and the result drops out easily. □

Choosing N large enough such that $2^{1/N} < \rho(\Phi_\xi)$ and noting that $\Phi_\xi - {}_H\Phi_\xi$ has finite-dimensional range $\left(\text{contained in } \ell^1(H)\right)$ and is hence compact, it follows that Φ_ξ is quasi-compact.

The spectrally essential index $a = (0,\nu_t)$:

We now establish the property that the spectral radius of Φ_ξ is strictly diminished by deleting the incident edges of merely <u>one</u> vertex, namely, the apex $a = (0,\nu_t)$. The resulting operator we write as ${}_a\Phi_\xi$. It is possible to verify that $\rho({}_a\Phi_\xi) < \rho(\Phi_\xi)$ directly in particular cases of the sequence ξ.

For instance if ξ is contained in $\{0\ 1\ 1\ 1\}$ then $\nu_t = \nu_1 = 2$ and one can verify that the vertex-return growth-rate g_a of the vertex a is at least as great as the largest solution of the equation $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ and hence greater than 1.8 whereas the spectral radius of ${}_a\Phi_\xi$ is most $(1+\sqrt{5})/2 = 1.618\ldots$.

It is considerably easier to directly verify the spectrally-essential property in particular cases if we deal with the "symmetrized" operator, where (m,n) is identified with (n,m) and the apex a is now $\{(0,\nu_t),(\nu_t,0)\}$. Here if ξ lies in $\{0\ 1\ 1\}$, for example, then g_a is bounded below by $(1+\sqrt{5})/2$ whereas $\rho\left({}_a\Phi_\xi\right)$ is easily seen to be at most $2^{1/3}$. Even so, the property becomes increasingly difficult to verify as ξ approaches type (A). For instance, given the initial segment for ξ :

$$\overbrace{0100010101000100010001000100} = [0\_]\bullet[1\_]\bullet[0\_]\bullet 100$$

where $\nu_t = \nu_4 = 16$, the lower bound on g_a is $\left((1+\sqrt{5})/2\right)^{1/8}$ $\left(\text{which is attained when}\right.$

$$\xi = [0100010\_]\bullet[10\_]\bullet\psi$$

$\left.\text{for any sequence } \psi \text{ with } \psi_1 = 0\right)$.

However, if

$$\xi = 0\,1\,0\,0\,0\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,0\,0\,1\,0\,0\,\left(0\,1\,0\,0\,1\right)^{\infty}$$

then the ${}_{a}\Phi_{\xi}$ -orbit of the index $\underline{s} = \{(5,0),(0,5)\}$ has first-return power series :

$$F_{\underline{s}\,\underline{s}}(z) = z^{20}/\left(1 - z^{5}\right)$$

and the spectral radius of ${}_{a}\Phi_{\xi}$ is at least the reciprocal of smallest positive solution of

$$F_{\underline{s}\,\underline{s}}(z) = 1$$

and this exceeds $\left((1+\sqrt{5})/2\right)^{1/8}$. It is needless to say that for this value of ξ the full growth-rate g_a will also exceed $\left((1+\sqrt{5})/2\right)^{1/8}$, but the point is made that information on ξ beyond ν_{t+1} is required for a direct proof that the index a is spectrally essential.

We therefore resort to a less constructive proof and show that the hypotheses of Theorem 5.5.2 are satisfied - in other words that, after choosing N large enough so that $2^{1/N} < \rho(\Phi_{\xi})$, we have that for all $n < N$, either $(0,n)$ communicates with $a = (0,\nu_t)$ or the vertex-return growth-rate of $(0,n)$ is smaller than that of a.

LEMMA 4.5.2 : *If the vertex-return growth-rate of the vertex* $(0,n)$ *is strictly greater than one then* a *leads to* $(0,n)$.

Proof. First suppose n is of the form $c_i\nu_i$ with $i < t$ and $1 \le c_i < \nu_{i+1}/\nu_i$. Suppose that for some k we have $\Phi_{\xi}^{k}(0,n) \ge 2\cdot(0,n)$, in other words that there are at least two paths of length k from $(0,n)$ to $(0,n)$. At most one of the paths can be disjoint from $\left\{(\nu_t,0),(0,\nu_t)\right\}$:- namely the "lower orbit" path whose intersection with the axes oscillates between $(0,n)$ and $(\mu(n),0)$. Any other path, after passing through one of the vertices $(0,\nu_{i+1})$ and $(\nu_{i+1},0)$ must remain in the set $\left\{(m',n') : \; m' = n' \pmod{\nu_{i+1}}\right\}$ until it enters $\left\{(\nu_t,0),(0,\nu_t)\right\}$. It follows that one of $(\nu_t,0)$ and $(0,\nu_t)$ $\left(\text{and hence both}\right)$ must lead to $(0,n)$.

Now suppose n is not of the form $c_i\nu_i$. Rewriting equation $\left(4.1\right)$ when $x = \sigma^{n}\xi$ we obtain

$$\Phi_{\xi}^{k}(0,n) = \left[\,\Phi_{\xi}^{k-1}(1,0) + \ldots + \Phi_{\xi}(1,0) + (1,0)\,\right] + (0,n+k) + u - v$$

where u, v are nonnegative vectors in $\ell^{1}\left(\mathbb{Z}_{+}\times\mathbb{Z}_{+}\right)$ with $\|u\|_1 \le k$. If for some $k \ge 1$ the coefficient of $(0,n)$ in $\Phi_{\xi}^{k}(0,n)$ is greater than $k+1$ it follows that the coefficient of $(0,n)$ in $\left[\,\Phi_{\xi}^{k-1}(1,0) + \ldots + \Phi_{\xi}(1,0) + (1,0)\,\right] + u$ is also greater than $k+1$ and so there is some path from $(1,0)$ to $(0,n)$. Since $n \notin \left\{\nu_i,\ \nu_{i+1} - \nu_i : \; 0 \le i < t\right\}$ it follows that this path must pass through the set $\left\{(\nu_t,0),(0,\nu_t)\right\}$. Hence one of $(\nu_t,0)$ and $(0,\nu_t)$ $\left(\text{and hence both}\right)$ must lead to $(0,n)$. □

From Theorem 4.4.4 we know that the only vertices $(0,n)$ that do not lead to a lie in the diagonal subspace Q corresponding to some nontrivial island factor of ξ. Since the vertex-return growth-rate of such an element $(0,n)$ is bounded by the spectral

radius $\rho(\Phi_\xi|_Q)$ which, by Proposition 4.2.1, is strictly less than $\rho(\Phi_\xi)$ it follows by the above result and Corollary 5.3.2 that $\rho(\Phi_\xi)$ must be the vertex-return growth-rate associated to the self-communicating class of a.

We now have all the ingredients of Theorem 5.5.2. We therefore conclude

THEOREM 4.5.3: $\rho({}_a\Phi_\xi) < \rho(\Phi_\xi)$. □

(Here ${}_a\Phi_\xi$ can be the operator obtained by removing from the graph associated to Φ_ξ those edges incident to either both vertices $(0,\nu_i)$, $(\nu_i,0)$ or just the one vertex $(0,\nu_i)$).

§4.6 NONNEGATIVE EIGENVECTORS OF Φ_ξ^*:

LEMMA 4.6.1: *If* $F\ (\neq 0)$ *is a nonnegative* (ℓ^∞-)*eigenvector of* Φ_ξ^* *corresponding to eigenvalue* $\lambda > 1$ *then* $F(0,\nu_i) > 0$ *for all* $i \in \{0, 1, \ldots, t\}$. *Consequently* $\lambda = \rho(\Phi_\xi)$.

Proof. If $i < t$ then $\nu(\mu(\nu_i)) = \nu(\nu_i) = \nu_{i+1}$. By the axis-return equation we have that $\Phi^{\nu_{i+1}-\nu_i}(0,\nu_i) = (\mu(\nu_i), 0) + (0,\nu_{i+1})$ and $\Phi^{\nu_i}(\mu(\nu_i), 0) = (\nu_{i+1}, 0) + (0,\nu_i)$. It follows that

$$\Phi^{\nu_{i+1}}(0,\nu_i) = (\nu_{i+1}, 0) + (0,\nu_i) + \Phi^{\nu_i}(0,\nu_{i+1}) .$$

Applying our (symmetric) eigenvector F we obtain

$$\lambda^{\nu_{i+1}}.\ F(0,\nu_i) = F(0,\nu_{i+1}) + F(0,\nu_i) + \lambda^{\nu_i}.\ F(0,\nu_{i+1})$$

and so

$$F(0,\nu_{i+1}) = \left(\lambda^{\nu_{i+1}}-1)/(\lambda^{\nu_i}+1)\right).\ F(0,\nu_i) .$$

By Proposition 4.1.4 we have that $F(0,\nu_0) = F(0,1) > 0$ and the result follows by induction. □

THEOREM 4.6.2 *Let* ξ *be a sequence in* Σ *which is not of type* (A). *There exists a nonnegative eigenvector* F *of* Φ_ξ^* *corresponding to eigenvalue* $\rho(\Phi_\xi)$. *Any nonnegative eigenvector of* Φ_ξ^* *corresponding to eigenvalue strictly greater than one is a scalar multiple of* F.

If ξ *is not of type* (B) *then* $F(0,n) > 0$ *for all* $n \in \mathbb{N}$. *If, on the other hand,* $\xi = \beta * \psi$ *where* β *is a nontrivial island factor of minimal length* ℓ *then* $F(0,n) = 0 \Leftrightarrow \ell$ *divides* n.

Proof. Let I be the index set $\mathbb{Z}_+ \times \mathbb{Z}_+$. Let $T = (t_{ij})$ be the infinite (0,1)-matrix defined by

$$\Phi_\xi(i) = \sum_{j \in I} t_{ij} \cdot j \qquad \text{for all } i \in I .$$

Since ξ is not of type (A) we have the index $a = (0,\nu_t)$ which, by Theorem 4.5.3, satisfies $\rho(_a\Phi_\xi) < \rho(\Phi_\xi)$. Theorem 5.4.2 then guarantees that there is a nonnegative eigenvector d, unique up to scalar multiples, of Φ_ξ^* satisfying $d(0,\nu_t) > 0$ and that the corresponding eigenvalue is $\rho(\Phi_\xi)$. By Lemma 4.6.1 any nonnegative eigenvector corresponding to eigenvalue greater than one must have its $(0,\nu_t)$-ordinate positive.

Now $d(0,n) > 0 \iff (0,n)$ leads to $(0,\nu_t)$. By Theorem 4.4.4, if the set of n which do not lead to $(0,\nu_t)$ is nonempty then it consists precisely of the multiples of some ℓ which is the length of a nontrivial island factor of ξ. By Lemma 4.4.3, the multiples of the length of any nontrivial island factor do not lead to $(0,\nu_t)$. □

Proof of Theorem M.

The existence and uniqueness of the required pseudo-metric d on Σ_ξ follows from Theorems 4.6.2 and 4.1.1. The extent to which d is a metric is determined by Propositions 4.1.3 and 4.2.2. □

§4.7 PERIODIC BEHAVIOUR OF $\Phi_\xi/\rho(\Phi_\xi)$:

Theorem 5.4.4 implies that the iterates of $\Phi_\xi/\rho(\Phi_\xi)$ converge in operator norm to a one-dimensional projection operator provided that the spectrally-essential index $a = (0,\nu_t)$ is aperiodic, in other words that $h := \mathrm{HCF}\left\{ n \in \mathbb{N} : \Phi_\xi^n(0,\nu_t) \geq (0,\nu_t) \right\}$ satisfies $h = 1$. If $h > 1$ then the "normalized" iterates no longer converge but instead approach a periodic behaviour (of period h). This is in accordance with the fact that the spectrum of Φ_ξ contains exactly h points on the circle $|\lambda| = \rho(\Phi_\xi)$, namely the solutions of $\lambda^h = \left(\rho(\Phi_\xi)\right)^h$.

We now determine h in terms of factorisability properties of the sequence ξ.

LEMMA 4.7.1: *If $\xi = \beta \bullet \psi$ where β is a block of length ℓ which is a continent factor then ℓ divides $h = \mathrm{HCF}\left\{ n \in \mathbb{N} : \Phi_\xi^n(0,\nu_t) \geq (0,\nu_t) \right\}$.*

Proof. It suffices to show that ν_t is a multiple of ℓ (since it follows that, starting from $(0,\nu_t)$, axis-returns only occur at multiple-of-ℓ iterates of Φ_ξ). By coalescing factors if necessary we may assume that successive atomic factors in the decomposition of β are of different symbols. This decomposition must be of the form $\beta = \left[\xi_1^{\nu_1 - 1}\_\right] \bullet \left[\xi_{\nu_1}^{\nu_2/\nu_1 - 1}\_\right] \bullet \ldots \bullet \left[\xi_{\nu_k}^{\ell/\nu_k - 1}\_\right]$ and so $\nu_i \mid \nu_{i+1}$ for $i \leq k-1$. Furthermore ν_k (and indeed ℓ) divides ν_j for all $j > k$. Thus $k \leq t-1$ and $\ell \mid \nu_t$. □

THEOREM 4.7.2: *The period* $h = \mathrm{HCF}\left\{ n \in \mathbb{N} : \Phi_\xi^n(0,\nu_t) \geq (0,\nu_t) \right\}$ *is actually equal to* $\mathrm{HCF}\left\{ \nu_j : j \geq k+1 \right\}$ *where* $k = \max\left\{ i \leq t : \nu_i \mid \nu_j \text{ for all } j \geq t \right\}$ *and is therefore the maximum length of a left continent* $*$*-factor of* ξ .

Proof. For $m \geq t$ put $k_m := \max\left\{ i \leq t : \nu_i \mid \nu_j \text{ whenever } t \leq j \leq m \right\}$. Thus (k_m) forms a decreasing sequence in m. Clearly $k_t = t$ and $k_m \leq t-1$ when $m > t$.

For $m \geq t$ put $h_m := \mathrm{HCF}\left\{ \nu_j : k_m+1 \leq j \leq m \right\}$. Thus (h_m) is a decreasing sequence in m. Here $h_t = +\infty$ and $h_m < \nu_t$ when $m > t$. Note that $\nu_{k_m} \mid h_m$ for all m.

Clearly, $k_m \to k$ and $h_m \to \ell := \mathrm{HCF}\left\{ \nu_j : j \geq k+1 \right\}$ as $m \to +\infty$. Since $\nu_k \mid \ell$ and $\ell < \nu_{k+1}$ it follows that $\nu(\ell) = \nu_{k+1}$ and so, by Lemma 4.3.2, we have that ξ has a left $*$-factor of length ℓ. This factor is a $*$-product of atomic factors. By Lemma 4.7.1 it is sufficient to prove that $h \mid h_m$ for all $m \geq t$. We show the inductive step

$$h \mid h_m \Rightarrow h \mid h_{m+1} .$$

If $h_m \mid \nu_{m+1}$ then $\nu_{k_m} \mid \nu_{m+1}$, $k_{m+1} = k_m$, $h_{m+1} = h_m$ and so the implication is trivial. Otherwise $h_m \nmid \nu_{m+1}$ and so $h_m \nmid \nu_{m+1} - \nu_m = \mu(\nu_m)$. For ease of notation we put $r := k_{m+1}\ (< t)$. Notice that for the case $m = t$ this agrees with the r that we used in the proof of Proposition 4.4.1. The remainder of the proof will be concerned with the establishing the assertion

$$(*) \qquad \Phi_\xi^{p\cdot\nu_{r+1}}\left(\mu(\nu_m), 0\right) \geq (0,\nu_{r+1}) \qquad \text{for all } p \in \mathbb{N} .$$

From this it will follow from the facts

$$\Phi_\xi^{\nu_{m+1}-\nu_t}(0,\nu_t) \geq \Phi_\xi^{\nu_{m+1}-\nu_m}(0,\nu_m) \geq \left(\mu(\nu_m), 0\right)$$

and

$$\Phi_\xi^{\nu_t-\nu_{r+1}}(0,\nu_{r+1}) \geq (0,\nu_t)$$

that

$$\Phi_\xi^{\nu_{m+1}+p\cdot\nu_{r+1}}(0,\nu_t) \geq (0,\nu_t)$$

for all $p \in \mathbb{N}\cup\{0\}$, and thus that h divides $\mathrm{HCF}\left\{ h_m, \nu_{m+1}, \nu_{r+1} \right\} = h_{m+1}$.

From the definition of r we have $\nu_r \mid \nu_{m+1}$ and $\nu_{r+1} \nmid \nu_{m+1}$. If $r = k_m$ then $h_m \mid \nu_{r+1}$. Otherwise $r+1 \leq k_m$ and $\nu_{r+1} \mid h_m$. The quantity $h_{m+\frac{1}{2}} := \mathrm{HCF}\left\{ h_m, \nu_{r+1} \right\}$ is thus either h_m or ν_{r+1}. Let B denote the set $\left\{ b \in \mathbb{N} : b \text{ is divisible by } \nu_r \text{ but not by } h_{m+\frac{1}{2}} \right\}$. Clearly $\nu_{m+1} \in B$ and it follows that $\mu(\nu_m)\left(= \nu_{m+1}-\nu_m\right)$ and the numbers $\mu(\nu_m) + p\cdot\nu_{r+1}$ (for $p \in \mathbb{N}$) all lie in B. It should be clear that $\xi_b = \xi_{\nu_r}$ whenever $b \in B$ and $b < \nu_{m+1}$.

Observe that there is a unique c in the range

$$\mu(\nu_m) < c < \nu_{m+1}$$

such that $c \equiv \nu_{r+1} \pmod{\nu_m}$. Clearly $h_{m+\frac{1}{2}}$ divides c.

Now suppose that $b \in B$ satisfies $b < c$. We are interested in computing $\nu(b)$. Since ξ has period ν_m as far as $c\left(\le \nu_{m+1}-1\right)$ it follows that

$$\xi_c = \xi_{\nu_{r+1}} \neq \xi_{\nu_r} = \xi_{c-b}$$

and so $\nu(b) \le c$. In fact $\nu(b) \le \min\left\{c,\, b+\nu_{r+1}\right\}$ since if $b+\nu_{r+1} \le c\left(< \nu_{m+1}\right)$ we have

$$\xi_{b+\nu_{r+1}} = \xi_{\nu_r} \neq \xi_{\nu_{r+1}}.$$

On the other hand if $j < \nu_{r+1}$ and $\nu_r \nmid j$ then $\xi_{b+j} = \xi_j$ $\left(= \xi_{\nu_i}\right.$ where i is such that $\nu_i \mid j$ but $\nu_{i+1} \nmid j$ $\left.\right)$. Consequently $\mu(b)$ is a multiple of ν_r. There are two cases :

CASE I : $\mu(b) < \nu_{r+1}$. In this case $\nu\big(\mu(b)\big) = \nu_{r+1}$ and $\mu\mu\big(\mu(b)\big) = \mu(b)$. Thus

$$\Phi_\xi^{\mu(b)}(b,0) \ge \big(0,\mu(b)\big);$$

$$\Phi_\xi^{\nu_{r+1}}\big(0,\mu(b)\big) \ge \Phi_\xi^{\mu(b)}\big(\nu_{r+1}-\mu(b),0\big) \ge \big(0,\mu(b)\big)$$

and

$$\Phi_\xi^{\nu_{r+1}-\mu(b)}\big(0,\mu(b)\big) \ge (0,\nu_{r+1}).$$

It follows that $\Phi_\xi^{p\cdot\nu_{r+1}}(b,0) \ge (0,\nu_{r+1})$ for $p \ge 1$.

CASE II : $\mu(b) = \nu_{r+1}$. In this case $\nu(b) = b+\nu_{r+1} \in B$ and

$$\Phi_\xi^{\nu_{r+1}}(b,0) = (b+\nu_{r+1},0) + (0,\nu_{r+1}).$$

and furthermore $\nu(b) < c$.

If CASE I occurs when $b = \mu(\nu_m)$ then we have proved assertion $(*)$. If CASE II occurs then repeat the argument with $b = \nu\big(\mu(\nu_m)\big)$, and so on. There exists $q \ge 0$ such that

$$\nu^i\big(\mu(\nu_m)\big) = \mu(\nu_m) + i\cdot\nu_{r+1}$$

for all $i \le q$, but $\mu\Big(\nu^q\big(\mu(\nu_m)\big)\Big) < \nu_{r+1}$. $\Big($In fact $q \le \left[\left(c-\mu(\nu_m)\right)/\nu_{r+1}\right]\Big)$. The statement

$$\Phi_\xi^{p\cdot\nu_{r+1}}\big(\nu^i\mu(\nu_m)\big) \ge (0,\nu_{r+1}) \quad \forall\, p \ge 1,$$

true for $i = q$, can be inductively verified for $i \le q$, and so holds, in particular, for $i = 0$. □

§4.8 VARIATION OF $\rho(\Phi_\xi)$ WITH ξ.

PROPOSITION 4.8.1 : *The map* $\xi \mapsto \rho(\Phi_\xi)$ *from* Σ *to* $\mathbb{R}$ *is continuous*.

Proof. We first show the map is lower semi-continuous at any given point ξ. If ξ is of type (A) then this is clear since $\rho(\Phi_\xi) = \rho(\varphi_\xi) = 1$. If ξ is not of type (A) then $\rho(\Phi_\xi)$ equals g_a the index-return growth-rate of the spectrally-essential index $a = (0,\nu_1)$. For any $\varepsilon > 0$ we can choose k such that $\left(t_{a\,a}{}^{(k)}(\xi)\right)^{1/k} > g_a(\xi) - \varepsilon$. For any ξ' the growth-rate $g_a(\xi')$ is bounded below by the number $t_{a\,a}{}^{(k)}(\xi')$ which is the coefficient of a in the vector $\Phi_{\xi'}{}^{k}(a)$. This vector remains constant if we vary ξ'

over the cylinder $\left[\xi_1 \ldots \xi_{\max(v_t+k,\, v_{t+1})}\right]$.

To prove upper-semi-continuity, take any $\varepsilon > 0$ and we can find $k \geq 1$ satisfying

$$\left(\sup_{(m,n)} \| \Phi_\xi{}^k(m,n) \|_1 \right)^{1/k} < \rho(\Phi_\xi) + \varepsilon/2$$

and also

$$\left(4 \cdot k - 2\right)^{1/k} < \left(\rho(\Phi_\xi) + \varepsilon\right)/\left(\rho(\Phi_\xi) + \varepsilon/2\right).$$

Hence in particular we have $\left(\| \Phi_\xi{}^k(1,0) \|_1\right)^{1/k} < \rho(\Phi_\xi) + \varepsilon/2$. Now for any ξ' in the cylinder $\left[\xi_1 \ldots \xi_{k+1}\right]$ we have $\Phi_{\xi'}{}^k(1,0) = \Phi_\xi{}^k(1,0)$. For any $r \leq k$ we have, from Equation (4.1), that for all $n \geq 1$,

$$\begin{aligned} \| \Phi_{\xi'}{}^r(0,n) \|_1 &\leq \| \Phi_{\xi'}{}^{r-1}(1,0) + \ldots + \Phi_{\xi'}(1,0) + (1,0) \|_1 + 1 + r \\ &\leq r \cdot \| \Phi_{\xi'}{}^k(1,0) \|_1 + 1 + r \\ &\leq \left(2 \cdot r + 1\right) \cdot \| \Phi_{\xi'}{}^k(1,0) \|_1 . \end{aligned}$$

Hence for all (m,n) we have

$$\| \Phi_{\xi'}{}^k(m,n) \|_1 \leq 2 \cdot \left(2 \cdot (k-1) + 1\right) \cdot \| \Phi_{\xi'}{}^k(1,0) \|_1 .$$

It follows that

$$\left(\sup_{(m,n)} \| \Phi_{\xi'}{}^k(m,n) \|_1 \right)^{1/k} < \rho(\Phi_\xi) + \varepsilon$$

whence $\rho(\Phi_{\xi'}) < \rho(\Phi_\xi) + \varepsilon$. □

We can see from Equation (4.2) that the map $\xi \mapsto \rho(\Phi_\xi)$ factors through the equivalence relation given in Chapter 2, Example 3 to give a continuous map $\Sigma_\Sigma \to \mathbb{R}$.

Monotonicity of $\rho(\Phi_\xi)$.

We now show that $\rho(\Phi_\xi)$ increases monotonically as ξ "radiates out" along the "filaments" of Σ_Σ away from the central equivalence class – "the continent molecule" – of type (A) sequences.

DEFINITION 4.8.1 : A cylinder $\left[y_1 \ldots y_k\right]$ is said to be *principal* if there exists $i \geq 0$ such that $\nu^i(1) = k$. (Here $\nu = \nu_y$ the nonperiodicity function for any sequence $y \in \left[y_1 \ldots y_k\right]$.)

Hence, by Theorem 2.2.2, a cylinder $\left[y_1 \ldots y_k\right]$, $k \geq 1$, is principal if and only if its branch $B_\Sigma\left[y_1 \ldots y_k\right]$ (in the glueing space (Σ, Γ_Σ)) does not contain the "root" – the set $\left\{\alpha^\infty : \alpha \in L\right\}$ of period one sequences.

LEMMA 4.8.2 : *If $\left[y_1 \ldots y_k\right]$ is a principal cylinder then the map $\xi \mapsto \rho(\Phi_\xi)$, restricted to $\xi \in \left[y_1 \ldots y_k\right]$, attains its minimum at the point $\xi = (y_1 \ldots y_k)^\infty$.*

Proof. If the sequence $\xi = (y_1 \ldots y_k)^\infty$ is of type (A) then $\rho(\Phi_\xi) = 1$ which is obviously the minimum value. Otherwise, by Prop 4.3.3, the fact that $\nu_\xi(k) = +\infty$ and the hypothesis that k is a principal nonperiodity (of ξ), we must have that $k = \nu^j(1)$ where $j \geq t+1$ and $\nu^j(1)$ is the first non-dividing principal nonperiodicity of ξ (or any

sequence ξ' in $[y_1 \ldots y_k]$).

Let ξ be in $[y_1 \ldots y_k]$. Write $S := \{ n \in \mathbb{N} : (\exists i \geq 1)(\nu^i(n) = k) \}$. Thus S is a subset of $\{1, 2, \ldots, k-1\}$, contains $\nu^i(1)$ and is independent of ξ. We see that S is "almost invariant" under applications of the maps μ and ν, and that the only "escape" from S is via $\{k\}$. If $n \in S$ then either $\nu(n) \in S$ or $\nu(n) = k$. In either case $\nu(n) \leq k$ and so, by Theorem 2.5.3, the nonperiodicities of $\mu(n)$ become principal on or before reaching k, and so $\mu(n) \in S$.

Now consider the symmetrised action of Φ_ξ for any $\xi \in [y_1 \ldots y_k]$. Define a set of vertices $W := \{ \{m, n+m\} : n \in S, m < \mu(n) \}$. Observe that this is independent of the choice of ξ in $[y_1 \ldots y_k]$ since $n + \mu(n) \leq k$ for all $n \in S$. Observe furthermore that if $w \in W$ then $\Phi_\xi(w)$ is independent of the choice of ξ in $[y_1 \ldots y_k]$ and is either an element of W or the sum of two vertices either both in W or one in W and the other equal to the vertex $\underline{k} = \{0, k\}$.

Writing a for the apex $\{0, \nu^k(1)\}$, we therefore have that the "restricted apex-return" power series ${}_{\underline{k}}T_{a\,a}(z) \left(= \sum_{i \geq 0} {}_{\underline{k}}t_{a\,a}^{(i)} z^i \right)$ remains constant as ξ varies over $[y_1 \ldots y_k]$. (Here ${}_{\underline{k}}t_{a\,a}^{(i)}$ denotes the number of paths of length i from a to a which avoid the vertex $\underline{k}$. Any such path must remain entirely within W). Clearly $t_{a\,a}^{(i)} \geq {}_{\underline{k}}t_{a\,a}^{(i)}$ for all i (where $t_{a\,a}^{(i)}$ denotes the full number of paths of length i from a to a). Furthermore equality must occur when $\xi = (y_1 \ldots y_k)^\infty$ since in this case there are no paths from $\underline{k}$ to a (since $(\Phi_\xi)^i \{0, k\} = \{i, k+i\}$ for all $i \geq 0$).

We have therefore established that $g_a = \lim_{i \to \infty} \left(t_{a\,a}^{(i)}\right)^{1/i}$ is minimised, for ξ in $[y_1 \ldots y_k]$, when $\xi = (y_1 \ldots y_k)^\infty$. The result follows as $\rho(\Phi_\xi) = g_a(\xi)$. □

COROLLARY 4.8.2: *If $[y_1 \ldots y_k]$ is a principal cylinder then the map $\xi \mapsto \rho(\Phi_\xi)$, restricted to $\xi \in B_\Sigma[y_1 \ldots y_k]$, attains its minimum at the point $\xi = (y_1 \ldots y_k)^\infty$.*

Proof. We prove this by induction on k. By the recursive definition of branch given in Definition 2.1.2 we have

$$B_\Sigma[y_1 \ldots y_k] = [y_1 \ldots y_k] \cup \bigcup \{ B_\Sigma[y_1 \ldots y_{r-1}; \beta] : 1 \leq r < k, (y_1 \ldots y_r)^\infty \in [y_1 \ldots y_k] \text{ and } \beta \neq y_r \}.$$

By inductive hypothesis we have, for each r with $1 \leq r < k$ and $(y_1 \ldots y_r)^\infty \in [y_1 \ldots y_k]$ and each $\beta \neq y_r$, that $\inf\{ \rho(\Phi_\xi) : \xi \in B_\Sigma[y_1 \ldots y_{r-1}; \beta] \} = \rho\left(\Phi_{(y_1 \ldots y_{r-1}\,\beta)^\infty}\right)$.

Observe that $(y_1 \ldots y_{r-1}\,\beta)^\infty = [y_1 \ldots y_{r-1}\,\_] * (\beta^\infty)$ and also that $(y_1 \ldots y_{r-1}\,y_r)^\infty = [y_1 \ldots y_{r-1}\,\_] * (y_r)^\infty$. Equation (4.2) then shows the corresponding ρ-values to be equal. Now $\rho\left(\Phi_{(y_1 \ldots y_k)^\infty}\right) \leq \rho(\Phi_\xi)$ whenever $\xi \in [y_1 \ldots y_k]$ and so, in particular, when $\xi = (y_1 \ldots y_{r-1}\,y_r)^\infty$, and therefore when $\xi = (y_1 \ldots y_{r-1}\,\beta)^\infty$.

The result follows since each cylinder $[y_1 \ldots y_{r-1}\,\beta]$ is principal. □

THEOREM 4.8.3 : *If* $x, y \in \Sigma$ *and* $y \in \Gamma_\Sigma\text{-arc}\left(x, (x_1)^\infty\right)$ *then* $\rho(\Phi_x) \geq \rho(\Phi_y)$.

Proof . If $y = (x_1)^\infty$ then $\rho(\Phi_y) = 1$ and result is obvious . Otherwise , by Proposition 2.3.3 , arbitrarily close to y , there exists a gluepoint $(y_1 \ldots y_k)^\infty$ such that $B_\Sigma\left[y_1 \ldots y_k\right]$ contains the point x but not the point $(x_1)^\infty$. Hence the cylinder $\left[y_1 \ldots y_k\right]$ is principal . It follows by the above Corollary that $\rho(\Phi_x) \geq \rho\left(\Phi_{(y_1 \ldots y_k)^\infty}\right)$. The (lower-semi-) continuity of $\xi \mapsto \rho(\Phi_\xi)$ at the point $\xi = y$ then gives the result . □

NOTE : If the map $\mathcal{M} \to \Sigma_\Sigma$, conjectured to exist at the end of Chapter 1 , were "monotone" in the sense that :

$$\left(\begin{array}{l} c_1 \text{ and } 0 \text{ not in the same component of } \mathcal{M}\backslash\{c_2\} \\ \qquad \text{implies} \quad \xi(f_{c_1}) \in \Gamma_\Sigma\text{-arc}\left(0^\infty, \xi(f_{c_2})\right) \end{array} \right)$$

for c_1 and c_2 , say , pre-periodic points of $\mathcal{M}$ then it should follow that $\rho\left(\Phi_{\xi(f_c)}\right)$ is "monotonic" in c along the filaments of $\mathcal{M}$. We conjecture that Douady and Hubbard's proof for the monotonicity of the kneading invariant for the real quadratic family $\left(\text{outlined in } [D]\right)$ can be generalised to the above statement .

CHAPTER 5

In this chapter we are interested in the Perron-Frobenius theorem for infinite non-negative matrices from the point of view of finding nonnegative eigenvectors. The basic approach comes from Vere-Jones however some modifications are required to relax the assumption of irreducibility. We restrict to a special class of matrices whose row sums are bounded.

§5.1 ℓ-SPACES :

Given an index set I we recall that $\ell^1(I)$ is the set of formal $\mathbb{C}$-linear combinations $v = \sum v_i \cdot i$ ($i \in I$) such that $\sum |v_i| < \infty$. This is a Banach space under the ℓ^1-norm $\| \ \|_1$ given by $\|v\|_1 = \sum |v_i|$. We shall adopt the convention of thinking of our typical element as a row vector (v_i) which in the case I is countable we can write $(v_1, v_2, v_3, \ldots)$. The dual space $(\ell^1(I))^*$ is naturally isomorphic to $\ell^\infty(I)$ the set of bounded functions b from I to $\mathbb{C}$, equipped with the ℓ^∞-norm $\| b \|_\infty = \sup |b(i)|$. We shall think of b as the column vector $(b_i)^T$ $\Big($ or, again, $(b_1, b_2, b_3, \ldots)^T$ if I is countable$\Big)$.

Note that $(\ell^\infty(I))^* \supsetneq \ell^1(I)$ for I infinite. We shall write V for $\ell^1(I)$.

Regarding "inner products" between row and column vectors, we have the following simple case of Hölder's inequality :

$$|b(v)| \le \| b \|_\infty \|v\|_1 \qquad \text{for } v \in V, b \in V^*.$$

ℓ^1-matrices :

Any bounded linear operator on V is determined by its action on the basis elements i, and is hence represented by a matrix whose rows are (the co-ordinates of) the images of these basis vectors. Conversely a matrix $T = (t_{ij})$ $(i,j \in I)$ is said to be (row) ℓ^1 if it represents such a bounded linear operator, via the equations :

$$(T(v))_j = ((v_i) \cdot T)_j = \sum_i v_i t_{ij}$$

(adopting the convention that the matrix acts by right-multiplication). Thus T is ℓ^1 if and only if :

$$\sup_i (\textstyle\sum_j |t_{ij}|) < \infty,$$

and the operator norm $\| T \|$ defined formally as $\sup (\|T(v)\|_1 / \|v\|_1)$ $(v \neq 0)$ can be obtained as :

$$\| T \| = \sup_i (\textstyle\sum_j |t_{ij}|).$$

We have :

$$\|T(v)\|_1 \le \| T \| \cdot \|v\|_1 \quad \text{for all } v \in V$$

and $$\|T^*(b)\|_\infty \le \|T\| \cdot \|b\|_\infty \quad \text{for all } b \in V^*$$

where T^* is the dual operator on V^*, represented by the action of the matrix T on ℓ^∞-column vectors:

$$(T^*(b))_i = (T \cdot (b_j)^T)_i = \sum_j t_{ij}\, b_j \quad .$$

For any two ℓ^1 matrices S and T, the product ST is a well-defined ℓ^1 matrix and:

$$\|ST\| \le \|S\| \|T\| \quad .$$

We write the matrix iterates $T^n = (t_{ij}^{(n)})$ for $n \ge 0$ (with the usual convention that $t_{ij}^{(0)} = \delta_{ij}$). The inequality:

$$\|T^{m+n}\| \le \|T^m\| \|T^n\| \qquad (\text{" sub-multiplicativity "})$$

implies, by a standard argument, that $\|T^n\|^{1/n}$ has a limit $G(T)$, the *spectral radius of* T, as $n \to \infty$ which is also the infimum over $n \ge 1$.

It is clear that no complex number λ with $|\lambda| > G(T)$ can be an eigenvalue corresponding to an ℓ^1 row eigenvector or to an ℓ^∞ column eigenvector of the matrix T.

§5.2 NONNEGATIVE MATRICES :

A matrix T is said to be *nonnegative* if all its entries t_{ij} are nonnegative. In this case when T is ℓ^1 we can write:

$$\|T^n\| = \sup_i (\sum_j t_{ij}^{(n)}) = \|T^{*n}(1)\|_\infty$$

where 1 is the ℓ^∞ vector given by $1(i) = 1$ for all $i \in I$.

If T has all entries 0 or 1 then it is best thought of as a directed graph with vertex set I where t_{ij} is the number of directed edges from i to j. It follows that $t_{ij}^{(n)}$ is the number of paths of length n from i to j and $G(T)$ is the ℓ^1 growth-rate of the graph, that is the growth-rate of the largest number of paths of length n from a vertex.

Nonnegative matrices can be characterised by the fact that they preserve the partial order on row (and column) vectors given by:

$$x \ge y \quad (\text{and } x^T \ge y^T) \quad \text{if} \quad x_i \ge y_i \text{ for all } i \in I \ .$$

A row (or column) vector is said to be *nonnegative* if all its entries are nonnegative. The *support* of a nonnegative row vector x (or its transpose) is the set of i for which $x_i > 0$.

The following sections can be applied to nonnegative matrices T which are not ℓ^1, but we need to assume always that the iterates T^n have finite entries.

Action on supports of row vectors :

A nonnegative matrix T acts on the collection of sets J of indices via the rule:

$$k \in T(J) \iff t_{jk} > 0 \text{ for some } j \in J \ .$$

Given indices $i, j \in I$ we say i *leads to* j and write $i \to j$ if there exists $n = n(i,j) \geq 1$ such that $j \in T^n\{i\}$ (or equivalently such that $t_{ij}^{(n)} > 0$).

We say i *communicates with* j and write $i \leftrightarrow j$ if $i \to j$ and $j \to i$.

We say an index i is *recurrent* if $i \to i$.

The index set I can now be decomposed into *irreducible classes* which are the equivalence classes ("self-communicating" classes) induced by the relation $\leftrightarrow$ on recurrent indices, together with the remaining indices in singleton sets. The relation $\to$ induces a corresponding relation on classes.

The *period* d of a recurrent index i, which is defined to be the H.C.F. of those n for which $t_{ii}^{(n)} > 0$, is common to indices in its irreducible class. The indices of such a class partition themselves into d subclasses permuted cyclically by the action of T.

If $i \to j$ for all $i, j \in I$ then we say the matrix T is *irreducible*.

Index-return growth-rates :

For any index i, the inequality :

$$t_{ii}^{(m+n)} \geq t_{ii}^{(m)} t_{ii}^{(n)} \qquad \text{("super-multiplicativity")}$$

implies that $(t_{ii}^{(n)})^{1/n}$ has a limsup $g_i(T)$ as $n \to \infty$ which is also the supremum over $n \geq 1$. This limsup is actually a limit when taken over those n which are multiples of the period $d(i)$ of the index i, ($t_{ii}^{(n)}$ being zero for all other n) (Seneta, Lemma A·4 of Appendix A).

We now observe that g_i gives a lower bound to eigenvalues corresponding to nonnegative eigenvectors whose i^{th} entries are positive:

LEMMA 5.2.1 : *If either of the inequalities* $x \cdot T \leq \lambda x$ *or* $T \cdot x^T \leq \lambda x^T$ *has a nonnegative solution* (x_i) *with* $x_i > 0$ *then* $\lambda \geq g_i$.

Proof. If $x \cdot T \leq \lambda x$ then $\lambda \geq 0$ and it follows inductively that

$$x \cdot T^n \leq \lambda^n x$$

since T is nonnegative. Taking i^{th} ordinates we obtain :

$$x_i t_{ii}^{(n)} \leq \sum_j x_j t_{ji}^{(n)} \leq \lambda^n x_i$$

and the result follows as $x_i > 0$.

(The dual result follows similarly.) □

Now $g_i(T)^{-1}$ is the radius $R_i(T)$ of convergence of the power series $T_{ii}(z) = \sum_{0 \leq n < \infty} t_{ii}^{(n)} z^n$. Since the coefficients are nonnegative it follows that any analytic continuation of this function must have a singularity at the point $z = R_i$ (Vivanti-Pringsheim theorem - see [VJ3]). We introduce some definitions from Vere-Jones relating to the nature of this singularity :

We say that an index i is R_i-*recurrent* if the series

$$\sum_n t_{ii}^{(n)} R_i^n \quad \text{diverges}$$

(or equivalently $\quad T_{ii}(x) \to +\infty \quad \text{as} \quad x \nearrow R_i$) .

We say that i is R_i-*positive* if , furthermore ,

$$t_{ii}^{(n)}.R_i^n \to 0 \quad \text{as} \quad n \to \infty .$$

Given two indices i, j with $i \leftrightarrow j$ it follows from such inequalities as

$$t_{jj}^{(k+m+n)} \geq t_{ji}^{(k)}.t_{ii}^{(m)}.t_{ij}^{(n)}$$

(letting m tend to infinity , keeping k and n fixed) that $g_i = g_j$.

Hence all the indices of an irreducible class have the same associated radius of convergence R and it is clear also that :

all or none of them are R-recurrent ;

all or none of them are R-positive .

In particular if T is an irreducible matrix we have corresponding definitions for T to be *R-recurrent* and *R-positive* .

We note some results of Vere-Jones [VJ1] (Theorem 4.1 and Criterion III with Lemma 5.2) regarding existence of nonnegative eigenvectors in the irreducible case :

THEOREM 5.2.2 : *Let* T *be a nonnegative irreducible matrix .*

(i) *If* T *is* R*-recurrent then there exist unique nonnegative row and column eigenvectors (up to scalar multiples) corresponding to eigenvalue* $1/R$.

(ii) *Any pair of nonnegative row and column eigenvectors have finite inner product if and only if* T *is* R*-positive and the corresponding eigenvalues are both equal to* $1/R$.

First-return 'probabilities' :

We now single out a 'special' index a (which we assume to be recurrent) . In order to analyse more closely the singularity $z = R_a$ of the function $T_{aa}(z)$ we must introduce quantities analogous to "first-entrance" and "last-exit" probabilities :

We define $f_{ia}^{(0)} = 0$, $f_{ia}^{(1)} = t_{ia}$ and then recursively

(5.2.1) $$f_{ia}^{(n)} = \sum_{j \neq a} t_{ij} f_{ja}^{(n-1)} \quad \text{for} \quad n = 2, 3, \ldots$$

Similarly we define $l_{aj}^{(0)} = 0$, $l_{aj}^{(1)} = t_{aj}$ and

(5.2.2) $$l_{aj}^{(n)} = \sum_{i \neq a} l_{ai}^{(n-1)} t_{ij} \quad \text{for} \quad n = 2, 3, \ldots$$

Clearly we have that $f_{ia}^{(n)} \leq t_{ia}^{(n)}$ and $l_{aj}^{(n)} \leq t_{aj}^{(n)}$.

Now the "first-return probabilities" $f_{aa}^{(n)}$ and $l_{aa}^{(n)}$ are equal for all n and we can recover the $t_{aa}^{(n)}$ by the rules :

$$t_{aa}^{(0)} = 1 \quad \text{and}$$

$$t_{aa}^{(n)} = \sum_{0 \leq k < n} t_{aa}^{(k)} l_{aa}^{(n-k)} \quad \text{for} \quad n = 1, 2, \ldots$$

(using the fact that any path of length n (≥ 1) from the vertex a to itself has a unique time k (with $0 \leq k < n$) for which there is a "last exit" from a). It is worth noting at this stage that the H.C.F. of those n for which $l_{aa}^{(n)} > 0$ is the same as the corresponding H.C.F. for the $t_{aa}^{(n)}$.

Multiplying by z^n and summing over $n \geq 0$ we obtain the power series identity:

$$T_{aa}(z) = 1 + T_{aa}(z) \cdot L_{aa}(z)$$

where the generating function $L_{aa}(z)$ $\left(= F_{aa}(z) \right) = \sum_{n \geq 0} l_{aa}^{(n)} z^n$ has radius of convergence at least R_a. Thus the fundamental equation

(5.2.3) $$T_{aa}(z) = 1/\left(1 - L_{aa}(z)\right)$$

holds initially over $|z| < R_a$ but in fact extends over any region for which either side has analytic continuation.

Observe now that the positivity of $T_{aa}(x)$ implies via (5.2.3) that $L_{aa}(x) < 1$ for x on the real interval $0 \leq x < R_a$. The nature of the singularity of $T_{aa}(z)$ at the point $z = R_a$ can be characterised by the limiting behaviour of $L_{aa}(x)$ as $x \nearrow R_a$.

The index a is R_a-recurrent if and only if

$$\sum_{n \geq 1} l_{aa}^{(n)} R_a^n \left(= L_{aa}(R_a^-) \right) = 1 .$$

In this case the Erdös-Feller-Pollard theorem [EFP] guarantees that the terms $t_{aa}^{(nd)} . R_a^{nd}$ actually converge with limit $d/R_a \cdot L_{aa}'(R_a^-)$ (where $d = d(a)$ is the period of a). Thus the index a is R_a-positive if and only if $L_{aa}(R_a^-) = 1$ and the "mean recurrence-time" $\sum_{n \geq 1} n \cdot l_{aa}^{(n)} . R_a^n \left(= R_a \cdot L_{aa}'(R_a^-) \right)$ is finite.

Meromorphic continuation :

We now see that meromorphic continuation of $T_{aa}(z)$ beyond the point $z = R_a$ is a sufficient condition for R_a-positivity.

LEMMA 5.2.3 : *The complex function* $T_{aa}(z)$ *has a pole at* $z = R_a$ *if and only if the power series* $L_{aa}(z)$ *has radius of convergence strictly greater than* R_a. *In this event* $L_{aa}(R_a) = 1$, *the pole is simple with residue* $-1/L_{aa}'(R_a)$ *and the index* a *is* R_a*-positive*.

Proof. If $T_{aa}(z)$ has a pole at $z = R_a$ then $L_{aa}(z)$ is certainly analytic (taking the value 1) at that point. That $L_{aa}(z)$ has strictly greater radius of convergence follows by the Vivanti-Pringsheim theorem as the coefficients $l_{aa}^{(n)}$ are nonnegative.

Conversely, if $L_{aa}(z)$ has radius of convergence R_a' then the monotonicity of $L_{aa}(x)$ (and its derivative) with real x in the range $0 \leq x < R_a'$ together with the

inequality

$$|L_{aa}(z)| \le L_{aa}(|z|) \quad \text{for} \quad |z| < R_a'$$

implies that either : $|L_{aa}(z)| < 1$ for all z $\left(\text{with } |z| < R_a'\right)$ in which case $R_a = R_a'$; or the function $L_{aa}(x) - 1$ has exactly one zero on the real interval $0 \le x < R_a'$ and this zero $x = R_a$ is simple .

The remaining assertions follow as $\left(L_{aa}(z) - 1\right)/\left(z - R_a\right)$ has a positive finite limit $L_{aa}'(R_a)$ as z tends to R_a . □

Continuing the situation of the Lemma we observe that as $T_{aa}(z)$ is a function of z^d it also has simple poles at $z = R_a\zeta$ for ζ the dth roots of unity . The function $T_{aa}(z)$ cannot have any other pole on the circle $|z| = R_a$ since this would entail $L_{aa}(z) = 1 = L_{aa}(|z|)$ which can only happen when the non-zero terms $t_{aa}^{(n)} z^n$ are all real and positive $\left(\text{requiring } z^d \text{ real and positive}\right)$.

For completeness , we give the geometric case of the Erdös-Feller-Pollard result .

LEMMA 5.2.4 : *If* $L_{aa}(z)$ *has radius of convergence strictly greater than* R_a *then* $t_{aa}^{(nd)} R_a^{nd}$ *tends geometrically to the limit* $d/R_a \cdot L_{aa}'(R_a)$.

Proof . Write $e_n = t_{aa}^{(nd)} R_a^{nd} - t_{aa}^{((n-1)d)} R_a^{(n-1)d}$ for $n \ge 1$ and put $e_0 = 1$. Form the power series $E(\zeta) = \sum_{n\ge 0} e_n \zeta^n$. So we have

$$E(\zeta^d) = \left(1 - \zeta^d\right) \cdot T_{aa}(R_a\zeta)$$

valid initially for $|\zeta| < 1$, but which , by hypothesis , extends meromorphically over a larger disk . Now the singularities on the circle $|\zeta| = 1$ of the right-hand side are removable . Consequently $E(\zeta^d)$ has radius of convergence greater than one and so e_n tends to zero geometrically . Hence $t_{aa}^{(nd)} R_a^{nd}$ converges geometrically with limit :

$$\begin{aligned} \textstyle\sum_{n\ge 0} e_n = E(1) &= \lim_{\zeta\to 1} \left(1 - \zeta^d\right) \cdot T_{aa}(R_a\zeta) \\ &= \lim_{\zeta\to 1} \left(1 - \zeta^d\right) / \left(1 - L_{aa}(R_a\zeta)\right) \\ &= d/R_a \cdot L_{aa}'(R_a) \end{aligned}$$

□

§5.3 RELATIONS BETWEEN VARIOUS GROWTH-RATES :

From now on T will be a nonnegative ℓ^1 matrix . For any index i the obvious inequality

$$t_{ii}^{(n)} \le \sup_k \left(\textstyle\sum_j t_{kj}^{(n)}\right)$$

implies

$$\beta_i(T) \le G(T) .$$

For finite matrices there is always an index i for which $g_i = G$ (see Corollary 5.3.2). However, this is not true in general for infinite ℓ^1, even irreducible, matrices. For instance, if $I = \mathbb{Z}$ and $t_{ij} = 1$ whenever $j - i \in \{-1,2\}$ and $t_{ij} = 0$ otherwise then $G(T) = \|T\| = 2$ whereas one can show (by considering positive eigenvectors x of the form $x_i = \alpha^i$ in conjunction with Lemma 5.2.1) that the g_i have a common value at most (in fact equal to) $3/(2^{2/3})$. The essential reason for this, viewing the matrix as a graph, acting on "mass-distributions" over $\mathbb{Z}$, is that although the total mass is doubling on each iteration the "centre of mass" has a "drift" velocity in the direction of the positive integers, and so it is to be expected that the mass remaining near a fixed vertex should grow at a slower rate.

If we replace $\{-1,2\}$ by $\{-1,1\}$ in the above example then there is no "drift" and $g_i = G = 2$. However, as a proportion of the total mass, the mass remaining near a given vertex i still decays to zero (albeit not now exponentially), so that

$$t_{ii}^{(n)}/\|T^n\| \to 0 \quad \text{as } n \to \infty .$$

The irreducible matrix here is *R-recurrent* but not *R-positive*.

For a general nonnegative ℓ^1 matrix T and a given index i, we have the inequalities:

$$t_{ii}^{(n)} \le g_i^n \le G(T)^n \le \|T^n\| .$$

So in order for the proportion $t_{ii}^{(n)}/\|T^n\|$ to be eventually bounded away from zero (as $n \to \infty$ through multiples of the period $d(i)$) it is necessary and sufficient that:

(i) the index i is R_i-positive

(ii) $g_i = G$

(iii) the normalised operator T/G has bounded iterates (i.e. $\|T^n\| \le K \cdot G^n$ for some constant K).

The relationship between these conditions and the existence of (nonnegative) ℓ^1 row and ℓ^∞ column eigenvectors is investigated, for the case T is irreducible, by Vere-Jones in [VJ2] (Theorem 2.2), where it is established that under assumption (iii) the existence of such eigenvectors corresponding to eigenvalue $G(T)$ is equivalent to conditions (i) and (ii).

Taboo growth-rates :

We shall see that conditions (i), (ii) and (iii) are satisfied in the special situation that removing the ith row and i^{th} column from the matrix T strictly decreases the spectral radius (Lemma 5.4.1). In the graph context this means that removing the edges leading immediately to and from the vertex i reduces the ℓ^1 growth-rate of the graph.

We specifically allow here the possibility that T is not irreducible and so we have to be careful when applying the Vere-Jones theory.

DEFINITIONS 5.3.1 : Given a nonnegative matrix T and a finite set H of indices we define ${}_HT$ to be the matrix obtained from T by deleting the rows H and columns H (i.e. replacing them by rows and columns of zeros). We write the iterates $({}_HT)^n = ({}_Ht_{ij}^{(n)})$.

In the graph situation this corresponds to "nullifying" the vertices H, and ${}_Ht_{ij}^{(n)}$ is the number paths length n from i to j which avoid the "taboo" set H.

Clearly, we have the inequalities:

$$G({}_HT) \le G(T)$$
$$g_i({}_HT) \le g_i(T) \quad .$$

Often H will be a single vertex $\{a\}$. To simplify notation we will write ${}_aT$ for ${}_{\{a\}}T$, etc. We form the generating function

$${}_aT(z) = \sum_{n\ge 0} \|({}_aT)^n\| z^n \quad .$$

DEFINITION 5.3.2 : If T is a nonnegative ℓ^1 matrix we say an index a ($\in I$) is *spectrally essential* if $G({}_aT) < G(T)$.

We see from the following Lemma that this is equivalent to the condition $G({}_aT) < g_a(T)$. This condition is 'stable' in the sense that it holds if and only if for some n, $\|({}_aT)^n\| < t_{aa}^{(n)}$. Thus, if we are dealing with a matrix with a complicated structure (for which the growth-rates G, etc. cannot be computed exactly and for which we may not even know the irreducible classes completely), we still have a chance of verifying this property (See Theorem 5.5.2).

Before entering into the proof of Lemma 5.3.2 we introduce the quantities $\gamma_{ia}^{(n)}$ and $\lambda_{aj}^{(n)}$ which can be thought of as "first-occurrence" and "last-occurrence" probabilities, given by:

$$\gamma_{ia}^{(0)} = \delta_{ia} \qquad \text{and} \qquad \gamma_{ia}^{(n)} = \sum_j {}_at_{ij}^{(n-1)}.t_{ja} \quad (n \ge 1)$$
$$\lambda_{aj}^{(0)} = \delta_{aj} \qquad \text{and} \qquad \lambda_{aj}^{(n)} = \sum_i t_{ai}\cdot{}_at_{ij}^{(n-1)} \quad (n \ge 1)$$

These differ from the respective quantities $f_{ia}^{(n)}$ and $l_{aj}^{(n)}$ only when $i = a$ or $j = a$. We note the following technical result.

PROPOSITION 5.3.1 : *The quantities* $\sup_i \gamma_{ia}^{(n)}$ *and* $\sum_j \lambda_{aj}^{(n)}$ *are bounded above by the coefficient of* z^n *in the power series* $1 + \|T\|\cdot z\cdot {}_aT(z)$.

Proof. We have $\sup_i \gamma_{ia}^{(0)} = 1$ and for $n \ge 1$:

$$\begin{aligned}\sup_i \gamma_{ia}^{(n)} &= \sup_i \left(\sum_j {}_at_{ij}^{(n-1)}.t_{ja}\right) \\ &\le \left(\sup_i \sum_j {}_at_{ij}^{(n-1)}\right)\left(\sup_j t_{ja}\right) \\ &\le \|({}_aT)^{n-1}\| \, \|T\| .\end{aligned}$$

In a similar fashion $\sum_j \gamma_{aj}^{(0)} = 1$ and for $n \geq 1$

$$\begin{aligned}\sum_j \gamma_{aj}^{(n)} &= \sum_i t_{ai} \cdot \left(\sum_j {}_a t_{ij}^{(n-1)}\right) \\ &\leq \left(\sum_i t_{ai}\right) \cdot \left(\sup_i \sum_j {}_a t_{ij}^{(n-1)}\right) \\ &\leq \| T \| \cdot \| ({}_a T)^{n-1} \| \ .\end{aligned}$$

□

LEMMA 5.3.2 : *Given a nonnegative ℓ^1 matrix* T *and an index* $a \in I$ *then*

$$G(T) = \max\left\{ G({}_a T) \, , \, g_a \right\} \ .$$

Proof. It is clear that we need only show the inequality

$$G(T) \leq \max\left\{ G({}_a T) \, , \, g_a \right\} \ .$$

For any pair of indices i , j and $n \geq 0$ any path of length n from i to j is either disjoint from $\{a\}$ or has unique "first-occurrence" time k and "last-occurrence" time m (with $0 \leq k \leq m \leq n$) . So we have :

$$(5.3) \qquad t_{ij}^{(n)} = {}_a t_{ij}^{(n)} + \sum_{0 \leq k \leq m \leq n} \gamma_{ia}^{(k)} \cdot t_{aa}^{(m-k)} \cdot \gamma_{aj}^{(n-m)}$$

Summing over j and taking the supremum over i we obtain :

$$\| T^n \| = \sup_i \sum_j t_{ij}^{(n)} \leq \| ({}_a T)^n \| + \sum_{0 \leq k \leq m \leq n} \sup_i \gamma_{ia}^{(k)} \cdot t_{aa}^{(m-k)} \cdot \sum_j \gamma_{aj}^{(n-m)} \ .$$

Thus $\| T^n \|$ is less than or equal to the nth term P_n of the power series

$$P(z) = {}_a T(z) + \left(1 + \| T \| z \cdot {}_a T(z) \right)^2 \cdot T_{aa}(z) \ .$$

Now ${}_a T(z)$ and $T_{aa}(z)$ have radii of convergence respectively $G({}_a T)^{-1}$ and g_a^{-1} , whence $P(z)$ has radius of convergence $\min\left\{ G({}_a T)^{-1} , g_a^{-1} \right\}$. The result follows as $G(T) = \lim \| T^n \|^{1/n} \leq \limsup (P_n)^{1/n}$. □

COROLLARY 5.3.2 : *If* T *is a nonnegative ℓ^1 matrix and* H *is a finite set of indices then*

$$G(T) = \max\left\{ G({}_H T) \, , \, \max\left\{ g_h(T) : h \in H \right\} \right\} \ .$$

Proof . We proceed by induction on $|H|$. Again , we clearly need only show

$$G(T) \leq \max\left\{ G({}_H T) \, , \, \max\left\{ g_h(T) : h \in H \right\} \right\} .$$

Choose $a \in H$. By inductive hypothesis :

$$\begin{aligned} G({}_a T) &= \max\left\{ G({}_H T) \, , \, \max\left\{ g_h({}_a T) : h \in H \backslash \{a\} \right\} \right\} \\ &\leq \max\left\{ G({}_H T) \, , \, \max\left\{ g_h(T) : h \in H \backslash \{a\} \right\} \right\} \end{aligned}$$

The result follows by the lemma . □

§5.4 MATRICES WITH A SPECTRALLY-ESSENTIAL INDEX :

LEMMA 5.4.1 : *If* T *is a nonnegative* ℓ^1 *matrix with a spectrally-essential index* a *then* $g_a = G$, *the index* a *is* R_a*-positive and the normalised operator* T/G *has bounded iterates .*

Proof. By assumption , we have $G({}_aT) < g_a(T)$. We observe that the power series $L_{aa}(z)$ has radius of convergence at least $G({}_aT)^{-1}$ and is therefore strictly greater than $g_a^{-1} = R_a$ the radius of convergence of $T_{aa}(z)$. This is because for $n \geq 2$:

$$\begin{aligned} t_{aa}^{(n)} &= \textstyle\sum_i \sum_j t_{ai} \cdot {}_at_{ij}^{(n-2)} \cdot t_{ja} \\ &\leq \left(\textstyle\sum_i t_{ai}\right) \cdot \left(\sup_i \textstyle\sum_j {}_at_{ij}^{(n-2)}\right) \left(\sup_j t_{ja}\right) \\ &\leq \| T \|^2 \cdot \| ({}_aT)^{n-2} \| \ . \end{aligned}$$

It follows by Lemma 5.2.3 that the index a is R_a-positive .

It remains to show that $\| T^n \| / g_a^{\,n}$ is bounded . Choose γ with

$$G({}_aT) < \gamma < g_a \ .$$

So there exists $C \geq 1$ such that for all $n \geq 1$, $\| T \| \| ({}_aT)^{n-1} \| \leq C \cdot \gamma^n$. From Proposition 5.3.1 and the proof of Lemma 5.3.2 we have :

$$\begin{aligned} \| T^n \| &\leq \| ({}_aT)^n \| + \textstyle\sum_{0 \leq k \leq m \leq n} C \gamma^k \cdot t_{aa}^{(m-k)} \cdot C \cdot \gamma^{n-m} \\ &= \| ({}_aT)^n \| + \textstyle\sum_{0 \leq r \leq n} (r+1)\, t_{aa}^{(n-r)} \cdot C^2 \cdot \gamma^r \\ &\leq \| ({}_aT)^n \| + C^2 \cdot g_a^{\,n} \cdot \textstyle\sum_{r \geq 0} (r+1) \left(\gamma / g_a\right)^r \ , \end{aligned}$$

and hence $\| T^n \| / g_a^{\,n} \leq C + C^2 / \left(1 - \gamma / g_a\right)^2$. □

We now establish the main result of this chapter .

THEOREM 5.4.2 : *Let* T *be a nonnegative* ℓ^1 *matrix with spectrally-essential index* a . *Then* T *has unique nonnegative* ℓ^1 *row eigenvector and* ℓ^∞ *column eigenvector whose* a^{th} *ordinates are* 1 . *The corresponding eigenvalue is* G(T) . *An index* j *is the support of the row eigenvector if and only if* a *leads to* j . *An index* i *is the support of the column eigenvector if and only if* i *leads* a .

Proof. We reiterate that the hypothesis gives us $G({}_aT) < g_a(T) = G(T)$.

"Uniqueness"

It is clear from Lemma 5.2.1 that the eigenvalue corresponding to a nonnegative ℓ^1 row eigenvector whose a^{th} ordinate is positive , must be G(T) . Now let x and y be two such eigenvectors normalised so that $x_a = y_a = 1$. Thus x - y is a ℓ^1 vector which is either zero or an eigenvector of T with a^{th} ordinate zero and so a ℓ^1 eigenvector of the matrix ${}_aT$. The latter case is not a possibility since the corresponding eigenvalue G(T) is

larger than the spectral radius of ${}_aT$.

A similar argument works for ℓ^∞ column vectors.

"Existence"

By Lemma 5.4.1 the index a is R_a-positive and so $L_{aa}(R_a) = F_{aa}(R_a) = 1$. We now consider the off-diagonal generating functions:

$$L_{aj}(z) = \sum_{n\geq 0} l_{aj}^{(n)} z^n$$

and

$$F_{ia}(z) = \sum_{n\geq 0} f_{ia}^{(n)} z^n$$

It is easily verified that these all have radius of convergence at least $G({}_aT)^{-1}$. Recalling the defining equations $(5.2.1)$ and $(5.2.2)$ for $f_{ia}^{(n)}$ and $l_{aj}^{(n)}$, multiplying by z^n and summing over $n \geq 1$ we obtain the identities:

$$F_{ia}(z) = z\cdot\sum_j t_{ij}\cdot F_{ja}(z) + z\cdot t_{ia}\cdot\left(1 - F_{aa}(z)\right) \quad \text{for all } i \in I$$

and

$$L_{aj}(z) = z\cdot\sum_i L_{ai}(z)\cdot t_{ij} + z\cdot t_{aj}\cdot\left(1 - L_{aa}(z)\right) \quad \text{for all } j \in I$$

which are valid for $|z| < G({}_aT)^{-1}$ (absolute convergence of the summations being guaranteed by the nonnegativity of all but one of the terms on the right for the case $z \geq 0$). Substituting $z = R_a$ we obtain that the row vector α given by $\alpha_j = L_{aj}(R_a)$ is a left eigenvector of the matrix T and also that the column vector β^T given by $\beta_i = F_{ia}(R_a)$ is a right eigenvector of T, both corresponding to eigenvalue g_a. These vectors are nonnegative with a^{th} ordinate 1.

It remains now to show that α and β lie in the appropriate ℓ^p spaces. We note from the technical Proposition 5.3.1 that for $n \geq 1$:

$$\sum_{j\neq a} l_{aj}^{(n)} \leq \| T \| \, ({}_aT)^{n-1} \|$$

$$\sup_{i\neq a} f_{ia}^{(n)} \leq \| ({}_aT)^{n-1} \| \, \| T \|$$

and so $\sum_j L_{aj}(R_a) \leq 1 + \| T \| \, {}_aT(R_a) < \infty$, whence α is ℓ^1. Likewise $\sup_i F_{ia}(R_a) \leq \max\left\{ 1 , \| T \| \, {}_aT(R_a) \right\} < \infty$, so that β is ℓ^∞. □

We now observe that the inner product $\alpha\cdot\beta^T$ (guaranteed finite) of the eigenvectors obtained above is none other than the "mean recurrence-time" of the index a.

LEMMA 5.4.3: *The equation*

$$\sum_i L_{ai}(z)\cdot F_{ia}(z) = z\, L_{aa}'(z) - L_{aa}(z)\left(1 - L_{aa}(z)\right)$$

is valid for $|z| < G({}_aT)^{-1}$.

Proof. We have

$$\begin{aligned}\sum_{i\neq a} L_{ai}(z)\cdot F_{ia}(z) &= \sum_{i\neq a}\sum_{n\geq 0}\left(\sum_{0\leq k\leq n} l_{ai}^{(k)} f_{ia}^{(n-k)}\right) z^n \\ &= \sum_{n\geq 2}\left(\sum_{0<k<n}\sum_{i\neq a} l_{ai}^{(k)} f_{ia}^{(n-k)}\right) z^n \\ &= \sum_{n\geq 1} (n-1)\, l_{aa}^{(n)} z^n \\ &= z\cdot L_{aa}'(z) - L_{aa}(z) .\end{aligned}$$

Adding $L_{aa}(z)\cdot F_{aa}(z)$ to both sides gives the result . □

Substituting $z = R_a$ gives $\sum_i \alpha_i \beta_i = R_a\cdot L_{aa}'(R_a)$.

Limiting behaviour of normalised iterates :

When the index a of a nonnegative ℓ^1 matrix T is spectrally-essential we have seen (via the Erdős-Feller-Pollard theorem) that the $(a,a)^{th}$ entry of the normalised matrix iterates $T^n/G(T)^n$ has an asymptotic behaviour given by :

$$t_{aa}^{(n)} R_a^n \to d(a)\cdot \beta_a\alpha_a/\sum_k \alpha_k\beta_k$$

as $n \to \infty$ through multiples of the period d(a) (the entry being zero for all other values of n). In general if the matrix T is irreducible with period d and is R-positive then the behaviour of the $(i,j)^{th}$ entry is given by :

$$t_{ij}^{(n)} R^n \to d\cdot \beta_i\alpha_j/\sum_k \alpha_k\beta_k$$

as n tends to infinity through the appropriate residue class (mod d) ([VJ1] Corollary to Lemma 5.3). If T fails to be irreducible then the asymptotic behaviour cannot be so easily described , owing to the varying possible periods among different irreducible classes of indices . However , if our spectrally essential index a is *aperiodic* , that is d(a) = 1 , then the normalised matrix iterates $T^n R_a^n$ converge , not only elementwise but also in the operator norm topology , to the projection operator Π , represented by the matrix :

$$(\Pi_{ij}) = \left(\beta_i\,\alpha_j/\sum_k \alpha_k\beta_k\right) = \left(1/\,\alpha\cdot\beta^T\right)\beta^T\cdot\alpha\ .$$

Note that : $(T/G)\cdot\Pi = \Pi\cdot(T/G) = \Pi = \Pi^2$.

THEOREM 5.4.4 : *Let T be a nonnegative ℓ^1 matrix with spectrally-essential index a . Let α and β^T be the eigenvectors of Theorem 5.4.2 with associated projection operator Π . If the index a is aperiodic then the ℓ^1 operator $T/G - \Pi$ has spectral radius less than 1*
(in other words, $\| T^n/G^n - \Pi \|$ tends to zero exponentially in n).

Proof. To simplify notation we will use R for R_a . From equation (5.3) we have for each (i,j) :

$$t_{ij}^{(n)} = {}_a t_{ij}^{(n)} + \sum_{0\le r\le n}\left(\sum_{0\le k\le r} T_{ia}^{(k)} T_{aj}^{(r-k)}\right)\cdot t_{aa}^{(n-r)}\ .$$

We have also :

$$\beta_i\,\alpha_j = F_{ia}(R)\cdot L_{aj}(R) = \left(\sum_{k\ge 0} T_{ia}^{(k)} R^k\right)\cdot\left(\sum_{k\ge 0} T_{aj}^{(k)} R^k\right)$$

(even where one of i and j coincides with a .)

So $$\beta_i\,\alpha_j = \sum_{r\ge 0}\left(\sum_{0\le k\le r} T_{ia}^{(k)} T_{aj}^{(r-k)}\right) R^r$$

Hence

$$t_{ij}^{(n)}R^n - \beta_i\alpha_j/\sum\alpha_k\beta_k = {}_at_{ij}^{(n)}R^n$$
$$+\sum_{0\le r\le n}\left(\sum_{0\le k\le r} T_{ia}^{(k)}T_{aj}^{(r-k)}\right)R^r\cdot\left(t_{aa}^{(n-r)}R^{n-r} - 1/\sum\alpha_k\beta_k\right)$$
$$-\left(1/\sum\alpha_k\beta_k\right)\sum_{r>n}\left(\sum_{0\le k\le r} T_{ia}^{(k)}T_{aj}^{(r-k)}\right)R^r \quad .$$

Now we are in the geometric case of the Erdös-Feller-Pollard theorem so there exists constants γ, K and $C \ge 1$ with $G({}_aT)/G(T) < \gamma < 1$ such that

$$|t_{aa}^{(n)}R^n - 1/\sum\alpha_k\beta_k| \le K\gamma^n \qquad \text{and}$$
$$\|T\|\,\|({}_aT)^{n-1}\| \le C\cdot(\gamma\cdot G)^n \quad \text{for all} \quad n \ge 1 \ .$$

Therefore on taking the absolute value, summing over j and taking the supremum over i we obtain :

$$\|T^n/G^n - \Pi\| \le \|({}_aT)^n\|/G^n$$
$$+\sum_{0\le r\le n}(r+1)\,C^2\gamma^r\cdot K\,\gamma^{n-r}$$
$$+\sum_{r>n}(r+1)\,C^2\gamma^r$$

and so the result follows with

$$\|T^n/G^n - \Pi\| \le C\gamma^n + C^2\cdot\gamma^n\left[K\cdot(n+1)(n+2)/2 + \gamma\cdot(n+1)/(1-\gamma) + \gamma/(1-\gamma)^2\right] .$$

□

§5.5 QUASI-FINITE MATRICES :

We say that a nonnegative ℓ^1 matrix T is *quasi-finite* if there exists a finite set H of indices such that $G({}_HT) < G(T)$. Note that this definition includes all finite nonnegative matrices (with the exception of nilpotent matrices, which have spectral radius zero). We note also that ${}_HT$ differs from T by a compact operator and hence the normalised operator T/G is "quasi-compact" in the sense of Vere-Jones in [VJ4].

Defining the *inner radius* of a quasi-finite matrix to be the infimum of $G({}_HT)$ over finite sets H of indices, it can be verified that the properties of the spectrum outside the disk of inner radius are similar to the spectral properties of compact operators. In particular the resolvent operator $(\lambda I - T)^{-1}$ behaves meromorphically in λ outside this disk.

LEMMA 5.5.1 : *Let J be a finite set of indices and let i be another index with the property that* $i \leftrightarrow j$ *for some* $j \in J$. *If i is an* $R_i({}_JT)$*-recurrent index of the matrix* ${}_JT$ *then* $g_i(T) > g_i({}_JT)$.

Proof. Comparing the first-return 'probabilities' for T and ${}_JT$ it is clear that $l_{ii}(T)^{(n)} \geq l_{ii}({}_JT)^{(n)}$ for all $n \geq 0$. The assumption $i \leftrightarrow j$ implies $\gamma_{ij}^{(m)} > 0$ and $\gamma_{ji}^{(n)} > 0$ for some $m, n > 0$ and so strict inequality occurs with

$$l_{ii}(T)^{(m+n)} \geq l_{ii}({}_JT)^{(m+n)} + \gamma_{ij}^{(m)}.\gamma_{ji}^{(n)}.$$

If the power series $L_{ii}(z)$ has radius of convergence less than $R_i({}_JT)$ then so does $T_{ii}(z)$. Otherwise $\lim_{x \nearrow R_i({}_JT)} L_{ii}(x)$ exists and has value at least $1+\gamma_{ij}^{(m)}.\gamma_{ji}^{(n)}.R_i({}_JT)^{m+n}$ which is greater than 1, forcing the function $T_{ii}(z)$ to have a pole at some positive value of z less than $R_i({}_JT)$. Either way the result follows. □

THEOREM 5.5.2 : *If* T *is a quasi-finite matrix with* $G(T) > G({}_HT)$ *then an index* a *is spectrally-essential if and only if for each* $h \in H$ *either* $h \leftrightarrow a$ *or* $g_a(T) > g_h(T)$.

Proof. If the index a is spectrally-essential and i is any index for which $i \not\leftrightarrow a$ then since any path from i to i cannot pass through a it follows that

$$g_i(T) = g_i({}_aT) \leq G({}_aT) < g_a(T) .$$

Conversely, if the set $K = \{h \in H : h \leftrightarrow a\}$ satisfies

$$g_h(T) < g_a(T) \quad \text{for all } h \in H \backslash K$$

then by Corollary 5.3.2 we have $G({}_KT) = \max\{G({}_HT), \max\{g_h({}_KT)\}\} < G(T)$. By enlarging H if necessary, to include a, we may assume that $a \in K$. If $G({}_aT) = G({}_KT)$ then certainly $G({}_aT) < G(T)$. Otherwise choose J, in K, maximal, containing a, such that $G({}_aT) = G({}_JT)$. So there exists $i \in K \backslash J$ such that $G({}_JT) > G({}_{J \cup \{i\}}T)$. In other words i is a spectrally-essential index of the matrix ${}_JT$. Thus certainly i is an $R_i({}_JT)$-recurrent index of ${}_JT$ and since also $i \leftrightarrow a \in J$, by Lemma 5.5.1, we have that $g_i(T) > g_i({}_JT) = G({}_JT) = G({}_aT)$. Hence $G(T) > G({}_aT)$.

□

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TITLE

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