Irreducibility in Exchange Economies

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A thesis submitted for the degree of Doctor of Philosophy in Economics

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June 2002

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[1]**This chapter is co-authored with Sayantan Ghosal.**

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Chapter 1

Introduction

1 Introduction

THIS THESIS CONSIDERS THE PROBLEM of non-existence of competitive equilibria in exchange economies. When equilibria fail to exist, prices fail to allocate commodities across individuals in such a way that markets clear. This is *the* most fundamental of all market failures.

The classical theorem on the existence of competitive equilibrium dates back to the 1950's, when it was proved in various guises by Arrow and Debreu (1954), Debreu (1956, 1959), Gale (1955), Kuhn (1956), McKenzie (1954, 1955, 1959), and Nikaido (1956). Major predecessors of these papers were Wald (1935, 1936) and von Neumann (1937). Wald was first to demonstrate the existence of a meaningful solution to the Walrasian system of equations. Using an assumption which has since been termed the weak axiom of revealed preference, and which effectively reduces the set of consumers to one, Wald's proof involves a simple maximisation problem. With many consumers with independent preferences however fixed point methods are needed (as shown in Uzawa (1962)). Although von Neumann's paper is not concerned with competitive equilibrium in the classical sense, he was the first to use a fixed point theorem for an existence argument in economics, and provided the generalisation of the Brouwer fixed point theorem, which is the major mathematical

tool used in the classical existence proofs.

Since the 1950's the proof of existence of competitive equilibria has undergone various major improvements, particularly in terms of the role and the formulation of the so called survival condition. The strong survival condition, or Slater assumption, states that every consumer can consume some bundle of goods in the interior of their consumption set without resorting to trade. Such a condition would be satisfied if each consumer's initial endowment lay in the interior of their consumption set, which, if the consumption set corresponds to the positive orthant, implies that each consumer is endowed with a positive amount of each commodity. Given the standard interpretation of a commodity in general equilibrium (that is, defined not only in terms of its physical characteristics, but also the date and location in which it is consumed), the division of labour, and the spatial dispersion of economies, it is unrealistic to assume that individuals be endowed with positive amounts of all commodities, or indeed desire all commodities. However, it is now well known (see Arrow (1951)) that exchange economies with boundary endowments (where individuals have zero endowments of some commodities) and weakly monotone preferences may fail to have competitive equilibria. The use of fixed point theorems in the classical proof of existence of competi-

tive equilibria requires that the demand correspondences of individuals be non-empty, convex-valued and upper hemi-continuous in prices.[2] To guarantee that demand correspondences possess these properties, it is sufficient that the utility functions of individuals be continuous and quasi-concave, and that budget sets be continuous in prices. If individuals have interior endowments, and prices lie in the unit simplex, budget correspondences are indeed continuous. However, when individuals have boundary endowments, the budget correspondence may fail to be lower hemi-continuous along those sequences of prices that converge to the boundary of the simplex. In particular, if at the limit an individual has zero income, this would imply that demand correspondences fail to be upper hemi-continuous in prices.

To illustrate the problem caused by boundary endowments and weakly monotone preferences, consider the following economy which, despite satisfying standard Arrow-Debreu assumptions, has no competitive equilibria.

Example 1 *Consider a pure exchange economy with no production, in which*

[2]This property is often referred to as the closed graph property. The graph of a correspondence $f : X \to Y$ is the set (x, y) such that $y \in f(x)$. Upper hemi-continuity requires that for any x, and for any open set O, that contains x there exists some neighbourhood N of x such that $f(x') \subseteq O$ if $x' \in N$. The closed graph property, on the other hand, requires that for any two sequences $x^m \to x \in X$ and $y^m \to y$, with $x^m \in X$ and $y^m \in f(x^m)$ for every m, we have $y \in f(x)$. These two concepts coincide if the range of f is compact and $f(x)$ is closed for each x, conditions which are generally satisfied when applying fixed point theorems (see e.g. Green and Heller (1981)).

there are two commodities, $L = \{1, 2\}$, *and two individuals,* $I = \{1, 2\}$, *with consumption sets* $X^1 = X^2 = R^2_+$, *preferences* $u^1 = x^1_1$, $u^2 = x^2_2$, *and endowments* $w^1 = (1, 0)$, $w^2 = (1, 1)$. *Let* p_l *denote the price of commodity* l. *Individual 2's demand for commodity 2 is* $\frac{p_1+p_2}{p_2}$, *and since* $w_1 = 1$, *there are no prices which clear markets.*

The problem here is that there are price vectors, $p \in R^2_+/\{0\}$, at which the set $\{x \in X^1 : px \leq pw^1\}$ has an empty interior. The strong survival condition prevents the possibility of minimum-wealth situations. We could modify the above example to satisfy the condition by increasing individual 1's endowment of commodity 2 by one unit, thereby ensuring upper hemi-continuity of demand correspondences. A competitive equilibrium would then exist with $p_1 = p_2$ and $x^1 = (2, 0), x^2 = (0, 2)$. In practice however, most consumers only have one commodity to sell, their labour. One could argue therefore that the strong survival condition is almost never satisfied. Alternatively, we could make individuals' preferences strongly monotone. This would imply that all prices must be strictly positive, thereby guaranteeing that all individuals have positive income. With the preferences $u^i = x^i_1 + x^i_2$ for example, a competitive equilibrium exists with $p_1 = p_2$ and $x^1 = (1, 0), x^2 = (1, 1)$. However, as already noted, it is not reasonable to assume that individuals

desire all commodities.

McKenzie (1959, 1961), generalising a concept of Gale (1955), proposed a way of overcoming the existence problem, which allows the possibility of weakly monotone preferences and boundary endowments. He studied irreducible exchange economies. "In loose terms, an economy is irreducible if it cannot be divided into two groups of consumers where one group is unable to supply any goods which the other group wants."[3] McKenzie presents an existence proof in two stages. First, he uses fixed point theorems to prove the existence of a quasi-equilibrium (formally defined by Debreu (1962)). A quasi-equilibrium differs from a competitive equilibrium only in so far as the utility maximising requirement of individuals is replaced by a cost minimising requirement. A competitive equilibrium is by definition a quasi-equilibrium, but the reverse is not necessarily true. Under standard assumptions, a quasi-equilibrium exists but a competitive equilibrium may not, since with weakly monotone preferences and boundary endowments there is no guarantee that every consumer can participate with a budget of positive value[4] in the neighbourhood of equilibrium. However, McKenzie shows that with the additional requirement that the economy be irreducible, any quasi-equilibrium is indeed

[3]McKenzie (1959)
[4]Negative value in the case of net trades.

a competitive equilibrium. The irreducibility assumption states that however the set of individuals be partitioned into two groups, $\{1,2\}$, then at any feasible allocation for the economy, there is some feasible net trade available to group 2, which when added to the allocation of group 1 leaves all individuals in group 1 at least as well off and at least one individual in group 1 strictly better off. (Note that the addition of the net trade may imply individuals in the first group supplying as well as receiving commodities). Since irreducibility must hold for any partition of the set of individuals, group 1 may consist of one arbitrarily chosen individual. Under standard assumptions, it can be shown that at a quasi-equilibrium there is some individual with positive income. Let group 1 consist of this individual and group 2 all other individuals. Irreducibility implies that there is a net trade available to group 2 which makes the individual in group 1 strictly better off. Therefore, there is some individual in group 2 with positive income. In fact, irreducibility ensures that if any one individual has positive income (which is true at a quasi-equilibrium), then so will all other individuals in the economy. That is, an irreducible economy has a competitive equilibrium.

Following McKenzie's seminal papers, Arrow and Hahn (1971) proposed an alternative to irreducibility, namely resource relatedness. "... household

h' is *resource related* to household h'' if some increase in those assets held by household h' in some positive amounts can be used in a reallocation of the entire economy so that no household is worse off and household h'' is strictly better off.[5]" The only property of household h' which is relevant in this definition is a list of the commodities which he/she can supply positive amounts of, and the only relevant property of household h'' is their utility function. It is possible, with this definition, that h'''s welfare be directly or only very indirectly improved through the addition of these commodities. It may be that these commodities enter h'''s utility function directly. Alternatively, they may be factors of production used to produce commodities that enter h'''s utility function. On the other hand, they may be employed to produce commodities which enter some individual h''''s utility function, allowing h''' to give up the consumption of some other commodity - thus maintaining utility of h''' constant - which does enter h'''s utility function directly. The definition implies that if any household h' is resource related to a household h'' with positive income, then h' itself has positive income. If any household h' is resource related to a household which is resource related to a household which is resource related to a household h'' - that is, if h'

[5]Arrow and Hahn (1971)

is indirectly resource related to h'' - which has positive income, then h' also has positive income. Arrow and Hahn show that if every household (or individual) is indirectly resource related to every other household, then if one household has positive income at prices p, then so do all other households. Resource relatedness therefore guarantees that under standard assumptions a competitive equilibrium exists.

Although relaxing the strong survival condition, both McKenzie and Arrow and Hahn rely on a weak survival assumption, otherwise known as the autarchy assumption, to prove existence of competitive equilibria. This assumption states that every consumer can consume some bundle of goods in their consumption set without trade. This implies that an individual's endowment is an element of their consumption set, clearly a restrictive assumption. Moore (1975) further weakens the endowment assumption, showing that the requirement that individuals may survive without trade is superfluous for an irreducible economy. He proves existence in economies in which consumers must rely on trade to achieve a feasible commodity bundle.

In economies with complete markets the irreducibility condition is always satisfied if preferences are strictly monotone. However, this is no longer true with incomplete asset markets, since the incompleteness of the market

may limit the possibility of agents attaining some feasible transactions. The majority of papers in the incomplete markets literature rely then, for existence, on the assumption that every agent's endowment lie in the interior of his/her consumption set. In finite pure exchange economies with incomplete asset markets and with no restriction on endowments, Gottardi and Hens (1996) propose an extension of the irreducibility condition to prove the existence of a competitive equilibrium. The key innovation of their paper is that the irreducibility condition is reformulated to account for the restrictions on transferring wealth across possible states of nature. Their condition requires that however the set of individuals be partitioned into two groups, $\{1,2\}$, then at any feasible allocation for the economy, there is some feasible net trade available to group 2, which, taking into account the constraints imposed by the asset market structure, can be added to the allocation of group 1, such that all individuals in group 1 are at least as well off and at least one individual in group 1 is strictly better off. The authors consider a pure exchange economy satisfying standard assumptions, in which individual preferences are strongly monotone, (an economy for which, with complete markets, competitive equilibria exist with no restriction on endowments). They show that with incomplete markets, their irreducibility condition is sufficient for exis-

tence; using illustrative examples to prove that in the absence of irreducibility existence of equilibria may fail.

In economies with a countable set of individuals and commodities Balasko, Cass and Shell (1980), Burke (1988), Geanakoplos and Polemarchakis (1991), and Wilson (1981) have also proposed different extensions of irreducibility, to prove the existence of competitive equilibria in infinite dimensional economies. The argument for the existence of competitive equilibria in infinite economies proceeds by considering a sequence of finite truncated economies which tend to the full economy in the limit. More specifically, the argument is to consider a convergent sequence of prices and a convergent sequence of feasible allocations, obtained from the competitive equilibria for a sequence of finite truncated economies, and to then show that the limits of these sequences constitute a competitive equilibrium for the full economy. To guarantee the existence of such sequences, one can impose some survival condition directly on the full economy (Burke (1988), Geanakoplos and Polemarchakis (1991)) and/or one can assume some form of irreducibility for the truncated economies (Balasko, Cass and Shell (1980), Wilson (1981)). Wilson (1981) imposes McKenzie's irreducibility condition on the entire economy, and also assumes the existence of a collection of finite sub-economies, each

of which is irreducible. This guarantees competitive equilibria for the sub-economies and ensures that their limit constitute a competitive equilibrium for the entire economy. Balasko, Cass and Shell (1980) assume irreducibility of the finite truncated economies only. However, although not explicitly brought out in their paper, their *intertemporal irreducibility* condition is such that it necessarily implies irreducibility of the full economy. Burke (1988) refines the existence proof techniques of these earlier authors to eliminate what he deems to be an "unnatural sub-economy irreducibility assumption.[6]" He assumes irreducibility of the full economy only. In order to guarantee the existence of competitive equilibria for a sequence of finite truncated economies, he perturbs individual endowments to be strictly positive, shrinking the perturbations to zero so that the limit of the perturbed economy equilibria constitutes a competitive equilibrium for the initial unperturbed economy. However, since the perturbation of endowments is not necessarily limited to finitely many commodities, showing that the perturbation vanishes at the limit is very involved. Geanakoplos and Polemarchakis (1991) follow Burke's approach of imposing irreducibility on the full economy alone, but simplify the existence argument by perturbing utility functions as well as endowments

[6]Burke (1988)

in the finite truncated economies. This allows them to perturb endowments in only one commodity, thus simplifying the limiting argument.

Each of the survival conditions proposed in the literature is framed very differently, and implies different restrictions on the underlying fundamentals of the economy. This raises the question of the relationship between alternative survival conditions, but more importantly, and this is the main contribution of the thesis, what are the key restrictions to guarantee the existence of competitive equilibria in each of these General Equilibrium frameworks.

The key innovation of this thesis is the use of graph theory to develop conditions which guarantee the existence of competitive equilibria in economies with boundary endowments. Moreover, we develop weaker irreducibility conditions than any proposed in the existing literature. The graph theoretic approach is very intuitive. To each exchange economy at each vector of prices, we associate a price graph, in which each individual is represented by a vertex, and directed arcs between vertices represent a particular relationship between the associated individuals. More specifically, the existence of a directed arc between vertices implies a certain coincidence of the preferences and endowments of the individuals concerned. Roughly speaking, given a price vector, a directed arc between individuals i and j exists when

(a) individual i is a member of some set of individuals who can supply commodities which j desires at any utility maximizing affordable feasible bundle, and (b) however we partition the set of individuals in (a) into two groups, the group containing i can supply commodities which are desired by some individual (at any of their utility maximizing affordable feasible bundles) in the other group. A graph is said to be strongly connected if for each pair of vertices, there is a path of arcs linking the first to the second and vice versa. Let C denote the collection of price graphs of an economy at every vector of prices. An economy is said to be C-irreducible if each member of C is strongly connected. Given that strong connectedness of the graph is required at all vectors of prices, it follows that C-irreducibility is a condition on preferences, consumption sets and technology, and *not* on prices. Under standard assumptions, C-irreducibility is a sufficient condition for the existence of a competitive equilibrium.

The added appeal of C-irreducibility over either irreducibility or resource relatedness is two-fold. Firstly, C-irreducibility does not require that individual preferences be strongly quasi-concave. Secondly, C-irreducibility allows an operational method for testing using computational algorithms. While McKenzie's definition needs to be checked at each of a continuum of

feasible allocations, C-irreducibility need only be checked for a finite set of graphs. For small models, these graphs may be inferred and strong connectedness determined by inspection. For large, more complex models, this procedure is obviously problematic. However, even with such large models, C-irreducibility lends itself to testability, where irreducibility does not. The adjacency matrix of an economy graph can easily be constructed using the relevant index sets of the economy. Entries in the matrix are zeros and ones, where a one implies the existence of a directed arc between associated vertices and zeros signify the absence of arcs. There are well-known efficient computational algorithms which can be used to test for the irreducibility of the adjacency matrix, which implies strong connectedness of the economy graph.

One of the virtues of a graph theoretic approach is that it allows one to abstract away from the economics. Translating the survival conditions presented in the literature into restrictions on graphs, one can determine relationships between them, previously obscured by their economic formulation. We characterize both McKenzie's (1961) irreducibility and Arrow and Hahn's (1971) resource relatedness in terms of restrictions on graphs. Our first observation is that irreducibility and C-irreducibility are differ-

ent conditions, in that neither implies the other. However, the nature of this difference suggests a particular modification in the definition of an arc in the price graph to generate an alternative condition - which we call C'-irreducibility - which is sufficient for the existence of competitive equilibria for a significantly larger class of economies than McKenzie's irreducibility. C'-irreducibility is obtained from C-irreducibility by adding a second type of arc, implying a slightly different relationship between associated individuals. While irreducibility requires a particular relationship to hold between every pair of individuals at all feasible allocations, C'-irreducibility only requires a relationship to hold between every pair of individuals for some subset of feasible allocations, which always contains the quasi-equilibrium allocations.

It is well known (see McKenzie (1981)) that resource relatedness implies irreducibility, but the reverse relationship is far from obvious. However, here we are able to provide conditions under which the two definitions are equivalent[7]. Therefore, we are able to conclude that under standard assumptions, C'-irreducibility is weaker than irreducibility, which is in turn weaker than resource relatedness.

In the third chapter, we extend the analysis to pure exchange economies

[7]This answers an open question in the existing literature (see McKenzie (1981)).

with incomplete asset markets. Although the notion of a price graph extends trivially to such economies, the incompleteness of asset markets presents several issues of substance. Even with the assumption of strictly monotone preferences, C-irreducibility does not guarantee the existence of competitive equilibria in economies with an incomplete set of asset markets, since the incompleteness of the market may limit the possibility of agents attaining some feasible transactions. Additional restrictions with respect to the complete market case are therefore required. These are joint restrictions on the asset structure and the distribution of endowments and the preferences of individuals. By modifying the definition of C'-irreducibility to allow for the attainability of trades given the asset market structure, we obtain an analogous condition, $\bar{C}'$-irreducibility, which guarantees existence for economies with incomplete markets. We show that $\bar{C}'$-irreducibility is weaker than Gottardi and Hens'(1996) alternative survival condition. Our final result in this section is to provide, using a graph theoretic characterisation, a sufficient condition for an exchange economy with incomplete asset markets to be effectively complete.

In the fourth chapter, we study the consequences for the existence of competitive equilibria, of individuals' participation in markets being restricted

in some way. In practice, individuals' participation in markets is subject to a wide range of diverse institutional restrictions. Age restrictions apply for the purchase of cigarettes and alcohol. Commodities such as weapons and motorcycle insurance can only be purchased by registered license holders. In some countries, foreign nationals may be legally prevented from buying commodities such as domestic property. Wholesale markets are restricted to registered traders. Traders in futures markets face margin calls, where they have to put up shares to the value of their trades, imposing limits on short sales.

Formally, participation in markets is restricted when some reallocations of commodities are not attainable by some individuals at some commodity prices. When participation is restricted, competitive equilibria may not exist, even if individuals have strongly monotone preferences. Inexistence is due to the discontinuous dependence of attainable allocations on prices. An economy with an incomplete set of asset markets is a special case of an economy with restricted participation, in which all individuals face the same participation constraints, and discontinuities in the correspondence of attainable allocations therefore occur at the same prices for all individuals. On the other hand, when market participation is restricted in such a way

that different individuals have access to different attainable allocations, discontinuities occur at different commodity prices for different individuals. In effect, prices must perform a dual role. Not only must they adjust to attain market clearing, but also to ensure that the net demands of unrestricted individuals in each market be consistent with the limits imposed by other individuals' participation in the market being restricted.

Existence of equilibria in economies with restricted participation in asset markets has been studied by Balasko, Cass and Siconolfi (1990), - who consider linear homogeneous constraints on asset holdings, - Polemarchakis and Siconolfi (1997), - who consider an incomplete asset market economy with asset payoffs denominated in multiple commodities with individuals facing asymmetric linear constraints on portfolio incomes, - and Cass, Siconolfi and Villanacci (2001), - who modify the Balasko, Cass and Siconolfi (1990) model to allow for any smooth, quasi-concave inequality constraints[8]. In each of these papers, existence of equilibrium is demonstrated for the case in which individuals have interior endowments. We consider a pure exchange economy without uncertainty, in which individuals face linear homogeneous constraints on participation in markets. Individuals are assumed to have

[8]Earlier contributions in this area are due to Siconolfi (1986, 1989).

strongly monotone preferences and boundary endowments. Our formulation therefore characterises economies in which, without restricted participation, competitive equilibria exist under standard assumptions. We show that when participation in markets is restricted, additional conditions are required to ensure existence with boundary endowments. In particular, some form of C-irreducibility condition is required. This condition must be formulated to take into account the restrictions on net trades imposed. Robust non-existence examples illustrate that our characterisation is tight, that is, without our survival condition, existence of competitive equilibria may fail.

In the final chapter we study irreducibility in pure exchange economies with countable sets of individuals and economies. The argument for the existence of competitive equilibria in infinite dimensional economies proceeds by considering a sequence of finite truncated economies which tend to the full economy in the limit. Existence is established by proving that the limit of a sequence of prices and allocations, corresponding to the competitive equilibria of the finite truncated economies, constitutes a competitive equilibrium for the full economy. In the literature, authors assumed irreducibility of the full economy (Burke (1988); Geanakoplos and Polemarchakis (1991)) and/or imposed some form of irreducibility condition on the finite truncated

economies, (Wilson (1981); Balasko, Cass, and Shell (1980)).

We consider a sequence of finite truncated economies which converge to the full exchange economy in the limit. To this sequence we associate a sequence of price graphs. Modifying the definition of C-irreducibility to allow for a countable infinity of individuals, we illustrate via an example that an economy which is the limit of some sequence of finite C-irreducible economies (a sequentially C-irreducible economy) will not necessarily be C-irreducible. The implication of this result is that it is not sufficient for the existence of competitive equilibria to impose the C-irreducibility condition on the finite truncated subeconomies alone. However, by deriving conditions under which an increasing sequence of strongly connected graphs will converge to a graph which is itself strongly connected, we are able to provide conditions under which sequential C-irreducibility does indeed imply C-irreducibility. Essentially, the condition implies that the preferences and endowments of individuals be such that the economy can be approximated by an increasing sequence of finite truncated economies in which links between individuals in any given finite economy cannot be arbitrarily broken as "new" individuals are introduced. Next we show by an example that a C-irreducible economy may not necessarily be approximated by any sequence of finite C-irreducible

economies. Finally, by deriving conditions under which a strongly connected infinite graph can be approximated by a sequence of finite strongly connected graphs, we are able to provide a condition on the collection of price graphs of the full economy to guarantee the existence of some sequence of finite C-irreducible economies, which has the full economy as its unique limit. The key restriction here is that there must be sufficient arcs *across* truncated economies. The results of this chapter, lead us to conclude that when working with infinite dimensional economies it is far less restrictive to prove the existence of competitive equilibria by imposing some form of irreducibility condition on the full economy alone.

The use of graph theory in general equilibrium analysis has been explored by Rosenblatt (1957), Eaves (1985), and more recently Maxfield (1997). Rosenblatt develops results for the properties of the graphs of Minkowski-Leontief matrices (a finite nonnegative square matrix A such that each row sum is less than or equal to 1) to provide a complete characterisation of solutions to linear input-output models. In an input-output model, entries in a Minkowski-Leontief matrix represent input-dependencies between industries. In the graph theoretic representation, vertices represent industries, and a directed arc $v_i v_j$ exists if and only if industry i purchases or procures input from

industry j. Circular flows - a la Quesnay ("Analyse du tableau economique") - and feed-back input-dependencies between industries are captured by what Rosenblatt calls cyclic nets, which are in fact strongly connected subgraphs. A cyclic net H of a graph G is said to be closed if (a) every cyclic net of G is either a subgraph of H or has no vertex in common with H, and (b) every vertex of G which is attainable from a vertex in H is contained in the vertex set of H. In figure 1 over leaf, although H is a closed cyclic net of G, H' is not since v_2 is not an element of the vertex set of H' but is attainable from vertices in H'.

Rosenblatt shows that if the Minkowski-Leontief matrix contains no null rows - which in terms of the graph-theoretic representation implies that each vertex is connected to at least one other vertex - then the associated graph contains at least one closed cyclic net. It follows then that either the graph itself is a closed cyclic net, in which case each industry is connected to every other industry, or else all closed cyclic nets are proper subgraphs, in which case the economy can be decomposed into distinct sub-economies, where at least one sub-economy is strongly connected. Using results from the theory of finite-dimensional stationary Markov chains (with a discrete time parameter), the author proves that solutions to the input-output system always exist

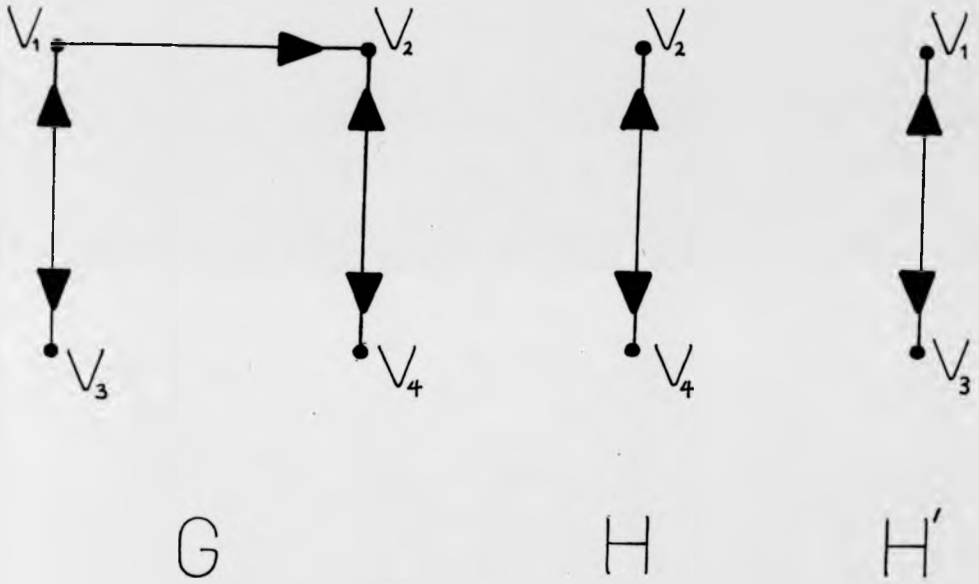

Figure 1:

in which only industries in strongly connected sub-economies operate at a positive level of activity.

Eaves (1985) provides necessary and sufficient conditions for the existence of competitive equilibria in pure exchange economies with Cobb-Douglas utilities. Although the author does not explicitly appeal to graph theoretic techniques, he develops properties of matrices which have a graph theoretic interpretation. He demonstrates that an equilibrium exists in his model if and only if the matrix formed from the product of the endowment matrix and the transpose of the matrix of utility exponents has what he calls "symmetric access." A commodity l is said to access a commodity $l' \neq l$ if there exists some sequence of commodities $\{l = g_1, g_2, ..., g_n, ..., g_N = l'\}$ and some sequence of individuals $\{i_1, i_2, ..., i_m, ..., i_M\}$ such that agent i_m is endowed with commodity g_n and desires commodity g_{n+1} for $n = 1, ..., N$. The matrix has symmetric access if for every pair of commodities either they access each other or neither accesses the other. A sufficient condition for the existence of a competitive equilibrium is that the matrix have full access, that is, each commodity accesses the other.

Work most closely related to this thesis is Maxfield (1997). Maxfield also uses graph theory to provide results for weakening the interior endowment

assumption of Arrow and Debreu, in finite economies. He obtains similar results to those reported here in section 2.3.1 of chapter 2, but for a limited class of economic models. He defines an *economy graph* in which vertices represent individuals and firms, and arcs link the sources of commodities and profits to users of commodities and recipients of profits. An arc exists between two individuals when one has a tradeable endowment of at least one commodity for which the other is non-satiable. (According to Maxfield, an individual is non-satiable on a set of commodities $\hat{L}$, if when the individual has strictly positive income and the price of any one commodity in the set $\hat{L}$ is non-positive, the individual demands an infinitely large amount of some commodity, not necessarily in $\hat{L}$). An arc exists from an individual i to a firm k whenever either i has a tradeable endowment of at least one commodity which is a possible input to production for firm k, or i owns a positive share of profits of firm k. An arc exists from firm k to firm k' whenever k has at least one output commodity which is a possible input for firm k'. Finally, an arc exists between firm k and individual i whenever firm k has at least one output commodity for which i is non-satiable. Maxfield shows that strong connectedness of the economy graph is a sufficient condition for the existence of a competitive equilibrium. Through examples, he shows

that his alternative condition is neither stronger nor weaker than McKenzie's irreducibility.

The layout of the rest of the thesis is as follows. In chapter 2, we present our results for finite production economies. In chapter 3 we extend the analysis to economies with an incomplete set of asset markets. In chapter 4, we study a more general model of restricted participation, with different individuals potentially having access to different subsets of markets. Finally, in chapter 5, we consider economies with countable sets of commodities and individuals.

Chapter 2

Finite Production Economies

2 Finite Production Economies[9]

2.1 Introduction

In this first chapter, we focus on finite production economies. In order to prove the existence of competitive equilibria when individuals have weakly monotone preferences and boundary endowments, some condition is required to ensure that at any given prices, if at least one individual in the economy has positive income, then so do all the others, which then guarantees that any quasi-equilibrium is also a competitive equilibrium for the economy. To this effect, McKenzie (1959,1961) proposes a condition he calls irreducibility. Arrow and Hahn (1971) appeal to a similar condition, resource relatedness.

We apply graph theoretic notions - from the theory of finite directed graphs - to finite production economies to develop an alternative survival condition, C-irreducibility, which is also sufficient to guarantee that at quasi-equilibrium prices each individual's budget set has a non-empty interior. C-irreducibility is therefore a sufficient condition for the existence of competitive equilibria, under standard Arrow-Debreu assumptions, when individuals have weakly monotone preferences and boundary endowments. The added appeal

[9]**This chapter is co-authored with Sayantan Ghosal.**

of C-irreducibility over either irreducibility or resource relatedness is two-fold. Firstly, C-irreducibility does not require that individual preferences be strongly quasi-concave. Secondly, C-irreducibility allows an operational method for testing using well-known efficient computational algorithms.

It is well known that resource relatedness implies irreducibility. However, the relative merits of C-irreducibility and irreducibility, in terms of the class of economies for which each is sufficient for existence, is unclear in that neither implies the other. By reinterpreting the irreducibility condition in terms of restrictions on graphs, we are able to highlight a particular modification in the definition of C-irreducibility, to generate an alternative survival condition (for want of a better name we call it C'-irreducibility), sufficient for the existence of competitive equilibria for a substantially larger class of economies than McKenzie's irreducibility. The sense in which C'-irreducibility is weaker than irreducibility is that whereas irreducibility requires a particular relationship to hold between every pair of individuals at every feasible allocation, C'-irreducibility only requires a relationship to hold between any pair of individuals for some subset of feasible allocations, which contains the set of quasi-equilibrium allocations.

The layout of the chapter is as follows. In section 2.2 we present our exten-

sions of certain basic definitions and results from the theory of finite directed graphs. Whilst many of the results of this section are standard, a number of the lemmas are original and have been developed especially for our analysis of exchange economies. In section 2.3.1, we apply the graph theoretic notions to a standard finite production economy. Here we develop our C-irreducibility condition, and use it to prove the existence of competitive equilibria. In section 2.3.2, we highlight the relationship between C-irreducibility and irreducibility, and present the weaker condition of C'-irreducibility. We show that irreducibility implies C'-irreducibility, and demonstrate by an example that the reverse is not necessarily true. In section 2.3.3 we compare irreducibility and resource relatedness. Characterising each of these conditions in terms of restrictions on graphs allows us to derive conditions under which they are in fact equivalent. Finally, in section 2.4 we briefly conclude the results of the chapter.

2.2 Properties of directed graphs

This section extends some basic definitions and results from the theory of finite directed graphs (see for example Harary, Norman and Cartwright (1965), or Busacker and Saaty (1965)).

Definition 2 *(directed graph) Let V be a non-empty set whose elements may be labelled $v_i, i = 1, ...N$ and let A be a binary relation on V, that is, a set of ordered pairs of elements of V. The pair $\Gamma = (V, A)$, where elements of V are vertices and elements of A are arcs, is called a directed graph. An arc directed from v_i to v_j is denoted $v_i v_j$.*

Definition 3 *(path) A path in a directed graph is an ordered collection of arcs and vertices $\{v_o, a_1, v_1, ..., a_n, v_n\}$ in which each arc a_i is $v_{i-1}v_i$, and all vertices are distinct.*

Definition 4 *(strongly connected graph) A directed graph is strongly connected if for every pair of distinct vertices (v_i, v_j) there exists a path connecting v_i to v_j and a path connecting v_j to v_i.*

Note that there are a number of ways in which a graph with vertex set V may be strongly connected. For example, figure 2 shows four distinct strongly connected graphs, each with the same vertex set $V = \{v_1, v_2, v_3\}$.

Consider a directed graph $\Gamma = (V, A)$ and let V^1 and V^2 be such that $V^k \subset V$, $k = 1, 2$, and $V_1 \cap V_2 = \emptyset$. An arc directed from $V_1$to V_2 is said to exist if there is $v_i \in V_1$ and $v_j \in V_2$ such that the arc $v_i v_j \in A$.

Lemma 5 *A directed graph is strongly connected if and only if for every*

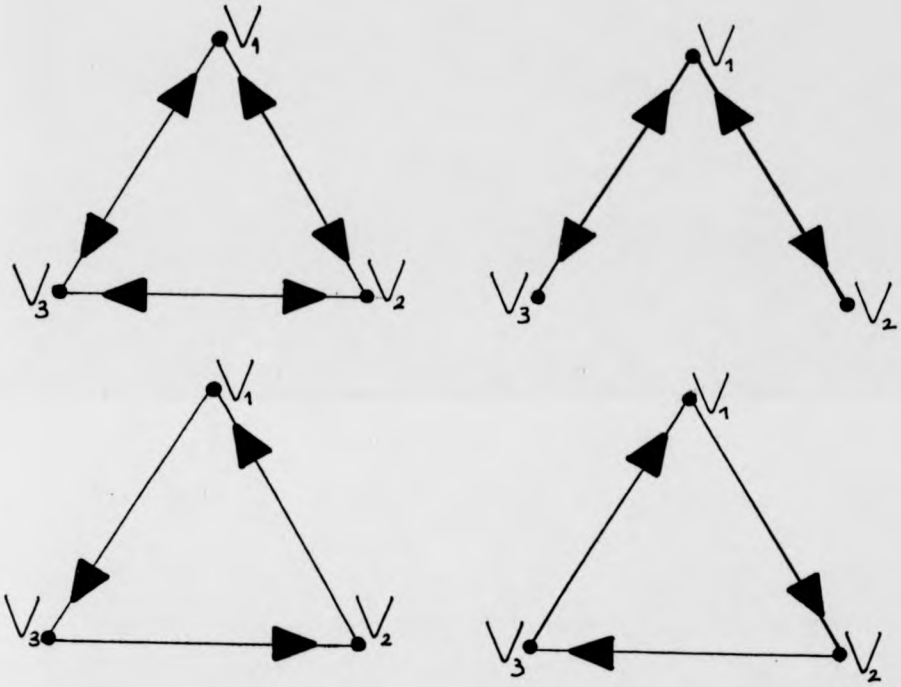

Figure 2:

non-trivial vertex partitioning $\{V^1, V^2\}$ *there exists at least one arc directed from* V^1 *to* V^2*and at least one arc directed from* V^2 *to* V^1.

Proof. Suppose that for some vertex partitioning $\{V^1, V^2\}$ there is no arc directed from V^1 to V^2. Then there can be no path from any vertex in V^1 to any vertex in V^2, and therefore the graph is not strongly connected. Conversely, suppose that the graph is not strongly connected. Then there are two vertices v_i and v_j in the graph such that no path exists from v_i to v_j. Consider the vertex partitioning $\{V^1, V^2\}$ such that V^1 be the set of vertices containing v_i and all vertices to which there exists a path directed from v_i. Then there exist no arcs directed from V^1 to V^2. ■

Definition 6 *(irreducible adjacency matrix) The adjacency matrix of a directed graph* Γ *is an* $N \times N$ *(where* N *is the number of vertices in* V*) nonnegative matrix* $M_\Gamma = [m_{ij}]$*, where* $m_{ij} = 0$ *if* $v_i v_j \notin \Gamma$*, and* $m_{ij} = 1$ *if* $v_i v_j \in \Gamma$*. Let* $M_\Gamma^k = [m_{ij}^k]$ *denote the kth power of* M_Γ*. The adjacency matrix* M_Γ *is irreducible if for every pair,* i, j*, of its index set, there exists a positive integer* $\alpha \equiv \alpha(i,j)$ *such that* $m_{ij}^\alpha > 0$*. Equivalently, the adjacency matrix* M_Γ *is irreducible if for every pair,* i, j*, there exists a sequence* $(i_0, i_1, i_2, ..., i_K)$*, for* $K \geq 1$*, (where* $i_0 = i$*, and* $i_K = j$*) from the index set* $\{1, 2, ..., N\}$ *such that* $m_{ii_1} m_{i_1 i_2} ... m_{i_{K-2} i_{K-1}} m_{i_{K-1} j} > 0$.

Lemma 7 *A directed graph Γ is strongly connected if and only if the adjacency matrix M_Γ is irreducible.*

Proof. By definition, the adjacency matrix is irreducible if and only if $m_{i_k i_{k+1}} = 1$, for all i_k in the sequence. Given the way the adjacency matrix is constructed, $m_{i_k i_{k+1}} = 1$ if and only if there exists a directed arc from i_k to i_{k+1} in the associated graph. Therefore, $m_{ii_1} m_{i_1 i_2} \ldots m_{i_{K-2} i_{K-1}} m_{i_{K-1} j} > 0$ implies that there exists a path from i to j. Since irreducibility of the adjacency matrix requires that this be the case for all i, j, the associated graph is strongly connected if and only if the adjacency matrix is irreducible.

■

Definition 8 *(subgraph) If $V' \subset V$ and $A' \subset A$, then $\Gamma' = (V', A')$ is a subgraph of $\Gamma = (V, A)$.*

Definition 9 *(spanning subgraph) A spanning subgraph $\Gamma' = (V', A') \subset \Gamma = (V, A)$ is a subgraph with the same set of vertices, i.e. $V' = V$.*

Lemma 10 *Every strongly connected directed graph has at least one spanning subgraph which is strongly connected.*

Proof. It suffices to note that every strongly connected directed graph contains the strongly connected spanning subgraph which is the graph itself.

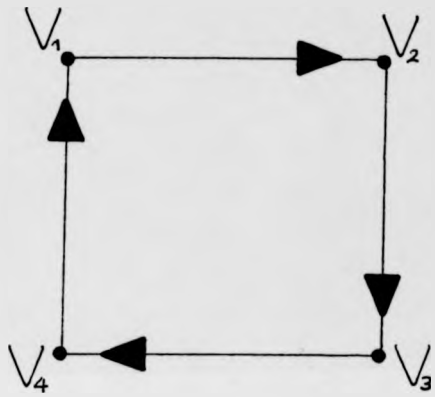

Figure 3:

Definition 11 *(cycle) A cycle is a path in which the initial vertex and the terminal vertex are the same.*

Definition 12 *(spanning cycle) A spanning cycle is a cycle which features all $v \in V$.*

Figure 3, illustrates a spanning cycle for a graph with vertex set $\{v_1, v_2, v_3, v_4\}$.

Lemma 13 *The least number of arcs required to strongly connect a directed graph with n vertices is n.*

Proof. Consider a set of vertices V. Let A_V be a collection of arcs which strongly connects V such that there is exactly one arc between any two distinct vertices v_i and v_j. Then $\Gamma = (V, A_V)$ is a spanning cycle and is strongly connected. Therefore, if any $v_i \in V$ is removed, then the arcs $v_{i-1}v_i$ and v_iv_{i+1} are also removed. The graph $\Gamma = (V/v_i, A_V/\{v_{i-1}v_i, v_iv_{i+1}\})$ fails to be strongly connected. ■

2.3 Economies and graphs

2.3.1 C-irreducibility and competitive equilibria

Consider an exchange economy E, with individuals denoted by $i \in I = \{1, ..., I\}$, firms denoted by $k \in K = \{1, ..., K\}$, and commodities denoted by $l \in L = \{1, ..., L\}$, where I, K, and L are non-empty, finite sets.

The commodity space, denoted $\Re^L$, is the Euclidean space with dimension equal to the number of commodities.

Trades in commodities are denoted by, $z = (z_1, ..., z_L) \in \Re^L$.

An individual, i, is characterized by a pair, (Z^i, u^i), of a feasible trade set, $Z^i \subset \Re^L$, and a utility function, $u^i : Z^i \to \Re$.

For any non-empty subset of the set of individuals $\bar{I} \subseteq I$, let $Z^{\bar{I}} = \Sigma_{i \in \bar{I}} Z^i$.

The aggregate domain of trades in commodities is denoted Z.

Each firm has a production set $X^k \subset \Re^l$, where an element $x^k \in X^k$ is a netput vector. Let $X = \sum_{k=1}^{K} X^k$ denote the aggregate production set.

At various points in the paper we will indicate the use of some or all of the following assumptions.

*A*1 For each $i \in I$, the set of feasible trades, Z^i, is closed, convex, bounded below, $(\exists \underline{z}^i \in Z^i : \underline{z}^i \leq 0,\ \underline{z}^i \neq 0,\ z^i \geq \underline{z}^i\ \forall z^i \in Z^i, and\ z^i \ll \underline{z}^i\ \forall z^i \notin Z^i)$ and allows for free disposal $(z \in Z^i,\ z' \geq z \Rightarrow z' \in Z^i)$.

*A*2 Autarchy is feasible $(0 \in Z^i)$.

*A*3 The utility function, u^i, is continuous.

*A*4 The utility function, u^i, is quasi-concave $(u^i(z') \geq u^i(z) \Rightarrow u^i(\lambda z' + (1 - \lambda)z) \geq u^i(z),\ 0 \leq \lambda \leq 1)$.

*A*5 The utility function, u^i, satisfies local non-satiation, $(\forall z \in Z^i,\ \forall \varepsilon > 0,\ \{z' \in Z^i : u^i(z') > u^i(z), \mid z' - z \mid < \varepsilon\} \neq \emptyset)$.

*A*6 The utility function, u^i, is weakly monotonically increasing $(\forall z \in Z^i,\ z' \geq z \Rightarrow u^i(z') \geq u^i(z))$.

*A*7 X is a closed convex cone.

The assumption that X be a closed convex cone can be shown to be mathematically equivalent to the assumption that firms production sets be convex, by introducing a firm specific input (McKenzie (1959), pp.66-67)

A8 $X \cap \Re^l_+ = \{0\}$.

Assumption A8 puts no real restriction on production. It simply implies that goods which can be produced at no cost be ignored.

A9 $Z^i \cap X \neq \emptyset$.

A9 (with A2) implies that any consumer can survive without trade.

A10 $0 \in IntZ \cap X$.

A10 implies that any price which supports X will have $pz < 0$ for some $z \in Z$. This means that, if prices are compatible with equilibrium, then some consumer has positive income, in the sense that they are not on the boundary of their net trade set.

An allocation is a pair, $(z^I, x) = \{z^i \in Z^i : i \in I,\ x^k \in X^k : k \in K\}$, of individuals' net trades, and an output vector. An allocation is feasible if and only if $\Sigma_{i \in I} z^i = x$.

Prices are $p \in \Re^L_+/\{0\}$.

Definition 14 *(competitive equilibrium (McKenzie (1981))A competitive equilibrium is a collection* (p^*, z^{*I}, x^*) *such that:*

(i) $z^{*i} \in Z^i$ and $p^* \cdot z^{*i} \leq 0$, and $u^i(z^{*i}) \geq u^i(z')$ for any $z' \in Z^i$ such that $p^* \cdot z' \leq 0$, $i = 1, ..., I$.

(ii) $x^* \in X$, $p^* \cdot x^* = 0$, and $p^* \cdot x' \leq 0$ for any $x' \in X$.

(iii)$\Sigma_{i \in I} z^{*i} = x^*$.

The definition of a quasi-equilibrium is as in definition 14, replacing (i) with (i') $z^{*i} \in Z^i$ and $p^* \cdot z^{*i} \leq 0$, and $u^i(z^{*i}) \geq u^i(z')$ for any $z' \in Z^i$ such that $p^* \cdot z' \leq 0$, or $p^* \cdot z^{*i} \leq p^* \cdot z'$ for all $z' \in Z^i$, $i = 1, ..., I$.

At prices $p \in \Re^L_+/\{0\}$, let $Z^i(p) = \{z^i \in Z^i \cap (-Z + X) : pz^i \leq 0\}$ denote the set of affordable trades for individual i which satisfy the aggregate feasibility constraint. $\hat{Z}^i(p) = \{z^i \in Z^i(p) : z^i \in \underset{z^i \in Z^i(p)}{\arg\max}\ u^i(z)\}$, denotes those affordable feasible trades for individual i which give him the most utility. We then define, $\Phi^i(p) = \{z \in Z^i : u^i(z^i + z) > u^i(z^i), z^i \in \hat{Z}^i(p)\}$ as the set of net trades which when added to some utility maximizing affordable feasible trade at prices p, make individual i better off. Local non-satiation (a sufficient condition) implies that this set is non-empty.

Definition 15 *(price graph) The price graph of the exchange economy* E *at prices* p*, denoted* $\Gamma(E(p))$*, is a collection of vertices* V *and arcs* A *such that*

each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$ and an arc directed from v_i to v_j exists whenever i can make j better off, in the sense that there is some $\bar{I} \subset I$[10]*, with $i \in \bar{I}$, $j \in I/\bar{I}$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(p) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(p) \neq \emptyset$.*

According to the above definition, in a three individual economy with $i = 1, 2, 3$, an arc $v_1 v_3$ exists if either (a) 1 can single handedly make 3 better off, (b) 1 and 2 can make 3 better off and 1 can make 2 better off. The following example illustrates when arcs exist in a price graph and when they do not.

Example 16 *Let $I = \{1, 2, 3\}$, $L = \{1, 2\}$, $X = \{0\}$, $u^1(z) = \min\{z_1^1 + 1, z_2^1 + 1\}$, $u^2(z) = \min\{z_1^2, z_2^2 + 1\}$, $u^3(z) = \min\{z_1^3 + 1, z_2^3\}$, $Z^1 = \{z : z_1^1 \geq -1, z_2^1 \geq -1\}$, $Z^2 = \{z : z_1^2 \geq 0, z_2^2 \geq -1\}$, $Z^3 = \{z : z_1^3 \geq -1, z_2^3 \geq 0\}$. At prices $p = (0, 1)$, the arc set of the price graph $\Gamma(E(p))$ is $\{v_1 v_2, v_1 v_3, v_2 v_1, v_2 v_3\}$. The arcs $v_1 v_2$ and $v_1 v_3$ exist because individual 1 can supply positive amounts of both commodities, and therefore make both individual 1 and individual 2 better off, in the required sense, at any feasible affordable utility maximising net trade (or indeed at any feasible net trade). To see that the arc $v_2 v_3$ exists,*

[10] *Here, without confusion, the symbol "$\subset$" denotes strict inclusion.*

note that at these prices $\bar{Z}^3(p) = \{(0,0)\}$. *Since* $(0,1) \in -Z^2$ *and* $u^3(0,1) > u^3(0,0)$, $(-Z^2 + X) \cap \Phi^3(p) = \emptyset$, *and the arc* v_2v_3 *exists. The arc* v_2v_1 *also exists because although* $(-Z^2 + X) \cap \Phi^1(p) = \emptyset$, $(-Z^{\{2,3\}} + X) \cap \Phi^1(p) \neq \emptyset$ *and* $(-Z^2 + X) \cap \Phi^3(p) \neq \emptyset$. *The arcs* v_3v_i, $i = 1, 2$, *do not exist because* $(-Z^3 + X) \cap \Phi^i(p) = \emptyset$.

Let C denote the collection of price graphs of economy E at all prices, $p \in \Re^L_+/\{0\}$. Note that the set of price graphs of an economy does not depend on the normalization chosen. That is, all p such that $\exists q$ with $p = \lambda q$ for some $\lambda > 0$ have the same price graph, $\Gamma(E(p))$.

We are now in a position to state the main definition of this section.

Definition 17 *(C-irreducibility) The economy is said to be C-irreducible if every member of the collection of price graphs C is strongly connected, that is, if* $\Gamma(E(p))$ *is strongly connected at each* $p \in \Re^L_+/\{0\}$.

Proposition 18 *If the economy E satisfies assumptions* $A1 - A10$ *above, and in addition is C-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $A1 - A10$[11], a quasi-equilibrium (p^*, z^{*I}, x^*) for the economy E exists (see Debreu (1962), McKenzie(1981)), at which

[11] Assumption ($A6$) is needed to ensure that at a quasi-equilibrium, prices are contained in $\Re^L_+/\{0\}$.

there is some individual $\hat{i}$ for whom $\exists z^{\hat{i}} \in Z^{\hat{i}}$ such that $p^* z^{\hat{i}} < 0$. Consider the partition of the set of vertices of $\Gamma(E(p^*))$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : \exists z^i \in Z^i \; p^* z^i < 0\}$, and $V^2 = \{v_i : \nexists z^i \in Z^i \text{ s.t. } p^* z^i < 0\}$. Note that, by definition of a quasi-equilibrium, $\forall v_i \in V^1$, $z^{*i} \in \hat{Z}^i(p^*)$. We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption, the graph $\Gamma(E(p^*))$ is strongly connected and so, by lemma 5, there exists an arc from V^2 to V^1, and vice versa. In particular, there exists $v_i \in V^2$ and $v_j \in V^1$ such that $v_i v_j$ exists. Given the way arcs are defined, the existence of $v_i v_j$ implies that $(-Z^{\tilde{I}} + X) \cap \Phi^j(p^*) \neq \emptyset$, with $i \in \tilde{I}$, $j \notin \tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $(-Z^{\tilde{I}'} + X) \cap \Phi^m(p) \neq \emptyset$. Since individual j is utility maximizing at z^{*I}, it follows that there is an $m_0 \in \tilde{I}$ such that $v_{m_0} \in V^1$. But then, there exists a sequence of individuals $(m_1, ..., m_K)$ such that $(-Z^i + X) \cap \Phi^{m_K}(p^*) \neq \emptyset$, and $(-Z^{m_k} + X) \cap \Phi^{m_{k-1}}(p^*) \neq \emptyset$. Therefore, each $v_{m_k} \in V^1$, the arcs $v_{m_k} v_{m_{k-1}}$, $v_i v_{m_K}$, $k = 1, ..., K$ exist, and thus $v_i \in V^1$, a contradiction. ■

To illustrate the role of C-irreducibility in the proof of existence, consider an economy with 4 individuals, $I = \{1, 2, 3, 4\}$. Assumptions A1-A10 guarantee the existence of a quasi-equilibrium $(\hat{p}, \hat{z}^I, \hat{x})$ at which at least one individual has positive income. To show that a quasi-equilibrium is a com-

petitive equilibrium, we must show that if one individual has positive income then so do all the other individuals in the economy. Suppose that individual 1 has positive income at $\hat{p}$. We show that if the economy is irreducible then individual 2 also has positive income, and since individual 2 was chosen arbitrarily then so do all others. C-irreducibility implies that at prices $\hat{p}$ there exists an arc in the graph $\Gamma(E(\hat{p}))$ from $V^2 = \{2, 3, 4\}$ to $V^1 = \{1\}$, which implies that either $2, 3$, or 4 has positive income. If it is individual 2, we are done. If not, then either 3 or 4 has positive income, and since C-irreducibility implies the existence of an arc from $V^2 = \{2\}$ to $V^1 = \{1, 3, 4\}$ in $\Gamma(E(\hat{p}))$ then 2 can make either 3 or 4 better off. Assume without loss of generality that player 3 has positive income. Then if 2 can make 3 better off we are done. If not, then 2 can make 4 better off. However, C-irreducibility also implies the existence of an arc from $V^2 = \{2, 4\}$ to $V^1 = \{1, 3\}$ in $\Gamma(E(\hat{p}))$. Therefore, either 2 and 4 can make 1 better off, or they can make 3 better off. Either scenario implies that either 2 or 4 has positive income. If it is 2 we are done. If it is 4, then since 2 can make 4 better off we are done.

Note that since V is finite, by assumption, C contains a finite number of graphs. For some practical models, it may be possible to infer all the members of C, in which case each graph could be tested for C-irreducibility. That is,

one could plot the graph and determine it's connectedness by inspection. It is however sufficient for existence of a competitive equilibrium to work with a collection which contains a strongly connected subgraph of each member of C.

Corollary 19 *If one can construct a set of strongly connected graphs $\widehat{C}$, such that each member of C has at least one member of $\widehat{C}$ as a spanning subgraph, then the economy E is C-irreducible.*

2.3.2 C-irreducibility and irreducibility

In this section, we show that C-irreducibility and McKenzie's (1961) irreducibility are different conditions, that is, neither implies the other. We then expand the definition of an arc in the price graph to define an alternative condition, C'-irreducibility, a sufficient condition for existence of competitive equilibria, which, we show, is weaker than McKenzie's irreducibility.

Let (z^I, x) be a feasible allocation and consider a trade z'which if added to individual i's allocation at z^I she is made better off i.e. $u^i(z^i + z') > u^i(z^i)$. Define $\Phi^i(z^I, x)$ as the collection of all such trades. Weak monotonicity (a necessary condition) and local non-satiation (a sufficient condition) imply that this set is non-empty. We define an allocation graph for the exchange

economy E as follows.

Definition 20 *(type 1 allocation graph) The type 1 allocation graph of economy E at allocation (z^I, x), denoted $\Gamma^1(E(z^I, x))$, is a collection of arcs A and vertices V, where each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$, and an arc directed from v_i to v_j exists whenever there is some $\tilde{I} \subset I$,with $i \in \tilde{I}$, $j \in I/\tilde{I}$ such that $(-Z^{\tilde{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $(-Z^{\tilde{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$.*

Let C^1 denote the collection of allocation graphs $\Gamma^1(E(z^I, x))$ at all feasible allocations.

McKenzie (1959, 1961) proposed the irreducibility condition as a way of overcoming the existence problem with boundary endowments. The formal definition of irreducibility is as follows.

Definition 21 *(irreducibility (McKenzie (1961)): Let $\{I^1, I^2\}$ be a non-trivial partition of the set of individuals such that $I^1 \cap I^2 = \emptyset$, $I^1 \cup I^2 = I$, $z^{I^k} = \Sigma_{i \in I^k} z^i$, and $Z^{I^k} = \Sigma_{i \in I^k} Z^i$, for $k = 1, 2$. Then the economy is irreducible if, however I^1 and I^2 may be selected, if $z^{I^1} = x - z^{I^2}$ with $x \in X$, $z^{I^1} \in Z^{I^1}$, and $z^{I^2} \in Z^{I^2}$, then there is also $x' \in X$, and $w \in Z^{I^2}$, such that $z'^{I^1} = x' - z^{I^2} - w$ and $u^i(z'^i) \geq u^i(z^i)$ for all $i \in I^1$, and $u^i(z'^i) > u^i(z^i)$ for*

some $i \in I^1$.

Under standard assumptions ($A1 - A10$) a quasi-equilibrium exists. Irreducibility is sufficient to guarantee that if one individual in the economy has positive income (true at a quasi-equilibrium) then so do all others, and quasi-equilibria are thus competitive equilibria. To see this, let $(\hat{p}, \hat{z}^I, \hat{x})$ be a quasi-equilibrium, and consider the partition $\{I^1, I^2\}$, where I^1 is the set of consumers such that there is $z^i \in Z^i$ with $\hat{p}z^i < 0$, and $I^2 = I/I^1$. We now show that irreducibility implies that I^2 is empty. Irreducibility implies that there is $x' \in X$, and $w \in Z^{I^2}$, such that $z'^{I^1} = x' - \hat{z}^{I^2} - w$ and $u^i(z'^i) \geq u^i(\hat{z}^i)$ for all $i \in I^1$, and $u^i(z'^{i_1}) > u^i(\hat{z}^{i_1})$ for some $i_1 \in I^1$. Since z'^{i_1} is preferred to $\hat{z}^{i_1}$ and i_1 is utility maximising, then $\hat{p}z'^{i_1} > 0$, and so $\hat{p}\ z'^{I^1} > 0$. But $\hat{p}\hat{z}^{I^1} = 0$ since individuals in I^1 are utility maximising, and $\hat{p}x = 0$ by definition at a quasi-equilibrium, which implies $\hat{p}z'^{I^1} = 0$. Therefore, since $\hat{p}x' \leq 0$, $pw < 0$. But $w \in Z^{I^2}$, so some individual in I^2 has positive income. This contradicts the definition of I^2, therefore I^2 is empty.

Lemma 22 *The economy E is irreducible if and only if every member of C^1 is strongly connected.*

Proof. Assume the economy E is irreducible. First notice that by setting

$I^2 = \{i\}$, $I^1 = \{I/i\}$, at any feasible allocation (z^I, x) there is some $i' \in I$ such that $(-Z^{i'} + X) \cap \Phi^{i'}(z^I, x) \neq \emptyset$. Now consider a non-trivial partition of the set of individuals $\{I^1, I^2\}$. At any feasible allocation (z^I, x), irreducibility implies that there exists $j \in I^1$ such that $(-Z^{I^2} + X) \cap \Phi^j(z^I, x) \neq \emptyset$. Next, we claim that there exists $i \in I^2$ such that the arc $v_i v_j$ exists in the associated allocation graph $\Gamma^1(E(z^I, x))$. That is, we claim that there is $\bar{I} \subseteq I^2$ with $i \in \bar{I}$, such that $(-Z^{\bar{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$. We can construct such a subset $\bar{I}$ as follows. First, recall that there is some $i' \in I$ such that $(-Z^{i'} + X) \cap \Phi^{i'}(z^I, x) \neq \emptyset$. If $i' \in I^1$, set $\bar{I} = i$, and $j = i'$. If on the other hand $i' \in I^2$, consider $\bar{I}_1 = \{i, i'\}$. By irreducibility, either $\exists\ j \in I^1$ such that $(-Z^{\bar{I}_1} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, in which case set $\bar{I} = \bar{I}_1$, or $\exists i'' \in I^2/\bar{I}_1$ such that $(-Z^{\bar{I}_1} + X) \cap \Phi^{i''}(z^I, x) \neq \emptyset$, in which case consider $\bar{I}_2 = \{i, i', i''\}$. By irreducibility, either $\exists\ j \in I^1$ such that $(-Z^{\bar{I}_2} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, in which case set $\bar{I} = \bar{I}_2$, or $\exists i''' \in I^2/\bar{I}_2$ such that $(-Z^{\bar{I}_2} + X) \cap \Phi^{i'''}(z^I, x) \neq \emptyset$, in which case consider $\bar{I}_3 = \{i, i', i'', i'''\}$. By repeating this process if necessary, we can find $\bar{I}_n$ such that $\exists\ j \in I^1$ such that $(-Z^{\bar{I}_n} + X) \cap \Phi^j(z^I, x) \neq \emptyset$ and set $\bar{I} = \bar{I}_n$. By construction, this subset satisfies the condition that for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in$

$\bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$. Therefore, the arc $v_i v_j$ exists in the associated allocation graph $\Gamma^1(E(z^I, x))$. By irreducibility, this is true for any non-trivial partition of the set of individuals $\{I^1, I^2\}$, and for any feasible allocation (z^I, x). Therefore, by lemma 5, the allocation graph $\Gamma^1(E(z^I, x))$ is strongly connected, for all feasible allocations (z^I, x). Conversely, assume that the allocation graph $\Gamma^1(E(z^I, x))$ is strongly connected, for all feasible allocations (z^I, x). By lemma 2.4, at a feasible allocation (z^I, x), for every non-trivial vertex partitioning $\{I^1, I^2\}$, we can find some $i \in I^2$ and $j \in I^1$, such that, $\exists \bar{I} \subset I$, with $i \in \bar{I}$, such that $(-Z^{\bar{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ with $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$. Therefore, there exists $\bar{I} \subseteq I^2$ and $j \in I^1$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$. But then, there exists $w \in Z^{I^2}$ and $x' \in X$ such that $\tilde{z}^j = z^j - w + x'$ and $u^j(\tilde{z}^j) > u^j(z^j)$. By assumption $z^{I^1} = x - z^{I^2}$ with $z^{I^1} \in Z^{I^1}$, $z^{I^2} \in Z^{I^2}$ and $x \in X$. Define $z'^{j'} = z^{j'} \forall j' \in I^1/j$ and $z'^j = \tilde{z}^j$. Then, $z'^{I^1} = x' - z^{I^2} - w$ while, by construction, $u^{j'}(z^{j'}) \geq u^{j'}(z^{j'})$ for all $j' \in I^1$, and $u^j(z'^j) > u^j(z^j)$ as required. As this construction works at any feasible allocation (z^I, x), and by assumption $\Gamma^1(E(z^I, x))$ is strongly connected at every feasible allocation (z^I, x), the economy E is irreducible.

■

Comparison of C-irreducibility and the graph theoretic representation of irreducibility highlights that although the two conditions are closely related, neither implies the other. To see that irreducibility does not necessarily imply C-irreducibility, suppose the economy E is irreducible and consider the price graph of the economy at prices $p \in \Re^L_+/\{0\}$. Consider a non-trivial partition of the set of individuals $\{I^1, I^2\}$ with $I^1 = \{i\}$, $I^2 = I/\{i\}$, and a feasible allocation (z^I, x) such that $z^i \in \hat{Z}^i(p)$. Irreducibility implies that $\exists j \in I^2$ such that $j \in \tilde{I}$, $i \in I/\tilde{I}$, $(-Z^{\tilde{I}} + X) \cap \Phi^i(z^I, x) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $j \in \tilde{I}$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $(-Z^{\tilde{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, and since $\Phi^i(z^I, x) \subseteq \Phi^i(p)$, by construction, an arc exists from I^2 to I^1. Now consider a feasible allocation (z'^I, x) which implies an affordable utility maximising net trade for some $j \in I^2$. Irreducibility implies that $\exists k \in I^2$ such that $i \in \tilde{I}$, $k \in I/\tilde{I}$, $(-Z^{\tilde{I}} + X) \cap \Phi^k(z^I, x) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $(-Z^{\tilde{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$. However, there is no guarantee that $k = j$, and no guarantee therefore that an arc exist from I^1 to I^2, so that the price graph at $p \in \Re^L_+/\{0\}$ may fail to be strongly connected, in which case C-irreducibility fails.

To see that the reverse is not true either, that is, C-irreducibility does not necessarily imply irreducibility, consider the following example:

Example 23 *Let* $I = \{1,2,3\}$, $L = \{1,2\}$, $X = \{0\}$, $u^1(z) = \min\{z_1^1 + 1, z_2^1+1\}$, $u^2(z) = z_2^2+1+\min\{\alpha z_1^2, z_2^2+1\}$, $u^3(z) = z_1^3+1+\min\{z_1^3+1, \alpha z_2^3\}$, $0 < \alpha < 1/4$, $Z^1 = \{z : z_1^1 \geq -1,\ z_2^1 \geq -1\}$, $Z^2 = \{z : z_1^2 \geq 0,\ z_2^2 \geq -1\}$, $Z^3 = \{z : z_1^3 \geq -1,\ z_2^3 \geq 0\}$. *Note that as individual* 1 *is endowed with positive amounts of both commodities, it follows that for all* $p \in \Re^2_+/\{0\}$ *the arcs* v_1v_2 *and* v_1v_3 *exist in the associated price graph. Also note that since* $(1,1) \in -Z^{\{2,3\}}$, $(-Z^{\{2,3\}}+X)\cap\Phi^1(p) \neq \emptyset$, *for all* $p \in \Re^2_+/\{0\}$. *When* $p_1 = 0$, $p_2 > 0$, *then* $\hat{Z}^3(p) = \{(0,0)\}$, *and since* $(0,1) \in -Z^2$ *and* $u^3(0,1) = 1+\alpha > 1 = u^3(0,0)$ *the arc* v_2v_3 *exists. Although individual* 2 *alone cannot make individual 1 better off at these prices, (as* $(-Z^2+X)\cap\Phi^1(p) = \emptyset$*), we have established that* $(-Z^{\{2,3\}}+X)\cap\Phi^1(p) \neq \emptyset$ *and* $(-Z^2+X)\cap\Phi^3(p) \neq \emptyset$, *so that the arc* v_2v_1 *does in fact exist. Since* $\hat{Z}^2(p) = \{(2,0)\}$, $(1,0) \in -Z^3$, *and* $u^2(3,0) = 1+3\alpha > 1+2\alpha = u^3(2,0)$, *the arcs* v_3v_2 *and* v_3v_1 *also exist, and the graph is therefore strongly connected. By a symmetric argument, when* $p_1 > 0$, $p_2 = 0$, *the graph is also strongly connected. It remains to check the arc set of the price graph when* $p_1 > 0$, $p_2 > 0$. *Note first that individual rationality implies that* $\alpha z_1^2 \leq z_2^2 + 1$. *Individual rationality further implies that if* $\alpha z_1^2 = z_2^2 + 1$, *then it must be that* $u^1(z^1) = 2(z_2^2+1) = 2\alpha z_1^2 \geq 1 = u^1(0,0)$, *a contradiction since* $0 < \alpha < 1/4$, *and at any feasible allocation*

$z_1^2 \leq 2$. It follows that at any feasible affordable utility maximising bundle $\alpha z_1^2 < z_2^2 + 1$, and therefore, since $(1,0) \in -Z^3$, the arcs v_3v_2 and v_3v_1 exist. A similar argument establishes the existence of v_2v_3 and v_2v_1, and the graph is therefore strongly connected. With reference to McKenzie's definition of irreducibility, consider the partition $I^1 = \{1,2\}$, $I^2 = \{3\}$. At the feasible allocation (z^I, x), with $z^1 = (0,0)$, $z^2 = (1,-1)$, $z^3 = (-1,1)$, $x = 0$, neither individual 1 nor individual 2 can be made better off by any feasible trade from $(-Z^3 + X)$, and thus the economy is not irreducible. However, a competitive equilibrium exists: $p^* = (1,1)$, $z^{*1} = (0,0)$, $z^{*2} = (0,0)$, $z^{*3} = (0,0)$.

Notice that what matters in this economy, other than the fact that individuals have Leontief preferences and boundary endowments, is the size of the preference parameter α relative to the aggregate endowment of commodities 1 and 2. Utility maximising individuals with these Leontief preferences would always wish to consume the two goods in some fixed proportions. When this is the case, they can not be made better off by receiving more of a single good. At such net trades, individuals 2 and 3 would not be able to make each other better off. However, the relationship of aggregate endowments to the preference parameter α is such that it is never possible for both 2 and 3 to consume the two goods in the desired proportions. C-irreducibility holds

then, because whatever the price vector, and for all i and j, i can make j better off (in the sense of definition 15) at any net trade which is individually rational for j. Irreducibility, on the other hand, fails because we are required to consider all feasible allocations.

By modifying the definition of an arc in the price graph, we can however derive an alternative condition, sufficient for existence, which is in fact weaker than irreducibility. We now define two types of arcs. The first type of arcs are just those used in the definition of C-irreducibility.

Definition 24 *(Type 1 arcs) Consider a collection of vertices V where each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$. At prices p a type 1 arc directed from v_i to v_j exists whenever there is some $\bar{I} \subset I$, with $i \in \bar{I}$, $j \in I/\bar{I}$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(p) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(p) \neq \emptyset$. At prices p, denote the set of such arcs as $A_1(p)$.*

The second type of arc exists under less restrictive conditions.

Definition 25 *(Type 2 arcs) Consider a collection of vertices V where each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$. At prices p a type 1 arc directed from v_i to v_j exists whenever there is some $k_0 \in \bar{I}$, $j \in I/\bar{I}$ such*

that $(-Z^{\bar{I}} + X) \cap \Phi^j(p) \neq \emptyset$, *and for any* $\bar{I}' \subset \bar{I}$ *such that* $k_0 \in \bar{I}'$, *there is some* $m \in \bar{I}/\bar{I}'$ *such that* $(-Z^{\bar{I}'} + X) \cap \Phi^m(p) \neq \emptyset$, *and in addition, for any feasible allocation* (z^I, x) *such that* $z^j \in \bar{Z}^j(p)$, *there is some sequence of individuals* $(k_1, k_2, ..., k_N)$, *such that (a)* $i \in \bar{I}$, $k_N \in I/\bar{I}$ *with* $(-Z^{\bar{I}} + X) \cap \Phi^{k_N}(z^I, x) \neq \emptyset$ *and for any* $\bar{I}' \subset \bar{I}$ *such that* $i \in \bar{I}'$, *there is some* $m \in \bar{I}/\bar{I}'$ *such that* $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x)$, *and (b)* $\forall k_n$, $k_n \in \bar{I}, k_{n-1} \in I/\bar{I}$ *with* $(-Z^{\bar{I}} + X) \cap \Phi^{k_{n-1}}(z^I, x) \neq \emptyset$ *and for any* $\bar{I}' \subset \bar{I}$ *such that* $k_n \in \bar{I}'$, *there is some* $m \in \bar{I}/\bar{I}'$ *such that* $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x)$. *At prices* p, *denote the set of such arcs as* $A_2(p)$.

Consider a 3 individual economy, with $i = 1, 2, 3$. For a type 1 arc to exists from v_1 to v_3 (at prices p), either individual 1 alone can make 3 better off at any feasible affordable utility maximising net trade, or else 1 and 2 can make 3 better off at any feasible affordable utility maximising net trade, and 1 can make 2 better off at any feasible affordable utility maximising net trade. The existence of a type 2 arc from 1 to 3 (at prices p) requires that individual 2 can make 3 better off at any feasible affordable utility maximising net trade, and that 1 can make 2 better off at any allocation which implies a feasible affordable utility maximising net trade for individual 3.

Definition 26 *(modified price graph) The modified price graph of the ex-*

change economy E at prices p, denoted $\Gamma'(E(p))$, is a collection of vertices V and arcs A such that each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$ and $A = A_1(p) \cup A_2(p)$.

Definition 27 *(C'-irreducibility) The economy is said to be C'-irreducible if $\Gamma'(E(p))$ is strongly connected at each $p \in \Re^L_+/\{0\}$.*

For ease of exposition, let $A_1(p)$ arcs be called *direct arcs*, and $A_2(p)$ arcs be called *indirect arcs*.

Proposition 28 *If the economy E satisfies assumptions $A1 - A10$ above, and in addition is C'-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $A1 - A10$, a quasi-equilibrium (p^*, z^{*I}, x^*) for the economy E exists (see Debreu(1962), McKenzie(1981)), at which there is some individual $\hat{i}$ for whom $\exists z^{\hat{i}} \in Z^{\hat{i}}$ such that $p^* z^{\hat{i}} < 0$. Consider the partition of the set of vertices of $\Gamma(E(p^*))$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : \exists z^i \in Z^i \; p^* z^i < 0\}$, and $V^2 = \{v_i : \nexists z^i \in Z^i \text{ s.t. } p^* z^i < 0\}$. Note that, by definition of a quasi-equilibrium, $\forall v_i \in V^1$, $z^{*i} \in \hat{Z}^i(p^*)$.We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption the graph $\Gamma(E(p^*))$ is strongly connected and so, by lemma 5, there exists an arc from V^2 to V^1, and vice versa. In particular, there exists $v_i \in V^2$ and

$v_j \in V^1$ such that $v_i v_j$ exists. Given the definition of C'-irreducibility, $v_i v_j$ may be a direct or an indirect arc. If $v_i v_j$ is a direct arc, this implies that $(-Z^{\bar{I}} + X) \cap \Phi^j(p^*) \neq \emptyset$, with $i \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(p) \neq \emptyset$. Since individual j is utility maximizing at z^{*I}, it follows that there is an $m_0 \in \bar{I}$ such that $v_{m_0} \in V^1$. But then, there exists a sequence of individuals $(m_1, ..., m_K)$ such that $(-Z^i + X) \cap \Phi^{m_K}(p^*) \neq \emptyset$, and $(-Z^{m_k} + X) \cap \Phi^{m_{k-1}}(p^*) \neq \emptyset$. Therefore, each $v_{m_k} \in V^1$, the arcs $v_{m_k} v_{m_{k-1}}$, $v_i v_{m_K}$, $k = 1, ..., K$ exist, and thus $v_i \in V^1$, a contradiction. If $v_i v_j$ is an indirect arc, then there exist $v_i, v_{k_0} \in V^2$ and $v_j \in V^1$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(p^*) \neq \emptyset$, with $k_0 \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $k_0 \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(p) \neq \emptyset$ (so that by the previous argument $v_{k_0} \in V^1$), and for any feasible allocation (z^I, x) such that $z^j \in \hat{Z}^j(p)$, there is some sequence of individuals $(k_1, k_2, ..., k_N)$, such that $(-Z^i + X) \cap \Phi^{k_N}(z^I, x) \neq \emptyset$ and $(-Z^{k_n} + X) \cap \Phi^{k_{n-1}}(z^I, x) \neq \emptyset$, so that each $v_{k_n} \in V^1$, $n = 1, ..., N$, and thus $v_i \in V^1$, a contradiction. ■

Notice then that C-irreducibility implies C'-irreducibility, but the reverse is not necessarily true. We now show that C'-irreducibility is weaker than McKenzie's irreducibility.

Proposition 29 *Irreducibility implies C'-irreducibility.*

Proof. Suppose the economy is irreducible. Consider prices $p \in \Re^L_+/\{0\}$, and a non-trivial partition of the set of vertices $\{V^1, V^2\}$. Consider the feasible allocation (z^I, x) which implies an affordable utility maximising net trade for individual j, where $v_j \in V^2$. Since p, $\{V^1, V^2\}$, and j were chosen arbitrarily, to establish that the economy is C'-irreducible, it is sufficient to show that there is some $v_i \in V^1$ such that the arc $v_i v_j$ exists in the modified price graph $\Gamma'(E(p))$. By irreducibility, either there exists $v_i \in V^1$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, with $i \in \bar{I}$, $j \in I/\bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, - in which case the arc $v_i v_j$ exists and we are done - or there exists $v_{k_0} \in V^2$ such that $(-Z^{\bar{I}} + X) \cap \Phi^j(z^I, x) \neq \emptyset$, with $k_0 \in \bar{I}$, $j \in I/\bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, in which case the arc $v_{k_0} v_j$ exists in the modified price graph $\Gamma'(E(p))$. Now, irreducibility implies that either there exists $v_i \in V^1$ such that $(-Z^{\bar{I}} + X) \cap \Phi^{k_0}(z^I, x) \neq \emptyset$, with $i \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, - in which case the arc $v_i v_j$ exists and we are done - or there exists $v_{k_1} \in V^2$ such that $(-Z^{\bar{I}} + X) \cap \Phi^{k_0}(z^I, x) \neq \emptyset$, with $k_1 \in \bar{I}$,

$k_0 \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $k_1 \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, in which case the arc $v_{k_1} v_j$ exists in the modified price graph $\Gamma'(E(p))$. If we repeat this process, we end up with a sequence of individuals $(k_0, k_1, k_2, ..., k_N)$, with $k_n \in I^1$, $n = 0, 1, ..., N$, such that $\forall k_n$, $k_n \in \bar{I}, k_{n-1} \in I/\bar{I}$ with $(-Z^{\bar{I}} + X) \cap \Phi^{k_{n-1}}(z^I, x) \neq \emptyset$ and for any $\bar{I}' \subset \bar{I}$ such that $k_n \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x)$. In addition, we have $k_0 \in \bar{I}$, $j \in I/\bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$. Consider the partition $\{I^{2_1}, I^{2_2}\}$ of I^2, where $I^{2_1} = \{k_0, k_1, k_2, ..., k_N\}$ and $I^{2_2} = I^2/I^{2_1}$. By construction, no arcs exist from I^{2_1} to I^{2_2} in the allocation graph $\Gamma^1(E(z^I, x))$. By irreducibility, an arc must exist from $I^1 \cap I^{2_2}$ to I^{2_1}. It follows therefore that there exists $v_i \in V^1$ such that $(-Z^{\bar{I}} + X) \cap \Phi^{k_n}(z^I, x) \neq \emptyset$, with $i \in \bar{I}$, $k_n \in I/\bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'} + X) \cap \Phi^m(z^I, x) \neq \emptyset$, for some k_n in the sequence, and therefore the arc $v_i v_j$ exists. ■

Given that C-irreducibility implies C'-irreducibility, example 23 also serves to illustrate that C'-irreducibility does not imply irreducibility. We can therefore conclude that C'-irreducibility is a weaker sufficient condition for the existence of competitive equilibria than is irreducibility. Intuitively, the sense

in which C'-irreducibility is weaker is that whereas irreducibility requires a particular relationship to hold between every pair of individuals at every feasible allocation, C'-irreducibility only requires a relationship to hold between any pair of individuals for some subset of feasible allocations, which contains the set of quasi-equilibrium allocations. While McKenzie's definition needs to be checked at each of a continuum of feasible allocations, C'-irreducibility needs to be checked at a continuum of possible prices (the dual of the commodity space). However, as corollary 19 makes clear, the added advantage of a graph theoretic representation is that C'-irreducibility need only be checked for a finite set of graphs. For small models, these graphs may be inferred and strong connectedness determined by inspection. For large, more complex models, this procedure is obviously problematic. However, even with such large models, C'-irreducibility lends itself to testability, where irreducibility does not. The adjacency matrix of an economy graph can easily be constructed using the relevant index sets of the economy. There are then well-known efficient computational algorithms which can be used to test for the irreducibility of the matrix (see for example Aho et al. (1983)). Also, C'-irreducibility works with quasi-concave utility functions, an added advantage over McKenzie (1959, 1981).

2.3.3 Irreducibility and resource relatedness

In this section we study the relationship between McKenzie's (1961) irreducibility and Arrow and Hahn's (1971) resource relatedness. It is well known that resource relatedness implies irreducibility (see McKenzie (1981)). The reverse relationship is however far from obvious, and remains an open question in the literature. However, by translating these two conditions into restrictions on appropriately defined economy graphs, we are able to highlight the relationship between the two, and in particular provide conditions under which the two conditions are equivalent.

Denote by $Z^i_- \subset Z^i$ the set of all feasible trades of individual i which are non-positive in all components, and strictly negative in some, $Z^i_- = \{z^i \in Z^i : z^i \leq 0,\ z^i_l \neq 0 \text{ for some } l \in L\}$. This set is non-empty by assumption. We now define a second type of allocation graph for the exchange economy E. This graph has an identical vertex set to the type 1 allocation graph, but differs in the way arcs are defined.

Definition 30 *(type 2 allocation graph) The type 2 allocation graph of economy E at allocation (z^I, x), denoted $\Gamma^2(E(z^I, x))$, is a collection of arcs A and vertices V, where each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$, and an arc directed from v_i to v_j exists whenever $-Z^i_- + X \cap \Phi^j(z^I, x) \neq \emptyset$.*

Let C^2 denote the collection of allocation graphs $\Gamma^2(E(z^I, x))$ at all feasible allocations $(z^I, x) \in F$.

Arrow and Hahn (1971) define a relationship between two individuals known as resource relatedness.

Definition 31 *(resource relatedness (Arrow and Hahn (1971)): Individual i is said to be resource related to individual j if, for every feasible allocation, (z^I, x), there exists an allocation (z'^I, x') and a vector $w \leq 0$, $w \neq 0$ such that,*

(i) $\Sigma_{i \in I} z'^i \leq x' - w$

(ii) $\Sigma_{i \in I} z'^i \in Z + w$

(iii) $u^i(z'^i) \geq u^i(z^i) \;\; \forall i \in I$

(iv) $u^j(z'^j) > u^i(z^j) \;\; j \in I$

(v) $w_l < 0 \Longleftrightarrow z_l^i < 0$

That is, (z'^I, x') would be a feasible allocation if the aggregate domain of trade were extended by w (where w is an increase only in those commodities of which individual i can supply positive amounts), all individuals are at least as well off under allocation (z'^I, x') as under (z^I, x), and individual j is made strictly better off.

We now show that if individual i is resource related to j, and j has positive income at quasi-equilibrium prices $\hat{p}$, then i also has positive income at $\hat{p}$. Resource relatedness implies that we can find a vector w and an allocation (z'^I, x') satisfying (i)-(v) with (z^I, x) replaced with the quasi-equilibrium allocation $(\hat{z}^I, \hat{x})$. Since $\hat{z}^i$ minimises $\hat{p}z^i$ subject to $u^i(\hat{z}^i) \geq u^i(z')$ for any $z' \in Z^i$, it follows from (iii) that $\hat{p}z'^i \geq \hat{p}\hat{z}^i$ for all $i \in I$. Since $\hat{p}z^j < 0$, it follows that $\hat{z}^j$ maximises $u^j(z^j)$ subject to $\hat{p}z^j \leq \hat{p}\hat{z}^j$. Therefore, $\hat{p}z'^j > \hat{p}\hat{z}^j$, and $\hat{p}\sum_{i\in I} z'^i > \hat{p}\sum_{i\in I} \hat{z}^i$. Profit maximisation implies that $\hat{p}\hat{x}^k \geq \hat{p}x'^k$, for all $k \in K$, thus $\hat{p}\sum_{k\in K} x'^k \leq \hat{p}\sum_{k\in K} \hat{x}^k$. At $(\hat{p}, \hat{z}^I, \hat{x})$, by definition, $\sum_{i\in I} \hat{z}^i = \sum_{k\in K} \hat{x}^k$, so $\hat{p}\sum_{i\in I} z'^i - \hat{p}\sum_{k\in K} x'^k > 0$. On the other hand, it follows from (i) that $\hat{p}\sum_{i\in I} z'^i - \hat{p}\sum_{k\in K} x'^k \leq -\hat{p}w$, so that $\hat{p}w < 0$. Since $\hat{p} > 0$, this means that for some l, $\hat{p}_l > 0$ and $w_l < 0$, which by (v) implies that $\underline{z}^i_l < 0$ and so $\hat{p}\underline{z}^i < 0$.

Individual i is said to be *indirectly resource related* to individual j if there is a sequence of individuals, i_n, $(n = 0, 1, .., N)$, with $i_0 = i$, $i_N = j$, and i_n resource related to i_{n+1}. The economy is said to be resource related if every individual is indirectly resource related to every other. If the economy E satisfies assumptions $A1 - A10$ and in addition is resource related, then a competitive equilibrium exists. That is, resource relatedness is sufficient to

guarantee that if one individual in the economy has positive income then so do all the others. To see this, let $(\bar{p}, \bar{z}^I, \bar{x})$ be a quasi-equilibrium at which individual j has positive income. It is enough to show that if i is indirectly resource related to j, and j has positive income, then i also has positive income. By definition, i_{N-1} is resource related to $i_N = j$, and so by the above argument i_{N-1} has positive income. Repeating this argument we have i_{N-2} with positive income, and so on and so forth, so that finally $i_0 = i$, has positive income.

Let us now restate the Arrow-Hahn condition in terms of partitions of the set of individuals. Consider a non-trivial partition of the set of individuals $\{I^1, I^2\}$, such that $I^1 \cap I^2 = \emptyset$, $I^1 \cup I^2 = I$. We say that I^1 is resource related to I^2 if there exists $i \in I^1$ and $j \in I^2$ such that i is resource related to j. We say that the economy is resource related if for any non-trivial partition of the set of individuals, I^1 is resource related to I^2.

Lemma 32 *Every individual is indirectly resource related to every other if and only if every member of C^2 has an identical strongly connected spanning subgraph.*

Proof. If there exists an arc directed from v_i to v_j in $\Gamma^2(E(z^I, x))$, $-Z^i_- + X \cap \Phi^j(z^I, x) \neq \emptyset$ at the feasible allocation (z^I, x). If there is an arc

directed from v_i to v_j in $\Gamma^2(E(z^I, x))$ for all feasible (z^I, x) then $-Z^i_- + X \cap \Phi^j(z^I, x) \neq \emptyset$ for all feasible (z^I, x). Setting $z'^{i'} = z^{i'}$ for all $i' \in I/\{j\}$ and $z'^j = z^j + w$, $w \in -Z^i_-$, individual i is then resource related to individual j. If i is resource related to j, $-Z^i_- + X \cap \Phi^j(z^I, x) \neq \emptyset$ for all (z^I, x); therefore the arc $v_i v_j$ exists in $\Gamma^2(E(z^I, x))$ for all (z^I, x). It follows that individual i is indirectly resource related to individual j if and only if there exists the same path directed from v_i to v_j in every allocation graph, $\Gamma^2(E(z^I, x))$, and every individual is therefore indirectly resource related to every other if and only if the allocation graph $\Gamma^2(E(z^I, x))$ is strongly connected and identical for all feasible allocations $(z^I, x) \in F$. ■

Lemma 33 *The economy is resource related if and only if every member of C^2 has an identical strongly connected spanning subgraph.*

Proof. By lemma 5, the allocation graph $\Gamma^2(E(z^I, x))$ is strongly connected if and only if for any non-trivial partitioning of the vertex set $\{V^1, V^2\}$ there exists at least one arc directed from V^1 to V^2 and at least one arc directed from V^2 to V^1. In the allocation graph $\Gamma^2(E(z^I, x))$ an arc exists from V^1 to V^2 if there exists $i \in I^1$ and $j \in I^2$ such that $-Z^i_- + X \cap \Phi^j(z^I, x) \neq \emptyset$, at the feasible allocation (z^I, x). By definition, therefore, the economy is

resource related if and only if the allocation graph $\Gamma^2(E(z^I, x))$ is strongly connected and identical for all feasible allocations (z^I, x). ■

Corollary 34 *The economy is resource related if and only if every individual is indirectly resource related to every other.*

It is well known (see McKenzie (1981)) that resource relatedness implies irreducibility. Indeed, it is evident that if an arc $v_i v_j$ exists in the graph $\Gamma^2(E(z^I, x))$, then $v_i v_j$ also exists in the graph $\Gamma^1(E(z^I, x))$. Comparison of the type 1 and 2 allocation graphs also highlights however that the reverse is not necessarily true. Nevertheless, the next proposition provides a set of sufficient conditions under which irreducibility and resource relatedness are equivalent. Let $\mathbf{1}_l = (0, ..., 1, ..., 0)$ be an L dimensional vector with 1 in the lth coordinate and zero elsewhere.

Proposition 35 *If utility functions are weakly monotone, and in addition satisfy $u^i(z+\mathbf{1}_l) > u^i(z)$ at some $z \in Z^i \Rightarrow u^i(z'+\mathbf{1}_l) > u^i(z')$ for all $z' \in Z^i$, then irreducibility and resource related are equivalent.*

Proof. It follows from the definition of an arc in each of the two types of allocation graph, that if an arc exists between two vertices in $\Gamma^2(E(z^I, x))$ then an arc exists between these vertices in $\Gamma^1(E(z^I, x))$, for any feasible

allocation (z^I, x). Strong connectedness of the allocation graph $\Gamma^2(E(z^I, x))$ therefore implies strong connectedness of the allocation graph $\Gamma^1(E(z^I, x))$, and hence, resource relatedness implies irreducibility. To see that when preferences are weakly monotone and for each $i \in I$, if $u^i(z + \mathbf{1}_l) > u^i(z)$ at some $z \in Z^i$ then $u^i(z' + \mathbf{1}_l) > u^i(z')$ for all $z' \in Z^i$, the reverse is also true, suppose that the economy is irreducible. Then, at any feasible allocation, (z^I, x), and for any partition of the set of individuals, $\{I^1, I^2\}$, there is some net trade, $z \in -Z^{\bar{I}}$, with $\bar{I} \subseteq I^2$, and some $x' \in X$ such that $(z + x') \in \Phi^j(z^I, x)$, where $j \in I^1$. With weak monotonicity if $(z + x') \in \Phi^j(z^I, x)$, then there exists some $z' \in -Z^{\bar{I}}_{-}$ such that $(z' + x') \in \Phi^j(z^I, x)$. But then there is some l such that $z'_l > 0$ and $u^i(z + \mathbf{1}_l) > u^i(z)$ for all $z \in Z^i$. This further implies that there is some $i \in \bar{I}$, with $z^j_l < 0$ and therefore some $\widehat{z} \in -Z^i$, and $x' \in X$ such that $(\widehat{z} + x') \in \Phi^j(z^I, x)$, at any feasible allocation (z^I, x), that is j is resource related to i, and therefore I^2 is resource related to I^1. Since irreducibility implies that this is true for every partition of the set of individuals, the economy is therefore resource related. ■

Note that the two conditions are not equivalent in economies with Leontief preferences since the condition that $u^i(z + \mathbf{1}_l) > u^i(z)$ at some $z \in Z^i \Rightarrow u^i(z' + \mathbf{1}_l) > u^i(z')$ for all $z' \in Z^i$ is violated. We conclude that under

standard assumptions, C'-irreducibility is weaker than irreducibility, which is weaker than resource relatedness.

2.4 Conclusion

The graph theoretic representation has allowed us to develop a weaker survival condition than any proposed in the existing literature to guarantee (that quasi-equilibria are competitive equilibria and therefore) the existence of competitive equilibria in finite production economies with boundary endowments. Whereas irreducibility requires a particular relationship to hold between every pair of individuals at every feasible allocation, C'-irreducibility only requires a relationship to hold between any pair of individuals for some subset of feasible allocations, which contains the set of quasi-equilibrium allocations. While McKenzie's definition needs to be checked at each of a continuum of feasible allocations, C'-irreducibility need only be checked for a finite set of graphs. An added advantage of C'-irreducibility over irreducibility is that it works with quasi-concave utility functions.

Chapter 3

Finite Pure Exchange Economies with Incomplete Markets

3 Finite Pure Exchange Economies with Incomplete Markets

3.1 Introduction

In economies with complete markets, C-irreducibility is trivially satisfied if preferences are strictly monotone. However, this is no longer true with incomplete asset markets, since the incompleteness of the market may limit the possibility of agents attaining some feasible trades. In this chapter, we extend the analysis of C-irreducibility to pure exchange economies with incomplete asset markets. Additional restrictions with respect to the complete market case are required. These are joint restrictions on the asset structure and the distribution of endowments and the preferences of individuals. By modifying the definition of C-irreducibility to allow for the attainability of trades given the asset market structure, we obtain an analogous condition, $\bar{C}$-irreducibility, which guarantees existence for economies with incomplete markets.

Gottardi and Hens (1996) propose an extension of McKenzie's irreducibility condition - which they refer to as resource relatedness - to prove the existence of competitive equilibria in exchange economies with incomplete mar-

kets and boundary endowments. With a complete set of markets, Gottardi and Hens' and McKenzie's conditions are in fact equivalent. We characterise Gottardi and Hens' condition in terms of restrictions on graphs, to illustrate the relationship with $\bar{C}$-irreducibility. We show, with the aid of an example, that neither condition implies the other. However, a modification in the definition of the arc set of a price graph determines an alternative condition, $\bar{C}'$-irreducibility, which is sufficient for existence of competitive equilibria for a larger class of economies than Gottardi and Hens' irreducibility.

The layout of the chapter is as follows. In section 3.2.1 we develop our $\bar{C}$-irreducibility condition, and use it to prove the existence of competitive equilibria in economies with incomplete asset markets and boundary endowments. In section 3.2.2 we highlight the relationship between $\bar{C}$-irreducibility and the irreducibility condition proposed by Gottardi and Hens, and present the weaker condition of $\bar{C}'$-irreducibility. We show that irreducibility implies $\bar{C}'$-irreducibility, and demonstrate by an example that the reverse is not necessarily true. In section 3.2.3, using a graph theoretic characterisation, we offer a sufficient condition for an exchange economy with incomplete asset markets to be effectively complete. Finally, in section 3.2.4 we briefly conclude the results of the chapter.

3.2 Asset markets and graphs

3.2.1 $\bar{C}$-irreducibility and competitive equilibria

In this section, we study exchange economies with incomplete asset markets. As in the previous chapter, there is a set I of individuals who consume and trade commodities in the set L. However, in this chapter we ignore production due to the conceptual difficulty that with incomplete markets it is no longer obvious what the objective function of the firm should be. Traditionally it is assumed that a firm's objective is to maximise profits. The foundation for profit maximisation is the Fisher separation theorem (see Milne, (1974)). If markets are complete, then the Fisher separation theorem implies that shareholders will be unanimous with respect to the optimal production plan for the firm, and the assumption of profit maximisation is therefore justified. However, if asset markets are incomplete, then generically there is disagreement amongst shareholders. The assumption that owners will wish to maximise profits is no longer appropriate. It is not clear however what the alternative should be. This question is beyond the scope of this thesis.

There is a set, S, of states of nature with $S = \{1, ..., S\}$, a finite non-

empty set. There are two periods $t = 0, 1$. In $t = 1$ some state $s \in S$ of the world is realized. Let l_s denote commodity l in state s. In state s, commodity 1_s is the designated numeraire commodity. At $t = 0$, individuals trade in real assets, denoted by $j = 1, ..., J$. Asset j pays R_s^j units of the numeraire commodity in state s. $R \in \Re^{S \times J}$denotes the asset returns matrix. Assets are in zero net supply and there is no consumption in period $t = 0$. We assume that the asset returns matrix R has full column rank i.e. there are no redundant assets. A trade in assets, an asset portfolio for an individual $i \in I$, is $y^i = (y_1^i, ..., y_J^i) \in \Re^J$. At $t = 1$, individuals trade in spot markets for all commodities in L in each state $s \in S$. A net trade in commodities for an individual $i \in I$ is $z^i \in \Re^{LS}$, describing his net trade for each commodity in each state of nature. The set of feasible commodity net trades for i is $Z^i \subset \Re^{LS}$. Each individual evaluates a net trade $z \in Z^i$ according to his utility function u^i where $u^i : Z^i \to \Re$.

We assume that u^i, Z^i and Z satisfy assumptions $A1 - A5$ of section 2.3.1. In addition, we assume strong monotonicity of individual preferences. Although this assumption could be relaxed, our motivations for maintaining the strong monotonicity of preferences are to focus on the implications of incompleteness for C-irreducibility, and for comparability with the exist-

ing literature. We show that even with strongly monotone preferences our condition is weaker than that proposed by Gottardi and Hens (1996).

B6 $\forall i \in I, \forall z, z' \in \Re^{LS}, z' > z \Longrightarrow u^i(z') > u^i(z)$.

An allocation of net trades is $z^I = (z^1, ..., z^I)$ where $z^i \in Z^i$ for each $i \in I$. A feasible allocation of net trades z^I satisfies in addition the condition that $\Sigma_{i \in I} z^i = 0$.

Let q denote the set of asset prices and let $p_s \in \Re^L_{++}$ denote spot prices in each state $s \in S$. Given prices $q, p_s, s = 1, ..., S$ each individual solves the following maximization problem:

$$Max_{\{y,z\}}\ u^i(z) \qquad s.t.\ \ qy \leq 0,\ p_s z_s \leq p_{s1} R_s y,\ s = 1, ..., S \quad (B)$$

A competitive equilibrium is a collection (q^*, p^*, y^*, z^{*I}) such that:

(i) Given (q^*, p^*), (y^{*i}, z^{*i}) solves (B) for each $i \in I$;

(ii) $\Sigma_{i \in I} y^{*i} = 0$, and $\Sigma_{i \in I} z^{*i} = 0$.

It is well known (see for instance Geanakoplos and Polemarchakis (1986)) that the maximization problem (B) has a solution if and only if there are no arbitrage opportunities i.e. if and only if the following condition is satisfied:

B7 (NAC) $\nexists y \in \Re^J$ *s.t.* $qy \leq 0$ *but* $Ry > 0$.

It is also well known (see for instance Geanakoplos and Polemarchakis (1986)) that (NAC) is equivalent to requiring that asset prices lie in the interior of the convex cone spanned by the rows of R. In other words we require that asset prices lie in the set $Q = \{q \in \Re^J : q = R^T\theta, \ for\ some\ \theta \in \Re^S_{++}\}$.

In order to rule out satiation in asset demand, we need the following assumption.

$B8$ $\exists y \in \Re^J$ such that $Ry > 0$.

With this additional assumption, Q is a proper subset of $\Re^J$ and therefore the boundary of Q, denoted by $\bar{Q}$, is non-empty. In addition, with this assumption, $0 \notin Q$. This implies that we can normalize both asset prices and spot commodity prices to lie on the unit sphere. For any k, let $S^k = \{x \in \Re^K : \|x\| = 1\}$.

In order to prove the existence of a competitive equilibrium we also need to assume, as in Gottardi and Hens (1996), that for at least one individual $i \in I$, the set of date 0 admissible trades has a non-empty interior.

$B9$ $\forall q \in \bar{Q} \cap S^J, \forall p_s \in \Re^L_{++} \cap S^L, s = 1, ..., S, \exists i \in I$ such that $Int(B_0(p, q, R) = \{y \in \Re^J : qy < 0, p\underline{z}^i_s + p_{s1}R_s y \geq 0, s = 1, ..., S\}) \neq \emptyset$.

Let $\Omega(p_{.1})$ be the $S \times S$ diagonal matrix formed out of the price of good 1 in the various states. Then, $\langle \Omega(p_{.1})R \rangle$ is the linear space generated by the column vectors of the matrix $\Omega(p_{.1})R$.

A net trade for individual i, $z^i \in Z^i$, is attainable if for all $p_s \in \Re^L_{++} \cap S^L, s = 1, ..., S$, $[(p_s z_s), s = 1, .., S] \in \langle \Omega(p_{.1})R \rangle$. Let Z^i_A denote the set of attainable trades of individual i. An allocation z^I is attainable if $z^i \in Z^i_A$ for all $i \in I$. Let $A = \{z^I : \Sigma_{i \in I} z^i = 0,\ z^i \in Z^i_A,\ \forall i \in I\}$ denote the set of all feasible attainable allocations. $Z^i_A(p,q) = \{z^i \in Z^i_A : \exists y \in \Re^J$ such that $qy \leq 0$, $p_s z_s \leq p_{s1} R_s y$, $s = 1, ..., S\} \cap A$, then denotes the set of feasible attainable trades of individual i which are affordable at prices (p,q), and $\hat{Z}^i_A(p,q) = \{z^i \in Z^i_A(p,q) : z^i \in \underset{z^i \in Z^i_A(p,q)}{\arg\max}\ u^i(z)\}$, denotes those affordable feasible attainable trades for individual i which give him/her the most utility. We then define, $\Phi^i(p,q) = \{z \in Z : u^i(\hat{z}^i + z) > u^i(\hat{z}^i), \forall \hat{z}^i \in \hat{Z}^i_A(p,q)\}$ as the set of trades which when added to any utility maximizing affordable feasible attainable trade at prices (p,q), make individual i better off. Strong monotonicity (a sufficient condition) and local non-satiation (a sufficient condition) imply that this set is non-empty.

Definition 36 *(price graph) The price graph of the exchange economy E with asset returns matrix R at prices (p,q), denoted $\Gamma(E(p,q),R)$, is a collec-*

tion of vertices V and arcs A such that each vertex v_i corresponds to consumer i for $i = 1, 2, ..., I$ and an arc directed from v_i to v_j exists whenever there is some $\bar{I} \subset I$,with $i \in \bar{I}$, $j \in I/\bar{I}$ such that $-Z_A^{\bar{I}} \cap \Phi^j(p,q) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z_A^{\bar{I}'} \cap \Phi^m(p,q) \neq \emptyset$.

Let $\widetilde{C}$ denote the collection price graphs of economy E, at all prices (p,q), $p_s \in \Re^L_{++} \cap S^L, s = 1, ..., S$, $q \in \bar{Q} \cap S^J$.

Definition 37 *($\widetilde{C}$-irreducibility) The economy E with asset returns matrix R is $\widetilde{C}$-irreducible if each member of $\widetilde{C}$ is strongly connected.*

Proposition 38 *If the economy E with incomplete markets satisfies assumptions $A1 - A5$ and $B6 - B9$, and in addition is $\widetilde{C}$-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $A1 - A5$ and $B6 - B8$ a quasi-equilibrium $(\overline{p}, \overline{q}, \overline{z}^I, \overline{y})$,exists with $\overline{p} \gg 0$, (Gottardi and Hens (1996)). From $B9$ it follows that at prices $(\bar{p}, \bar{q})$ there is some $\widehat{i}$ for whom $IntB_0(\bar{p}, \bar{q}, R) \neq \emptyset$ and therefore cost minimization implies utility maximization. Consider the partition of the set of vertices of $\Gamma(E(\overline{p}, \overline{q}), R)$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : IntB_0(\overline{p}, \overline{q}, R) \neq \emptyset\}$, and $V^2 = \{v_i : IntB_0(\overline{p}, \overline{q}, R) = \emptyset\}$. We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption

the graph $\Gamma(E(\overline{p},\overline{q}), R)$ is strongly connected and so, by lemma 5, there exists an arc from V^1 to V^2, and an arc from V^2 to V^1. It follows from the definition of $\tilde{C}$-irreducibility that there exists $v_i \in V^2$ and $v_j \in V^1$ such that $-Z_A^{\tilde{I}} \cap \Phi^j(\overline{p},\overline{q}) \neq \emptyset$, with $i \in \tilde{I}$, $j \notin \tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi^m(\overline{p},\overline{q}) \neq \emptyset$. Since individual j is utility maximizing at $\overline{z}^I$, it follows that there is an $m_0 \in \tilde{I}$ such that $v_{m_0} \in V^1$. But then, there exists a sequence of individuals $(m_1, ..., m_K)$ such that each $v_{m_k} \in V^1$and the arcs $v_{m_{k-1}}v_{m_k}$, $v_{m_k}v_i$, $k = 1, ..., K$ exist. Therefore $v_i \in V^1$, a contradiction. ■

3.2.2 $\tilde{C}$-irreducibility and irreducibility

In this section we establish the relationship between $\tilde{C}$-irreducibility and Gottardi and Hens' (1996) irreducibility condition for economies with incomplete asset markets. We show that $\tilde{C}$-irreducibility is a different condition than irreducibility, that is, neither implies the other. Subsequently, we define an alternative condition, $\tilde{C}'$-irreducibility - sufficient for existence - which we show to be weaker than Gottardi and Hens' (1996) irreducibility.

Let $z^I \in A$ be a feasible allocation and consider a net trade z' which if added to individual i's allocation at z^I he/she is made better off i.e.

$u^i(z^i + z') > u^i(z^i)$. Define $\Phi^i_A(z^I)$ as the collection of all such trades. Weak monotonicity (a necessary condition) and local non-satiation (a sufficient condition) imply that this set is non-empty.

Definition 39 *(allocation graph) The allocation graph of economy E with asset returns matrix R at allocation z^I, denoted $\Gamma(E(z^I), R)$, is a collection of arcs A and vertices V, where each vertex v_i corresponds to consumer i for $i = 1, 2, ...I$, and an arc directed from v_i to v_j exists whenever there is some $\bar{I} \subset I$,with $i \in \bar{I}$, $j \in I/\bar{I}$ such that $-Z^{\bar{I}}_A \cap \Phi^j_A(z^I) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z^{\bar{I}'}_A \cap \Phi^m_A(z^I) \neq \emptyset$.*

Let C^3 denote the collection of allocation graphs $\Gamma(E(z^I), R)$ for all feasible attainable allocations $z^I \in A$.

Definition 40 *(irreducibility with incomplete asset markets (Gottardi and Hens (1996))[12]) Let $\{I^1, I^2\}$ be a non-trivial partition of the set of individuals such that $I^1 \cap I^2 = \emptyset$, $I^1 \cup I^2 = I$. $z^{I^k} = \Sigma_{i \in I^k} z^i$, and $Z^{I^k}_A = \Sigma_{i \in I^k} Z^i_A$, for $k = 1, 2$. Then the economy is irreducible if, however I^1 and I^2 may be selected, if $z^{I^1} = -z^{I^2}$ with $z^{I^1} \in Z^{I^1}_A$, and $z^{I^2} \in Z^{I^2}_A$, then there is also*

[12] *Definition 40 is identical to Gottardi and Hens' (1996) assumption A5 which they refer to as resource relatedness. We express it differently here to highlight the relationship to McKenzie's (1961) definition of irreducibility.*

$w \in Z_A^{I^2}$, such that $z'^{I^1} = -z^{I^2} - w$, and $u^i(z'^i) \geq u^i(z^i)$ for all $i \in I^1$, and $u^i(z'^i) > u^i(z^i)$ for some $i \in I^1$.

Gottardi and Hens' definition of irreducibility generalizes McKenzie's definition of irreducibility to the case with incomplete asset markets. Note that when asset markets are complete - when the asset returns matrix R has full rank - $Z^i = Z_A^i$ for each $i \in I$ and the two definitions coincide.

Lemma 41 *The economy E with asset returns matrix R is irreducible if and only if every member of C^3 is strongly connected.*

Proof. Suppose the economy E with asset returns matrix R is irreducible. First notice that by setting $I^2 = \{i\}$, $I^1 = \{I/i\}$, at any feasible attainable allocation z^I there is some $i'_{(z^I)} \in I$ such that $-Z_A^i \cap \Phi_A^{i'}(z^I) \neq \emptyset$. Now consider a non-trivial partition of the set of individuals $\{I^1, I^2\}$. At any feasible attainable allocation z^I, irreducibility implies that there exists $j \in I^1$ such that $-Z_A^{I^2} \cap \Phi_A^j(z^I) \neq \emptyset$. Next, we claim that there exists $i \in I^2$ such that the arc $v_i v_j$ exists in the associated allocation graph $\Gamma(E(z^I), R)$. That is, we claim that there is $\bar{I} \subseteq I^2$ with $i \in \bar{I}$, such that $-Z_A^{\bar{I}} \cap \Phi_A^j(z^I) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z_A^{\bar{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$. We can construct such a subset $\bar{I}$ as follows. First, recall that there is some

$i' \in I$ such that $-Z_A^i \cap \Phi_A^{i'}(z^I) \neq \emptyset$. If $i' \in I^1$, set $\bar{I} = i$, and $j = i'$. If on the other hand $i' \in I^2$, consider $\bar{I}_1 = \{i, i'\}$. By irreducibility, either $\exists$ $j \in I^1$ such that $-Z_A^{\bar{I}_1} \cap \Phi_A^j(z^I) \neq \emptyset$, in which case set $\bar{I} = \bar{I}_1$, or $\exists i'' \in I^2/\bar{I}_1$ such that $-Z_A^{\bar{I}_1} \cap \Phi_A^{i''}(z^I) \neq \emptyset$, in which case consider $\bar{I}_2 = \{i, i', i''\}$. By irreducibility, either $\exists$ $j \in I^1$ such that $-Z_A^{\bar{I}_2} \cap \Phi_A^j(z^I) \neq \emptyset$, in which case set $\bar{I} = \bar{I}_2$, or $\exists i''' \in I^2/\bar{I}_2$ such that $-Z_A^{\bar{I}_2} \cap \Phi_A^{i'''}(z^I) \neq \emptyset$, in which case consider $\bar{I}_3 = \{i, i', i'', i'''\}$. By repeating this process if necessary, we can find $\bar{I}_n$ such that $\exists$ $j \in I^1$ such that $-Z_A^{\bar{I}_n} \cap \Phi_A^j(z^I) \neq \emptyset$ and set $\bar{I} = \bar{I}_n$. By construction, this subset satisfies the condition that for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z_A^{\bar{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$. Therefore, the arc $v_i v_j$ exists in the associated allocation graph $\Gamma(E(z^I), R)$. By irreducibility, this is true for any non-trivial partition of the set of individuals $\{I^1, I^2\}$, and for any feasible attainable allocation $z^I \in A$. Therefore, by lemma 5, the allocation graph $\Gamma(E(z^I), R)$ is strongly connected, for all feasible attainable allocations $z^I \in A$. Conversely, assume that the allocation graph $\Gamma(E(z^I), R)$ is strongly connected, for all feasible attainable allocations $z^I \in A$. By lemma 5, at a feasible attainable allocation z^I, for every non-trivial vertex partitioning $\{I^1, I^2\}$, we can find some $i \in I^2$ and $j \in \bar{I}$, such that, $\exists \bar{I} \subset I$, with $i \in \bar{I}$, such that $-Z_A^{\bar{I}} \cap \Phi_A^j(z^I) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ with $i \in \bar{I}'$, there is some

$m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$. Therefore, there exists $\tilde{I} \subseteq I^2$ and $j \in I^1$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^j(z^I) \neq \emptyset$. But then, there exists $w \in Z_A^{I^2}$ such that $\bar{z}^j = z^j - w$ and $u^j(\bar{z}^j) > u^j(z^j)$. By assumption $z^{I^1} = -z^{I^2}$ with $z^{I^1} \in Z_A^{I^1}$, $z^{I^2} \in Z_A^{I^2}$. Define $z'^{j'} = z^{j'}\ \forall j' \in I^1/j$ and $z'^j = \bar{z}^j$. Then, $z'^{I^1} = -z^{I^2} - w$ while, by construction, $u^{j'}(z^{j'}) \geq u^{j'}(z^{j'})$ for all $j' \in I^1$, and $u^j(z'^j) > u^j(z^j)$ as required. As this construction works at any feasible attainable allocation z^I, and by assumption $\Gamma(E(z^I), R)$ is strongly connected at every feasible attainable allocation $z^I \in A$, the economy E is irreducible. ■

We now show that $\bar{\mathrm{C}}$-irreducibility and Gottardi and Hens' irreducibility are different conditions, that is, neither condition implies the other.

To see that irreducibility does not necessarily imply $\bar{\mathrm{C}}$-irreducibility, suppose the economy E with asset returns matrix R is irreducible and consider the price graph of the economy at prices $(p, q), p_s \in \Re^L_{++} \cap S^L, s = 1, ..., S, q \in \bar{Q} \cap S^J$. Consider a non-trivial partition of the set of individuals $\{I^1, I^2\}$ with $I^1 = \{i\}$, $I^2 = I/\{i\}$, and a feasible allocation z^I which implies an attainable affordable utility maximising net trade for individual i. Irreducibility implies that $\exists j \in I^2$ such that $j \in \tilde{I}$, $i \in I/\tilde{I}$, $-Z_A^{\tilde{I}} \cap \Phi^i(p, q) \neq \emptyset$ and for any $\tilde{I}'$ such that $j \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$such that $-Z_A^{\tilde{I}} \cap \Phi_A^j(z^I) \neq \emptyset$, and since $\Phi_A^i(z^I) \subseteq \Phi^i(p, q)$, by construction, an arc exists from I^2 to I^1. Now consider

a feasible allocation z'^I, which implies an attainable affordable utility maximising net trade for some $j \in I^2$. Irreducibility implies that $\exists k \in I^2$ such that $i \in \tilde{I}$, $k \in I/\tilde{I}$, $-Z_A^{\tilde{I}} \cap \Phi_A^k(z^I) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$. However, there is no guarantee that $k = j$, and no guarantee therefore that an arc exist from I^1 to I^2, so that the price graph at $(p, q), p_s \in \Re_{++}^L \cap S^L, s = 1, ..., S, q \in \bar{Q} \cap S^J$ may fail to be strongly connected, in which case $\tilde{C}$-irreducibility fails.

To see that the reverse is not true either, that is, $\tilde{C}$-irreducibility does not necessarily imply irreducibility, consider the following example:

Example 42 *Let* $I = \{1, 2, 3\}$, $S = \{1, 2, 3\}$, $L = \{1\}$, $u^1 = z_1 + z_2 + z_3 + \frac{7}{3}$, $u^2 = z_1 + z_2 + 3z_3 + 9$, $u^3 = z_1 + z_2 + z_3 + \frac{4}{3}$, $Z^1 = \{z : z_1^1 \geq -1, z_2^1 \geq -1, z_3^1 \geq -\frac{1}{3}\}$, $Z^2 = \{z : z_1^2 \geq -3, z_2^2 \geq -1, z_3^2 \geq -\frac{1}{3}\}$, $Z^3 = \{z : z_1^3 \geq -1, z_2^3 \geq 0, z_3^3 \geq -\frac{1}{3}\}$, *and* $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$. *Set* $p_s = 1$, $s = 1, 2, 3$. *From the no arbitrage condition,* $\overline{Q} = \Re_+^2$. *Therefore, asset prices are restricted to* $(q_1, q_2) \in \Re_+^2 \cap S^2$, *or* $q = \frac{q_1}{q_2} \geq 0$. *At any* $q \geq 0$, *the arcs* v_1v_2, v_1v_3, v_2v_1 *always exist as preferences are strongly monotone over consumption in each state* s. *Any transfer of the commodity in state 3 from individual 3 to in-*

dividual 2, reduces the amount of the good in state 1 for individual 2. For individual 2 we can write $z_1^2 = y_1^2, z_2^2 = y_1^2 + y_2^2, z_3^2 = y_2^2$, *and therefore, at any* $q \geq 0$, *individual 2 maximises* $2(y_1^2 + 2y_2^2)$, *subject to* $y_1^2 + qy_2^2 = 0$, $-3 \leq y_1^2 \leq 2$, $-1 \leq y_1^2 + y_2^2 \leq 1$, $-\frac{1}{3} \leq y_2^2 \leq \frac{2}{3}$. *When* $q < 2$, *at any optimum for individual 2,* $y_2^2 = \frac{2}{3}$ *and* $y_1^2 = -\frac{2q}{3} < -3$. *If* $q = 2$, *any feasible combination of* y_1^2 *and* y_2^2 *such that* $y_1^2 + qy_2^2 = 0$, *is an optimum. If* $q > 2$, *at any optimum* $y_2^2 = -\frac{1}{3}$ *and* $y_1^2 = \frac{q}{3}$. *At each* $q \geq 0$, *individual 3 can make individual 2 better off by reducing* y_1^2 *slightly, and increasing* y_2^2. *Thus, for any* (p, q), *the arc* v_3v_2 *exists, the price graph* $\Gamma(E(p,q), R)$ *is strongly connected, and the economy is therefore* $\widetilde{C}$*-irreducible. However, the economy is not irreducible. To see this, consider the partition* $I^1 = \{1, 2\}, I^2 = \{3\}$. *Individual 3 is unable to make individual 1 better off at any feasible allocation. Moreover, at the feasible allocation* $z^1 = (3, 0, 0)$, $z^2 = (-3, 0, 0)$, $z^3 = (0, 0, 0)$, *any transfer of the commodity in state 3 from individual 3 to individual 2 results in a net trade that lies outside* Z_A^2. *Nevertheless, a competitive equilibrium exists:* $q^* = 1, y^{*1} = (\frac{1}{2}, -\frac{1}{3}), y^{*2} = (-1, \frac{2}{3}), y^{*3} = (\frac{1}{2}, -\frac{1}{3}), p^* = (1, 1, 1), z^{*1} = (\frac{1}{2}, \frac{1}{6}, -\frac{1}{3}), z^{*2} = (-1, -\frac{1}{3}, \frac{2}{3}), z^{*1} = (\frac{1}{2}, \frac{1}{6}, -\frac{1}{3})$.

By modifying the definition of an arc in the price graph, we can however derive an alternative condition, sufficient for existence, which is in fact weaker

than irreducibility.

Definition 43 *(modified price graph) The modified price graph of the exchange economy E with asset returns matrix R at prices (p,q), denoted $\Gamma'(E(p,q),R)$, is a collection of vertices V and arcs A such that each vertex v_i corresponds to consumer i for $i = 1,2,...I$ and an arc directed from v_i to v_j exists whenever either (i) i can make j better off, in the sense that there is some $\tilde{I} \subset I$,with $i \in \tilde{I}$, $j \in I/\tilde{I}$ such that $-Z_A^{\tilde{I}} \cap \Phi^j(p,q) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi^m(p,q) \neq \emptyset$, or (ii) there is some k_0, who can make j better off in the sense outlined in (i), and in addition, for any feasible attainable allocation z^I such that $z^j \in \tilde{Z}_A^j(p,q)$, there is some sequence of individuals $(k_1,k_2,...,k_N)$, such that (a) $i \in \tilde{I}$, $k_N \in I/\tilde{I}$ with $-Z_A^{\tilde{I}} \cap \Phi_A^{k_N}(z^I) \neq \emptyset$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I)$, and (b) $\forall k_n$, $k_n \in \tilde{I}, k_{n-1} \in I/\tilde{I}$ with $-Z_A^{\tilde{I}} \cap \Phi_A^{k_{n-1}}(z^I) \neq \emptyset$ and for any $\tilde{I}' \subset \tilde{I}$ such that $k_n \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I)$.*

Definition 44 *($\tilde{C}'$-irreducibility) The economy E with asset returns matrix R is $\tilde{C}'$-irreducible if $\Gamma'(E(p,q),R)$ is strongly connected for all $(p,q),p_s \in \Re_{++}^L \cap S^L, s = 1,...,S, q \in \bar{Q} \cap S^J$.*

For ease of exposition, let arcs defined as per (i) in the above definition of a modified price graph be called *direct arcs*, and arcs defined as in (ii) be called *indirect arcs*.

Proposition 45 *If the economy E with incomplete markets satisfies assumptions $A1 - A5$ and $B6 - B9$, and in addition is $\tilde{C}'$-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $A1 - A5$ and $B6 - B8$ a quasi-equilibrium $(\overline{p}, \overline{q}, \overline{z}^I, \overline{y})$,exists with $\overline{p} \gg 0$, (Gottardi and Hens (1996)). From $B9$ it follows that at prices $(\overline{p}, \overline{q})$ there is some $\widehat{i}$ for whom $IntB_0(\overline{p}, \overline{q}, R) \neq \emptyset$ and therefore cost minimization implies utility maximization. Consider the partition of the set of vertices of $\Gamma(E(\overline{p}, \overline{q}), R)$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : IntB_0(\overline{p}, \overline{q}, R) \neq \emptyset\}$, and $V^2 = \{v_i : IntB_0(\overline{p}, \overline{q}, R) = \emptyset\}$. We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption the graph $\Gamma'(E(\overline{p}, \overline{q}), R)$ is strongly connected and so, by lemma 5, there exists an arc from V^1 to V^2, and an arc from V^2 to V^1. It follows from the definition of $\tilde{C}$-irreducibility that there exists $v_i \in V^2$ and $v_j \in V^1$ such that the $v_i v_j$ exists in the graph $\Gamma'(E(\overline{p}, \overline{q}), R)$. Given the definition of $\tilde{C}'$-irreducibility, $v_i v_j$ may be a direct or an indirect arc. If $v_i v_j$ is a direct arc then $-Z_A^{\tilde{I}} \cap \Phi^j(\overline{p}, \overline{q}) \neq \emptyset$, with $i \in \tilde{I}$, $j \notin \tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$

such that $-Z_A^{\bar{I}'} \cap \Phi^m(\bar{p}, \bar{q}) \neq \emptyset$. Since individual j is utility maximizing at $\bar{z}^I$, it follows that there is an $m_0 \in \bar{I}$ such that $v_{m_0} \in V^1$. But then, there exists a sequence of individuals $(m_1, ..., m_K)$ such that each $v_{m_k} \in V^1$and the arcs $v_{m_{k-1}} v_{m_k}$, $v_{m_k} v_i$, $k = 1, ..., K$ exist. Therefore $v_i \in V^1$, a contradiction. If on the other hand $v_i v_j$ is an indirect arc then there exists $v_i, v_{k_0} \in V^2$ and $v_j \in V^1$ such that $-Z_A^{\bar{I}} \cap \Phi^j(\bar{p}, \bar{q}) \neq \emptyset$, with $k_0 \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $k_0 \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z_A^{\bar{I}'} \cap \Phi^m(\bar{p}, \bar{q}) \neq \emptyset$ (so that by the previous argument $v_{k_0} \in V^1$), and for any feasible attainable allocation z^I such that $z^j \in \bar{Z}_A^j(\bar{p}, \bar{q})$, there is some sequence of individuals $(k_1, k_2, ..., k_N)$, such that $-Z_A^i \cap \Phi_A^{k_N}(z^I) \neq \emptyset$ and $-Z_A^{k_n} \cap \Phi_A^{k_{n-1}}(z^I) \neq \emptyset$, so that each $v_{k_n} \in V^1$, $n = 1, ..., N$, and thus $v_i \in V^1$, a contradiction. ■

We now show that $\bar{C}'$-irreducibility is weaker than Gottardi and Hens' irreducibility.

Proposition 46 *Irreducibility implies $\bar{C}'$-irreducibility.*

Proof. Suppose the economy is irreducible. Consider prices $(p, q), p_s \in \Re^L_{++} \cap S^L, s = 1, ..., S, q \in \bar{Q} \cap S^J$, and a non-trivial partition of the set of vertices $\{V^1, V^2\}$. Consider the feasible attainable allocation z^I, which implies an attainable affordable utility maximising net trade for individual j, where $v_j \in V^2$. Since (p, q), $\{V^1, V^2\}$, and j were chosen arbitrarily, to

establish that the economy is $\tilde{C}'$-irreducible, it is sufficient to show that there is some $v_i \in V^1$ such that the arc $v_i v_j$ exists in the modified price graph $\Gamma'(E(p,q),R)$. By irreducibility, either there exists $v_i \in V^1$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^j(z^I) \neq \emptyset$, with $i \in \tilde{I}$, $j \in I/\tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$, - in which case the arc $v_i v_j$ exists and we are done - or there exists $v_{k_0} \in V^2$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^j(z^I) \neq \emptyset$, with $k_0 \in \tilde{I}$, $j \in I/\tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$, in which case the arc $v_{k_0} v_j$ exists in the modified price graph $\Gamma'(E(p,q),R)$. Now, irreducibility implies that either there exists $v_i \in V^1$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^{k_0}(z^I) \neq \emptyset$, with $i \in \tilde{I}$, $j \notin \tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$, - in which case the arc $v_i v_j$ exists and we are done - or there exists $v_{k_1} \in V^2$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^{k_0}(z^I) \neq \emptyset$, with $k_1 \in \tilde{I}$, $k_0 \notin \tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $k_1 \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$, in which case the arc $v_{k_1} v_j$ exists in the modified price graph $\Gamma'(E(p,q),R)$. If we repeat this process, we end up with a sequence of individuals $(k_0, k_1, k_2, ..., k_N)$, with $k_n \in I^1$, $n = 0, 1, ..., N$, such that $\forall k_n$, $k_n \in \tilde{I}, k_{n-1} \in I/\tilde{I}$ with $-Z_A^{\tilde{I}} \cap \Phi_A^{k_{n-1}}(z^I) \neq \emptyset$ and for any $\tilde{I}' \subset \tilde{I}$ such that $k_n \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I)$. In addition, we

have $k_0 \in \tilde{I}$, $j \in I/\tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$. Consider the partition $\{I^{2_1}, I^{2_2}\}$ of I^2, where $I^{2_1} = \{k_0, k_1, k_2, ..., k_N\}$ and $I^{2_2} = I^2/I^{2_1}$. By construction, no arcs exist from I^{2_1} to I^{2_2} in the allocation graph $\Gamma(E(z^I), R)$. By irreducibility, an arc must exist from $I^1 \cap I^{2_2}$ to I^{2_1}. It follows therefore that there exists $v_i \in V^1$ such that $-Z_A^{\tilde{I}} \cap \Phi_A^{k_n}(z^I) \neq \emptyset$, with $i \in \tilde{I}$, $k_n \in I/\tilde{I}$ and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi_A^m(z^I) \neq \emptyset$, for some k_n in the sequence, and therefore the arc $v_i v_j$ exists. ■

Example 42 illustrates that the reverse is not true, that is, $\tilde{C}'$-irreducibility does not imply irreducibility. Intuitively, the sense in which $\tilde{C}'$-irreducibility is weaker than irreducibility is that whereas irreducibility requires a particular relationship to hold between every pair of individuals at every feasible allocation, $\tilde{C}'$-irreducibility only requires a relationship to hold between any pair of individuals for some subset of feasible allocations, which contains the set of quasi-equilibrium allocations.

3.3 Effectively complete markets

In this section, we study the restrictions on the collection of allocation graphs of an economy with incomplete asset markets, sufficient to guarantee effective

completeness.

Let $\hat{\mathbf{R}}$ denote the set of all asset returns matrices whose rank is S. Let $(\hat{q}, \hat{p}, \hat{y}, \hat{z}^I)$ denote a competitive equilibrium with complete markets, that is, when the asset returns matrix $\hat{R}$ is in $\hat{\mathbf{R}}$. Observe that for $\hat{R}$, $\hat{R}'$ in $\hat{\mathbf{R}}$, at any feasible attainable allocation z^I, $\Gamma(E, z^I, \hat{R}) = \Gamma(E, z^I, \hat{R}')$.

Definition 47 *(Arc preserving asset markets) The asset returns matrix R is arc preserving if for every feasible allocation z^I, $\exists$ a subset $\bar{A}$ of arcs of $\Gamma(E(z^I), \hat{R})$ such that (i) arcs in $\bar{A}$ strongly connect $\Gamma(E(z^I), \hat{R})$ (ii) if $v_i v_j$ is an arc in $\bar{A}$, then $v_i v_j$ is an arc in $\Gamma(E(z^I), R)$.*

It follows that if the asset returns matrix R is arc preserving and $\Gamma(E(z^I), \hat{R})$ is strongly connected at every feasible attainable allocation z^I, then so is $\Gamma(E(z^I), R)$. Note that $\hat{z}$ continues to be a competitive equilibrium commodity allocation for any asset returns matrix $\hat{R}$ in $\hat{\mathbf{R}}$. Let $\hat{Z}$ be the set of all competitive equilibrium commodity allocations for any asset return matrix $\hat{R}$ in $\hat{\mathbf{R}}$.

Definition 48 *(effectively complete asset markets) Consider an asset returns matrix $R \notin \hat{\mathbf{R}}$ and let Z_R^* be the associated set of all competitive equilibrium commodity allocations. If $\hat{Z} = Z_R^*$, then the asset returns matrix R*

is effectively complete.

Proposition 49 *Consider an asset returns matrix $R \notin \hat{\mathbf{R}}$. If R is arc preserving, then markets are effectively complete.*

Proof. As R is arc preserving, at each $\hat{z}^I \in \hat{Z}$ there is subset of arcs $\bar{A}$ of the allocation graph $\Gamma(E(\hat{z}^I), \hat{R})$ such that if $v_i v_j$ is an arc in $\bar{A}$, then $v_i v_j$ is an arc in $\Gamma(E(\hat{z}^I), R)$. This implies that at each feasible allocation $\hat{z}^I \in \hat{Z}$, $\hat{z}^i \in Z^i_A$, for all $i \in I$, that is, each $\hat{z}^I \in \hat{Z}$ is attainable with the asset returns matrix R. For any $\hat{z}^I \in \hat{Z}$, let $(\hat{q}, \hat{p}, \hat{y}, \hat{z}^I)$ be a competitive equilibrium for some $\hat{R}$ in $\hat{\mathbf{R}}$. By (NAC), $\hat{q} = \hat{R}^T\hat{\theta}$, for some $\hat{\theta} \in \Re^S_{++}$. Let $\hat{q}' = R^T\hat{\theta}$. For each $i \in I$, define $\tau^i = (\hat{p}_1\hat{z}^i_1, ..., \hat{p}_S\hat{z}^i_S)^T$. As $\hat{z}^I \in \hat{Z}$ is attainable, $\tau^i \in$ *Range* R. Choose $\hat{y}'^1, ..., \hat{y}'^{(I-1)}$ such that $\tau^i = \Omega(\hat{p}_{1.})R\hat{y}'^i$ for each $i = 1, ..., I-1$, and $\hat{y}'^I = -\hat{y}'^1 - ... - \hat{y}'^{(I-1)}$. Then, $(\hat{q}', \hat{p}, \hat{y}', \hat{z}^I)$ is a competitive equilibrium with the asset returns matrix R. Therefore, $\hat{Z} \subset Z^*_R$. As R is arc preserving, *Range* $R \subset$ *Range* $\hat{R}$, for all $\hat{R}$ in $\hat{\mathbf{R}}$. For any $\bar{z}^I \in Z^*_R$, let $(\bar{q}, \bar{p}, \bar{y}, \bar{z}^I)$ be a competitive equilibrium with R. By (NAC), set $\bar{q} = \hat{R}\bar{\theta}$, for some $\bar{\theta} \in \Re^S_{++}$. Let $\bar{q}'_{\hat{R}} = \hat{R}^T\bar{\theta}$. For each $i \in I$, define $\tau^i_{\hat{R}} = (\bar{p}_1\bar{z}^i_1, ..., \bar{p}_S\bar{z}^i_S)^T$. As $\bar{z}^I \in \hat{Z}$ is attainable, $\tau^i_{\hat{R}} \in$ *Range* $\hat{R}$. Choose $\bar{y}'^1, ..., \bar{y}'^{(I-1)}$ such that $\tau^i_{\hat{R}} = \Omega(\bar{p}_{.1})\hat{R}\bar{y}'^i$ for each $i = 1, ..., I-1$, and $\bar{y}'^I = -\bar{y}'^1 - ... - \bar{y}'^{(I-1)}$. Then, $(\bar{q}'_{\hat{R}}, \bar{p}, \bar{y}', \bar{z}^I)$ is a competitive equilibrium

with the asset returns matrix $\hat{R}$. By construction, this true for each $\hat{R}$ in $\hat{\mathbf{R}}$. Therefore, $Z_R^* \subset \hat{Z}$. ■

3.4 Conclusion

Although strongly monotone preferences are sufficient to circumvent the minimum-wealth problem brought about by boundary endowments in economies with a complete set of asset markets, with incomplete markets this is no longer true. The incompleteness of markets may limit the possibility of agents attaining some feasible net trades. In order to prove existence of competitive equilibria, one must therefore appeal to some form of irreducibility condition which takes account of the attainability of trades. We propose a sufficient condition, $\bar{C}$-irreducibility, a joint restriction on preferences, endowments, and the asset structure. This condition is neither stronger nor weaker than the alternative survival assumption presented by Gottardi and Hens. However, by a modification in the definition of an arc, we are indeed able to provide a weaker sufficient condition, $\bar{C}'$-irreducibility. Finally, we apply our graph theoretic techniques to provide a sufficient condition for asset markets to be effectively complete.

Chapter 4

Finite Pure Exchange Economies with Restricted Participation

4 Finite Pure Exchange Economies with Restricted Participation

4.1 Introduction

In this chapter, we study the impact of restricted participation in markets on the existence of competitive equilibria. In practice, individuals' participation in markets is subject to a wide range of diverse institutional restrictions such as age restrictions for the purchase of cigarettes and alcohol.

With restricted participation, the set of net trades which are *attainable* for an individual i at prices p may be only a subset of their budget set, and may vary discontinuously with prices. This discontinuous dependence of attainable allocations on prices may cause the existence of competitive equilibria to fail. An economy with an incomplete set of asset markets is a special case of an economy with restricted participation, in which all individuals face the same participation constraints generated by the asset market structure. Discontinuities in the correspondence of attainable allocations therefore occur at the same prices for all individuals. When participation constraints are heterogeneous across agents, discontinuities in the correspondence of attainable allocations will of course occur at different commodity prices for

different individuals. In effect, equilibrium prices must be able to perform a dual role. Not only must they adjust to attain market clearing, but also to ensure that the net demands of unrestricted individuals in each market be consistent with the limits imposed by the restrictions on other individuals in that market.

The existence of competitive equilibria in economies with restricted participation in asset markets has been studied in the literature by Balasko, Cass and Siconolfi (1990), Polemarchakis and Siconolfi (1997), and Cass, Siconolfi and Villanacci (2001), with the assumption that individuals have interior endowments. We consider a pure exchange economy without uncertainty, in which individuals have strongly monotone preferences and boundary endowments. We assume linear homogeneous constraints on participation in markets (as in Balasko, Cass and Siconolfi (1990)), that is, an individual either can or can not participate in any particular market. In the absence of market restrictions, competitive equilibria can be shown to exist in our framework, under standard Arrow-Debreu assumptions. However, as we illustrate via an example, when participation in markets is restricted, additional conditions are required to ensure existence with boundary endowments. In particular, some form of C-irreducibility condition is required. This condition must be

formulated to take into account the restrictions on net trades imposed, and is therefore a joint restriction on the attainability structure of the economy and the distribution of endowments.

4.2 The Economy

Consider a pure exchange economy E, with individuals denoted by $i \in \mathrm{I} = \{1, ..., I\}$, and commodities denoted by $l \in \mathrm{L} = \{1, ..., L\}$, where I, and L are non-empty, finite sets.

The commodity space, denoted $\Re^L$, is the Euclidean space with dimension equal to the number of commodities.

Trades in commodities are denoted by, $z = (z_1, ..., z_L) \in \Re^L$.

An individual, i, is characterized by (i) a feasible trade set, $Z^i \subset \Re^L$, (ii) a utility function, $u^i : Z^i \to \Re$, and (iii) a participation constraint $M^i \subset \Re^L$, where M^i is an L^i dimensional linear subspace of $\Re^L$ with $0 < L^i \leq L$. Note that $L^i > 0$ implies that each individual can participate in at least one market. Let $\mathbf{M} = (M^1, ..., M^i, ..., M^I)$ denote the attainability structure of the economy.

For $i \in I$, let Z^i_+ denote the restriction of individual i's net trade set to the non-negative orthant, and let Z^i_l denote the projection of Z^i on the lth

coordinate.

For any non-empty subset of the set of individuals $\bar{I} \subseteq \mathbf{I}$, let $Z^{\bar{I}} = \Sigma_{i \in \bar{I}} Z^i$. The aggregate domain of trades in commodities is denoted Z.

*A*1 For each $i \in I$, the set of feasible trades, Z^i, is closed, convex, bounded below, ($\exists \underline{z}^i \in Z^i : \underline{z}^i \leq 0,\ \underline{z}^i \neq 0,\ z^i \geq \underline{z}^i\ \forall z^i \in Z^i, and\ z^i \ll \underline{z}^i\ \forall z^i \notin Z^i$) and allows for free disposal ($z \in Z^i,\ z' \geq z \Rightarrow z' \in Z^i$).

*A*2 Autarchy is feasible ($0 \in Z^i$). $0 \in Int.Z$.

*A*3 The utility function, u^i, is continuous.

*A*4 The utility function, u^i, is quasi-concave ($u^i(z') \geq u^i(z) \Rightarrow u^i(\lambda z' + (1 - \lambda)z) \geq u^i(z),\ 0 \leq \lambda \leq 1$).

*A*5 The utility function, u^i, satisfies local non-satiation ($\{z' \in Z^i : u^i(z') > u^i(z),\ \mid z' - z \mid < \varepsilon,\ \forall \varepsilon > 0\} \neq \emptyset$).

*A*6 The utility function, u^i, is strictly monotonically increasing ($z' > z \Rightarrow u^i(z') > u^i(z)$).

*A*7 For each $l \in \mathbf{L}$, $\exists i \in I$ such that $Z^i_l \cap M^i \neq \emptyset$.

*A*8 $\exists l \in \mathbf{L}$ such that $Z^i_l / Z^i_+ \cap M^i \neq \emptyset$ and $Z^j_l \cap M^j \neq \emptyset$ for some $i, j \in \mathbf{I}$, with $i \neq j$.

Assumption $A7$ ensures that in each market there is at least one individual whose participation is not restricted. Although this assumption is not directly employed in the proof of existence of equilibrium, without it it is difficult to see why one would want to differentiate a particular good as an economic object. Assumption $A8$ simply states that there is at least one market in which two individuals can trade, with one of those individuals able to supply positive amounts of the good.

An allocation is a vector, $(z^I) = \{z^i \in Z^i : i \in \mathbf{I}\}$, of individuals' net trades. An allocation is feasible if and only if $\Sigma_{i \in I} z^i = 0$. A net trade for individual i, $z^i \in Z^i$, is attainable if $z^i \in M^i \cap Z^i$. Let $Z^i_A = M^i \cap Z^i$ denote the set of attainable trades of individual i. An allocation z^I is attainable if $z^i \in Z^i_A$ for all $i \in I$. Let $Z_A = \{z^I : \Sigma_{i \in I} z^i = 0,\ z^i \in Z^i_A,\ \forall i \in I\}$ denote the set of all feasible attainable allocations.

Prices are $p \in \Re^L_+/\{0\}$.

Definition 50 *A competitive equilibrium in an economy with restricted participation is a pair (p^*, z^{*I}) such that,*

(i) $\forall i \in \mathbf{I}$, $p \cdot z^{*i} \leq 0$, and $u^i(z^{*i}) \geq u^i(z^i)$ for any $z^i \in Z^i_A$ such that $p \cdot z^i \leq 0$,

(ii) $z^{*I} \in Z_A$.

The definition of a quasi-equilibrium is as in definition X.1, replacing (i)

with (i′) $p \cdot z^{*i} \leq 0$, and $u^i(z^{*i}) \geq u^i(z')$ for any $z' \in Z_A^i$ such that $p \cdot z' \leq 0$, or $p \cdot z^{*i} \leq p \cdot z'$ for all $z' \in Z_A^i$, $i = 1, ..., I$.

4.3 Existence of equilibrium

In an economy without restricted participation, assumptions A1-A6 are sufficient to guarantee the existence of a competitive equilibrium. Although individuals may have boundary endowments, assumption A6 ensures that the economy is irreducible. However, with restricted participation, A1-A6 are no longer sufficient for existence, as the following example illustrates.

Example 51 *Let* $I = \{1, 2\}$, $L = \{1, 2, 3\}$, $u^1(z) = z_1^1 + 2z_2^1 + 2z_3^1 + 5$, $u^2(z) = z_1^2 + z_2^2 + z_3^2 + 1$, $Z^1 = \{z : z_1^1 \geq -1,\ z_2^1 \geq -1, z_3^1 \geq -1\}$, $Z^2 = \{z : z_1^2 \geq 0,\ z_2^2 \geq 0, z_3^2 \geq -1\}$, $M^2 = \Re^3$, $M^1 = \{z \in \Re^3 : z_3^1 = 0\}$. *Without any restriction on individuals' participation in markets, a competitive equilibrium in this economy* (p^*, z^{*I}) *exists with* $p_1^* = p_2^* = p_3^*$, $z^{*1} = (-1, 0, 1)$, *and* $z^{*2} = (1, 0, -1)$. *However, given the restriction on participation in markets, the allocation* z^{*I} *is not attainable. Since individual 1 cannot trade in the market for good 3, equilibrium prices must be such that individual 2's excess demand for each good be zero. Given individual 2's preferences, this is only*

true for prices $p_1 = p_2 = p_3$. *However, at these prices markets do not clear since* $z_1^1 = -1$, *whereas* $z_1^2 = 0$. *With restricted participation therefore there are no prices which clear markets, and a competitive equilibrium for this economy fails to exist.*

Despite strongly monotone preferences, with restrictions on participation irreducibility may fail since the restriction may limit the net trades which are attainable. In example 51, although there are net trades $z \in -(Z^2 \cap M^2)$ such that $u^1(z^1 + z) > u^1(z^1)$ for all $z^1 \in Z^1 \cap M^1$, given the restriction on individual 1's participation in the market for good 3, $(z^1 + z) \notin Z^1 \cap M^1$, for all $z \in -(Z^2 \cap M^2)$, $z^1 \in Z^1 \cap M^1$. Therefore, individual 2 can never make individual 1 better off. With restricted participation, the C-irreducibility condition must be reformulated to take into account the restrictions on net trades.

Let $Z_A^i(p) = \{z^i \in Z_A^i \cap Z_A : pz^i \leq 0\}$, denote the set of feasible attainable trades of individual i which are affordable at prices p, and $\hat{Z}_A^i(p) = \{z^i \in Z_A^i(p) : z^i \in \arg\max_{z^i \in Z_A^i(p)} u^i(z)\}$, denote those affordable feasible attainable trades for individual i which yield the most utility. We then define, $\Phi^i(p) = \{z \in Z_A^i : u^i(\hat{z}^i + z) > u^i(\hat{z}^i), \forall \hat{z}^i \in \hat{Z}_A^i(p)\}$ as the set of attainable trades which when added to any utility maximizing affordable feasible attainable trade

at prices p, make individual i better off. Weak monotonicity (a sufficient condition) and local non-satiation (a sufficient condition) imply that this set is non-empty.

Definition 52 *(price graph) The price graph of the exchange economy E with attainability structure M at prices p, denoted $\Gamma(E(p), M)$, is a collection of vertices V and arcs A such that each vertex v_i corresponds to consumer i for $i = 1, 2, ..., I$ and an arc directed from v_i to v_j exists whenever there is some $\tilde{I} \subset I$, with $i \in \tilde{I}$, $j \in I/\tilde{I}$ such that $-Z_A^{\tilde{I}} \cap \Phi^j(p) \neq \emptyset$, and for any $\tilde{I}' \subset \tilde{I}$ such that $i \in \tilde{I}'$, there is some $m \in \tilde{I}/\tilde{I}'$ such that $-Z_A^{\tilde{I}'} \cap \Phi^m(p) \neq \emptyset$.*

Let $\dot{C}$ denote the collection price graphs of economy E, at all prices $p \in \Re_+^L/\{0\}$.

Definition 53 *($\dot{C}$-irreducibility) The economy E with attainability structure M is $\dot{C}$-irreducible if each member of $\dot{C}$ is strongly connected.*

Note that the above definition differs from the definition of C-irreducibility in the way the index sets are defined.

Proposition 54 *If the economy E with attainability structure M satisfies assumptions $A1 - A8$, and in addition is $\dot{C}$-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $A1 - A6$ a quasi-equilibrium $(\bar{p}, \bar{z}^I)$,exists. Assumption A8 ensures that at prices $\bar{p}$ there is some $\hat{i}$ for whom $\exists z^{\hat{i}} \in Z_A^{\hat{i}} \cap Z_A$ such that $\bar{p}z^{\hat{i}} < 0$. Consider the partition of the set of vertices of $\Gamma(E(\bar{p}), M)$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : \exists z^i \in Z_A^i \text{ s.t. } p^* z^i < 0\}$, and $V^2 = \{v_i : \nexists z^i \in Z_A^i \text{ s.t. } p^* z^i < 0\}$. We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption the graph $\Gamma(E(\bar{p}), M)$ is strongly connected and so, by lemma 5, there exists an arc from V^2 to V^1, and vice versa. In particular, there exists $v_i \in V^2$ and $v_j \in V^1$ such that $v_i v_j$ exists. Given the way arcs are defined, the existence of $v_i v_j$ implies that $-Z_A^{\bar{I}} \cap \Phi^j(\bar{p}) \neq \emptyset$, with $i \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z_A^{\bar{I}'} \cap \Phi^m(p) \neq \emptyset$. Since individual j is utility maximizing at $\bar{z}^I$, it follows that there is an $m_0 \in \bar{I}$ such that $v_{m_0} \in V^1$. But then, there exists a sequence of individuals $(m_1, ..., m_K)$ such that $-Z_A^i \cap \Phi^{m_K}(\bar{p}) \neq \emptyset$, and $-Z_A^{m_k} \cap \Phi^{m_{k-1}}(\bar{p}) \neq \emptyset$. Therefore, each $v_{m_k} \in V^1$, the arcs $v_{m_k} v_{m_{k-1}}$, $v_i v_{m_K}$, $k = 1, ..., K$ exist, and thus $v_i \in V^1$, a contradiction. ■

Note that the economy of example 51 is not Ċ-irreducible. At any vector of prices $p \in \Re_+^L/\{0\}$, the price graph $\Gamma(E(p), M)$ has a unique vertex $v_1 v_2$.

4.4 Conclusion

In an economy with participation constraints and boundary endowments competitive equilibria may fail to exist, even with the assumption of strongly monotone preferences. However, we develop a condition, $\hat{C}$-irreducibility which is sufficient to guarantee existence, a condition which implies joint restrictions on the attainability structure of the economy and the distribution of endowments.

Chapter 5

Pure Exchange Economies with Countable Sets of Individuals and Commodities

5 Pure Exchange Economies with Countable Sets of Individuals and Commodities

5.1 Introduction

In this chapter we extend our analysis to pure exchange economies with countable sets of individuals and economies. The argument for the existence of competitive equilibria in infinite dimensional economies proceeds by considering a sequence of finite truncated economies which tend to the full economy in the limit. The first step in the proof is to demonstrate the existence of competitive equilibria for each of the finite truncated economies in the sequence, which would require that each economy in the sequence satisfy some form of irreducibility condition. The second step is to show that the limit of the sequence of competitive equilibria for the finite economies is a quasi-equilibrium for the full economy. Assuming some form of irreducibility of the full economy is then sufficient for a quasi-equilibrium to constitute a competitive equilibrium for the full economy. In the literature, some authors prove existence of competitive equilibria by assuming irreducibility of the full economy alone (Burke (1988); Geanakoplos and Polemarchakis (1991)). Others (Wilson (1981); Balasko, Cass, and Shell (1980)) assume some form

of irreducibility of the truncated economies.

We modify the definition of C-irreducibility to allow for a countable infinity of individuals. We show that an economy which is approximated by some sequence of finite C-irreducible economies will not necessarily be itself C-irreducible. Assuming C-irreducibility of the finite truncated economies alone is therefore not sufficient to prove existence of competitive equilibria in infinite dimensional economies, unless, that is, one imposes some additional restriction on the sequence of truncated economies. We characterise some such condition in terms of restrictions on price graphs. Broadly speaking, the restriction implies that links between individuals in any given finite economy cannot be arbitrarily broken as "new" individuals are introduced.

Subsequently, we show that C-irreducibility of an infinite dimensional economy does not necessarily imply the existence of an increasing sequence of finite truncated C-irreducible economies which tend to the full economy in the limit. We develop an additional restriction on the preferences and endowments of individuals in the infinite dimensional economy such that it may indeed be approximated by an increasing sequence of finite truncated C-irreducible economies. The key restriction here is that there must be sufficient arcs across truncated economies.

The results of this chapter, lead us to conclude that when working with infinite dimensional economies it is far less restrictive to prove the existence of competitive equilibria by imposing some form of irreducibility condition on the full economy alone.

In order to prove our results for infinite dimensional economies, we needed results for infinite dimensional graphs which we were unable to find in the literature. In section two, we present our results for countably infinite directed graphs. In section three, we apply our results for infinite directed graphs to study C-irreducibility in pure exchange economies with countable sets of commodities and individuals. Finally, in section 4, we conclude.

5.2 Results for infinite graphs

Definition 55 *(infinite directed graph) An infinite directed graph is a pair* $\Gamma = (V, A)$ *of a vertex set,* V*, and a binary relation,* A*, on* V*, such that* $V \cup A$ *is infinite.*

Note that, by definition, an arc set can only be infinite if the vertex set is infinite. Hence, an infinite graph is essentially a graph with an infinite vertex set. We restrict attention to graphs with a countable infinity of vertices.

Observe that the definitions of a path, a cycle, a subgraph, and a strongly

directed graph, presented in chapter 2 for finite graphs, extend trivially to infinite directed graphs, with obvious modifications.

We first define an increasing sequence of graphs in terms of set inclusion.

Definition 56 *(increasing sequence of finite graphs) A sequence of finite graphs* $\langle \Gamma_m = (V_m, A_m) \rangle$ *is increasing if* $V_1 \subset V_2 \subset \ldots \subset V_m \subset V_{m+1} \subset \ldots$, *and* $(A_1 \cap A_2) \subset (A_2 \cap A_3) \subset \ldots (A_m \cap A_{m+1}) \subset \ldots$, *where* $A_m \cap A_{m+1} \neq \emptyset$, *for all* $m > 1$.

Note that the above definition implies that if two subsequent graphs, Γ_m and Γ_{m+1}, have a common subgraph, Γ'_m, then Γ'_m is also a subgraph of all Γ_n, $n > m + 1$.

Definition 57 *(convergence of an increasing sequence of finite graphs) An increasing sequence of finite graphs* $\langle \Gamma_m = (V_m, A_m) \rangle$, *converges to the infinite graph* $\Gamma = (V, A)$ *if and only if* $V_1 \subset V_2 \subset \ldots \subset V_m \subset V_{m+1} \subset \ldots \subset V = \bigcup_{m=1}^{\infty} V_m$, *and* $A = \bigcup_{m=1}^{\infty} (A_m \cap A_{m+1})$.

Note that there may be more than one increasing sequence of finite graphs which converges to some infinite graph Γ. Our first observation is that a sequence of finite strong connected graphs may converge to an infinite graph which itself fails to be strongly connected.

Example 58 *Consider the increasing sequence of finite strongly connected graphs* $\langle\ \Gamma_m = (V_m, A_m)\rangle$, *where* $V_m = \{1, 2, ..., m\}$ *and* $A_m = \{v_m v_1, v_i v_{i+1} : i = 1, 2, ..., m-1\}$. *Note that* $\langle\ \Gamma_m = (V_m, A_m)\rangle$ *satisfies the definition of an increasing sequence of graphs. That is,* $V_m \subset V_{m+1}$ *for all* m, *and* $(A_m \cap A_{m+1}) = \{v_i v_{i+1} : i = 1, .., m-1\} \subset \{v_i v_{i+1} : i = 1, .., m\} = (A_{m+1} \cap A_{m+2})$. *Each graph* Γ_m *is strongly connected, and therefore - by lemma 4 of chapter 2 - has a strongly connected spanning subgraph with vertex set* E_m. *In this case,* $E_m = A_m$, *that is, there is exactly one arc between any two distinct vertices* v_i *and* v_j, *and* Γ_m *is a spanning cycle. In each of the finite graphs in the sequence, strong connectedness depends on there being an arc from the "last" vertex to the "first." The infinite graph to which the sequence converges has no "last" vertex, and hence it fails to be strongly connected.*

We can however impose conditions on an increasing sequence of finite strongly connected graphs, to guarantee that its limit be a strongly connected graph.

Lemma 59 *The infinite graph,* $\Gamma = (V, A)$, *to which the sequence of finite strongly connected graphs,* $\langle\ \Gamma_m = (V_m, A_m)\rangle$, *converges, is itself strongly connected if for all* m *there exist strongly connected spanning subgraphs,* Γ'_m, *with arc set* $E_m \subseteq A_m$, *such that* $E_m \subseteq A_{m+1}$.

Proof. Take any two distinct vertices v_i, v_j in the infinite graph, Γ. Then there exists some finite graph Γ_m in the sequence such that $v_i, v_j \in V_m$. By assumption, Γ_m is strongly connected, and therefore has at least one spanning subgraph. By definition, each strongly connected spanning subgraph contains paths of arcs connecting v_i to v_j, and vice versa. By assumption, some such spanning subgraph, Γ'_m, has arc set $E_m \subseteq A_{m+1}$, and so, by definition of an increasing sequence of graphs, $E_m \subseteq A_n$, for any $n \geq m+1$. In other words, the arcs connecting v_i to v_j in Γ'_m, are also contained in all graphs Γ_n, for any $n \geq m+1$, and therefore in the infinite graph Γ. Since v_i and v_j were chosen arbitrarily, this is true for any pair of vertices in the infinite graph, and hence Γ is strongly connected. ■

The restriction imposed in lemma 59 implies that any arcs required to strongly connect a given finite graph in the sequence should be included in the vertex set of any subsequent graph in the sequence. Observe that in the increasing sequence of finite strongly connected graphs in example 58, $v_m v_1 \in E_m$, but $v_m v_1 \notin E_n$ for any $n > m$, and hence $E_m \nsubseteq A_{m+1}$, and so the condition of lemma 59 is violated.

In fact, it is enough for strong connectedness of the infinite graph that all arcs required to strongly connect a given finite graph in the increasing

reappear in the arc set of some larger graph later in the sequence and remain for ever after in the sequence. The following lemma presents a more formal statement of this requirement.

Lemma 60 *The infinite graph, $\Gamma = (V, A)$, to which the sequence of finite strongly connected graphs, $\langle\ \Gamma_m = (V_m, A_m)\rangle$, converges, is itself strongly connected if there exist strongly connected spanning subgraphs, Γ'_m, with arc set $E_m \subseteq A_m$, for all m, such that for all $v_i v_j \in E_m$, there exists n such that $v_i v_j \in A_n \cap A_{n'}$, for all $n' > n$.*

Proof. Take any two distinct vertices v_i, v_j in the infinite graph, Γ. Then there exists some strongly connected finite graph Γ_m in the sequence such that $v_i, v_j \in V_m$. Let Γ'_m be a strongly connected spanning subgraph of Γ_m. There exists a path of arcs in Γ'_m which connect v_i to v_j, and vice versa. By assumption, these arcs also connect v_i and v_j in the graphs Γ_n and Γ'_n, or all $n' > n$. Therefore, a path of arcs from v_i to v_j, and vice versa, exists in the infinite graph Γ. Since v_i and v_j were chosen arbitrarily, this is true for any pair of vertices in the infinite graph, and hence Γ is strongly connected. ■

We now show, by example, that a strongly connected infinite graph Γ cannot necessarily be approximated by an increasing sequence of strongly connected finite subgraphs $\langle\ \Gamma_m\rangle$.

Example 61 *Consider the strongly connected infinite graph which is a spanning cycle. A spanning cycle has the property that if any* $v_i \in V_m$ *is removed, then the arcs* $v_{i-1}v_i$ *and* v_iv_{i+1} *are also removed. Therefore, the graph* $\Gamma = (V/v_i, A_V/\{v_{i-1}v_i, v_iv_{i+1}\})$ *fails to be strongly connected. The only candidate sequence of strongly connected finite subgraphs to approximate* Γ *is the sequence* $\langle\ \Gamma_m = (V_m, A_m)\rangle$, *where* $V_m = \{1, 2, ..., m\}$ *and* $A_m = \{v_mv_1, v_iv_{i+1} : i = 1, 2, ..., m-1\}$. *However, as illustrated in example 58 above, the infinite graph to which this sequence converges is not strongly connected.*

Our final lemma in this section provides a sufficient condition for a strongly connected infinite graph Γ to be the limit of an increasing sequence of finite strongly connected graphs.

Lemma 62 *Let* $\Gamma = (V, A)$ *be a strongly connected infinite graph. If there exists a partition of* V, $\mathbf{V} = \{\widetilde{V}_1, \widetilde{V}_2, ...\}$, *and a partition of* A, $\mathbf{A} = \{\widetilde{A}_1, \widetilde{A}_2, ...\}$, *such that* $\Gamma_n = (\widetilde{V}_n, \widetilde{A}_n)$ *is a strongly connected subgraph of* Γ, *for all* n, *and in addition there exist arcs from* $\widetilde{V}_n$ *to* $\widetilde{V}_{n+1}$, *and from* $\widetilde{V}_{n+1}$ *to* $\widetilde{V}_n$, *then* Γ *can be approximated by an increasing sequence of finite strongly connected graphs.*

Proof. *We can construct such a sequence as follows:* $\Gamma_1 = (V_1 = \widetilde{V}_1, A_1 = \widetilde{A}_1), \Gamma_2 = (V_2 = \widetilde{V}_1 \cup \widetilde{V}_2, A_2 = \widetilde{A}_1 \cup \widetilde{A}_2 \cup \{v_iv_j, v_jv_i : v_i \in \widetilde{V}_1, v_j \in$

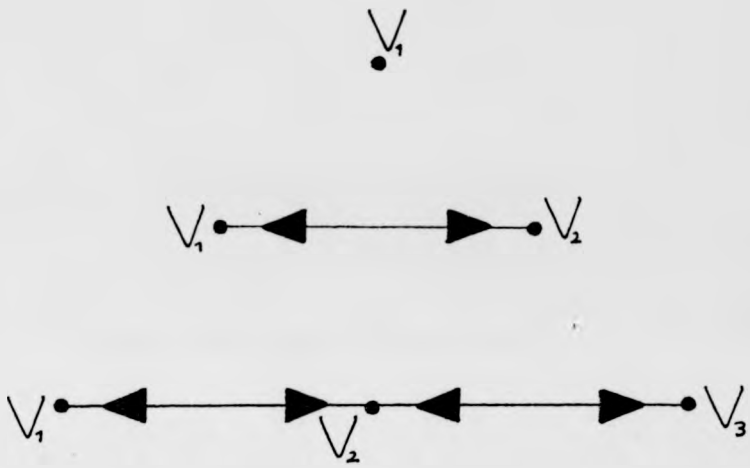

Figure 4:

$\widetilde{V}_2\}), \Gamma_3 = (V_3 = V_2 \cup \widetilde{V}_3, A_3 = A_2 \cup \widetilde{A}_3 \cup \{v_i v_j, v_j v_i : v_i \in V_2, v_j \in \widetilde{V}_3\}), \Gamma_4 = (V_4 = V_3 \cup \widetilde{V}_4, A_4 = A_3 \cup \widetilde{A}_4 \cup \{v_i v_j, v_j v_i : v_i \in V_3, v_j \in \widetilde{V}_4\}), \Gamma_n = (V_n = V_{n-1} \cup \widetilde{V}_n, A_n = A_{n-1} \cup \widetilde{A}_n \cup \{v_i v_j, v_j v_i : v_i \in V_{n-1}, v_j \in \widetilde{V}_n\})$ ■

As an example of an infinite strongly connected graph which satisfies the condition of lemma 62, consider $\Gamma = (V, A)$, where $V = \{1, ...\}$, and $A = \{v_i v_{i+1}, v_{i+1} v_i : i = 1, ...\}$, which can be approximated by the increasing sequence of finite strongly connected graphs $\Gamma_n = \langle (V_n, A_n) \rangle$, where $V_n = \{1, ..., n\}$ and $A_n = \{v_i v_{i+1}, v_{i+1} v_i : i = 1, ..., n-1\}$. The first 3 graphs in

this sequence are illustrated in figure 4.

5.3 Truncations and graphs

In this section, we apply our results for infinite directed graphs, from the previous section, to study C-irreducibility in pure exchange economies with countable sets of commodities and individuals. We ignore production due to conceptual problems in the specification of the objective function of firms. Consider a pure exchange economy E, with individuals denoted by $i \in I = \{1, ...\}$, and commodities by $l \in L = \{1, ...\}$, where I and L are both non-empty, countable sets.

The commodity space, denoted Λ, is the Euclidean space with dimension equal to the number of commodities. The vector of units is $\widetilde{1}$, and $\widetilde{1}_l$ is the vector with 1 in the lth coordinate and zero elsewhere.

The commodity space is a topological vector space with the product topology. A base for the product topology consists of the sets $\Pi_{l \in L_F} O_l \times \Pi_{l \in L/L_F} \Lambda_l$, where $O_l \subseteq \Lambda_l$ are open and $L_F \subset L$ is finite.

Trades in commodities are denoted by, $z = (z_1, ..., z_l, ...) \in \Lambda$.

An individual, i, is characterized by a pair, (Z^i, u^i), for all $i \in I$, of a feasible trade set, $Z^i \subset \Lambda$, and a utility function, $u^i : Z^i \to \Re$.

An economy is thus a collection, $E = \{I, L, (Z^i, u^i) : i \in I\}$.

Overlapping generations economies are a special case of the above framework. Time periods are denoted by $t \in T_{\underline{t},\bar{t}} = \{\underline{t}, ..., \bar{t}\}$. If time extends infinitely into the future but not into the past, that is $T_{\underline{t},\bar{t}} = T_{1,\infty}$, we say the temporal structure is simple. An individual is identified by the period in which he/she is born, that is, individual i born at t is denoted (i, t). The set or "generation"of individuals born at t is denoted I_t, and the set of all individuals is thus $I = \cup_{t \in T_{\underline{t},\bar{t}}} I_t$. A commodity l available at t is denoted (l, t). The set of commodities available at t is denoted L_t, and the set of all commodities is $L = \cup_{t \in T_{\underline{t},\bar{t}}} L_t$. Periods of trade of an individual are $T^{(h,t)} = \{\underline{t}^{(h,t)}, ..., \bar{t}^{(h,t)}\} \subseteq T_{\underline{t},\bar{t}}$, such that $t' \notin T^{(h,t)} \Longrightarrow L_{t'} \cap L^{(h,t)} = \emptyset$. The trading span of an individual is $1 \leq \mathbf{T}^{(h,t)} \leq \bar{t}^{(h,t)} - \underline{t}^{(h,t)} + 1$. The trading span of an individual may therefore be infinite.

Assumptions $A1 - A6$ from section 2.3.1 extend to infinite dimensional pure exchange economies with the following necessary modifications to assumptions $A1$ and $A3$. Assumption $A1$ is replaced by the following assumption.

$C1$ Z^i is closed in the product topology.

Assumption $A3$ is replaced by the following assumption.

$C2$ $u^i : Z^i \to \Re$ is continuous in the product topology.

In addition, we require two further assumptions.

$C3$ $0 > \Sigma_{i \in I} z_l^i > -\infty$, for all $l \in L$.

An individual does not desire a commodity l at $z \in Z^i$ if $u^i(z) = u^i(z - z_l \bar{1}_l)$, where $z_l > 0$ and $(z - z_l \bar{1}_l) \in Z^i$. Let $L^i \subseteq L$ denote the set of commodities desired by individual i, somewhere in his/her net trade set. Let $D_l = \{i \in I : l \in L^i\}$ denote the set of individuals who desire commodity l somewhere in their net trade sets.

$C4$ D_l is finite, for $l \in L$.

An allocation is a profile of individual net trades, $z^I = \{z^i \in Z^i : i \in I\}$. An allocation is feasible if and only if $\Sigma_{i \in I} z^i = 0$. Let $F = \{z^I : \Sigma_{i \in I} z^i = 0, z^i \in Z^i, \forall i \in I\}$ denote the set of all feasible allocations.

Prices are $p \in P = \Lambda_+ / \{0\}$.

Given prices p, an individual $i \in I$ solves the following maximization problem:

$$Max_{\{z\}}\ u^i(z) \qquad s.t.\ pz \leq 0,\ z \in Z \tag{B}$$

Definition 63 *A competitive equilibrium is a collection* (p^*, z^{*I}) *such that,*

(i) Given p^*, z^{*i} *solves* (B) *for each* $i \in I$;

(ii)$\Sigma_{i \in I} z^{*i} = 0$.

The aggregate domain of trades in commodities is $Z = \Sigma_{i \in I} Z^i$. At prices $p \in P$, let $Z^i(p) = \{z^i \in Z^i \cap -Z : pz^i \leq 0\}$ denote the set of affordable trades for individual i which satisfy the aggregate feasibility constraint. $\hat{Z}^i(p) = \{z^i \in Z^i(p) : z^i \in \arg\max_{z^i \in Z^i(p)} u^i(z)\}$, denotes those affordable feasible trades for individual i which give him/her the most utility. We then define, $\Phi^i(p) = \{z \in Z^i : u^i(z^i + z) > u^i(z^i), z^i \in \hat{Z}^i(p)\}$ as the set of net trades which when added to some utility maximizing affordable feasible trade at prices p, make individual i better off. Local non-satiation (a sufficient condition) implies that this set is non-empty.

Definition 64 *(price graph) The price graph of the exchange economy* E *at prices* p*, denoted* $\Gamma(E(p))$*, is a collection of vertices* V *and arcs* A *such that each vertex* v_i *corresponds to consumer* i *for* $i = 1, 2, \ldots$ *and an arc directed from* v_i *to* v_j *exists whenever there is some* $\bar{I} \subset I$*,with* $i \in \bar{I}$, $j \in I/\bar{I}$ *such that* $-Z^{\bar{I}} \cap \Phi^j(p) \neq \emptyset$*, and for any* $\bar{I}' \subset \bar{I}$ *such that* $i \in \bar{I}'$*, there is some* $m \in \bar{I}/\bar{I}'$ *such that* $-Z^{\bar{I}'} \cap \Phi^m(p) \neq \emptyset$.

Note that the above definition allows both finite and countably infinite economies. Let C denote the collection of price graphs at all prices $p \in P$.

Definition 65 *(C-irreducibility) The economy is said to be C-irreducible if every member of the collection of price graphs C is strongly connected, that is, if $\Gamma(E(p))$ is strongly connected at all $p \in P$.*

In contrast to the finite case, the assumption of continuity of the utility functions in the product topology restricts the set of economies which may be C-irreducible. For example, when individuals and commodities are indexed by calendar time which extends infinitely into the future, continuity in the product topology suggests "impatience," which may prevent commodities whose indices are sufficiently far away from that of an individual with index t from entering $\Phi^t(p)$.

In order to draw comparisons between C-irreducible and irreducible economies with countable sets of individuals and commodities, we extend the definition of an allocation graph presented in section 2.3.2, to allow for an infinite dimensional vertex set.

Definition 66 *(allocation graph) The allocation graph of economy E at allocation (z^I), denoted $\Gamma(E(z^I))$, is a collection of arcs A and vertices V,*

where each vertex v_i corresponds to consumer i for $i \in I$, and an arc directed from v_i to v_j exists whenever there is some $\bar{I} \subset I$,with $i \in \bar{I}$, $j \in I/\bar{I}$ such that $(-Z^{\bar{I}}) \cap \Phi^j(z^I) \neq \emptyset$, and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $(-Z^{\bar{I}'}) \cap \Phi^m(z^I) \neq \emptyset$.

We now define a truncated economy. Consider an economy E. To this economy we can associate a truncated economy $E^n = \{I^n, L^n, (Z^{in}, u^{in}) : i \in I^n\}$, constructed as follows. Consider a finite subset of the set of commodities $L^n \subset L$. Let $\Lambda^n \subset \Lambda$ be the commodity subspace with dimension equal to the cardinality of L^n. Let $I^n = \{i \in I : Z^i \cap \Lambda^n \neq 0\}$. A truncation is said to be non-trivial if either L^n is a proper subset of L, or I^n is a proper subset of I, for some n. For a net trade $z \in \Lambda$, $z^n = proj_{\Lambda^n} z$. For a net trade $z^n \in \Lambda^n$, and an individual $i \in I$, let $\hat{z}^{ni} \in \Lambda$ be the commodity bundle defined by $\hat{z}^{ni}_l$, for $l \in L^n$, and $\hat{z}^{ni}_l = 0$ for $l \notin L^n$. For prices $p \in P$, $p^n = proj_{\Lambda^n} p$. For prices $p^n \in P^n = \Lambda^n_+/\{0\}$, we write $\hat{p}^n \in P$ for the prices defined by $\hat{p}^n_l$ for $l \in L^n$ and $\hat{p}^n_l = 0$ for $l \notin L^n$. Now define $Z^{ni} \subset \Lambda$ as $Z^{ni} = \{z^n \in \Lambda^n : \hat{z}^{ni} \in Z^i\}$ with $u^{ni} : Z^{ni} \to \Re$. Observe that $u^{ni}(z^n) = u^i(\hat{z}^{ni})$. An allocation in the truncated economy is z^{I^n} such that $z^{ni} \in Z^{ni}$ for all $i \in I^n$. An allocation is feasible if and only if $\Sigma_{i \in I^n} z^i = 0$.

Consider a sequence $(L^n : L^n \subseteq L^{n+1},\ n = 1, 2, ...)$ such that $\cup_{n=1}^{\infty} L^n = L$.

Consider the associated sequence of truncated economies $(E^n : n = 1, 2, ...)$. Then, $I^n \subseteq I^{n+1}$, for all $n = 1, 2...$ and $\cup_{n=1}^{\infty} I^n = I$. We say that the sequence of truncated economies $(E^n : n = 1, 2, ...)$ converges to the economy E, and the corresponding sequence of prices $(p^n : n = 1, 2...)$ converges to $p \in \Lambda_+/\{0\}$. Observe that by construction, each sequence of economies $(E^n : n = 1, 2, ...)$ has a unique limit economy E, and each sequence of prices $(p^n : n = 1, 2...)$ has a unique limit $p \in P$.

Now consider the truncated economy E^n and prices p^n. We can associate to this economy the price graph $\Gamma(E^n(p^n)$. For notational simplicity, without ambiguity let $\Gamma_n = \Gamma(E^n(p^n)$. For any sequence of truncated economies $(E^n : n = 1, 2, ...)$ and corresponding sequence of prices $(p^n : n = 1, 2...)$ there is a corresponding sequence of price graphs $(\Gamma_n : n = 1, 2...)$. If E and p are the corresponding limits, let $\Gamma(E(p))$, the price graph corresponding to E and p, be defined as the limit of the sequence $(\Gamma_n : n = 1, 2...)$. By construction, this limit exists and is unique.

Alternatively, consider the truncated economy E^n and allocation z^{I^n}. We can associate to this economy the allocation graph $\Gamma(E^n(z^{I^n})$. For any sequence of truncated economies $(E^n : n = 1, 2, ...)$ and corresponding sequence of allocations $(z^{I^n} : n = 1, 2...)$ there is a corresponding sequence of allocation

graphs $(\Gamma(E^n(z^{I^n}) : n = 1, 2...)$. If E and z^I are the corresponding limits, let $\Gamma(E(z^I))$, the allocation graph corresponding to E and z^I, be defined as the limit of the sequence $(\Gamma(E^n(z^{I^n}) : n = 1, 2...)$. By construction, this limit exists and is unique.

The argument for the existence of competitive equilibria in infinite dimensional economies proceeds by considering a sequence of finite truncated economies which tend to the full economy in the limit. The first step of the proof is to demonstrate the existence of competitive equilibria for each of the finite truncated economies. This requires that each finite truncated economy be irreducible/C-irreducible. The second step of the proof is to show that the limit of the sequence of competitive equilibria for the finite economies constitutes a quasi-equilibrium for the full infinite dimensional economy. Irreducibility/C-irreducibility of the full economy then guarantees the existence of a competitive equilibrium. One can impose the C-irreducibility condition directly on the full economy, E, and/or one can assume C-irreducibility of the truncated economies, $(E^n : n = 1, 2, ...)$.

Definition 67 *(sequential C-irreducibility) An economy is sequentially C-irreducible if it is the limit of a sequence of truncated C-irreducible economies.*

Similarly, an economy is said to be sequentially irreducible if it is the

limit of a sequence of truncated irreducible economies.

In view of example 58, we might expect that an economy which is sequentially C-irreducibility may nevertheless fail to be C-irreducible. The following example illustrates that this is indeed the case.

Example 68 *Consider an overlapping generations economy in which time extends infinitely into the future. There are two commodities available each period, (l,t), for $l = 1,2$. Each period, an individual is born, $(1,t)$, with preferences*

$$u^{(1,t)} = \begin{cases} = z_{(1,t)} + z_{(2,t+1)} - 1 & \text{for } 0 \leq z_{(2,t+1)},\ z_{(1,t)} \leq 1,\ z_{(2,t+1)} + z_{(1,t)} \leq 1 \\ = \frac{2z_{(2,t+1)}+2z_{(1,t)}-2}{1+z_{(1,t)}} & \text{for } 0 \leq z_{(2,t+1)} \leq 2, 0 \leq z_{(1,t)} \leq 1, z_{(2,t+1)} + z_{(1,t)} \geq 1 \\ = z_{(2,t+1)} & \text{for } z_{(2,t+1)} \geq 2 \text{ or } z_{(1,t)} \geq 1 \end{cases}$$

and net trade set $Z^{(1,t)} = \{z : z^{(1,t)}_{(2,t)} \geq -2,\ z^{(1,t)}_{(l,s)} \geq 0 \text{ for } (l,s) \neq (2,t)\}$. An additional individual is born in period 1, with preferences $u^2 = z_{(2,1)}$ and net trade set $Z^2 = \{z : z^2_{(1,t)} \geq -1/2,\ z^2_{(2,t)} \geq 0 \text{ for all } t\}$. Observe that individual $(1,t)$ always desires commodity $(2,t+1)$ but desires commodity $(1,t)$ only as long as $z_{(2,t+1)} < 2$.

Any finite truncated economy $E^{\bar{t}}$ associated with the set of commodities $L^{\bar{t}} = \{(l,t) : t = 1,...,\bar{t}\}$ is C-irreducible. To see this, note that at any prices p, the arcs $v_{(1,t)}v_{(1,t-1)}$ and $v_2v_{(1,t)}$ exist in the associated price graphs. In other

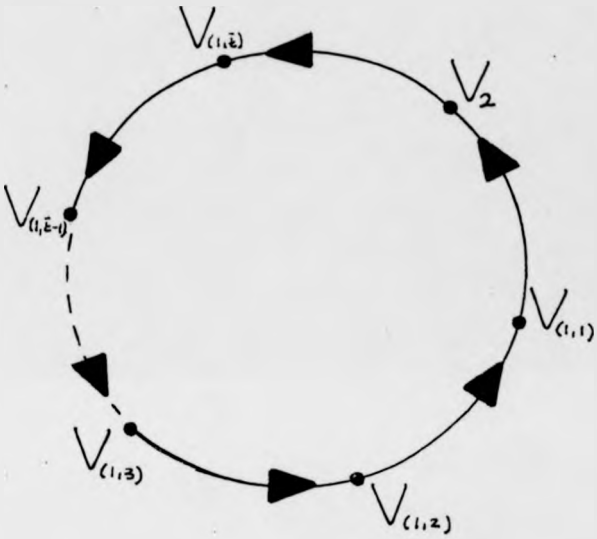

Figure 5:

words, each price graph has the strongly connected graph pictured in figure 5 as a spanning subgraph. Competitive equilibrium prices for the truncated economy $E^{\bar{t}}$ *are* $p^{*\bar{t}} = (p_1^{*\bar{t}} = (0,1), p_2^{*\bar{t}} = (0,1), ..., p_{\bar{t}-1}^{*\bar{t}} = (0,1), p_{\bar{t}}^{*\bar{t}} = (4,1))$ *and the associated equilibrium allocation is* $z^{*\bar{t},2} = (z_1^{*\bar{t},2} = (-1/2, 2), z_t^{*\bar{t},2} = (0,0)$ *for* $t \neq 1)$, $z^{*\bar{t},(1,t)} = (z_t^{*\bar{t},(1,t)} = (0,-2), z_{t+1}^{*\bar{t},(1,t)} = (0,2), z_s^{*\bar{t},2} = (0,0)$ *for* $s \neq t, t+1)$, *for* $t = 1, 2, ...\bar{t}-1$ *and* $z^{*\bar{t},(1,\bar{t})} = (z_{\bar{t}}^{*\bar{t},(1,\bar{t})} = (1/2,-2), z_t^{*\bar{t},(1,\bar{t})} = (0,0)$ *for* $t \neq \bar{t})$.

The full economy E however, fails to be C-irreducible. At prices $p = (p_t = (1,1)$, *for all* t), *each individual* $(1,t)$ *has the utility maximising feasible affordable net trade* $z^{(1,t)} = (z_t^{(1,t)} = (0,-2), z_{t+1}^{(1,t)} = (0,2), z_s^{*\bar{t},2} = (0,0)$ *for* $s \neq t, t+1)$. *This means that no individual can be made better of by a net trade from individual 2. Thus, if we consider the partition* $\{V^1 = v_2, V^2 = V/\{v_2\}\}$ *of the vertex set of the associated price graph, there is no arc from* V^1 *to* V^2, *hence the price graph is not strongly connected.*

Note also that no competitive equilibria exist for the full economy. To see this, suppose that p^* *were equilibrium prices. Since individuals* 2 *and* $(1,t)$ *always desire commodities* $(2,1)$ *and* $(2,t+1)$ *respectively* $p^*_{(2,t)} > 0$. *Since only individual* 2 *desires commodity* $(2,1)$, *equilibrium requires that individual* 2 *have positive income at* p^*, *that is* $p^*_{(1,t)} > 0$ *for some* t. *However, since commodity* $(2,t)$ *is only desired by individual* $(1,t-1)$, *equilibrium requires that* $z^{*(1,t)}_{(2,t+1)} = 2$, *which in turn implies that nobody desires* $(1,t)$ *and so* $p_{(1,t)} = 0$, *a contradiction. Note that as the period of truncation tends to infinity, equilibrium prices tend to* $\lim_{\bar{t}\to\infty} p^{*\bar{t}} = ((0,1),(0,1),...,(0,1),...)$ *at which individual 2's revenue vanishes to zero.*

Note that in the above example, each finite truncated economy is irreducible in the McKenzie sense, whereas the full economy is not. In other

words, the example also serves to illustrate that sequentially irreducibility does not necessarily imply irreducibility.

Here, we apply lemma 60 to provide a sufficient condition on the sequence of truncated economies such that sequential C-irreducibility does indeed imply C-irreducibility of the full economy.

Proposition 69 *Consider a sequence of strongly connected price graphs* $(\Gamma_n = (V_n, A_n) : n = 1, 2, ...)$ *associated with the sequence of truncated economies* $(E^n : n = 1, 2, ...)$ *and the corresponding sequence of prices* $(p^n : n = 1, 2...)$. $\Gamma(E(p))$ *is strongly connected if for any* n, *there is a strongly connected spanning subgraph* $\Gamma'_n = (V_n, A'_n) \subseteq \Gamma_n$ *such that for all* $v_i v_j \in A'_n$, *there exists* k *such that* $v_i v_j \in A_k \cap A_{k'}$, *for all* $k' > k$.

Proof. By assumption, Γ'_n is strongly connected and therefore contains paths which connect v_i to v_j, and vice versa for all $v_i, v_j \in V_n$. By assumption, for some k, a path connecting v_i and v_j in Γ'_n is also contained in all graphs $\Gamma_{k'}$ for all $k' > k$, and therefore in the graph $\Gamma(E(p))$. Since v_i and v_j were chosen arbitrarily, this is true for any pair of vertices in the infinite graph, and hence the infinite graph is strongly connected. ■

This restriction implies that any arc which is needed to strongly connect the price graph of some finite truncated economy reappears in the arc set

of some price graph later in the sequence, and remains in the arc set of all subsequent price graphs. This rules out the possibility that the arcs which strongly connect the price graph in a finite truncated economy are artificially created by the way the economy is truncated. This condition is violated in example 67.

Despite the result that sequential irreducibility does not necessarily imply irreducibility of the full economy, Balasko, Cass, and Shell (1980) prove the existence of competitive equilibria in infinite dimensional economies, imposing an irreducibility condition on the finite truncated economies alone, namely intertemporal irreducibility.

Definition 70 *(intertemporal irreducibility (adapted from Balasko, Cass, and Shell (1980))) Let a t-economy be specified by the tastes and feasible net trade sets of all individuals born up to and including period t,* $E^t = \{I_t, (Z^i, u^i) : i \in I_t\}$. *The infinite economy is intertemporally irreducible if there exists a subsequence of periods* $\{t_v\}$ *such that, given any* $t_v -$ *economy and any feasible allocation sequence* $z \in Z$ *for the infinite economy, then for any non-trivial partition of the set of individuals* $\{I^1_{t_v}, I^2_{t_v}\}$, *there exists* $x^i \leq 0$ *for* $i \in I^1_{t_v}$ *and* $z'^i \geq 0$ *for* $i \in I^2_{t_v}$ *such that,*

$\sum_{i \in I^1_{t_v}} x^i_{t,l} = 0$ *whenever* $\sum_{i \in I^1_{t_v}} \underline{z}^i_{t,l} = 0$ *for* $1 \leq t \leq t_v + 1$

$\sum_{i \in I^2_{t_v}} z'^i \leq \sum_{i \in I^1_{t_v}} \underline{z}^i_{t,l} + \sum_{i \in I^1_{t_v}} x^i_{t,l}$

$u^i(z'^i) \geq u^i(z^i)$ *for all* $i \in I^2_{t_v}$

$u^i(z'^i) > u^i(z^i)$ *for some* $i \in I^2_{t_v}$

Note that intertemporal irreducibility implies that in the sequence of finite truncated economies, there is some subsequence of finite economies which are irreducible. This guarantees the existence of sequences of prices and allocations which constitute competitive equilibria for those finite economies. The limits of these sequences constitute a quasi-equilibrium for the full economy. In order to see why intertemporal irreducibility is sufficient for existence (without the assumption that the full economy E be irreducible) we restate proposition 68 in terms of restrictions on allocation graphs.

Proposition 71 *Consider a sequence of strongly connected allocation graphs* $(\Gamma_n = (V_n, A_n) : n = 1, 2, ...)$ *associated with the sequence of truncated economies* $(E^n : n = 1, 2, ...)$ *and the corresponding sequence of prices* $(z^{I_n} : n = 1, 2...)$. $\Gamma(E(z^I))$ *is strongly connected if for any* n, *there is a strongly connected spanning subgraph* $\Gamma'_n = (V_n, A'_n) \subseteq \Gamma_n$ *such that for all* $v_i v_j \in A'_n$, *there exists* k *such that* $v_i v_j \in A_k \cap A_{k'}$, *for all* $k' > k$.

Intertemporal irreducibility implies sufficient restrictions on the sequence of finite truncated economies such that the limit of the sequence is itself an irreducible economy. Since the allocation sequence, $z \in Z$, is fixed in definition 69, any element of the vertex set of $\Gamma(E^{t_v}(z^{It_v}))$ is necessarily also an element of the vertex set of the allocation graph of any larger truncated economy in the sequence. The $t_v - economy$ is assumed to be irreducible and hence, the allocation graph $\Gamma(E^{t_v}(z^{It_v}))$, has a spanning subgraph, the vertices of which are all members of the vertex sets of the allocation graph of any larger truncated economy in the sequence. Since t was chosen arbitrarily, this holds generically for any t_v. By proposition 71 therefore, the full economy is irreducible.

Burke (1988), and Geanakoplos and Polemarchakis (1986) prove the existence of competitive equilibria in infinite dimensional economies assuming irreducibility of the full economy alone. However, as the following example illustrates, an economy which is C-irreducible may not be the limit of any non-trivial sequence of truncated C-irreducible economies, that is, C-irreducibility does not imply sequential C-irreducibility.

Example 72 *There are time periods $t = 1, 2, \ldots\ldots, \aleph_0$, where $\aleph_0$ is the number of elements in any countably infinite set (see Simmons (1963) page 34-35).*

In each period t there are two commodities, $l = 1t, 2t$, and one individual, $i = t$, is born with preferences, $u^t = \begin{Bmatrix} z^t_{1t+1} & for\ t < \aleph_0 \\ z^t_{2t} & for\ t = \aleph_0 \end{Bmatrix}$, and feasible trade sets, $Z^t = \{z : z^t_{1t} \geq -1,\ z^t_{1t'} \geq 0,\ for\ all\ t' \neq t,\ z^t_{2t} \geq 0,\ for\ all\ t\}$ for t. Additionally, there is an individual $i = 0$ born in period $t = 1$ with preferences $u^0 = z^0_{11}$, and feasible trade set, $Z^0 = \{z : z^0_{2t} \geq -1,\ z^0_{1t} \geq 0,\ for\ all\ t\}$. At all prices $p \in \Lambda_+/\{0\}$, the price graph of this economy is a spanning cycle and the economy is therefore C-irreducible. A competitive equilibrium for this economy is a pair (p^*, z^*), where $p^*_t = (1, 0)$ for all $t < \aleph_0$, $p^*_{\aleph_0} = (1, 1)$, $z^{*t} = (z^{*t}_{1t} = -1, z^{*t}_{1t+1} = 1, z^{*t}_{lt} = 0\ for\ all\ lt \neq 1t, 1t+1)$ for $t < \aleph_0$, $z^{*0} = (z^{*0}_{11} = 1, z^{*0}_{2t} = -1,\ for\ all\ t,\ z^{*0}_{1t} = 0\ for\ all\ t > 1)$, $z^{*\aleph_0} = (z^{*\aleph_0}_{2t} = 1$ $for\ all\ t, z^{*\aleph_0}_{1\aleph_0} = -1, z^{*\aleph_0}_{1t} = 0\ for\ all\ t \neq \aleph_0)$. However, the infinite economy cannot be approximated by any increasing sequence of finite C-irreducible economies. By definition, truncating the economy involves removing vertices, and therefore arcs. Removing arcs from a spanning cycle disconnects the graph, and therefore any truncated economy, E^n, fails to be C-irreducible.

The following proposition provides a sufficient condition on the set of price graphs $\Gamma(E(p))$ of economy E at all prices $p \in P$ so that a C-irreducible economy can be approximated by a sequence of truncated C-irreducible economies. First, we need the following notation. Let $\mathbf{I} = \{\widetilde{I}_1, \widetilde{I}_2, ...\}$ be

a collection of non-empty finite subsets $\widetilde{I}_n$ of I, with $\widetilde{I}_n \cap \widetilde{I}_{n'} = \emptyset$, $\cup_n \widetilde{I}_n = I$. Let $\mathbf{V} = \{\widetilde{V}_1, \widetilde{V}_2, ...\}$ be the corresponding partition of the vertex set of the price graph $\Gamma(E(p))$. For any $n = 1, 2, ...$, let $L_j^n \subset L$ be a set of commodities such that $Z^i \subset \Lambda_j^n$, for all $i \in \cup_{n' \leq n} \widetilde{I}_n$. Let $L^n = \cap_j L_j^n$, and let $E^n = \{I^n, L^n, (Z^{in}, u^{in}) : i \in I^n\}$, be the associated truncated economy, with Γ_n the associated price graph at p^n.

Definition 73 ***Condition 74*** *(Sufficient arcs condition): For each price vector $p \in \Lambda_+/\{0\}$, there exists a partition of A, $\mathbf{A} = \{\widetilde{A}_1, \widetilde{A}_2, ...\}$, such that $\Gamma_n = (\widetilde{V}_n, \widetilde{A}_n)$ is a strongly connected subgraph of $\Gamma(E(p))$. For any $n = 1, 2, ...$, (i) There exists an arc from $\widetilde{V}_n$ to $\widetilde{V}_{n+1}$, and from $\widetilde{V}_{n+1}$ to $\widetilde{V}_n$, (ii) If $i, j \in I^n$ and there exists a path between v_i and v_j in $\Gamma(E(p))$, then there exists a path between v_i and v_j in Γ_n.*

Note that the economy described in example 72 fails to satisfy the sufficient arcs condition. An example of an economy which does satisfy the sufficient arcs condition is the Samuelson overlapping generations economy. In his model, a generation of individuals is born each period and lives for two periods. There is one commodity in each period. Individuals are endowed with 1 unit of the commodity in the first period of their lives, and derive utility from consumption in both periods. The key restriction of the sufficient

arcs condition, applied to overlapping generations economies, is that there must be enough arcs across individuals in different generations.

Proposition 75 *Suppose that for all prices $p \in P$, $\Gamma(E(p))$ is such that the sufficient arcs condition is satisfied. Then, if E is C-irreducible, it is the limit of some sequence of truncated C-irreducible economies $(E^n : n = 1, 2...)$.*

Proof. We construct a sequence of truncated C-irreducible economies that converge to E. Let $\mathbf{I} = \{\widetilde{I}_1, \widetilde{I}_2, ...\}$ be a collection of non-empty finite subsets $\widetilde{I}_n$ of I, with $\widetilde{I}_n \cap \widetilde{I}_{n'} = \emptyset$, $\cup_n \widetilde{I}_n = I$. Let $\mathbf{V} = \{\widetilde{V}_1, \widetilde{V}_2, ...\}$ be the corresponding partition of the vertex set of the price graph $\Gamma(E(p))$. For any $n = 1, 2, ...$, let $L_j^n \subset L$ be a set of commodities such that $Z^i \subset \Lambda_j^n$, for all $i \in \cup_{n' \leq n} \widetilde{I}_n$. Let $L^n = \cap_j L_j^n$, and let $E^n = \{I^n, L^n, (Z^{in}, u^{in}) : i \in I^n\}$, be the associated truncated economy, with Γ_n the associated price graph at p^n. By construction , $L^n \subseteq L^{n+1}$, $\cup_n L^n = L$. Therefore, $(E^n : n = 1, 2...)$ converges to E. As $\Gamma(E(p))$ satisfies the sufficient arcs condition, $\Gamma_n = (\widetilde{V}_n, \widetilde{A}_n)$ is strongly connected, for all n. By construction the sequence $(\Gamma_n : n = 1, 2...)$ converges to $\Gamma(E(p))$. ■

Despite the possible failure of a C-irreducible economy to be sequentially C-irreducible, it is nevertheless sufficient for existence to assume that the full economy is C-irreducible, without requiring that the sufficient arcs condition

hold. Following Geanakoplos and Polemarchakis (1991), we can modify each finite truncated economy in the sequence by perturbing the utility functions and net trade sets of individuals, such that each modified truncated economy is C-irreducible, and the perturbation vanishes at the limit. Note that, if we were to perturb only individuals' net trade sets, this may involve infinitely many commodities. It is then rather involved to show that the perturbation vanishes at the limit (Burke (1988)). However, by perturbing both utility functions and net trade sets, we are able to perturb net trade sets along one dimension only, thus simplifying the limiting argument.

The modified truncated economy E'^n is obtained by perturbing fundamentals as follows,

$$u'^{n,i}(z^i) = u^i(\bar{z}^{n,i} + \tfrac{1}{n} z_i \widetilde{1}) \quad \text{for } i \in I^n$$

$$Z'^{n,i} = Z^i - \tfrac{1}{n}\widetilde{1}_{i+1} \quad \text{for } i \in I^n/\{n\}$$

$$Z'^{n,n} = Z^n - \tfrac{1}{n}\widetilde{1}_1$$

Since u^i is weakly monotone, $u'^{n,i}$ is strictly increasing in commodity i. Individual $i < n$ can supply positive amounts of commodity $i+1$ and individual n can supply positive amounts of commodity 1. Therefore, each modified truncated economy is C-irreducible, and a competitive equilibrium for E'^n, (p^{*n}, z^{*n,I^N}), exists. Geanakoplos and Polemarchakis (1991) show

that the limit of the sequence $\langle (p^{*n}, z^{*n,I^N}) \rangle_{n=1,\ldots}$ is a quasi-equilibrium for the economy E.

Proposition 76 *If the infinite dimensional economy E satisfies $C1 - C4$, $A2$, and $A4 - A6$, and in addition is C-irreducible, then a competitive equilibrium exists.*

Proof. Under assumptions $C1 - C4$, $A2$, and $A4 - A6$ a quasi equilibrium (p^*, z^{*I}) for the economy exists (see Geanakoplos and Polemarchakis (1991)). $A2$ guarantees that at a quasi-equilibrium there is at least one individual $\widehat{i}$ for whom $\exists z^{\widehat{i}} \in Z^{\widehat{i}}$ s.t. $p^* z^{\widehat{i}} < 0$. Consider the partition of the set of vertices of $\Gamma(E, z^{*I})$, $\{V^1, V^2\}$ such that $V^1 = \{v_i : \exists z^i \in Z^i \text{ s.t. } p^* z^i < 0\}$, and $V^2 = \{v_i : \nexists z^i \in Z^i \text{ s.t. } p^* z^i < 0\}$. Note that, by definition of a quasi-equilibrium, $\forall v_i \in V^1$, $z^{*i} \in \hat{Z}^i(p^*)$.We prove that V^2 is empty. Suppose by contradiction that V^2 is non-empty. By assumption the graph $\Gamma(E(p^*))$ is strongly connected and so, by lemma 5, there exists an arc from V^1 to V^2, and an arc from V^2 to V^1. It follows from the definition of C-irreducibility that there exists $v_i \in V^2$ and $v_j \in V^1$ such that $-Z^{\bar{I}} \cap \Phi^j(p^*) \neq \emptyset$, with $i \in \bar{I}$, $j \notin \bar{I}$ and for any $\bar{I}' \subset \bar{I}$ such that $i \in \bar{I}'$, there is some $m \in \bar{I}/\bar{I}'$ such that $-Z^{\bar{I}'} \cap \Phi^m(p) \neq \emptyset$. Since individual j is utility maximizing at z^{*I}, it follows that there is an $m_0 \in \bar{I}$ such that $v_{m_0} \in V^1$. But then, there exists

a sequence of individuals $(m_1, ..., m_K)$ such that each $v_{m_k} \in V^1$and the arcs $v_{k-1}v_k$, $v_K v_i$, $k = 1, ..., K$ exist. Therefore $v_i \in V^1$, a contradiction. ■

5.4 Conclusion

The main implication of the results proved in this section is that assuming C-irreducibility for the economy E is a weaker assumption than assuming sequential C-irreducibility, and therefore when working with infinite dimensional economies it is much less restrictive to prove the existence of competitive equilibria by imposing C-irreducibility on the full economy alone. Note that by using the graph theoretic characterization of irreducibility, the results of this section extend to irreducible exchange economies with countable sets of individuals and commodities. In particular, assuming irreducibility of the full economy, E, (Burke (1988); Geanakoplos and Polemarchakis (1991)), is weaker than assuming irreducibility of the truncated economies, $(E^n : n = 1, 2, ...)$, (Wilson (1981); Balasko, Cass, and Shell (1980)).

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