Abelian orbifolds in dimension four and crepant resolutions via G-Hilbert schemes

by

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Contents

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Declarations

I declare that, to the best of my knowledge, the material in this thesis is the original work of the author except where stated explicitly in the text.

The material in this thesis is submitted for the degree of Ph.D. to the University of Warwick only.

Abstract

We study the quotient $X = \mathbb{C}^4/G$, where the group $G \cong (\mathbb{Z}/r)^{\oplus 3} \subset \mathrm{SL}\,(4, \mathbb{C})$ acts by $\frac{1}{r}\,(1, -1, 0, 0) \oplus \frac{1}{r}\,(1, 0, -1, 0) \oplus (1, 0, 0, -1)$. The affine quotient $X = \mathbb{C}^4/G$ is a Gorenstein hypersurface singularity $(x_1 x_2 x_3 x_4 = y^r)$. In this thesis, we give an explicit description of the G-Hilbert scheme G-Hilb $\mathbb{C}^4$ through its toric fan. We show that it is an irreducible toric variety that is a discrepant resolution of singularities of X. Furthermore, we construct a certain class of crepant resolutions of X, called the special crepant resolutions, that are obtained from the G-Hilb $\mathbb{C}^4$ by a series of contractions of curves.

Chapter 1

Introduction

In this thesis we study a class of abelian quotient singularities in dimension four and their resolutions. We pay special attention to the G-Hilbert scheme G-Hilb $\left(\mathbb{C}^4\right)$ and its relation to certain "special" crepant resolutions.

For a finite group $G \subset \mathrm{SL}\,(n, \mathbb{C})$, a G-cluster is a G-invariant subscheme $Z \subset \mathbb{C}^n$ such that $H^0\left(\mathcal{O}_Z\right)$ is the regular representation of G. The G-Hilbert scheme G-Hilb $(\mathbb{C}^n)$ is the moduli space of G-clusters. The G-Hilbert scheme was introduced by Ito and Nakamura [18] for finite groups $G \subset \mathrm{SL}\,(2, \mathbb{C})$. In the same work, they proved that the G-Hilbert scheme is the minimal resolution of the quotient $\mathbb{C}^2/G$. In dimension three, the crepant resolutions are no longer unique. Nakamura [24] and subsequently Craw and Reid [9] proved that for a finite abelian group $G \subset \mathrm{SL}\,(3, \mathbb{C})$, the G-Hilbert scheme is a crepant resolution of singularities. Bridgeland, King and Reid [2] generalised this result for all finite (including nonabelian) groups $G \subset \mathrm{SL}\,(3, \mathbb{C})$. After King [22] introduced the theory of McKay quiver representations, equivalent to the theory of G-constellations, Craw and Ishii [6] proved that for a finite abelian $G \subset \mathrm{SL}\,(3, \mathbb{C})$, every projective crepant resolution is isomorphic to a fine moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations.

In dimension four however, Gorenstein singularities do not necessarily have crepant resolutions. In her PhD thesis, Davis [13] gives a sufficient and necessary condition for existence of crepant resolutions for a quotient by the group action of

type $\frac{1}{r}(1,1,a,r-a-2)$. In the cases where crepant resolutions exist, she shows that, although G-Hilb $\mathbb{C}^4$ is not crepant, it paves the way for constructing a number of crepant resolutions. This phenomenon of traps, further described by Davis, Logvinenko and Reid [12], exhibits some similarities to the subdivisions of the octahedra in this thesis.

1.1 Statement of the results

Unless otherwise stated, throughout this thesis, $G \cong (\mathbb{Z}/r)^{\oplus 3}$ acts on $\mathbb{C}^4$ by

$$\tfrac{1}{r}(1,-1,0,0) \oplus \tfrac{1}{r}(1,0,-1,0) \oplus \tfrac{1}{r}(1,0,0,-1). \tag{1.1}$$

The action of type (1.1) simply means that the representation $G \to \mathrm{SL}(4,\mathbb{C})$ induced by this action is defined by

$$\alpha \mapsto \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \beta \mapsto \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ \gamma \mapsto \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon^{-1} \end{bmatrix},$$

where $\varepsilon = e^{\frac{2\pi \mathrm{i}}{r}}$ is a primitive r-th root of unity and $\{\alpha,\beta,\gamma\}$ is a choice of group generators. Notice that the group G is the maximal diagonal abelian group of exponent r in $\mathrm{SL}(4,\mathbb{C})$:

$$G \cong \{g = \mathrm{diag}(g_1,g_2,g_3,g_4) \mid g^r = I_4\} \subset \mathrm{SL}(4,\mathbb{C}).$$

The dual action of G on V^* defined by $(g \cdot f)(z) := f(g \cdot z)$, for all $g \in G$, $f \in V^*$, and $z \in V$ extends to a G-action on $\bigoplus_{k\geq 0} \mathrm{Sym}^k(V^*) = \mathbb{C}[x_1,x_2,x_3,x_4]$. Let R be the invariant ring $\mathbb{C}[x_1,x_2,x_3,x_4]^G$. Then the **affine quotient** is defined as $X = \mathbb{C}^4/G := \mathrm{Spec}(R)$, and it parametrises all the G-orbits in $\mathbb{C}^4$, see Proposition 2.4 of [5]. As

$$R = \mathbb{C}\left[x_1^r, x_2^r, x_3^r, x_4^r, x_1x_2x_3x_4\right] \cong \mathbb{C}\left[y_1, y_2, y_3, y_4, w\right] / \left(y_1y_2y_3y_4 - w^r\right),$$

the quotient $X = \mathbb{C}^4/G$ is a hypersurface ($w^r = y_1y_2y_3y_4$) in $\mathbb{C}^5_{\langle y_1,\ldots,y_4,w\rangle}$, singular along the six coordinate planes ($w = y_i = y_j = 0$) for distinct $i, j \in \{1, 2, 3, 4\}$.

The main result of this thesis is the explicit description of the G-Hilbert scheme in terms of its toric fan. Similar to the other known result (see [13]) in dimension four, the G-Hilbert scheme is not a crepant resolution of singularities, but it is closely related to a certain class of crepant resolutions, which we call **special crepant resolutions**.

The special crepant resolutions are constructed in Chapter 2 using the methods of toric geometry. In the last section of the chapter, we show that we can always go from one special crepant resolution to the other by a sequence of ordinary flops. Chapter 3 focuses on the G-Hilbert scheme. The chapter starts by constructing the fan $\Sigma_{\text{G-Hilb}}$ (see Definition 3.1.1) and stating the main theorem:

Theorem (3.1.1)**.** *The G-Hilbert scheme* $\text{G-Hilb}\left(\mathbb{C}^4\right)$ *is an irreducible toric variety defined by the fan* $\Sigma_{\text{G-Hilb}}$ *from Definition 3.1.1.*

Using its toric fan, we show that $\text{G-Hilb}\left(\mathbb{C}^4\right)$ is a resolution of singularities (see Corollary 3.1.3) of X, but it is not crepant: there are exactly $\binom{r+1}{3}$ discrepant exceptional divisors all of which are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (Corollary 3.1.4). An observation that contracting a single factor $\mathbb{P}^1$ of a discrepant exceptional divisor $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ preserves the smoothness, gives rise to the special crepant resolutions of Chapter 2. See Corollary 3.1.5.

Chapter 4 contains several open problems and worked examples for small r. In Section 4.1 we conjecture that for every projective special crepant resolution Y there exists a stability parameter θ such that $Y \cong \mathcal{M}_\theta$. The worked examples of Section 4.3 seem to support the conjecture when $Y = Y_{[12\text{-}34]}$. The worked examples of Section 4.4 are here as an early attempt to understand the ideas of Reid's recipe,

applied to the special crepant resolution $Y_{[12\text{-}34]}$.

Section 4.5 provides an interesting application of the results of this thesis. If Conjecture 4.5.1 is true, the existence of crepant resolutions and a clear description how to construct them for $X = \mathbb{C}^4 / (\mathbb{Z}/r)^{\oplus 3}$ implies existence and constructibility of crepant resolutions for a potentially large class of finite abelian groups in $\mathrm{SL}(4, \mathbb{C})$.

Chapter 2

Crepant resolutions

In this chapter we construct certain crepant resolutions of the singularity $\mathbb{C}^4/G$, for $G \cong (\mathbb{Z}/r)^{\oplus 3}$ acting by $\frac{1}{r}(1,-1,0,0) \oplus \frac{1}{r}(1,0,-1,0) \oplus \frac{1}{r}(1,0,0,-1)$ using the methods of toric geometry. The main theorem of the chapter is Theorem 2.3.1 which gives an explicit description of all the special crepant resolutions.

Most of the necessary background in toric geometry can be found in first three chapters of [15] and first two chapters of [11]. Another good source is [3], which is a more detailed treatment of the subject and also a great source of worked examples. The notation in this thesis is compatible with these resources.

2.1 X is toric

The quotient variety $X = \mathbb{C}^4/G$ is a hypersurface with the corresponding polynomial ring $\mathbb{C}[x_1^r, x_2^r, x_3^r, x_4^r, x_1x_2x_3x_4]$. In this section we observe that X is a toric variety.

Define the lattice

$$N = \mathbb{Z}^4_{\langle e_1,e_2,e_3,e_4\rangle} + \mathbb{Z}\cdot\tfrac{1}{r}(1,-1,0,0) + \mathbb{Z}\cdot\tfrac{1}{r}(1,0,-1,0) + \mathbb{Z}\cdot\tfrac{1}{r}(1,0,0,-1), \quad (2.1)$$

where $\{e_1, e_2, e_3, e_4\}$ is the standard basis for the sublattice $\mathbb{Z}^4 \subseteq N$. Let M be its dual lattice, i.e. $M = \mathrm{Hom}(N, \mathbb{Z})$, also referred to as the monomial lattice. It is easy

to see that

$$M = r \cdot \mathbb{Z}^4 + \mathbb{Z} \cdot (1,1,1,1) \subseteq \mathbb{Z}^4_{\langle m_1, m_2, m_3, m_4 \rangle},$$

with $\{m_1, m_2, m_3, m_4\}$ the basis of the overlattice $\mathbb{Z}^4$. As is standard in toric geometry, set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, the ambient space of the lattice N, and similarly for the dual lattice, set $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 2.1.1. *The quotient variety $X = \mathbb{C}^4/G$ is isomorphic to the affine toric variety X_σ, defined by the cone $\sigma = \mathrm{Cone}\,(e_1, e_2, e_3, e_4) \subseteq N_{\mathbb{R}}$.*

Proof. From the definition of the toric variety given by the cone, we have that $X_\sigma = \mathrm{Spec}\,\mathbb{C}\,[\sigma^\vee \cap M]$. The dual cone $\sigma^\vee$ is equal to the first octant in $M_{\mathbb{R}}$ and $\sigma^\vee \cap M$ is generated as a semigroup by the lattice points rm_1, rm_2, rm_3, rm_4 and $m_1 + m_2 + m_3 + m_4$, in other words $\mathbb{C}\,[\sigma^\vee \cap M] = \mathbb{C}\,[x_1^r, x_2^r, x_3^r, x_4^r, x_1x_2x_3x_4]$ which is exactly the ring defining X. $\square$

Remark 2.1.2. Notice that, the cone σ from the proposition above is not smooth, as expected. For example, the point $\frac{1}{r}\,(1, r-1, 0, 0) \in \sigma$ cannot be written as an integral linear combination of the minimal generators e_1, e_2, e_3 and e_4 of σ, i.e. the minimal generators of the cone do not form a basis for the lattice.

The rest of the chapter deals with the construction of a crepant resolution of X using the tools of toric geometry.

Definition 2.1.3. Let X be a singular variety. A proper algebraic morphism $f : Y \to X$ is a **crepant resolution of singularities** if:

- Y is smooth,
- f is a birational map, and
- $K_Y = f^*\,(K_X)$.

If only first two of the above conditions are satisfied, then the map f is a **resolution of singularities**.

Suppose a fan Σ subdivides the singular cone σ, i.e. Σ is a fan such that $|\Sigma| = |\sigma|$. Then this subdivision defines a birational map $f : X_\Sigma \to X_\sigma$ so we have

$$K_{X_\Sigma} = f^*\left(K_{X_\sigma}\right) + \sum_{\tau \in \Sigma(1) \setminus \sigma(1)} a_\tau D_\tau,$$

In the formula above, D_τ is a divisor corresponding to the ray $\tau \in \Sigma$ and the values $a_\tau \in \mathbb{Q}$ are called **the discrepancies**. From the discrepancy calculation in [26], we get the formula:

$$a_\tau = \left(\sum_{i=1}^{4} \alpha_i\right) - 1, \tag{2.2}$$

where $A_\tau = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the primitive generator of the ray τ. Hence, in order for a torus invariant divisor D_τ to have the zero discrepancy, the sprimitive generator of the ray τ must lie in the hyperplane $H_1 = \left(\sum_{i=1}^{4} p_i = 1\right) \subseteq N_\mathbb{R}$. The value $\sum_{i=1}^{4} \alpha_i$ from (2.2) is called the **age** of the lattice point A_τ.

The question of finding a crepant resolution of the orbifold $X = X_\sigma$ now translates to the question of subdividing the **junior simplex**

$$\Delta = \sigma \cap H_1 = \left\{ \sum_{i=1}^{4} \alpha_i e_i \;\middle|\; \alpha_i \geq 0,\ \sum_{i=1}^{4} \alpha_i = 1 \right\} \subseteq N_\mathbb{R}$$

into smaller tetrahedra whose vertices are lattice points that base the whole lattice. The junior simplex is the locus containing all the lattice points of age one that lie within the cone σ.

Notice that Δ is a regular tetrahedron with vertices e_1, e_2, e_3, e_4. The lattice points it contains are $\Delta \cap N = \left\{ \frac{1}{r}\left(v_1, v_2, v_3, v_4\right) \;\middle|\; v_i \in \mathbb{Z}_{\geq 0},\ \sum_{i=1}^{4} v_i = r \right\}$.

2.2 Tetrahedral-octahedral fan

This section is a stepping stone to the construction of certain special crepant resolutions, as well as the fan of the G-Hilb $\left(\mathbb{C}^4\right)$ which is the main topic of the following chapter. Although the fan Σ_{TO} introduced in this section has a geometric interpretation: the variety it describes is a partial resolution of singularities of X, we mostly use it to make the construction of crepant resolutions more clear to understand.

In the three dimensional case ([9]), the junior simplex is an equilateral triangle and a canonical choice of crepant resolution subdivides it into a number of equilateral triangles of side length $\frac{1}{r}$th of the side length of the junior simplex. This subdivision arises from the regular tessellation of the two-dimensional plane by the equilateral triangles so a natural question one might ask is can we fill up 3-space (and then also the tetrahedron λ into regular tetrahedra of same side length). The answer to this question is negative as the dihedral angle along an edge of the tetrahedron is an irrational multiple of π. However, 3-space can be tessellated by regular tetrahedra and regular octahedra of the same side length. This tessellation of 3-space is known as the **Alternated cubic honeycomb** or **Tetrahedral-octahedral honeycomb**. This tessellation of 3-space gives rise to a subdivision of a tetrahedron consisting of smaller tetrahedra and octahedra.

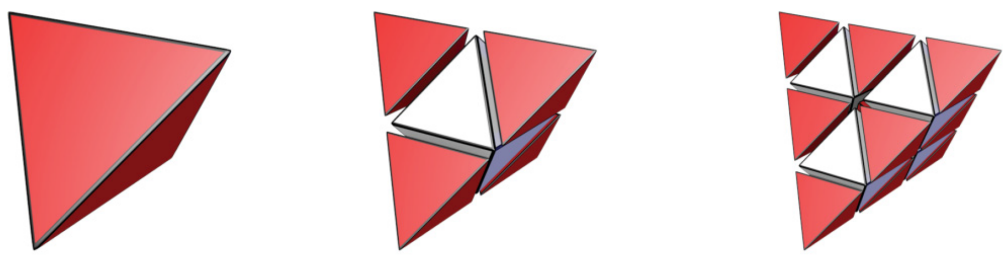


Figure 2.1: The junior simplex Δ and its tetrahedral-octahedral subdivisions for $r = 2$ and 3. Image taken from [16]

Proposition 2.2.1. *The junior simplex* $\Delta = \operatorname{conv}(e_1, e_2, e_3, e_4)$ *can be subdivided*

into:

- $\binom{r+2}{3}$ ***"up" tetrahedra*** U_p*, for all* $p \in \mathbb{Z}^4_{\geq 0}$ *such that* $\sum_{i=1}^4 p_i = r-1$*, defined by*

$$U_p = \operatorname{conv}\left(u_1^p, u_2^p, u_3^p, u_4^p\right) = \operatorname{conv}\begin{pmatrix} \frac{1}{r}\left(p_1+1, p_2, p_3, p_4\right), \\ \frac{1}{r}\left(p_1, p_2+1, p_3, p_4\right), \\ \frac{1}{r}\left(p_1, p_2, p_3+1, p_4\right), \\ \frac{1}{r}\left(p_1, p_2, p_3, p_4+1,\right) \end{pmatrix},$$

  where $u_i^p = \frac{1}{r}\left(p + e_i\right)$.

- $\binom{r+1}{3}$ ***octahedra*** O_p *centred at a point* $\frac{1}{r}\left(p_1+\frac{1}{2}, p_2+\frac{1}{2}, p_3+\frac{1}{2}, p_4+\frac{1}{2}\right)$*, for all* $p \in \mathbb{Z}^4_{\geq 0}$ *such that* $\sum_{i=1}^4 p_i = r-2$*:*

$$O_p = \operatorname{conv}\left(o_{12}^p, o_{13}^p, o_{14}^p, o_{23}^p, o_{24}^p, o_{34}^p\right) = \operatorname{conv}\begin{pmatrix} \frac{1}{r}\left(p_1+1, p_2+1, p_3, p_4\right), \\ \frac{1}{r}\left(p_1+1, p_2, p_3+1, p_4\right), \\ \frac{1}{r}\left(p_1+1, p_2, p_3, p_4+1\right), \\ \frac{1}{r}\left(p_1, p_2+1, p_3+1, p_4\right), \\ \frac{1}{r}\left(p_1, p_2+1, p_3, p_4+1\right), \\ \frac{1}{r}\left(p_1, p_2, p_3+1, p_4+1\right) \end{pmatrix},$$

  where $o_{ij}^p = \frac{1}{r}\left(p + e_i + e_j\right)$.

- $\binom{r}{3}$ ***"down" tetrahedra*** D_p*, for all* $p \in \mathbb{Z}^4_{\geq 0}$ *such that* $\sum_{i=1}^4 p_i = r-3$*, defined by*

$$D_p = \operatorname{conv}\left(d_1^p, d_2^p, d_3^p, d_4^p\right) = \operatorname{conv}\begin{pmatrix} \frac{1}{r}\left(p_1, p_2+1, p_3+1, p_4+1\right), \\ \frac{1}{r}\left(p_1+1, p_2, p_3+1, p_4+1\right), \\ \frac{1}{r}\left(p_1+1, p_2+1, p_3, p_4+1\right), \\ \frac{1}{r}\left(p_1+1, p_2+1, p_3+1, p_4\right) \end{pmatrix},$$

  where $d_i^p = \frac{1}{r}\left(p + (1,1,1,1) - e_i\right)$.

Before the proof, notice that an "up" tetrahedron is a scaled-down version of the junior simplex itself with the same orientation, whereas a "down" tetrahedron has the antipodal orientation to the junior simplex. In other words, up to translation, an "up" tetrahedron is equal to $\frac{1}{r}\Delta$ and a "down" tetrahedron to $-\frac{1}{r}\Delta$.

When $r = 2$, there are four "up" tetrahedra, obtained by slicing the corners of the junior simplex along the planes $(x_i = x_j + x_k + x_l)$, for all the permutations (i, j, k, l) of the tuple $(1, 2, 3, 4)$ and a single octahedron left in the middle. In this example, there are no "down" tetrahedra in the tetrahedral-octahedral subdivision. Going one case up, when $r = 3$, there are ten "up" tetrahedra, four octahedra, and a single "down" tetrahedron in the middle. The "down" tetrahedron is the only polyhedron of the subdivision not visible from the outside of the junior simplex, see Figure 2.2.

Proof. First notice that every polyhedron from the proposition is a subset of Δ. This follows from the fact that every polyhedron is defined as a convex hull of four or six lattice points, all of which lie within Δ. To show that they subdivide Δ, one needs to show that every two polyhedra either do not intersect, or they intersect at a common face and that the union of all the polyhedra covers Δ.

UP-UP Suppose a point $t \in \mathbb{R}^4$ lies in the intersection of two "up" tetrahedra U_p and U_q. Then it can be expressed both as the convex sum of the vertices $u_i^p, i = 1, 2, 3, 4$ and as a convex sum of $u_i^q, i = 1, 2, 3, 4$:

$$t = \sum_{i=1}^{4} \alpha_i u_i^p = \sum_{i=1}^{4} \beta_i u_i^q, \quad \text{where} \quad \alpha_i, \beta_i \in [0, 1] \ \text{ and } \ \sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{4} \beta_i = 1.$$

So $r \cdot t = \sum_{i=1}^{4} \alpha_i (p + e_i) = (\sum_{i=1}^{4} \alpha_i)p + \sum_{i=1}^{4} \alpha_i e_i$ and this is the point with coordinates $(p_1 + \alpha_1, p_2 + \alpha_2, p_3 + \alpha_3, p_4 + \alpha_4)$. Equally, the same point can be written as $r \cdot t = (q_1 + \beta_1, q_2 + \beta_2, q_3 + \beta_3, q_4 + \beta_4)$. Hence, $p_i + \alpha_i = q_i + \beta_i$ for all i and p_i, and q_i are integers, while α_i and β_i are real numbers from the interval $[0, 1]$.

Suppose $\alpha_1 = 1$. Then $\alpha_i = 0$ for $i \neq 1$ and the point $r \cdot t$ has integral entries, so β_j also have to be integers which means that $\beta_j = 1$ for exactly one $1 \leq j \leq 4$ and the others are zero. So $r = u_1^p = u_j^q$, i.e. the vertex of both tetrahedra. Notice that $j = 1$ implies $p = q$.

If $0 < \alpha_1 < 1$, then $\beta_1 = \alpha_1$ and $p_1 = q_1$. This also means that no other α_j nor β_j can be equal to 1. If some $\alpha_j > 0$ then again $\beta_j = \alpha_j$ so $p_j = q_j$, and if $\alpha_j = 0$, the entry $p_j + \alpha_j$ is an integer so β_j must also be zero, again implying $p_j = q_j$. So in this case the two tetrahedra U_p and U_q are equal.

Finally if $\alpha_1 = 0$, then at least one other $\alpha_j > 0$ which reduces this case to one of the above two cases. The conclusion is that two distinct "up" tetrahedra intersect at most at a common vertex.

UP-DOWN Let $t \in U_p \cap D_q$. Then $t = \sum_{i=1}^4 \alpha_i u_i^p = \sum_{i=1}^4 \beta_i d_i^q$ where $\alpha_i, \beta_i \in [0,1]$ and $\sum_{i=1}^4 \alpha_i = \sum_{i=1}^4 \beta_i = 1$. As above $r \cdot t = (p_1 + \alpha_1, p_2 + \alpha_2, p_3 + \alpha_3, p_4 + \alpha_4)$. Similarly,

$$\begin{aligned} r \cdot t &= \sum_{i=1}^4 \beta_i \left(q + (1,1,1,1) - e_i\right) = q + (1,1,1,1) - \sum_{i=1}^4 \beta_i e_i \\ &= (q_1 + 1 - \beta_1, q_2 + 1 - \beta_2, q_3 + 1 - \beta_3, q_4 + 1 - \beta_4). \end{aligned}$$

So for each $1 \leq i \leq 4$, there is the equality $p_i + \alpha_i = q_i + 1 - \beta_i$.

If $\alpha_1 = 1$, then $\alpha_i = 0$ for $i \neq 1$. Every entry of $r \cdot t$ is an integer, so that all β's have to be integers, but as they sum up to 1 it follows that there is an index j such that $\beta_j = 1$, and $\beta_k = 0$ for $k \neq j$. The point of intersection is $t = u_i^p = d_j^q$, i.e. the vertex of both tetrahedra.

In the case $0 < \alpha_1 < 1$, from the equality $p_1 + \alpha_1 = q_1 + 1 - \beta_1$ it follows that $\beta_1 = 1 - \alpha_1$ and $p_1 = q_1$. If for any other index i it holds that $\alpha_i \in (0,1)$, then by the same logic $p_i = q_i$ and $\alpha_i = 1 - \beta_i$. If $\alpha_i = 0$, then $\beta_i = 0$ as it also has to be an integer, but cannot be 1, as $\sum_{j=1}^4 \beta_i = 1$. In this case, $p_i = q_i + 1$. As $\sum_{j=1}^4 p_i = \sum_{j=1}^4 q_i + 2$, $\alpha_i = 0$ (and then $\beta_i = 0$ for exactly two elements

$i \in \{1,2,3,4\}$. So the point of intersection lies along an edge of both U_p and D_q.

The case $\alpha_i = 0$ reduces to one of the above cases and it follows that an "up" and a "down" tetrahedron intersect at most at a common edge (if at all).

DOWN-DOWN If a point t lies in the intersection of two "down" tetrahedra D_p and D_q, then

$$t = \sum_{i=1}^{4} \alpha_i d_i^p = \sum_{i=1}^{4} \beta_i d_i^q.$$

Hence, the equality $p_i + 1 - \alpha_i = q_i + 1 - \beta_1$ holds for each $1 \leq i \leq 4$. This is now analogous to the case regarding the intersection of two "up" tetrahedra – the only intersection is a common vertex $d_i^p = d_j^q$, for some i and j.

UP-O Assume now a point t lies in the intersection of an "up" tetrahedron U_p and an octahedron O_q. Then the point t can be written as convex combination in two ways:

$$t = \sum_{i=1}^{4} \alpha_i u_i^p = \sum_{1 \leq i < j \leq 4} \beta_{ij}\, o_{ij}^q,$$

where $\alpha_i, \beta_{ij} \in [0,1]$ and $\sum_{i=1}^{4} \alpha_i = \sum_{1 \leq i < j \leq 4} \beta_{ij} = 1$. As before $r \cdot t$ can be written as $(p_1 + \alpha_1, p_2 + \alpha_2, p_3 + \alpha_3, p_4 + \alpha_4)$ and also

$$\begin{aligned} r \cdot t &= \sum_{1 \leq i < j \leq 4} \beta_{ij} \left(q + e_i + e_j\right) \\ &= \sum_{i=1}^{4} \left(q_i + \beta_{ij} + \beta_{ik} + \beta_{jk}\right) e_i, \text{ where } \{j,k,l\} = \{1,2,3,4\} \setminus \{i\}. \end{aligned}$$

Let $0 < \alpha_1 < 1$. Since $p_1 + \alpha_1 = q_1 + \beta_{12} + \beta_{13} + \beta_{14}$, it follows that $p_1 = q_1$ and $\alpha_1 = \beta_{12} + \beta_{13} + \beta_{14}$. If any other α_i is nonzero, then $p_i = q_i$ and for distinct $j,k,l \in \{1,2,3,4\} \setminus \{i\}$ it holds $\alpha_i = \beta_{ij} + \beta_{ik} + \beta_{il}$. If however, any $\alpha_i = 0$, then $\beta_{ij} + \beta_{ik} + \beta_{il}$ is an integer (either 0 or 1). Notice that at least

one α_i must be zero. Otherwise,

$$1 = \sum_{i=1}^{4} \alpha_i = 2 \cdot \sum_{1 \le i < j \le 4} \beta_{ij} = 2 \quad \text{a contradiction.}$$

Without loss of generality, assume $0 < \alpha_1, \alpha_2 < 1$ and $\alpha_3 = 0$. Then t lies on the side of the tetrahedron U_p spanned by the three vertices u_1^p, u_2^p and u_4^p and

$$\begin{aligned}
\alpha_1 &= \beta_{12} + \beta_{13} + \beta_{14}, \\
\alpha_2 &= \beta_{12} + \beta_{23} + \beta_{24}, \\
\alpha_3 &= 0 \Rightarrow \beta_{13} + \beta_{23} + \beta_{34} \in \{0, 1\}.
\end{aligned}$$

If the sum from the last row above is zero, then all three summands must be zero. But then t lies on the side of the octahedron O_q spanned by o_{12}^q, o_{14}^q and o_{24}^q. If the same sum is equal to 1, then $\beta_{12} = \beta_{14} = \beta_{24} = 0$, so t lies on the opposite side of O_q.

The case $\alpha_1 = 0$ reduces to one of the above cases. The conclusion is that if an "up" tetrahedron and an octahedron intersect, the intersection is a side of both.

DOWN-O Analogous to the above case. A "down" tetrahedron and an octahedron intersect at a common side (or they do not intersect at all).

O-O Finally, if a point t lies in the intersection of two octahedra O_p and O_q, then it can be written in two ways as a convex sum:

$$\sum_{1 \le i < j \le 4} \alpha_{ij}\, o_{ij}^p = \sum_{1 \le i < j \le 4} \beta_{ij}\, o_{ij}^q,$$

where $\alpha_{ij}, \beta_{ij} \in [0, 1]$ for all i, j and $\sum_{1 \le i < j \le 4} \alpha_{ij} = \sum_{1 \le i < j \le 4} \beta_{ij} = 1$. This

imposes the equations:

$$p_1 + \alpha_{12} + \alpha_{13} + \alpha_{14} = q_1 + \beta_{12} + \beta_{13} + \beta_{14}$$
$$p_2 + \alpha_{12} + \alpha_{23} + \alpha_{24} = q_2 + \beta_{12} + \beta_{23} + \beta_{24}$$
$$p_3 + \alpha_{13} + \alpha_{23} + \alpha_{34} = q_3 + \beta_{13} + \beta_{23} + \beta_{34}$$
$$p_4 + \alpha_{14} + \alpha_{24} + \alpha_{34} = q_4 + \beta_{14} + \beta_{24} + \beta_{34}$$

If for all $i \in \{1,2,3,4\}$ and $\{j,k,l\} = \{1,2,3,4\} \setminus \{i\}$, both $\alpha_{ij} + \alpha_{il} + \alpha_{il}$ and $\beta_{ij} + \beta_{ik} + \beta_{il}$ lie in the interval $[0,1)$, or if both lie in the interval $(0,1]$, then they must be equal, so it also follows that $p_i = q_i$. But then the two octahedra O_p and O_q are equal.

Without loss of generality, assume that the sum $\alpha_{12} + \alpha_{13} + \alpha_{14} = 0$, but $\beta_{12} + \beta_{13} + \beta_{14} = 1$. So $\alpha_{1i} = 0, \beta_{ij} = 0$ for all $i, j \neq 1$ and the four equations from above become:

$$p_1 = q_1 + 1$$
$$p_2 + \alpha_{23} + \alpha_{24} = q_2 + \beta_{12}$$
$$p_3 + \alpha_{23} + \alpha_{34} = q_3 + \beta_{13}$$
$$p_4 + \alpha_{24} + \alpha_{34} = q_4 + \beta_{14}$$

If $\beta_{12} = 1$, then $\beta_{13} = \beta_{14} = 0$. Hence, t is a vertex o_{12}^q of O_q. But then the sums $\alpha_{ij} + \alpha_{ik}$, for all distinct $i, j, k \in \{1,2,3\}$, must be integral. Since $\alpha_{23} + \alpha_{23} + \alpha_{34} = 1$, it follows that every summand is an integer, and exactly one summand is equal to 1. The point t is then also a vertex of O_p.

If $0 < \beta_{12} < 1$, then $\beta_{12} = \alpha_{23} + \alpha_{24}$ and $p_2 = q_2$. If both β_{13} and β_{14} were not integers then

$$1 = \beta_{12} + \beta_{13} + \beta_{14} = 2\left(\alpha_{23} + \alpha_{24} + \alpha_{34}\right) = 2,$$

which is a contradiction. So either $\beta_{13} = 0$ or $\beta_{14} = 0$ – they cannot both be zero, since β_{12} would be the only nonzero coefficient in the convex sum. Assume $\beta_{13} \in (0, 1)$ and $\beta_{14} = 0$ (the analysis of the opposite case goes the same). Then $p_3 = q_3$. Also, $\alpha_{23} + \alpha_{34} = \beta_{13}$, and $\alpha_{23} + \alpha_{34}$ is an integer. If the latter sum were equal to 0, then $p_4 = q_4$, but this is not possible as then $\sum p_i$ would not be the same as $\sum q_i$. So $\alpha_{24} + \alpha_{34} = 1$, which means that α_{23} must be zero. This shows that t lies on the edge $\operatorname{conv}(o^p_{24} o^p_{34})$ of O_p and on the edge $\operatorname{conv}(o^q_{12} o^q_{13})$ of O_q.

If $\beta_{12} = 0$, then some other β_{ij} is either 1 or in $(0, 1)$ so this case reduces to the one of the above two and it has been shown that the two octahedra intersect at a common edge.

The number of "up" tetrahedra in the subdivision of the junior simplex is equal to number of points $p \in \mathbb{Z}^4_{\geq 0}$ with the property $\sum_{i=1}^4 p_i = r - 1$ i.e. the number of ways to express $r - 1$ as an ordered sum of four nonnegative integers. But this number is the same as the number of ways to put $r - 1$ unlabelled marbles into four labelled boxes which is $\binom{(r-1)+4-1}{4-1} = \binom{r+2}{3}$. Similarly, the number of octahedra is the number of ways to put $r - 2$ unlabelled marbles into four labelled boxes: $\binom{(r-2)+4-1}{4-1} = \binom{r+1}{3}$. Number of "down" tetrahedra is $\binom{(r-3)+4-1}{4-1} = \binom{r}{3}$.

The only thing left to prove is that the polyhedra from the above actually cover the junior simplex Δ. The volume of a tetrahedron of side length a is $\frac{\sqrt{2}}{12} a^3$ and the volume of an octahedron of the side length a is $\frac{\sqrt{2}}{3} a^3$. The side length of the junior simplex Δ is r times the side of the regular polyhedra that form the subdivision. If we label the side length of "up" and "down" tetrahedra, and octahedra by a, the sum of volumes of all of the polyhedra from the proposition

statement is

$$\left[\binom{r+2}{3}+\binom{r}{3}\right]\cdot\frac{\sqrt{2}}{12}a^3+\binom{r+1}{3}\frac{\sqrt{2}}{3}a^3$$
$$=\frac{\sqrt{2}}{12}a^3\left[\binom{r+2}{3}+4\binom{r+1}{3}+\binom{r}{3}\right]$$
$$=\frac{\sqrt{2}}{12}a^3\cdot r^3=\frac{\sqrt{2}}{12}(ra)^3,$$

which is exactly the volume of the junior simplex. This finishes the proof. □

As the two polyhedra in the subdivision of the junior simplex Δ intersect at a common face, the cones above these polyhedra also intersect at a common face so the following definition makes sense.

Definition 2.2.2. Using the notation from the Proposition 2.2.1, **the tetrahedral-octahedral fan $\boldsymbol{\Sigma}_{\mathrm{TO}}$** is defined to be a fan that consists of the following cones and their faces:

- $\mathrm{Cone}\,(U_p)=\mathrm{Cone}\,(u_1^p,u_2^p,u_3^p,u_4^p)$, for all $p\in\mathbb{Z}^4_{\geq 0}$ such that $\sum p_i=r-1$,
- $\mathrm{Cone}\,(O_p)=\mathrm{Cone}\,(o_{12}^p,o_{13}^p,o_{14}^p,o_{23}^p,o_{24}^p,o_{34}^p)$, for all $p\in\mathbb{Z}^4_{\geq 0}$ such that $\sum p_i=r-2$,
- $\mathrm{Cone}\,(D_p)=\mathrm{Cone}\,(d_1^p,d_2^p,d_3^p,d_4^p)$, for all $p\in\mathbb{Z}^4_{\geq 0}$ such that $\sum p_i=r-3$.

The case $r=2$ is the only case where the the TO-fan does not have any "down" tetrahedra: there are four "up" tetrahedra obtained by chopping off the corners of the junior simplex Δ, and a single octahedron in the middle. If we go one case up ($r=3$), there are ten "up" tetrahedra, four octahedra and a single "down" tetrahedron in the middle, surrounded by the octahedra. In all the other cases all three types of cones appear.

Proposition 2.2.3. *The cones corresponding to the "up" and "down" tetrahedra are smooth.*

Proof. A cone is smooth if the primitive generators of its rays form the $\mathbb{Z}$-basis for the lattice N, defined by (2.1). Since N is defined through seven generators: four generators e_1, e_2, e_3, e_4 of the sublattice $\mathbb{Z}^4 \subset N$, and the three fractional generators, it is enough to show that each one of them can be written as an integral linear combination of the primitive generators of the rays of the given cone. Then the four primitive generators of the rays of the cone span N, but as there are exactly four of them it follows that they must be the basis for the four-dimensional lattice N.

Choose a cone over an "up" tetrahedron U_p. Then the point $p \in \mathbb{Z}^4_{\geq 0}$ with $\sum_{i=1}^4 p_i = r - 1$ and the primitive generators of the rays of the cone $\mathrm{Cone}\,(U_p)$ are the lattice points:

$$\begin{aligned} u_1^p &= \tfrac{1}{r}\,(p_1 + 1, p_2, p_3, p_4) \\ u_2^p &= \tfrac{1}{r}\,(p_1, p_2 + 1, p_3, p_4) \\ u_3^p &= \tfrac{1}{r}\,(p_1, p_2, p_3 + 1, p_4) \\ u_4^p &= \tfrac{1}{r}\,(p_1, p_2, p_3, p_4 + 1)\,. \end{aligned}$$

For each $i \in \{1, 2, 3, 4\}$ we can write $e_i = (r - p_i)\, u_i^p - \sum_{j \neq i} p_j u_j^p$ and the three fractional generators of N can be written as:

$$\begin{aligned} \tfrac{1}{r}\,(1, -1, 0, 0) &= u_1^p - u_2^p, \\ \tfrac{1}{r}\,(1, 0, -1, 0) &= u_1^p - u_3^p, \\ \tfrac{1}{r}\,(1, 0, 0, -1) &= u_1^p - u_4^p. \end{aligned}$$

Similarly, if D_q is a "down" tetrahedron, then $q \in \mathbb{Z}^4_{\geq 0}$ satisfies $\sum_{i=1}^4 q_i = p - 3$ and

the primitive generators of the rays of D_q are

$$\begin{aligned}
d_1^q &= \tfrac{1}{r}\left(q_1, q_2+1, q_3+1, q_4+1\right)\\
d_2^q &= \tfrac{1}{r}\left(q_1+1, q_2, q_3+1, q_4+1\right)\\
d_3^q &= \tfrac{1}{r}\left(q_1+1, q_2+1, q_3, q_4+1\right)\\
d_4^q &= \tfrac{1}{r}\left(q_1+1, q_2+1, q_3+1, q_4\right).
\end{aligned}$$

Now direct computation shows that $e_i = (q_i + 1 - r)\, d_i^q + \sum_{j\neq i} (q_j + 1)\, d_j^q$ and each of the three fractional generators is just a difference of a pair of primitive ray generators:

$$\begin{aligned}
\tfrac{1}{r}\left(1,-1,0,0\right) &= d_2^q - d_1^q,\\
\tfrac{1}{r}\left(1,0,-1,0\right) &= d_3^q - d_1^q,\\
\tfrac{1}{r}\left(1,0,0,-1\right) &= d_4^q - d_1^q,
\end{aligned}$$

Hence both "up" and "down" tetrahedra generate smooth cones. □

Obviously, octahedral cones cannot be smooth as each $\mathrm{Cone}(O_p)$ has six vertices, and a four-dimensional lattice cannot have a basis consisting of six elements.

Corollary 2.2.4. *The subdivision $\Sigma_{\mathrm{TO}} \to \sigma$ induces a partial resolution of singularities $X_{\mathrm{TO}} \to X$. The variety X_{TO} has $\binom{r+2}{3}$ singular affine pieces isomorphic to the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

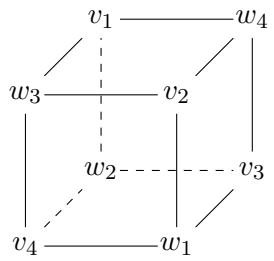
Proof. Fix a point $P \in \mathbb{Z}_{\geq 0}$ such that $\sum p_i = r - 2$. Then the variety given by $\mathrm{Cone}\left(O_P\right)$ is $\operatorname{Spec}\mathbb{C}\left[\mathrm{Cone}\left(O_P\right)^\vee \cap M\right]$. The semigroup $\mathrm{Cone}\left(O_P\right)^\vee \cap M$ is generated by eight elements of the monomial lattice

$$\begin{aligned}
v_i &= \frac{x_i^{r-p_i}}{\left(x_j x_k x_l\right)^{p_i}}, \quad \text{for all } \{i,j,k,l\} = \{1,2,3,4\}\\
w_i &= \frac{\left(x_j x_k x_l\right)^{p_i+1}}{x_i^{r-p_i-1}}, \quad \text{for all } \{i,j,k,l\} = \{1,2,3,4\}
\end{aligned}$$

with the ideal of relations

$$\langle v_i w_i = v_j w_j,\ v_i v_j = w_k w_l \ \mid \ \text{for all } \{i,j,k,l\} = \{1,2,3,4\}\rangle\,.$$

The same relations can be viewed as the 2×2 minors of the array



But these are exactly the relations that define the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^7_{\langle v_1,v_2,v_3,v_4,w_1,w_2,w_3,w_4\rangle}$ so our affine piece U_{O_P} is a cone over it. $\square$

2.3 Special crepant resolutions of $\mathbb{C}^4/(\mathbb{Z}/r)^3$

Although the fan Σ_{TO} is not a crepant resolution of singularities, it is not too far from it – the cones over the "up" tetrahedra U_p and "down" tetrahedra D_q for all $p, q \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^4 p_i = r-1$ and $\sum_{i=1}^4 q_i = r-3$ are already smooth. One only needs to further subdivide the remaining octahedra.

Fix a point $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum_{i=1}^4 p_i = r-2$. The point p defines an octahedron O_p with its six vertices o^p_{ij}, $1 \leq i < j \leq 4$. We can choose a pair of antipodal vertices, for example o^p_{12} and o^p_{34}, and slice the octahedron by two planes parallel to coordinate planes – each passing through o^p_{12} and o^p_{34} and additional two antipodal points of the octahedron. This method gives four new tetrahedra.

Of course, there are other choices of the axis through which the two slicing planes pass – any of the three antipodal pairs of points could have been chosen.

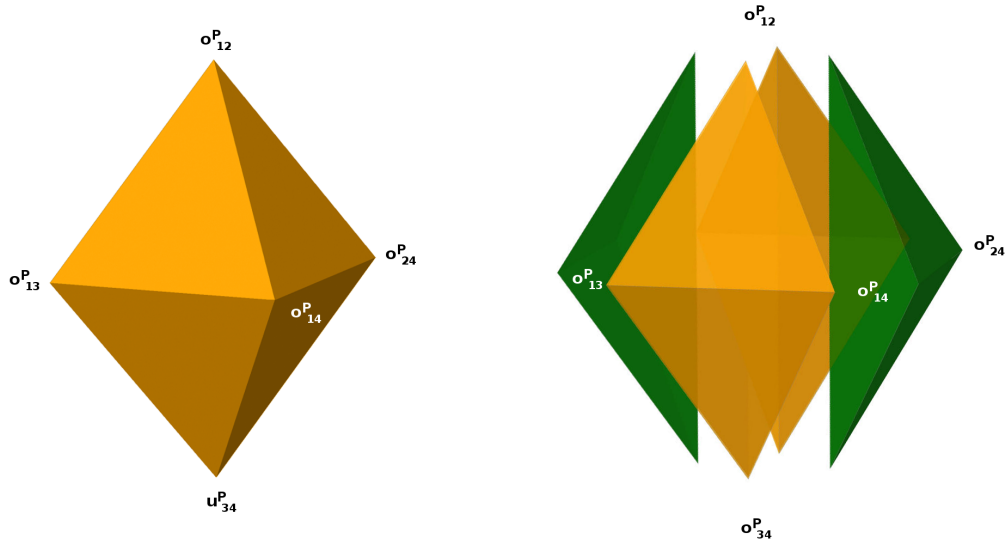


Figure 2.2: An octahedron O_P and the four orange-slice tetrahedra obtained by slicing along the 12-34 axis.

Definition 2.3.1. For an octahedron O_p and a fixed permutation (i, j, k, l) of the set of indices $\{1, 2, 3, 4\}$, the tetrahedra

$$\begin{aligned}
&\operatorname{conv}\left(o^p_{ij}, o^p_{kl}, o^p_{ik}, o^p_{il}\right)\\
&\operatorname{conv}\left(o^p_{ij}, o^p_{kl}, o^p_{il}, o^p_{jl}\right)\\
&\operatorname{conv}\left(o^p_{ij}, o^p_{kl}, o^p_{jl}, o^p_{jk}\right)\\
&\operatorname{conv}\left(o^p_{ij}, o^p_{kl}, o^p_{jk}, o^p_{ik}\right)
\end{aligned}$$

obtained as above are called **orange-slice tetrahedra** in direction $[ij - kl]$ corresponding to an octahedron O_p.

Proposition 2.3.2. *Every orange-slice tetrahedron corresponding to an octahedron O_p defines a smooth cone.*

Proof. It is enough to show that one of the four orange-slice tetrahedra generates a smooth cone, we shall prove it for the cone:

$$\operatorname{Cone}\left(o^p_{ij}, o^p_{kl}, o^p_{ik}, o^p_{il}\right).$$

As in Proposition 2.2.3, we need to show that the seven generators of the lattice N can be written as integral combinations of $o^p_{ij}, o^p_{kl}, o^p_{ik}$ and o^p_{il}.

$$\begin{aligned}
e_i &= -p_j o^p_{ij} - (r - p_i - 1)\, o^p_{kl} + (p_j + p_l)\, o^p_{ik} + (p_j + p_k)\, o^p_{il} \\
e_j &= (r - p_j)\, o^p_{ij} + (p_i + 1)\, o^p_{kl} - (p_i + p_k + 1)\, o^p_{ik} - (p_i + p_l + 1)\, o^p_{il} \\
e_k &= -p_j o^p_{ij} + (p_i + 1)\, o^p_{kl} + (p_j + p_l + 1)\, o^p_{ik} - (p_i + p_l)\, o^p_{il} \\
e_l &= -p_j o^p_{ij} + (p_i + 1)\, o^p_{kl} - (p_i + p_k + 1)\, o^p_{ik} + (p_j + p_k)\, o^p_{il}
\end{aligned}$$

As for the fractional generators, they will be, up to the sign, among the following lattice points, all of which are integral combinations of the primitive generators of the rays of the cone:

$$\begin{aligned}
\tfrac{1}{r}(e_i - e_j) &= o_{ik} - o_{kl} + o_{il} - o_{ij} \\
\tfrac{1}{r}(e_i - e_k) &= o_{il} - o_{kl} \\
\tfrac{1}{r}(e_i - e_l) &= o_{ik} - o_{kl} \\
\tfrac{1}{r}(e_j - e_k) &= o_{ij} - o_{ik} \\
\tfrac{1}{r}(e_j - e_l) &= o_{ij} - o_{il} \\
\tfrac{1}{r}(e_k - e_l) &= o_{ik} - o_{il}
\end{aligned}$$

The rays $o^p_{ij}, o^p_{kl}, o^p_{ik}$ and o^p_{il} generate the four dimensional lattice N, so they form a basis for the lattice, i.e. the cone is smooth. $\square$

These are all the ingredients needed for the construction of crepant resolutions. Define an **orientation function** going from the space of parameters p that define the octahedra to the set pairs of indices indicating the pairs of antipodal points of an octahedron:

$$\varphi : \left\{ p \in \mathbb{Z}^4_{\geq 0} \mid \sum_{i=1}^{4} p_i = r - 2 \right\} \to \left\{ \begin{array}{c} (\{1,2\},\{3,4\}) \\ (\{1,3\},\{2,4\}) \\ (\{1,4\},\{2,3\}) \end{array} \right\}$$

Definition 2.3.3. Let φ be an orientation function. Σ_φ is defined to be the fan consisting of the following cones and their faces:

- The "up" tetrahedra:

$$\mathrm{Cone}\left(U_p\right) = \mathrm{Cone}\left(u_1^p, u_2^p, u_3^p, u_4^p\right),$$

  for all $p \in \mathbb{Z}_{\geq 0}^4$ such that $\sum p_i = r - 1$.

- The "down" tetrahedra:

$$\mathrm{Cone}\left(D_p\right) = \mathrm{Cone}\left(d_1^p, d_2^p, d_3^p, d_4^p\right),$$

  for all $p \in \mathbb{Z}_{\geq 0}^4$ such that $\sum p_i = r - 3$.

- The orange-slice tetrahedra: for every $p \in \mathbb{Z}_{\geq 0}^4$ such that $\sum p_i = r - 2$, set $\{i, j\}$ to be the first projection of $\varphi(p)$ and $\{k, l\}$ the second projection, or equivalently the complement of $\{i, j\}$. Then the orange-slice cones are given by

$$\begin{aligned}
&\mathrm{Cone}\left(o_{ij}^p, o_{kl}^p, o_{ik}^p, o_{il}^p\right), \\
&\mathrm{Cone}\left(o_{ij}^p, o_{kl}^p, o_{il}^p, o_{jl}^p\right), \\
&\mathrm{Cone}\left(o_{ij}^p, o_{kl}^p, o_{jl}^p, o_{jk}^p\right), \\
&\mathrm{Cone}\left(o_{ij}^p, o_{kl}^p, o_{jk}^p, o_{ik}^p\right).
\end{aligned}$$

Theorem 2.3.1. *The variety Y_φ given by the fan Σ_φ is a crepant resolution of singularities of $X = \mathbb{C}^4/G$.*

Proof. Σ_φ is a smooth fan: the "up" and "down" tetrahedra are smooth by Proposition 2.2.3 and the orange-slice tetrahedra are smooth by Proposition 2.3.2. The fan obviously subdivides the fan Σ_{TO} which subdivides the cone σ, hence the resulting map $X_\varphi \to X_\sigma$ is birational. Finally, every torus-invariant exceptional divisor has

discrepancy zero since all the rays of the fan are generated by points of age 1, see (2.2). $\square$

Remark 2.3.4. There are $\binom{r+2}{3} + \binom{r}{3} + 4 \cdot \binom{r+1}{3} = r^3$ copies of $\mathbb{C}^4$ that cover these crepant resolutions, the number equal to the order of the group $G \cong (\mathbb{Z}/r)^{\oplus 3}$.

If the orientation function φ is constant, with image $(\{i,j\}, \{k,l\})$, then the fan Σ_φ is also denoted by $\Sigma_{[ij\text{-}kl]}$ and the corresponding crepant resolution, $Y_{[ij\text{-}kl]}$, is referred to as the $[ij\text{-}kl]$-crepant resolution.

2.4 Birational maps between the special crepant resolutions

In this section we study the maps leading from one special crepant resolution of $X = \mathbb{C}^4/G$ to the other. We explained in the previous section how to subdivide an octahedron O_p into four orange-slice tetrahedra, we must slice the octahedron along two planes parallel to coordinate planes and intersecting the interior of O_p. We also mentioned how there are three different ways of doing so. Comparing two different subdivisions, we notice that they must have one of planes in common. To understand maps between different choices of special crepant resolutions, we first explain what happens when an octahedron of the fan Σ_{TO} is sliced along a single plane parallel to a coordinate plane – a plane containing two pairs of its antipodal points.

Fix an $r \geq 2$ and an octahedron O_p of the fan Σ_{TO} centered at a point $p = \frac{1}{r}\left(p_1 + \frac{1}{2}, p_2 + \frac{1}{2}, p_3 + \frac{1}{2}, p_4 + \frac{1}{2}\right)$, where $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum p_i = r - 2$. Let φ and φ' be two orientation functions defining two special crepant resolutions Y_φ and $Y_{\varphi'}$. If $\varphi(p) \neq \varphi'(p)$, we can assume without loss of generality that

$$\varphi(p) = (\{13\}, \{24\}) \quad \text{and} \quad \varphi'(p) = (\{14\}, \{23\}).$$

Let Σ^p_φ be a subfan of the fan Σ_φ consisting of the orange-slice tetrahedra centered

at the point $\frac{1}{2}p$ and their faces. Similarly, let $\Sigma^p_{\varphi'}$ be a subfan of $\Sigma_{\varphi'}$ with the same support as Σ^p_{φ}. Both of these two fans are obtained by slicing O_p by plane through the points $o^p_{13}, o^p_{14}, o^p_{23}, o^p_{24}$ and one additional plane.

Now fan Σ^p_{φ} defines a toric variety with a compact subvariety defined by $\mathrm{Cone}\,(o^p_{13}, o^p_{24})$. This subvariety is a surface $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates

$$(x_1x_4)^{p_2+p_3+1} : (x_2x_3)^{p_1+p_4+1}\,, \quad (x_1x_2)^{p_3+p_4+1} : (x_3x_4)^{p_1+p_2+1}\,.$$

Similarly, the toric variety corresponding to $\mathrm{Cone}\,(o^p_{14}, o^p_{23}) \subset \Sigma^p_{\varphi'}$ is also a surface $\mathbb{P}^1 \times \mathbb{P}^1$, but with coordinates given by ratios:

$$(x_1x_3)^{p_2+p_4+1} : (x_2x_4)^{p_1+p_3+1}\,, \quad (x_1x_2)^{p_3+p_4+1} : (x_3x_4)^{p_1+p_2+1}\,.$$

We see that the second curve $\mathbb{P}^1$ in both varieties has the same coordinates, so the they only differ in the first curve $\mathbb{P}^1$. Hence, the birational map $\Phi^p_{[12-34]}$ from $U(\Sigma^p_{\varphi})$ to $U(\Sigma^p_{\varphi'})$ has to replace one copy of $\mathbb{P}^1$ with a different one. This map is known in the threefold setting as the ordinary flop.

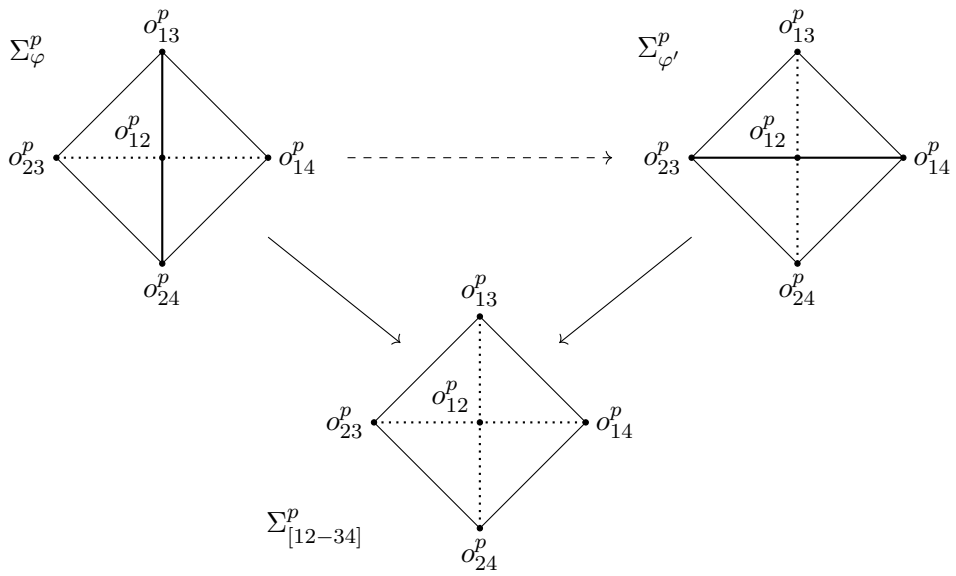


Figure 2.3: Two orange-slice subdivisions of O_p and the fan $\Sigma^P_{[12-34]}$ viewed from vertex o^p_{12}

Define a fan $\Sigma^p_{[12-34]}$ obtained from the cone O_p by slicing along the plane containing the four vertices $o^p_{12}, o^p_{13}, o^p_{34}, o^p_{23}$. Fan $\Sigma^p_{[12-34]}$ consists of two four-dimensional cones $C_1 = \mathrm{Cone}\left(o^p_{12}, o^p_{13}, o^p_{24}, o^p_{14}, o^p_{23}\right)$ and $C_2 = \mathrm{Cone}\left(o^p_{34}, o^p_{13}, o^p_{24}, o^p_{14}, o^p_{23}\right)$, as well as their faces. Semigroup $C_1^\vee \cap M$ has five generators, with a single relation $uv = wt$.

$$u = \frac{x_3^{r-p_3}}{(x_1x_2x_4)^{p_3}}, \qquad w = \frac{(x_2x_3x_4)^{r-p_1-1}}{x_1^{p_1+1}}, \qquad p = \frac{(x_1x_2)^{p_3+p_4+1}}{(x_3x_4)^{p_1+p_2+1}}$$
$$v = \frac{x_4^{r-p_4}}{(x_1x_2cx_3)^{p_4}}, \qquad t = \frac{(x_1x_3x_4)^{r-p_2-1}}{x_2^{p_2+1}},$$

The affine variety defined by this cone is $U(C_1) \cong \mathrm{Spec}\,\mathbb{C}\left[u, v, w, t, p\right] / \left(uv - wt\right)$. Similarly, cone C_2 defines the affine variety $\mathrm{Spec}\,\mathbb{C}\left[u', v', w', t', q\right] / \left(u'v' - w't'\right)$, where $q = p^{-1}, u' = pu, v' = pv, w' = pw$ and $t' = pt$. Therefore, the toric variety defined by the fan $\Sigma^p_{[12-34]}$ is a line of ordinary nodes along $\mathbb{P}^1$.

As Σ^p_φ and $\Sigma^p_{\varphi'}$ subdivide the fan $\Sigma^p_{[12-34]}$, there are two induced maps f and f' on the corresponding toric varieties as in the diagram below. Each of this maps contracts a $\mathbb{P}^1$ and the birational map $\Sigma^p_{[12-34]}$ is indeed an ordinary flop.

$$\begin{tikzcd}
U(\Sigma^p_\varphi) \arrow[rr, dashed, "\Phi^p_{[12-34]}"] \arrow[dr, "f"'] & & U(\Sigma^p_{\varphi'}) \arrow[dl, "f'"] \\
& U(\Sigma^p_{[12-34]}) &
\end{tikzcd}$$

We can repeat the above procedure for every octahedron O_P of the partial resolution Σ_{TO}, where $P \in \mathbb{Z}^4_{\geq 0}$, such that $\sum p_i = r - 2$, and thus we have

Theorem 2.4.1. *If φ, φ' are two orientation functions, a birational map Y_φ - $\to$ $Y_{\varphi'}$ can be obtained as a sequence of at most $\binom{r+1}{3}$ ordinary flops.*

Chapter 3

G-Hilb$(\mathbb{C}^4)$

In the previous chapter, we defined the special crepant resolutions of the quotient variety $X = \mathbb{C}^4/G$, for the group $G \simeq (\mathbb{Z}/r)^{\oplus 3}$ acting on $\mathbb{C}^4$, by

$$\tfrac{1}{r}(1,-1,0,0) \oplus \tfrac{1}{r}(1,0,-1,0) \oplus \tfrac{1}{r}(1,0,0,-1)\,. \tag{3.1}$$

The construction we presented is purely toric: the fans of the resolutions all come from a certain tesselation of 3-space. These crepant resolutions are in fact "special" not only because they have almost symmetric descriptions, but also because of their relation to the Hilbert scheme of G-orbits.

This chapter gives an explicit description of G-Hilb $\left(\mathbb{C}^4\right)$ in terms of its toric fan. In Section 3.1, we define the fan $\Sigma_{\text{G-Hilb}}$ and state the main theorem 3.1.1 which says that the G-Hilbert scheme is isomorphic to the toric variety given by the fan $\Sigma_{\text{G-Hilb}}$. The following two sections introduce the terminology used in the rest of the chapter.

In Section 3.4 we express all the G-clusters in terms of their defining ideals, see Theorem 3.4.1. Part one of this theorem is proven in Section 3.5, and part two in Section 3.6. Sections 3.7 and 3.8 describe the birational component $\text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$ of the G-Hilbert scheme. In the final section we prove that every G-cluster lies in one of the families parametrised by $\text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$ implying that G-Hilb $\left(\mathbb{C}^4\right)$ is in fact irreducible and equal to the birational component, proving the main theorem 3.1.1.

c

3.1 Toric fan of the G-Hilbert scheme

Using the notation from the previous chapter, we define a new toric fan $\Sigma_{\text{G-Hilb}}$ in the lattice $N = \mathbb{Z}^4 + \mathbb{Z} \cdot \frac{1}{r}(1,-1,0,0) + \mathbb{Z} \cdot \frac{1}{r}(1,0,-1,0) + \mathbb{Z} \cdot \frac{1}{r}(1,0,0,-1)$.

Definition 3.1.1. The toric fan $\Sigma_{\text{G-Hilb}}$ is defined by the collection of the following cones together with their faces:

1. For every $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum p_i = r-1$, there is an "up" tetrahedron:

$$\text{Cone}(U_p) = \text{Cone}(u_1^p, u_2^p, u_3^p, u_4^p),$$

   where $u_i^p = \frac{1}{r}(p + e_i)$, for $i \in \{1,2,3,4\}$.

2. For all $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum p_i = r-3$, there is a "down" tetrahedron

$$\text{Cone}(D_p) = \text{Cone}(d_1^p, d_2^p, d_3^p, d_4^p),$$

   where $d_i^p = \frac{1}{r}(p + (1,1,1,1) - e_i)$, for all $i \in \{1,2,3,4\}$.

3. The "halfway up" tetrahedra: for every $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum p_i = r-2$ there are four cones

$$\begin{aligned}
&\text{Cone}(o_{23}^p, o_{24}^p, o_{34}^p, m_p)\\
&\text{Cone}(o_{13}^p, o_{14}^p, o_{34}^p, m_p)\\
&\text{Cone}(o_{12}^p, o_{14}^p, o_{24}^p, m_p)\\
&\text{Cone}(o_{12}^p, o_{13}^p, o_{23}^p, m_p)
\end{aligned}$$

   where $o_{ij}^p = \frac{1}{r}(p + e_i + e_j)$, for all the distinct $i,j \in \{1,2,3,4\}$ and the central point $m_p = \frac{1}{r}(2p_1+1, 2p_2+1, 2p_3+1, 2p_4+1)$.

4. The "halfway down" tetrahedra: for every $p \in \mathbb{Z}^4_{\geq 0}$ such that $\sum p_i = r-2$ there are four cones

$$\begin{aligned}
&\mathrm{Cone}\left(o^p_{12}, o^p_{13}, o^p_{14}, m_p\right)\\
&\mathrm{Cone}\left(o^p_{12}, o^p_{23}, o^p_{24}, m_p\right)\\
&\mathrm{Cone}\left(o^p_{13}, o^p_{23}, o^p_{34}, m_p\right)\\
&\mathrm{Cone}\left(o^p_{14}, o^p_{24}, o^p_{34}, m_p\right)
\end{aligned}$$

where the lattice points m_p and o^p_{ij} are as in the "halfway up" case.

The fan $\Sigma_{\text{G-Hilb}}$ from the definition is a refinement of each special crepant resolution fan, and we prove in Section 3.9 of this chapter that it is the fan defining the G-Hilbert scheme G-Hilb $\left(\mathbb{C}^4\right)$. Remember that the special crepant resolution fans required a choice of direction: the octahedra were cut along two out of possible three axes. Here we slice along **all three** of the axes spanned by two pairs of antipodal vertices of an octahedron, and subdivide every octahedron into eight smaller tetrahedra. This is a nicer refinement in the sense that it is symmetric, but doing so creates a number of rays through points m_p which have age two. Such a ray intersects the junior simplex in a point

$$\tfrac{1}{2}m_p = \tfrac{1}{2r}\left(2p_1+1, 2p_2+1, 2p_3+3, 2p_4+1\right), \quad \text{where } \sum_{i=1}^{4} p_i = r-2,$$

which does not lie in the lattice N. Hence the point m_p is the principal generator of its ray and because of this, the fan $\Sigma_{\text{G-Hilb}}$ does not yield a crepant resolution of singularities.

Remark 3.1.2. A "halfway down" tetrahedron has a common face with an "up" tetrahedron, whereas a "half-way up" tetrahedron either shares a face with a "down" tetrahedron, or has an external face (a face that is a subset of a face of the junior simplex). This way an internal face always connects an "up" or "halfway up" tetrahedron with a "down" or "halfway down" tetrahedron. When $r = 2$, there are four "up" tetrahedra, each of them joined with a single "halfway down" tetrahedron.

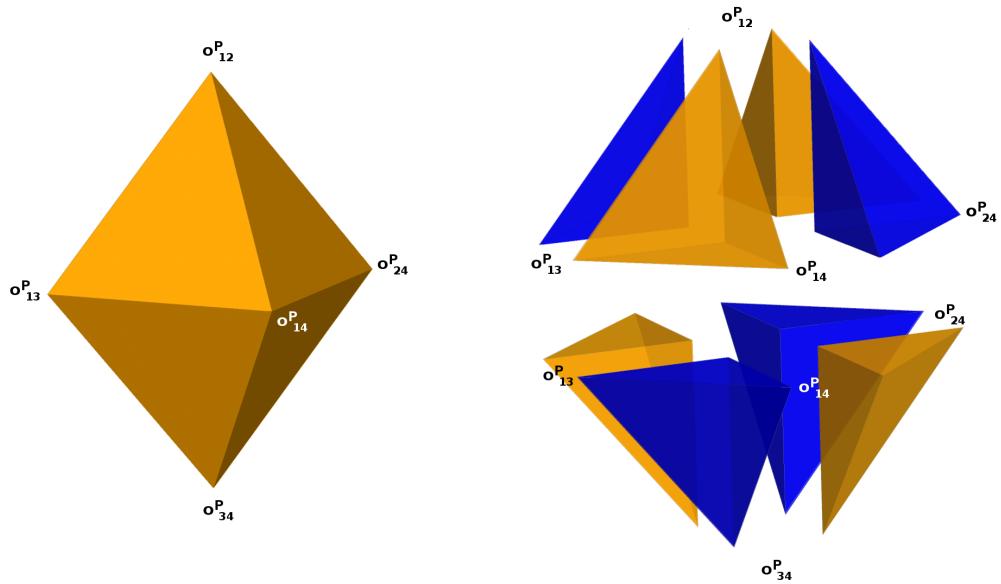


Figure 3.1: An octahedron O_P and the eight tetrahedra: "halfway up" are coloured blue and "halfway down" are coloured orange.

The four remaining "halfway down" tetrahedra all have an external face.

Theorem 3.1.1. *The scheme* G-Hilb $\left(\mathbb{C}^4\right)$ *is an irreducible toric variety given by fan* $\Sigma_{\text{G-Hilb}}$. *Consequently, it admits a projective birational morphism* G-Hilb $\left(\mathbb{C}^4\right) \to \mathbb{C}^4/G$.

The proof of this theorem is provided in the last section of this chapter. Notice that the result of Craw-Maclagan-Thomas, [7, Theorem 1.1.], shows that irreducibility implies the existence of a projective morphism to the quotient variety. Before we start paving the path towards the proof, we present several observations about the G-Hilb $\left(\mathbb{C}^4\right)$ that follow from its defining fan.

Corollary 3.1.3. G-Hilb $\to X$ *is a resolution of singularities.*

Proof. The "up" and "down" cones are smooth from the result in Proposition 2.2.3. Fix a "halfway up" tetrahedron: the primitive generators of its rays are $o^p_{ij}, o^p_{ik}, o^p_{jk}$ and m_p, for some $p \in \mathbb{Z}_{\geq 0}$, $\sum p_i = r-2$ and a permutation (i,j,k,l) of $\{1,2,3,4\}$. Since the lattice point o^p_{kl} can be written as $m_p - o^p_{ij}$, it is enough to show that the

points $o_{ij}^p, o_{ik}^p, o_{jk}^p$ and o_{kl}^p (instead of m_p) form a basis for the lattice N. But this follows from Proposition 2.3.2, when the orientation function φ is chosen so that $\varphi(p) = (\{i,j\}, \{k,l\})$.

Similarly, any "halfway down" tetrahedron has the vertices $o_{ij}^p, o_{ik}^p, o_{il}^p$ and m_p as the primitive generators of its rays, for some $p \in \mathbb{Z}_{\geq 0}$, $\sum p_i = r-2$ and some permutation (i,j,k,l) of $\{1,2,3,4\}$. We again use the result of Proposition 2.3.2 to show that these points are a basis for N.

The fan $\Sigma_{\text{G-Hilb}}$ is a subdivision of the fan Σ_{TO}, and $|\Sigma_{\text{G-Hilb}}| = |\Sigma_{\text{TO}}| = |\sigma|$ so the induced map of toric varieties $\text{G-Hilb}\left(\mathbb{C}^4\right) \to X_\sigma$ is a proper birational morphism. $\square$

Now that we have the fan structure for the toric variety $\text{G-Hilb}\left(\mathbb{C}^4\right)$, it is easy to answer the question of discrepancy of the resolution $\text{G-Hilb}\left(\mathbb{C}^4\right) \to X_\sigma$. All the rays in the fan $\Sigma_{\text{G-Hilb}}$ are the same as in any of the special crepant resolutions, with addition of rays through the points

$$m_p = \tfrac{1}{r}\left(2p_1+1, 2p_2+1, 2p_3+1, 2p_4+1\right),$$

for all $p \in \mathbb{Z}_{\geq 0}$ such that $\sum p_i = r-2$. The age of every such point is

$$\operatorname{age}(m_p) = \tfrac{1}{r}\sum_{i=1}^{4}(2p_i+1) = 2$$

and the point m_p is the primitive generator of its ray, since the point $\frac{1}{2}m_p$ is not a lattice point. The number of rays generated by such m_p is the same as the number of octahedra: $\binom{r+1}{3}$. This leads to the following corollary:

Corollary 3.1.4. $\text{G-Hilb} \to X$ *has* $\binom{r+1}{3}$ *discrepant divisors, all isomorphic to* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Proof. The only discrepant divisors correspond to the rays through lattice points $(2p_1+1, 2p_2+1, 2p_3+1, 2p_4+1)$, for $p \in \mathbb{Z}_{\geq 0}^4$ such that $\sum p_i = r-2$. The fan $\Sigma_{\text{G-Hilb}}$ shows that each one of these rays defines a copy of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. $\square$

Corollary 3.1.5. *A special crepant resolution Y_φ, where φ is the orientation function, is obtained from* $\text{G-Hilb}\left(\mathbb{C}^3\right)$ *by contracting a single $\mathbb{P}^1$ in each discrepant divisor of* $\text{G-Hilb}\left(\mathbb{C}^3\right)$.

The orientation function φ of the previous can now be regarded as the choice of projections of the $\binom{r+1}{3}$ copies of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ to two of its factors.

3.2 G-clusters and $\text{G-Hilb}(\mathbb{C}^n)$ for an Abelian group G

Definition 3.2.1. Let G be an Abelian group acting on $\mathbb{C}^n$. A **G-cluster** is a G-invariant subscheme $Z \subset \mathbb{C}^n$ such that $H^0\left(Z, \mathcal{O}_Z\right)$ is the regular representation of G. The **G-Hilbert scheme** $\text{G-Hilb}\left(\mathbb{C}^{\mathbf{n}}\right)$ is the moduli space of G-clusters.

As $\mathcal{O}_Z = \mathbb{C}\left[x_1, \ldots, x_4\right] / \mathcal{I}_Z$ is the regular representation of the Abelian group G, it breaks down into $|G|$ 1-dimensional eigenspaces – one for each character. On the other hand, since the polynomial ring $\mathbb{C}\left[x_1, \ldots, x_4\right]$ viewed as a vector space over $\mathbb{C}$ has a basis consisting of monomials, its quotient $\mathcal{O}_Z$ will also have a monomial basis. Henceforth, the image of a monomial $\mathbf{m}$ that is nonzero in $\mathcal{O}_Z$, and such that $\mathbf{m}$ is not a multiple of an invariant monomial, will be referred to as a **basic monomial**.

From the discussion above, it follows that there is at least one basic monomial in every eigenspace, and every monomial is a multiple of a basic monomial from the same eigenspace. Notice the difference between the notions of a "member of a basis for $\mathcal{O}_Z$" and a basic monomial: it is perfectly possible for two monomials from the same eigenspace to be basic. Being a basic monomial simply means that it is a good candidate to be a member of a basis for $\mathcal{O}_Z$.

Lemma 3.2.2. *The following statements hold:*

- $\mathbf{1}$ *is basic.*
- *If* $\mathbf{m}$ *is a basic monomial in* $\mathcal{O}_Z$ *and* $\mathbf{m} = \mathbf{m_1}\mathbf{m_2}$ *for some monomials* $\mathbf{m_1}$ *and* $\mathbf{m_2}$, *then* $\mathbf{m_1}$ *and* $\mathbf{m_2}$ *must also be basic.*

Proof. If $\mathbf{1}$ were zero in $\mathcal{O}_Z$, then every monomial would also have to be zero, i.e. $\mathcal{O}_Z$ would be a zero ring which is not possible. For the other statement, if $\mathbf{m_1}$ is zero, then so is $\mathbf{m}$. $\square$

3.3 Eigenspaces of the action of $(\mathbb{Z}/r)^{\oplus 3}$ on $\mathbb{C}^4$

Coming back to the group $G = (\mathbb{Z}/r)^{\oplus 3}$ acting on the four-dimensional complex affine space by $\frac{1}{r}(1,-1,0,0) \oplus \frac{1}{r}(1,0,-1,0) \oplus \frac{1}{r}(1,0,0,-1)$, the eigenspaces of its action on the polynomial ring $\mathcal{O}_{\mathbb{C}^4} = \mathbb{C}[x_1, x_2, x_3, x_4]$ can be described explicitly as R-modules, where $R = \mathbb{C}[x_1, x_2, x_3, x_4]^G$ is the ring of G-invariants.

Lemma 3.3.1. *Let α, β, γ be the three chosen generators of the group $G = (\mathbb{Z}/r)^{\oplus 3}$ defining the group action* (3.1)*. For every $s, u, v \in \{0, 1, \ldots, r-1\}$, the eigenspace*

$$\mathcal{L}_{suv} = \left\{ \mathbf{m} \in \mathbb{C}[x_1, \ldots, x_4] \ : \ \begin{array}{r} \alpha \cdot \mathbf{m} = \varepsilon^s \mathbf{m}, \\ \beta \cdot \mathbf{m} = \varepsilon^u \mathbf{m}, \\ \gamma \cdot \mathbf{m} = \varepsilon^v \mathbf{m} \end{array} \right\}$$

is an R-module represented by

$$\mathcal{L}_{suv} = x_1^v x_2^{\overline{v-s}} x_3^{\overline{v-u}} \cdot R + x_1^u x_2^{\overline{u-s}} x_4^{\overline{u-v}} \cdot R + x_1^s x_3^{\overline{s-u}} x_4^{\overline{s-v}} \cdot R + x_2^{\overline{-s}} x_3^{\overline{-u}} x_4^{\overline{-v}} \cdot R$$

where for an integer n, the number $\overline{n} \in \{0, 1, \ldots, r-1\}$ satisfies $\overline{n} \equiv n \pmod{r}$. Furthermore, set $S_{suv} := \left\{ x_1^v x_2^{\overline{v-s}} x_3^{\overline{v-u}},\ x_1^u x_2^{\overline{u-s}} x_4^{\overline{u-v}},\ x_1^s x_3^{\overline{s-u}} x_4^{\overline{s-v}},\ x_2^{\overline{-s}} x_3^{\overline{-u}} x_4^{\overline{-v}} \right\}$ is the set of minimal generators of $\mathcal{L}_{suv}$ over R.

Remark 3.3.2. Notice that $\mathcal{L}_{000}$ is a module over R generated by the unit element, so it is just the ring of invariants R.

Proof. The standard monomial multiplication gives the map $R \times \mathcal{L}_{suv} \to \mathcal{L}_{suv}$. It is well defined since multiplying by a G-invariant monomial keeps the eigenspace unchanged. This map gives the ring $\mathcal{L}_{suv}$ the structure of an R-module.

For the second statement, assume $\mathbf{m} = x_1^a x_2^b x_3^c x_4^d \in \mathcal{L}_{suv}$. Then $\alpha \cdot \mathbf{m} = \varepsilon^s \mathbf{m}$, and

$$\alpha \cdot \mathbf{m} = \alpha \cdot x_1^a x_2^b x_3^c x_4^d = \varepsilon^{a-b} x_1^a x_2^b x_3^c x_4^d = \varepsilon^{a-b} \mathbf{m},$$

so $a - b \equiv s \pmod{r}$. Similarly, $a - c \equiv u \pmod{r}$ and $a - d \equiv v \pmod{r}$.

Since $x_1 x_2 x_3 x_4$ and x_i^r are G-invariant monomials for $i \in \{1, 2, 3, 4\}$, the generators of the module $\mathcal{L}_{suv}$ over R are the monomials in at most three variables with exponents nonnegative and strictly smaller than r. So for every generator $\mathbf{m} = x_1^a x_2^b x_3^c x_4^d$, at least one of a, b, c and d is zero.

- If $a = 0$, then $b \equiv -s \pmod{r}$ which implies $b = \overline{-s}$. Similarly, $c = \overline{-u}$ and $d = \overline{-v}$.
- If $b = 0$, then $a = \overline{s} = s$ and $c \equiv a - u \pmod{r}$ so $c = \overline{s - u}$, $d = \overline{s - v}$.
- If $c = 0$, then $a = u$, $b = \overline{u - s}$ and $d = \overline{u - v}$.
- If $d = 0$, then $a = u$, $b = \overline{v - s}$ and $c = \overline{v - u}$.

Suppose that S_{suv} is not a minimal set of generators of the R-module $\mathcal{L}_{suv}$. Then there exist monomials $\mathbf{m}, \mathbf{n} \in S_{suv}$ such that $\mathbf{m}$ is divisible by $\mathbf{n}$. Without the loss of generality, there are three cases. First case is that $\mathbf{m} = x_1^v x_2^{\overline{v-s}} x_3^{\overline{v-u}}$ and $\mathbf{n} = x_1^u x_2^{\overline{u-s}} x_4^{\overline{s-v}}$. Since $\mathbf{n}$ divides $\mathbf{m}$, we must have $u = v$, which means that $\mathbf{m} = \mathbf{n}$. For the second case, $\mathbf{m}$ stays the same, but $\mathbf{n} = x_2^{\overline{-s}} x_3^{\overline{-u}} x_4^{\overline{-v}}$. Similarly as before, v must be zero and the two monomials are again equal. The third case is obtained by swapping the expressions $\mathbf{m}$ and $\mathbf{n}$ and imply in the same manner as beore that $v = 0$. So the set S_{suv} is indeed the minimal set of generators of $\mathcal{L}_{suv}$ and its cardinality is at most four. $\square$

As the group G has exactly three generators and it is acting on a four-dimensional space, the definition of the group action is seemingly breaking the symmetry. The following corollary describes the generators of the eigenspaces in more symmetric terms, which will simplify the proofs of the results that follow.

Corollary 3.3.3. *If the monomial $x_i^{a_i} x_j^{a_j} x_k^{a_k}$ is one of the minimal R-module generators of the eigenspace $\mathcal{L}_{suv}$, where $i, j, k \in \{1, 2, 3, 4\}$ are distinct, then the other three generators are*

$$x_i^{\overline{a_i - a_k}} x_j^{\overline{a_j - a_k}} x_l^{\overline{-a_k}}$$
$$x_i^{\overline{a_i - a_j}} x_k^{\overline{a_k - a_j}} x_l^{\overline{-a_j}}$$
$$x_j^{\overline{a_j - a_i}} x_k^{\overline{a_k - a_i}} x_l^{\overline{-a_i}}$$

where l is the element of $\{1, 2, 3, 4\}$ distinct from i, j, k.

Proof. If $\{i, j, k\} = \{1, 2, 3\}$ and the eigenspace in question is $\mathcal{L}_{suv}$, then using the notation of Lemma 3.3.1 above, we get $a_1 = v$, $a_2 = \overline{v - s}$ and $a_3 = \overline{v - u}$. and we just need to read out the other three generators:

$$x_1^u x_2^{\overline{u-s}} x_4^{\overline{u-v}} = x_1^{\overline{a_1 - a_3}} x_2^{\overline{a_2 - a_3}} x_4^{\overline{-a_3}}$$
$$x_1^s x_3^{\overline{s-u}} x_4^{\overline{s-v}} = x_1^{\overline{a_1 - a_2}} x_3^{\overline{a_3 - a_2}} x_4^{\overline{-a_2}}$$
$$x_2^{\overline{-u}} x_3^{\overline{-s}} x_4^{\overline{-v}} = x_2^{\overline{a_2 - a_1}} x_3^{\overline{a_3 - a_1}} x_4^{\overline{-a_1}}.$$

If $1 \in \{i, j, k\}$, the situation is symmetric to the case above, so the only remaining case to consider is $\{i, j, k\} = \{2, 3, 4\}$. The correctness of the statement in this case is equally easy to check. If $x_2^{a_2} x_3^{a_3} x_4^{a_4}$ is a minimal R-module generator of $\mathcal{L}_{suv}$, then $a_2 = \overline{-s}$, $a_3 = \overline{-u}$ and $a_4 = \overline{-v}$. The other three generators are:

$$x_1^v x_2^{\overline{v-s}} x_3^{\overline{v-u}} = x_1^{\overline{-a_4}} x_2^{\overline{a_2 - a_4}} x_3^{\overline{a_3 - a_4}}$$
$$x_1^u x_2^{\overline{u-s}} x_4^{\overline{u-v}} = x_1^{\overline{-a_3}} x_2^{\overline{a_2 - a_3}} x_4^{\overline{a_4 - a_3}}$$
$$x_1^s x_3^{\overline{s-u}} x_4^{\overline{s-v}} = x_1^{\overline{-a_2}} x_3^{\overline{a_3 - a_2}} x_4^{\overline{a_4 - a_2}}.$$

□

Corollary 3.3.4. *There are two special classes of eigenspaces for the G-action that are generated as R-modules by only two monomials. If $x_i^s \in \mathcal{L}'$, for an eigenspace*

$\mathcal{L}'$ *and* $1 \leq s \leq r-1$, *then*

$$\mathcal{L}' = x_i^s \cdot R + (x_j x_k x_l)^{r-s} \cdot R.$$

If $(x_i x_j)^u \in \mathcal{L}''$, *for an eigenspace* $\mathcal{L}''$ *and* $1 \leq u \leq r-1$, *then*

$$\mathcal{L}'' = (x_i x_j)^u \cdot R + (x_k x_l)^{r-u} \cdot R.$$

Having understood the action of G on $\mathbb{C}[x_1, x_2, x_3, x_4]$, we go back to G-clusters and describe their defining ideals a little better.

Lemma 3.3.5. *For a G-cluster Z, the ideal $\mathcal{I}_Z$ is generated by binomial relations of the form $\mathbf{n} - \lambda \mathbf{m}_{suv}$, where $\lambda \in \mathbb{C}$ and $\mathbf{n}, \mathbf{m}_{suv}$ are monomials in $\mathcal{O}_Z$ that lie in the same eigenspace $\mathcal{L}_{suv}$ of the G-action, such that $\mathbf{m}_{suv}$ is basic.*

Proof. Fix a G-cluster Z. The proof consists of three steps: we first choose a $\mathbb{C}$-basis for $\mathcal{O}_Z = \mathbb{C}[x_1, x_2, x_3, x_4]/\mathcal{I}_Z$ by choosing one basic monomial for each eigenspace of G-action on the polynomial ring $\mathbb{C}[x_1, x_2, x_3, x_4]$. Using this basis we construct a set of relations $B \subset \mathcal{I}_Z$ and show that the ideal generated by B defines a G-cluster. Finally, we observe that this G-cluster is exactly Z and that $\mathcal{I}_Z$ is generated by relations B.

Since $\mathbf{1}$ is a basic monomial, $x_i^r \in \mathcal{O}_Z$ is its multiple (possibly the zero multiple) for all $i \in \{1, 2, 3, 4\}$. By the same reasoning $x_1 x_2 x_3 x_4$ is a multiple of $\mathbf{1}$. Thus we get five relations contained in $\mathcal{I}_Z$:

$$\begin{aligned} x_i^r &- \zeta_i \cdot \mathbf{1}, \qquad \text{for } i \in \{1, 2, 3, 4\} \text{ and} \\ x_1 x_2 x_3 x_4 &- \xi \cdot \mathbf{1}, \end{aligned} \tag{3.2}$$

for some $\zeta_i, \xi \in \mathbb{C}$. A nontrivial eigenspace $\mathcal{L}_{suv}$, where $s, u, v \in \{0, 1, 2, \ldots, r-1\}$ are not all zero, is generated as an R-module by the monomials from set S_{suv}. From the definition of the G-constellation, at least one of those monomials is nonzero. Choose one such nonzero monomial and label it by $\mathbf{m}_{suv}$. As eigenspace $\mathcal{L}_{suv}$ is a one-dimensional vector subspace of $\mathcal{O}_Z$, all the other minimal generators are mul-

tiples of it. In other words, for every generator $\mathbf{n}$ of $\mathcal{L}_{suv}$ there exists a complex number $\zeta_{\mathbf{n}}$ such that $\mathbf{n} = \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv}$. This way we get a set of relations

$$\left\{ \mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv} \in \mathcal{I}_Z \;\middle|\; \begin{array}{l} \mathbf{n} \in S_{suv},\ \mathbf{n} \neq \mathbf{m}_{suv}, \\ s, u, v \in \{0, 1, \ldots, r-1\},\ s+u+v \neq 0 \end{array} \right\} \subset \mathcal{I}_Z. \quad (3.3)$$

Now define $\mathcal{J}$ to be an ideal generated by the relations (3.2) and (3.3). Clearly, $\mathcal{J}$ is a subideal of $\mathcal{I}_Z$.

We now prove that $\mathcal{J}$ defines a G-cluster. The affine scheme T defined by the ideal $\mathcal{I}$ is G-invariant, as $(\mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv})(z) = 0$ implies

$$(\mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv})(g \cdot z) = \varepsilon^t \cdot (\mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv})(z) = 0,$$

for all $\mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv}$ generating $\mathcal{J}$ and all $z \in T$ and $g \in G$, where t is some integer depending on the group element g and the eigenspace $\mathcal{L}_{suv}$ containing $\mathbf{n}$ and $\mathbf{m}_{suv}$.

Secondly, any monomial (not only a minimal generator of the R-module) $\mathbf{n} \in \mathcal{L}_{suv}$ reduces to a multiple of $\mathbf{m}_{suv}$ in the following way. We first write $\mathbf{n}$ as a product of G-invariant monomial $\mathbf{n}_{\mathrm{i}}$ and a generator of its eigenspace $\mathbf{n}'$. Then relations (3.2) reduce $\mathbf{n}_{\mathrm{i}}$ to a multiple of $\mathbf{1}$ and relations (3.3) reduce $\mathbf{n}'$ to a multiple of $\mathbf{m}_{suv}$. Therefore the elements of an eigenspace $\mathcal{L}_{suv}$ are all linearly dependent, i.e. they form a 1-dimensional subspace of $\mathcal{O}_T = \mathbb{C}[x_1, x_2, x_3, x_4]/\mathcal{J}$. In conclusion, $\mathcal{O}_T$ breaks down into $|G|$ distinct irreducible representations so it is a regular representation of G and $\mathcal{J}$ defines a G-cluster.

To finish the proof, we need to show that $\mathcal{J} = \mathcal{I}_Z$. Since $\mathcal{J} \subseteq \mathcal{I}_Z$, there is a natural surjective homomorphism of algebras $\Phi : R/\mathcal{J} \to R/\mathcal{I}_Z$, mapping the class $[m]_{\mathcal{J}}$ of an element $m \in R$ from the domain to the class $[m]_{\mathcal{I}_Z}$ in the codomain. As this is especially an epimorphism of the vector spaces of the same dimension $|G|$ over $\mathbb{C}$, this map is bijective. So Φ is an isomorphism of algebras. Finally, suppose $m \in \mathcal{I}_Z$. Then $\Phi\left([m]_{\mathcal{J}}\right) = [m]_{\mathcal{I}_Z} = [0]_{\mathcal{I}_Z}$ implies $[m]_{\mathcal{J}} = [0]_{\mathcal{J}}$ so $m \in \mathcal{J}$. $\square$

3.4 The equations of the G-clusters

This section contains the key result of this thesis. Theorem 3.4.1 below gives the explicit description of G-clusters which are, as will be shown in the later sections, parametrised by the variety given by $\Sigma_{\text{G-Hilb}}$ defined in 3.1.1.

From now on, the multiindex notation will be used in the following fashion: $b_{ij} = b_{ji}$ is a shorthand for $b_{\{i,j\}}$, and similarly, $\mu_{ij} = \mu_{\{i,j\}}$, $\nu_{ijk} = \nu_{\{i,j,k\}}$.

Theorem 3.4.1. ***1.*** *For every G-cluster $Z \subset \mathbb{C}^4$, generators of the ideal $\mathcal{I}_Z$ can be chosen as the system of fifteen equations:*

$$\begin{cases} x_i^{a_i} & = & \lambda_i \ (x_j x_k x_l)^{r-a_i}, & \\ (x_i x_j)^{b_{ij}} & = & \mu_{ij} \ \ (x_k x_l)^{r-b_{ij}}, & \textit{for all the permutations} \\ (x_i x_j x_k)^{r-a_l+1} & = & \nu_{ijk} \quad x_l^{a_l-1}, & (i,j,k,l) \textit{ of } \{1,2,3,4\} \\ x_1 x_2 x_3 x_4 & = & \xi & \end{cases} \tag{3.4}$$

where a_i and b_{ij} are integers such that

(i) $1 \leq b_{ij} \leq a_i \leq r$,

(ii) $b_{ij} + b_{kl} = r + 1$,

(iii) $b_{ij} \in \{a_i + a_j - r - 1, a_i + a_j - r\}$,

for all $\{i,j,k,l\} = \{1,2,3,4\}$, while $\lambda_i, \mu_{ij}, \nu_{ijk}$ and ξ are complex numbers satisfying

$$\lambda_i \nu_{jkl} = \mu_{ij} \mu_{kl} = \xi. \tag{3.5}$$

2. *Furthermore, exactly one of the following four cases holds:*

UP *For all* $1 \le i < j \le 4$ *it holds that* $b_{ij} = a_i + a_j - r$. *Then*

$$\sum_{i=1}^{4} a_i = 3r+1 \qquad \textit{and} \qquad \begin{aligned} \mu_{ij} &= \lambda_i \lambda_j, \\ \nu_{ijk} &= \lambda_i \lambda_j \lambda_k, \\ \xi &= \lambda_1 \lambda_2 \lambda_3 \lambda_4. \end{aligned} \tag{3.6}$$

HWU *For a fixed* $i \in \{1, \ldots, 4\}$ *and all the permutations* (j, k, l) *of the elements of* $\{1, 2, 3, 4\} \setminus \{i\}$, *it holds that* $b_{ij} = a_i + a_j - r$, *but* $b_{jk} = a_j + a_k - r - 1$. *Then*

$$\sum_{i=1}^{4} a_i = 3r+2 \qquad \textit{and} \qquad \begin{aligned} \lambda_j &= \lambda_i \mu_{jk} \mu_{jl}, \\ \mu_{ij} &= \lambda_i^2 \mu_{jk} \mu_{jl}, \\ \nu_{ijk} &= \lambda_i \, \mu_{jk}, \\ \nu_{jkl} &= \lambda_i \mu_{jk} \mu_{jl} \mu_{kl}, \\ \xi &= \lambda_i^2 \mu_{jk} \mu_{jl} \mu_{kl}. \end{aligned} \tag{3.7}$$

HWD *For a fixed* $i \in \{1, \ldots, 4\}$ *and all the permutations* (j, k, l) *of the elements of* $\{1, 2, 3, 4\} \setminus \{i\}$, *it holds that* $b_{ij} = a_i + a_j - r - 1$, *but* $b_{jk} = a_j + a_k - r$. *Then*

$$\sum_{i=1}^{4} a_i = 3r+2 \qquad \textit{and} \qquad \begin{aligned} \nu_{ijk} &= \nu_{jkl} \mu_{ij} \mu_{ik}, \\ \mu_{jk} &= \nu_{jkl}^2 \mu_{ij} \mu_{ij}, \\ \lambda_j &= \nu_{jkl} \mu_{ij}, \\ \lambda_i &= \nu_{jkl} \mu_{ij} \mu_{ik} \mu_{il}, \\ \xi &= \nu_{jkl}^2 \mu_{ij} \mu_{ik} \mu_{il}. \end{aligned} \tag{3.8}$$

DOWN *For all* $1 \le i < j \le 4$ *it holds that* $b_{ij} = a_i + a_j - r - 1$. *Then*

$$\sum a_i = 3r+3 \qquad \textit{and} \qquad \begin{aligned} \lambda_i &= \nu_{ijk} \nu_{ijl} \nu_{ikl}, \\ \mu_{ij} &= \nu_{ijk} \nu_{ijl}, \\ \xi &= \nu_{123} \nu_{124} \nu_{134} \nu_{234}. \end{aligned} \tag{3.9}$$

Part **1.** of the theorem is proved in Section 3.5 in several steps. Part **2.** is proved in Section 3.6.

3.5 Generating set of relations for $\mathcal{I}_Z$

In this section we prove the part one of Theorem 3.4.1. We start by proving a weaker result, see Proposition 3.5.2, which we then gradually strenghten. In more detail, we first show that the relations (3.4) of the theorem can be chosen in $\mathcal{I}_Z$ so that they satisfy fewer restrictions, namely, we only require conditions *(i)* and *(ii)* from the statement. This is the statement of Propostion 3.5.2. Then, having obtained this result, the proof of Proposition 3.5.6 shows that the existing set of relations 3.10 can be modified to also satisfy a new condition *(iii')*, still weaker than condition *(iii)* of the theorem . Finally, in the last subsection we show that the relations of Proposition 3.5.6 can again be adapted to satisfy the condition *(iii)* which finishes the proof.

3.5.1 Weak version

First we prove a weak version of the first statement of Theorem 3.4.1: we show that a set of generators of $\mathcal{I}_Z$ can be chosen as (3.4), but satisfying fewer conditions, see Proposition 3.5.2 for details. The following lemma is the starting point in proving the existence of relations (3.4) from the statement of Theorem 3.1.1.

Lemma 3.5.1. *1. For every $i \in \{1,2,3,4\}$, there is at least one $a_i \in \{1,2,\dots,r\}$ such that $1, x_i, x_i^2, \dots, x_i^{a_i-1}$ are basic monomials, and $x_i^{a_i}$ is a multiple (possibly the zero multiple) of a basic monomial $(x_j x_k x_l)^{r-a_i}$.*

2. For every pair of distinct indices $i,j \in \{1,2,3,4\}$, there exists an integer $b_{ij} \in \{1,2,\dots,\min\{a_i,a_j\}\}$ such that every monomial that divides $(x_i x_j)^{b_{ij}-1}$ is basic and $(x_i x_j)^{b_{ij}}$ is a multiple (possibly zero) of basic monomial $(x_k x_l)^{r-b_{ij}}$.

Proof. The key observation here is that for an eigenspace containing x_i^s, $1 \le s \le r$, there is only one other generator that is not a multiple of an invariant monomial,

namely $(x_j x_k x_l)^{r-s}$, see Corollary 3.3.4 So there are at most two choices of for a $\mathbb{C}$-basis of such an eigenspace.

Fix an $i \in \{1, 2, 3, 4\}$. If all the $1, x_i, x_i^2, \ldots, x_i^{r-1}$ are basic, then a_i can be set to any value from 1 to r, provided that $x_j x_k x_l{}^{r-a_i}$ is also basic. One such choice is $a_i = r$, as $\mathbf{1} = (x_j x_k x_l)^0$ is always basic by Lemma 3.2.2.

If, on the other hand, there exists an exponent $1 \leq s \leq r$ such that the monomial x_i^s is zero in $\mathcal{O}_Z$, then $(x_j x_k x_l)^{r-s}$ must be nonzero, hence basic. In this case, set a_i to be the minimal such integer s and we get a relation $x_i^{a_i} = 0 \cdot (x_j x_k x_l)^{r-a_i}$.

The proof of the second claim follows similar logic. Fix $i \neq j$ from the set $\{1, 2, 3, 4\}$ and assume $a_i \leq a_j$. If $(x_i x_j)^{a_i}$ is nonzero, then from the way the exponents a_i and a_j were chosen, it follows that the monomial $(x_j x_k x_l)^{r-a_i}$ is basic and so is $(x_k x_l)^{r-a_i}$, by Lemma 3.2.2. As they are also in the same eigenspace, $(x_i x_j)^{a_i}$ is a multiple of $(x_k x_l)^{r-a_i}$.

Otherwise, set b_{ij} to be the minimal exponent such that $(x_i x_j)^{b_{ij}} = 0$ in $\mathcal{O}_Z$. It is clear that such $b_{ij} \leq a_i, a_j$. Since its eigenspace is generated by only two elements as a R-module, its other generator $(x_k x_l)^{r-b_{ij}}$ is nonzero, i.e. basic so $(x_i x_j)^{b_{ij}}$ is a zero multiple of it.

□

What we prove next looks almost identical to part one of Theorem 3.4.1. The only difference is that for now, relations (3.4) only have to satisfy conditions (i) and (ii), but not necessarily condition (iii), thus proving a weaker statement to the one of the theorem.

Proposition 3.5.2. *For every G-cluster $Z \subset \mathbb{C}^4$, generators of the ideal $\mathcal{I}_Z$ can be*

chosen as the system of fifteen equations:

$$\begin{cases} x_i^{a_i} & = & \lambda_i \, (x_j x_k x_l)^{r-a_i} \, , & \\ (x_i x_j)^{b_{ij}} & = & \mu_{ij} \;\; (x_k x_l)^{r-b_{ij}} \, , & \textit{for all the permutations} \\ (x_i x_j x_k)^{r-a_l+1} & = & \nu_{ijk} \quad x_l^{a_l-1}, & (i,j,k,l) \textit{ of } \{1,2,3,4\} \\ x_1 x_2 x_3 x_4 & = & \xi & \end{cases} \tag{3.10}$$

where a_i and b_{ij} are integers satisfying conditions

(i) $1 \leq b_{ij} \leq a_i \leq r$ *and*

(ii) $b_{ij} + b_{kl} = r + 1$, *for all* $\{i,j,k,l\} = \{1,2,3,4\}$.

Furthermore, the complex coefficients $\lambda_i, \mu_{ij}, \nu_{ijk}, \xi$ satisfy equations (3.5)*, that is the equations $\lambda_i \nu_{jkl} = \mu_{ij}\mu_{kl} = \xi$ for all indices i,j,k,l.*

Proof. First, notice that $x_1x_2x_3x_4$ is an invariant monomial, so it has to be a multiple of the basic monomial $\mathbf{1}$ and this gives the relation $x_1x_2x_3x_4 = \xi \cdot \mathbf{1}$.

Fix a permutation (i,j,k,l) of $\{1,2,3,4\}$. Lemma 3.5.1 shows that the relation $x_i^{a_i} = \lambda_i \cdot (x_j x_k x_l)^{r-a_i}$ can always be chosen in $\mathcal{I}_Z$ in a way that the monomial $x_i^{a_i-1}$ is basic. As the monomial $(x_j x_k x_l)^{r-a_i+1}$ lies in the same eigenspace by Corollary 3.3.4, it must be a multiple of $x_i^{a_i-1}$. This gives the four relations of type $(x_j x_k x_l)^{r-a_i+1} = \nu_{jkl} \cdot x_i^{a_i-1}$.

Following the second part of the lemma, one obtains the existence of the remaining six relations $(x_i x_j)^{b_{ij}} = \mu_{ij} \, (x_k x_l)^{r-b_{ij}}$ which satisfy $b_{ij} \leq a_i$ for all distinct $i,j \in \{1,2,3,4\}$. However, the lemma does not ensure that $b_{ij} + b_{kl} = r+1$ for all the distinct i,j,k,l. Suppose that $b_{12} + b_{34} > r+1$. As $(x_1x_2)^{b_{12}-1}$ is basic, the monomial $(x_3x_4)^{r-b_{12}+1}$ is a multiple of it, and as $r - b_{12} + 1 < b_{34} \leq a_3, a_4$, the relation $(x_3x_4)^{b_{34}} = \mu_{34} \, (x_1x_2)^{r-b_{34}}$ can be replaced by a new relation

$$(x_3x_4)^{b'_{34}} = \mu'_{34} \cdot (x_1x_2)^{r-b'_{34}} \, , \quad \text{where} \quad b'_{34} = r - b_{12} + 1.$$

Suppose now that $b_{12} + b_{34} < r+1$. Again, it follows from the choice of b_{12} that the monomial $(x_3x_4)^{r-b_{12}}$ is basic (and so are all of its divisors). As $(x_1x_2)^{b_{12}-1}$ is

basic, $(x_3x_4)^{r-b_{12}+1}$ is a multiple of it so b_{34} could have been chosen to be exactly $r-b_{12}+1$. The only issue with this new relation is that b_{34} could now be greater than a_3 or a_4. But in this case, say $b_{34} > a_3$, the exponent a_3 can be replaced with a larger one as follows: as $(x_3x_4)^{b_{34}-1}$ is basic, it follows that $x_3^{b_{34}-1}$ is also basic. Also, $(x_1x_2x_4)^{r-b_{34}}$ is basic since it divides a basic monomial $(x_1x_2x_4)^{r-a_3}$ which means that $x_3^{b_{34}}$ must be a multiple of $(x_1x_2x_4)^{r-b_{34}}$. Hence, the pair of relations $x_3^{a_3} = \lambda_3 (x_1x_2x_4)^{r-a_3}$, $(x_1x_2x_4)^{r-a_3+1} = \nu_{124}\, x_3^{a_3-1}$ can be replaced by

$$\begin{array}{rcl} x_3^{a_3'} & = & \lambda_3' \cdot (x_1x_2x_4)^{r-a_3'}, \\ (x_1x_2x_4)^{r-a_3'+1} & = & \nu_{124} \cdot x_3^{a_3'-1}, \end{array} \qquad \text{where} \quad a_3' = b_{34}.$$

The same can be done in case $b_{34} < a_4$. This proves that the equations (3.10) can be chosen from the ideal $\mathcal{I}_Z$ in such a way that $b_{ij} \leq a_i$ and $b_{ij} + b_{kl} = r+1$, for all distinct i, j, k, l.

Now that we have shown that the relations (3.10) satisfying conditions (i) and (ii) can always be *chosen* in $\mathcal{I}_Z$, it is left to prove that such relations generate the ideal, but this follows from Lemma 3.5.3. □

Lemma 3.5.3. *Suppose Z is a G-cluster whose defining ideal $\mathcal{I}_Z$ contains relations* (3.10) *that satisfy conditions (i) and (ii) of Theorem 3.4.1. Then the relations* (3.10) *generate $\mathcal{I}_Z$.*

Proof. By Lemma 3.3.5, $\mathcal{I}_Z$ is generated by the binomial relations of the form:

$$\mathbf{n} - \zeta_{\mathbf{n}} \cdot \mathbf{m}_{suv}, \quad \text{where} \quad \mathbf{n}, \mathbf{m}_{suv} \in \mathcal{L}_{suv},\ \mathbf{m}_{suv} \neq 0,\ \zeta_{\mathbf{n}} \in \mathbb{C}.$$

Here, we assume $\mathbf{m}_{000} = \mathbf{1}$. Because of this, it is enough to show that every binomial relation of this form can be written as $f \cdot r$, where r one of relations from (3.10) and $f \in \mathbb{C}[x_1, x_2, x_3, x_4]$. Define $\mathcal{I}_\star$ to be the ideal in R generated by the relations (3.10). Then $\mathcal{I}_\star \subseteq \mathcal{I}$. We prove this lemma in three steps.

First let us show that all the relations of the form $x_i^v - \alpha (x_jx_kx_l)^{r-v}$ lie in

$\mathcal{I}_\star$. If $v \geq a_i$, then we can use the relations (3.10) to reduce the monomial x_i^v:

$$\begin{aligned} x_i^v &= x_i^{a_i} x_i^{v-a_i} = \lambda_i \left(x_j x_k x_l\right)^{r-a_i} x_i^{v-a_i} = \\ &= \lambda_i \left(x_i x_j x_k x_l\right)^{v-a_i} \left(x_j x_k x_l\right)^{r-v} = \\ &= \lambda_i \xi^{v-a_i} \left(x_j x_k x_l\right)^{r-v}, \end{aligned}$$

As these two monomials lie in the same (one-dimensional) eigenspace, there is one and only one relation between them so $\alpha = \lambda_i \xi^{v-a_i}$.

If, on the other hand $v < a_i$, or equivalently $r - v \geq r - a_i + 1$, then α has to be nonzero, as $x_i^v = 0$ would contradict the choice of a_i. The relation can then also be viewed as $\left(x_j x_k x_l\right)^{r-v} - 1/\alpha\, x_i^v$. But in this case the equations (3.10) give the reduction:

$$\begin{aligned} \left(x_j x_k x_l\right)^{r-v} &= \left(x_j x_k x_l\right)^{r-a_i+1} \left(x_j x_k x_l\right)^{a_i-1-v} = \nu_{jkl} x_i^{a_i-1} \left(x_j x_k x_l\right)^{a_i-1-v} = \\ &= \nu_{jkl} \left(x_i x_j x_k x_l\right)^{a_i-1-v} x_i^v = \\ &= \nu_{jkl} \xi^{a_i-1-v} x_i^v \end{aligned}$$

It follows that $1/\alpha = \nu jkl \xi^{a_i-1-v}$, or $\alpha = 1/\left(\nu_{jkl}\xi^{a_i-1-v}\right)$ and again, the relation is a multiple of relations (3.10). This shows that every relation $x_i^v - \alpha \left(x_j x_k x_l\right)^{r-v}$ from $\mathcal{I}_Z$, for some $1 \leq v \leq r$ and $\alpha \in \mathbb{C}$, lies in $\mathcal{I}_\star$.

Exactly the same reasoning leads to other two "diagonal" types of relations:

$$\begin{aligned} &\left(x_i x_j\right)^u - \beta \left(x_l x_k\right)^{r-u} && \in \mathcal{I}_Z \implies && \left(x_i x_j\right)^u - \beta \left(x_l x_k\right)^{r-u} && \in \mathcal{I}_\star \\ &\left(x_i x_j x_k\right)^v - \gamma\, x_l^{r-v} && \in \mathcal{I}_Z \implies && \left(x_i x_j x_k\right)^v - \gamma\, x_l^{r-v} && \in \mathcal{I}_\star \end{aligned}$$

Assume now there is a "nondiagonal" relation: $x_i^s x_j^u - \alpha \cdot \mathbf{m} \in \mathcal{I}_Z$, for some $1 \leq s, u \leq r$ and $\alpha \in \mathbb{C}$. Then by Corollary 3.3.3, $\mathbf{m}$ can only be one of $x_i^{\overline{s-u}} x_k^{r-u} x_l^{r-u}$ and $x_j^{\overline{u-s}} x_k^{r-s} x_l^{r-s}$. Without loss of generality we can assume it is the first one. If $s \geq u$, then $\overline{s-u} = s - u$, The relation decomposes as

$$x_i^s x_j^u - \alpha\, x_i^{s-u} \left(x_k x_l\right)^{r-u} = x_i^{s-u} \left(\left(x_i x_j\right)^u - \alpha\, \left(x_k x_l\right)^{r-u}\right),$$

so it is a multiple of a relation from the ideal $\mathcal{I}_\star$ by the above. If, on the other hand, $s < u$, the remainder of $s - u$ modulo r is $r + s - u$ and the relation is again a multiple of a monomial that must lie in $\mathcal{I}_\star$:

$$x_i^s x_j^u - \alpha\, x_i^{r+s-u} \left(x_k x_l\right)^{r-u} = x_i^s \left(x_j^u - \alpha \left(x_i x_k x_l\right)^{r-u}\right).$$

Finally, assume there is a relation of the form $x_i^s x_j^u x_k^v - \alpha \cdot \mathbf{m} \in \mathcal{I}_Z$. As $\mathbf{m}$ has to be in the same eigenspace as the monomial on the left hand side of the equation, and because we can assume that it is not divisible by an invariant, $\mathbf{m}$ can only be one of the three values:

$$\left\{ x_i^{\overline{s-v}} x_j^{\overline{u-v}} x_l^{r-v}, \quad x_i^{\overline{s-u}} x_k^{\overline{v-u}} x_l^{r-u}, \quad x_j^{\overline{u-s}} x_k^{\overline{v-s}} x_l^{r-s} \right\}.$$

Without loss of generality, we can assume it is the first one and obtain the relation $x_i^s x_j^u x_k^v - \alpha\, x_i^{\overline{s-v}} x_j^{\overline{u-v}} x_l^{r-v} \in \mathcal{I}_Z$. There are three cases to consider:

1. $s \geq v$ and $u \geq v$. In this case $\overline{s-v} = s - v$ and $\overline{u-v} = u - v$ and our relation decomposes as

$$x_i^s x_j^u x_k^v - \alpha\, x_i^{s-v} x_j^{u-v} x_l^{r-v} = x_i^{s-v} x_j^{u-v} \left(\left(x_i x_j x_k\right)^v - \alpha\, x_l^{r-v}\right)$$

2. $s \geq v$ and $u < v$ implies $\overline{u-v} = r + u - v$:

$$x_i^s x_j^u x_k^v - \alpha\, x_i^{s-v} x_j^{r+u-v} x_l^{r-v} = x_i^{s-v} x_j^u \left(\left(x_i x_k\right)^v - \alpha\, \left(x_j x_l\right)^{r-v}\right)$$

3. $s < v$ and $u \geq v$ is symmetric to the previous case.

4. $s < v$ and $u < v$:

$$x_i^s x_j^u x_k^v - \alpha x_i^{r+s-v} x_j^{r+u-v} x_l^{r-v} = x_i^s x_j^u \left(x_k^v - \alpha\, \left(x_i x_j x_l\right)^{r-v}\right)$$

In all of the four cases the relation is a multiple of a relation from $\mathcal{I}_\star$. We conclude that $\mathcal{I}_Z = \mathcal{I}_\star$. The relations (3.10) generate the ideal of the G-cluster.

To obtain the relations between the complex parameters $\lambda_*, \mu_{**}, \nu_{***}$ and ξ use the generators of the ideal to get the equality in $\mathcal{O}_Z$:

$$\xi\, x_i^{a_i-1} = x_i^{a_i} x_j x_k x_l = \lambda_i\, (x_j x_k x_l)^{r-a_i+1} = \lambda_i \nu_{jkl}\, x_i^{a_i-1}.$$

As $x_i^{a_i-1}$ is a basic element of the vector space $\mathcal{O}_Z$, the equality $\lambda_i \nu_{jkl} = \xi$ must hold. Similarly,

$$\xi\, (x_i x_j)^{b_{ij}-1} = (x_i x_j)^{b_{ij}}\, x_k x_l = \mu_{ij}\, (x_k x_l)^{r-b_{ij}+1} = \mu_{ij}\, (x_k x_l)^{b_{kl}} = \mu_{ij}\mu_{kl}\, (x_i x_j)^{r-b_{kl}}\,.$$

Again, the monomial $(x_i x_j)^{b_{ij}-1} = (x_i x_j)^{r-b_{kl}}$ is basic so $\mu_{ij}\mu_{kl} = \xi$. □

Remark 3.5.4. As $b_{ij} \leq a_i$ for all the pairs $1 \leq i \neq j \leq 4$ and $b_{ij} + b_{kl} = r + 1$, it follows that

$$r + 1 - a_j \leq r + 1 - b_{jk} = b_{il} \leq a_i.$$

Especially, the inequality $a_i + a_j \geq r + 1$ holds for all the $1 \leq i \neq j \leq 4$.

3.5.2 Intermediate step

To show first part of Theorem 3.4.1, we need to choose the system of relations from Proposition 3.5.2 a bit more carefully: we need to show that for a G-cluster Z, the choice of the relations (3.10) can be made in a way that every exponent b_{ij} is equal to either $a_i + a_j - r$ or to $a_i + a_j - r - 1$, while also preserving the properties (i) and (ii). However, we are still not ready to prove the full statement and in this subsection we modify the generators of $\mathcal{I}_Z$ from Proposition 3.5.2 to satisfy slightly stronger requrements: see conditions (i), (ii) and (iii') of Proposition 3.5.6. That is, we make an intermediate step between the weak result of Proposition 3.5.2 and Theorem 3.4.1, part **1**.

The following lemma is a crucial part of Proposition 3.5.6 below – whenever an exponent b_{ij} is not one of the four values of condition (iii'), it turns out that

$\xi \neq 0$. However, if (3.10) are generators of $\mathcal{I}_Z$ satisfying (i) and (ii) such that $\xi \neq 0$, then all the other complex coefficients $\lambda_i, \mu_{ij}, \nu_{ijk}$ also must be nonzero, for all $i, j, k \in \{1, 2, 3, 4\}$. But then all the monomials in $\mathcal{O}_Z$ are non-zero so we have a full freedom in choosing the generators of ideal $\mathcal{I}_Z$. In explanation, we may choose any collection of exponents a_i, b_{ij}, for all distinct $i, j \in \{1, 2, 3, 4\}$ satisfying (i), (ii), (iii) of Theorem 3.4.1. Then we can find relations (3.4) in $\mathcal{I}_Z$ with these exponents and by Lemma 3.5.3 they also must generate the ideal. Here we explicitly describe one such choice.

Lemma 3.5.5. *Suppose that for a cluster Z, there exists a set of generating relations* (3.10) *of its ideal $\mathcal{I}_Z$ satisfying (i) and (ii) of Proposition 3.5.2, such that $\xi \neq 0$. Then the relations*

$$\begin{array}{lll} & (x_1x_2)^r = \widetilde{\mu_{12}} \cdot 1 & \\ x_1^r = \widetilde{\lambda_1} \cdot 1 & (x_1x_3)^r = \widetilde{\mu_{13}} \cdot 1 & x_2x_3x_4 = \widetilde{\nu_{234}} \cdot x_1^{r-1} \\ x_2^r = \widetilde{\lambda_2} \cdot 1 & (x_2x_3)^r = \widetilde{\mu_{23}} \cdot 1 & x_2x_3x_4 = \widetilde{\nu_{134}} \cdot x_2^{r-1} \\ x_3^r = \widetilde{\lambda_3} \cdot 1 & x_1x_4 = \widetilde{\mu_{14}} \cdot (x_2x_3)^{r-1} & x_2x_3x_4 = \widetilde{\nu_{124}} \cdot x_3^{r-1} \\ x_4 = \widetilde{\lambda_4} \cdot (x_1x_2x_3)^{r-1} & x_2x_4 = \widetilde{\mu_{24}} \cdot (x_1x_3)^{r-1} & (x_1x_2x_3)^r = \widetilde{\nu_{123}} \cdot 1 \\ & x_3x_4 = \widetilde{\mu_{34}} \cdot (x_1x_2)^{r-1} & \end{array} \tag{3.11}$$

also generate the ideal $\mathcal{I}_Z$.

Proof. As $\xi \neq 0$, and $(x_1x_2x_3)^r = \nu_{123}\ (x_1x_2x_3)^{a_4-1}\, x_4^{a_4-1} = \nu_{123}\, \xi^{a_4-1} \neq 0$, all the monomials that divide it are basic, hence, all the monomials on the right hand side of the equations in (3.11) are basic. As each equation in (3.11) contains two monomials from the same eigenspace, the left-hand side monomial must be a multiple of a right hand side monomial. In conclusion, all fourteen equations from the lemma statement exist in $\mathcal{I}_Z$. But as (3.11) is just a special case of the set of relations of the form (3.4), by Lemma 3.5.3 they must generate the ideal $\mathcal{I}_Z$. □

Proposition 3.5.6 (The intermediate step)**.** *For every G-cluster $Z \subset \mathbb{C}^4$, the generators of the ideal $\mathcal{I}_Z$ can be chosen as the system of equations* (3.10) *satisfying properties (i), (ii) of Proposition 3.5.2 and*

(iii') $\quad b_{ij} \in \{a_i + a_j - r - 2, a_i + a_j - r - 1, a_i + a_j - r, a_i + a_j - r + 1\}$,

for all $i, j \in \{1, 2, 3, 4\}$ *such that* $i \neq j$. *Furthermore,*

$$\begin{aligned} b_{ij} = a_i + a_j - r - 2 \quad &\Longrightarrow \quad \mu_{ij}\, \nu_{ikl}\, \nu_{jkl} \neq 0 \\ b_{ij} = a_i + a_j - r + 1 \quad &\Longrightarrow \quad \mu_{kl}\, \lambda_i\, \lambda_j \neq 0 \end{aligned}$$

Proof. Assume we have chosen a set of generators of the ideal $\mathcal{I}_z$ that satisfy the conditions of Proposition 3.5.2. Suppose first that $b_{ij} \leq a_i + a_j - r - 2$. Focus on the eigenspace $\mathcal{L}$, containing the monomial $x_i^{b_{ij}} x_j^{b_{ij}+r-a_i+1}$. From Lemma 3.3.3, $\mathcal{L}$ is generated by three monomials as a module over R:

$$\begin{aligned} m_1 &= x_i^{\overline{b_{ij}}} x_j^{\overline{b_{ij}+r-a_i+1}} &&= x_i^{b_{ij}} x_j^{b_{ij}+r-a_i+1}, \\ m_2 &= x_j^{\overline{r-a_i+1}} (x_k x_l)^{\overline{r-b_{ij}}} &&= x_j^{r-a_i+1} (x_k x_l)^{r-b_{ij}}, \\ m_3 &= x_i^{\overline{a_i-1}} (x_k x_l)^{\overline{a_i-b_{ij}-1}} &&= x_i^{a_i-1} (x_k x_l)^{a_i-b_{ij}-1}. \end{aligned}$$

The second equality signs in each row above are true for the following reason. Notice that the inequality from our assumption implies that $b_{ij} < a_i$, since $b_{ij} = a_i$ would mean that $a_j \geq r + 2$, which is impossible from Proposition 3.5.2. By the same logic, $b_{ij} < a_j$. But then we also have $b_{ij} < r$ and $a_i, a_j > 1$. Because of this, the exponents appearing in the monomials m_1, m_2 and m_3 are all nonnegative and strictly smaller than r:

$$\begin{array}{ccccc} 0 & \leq & b_{ij} & < & r \\ 0 & \leq & b_{ij} + r - a_i + 1 & \leq & a_j - 1 < r - 1, \\ 0 & \leq & r - a_i + 1 & < & r - 1 + 1 = r \\ 0 & \leq & a_i - 1 & \leq & r - 1 \\ 0 & \leq & a_i - b_{ij} - 1 & < & a_i - 1 < r. \end{array}$$

Now we can use the relations (3.10) to get

$$\begin{aligned}
m_1 &= \mu_{ij} \ \cdot \ x_j^{r-a_i+1} \left(x_k x_l\right)^{r-b_{ij}} = \\
&= \mu_{ij}\, \nu_{jkl} \ \cdot \ x_i^{a_i-1} \left(x_k x_l\right)^{a_i-b_{ij}-1} = \\
&= \mu_{ij}\, \nu_{jkl}\, \nu_{ikl} \ \cdot \ x_i^{a_i+a_j-r-2} x_j^{a_j-1} \left(x_k x_l\right)^{a_i+a_j-r-2-b_{ij}} = \\
&= \mu_{ij}\, \nu_{jkl}\, \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} \ \cdot \ x_i^{b_{ij}} x_j^{b_{ij}+r-a_i+1} = \\
&= \mu_{ij}\, \nu_{jkl}\, \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} \ \cdot \ m_1
\end{aligned}$$

From the computation above, we can see that $m_1 = \mu_{ij} \cdot m_2$, $m_2 = \nu_{jkl} \cdot m_3$ and $m_3 = \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} \cdot m_1$. We can combine these, or repeat the similar computation as above, but starting with m_2 and m_3, to get

$$\begin{aligned}
m_2 &= \mu_{ij}\, \nu_{jkl}\, \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} \cdot m_2 \quad \text{and} \\
m_3 &= \mu_{ij}\, \nu_{jkl}\, \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} \cdot m_3.
\end{aligned}$$

Since at least one of the three generating monomials of the eigenspace $\mathcal{L}$ must be nonzero, we have

$$\mu_{ij}\, \nu_{jkl}\, \nu_{ikl}\, \xi^{a_i+a_j-r-2-b_{ij}} = 1. \tag{3.12}$$

Now, if $b_{ij} = a_i + a_j - r - 2$, then the exponent of ξ in (3.12) is zero, so we only have $\mu_{ij}\, \nu_{jkl}\, \nu_{ikl} = 1$. More importantly, all three coefficients are nonzero and we are finished with the proof. Otherwise, if $b_{ij} < a_i + a_j - r - 2$, and then also $\xi \neq 0$. But if ξ is nonzero, we can replace the chosen set of generators of the ideal $\mathcal{I}_Z$ with a different one, as in Lemma 3.5.5: the set of relations (3.11) has $b_{ij} = a_i + a_j - r$, for all distinct $i, j \in \{1, 2, 3, 4\}$.

Suppose now that $b_{ij} \geq a_i + a_j - r + 1$. We will take a closer look at the eigenspace containing the monomial $x_i^{a_i} x_j^{b_{ij}-1}$. Again using Lemma 3.3.3, the said

eigenspace is generated by the monomials

$$\begin{aligned}
n_1 &= x_i^{\overline{a_i}} x_j^{\overline{b_{ij}-1}} &&= x_i^{a_i} x_j^{b_{ij}-1} \\
n_2 &= x_i^{\overline{a_i-b_{ij}+1}} (x_k x_l)^{\overline{r-b_{ij}+1}} &&= x_i^{a_i-b_{ij}+1} (x_k x_l)^{r-b_{ij}+1} \\
n_3 &= x_j^{\overline{r-a_i+b_{ij}-1}} (x_k x_l)^{\overline{r-a_i}} &&= x_j^{r-a_i+b_{ij}-1} (x_k x_l)^{r-a_i}
\end{aligned}$$

All the exponents on the right hand side of the equations above are in the interval $[0, r-1]$. First, $a_i < r$ since otherwise the inequality from the assumption would imply $b_{ij} \geq r+1$. Similarly, $b_{ij} = 1$ would impose the inequality $a_i + a_j \leq r$ which is impossible, see Remark 3.5.4. Hence $b_{ij} > 1$.

$$\begin{array}{lllllll}
1 & \leq & b_{ij}-1 & < & a_i & < & r, \\
1 & \leq & a_i - b_{ij} + 1 & < & r - b_{ij} + 1 & < & r, \\
1 & \leq & r - a_i & < & r - a_i + b_{ij} - 1 & < & r, \qquad \text{as } b_{ij} < a_i + 1.
\end{array}$$

Now we can use the relations (3.10):

$$\begin{aligned}
n_1 = x_i^{a_i} x_j^{b_{ij}-1} &= \lambda_i \cdot x_j^{r-a_i+b_{ij}-1} (x_k x_l)^{r-a_i} = \\
&= \lambda_i \lambda_j \cdot x_i^{r-a_j} x_j^{b_{ij}-a_i-a_j+r-1} (x_k x_l)^{2r-a_i-a_j} = \\
&= \lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \cdot x_i^{a_i-b_{ij}+1} (x_k x_l)^{r-b_{ij}+1} = \\
&= \lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \mu_{kl} \cdot x_i^{a_i} x_j^{b_{ij}-1} = \\
&= \lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \mu_{kl} \cdot n_1
\end{aligned}$$

A similar calculation gives:

$$n_2 = \lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \mu_{kl} \cdot n_2 \quad \text{and} \quad n_3 = \lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \mu_{kl} \cdot n_3.$$

As one of the three monomials n_1, n_2 and n_3 must be nonzero, the multiple of the coefficients $\lambda_i \lambda_j \xi^{b_{ij}-a_i-a_j+r-1} \mu_{kl}$ is equal to 1. When $b_{ij} = a_i + a_j - r + 1$, the ξ vanishes from the expression, and we have $\lambda_i \lambda_j \mu_{kl} \neq 0$. If $b_{ij} > a_i + a_j - r + 1$, then ξ must be nonzero, in which case we can replace the chosen relations (3.10) with other set of relations, as in Lemma 3.5.5, which satisfy $b_{ij} = a_i + a_i - r$ for all

distinct $i, j \in \{1, 2, 3, 4\}$. $\square$

3.5.3 Proof of first part of Theorem 3.4.1

So far we know that for a cluster Z, we can choose the relations (3.4) so that for all indices $i \neq j$, the number b_{ij} satisfies $b_{ij} = a_i + a_j - r + \eta_{ij}$, where $\eta_{ij} \in \{-2, -1, 0, 1\}$. We refer to the number η_{ij} as the **type of exponent $\mathbf{b_{ij}}$**. In the proof of Proposition 3.5.6 we use Lemma 3.5.5 to show that if (3.4) were chosen so that b_{ij} were not of one of the four types, we can replace all the relations with the relations (3.11), for which all the b_{ij}'s are of type 0.

Remark 3.5.7. The condition $b_{ij} + b_{kl} = r + 1$, for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$ implies

$$\begin{aligned} r + 1 = b_{ij} + b_{kl} &= (a_i + a_j - r + \eta_{ij}) + (a_k + a_l - r + \eta_{kl}) = \\ &= \sum_{i=1}^{4} a_i - 2r + (\eta_{ij} + \eta_{kl}), \qquad \text{where } \eta_{ij}, \eta_{kl} \in \{-2, -1, 0, 1\}. \end{aligned}$$

Hence, $\sum_{i=1}^{4} a_i = 3r + 1 - (\eta_{ij} + \eta_{kl})$, which also implies that the sum $\eta = \eta_{ij} + \eta_{kl}$ remains constant, regardless of the chosen permutation (i, j, k, l) of $\{1, 2, 3, 4\}$.

Lemma 3.5.8. *Let Z be a G-cluster and assume* (3.10) *are the chosen generators if $\mathcal{I}_Z$ satisfying (i), (ii) and (iii') of Proposition 3.5.6. If $b_{ij} = a_i + a_j - r - 2$, the pair of relations from* (3.4)

$$(x_i x_j)^{b_{ij}} = \mu_{ij} (x_k x_l)^{r - b_{ij}} \qquad (x_k x_l)^{b_{kl}} = \mu_{kl} (x_i x_j)^{r - b_{kl}}$$

can be replaced by a new pair of relations:

$$(x_i x_j)^{b_{ij}+1} = \widetilde{\mu_{ij}} (x_k x_l)^{r - b_{ij} - 1}, \quad (x_k x_l)^{b_{kl}-1} = \widetilde{\mu_{kl}} (x_i x_j)^{r - b_{kl} + 1},$$

so that the set of relations adapted in such a way are still of the form (3.10) *satisfying (i) and (ii).*

Proof. We need to show that the two new relations exist in the ideal $\mathcal{I}_Z$ and that the restrictions on the exponents, namely $b_{ij} + b_{kl} = r + 1$ and $b_{ij} \leq a_i$, are still satisfied after the replacement.

The monomial $(x_k x_l)^{r-b_{ij}}$ is basic and so are all the monomials that divide it. Especially, the monomial $(x_k x_l)^{r-b_{ij}-1}$ is basic, so $(x_i x_j)^{b_{ij}+1}$ from the same eigenspace must be a multiple of it, so:

$$(x_i x_j)^{b_{ij}+1} = \widetilde{\mu_{ij}} \cdot (x_k x_l)^{r-b_{ij}-1},$$

for some complex number $\widetilde{\mu_{ij}}$. As $\mu_{ij} \neq 0$, the monomial $(x_i x_j)^{b_{ij}} = (x_i x_j)^{r-b_{kl}+1}$ is nonzero, and then as above, the monomial $(x_k x_l)^{r-b_{ij}} = (x_k x_l)^{b_{kl}-1}$ from the same eigenspace has to be a multiple of it. Hence, there exists a complex number $\widetilde{\mu_{kl}}$ such that

$$(x_k x_l)^{b_{kl}-1} = \widetilde{\mu_{kl}} \cdot (x_i x_j)^{r-b_{kl}+1} \in \mathcal{I}_Z.$$

Decreasing an exponent b_{kl} is not a problem as: $\widetilde{b_{kl}} := b_{kl} - 1 < b_{kl} \leq \min\{a_i, a_j\}$. Increasing the b_{ij} needs more care, but as $b_{ij} = a_i + a_j - r + 2$, it follows that $b_{ij} < \min\{a_i, a_j\}$ and then $\widetilde{b_{ij}} := b_{ij} + 1 \leq \min\{a_i, a_j\}$. Finally, a fairly obvious observation $\widetilde{b_{ij}} + \widetilde{b_{kl}} = b_{ij} + b_{kl} = r + 1$, finishes the proof. □

When using Lemma 3.5.8 to change the pairs of generators of $\mathcal{I}_Z$ we shall refer to it as **increasing the $\mathbf{b_{ij}}$** (which automatically **decreases the $\mathbf{b_{kl}}$**). The whole point is that we can manipulate the generating set (3.4) so that if b_{ij} is of type -2, we can instead turn to work with a new generating set of the same form (3.4), but whose b_{ij} is now of type -1.

Before we limit the possibilities for b_{ij}'s to only two values, we will pay attention to the **two boundary cases**. From Remark 3.5.7, $\sum_{i=1}^{4} a_i \in [3r-1, 3r+5]$. If $\sum_{i=1}^{4} a_i = 3r+5$, it must follow that all the b_{ij} are of type -2. Using Lemma 3.5.8, we can increase b_{12}, and decrease b_{34}. Now, the new b_{12} is of type -1, but the new b_{34} is equal to $a_3 + a_4 - r - 3$. From the proof of Proposition 3.5.6, it then follows that $\xi \neq 0$ and we can use Lemma 3.5.5 to yield a completely new set (3.11) of

generators for $\mathcal{I}_Z$. So, in this case the cluster Z can be parametrised by the set of relations of the form (3.4) satsfyingall the three conditions of Theorem 3.4.1, but more specifically it holds that $b_{ij} = a_i + a_j - r$ for all the i, j.

On the other hand, $\sum_{i=1}^{4} a_i = 3r - 1$ implies that all the b_{ij}'s are of type 1. Now, if $b_{ij} = a_i + a_j - r + 1$, the a_i cannot be equal to r. Hence $a_i < r$, for all $i \in \{1, 2, 3, 4\}$. Also, by Proposition 3.5.6, λ_i are nonzero, for $i \in \{1, 2, 3, 4\}$. This especially means, $x_i^{a_i}$ is a basic monomial for all i so the pair of relations

$$x_i^{a_i} = \lambda_i \cdot (x_j x_k x_l)^{r-a_i}, \quad (x_j x_k x_l)^{r-a_i+1} = \nu_{ijk} \cdot x_i^{a_i-1}$$

can be replaced by

$$x_i^{a_i+1} = \widetilde{\lambda}_i \cdot (x_j x_k x_l)^{r-a_i-1}, \quad (x_j x_k x_l)^{r-a_i} = \widetilde{\nu_{ijk}} \cdot x_i^{a_i}$$

in the set (3.4), as $\widetilde{a}_i := a_i + 1 \leq r$ and $b_{ij} \leq a_i < \widetilde{a}_i$. In other words, we can increase any of the a_i's, and still have the set of relations of the form (3.4) that generates $\mathcal{I}_Z$. If we increase three of them, say a_1, a_2 and a_3, the exponents b_{ij} will, relative to the new $\widetilde{a}_i = a_i + 1$, $i = 1, 2, 3$, change their types to -1 and 0:

$$\begin{aligned} b_{ij} &= (a_i + 1) + (a_j + 1) - r - 1 = \widetilde{a}_i + \widetilde{a}_j - r - 1, \quad \text{and} \\ b_{i4} &= (a_i + 1) + a_4 - r = \widetilde{a}_i + a_4 - r, \end{aligned}$$

for all distinct $i, j \in \{1, 2, 3\}$. With this in mind, we can prove the final ingredient that proves the first part of Theorem 3.4.1.

Proposition 3.5.9. *Theorem 3.4.1, part **1.** holds. Furthermore,*

$$\begin{aligned} b_{ij} = a_i + a_j - r \quad &\Longrightarrow \quad \begin{cases} \nu_{ikl} = \lambda_i \, \mu_{kl}, & \textit{for } k \neq i, j \\ \mu_{ij} = \lambda_i \lambda_j, & \textit{for } i \neq j \end{cases} \\ b_{ij} = a_i + a_j - r - 1 \quad &\Longrightarrow \quad \lambda_i = \nu_{ikl} \, \mu_{ij}, \quad \textit{for } \{k, l\} \cap \{i, j\} = \emptyset \end{aligned}$$

Proof. Proposition 3.5.6 ensures that there exists a generating set (3.4) of the ideal

$\mathcal{I}_Z$ satisfying conditions *(i)*, *(ii)* and *(iii')*. Based on the discussion above, we may assume that we are not in one of the boundary cases $\sum_{i=1}^{4} a_i \in \{3r-3, 3r+3\}$. Also, notice that if a pair $\{b_{ij}, b_{kl}\}$, where $\{i,j,k,l\} = \{1,2,3,4\}$ satisfies

$$b_{ij} = a_i + a_j - r - 2, \quad b_{kl} = a_k + a_l - r + 1,$$

we can increase b_{ij} to $\widetilde{b_{ij}} = b_{ij} + 1$ which is of type -1, and decrease b_{kl} to a type 0 exponent $\widetilde{b_{kl}} = b_{kl} - 1$ by applying Lemma 3.5.8. Hence, we may assume that al least one exponent from each of the three antipodal pairs

$$\{b_{12}, b_{34}\}, \quad \{b_{13}, b_{24}\} \quad \text{and} \quad \{b_{14}, b_{23}\}$$

is of type -1 or 0. Without loss of generality, assume that b_{12}, b_{13} and b_{14} are of types 0 or -1, while other three η_{ij}, $i,j \in \{2,3,4\}$, stay in $\{-2,-1,0,1\}$. Up to a permutation, there are four cases to consider, each with its own four subcases:

1. $\boldsymbol{\eta_{12} = \eta_{13} = \eta_{14} = -1}$. From Remark 3.5.7, it follows that $\eta_{34} = \eta_{24} = \eta_{23}$, and there are four possibilities for this value: $-2, -1, 0$ and 1.

   i. $\eta_{34} = \eta_{24} = \eta_{23} = -2$. From Proposition 3.5.6, it follows that ν_{134} and ν_{124} are nonzero. Then it is possible to replace the four relations from (3.4)

   $$\begin{aligned} x_2^{a_2} &= \lambda_2 \cdot (x_1x_3x_4)^{r-a_2}, \quad (x_1x_3x_4)^{r-a_2+1} = \nu_{134} \cdot x_2^{a_2-1} \\ x_3^{a_3} &= \lambda_3 \cdot (x_1x_2x_4)^{r-a_3}, \quad (x_1x_2x_4)^{r-a_3+1} = \nu_{124} \cdot x_3^{a_3-1} \end{aligned}$$

   with

   $$\begin{aligned} x_2^{a_2-1} &= \widetilde{\lambda_2} \cdot (x_1x_3x_4)^{r-a_2+1}, \quad (x_1x_3x_4)^{r-a_2+2} = \widetilde{\nu_{134}} \cdot x_2^{a_2-2} \\ x_3^{a_3-1} &= \widetilde{\lambda_3} \cdot (x_1x_2x_4)^{r-a_3+1}, \quad (x_1x_2x_4)^{r-a_3+2} = \widetilde{\nu_{124}} \cdot x_3^{a_3-2}, \end{aligned}$$

   where $\widetilde{\lambda_2}, \widetilde{\lambda_3}, \widetilde{\nu_{124}}, \widetilde{\nu_{134}}$ are complex numbers. In other words, we are decreasing the a_2 and a_3. This works because in this case $b_{ij} < a_i$ for all

$i, j \in \{1, 2, 3, 4\}$ so whenever $i = 2$ or 3 we have $b_{ij} \leq= a_i - 1 =: \widetilde{a_i}$. So by Lemma 3.5.3, this way we have obtained a new set of generators for $\mathcal{I}_Z$, but which satisfies:

$$
\begin{aligned}
b_{12} &= a_1 + a_2 - r - 1 = a_1 + \widetilde{a_2} - r, \\
b_{13} &= a_1 + a_3 - r - 1 = a_1 + \widetilde{a_3} - r, \\
b_{14} &= a_1 + a_4 - r - 1, \\
b_{34} &= a_3 + a_4 - r - 2 = \widetilde{a_3} + a_4 - r - 1, \\
b_{24} &= a_2 + a_4 - r - 2 = \widetilde{a_2} + a_4 - r - 1, \\
b_{23} &= a_2 + a_3 - r - 2 = \widetilde{a_2} + \widetilde{a_3} - r.
\end{aligned}
$$

ii. $\eta_{34} = \eta_{24} = \eta_{23} = -1$. In this case, there is nothing to do.

iii. $\eta_{34} = \eta_{24} = \eta_{23} = 0$. In this case, there is nothing to do.

iv. $\eta_{34} = \eta_{24} = \eta_{23} = 1$. We can apply Lemma 3.5.8 on all three pairs of exponents to increase b_{1i}, $i \in \{1, 2, 3\}$ by one, and decrease b_{jk}, $j, k \in \{2, 3, 4\}$. So we end up with a set of generators of $\mathcal{I}_Z$ such that $\widetilde{b_{1i}} := b_{1i} + 1$ are of type -1 for $i \in \{1, 2, 3\}$ and $\widetilde{b_{jk}} := b_{jk} - 1$ are of type 0, for $j, k \in \{1, 2, 3\}$.

2. $\boldsymbol{\eta_{12} = \eta_{13} = -1}$, $\boldsymbol{\eta_{14} = 0}$.

i. If $\eta_{34} = -2$, then $\eta_{24} = -2$ and $\eta_{23} = -3$, which is impossible due to our assumption.

ii. If $\eta_{34} = -1$, then $\eta_{24} = -1$ and $\eta_{23} = -2$, we can apply Lemma 3.5.8 on the pair of relations involving b_{14} and b_{23}: we get a set of relations (3.4) with increased b_{23} and decreased b_{14}, so that all of them are of type -1.

iii. If $\eta_{34} = 0$, then $\eta_{24} = 0$ and $\eta_{23} = -1$, and we are done.

iv. If $\eta_{34} = 1$, then $\eta_{24} = 1$ and $\eta_{23} = 0$, use Proposition 3.5.6 to get μ_{12} and μ_{13} are nonzero. Because of that, $(x_1x_2)^{b_{12}}$ and $(x_1x_3)^{b_{13}}$ are basic monomials. Also, as $b_{12} = a_1 + a_2 - r - 1$, we must also have a strict inequality $b_{12} < \min\{a_1, a_2\}$. Similarly, $b_{13} < \min\{a_1, a_3\}$. Because all of this, we

can replace the four relations from (3.4)

$$(x_1x_2)^{b_{12}} = \mu_{12} \cdot (x_3x_4)^{r-b_{12}}\,, \qquad (x_3x_4)^{b_{34}} = \mu_{34} \cdot (x_1x_1)^{r-b_{34}}\,,$$
$$(x_1x_3)^{b_{13}} = \mu_{13} \cdot (x_2x_4)^{r-b_{13}}\,, \qquad (x_2x_4)^{b_{24}} = \mu_{24} \cdot (x_1x_3)^{r-b_{24}}$$

with

$$(x_1x_2)^{b_{12}+1} = \widetilde{\mu_{12}} \cdot (x_3x_4)^{r-b_{12}-1}\,, \qquad (x_3x_4)^{b_{34}-1} = \widetilde{\mu_{34}} \cdot (x_1x_1)^{r-b_{34}+1}\,,$$
$$(x_1x_3)^{b_{13}+1} = \widetilde{\mu_{13}} \cdot (x_2x_4)^{r-b_{13}-1}\,, \qquad (x_2x_4)^{b_{24}-1} = \widetilde{\mu_{24}} \cdot (x_1x_3)^{r-b_{24}+1}\,,$$

for some complex numbers $\widetilde{b_{12}}, \widetilde{b_{13}}, \widetilde{b_{34}}, \widetilde{b_{24}}$. This way we have obtained a new set of generators of the ideal $\mathcal{I}_Z$ of the form (3.4), such that all of its exponents b_{ij} are of type 0.

3. $\boldsymbol{\eta_{12} = -1}$, $\boldsymbol{\eta_{13} = \eta_{14} = 0}$.

   i. If $\eta_{34} = -2$, then $\eta_{24} = -3$ and $\eta_{23} = -3$, which is impossible due to our assumption.

   ii. If $\eta_{34} = -1$, then $\eta_{24} = -2$ and $\eta_{23} = -2$, we can apply Lemma 3.5.8 on the pairs of relations involving b_{13}, b_{24} and b_{14}, b_{23}: we get a set of relations (3.4) with increased b_{23}, b_{24} and decreased b_{13}, b_{14} so that all of them are of type -1.

   iii. If $\eta_{34} = 0$, then $\eta_{24} = -1$ and $\eta_{23} = -1$, and we are done.

   iv. If $\eta_{34} = 1$, then $\eta_{24} = 0$ and $\eta_{23} = 0$, then by Proposition 3.5.6, it follows $\mu_{12} \neq 0$ so the monomial $(x_1x_2)^{b_{12}}$ is basic. Also, $b_{12} = a_1 + a_2 - r - 1$ implies that $b_{12} < \min\{a_1, a_2\}$. We can then replace the pair of relations from (3.4)

$$(x_1x_2)^{b_{12}} = \mu_{12} \cdot (x_3x_4)^{r-b_{12}} \quad \text{and} \quad (x_3x_4)^{b_{34}} = \mu_{34} \cdot (x_1x_2)^{r-b_{34}}$$

by relations

$$(x_1x_2)^{b_{12}+1} = \widetilde{\mu_{12}} \cdot (x_3x_4)^{r-b_{12}-1} \quad \text{and} \quad (x_3x_4)^{b_{34}-1} = \widetilde{\mu_{34}} \cdot (x_1x_2)^{r-b_{34}+1},$$

where $\widetilde{\mu_{12}}$ and $\widetilde{\mu_{34}}$ are complex numbers. The new set of generators of $\mathcal{I}_Z$ obtained in such a way is again of the form (3.4), but such that all of the exponents b_{ij} are of type 0.

4. $\boldsymbol{\eta_{12} = \eta_{13} = \eta_{14} = 0}$.

   i. $\eta_{34} = \eta_{24} = \eta_{23} = -2$. Apply Lemma 3.5.8 on all the three pairs of antipodal exponents $\{b_{ij}, b_{kl}\}$. The new set of generators has all the exponents b_{ij} of type -1.

   ii. $\eta_{34} = \eta_{24} = \eta_{23} = -1$. In this case, there is nothing to do.

   iii. $\eta_{34} = \eta_{24} = \eta_{23} = 0$. In this case, there is nothing to do.

   iv. $\eta_{34} = \eta_{24} = \eta_{23} = 1$. From Proposition 3.5.6, it follows that λ_2, λ_3 and λ_4 are all nonzero. Especially, the monomials $x_2^{a_2}$ and $x_3^{a_3}$ are basic. Also, $b_{23} = a_3 + a_4 - r + 1$ implies that a_2 and a_3 are strictly smaller than r. Then it is possible to replace the four relations from (3.4)

$$\begin{aligned} x_2^{a_2} &= \lambda_2 \cdot (x_1x_3x_4)^{r-a_2}, \quad (x_1x_3x_4)^{r-a_2+1} = \nu_{134} \cdot x_2^{a_2-1} \\ x_3^{a_3} &= \lambda_3 \cdot (x_1x_2x_4)^{r-a_3}, \quad (x_1x_2x_4)^{r-a_3+1} = \nu_{124} \cdot x_3^{a_3-1} \end{aligned}$$

   with

$$\begin{aligned} x_2^{a_2+1} &= \widetilde{\lambda_2} \cdot (x_1x_3x_4)^{r-a_2-1}, \quad (x_1x_3x_4)^{r-a_2} = \widetilde{\nu_{134}} \cdot x_2^{a_2} \\ x_3^{a_3+1} &= \widetilde{\lambda_3} \cdot (x_1x_2x_4)^{r-a_3-1}, \quad (x_1x_2x_4)^{r-a_3} = \widetilde{\nu_{124}} \cdot x_3^{a_3}, \end{aligned}$$

   for some complex numbers $\widetilde{\lambda_2}, \widetilde{\lambda_3}, \widetilde{\nu_{124}}, \widetilde{\nu_{134}}$. Set $\widetilde{a_2} = a_2 + 1$ and $\widetilde{a_3} = a_3 + 1$. This way we have obtained a new set of generators for $\mathcal{I}_Z$, but which sat-

isfies:

$$
\begin{aligned}
b_{12} &= a_1 + a_2 - r = a_1 + \widetilde{a_2} - r - 1, \\
b_{13} &= a_1 + a_3 - r = a_1 + \widetilde{a_3} - r - 1, \\
b_{14} &= a_1 + a_4 - r, \\
b_{34} &= a_3 + a_4 - r + 1 = \widetilde{a_3} + a_4 - r, \\
b_{24} &= a_2 + a_4 - r + 1 = \widetilde{a_2} + a_4 - r, \\
b_{23} &= a_2 + a_3 - r + 1 = \widetilde{a_2} + \widetilde{a_3} - r - 1.
\end{aligned}
$$

The first of part of the proof is now finished: the relations (3.4) can indeed be chosen so that $b_{ij} \in \{a_i + a_j - r - 1, a_i + a_j - r\}$ for all $i, j \in \{1, 2, 3, 4\}$. Suppose now that $b_{ij} = a_i + a_j - r - 1$.

$$
\begin{aligned}
\mu_{ij}\, \nu_{ikl} \cdot x_j^{a_j - 1} (x_k x_l)^{r - a_i} &= \mu_{ij}\, (x_i x_k x_l)^{r - a_j + 1} (x_k x_l)^{r - a_i} = \\
&= \mu_{ij} \cdot (x_k x_l)^{2r - a_i - a_j + 1}\, x_i^{r - a_j + 1} = \\
&= \mu_{ij} \cdot (x_k x_l)^{r - b_{ij}}\, x_i^{r - a_j + 1} = \\
&= (x_i x_j)^{b_{ij}}\, x_i^{r - a_j + 1} = \\
&= x_i^{a_i} x_j^{b_{ij}} = \\
&= \lambda_i \cdot x_j^{b_{ij}} (x_j x_k x_l)^{r - a_i} = \\
&= \lambda_i \cdot x_j^{a_j - 1} (x_k x_l)^{r - a_i}
\end{aligned}
$$

As the monomial $x_j^{a_j - 1} (x_k x_l)^{r - a_i}$ is basic, it follows that $\lambda_i = \mu_{ij} \nu_{ikl}$ and similarly, $\lambda_j = \mu_{ij} \nu_{jkl}$ when we swap indices i and j in the above calculation. If

$b_{ij} = a_i + a_j - r$, or equivalently $b_{kl} = r + 1 - b_{ij} = 2r - a_i - a_j + 1$, we get

$$\begin{aligned}
\lambda_i \mu_{kl} \cdot x_i^{b_{ij}-1} x_j^{a_j-1} &= \lambda_i \mu_{kl} \cdot (x_i x_j)^{r-b_{kl}} x_j^{r-a_i} = \\
&= \lambda_i \cdot (x_k x_l)^{b_{kl}} x_j^{r-a_i} = \\
&= \lambda_i (x_j x_k x_l)^{r-a_i} (x_k x_l)^{r-a_j+1} = \\
&= x_i^{a_i} (x_k x_l)^{r-a_j+1} = \\
&= (x_i x_k x_l)^{r-a_j+1} x_i^{b_{ij}-1} = \\
&= \nu_{ikl} \cdot x_i^{b_{ij}-1} x_j^{a_j-1}.
\end{aligned}$$

The monomial $x_i^{b_{ij}-1} x_j^{a_j-1}$ is basic so $\nu_{ikl} = \lambda_i \mu_{kl}$ and analogously by swapping i and j, it also holds that $\nu_{jkl} = \lambda_j \mu_{kl}$. Similarly,

$$\begin{aligned}
\mu_{ij} \cdot x_i^{r-a_j} (x_k x_l)^{b_{kl}-1} &= \mu_{ij} \cdot x_i^{r-a_j} (x_k x_l)^{r-b_{ij}} \\
&= x_i^{r-a_j} (x_i x_j)^{b_{ij}} = \\
&= x_i^{a_i} x_j^{b_{ij}} = \\
&= \lambda_i \cdot (x_j x_k x_l)^{r-a_i} x_j^{b_{ij}} = \\
&= \lambda_i \cdot x_j^{a_j} (x_k x_l)^{r-a_i} = \\
&= \lambda_i \lambda_j \cdot x_i^{r-a_j} (x_k x_l)^{r-b_{ij}}
\end{aligned}$$

Since the monomial $x_i^{r-a_j} (x_k x_l)^{b_{kl}-1}$ is basic, it follows that $\mu_{ij} = \lambda_i \lambda_j$.

□

3.6 Four ways to parametrise a G-cluster

Having proven the part one, we are ready to prove the second claim of Theorem 3.4.1

Proof of the second part of Theorem 3.4.1. Let Z be a G-cluster and suppose we have chosen relations (3.4) as generators of the ideal $\mathcal{I}_Z$, applying the result of part one of the theorem. Because of Remark 3.5.7, if some disjoint b_{ij} and b_{kl} are of the

same type, then η is either 0 or -2. If we consider another pair $\{b_{ik}, b_{jl}\}$, the sum $\eta_{ik} + \eta_{jl}$ is also either 0 or 2, so $\eta_{ik} = \eta_{jl} = \eta_{ij}$. In other words, all the other b_{**} will also have to be of the same type as b_{ij}.

If however, b_{ij} and b_{kl} are of different types, it follows that $\eta = -1$ and the the remaining two "antipodal" pairs b_{ik}, b_{jl} and b_{il}, b_{jk} must also be of different types. This analysis shows that there are four cases to consider, depending on the value of $\eta = 3r + 1 - \sum_{i=1}^{4} a_i$, or equivalently, the value of $\sum_{i=1}^{4} a_i$:

UP $\mathbf{\sum_{i=1}^{4} a_i = 3r + 1}$. Equivalently, all b_{ij} are of type 0, for distinct $i, j \in \{1, 2, 3, 4\}$. Proposition 3.5.9 gives $\mu_{ij} = \lambda_i \lambda_j$ and $\nu_{ijk} = \mu_{ij}\lambda_k = \lambda_i\lambda_j\lambda_k$, for all i, j, k. Since $\xi = \lambda_i \nu_{jkl} = \prod_{t=1}^{4} \lambda_t$, the cluster is completely determined by the four parameters $\lambda_1, \lambda_2, \lambda_3$ and λ_4.

HWU $\mathbf{\sum_{i=1}^{4} a_i = 3r + 2}$ **and for a fixed** $\mathbf{i \in \{1, 2, 3, 4\}}$**, the exponents** $\mathbf{b_{ij}, b_{ik}, b_{il}}$ **are of type 0**. It then follows that b_{jk}, b_{kl}, b_{jl} are of type -1. Then Proposition 3.5.9 applied on the type -1 exponents gives $\nu_{ijk} = \lambda_i \mu_{jk}$, $\nu_{ikl} = \lambda_i \mu_{kl}$, $\nu_{ijl} = \lambda_i \mu_{jl}$. As b_{il} and b_{ij} are of type 0, the same lemma together with the above gives:

$$\lambda_j = \mu_{jk}\nu_{ijl} = \lambda_i \mu_{jk}\mu_{jl} \qquad \mu_{ij} = \lambda_i\lambda_j = \lambda_i^2 \mu_{jk}\mu_{jl}$$

$$\lambda_k = \mu_{jk}\nu_{ikl} = \lambda_i \mu_{jk}\mu_{kl} \qquad \mu_{ik} = \lambda_i\lambda_k = \lambda_i^2 \mu_{jk}\mu_{kl}$$

$$\lambda_l = \mu_{kl}\nu_{ijl} = \lambda_i \mu_{kl}\mu_{jl} \qquad \mu_{il} = \lambda_i\lambda_l = \lambda_i^2 \mu_{jl}\mu_{kl}$$

Finally, we have $\nu_{jkl} = \lambda_j \mu_{kl} = \lambda_i \mu_{jk}\mu_{jl}\mu_{kl}$ and $\xi = \lambda_i \nu_{jkl} = \lambda_i^2 \mu_{jk}\mu_{jl}\mu_{kl}$. We conclude that, in this case, G-cluster is fully determined by λ_i and μ_{st} where $i \notin \{s, t\}$.

HWD $\mathbf{\sum_{i=1}^{4} a_i = 3r + 2}$ **and for a fixed** $\mathbf{i \in \{1, 2, 3, 4\}}$**, the exponents** $\mathbf{b_{ij}, b_{ik}, b_{il}}$ **are of type -1**. It then follows that b_{jk}, b_{kl}, b_{jl} are of type 0. The same lemma reduces λ_*'s: $\lambda_j = \mu_{ij}\nu_{jkl}$, $\lambda_k = \mu_{ik}\nu_{jkl}$ and $\lambda_l = \mu_{il}\nu_{jkl}$. Use this and the type-

0 relations from the lemma to get:

$$\nu_{ijk} = \lambda_k \mu_{ij} = \mu_{ij}\mu_{ik}\nu_{jkl} \qquad \mu_{jk} = \lambda_j \lambda_k = \mu_{ij}\mu_{ik}\nu_{jkl}^2$$
$$\nu_{ijl} = \lambda_l \mu_{ij} = \mu_{ij}\mu_{il}\nu_{jkl} \qquad \mu_{jl} = \lambda_j \lambda_l = \mu_{ij}\mu_{il}\nu_{jkl}^2$$
$$\nu_{ikl} = \lambda_k \mu_{il} = \mu_{ik}\mu_{il}\nu_{jkl} \qquad \mu_{kl} = \lambda_k \lambda_l = \mu_{ik}\mu_{il}\nu_{jkl}^2$$

Finally $\lambda_i = \mu_{ij}\nu_{ikl} = \mu_{ij}\mu_{ik}\mu_{il}\nu_{jkl}$ and $\xi = \lambda_i \nu_{jkl} = \mu_{ij}\mu_{ik}\mu_{il}\nu_{jkl}^2$ So all the parameters needed to describe G-cluster in this case are $\nu_{jkl}, \mu_{ij}, \mu_{ik}, \mu_{il}$.

DOWN $\sum_{\mathbf{i=1}}^{\mathbf{4}} \mathbf{a_i} = \mathbf{3r} + \mathbf{3}$. Then all the exponents b_{ij} are of type -1. For any permutation (i, j, k, l) of the set $\{1, 2, 3, 4\}$, use $b_{kl} = a_k + a_l - r - 1$ to get the equality:

$$\begin{aligned}
\nu_{ijk}\nu_{ijl} \cdot x_k^{a_k-1} x_l^{b_{kl}-1} &= (x_i x_j x_l)^{r-a_k+1}\, x_l^{b_{kl}-1} = \\
&= \nu_{ijk} \cdot x_l^{a_l-1}\, (x_i x_j)^{r-a_k+1} = \\
&= (x_i x_j x_k)^{r-a_l+1}\, (x_i x_j)^{r-a_k+1} = \\
&= (x_i x_j)^{r+1-b_{kl}}\, x_k^{r-a_l+1} = \\
&= (x_i x_j)^{b_{ij}}\, x_k^{r-a_l+1} = \\
&= \mu_{ij} \cdot (x_k x_l)^{r-b_{ij}}\, x_k^{r-a_l+1} = \\
&= \mu_{ij} \cdot x_k^{a_k-1} x_l^{b_{kl}-1}
\end{aligned}$$

As the monomial $x_k^{a_k-1} x_l^{b_{kl}-1}$ is basic, it follows that $\mu_{ij} = \nu_{ijk}\nu_{ijl}$, for all i, j, k, l. Furthermore, Proposition 3.5.9 gives $\lambda_i = \mu_{ij}\nu_{ikl} = \nu_{ijk}\nu_{ijl}\nu_{ikl}$, for all i. We also get $\xi = \nu_{123}\nu_{124}\nu_{134}\nu_{234}$. Hence, the G-cluster is determined by the four parameters $\nu_{123}, \nu_{124}, \nu_{134}$ and ν_{234}.

□

3.7 Nakamura's G-graphs

Theorem 3.4.1 gives us a description of every G-cluster in terms of its defining ideal. In particular, it states that every **monomial** ideal defining a G-cluster is generated by relations (3.4). Notice that its defining relations (3.4) then satisfy $\lambda_i = \mu_{ij} = \nu_{ijk} = \xi_{1234} = 0$, for all indices $i, j, k \in \{1, 2, 3, 4\}$.

Proposition 3.7.1 below gives the opposite direction: every monomial ideal generated by relations of type (3.4) of Theorem 3.4.1 defines a G-cluster. Thus we obtain the set of all the monomial G-clusters.

Proposition 3.7.1. *Let $\mathcal{I}$ be a monomial ideal generated by*

$$\left\{x_i^{a_i},\, (x_i x_j)^{b_{ij}}\,,\, (x_j x_k x_l)^{r-a_i+1}\,,\, x_1x_2x_3x_4 \quad \middle| \quad \{i,j,k,l\} = \{1,2,3,4\}\right\},$$

where the integers a_i, b_{ij}, for all $1 \le i < j \le 4$, satisfy the conditions

(i) $1 \le b_{ij} \le a_i \le r$,

(ii) $b_{ij} + b_{kl} = r + 1$,

(iii) $b_{ij} \in \{a_i + a_j - r - 1, a_i + a_j - r\}$.

Then the affine scheme defined by $\mathcal{I}$ is a G-cluster.

Proof. The ideal $\mathcal{I}$ clearly defines a G-invariant scheme $Z \in \mathbb{C}^4$, as $p \in Z$ implies $\mathbf{m}(p) = 0$, for every generating monomial $\mathbf{m}$ of $\mathcal{I}$, and then for every $g \in G$ we have $\mathbf{m}(g \cdot p) = g \cdot \mathbf{m}(p) = \varepsilon^t \mathbf{m}(p) = 0$.

Next we show that every eigenspace $\mathcal{L}$ of the group action on the ring $\mathbb{C}[x_1, x_2, x_3, x_4]$ induces a vector subspace in $\mathbb{C}[x_1, x_2, x_3, x_4]/\mathcal{I}$ of dimension **at most** one. More specifically, let Γ be the set of all the monomials not in the ideal $\mathcal{I}$. Figures 3.2 and 3.3 show two and three-dimensional slices of Γ in the monomial lattice M. We show that every eigenspace contains at most one monomial in set Γ.

The only monomial in the invariant ring $R = \mathbb{C}[x_1^r, x_2^r, x_3^r, x_4^r, x_1x_2x_3x_4]$ that is not in the ideal $\mathcal{I}$ is $\mathbf{1}$. As a consequence, the eigenspace $\mathcal{L}_{000}$ contains exactly

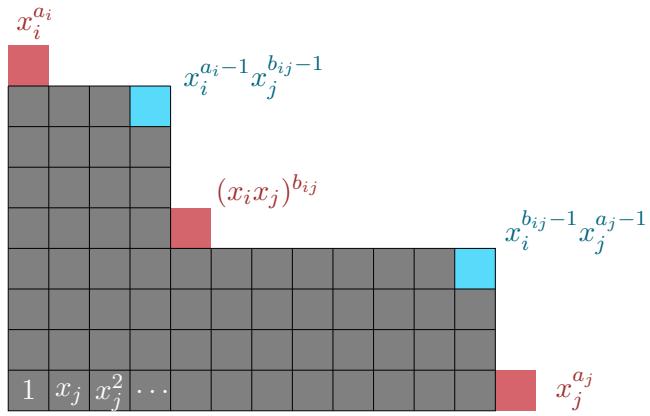


Figure 3.2: Two dimensional slice of Γ : monomials contained in (i,j)-plane of $M \cap \Gamma$

one member in Γ.

Suppose $x_i^s \in \Gamma$, for an integer $1 \leq s < r$. That means that $s < a_i$ and then $r - s \geq r - a_i + 1$, so $(x_jx_kx_l)^{r-s} \in \mathcal{I}$. By Corollary 3.3.4, an eigenspace $\mathcal{L}$ containing x_i^s is generated by x_i^s and $(x_jx_kx_l)^{r-s}$, only one of which is Γ. Similarly, if a monomial $(x_jx_kx_l)^s \in \Gamma$, then $s \leq r - a_i$, or equivalently, $r - s \geq a_i$, which implies that the only other generator of its eigenspace x_i^{r-s} lies in ideal $\mathcal{I}$.

If $(x_ix_j)^s \in \Gamma$, for $1 \leq s < r$, then $(x_kx_l)^{r-s} \in \mathcal{I}$. This is because $s \leq b_{ij} - 1$ implies $r - s \geq r - b_{ij} + 1 = b_{kl}$. Therefore, the second type of eigenspace from Corollary 3.3.4 also contains at most one monomial in Γ.

Now we consider all the other ("non-diagonal") monomials and their containing eigenspaces. Suppose $x_i^s x_j^t \in \Gamma$ for $1 \leq s < t < r$. Then the smaller coefficient satisfies $s < b_{ij}$ and the larger one satisfies $t < a_j$, see Figure 3.2. By Corollary 3.3.3, the eigenspace containing $m_1 = x_i^s x_j^t$ is generated by m_1 and two other monomials:

$$m_2 = x_i^{r+s-t}\,(x_kx_l)^{r-t} \quad \text{and} \quad m_3 = x_j^{t-s}\,(x_kx_l)^{r-s}\,,$$

both of which are contained in $\mathcal{I}$. To see this is true, notice first that the inequality $r + s - t > r - t > r - a_j$, means that $(x_ix_kx_l)^{r-a_j+1} \in \mathcal{I}$ divides m_2. We also have $r - s > r - b_{ij} = b_{kl} - 1$ or equivalently $r - s \geq b_{kl} - 1$ implying that $(x_kx_l)^{b_{kl}} \in \mathcal{I}$ divides m_3. Hence $m_2, m_3 \in \mathcal{I}$.

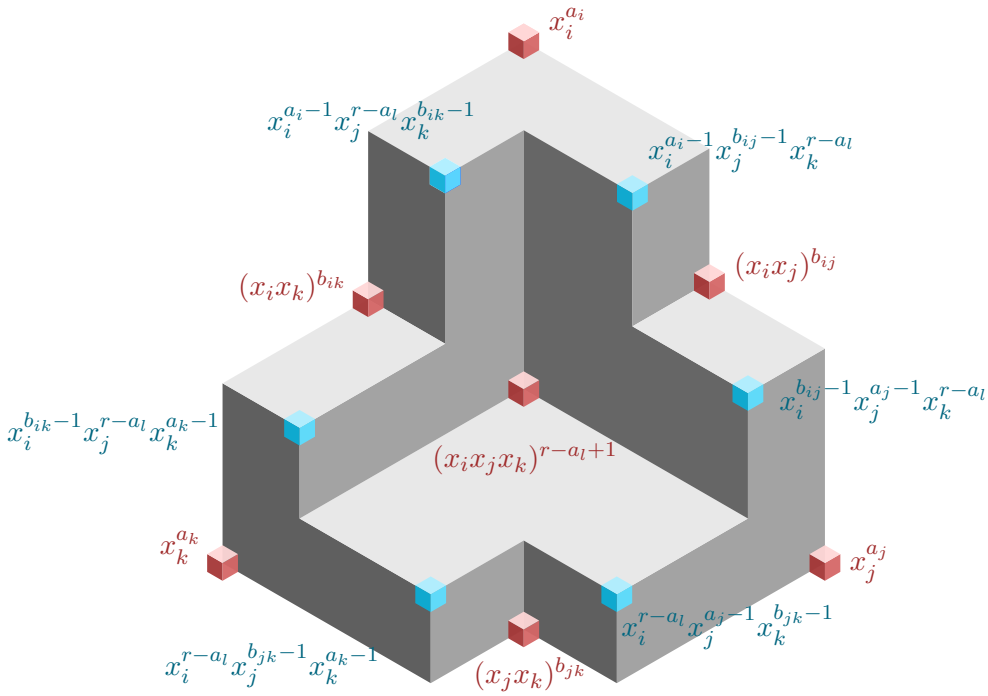


Figure 3.3: Three dimensional slice of Γ : monomials contained in (i,j,k)-corner of $M \cap \Gamma$

Again by Corollary 3.3.3, the eigenspace containing $m_1 = x_i^s (x_jx_k)^t$ for distinct $1 \leq s,t < r$ is generated by m_1 and

$$m_2 = x_i^{\overline{s-t}} x_l^{r-t}, \quad m_3 = (x_jx_k)^{\overline{t-s}} x_l^{r-s}.$$

Suppose $m_1 \in \Gamma$. In the case when $s < t$, it must hold that $s \leq r - a_l$ and $t \leq b_{jk} - 1$. But then $m_2 = x_i^{r+s-t} x_l^{r-t}$ is a multiple of the monomial $(x_ix_l)^{b_{il}} \in \mathcal{I}$ because we have $r + s - t > r - t \geq r + 1 - b_{jk} = b_{il}$. As for $m_3 = (x_jx_k)^{t-s} x_l^{r-s}$, it is a multiple of $x_l^{a_l} \in \mathcal{I}$ since $r - s \geq a_l$. In the opposite case, when $s > t$, having $m_1 \in \Gamma$ implies that $t \leq r - a_l$ and $s < a_i$. Then $m_2 = x_i^{s-t} x_l^{r-t}$ is divisible by $x_l^{a_l} \in \mathcal{I}$ and $m_3 = (x_jx_k)^{r+t-s} x_l^{r-s}$ by $(x_jx_kx_l)^{r-a_i+1} \in \mathcal{I}$. So an eigenspace containing m_1 contains no other monomials in Γ.

Finally, let $m_1 = x_i^s x_j^t x_k^u \in \Gamma$ for $1 \leq s < t < u < r$. Then the smallest exponent s must be smaller than $r - a_l + 1$, the middle one satisfies $t < b_{jk}$ and $u < a_k$.

Its eigenspace is generated by four monomials:

$$
\begin{aligned}
m_1 &= x_i^s x_j^t x_k^u \in \Gamma, \\
m_2 &= x_i^{r+s-u} x_j^{r+t-u} x_l^{r-u} = m_2' \cdot (x_i x_j x_l)^{r-a_k+1} \in \mathcal{I}, \\
m_3 &= x_i^{r+s-t} x_k^{u-t} x_l^{r-t} = m_3' \cdot (x_i x_l)^{b_{il}} \in \mathcal{I}, \\
m_4 &= x_j^{t-s} x_k^{u-s} x_l^{r-s} = m_4' \cdot x_l^{a_l} \in \mathcal{I},
\end{aligned}
$$

for some monomials m_2', m_3' and m_4' in $\mathbb{C}[x_1, x_2, x_3, x_4]$. This is the consequence of inequalities

$$
\begin{aligned}
r+t-u > r+s-u > r-u \geq r-a_k+1 \qquad & \text{for} \quad m_2, \\
r+s-t > r-t \geq r-b_{jk}+1 = b_{il} \qquad & \text{for} \quad m_3, \\
r-s \geq r-(r-a_l+1)+1 = a_l \qquad & \text{for} \quad m_4.
\end{aligned}
$$

We have shown so far that for every monomial in Γ, it is the only monomial from its eigenspace that is in Γ. Now we prove that Γ contains, not at most one, but **exactly one** monomial from each eigenspace. This will follow from $|\Gamma| = r^3$, the cardinality of group G.

Let $\Gamma_{ijk} \subseteq \Gamma$ be the subset of all the monomials in x_i, x_j and x_k, as shown in Figure 3.3. Similarly let $\Gamma_{ij} \subset \Gamma$ be the subset of all the monomials in x_i and x_j, see Figure 3.2 and $\Gamma_i = \{1, x_i, x_i^2, \ldots, x_i^{a_i-1}\}$, the subset of all the monomials in x_i only. Clearly,

$$
\begin{aligned}
\Gamma_{ijk} \cap \Gamma_{ijl} &= \Gamma_{ij} \\
\Gamma_{ijk} \cap \Gamma_{ijl} \cap \Gamma_{ikl} &= \Gamma_i \\
\bigcap_{1 \leq i < j < k \leq 4} \Gamma_{ijk} &= \{1\}.
\end{aligned}
$$

Then, by the inclusion-exclusion principle

$$|\Gamma| = \left(\sum_{1 \le i < j < k \le 4} |\Gamma_{ijk}| \right) - \left(\sum_{1 \le i < j \le 4} |\Gamma_{ij}| \right) + \left(\sum_{1 \le i \le 4} |\Gamma_i| \right) - 1. \tag{3.13}$$

Remember the notation from Section 3.5.3. For all distinct $i, j \in \{1, 2, 3, 4\}$ we have that $b_{ij} = a_i + a_j - r + \eta_{ij}$, where η_{ij} is either 0 or -1. We also introduced $\eta := \eta_{ij} + \eta_{kl}$ for all $\{i, j, k, l\} = \{1, 2, 3, 4\}$, and by Remark 3.5.7 we have $\sum_{i=1}^{4} a_i = 3r + 1 - \eta$. As $|\Gamma_i| = a_i$ for all $i \in \{1, 2, 3, 4\}$, the third summand in (3.13) is

$$\sum_{1 \le i \le 4} |\Gamma_i| = \sum_{i=1}^{4} a_i = 3r + 1 - \eta \tag{3.14}$$

The number of monomials in Γ that are factors of $(x_i x_j)^r$ (Figure 4.1) is

$$\begin{aligned} |\Gamma_{ij}| &= a_i b_{ij} + a_j b_{ij} - b_{ij}^2 \\ &= b_{ij} \left(a_i + a_j - b_{ij} \right) \\ &= \left(a_i + a_j - r + \eta_{ij} \right) \left(r - \eta_{ij} \right) \\ &= r \left(a_i + a_j \right) - r^2 + \eta_{ij} \left(2r - a_i - a_j - \eta_{ij} \right). \end{aligned}$$

When we sum up the six values $|\Gamma_{ij}|$, for $1 \le i < j \le 4$, we get the second summand in (3.13)

$$\begin{aligned} \sum_{1 \le i < j \le 4} |\Gamma_{ij}| &= 3r(\sum_{i=1}^{4} a_i) - 6r^2 + \sum_{1 \le i < j \le 4} \eta_{ij} \left(2r - a_i - a_j - \eta_{ij} \right) \\ &= 3r(3r + 1 - \eta) - 6r^2 + \sum_{1 \le i < j \le 4} \eta_{ij} \left(2r - a_i - a_j - \eta_{ij} \right) \\ &= 3r^2 + 3r - 3r\eta + \sum_{1 \le i < j \le 4} \eta_{ij} \left(2r - a_i - a_j - \eta_{ij} \right). \end{aligned} \tag{3.15}$$

To get the number of elements in a three-dimensional slice Γ_{ijk} we use the inclusion-

exclusion principle once again. Γ_{ijk} is covered by three sets

$$\bigcup_{s=0}^{r-a_l}(x_i^s\cdot\Gamma_{jk}),\qquad \bigcup_{s=0}^{r-a_l}(x_j^s\cdot\Gamma_{ik}),\qquad \bigcup_{s=0}^{r-a_l}(x_k^s\cdot\Gamma_{ij},)$$

which intersect pairwise in sets with cardinalities

$$\begin{aligned}
\left|\left\{x_i^s x_j^u x_k^v \quad\middle|\quad 0\le s\le a_i-1,\ \ 0\le u,v\le r-a_l\right\}\right| &= a_i\left(r-a_l+1\right)^2,\\
\left|\left\{x_i^s x_j^u x_k^v \quad\middle|\quad 0\le u\le a_j-1,\ \ 0\le s,v\le r-a_l\right\}\right| &= a_j\left(r-a_l+1\right)^2,\\
\left|\left\{x_i^s x_j^u x_k^v \quad\middle|\quad 0\le v\le a_k-1,\ \ 0\le s,u\le r-a_l\right\}\right| &= a_k\left(r-a_l+1\right)^2,
\end{aligned}$$

and the intersection of all three is a set of all the monomials dividing $(x_ix_jx_k)^{r-a_l}$ which has cardinality $(r-a_l+1)^3$. So we have

$$\begin{aligned}
|\Gamma_{ijk}| &= (r-a_l+1)\left(|\Gamma_{ij}|+|\Gamma_{jk}|+|\Gamma_{ik}|\right)-(r-a_l+1)^2(a_i+a_j+a_k)+(r-a_l+1)^3\\
&= (r-a_l+1)\ \Big[r\left(a_i+a_j\right)-r^2+\eta_{ij}\left(2r-a_i-a_j-\eta_{ij}\right)\\
&\qquad\qquad +r\left(a_j+a_k\right)-r^2+\eta_{jk}\left(2r-a_j-a_k-\eta_{jk}\right)\\
&\qquad\qquad +r\left(a_i+a_k\right)-r^2+\eta_{ik}\left(2r-a_i-a_k-\eta_{ik}\right)\Big]\\
&\qquad -(r-a_l+1)^2\left(a_i+a_j+a_k+a_l-r-1\right)\\
&= (r-a_l+1)\left[2r(a_i+a_j+a_k)-3r^2+\sum_{\substack{1\le i<j\le 4\\ i,j\ne l}}\eta_{ij}\left(2r-a_i-a_j-\eta_{ij}\right)\right]\\
&\qquad -(r-a_l+1)^2(2r-\eta)\\
&= (r-a_l+1)\ \left[2r(3r+1-\eta-a_l)-3r^2-(r-a_l+1)(2r-\eta)\right]\\
&\qquad +(r-a_l+1)\left[\sum_{\substack{1\le i<j\le 4\\ i,j\ne l}}\eta_{ij}\left(2r-a_i-a_j-\eta_{ij}\right)\right]\\
&= r^3+r^2-r^2a_l-r^2\eta+a_l^2\eta-2a_l\eta+\eta\\
&\qquad +(r-a_l+1)\left[\sum_{\substack{1\le i<j\le 4\\ i,j\ne l}}\eta_{ij}\left(2r-a_i-a_j-\eta_{ij}\right)\right]
\end{aligned}$$

The first summand of (3.13) is

$$\begin{aligned}
\sum_{1\le i<j<k\le 4} |\Gamma_{ijk}| &= 4r^3 + 4r^2 - r^2(\sum a_l) - 4r^2\eta + \eta(\sum a_l^2) - 2\eta(\sum a_l) + 4\eta \\
&\quad + \sum_{\substack{1\le i<j\le 4 \\ \{i,j,k,l\}=\{1,2,3,4\} \\ k<l}} \eta_{ij}\,(2r - a_i - a_j - \eta_{ij})\,(2r - a_k - a_l + 2) \\
&= 4r^3 + 4r^2 - r^2(3r + 1 - \eta) - 4r^2\eta + \eta(\sum_{i=1}^{4} a_i^2) \\
&\quad - 2\eta(3r + 1 - \eta) + 4\eta \\
&\quad + \sum_{\substack{1\le i<j\le 4 \\ \{i,j,k,l\}=\{1,2,3,4\} \\ k<l}} \eta_{ij}\,(2r - a_i - a_j - \eta_{ij})\,(2r - a_k - a_l + 2) \\
&= r^3 - 3r^2\eta + 3r^2 - 6r\eta + 2\eta^2 + 2\eta + \eta(\sum_{i=1}^{4} a_i^2) \\
&\quad + \sum_{\substack{1\le i<j\le 4 \\ \{i,j,k,l\}=\{1,2,3,4\} \\ k<l}} \eta_{ij}\,(2r - a_i - a_j - \eta_{ij})\,(2r - a_k - a_l + 2)
\end{aligned} \tag{3.16}$$

Using the second part of Theorem 3.4.1, set Γ is of one of four types.

UP In this case $\sum_{i=1}^{4} a_i = 3r + 1$ and $\eta_{ij} = \eta = 0$, for all indices i and j. Then most of sumands in (3.14) (3.15) and (3.16) are zero and the total number of elements in Γ is

$$|\Gamma| = \left(r^3 + 3r^2\right) - \left(3r^2 + 3r\right) + (3r + 1) - 1 = r^3 = |G|\,.$$

HWU In this case $\sum_{i=1}^{4} a_i = 3r + 2$ and for a fixed $i \in \{1, 2, 3, 4\}$ and the indices $\{j, k, l\} = \{1, 2, 3, 4\} \setminus \{i\}$ we have $\eta_{ij} = \eta_{ik} = \eta_{il} = 0$, $\eta_{jk} = \eta_{jl} = \eta_{kl} = -1$ and

$\eta = -1$.

$$
\begin{aligned}
|\Gamma| &= r^3 + 3r^2 + 3r^2 + 6r + 2 - 2 - (\sum_{i=1}^{4} a_i^2) \\
&\quad - (2r - a_j - a_k + 1)(2r - a_i - a_l + 2) \\
&\quad - (2r - a_j - a_l + 1)(2r - a_i - a_k + 2) \\
&\quad - (2r - a_k - a_l + 1)(2r - a_i - a_j + 2) \\
&\quad - \big[3r^2 + 3r + 3r - (2r - a_j - a_k + 1) - (2r - a_j - a_l + 1) \\
&\qquad\qquad - (2r - a_k - a_l + 1)\big] \\
&\quad + (3r + 2) - 1 \\
&= r^3 + 3r^2 + 3r + 1 - \sum_{i=1}^{4} a_i^2 - (2r - a_j - a_k + 1)(2r - a_i - a_l + 1) \\
&\qquad - (2r - a_j - a_l + 1)(2r - a_i - a_k + 1) \\
&\qquad - (2r - a_k - a_l + 1)(2r - a_i - a_j + 1) \\
&= r^3 + 3r^2 + 3r + 1 - 12r^2 + 6r \sum_{i=1}^{4} a_i - 12r + 3 \sum_{i=1}^{4} a_i - 3 \\
&\quad - \sum_{i=1}^{4} a_i^2 - 2 \sum_{1 \le i < j \le 4} a_i a_j \\
&= r^3 - 9r^2 - 9r - 2 + 6r(3r + 2) + 3(3r + 2) - \left(\sum_{i=1}^{4} a_i\right)^2 \\
&= r^3 + 9r^2 + 12r + 4 - (3r + 2)^2 \\
&= r^3
\end{aligned}
$$

HWD In this case $\sum_{i=1}^{4} a_i = 3r + 2$ and for a fixed $i \in \{1, 2, 3, 4\}$ and the indices $\{j, k, l\} = \{1, 2, 3, 4\} \setminus \{i\}$ we have $\eta_{ij} = \eta_{ik} = \eta_{il} = -1, \eta_{jk} = \eta_{jl} = \eta_{kl} = 0$ and $\eta = -1$. Computation here is identical to the HWU case, and we again get $|\Gamma| = r^3$.

DOWN In this case $\sum_{i=1}^{4} a_i = 3r+3$ and $\eta_{ij} = -1$, for all indices i and j so $\eta = -2$.

$$
\begin{aligned}
|\Gamma| &= 4r^3 + 4r^2 - r^2(3r+3) + 8r^2 - 2\sum_{i=1}^{4} a_i^2 + 4(3r+3) - 8 \\
&\quad - \sum_{\substack{1\le i<j\le 4 \\ \{i,j,k,l\}=\{1,2,3,4\} \\ k<l}} (2r - a_i - a_j + 1)\,(2r - a_k - a_l + 2) \\
&\quad - \left[3r^2 + 3r + 6r - \sum_{1\le i<j\le 4} (2r - a_i - a_j + 1) \right] \\
&\quad + (3r+3) - 1 \\
&= r^3 + 6r^2 + 6r + 6 - 2\sum_{i=1}^{4} a_i^2 \\
&\quad - \sum_{\substack{1\le i<j\le 4 \\ \{i,j,k,l\}=\{1,2,3,4\} \\ k<l}} (2r - a_i - a_j + 1)\,(2r - a_k - a_l + 1) \\
&= r^3 + 6r^2 + 6r + 6 - 24r^2 + 12r\sum_{i=1}^{4} a_i - 24r + 6\sum_{i=1}^{4} a_i - 6 \\
&\quad - 2\sum_{i=1}^{4} a_i^2 - 4\sum_{1\le i<j\le 4} a_i a_j \\
&= r^3 + 18r^2 + 36r + 18 - 2\left(\sum_{i=1}^{4} a_i\right)^2 \\
&= r^3.
\end{aligned}
$$

□

For a monomial G-cluster ideal $\mathcal{I}$, the set $\Gamma = \mathbb{C}[x_1, x_2, x_3, x_4] \setminus \mathcal{I}$ is the Nakamura's G-graph [24]. Craw, Maclagan and Thomas [8] refer to it as the set of standard monomials for $\mathcal{I}$.

3.8 The birational component $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^4)$

The moduli space definition of the G-Hilbert scheme that we use is one of the two definitions used in the literature. The other one, denoted by $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^n)$, was introduced by Ito-Nakamura [18] and it is defined as the irreducible component of $\left(\mathrm{Hilb}^{|\mathrm{G}|}(\mathbb{C}^n)\right)^G$ containing the general G-orbit. The two definitions are not equivalent in general: we have $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^n) \subset \text{G-Hilb}(\mathbb{C}^n)$ and $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^n)$ is birational to the quotient variety $\mathbb{C}^n/G$. We use the results of Nakamura [24] and Craw-Maclagan-Thomas [8] to describe the birational component $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^4)$ of $\text{G-Hilb}(\mathbb{C}^4)$ for $G \cong (\mathbb{Z}/r)^{\oplus 3}$.

To every monomial G-cluster ideal $\mathcal{I}$, we can associate a semigroup

$$A_{\mathcal{I}} := \left\langle m - n \in M \;\mid\; m, n \in \mathbb{Z}^4_{\geq 0},\, x^m \in \mathcal{I},\, x^n \notin \mathcal{I} \right\rangle.$$

Here, $M = \mathrm{Hom}(N, \mathbb{Z})$ is the monomial lattice with an overlattice $\mathbb{Z}^4$. As stated in the first chapter, it is of the form $M = r \cdot \mathbb{Z}^4 + \mathbb{Z} \cdot (1,1,1,1) \subset \mathbb{Z}^4$. To every element $n = (n_1, n_2, n_3, n_4)$ of $\mathbb{Z}^4_{\geq 0}$ (resp. $\mathbb{Z}^4$) we can associate a monomial (resp. Laurent monomial) $x^n = x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4}$, and conversely, the exponents of every monomial correspond to a tuple in $\mathbb{Z}^4$. The elements of $M \cap \mathbb{Z}^4_{\geq 0}$ then correspond to the invariant monomials, i.e. monomials in $\mathbb{C}[x_1, x_2, x_3, x_4]^G = \mathbb{C}[x_1^r, x_2^r, x_3^r, x_4^r, x_1x_2x_3x_4]$. Thus, $A_{\mathcal{I}}$ can also be viewed as a multiplicative semigroup generated by all $\frac{\mathbf{m}}{\mathbf{n}}$, where monomials $\mathbf{m}$ and $\mathbf{n}$ belong to the same eigenspace and $\mathbf{m} \in \mathcal{I}$, $\mathbf{n} \notin \mathcal{I}$. In Nakamura's paper [24], this semigroup is denoted by $S(\Gamma)$, where Γ is the G-graph associated to $\mathcal{I}$.

Lemma 3.8.1. *For every monomial G-cluster $\mathcal{I}$, semigroup $A_{\mathcal{I}}$ is generated by the*

set

$$\left\{ \begin{array}{l} \boldsymbol{\lambda_i} := x_i^{a_i} \,/\, (x_j x_k x_l)^{r-a_i}\,, \\ \boldsymbol{\mu_{ij}} := (x_i x_j)^{b_{ij}} \,/\, (x_k x_l)^{r-b_{ij}}\,, \\ \boldsymbol{\nu_{jkl}} := (x_j x_k x_l)^{r-a_i+1} \,/\, x_i^{a_i-1}, \\ \boldsymbol{\xi} := x_1 x_2 x_3 x_4 \end{array} \middle| \quad \textit{for all} \quad \{i,j,k,l\} = \{1,2,3,4\} \right\} \tag{3.17}$$

Proof. This follows from the proof of Lemma 3.5.3. We have shown there that if $\mathbf{m} - \alpha\mathbf{n} \in \mathcal{I}$, for $\mathbf{m}$ and $\mathbf{n}$ monomials in the same eigenspace that are not divisible by an invariant monomial, and $\alpha \in \mathbb{C}$, then there exists a monomial $\mathbf{f}$ such that $\mathbf{m} = \mathbf{f}\mathbf{m}'$ and $\mathbf{n} = \mathbf{f}\mathbf{n}'$, where $\frac{\mathbf{m}'}{\mathbf{n}'}$ is one of the Laurent monomials from (3.17). But then $\frac{\mathbf{m}}{\mathbf{n}} = \frac{\mathbf{f}\mathbf{m}'}{\mathbf{f}\mathbf{n}'} = \frac{\mathbf{m}'}{\mathbf{n}'}$ is also in (3.17). The value of α does not play a role here and in our case, $\mathcal{I}$ contains all $\mathbf{m} - 0 \cdot \mathbf{n}$, where $\mathbf{n} \notin \mathcal{I}$ is in the same eigenspace as monomial $\mathbf{m}$.

Let $\mathbf{m} \in \mathcal{I}$ and $\mathbf{n} \notin \mathcal{I}$ be monomials from the same eigenspace $\mathcal{L}$. Suppose $\mathbf{m} = \mathbf{m_0}\mathbf{m}'$, where $\mathbf{m_0}$ is invariant monomial and $\mathbf{m}'$ is not divisible by an invariant, then $\frac{\mathbf{m}}{\mathbf{n}} = \frac{\mathbf{m_0}\mathbf{m}'}{\mathbf{n}}$. As $\mathbf{m}' \in \mathcal{L}$ then $\frac{\mathbf{m}'}{\mathbf{n}}$ is in the set (3.17). We only need to check that $\mathbf{m_0}$ is always a multiple of elements in (3.17). Since it is invariant, it is a multiple of $x_1^r, x_2^r, x_3^r, x_4^r$ and $x_1x_2x_3x_4$, but the latter monomial is $\boldsymbol{\xi}$ and for all i we can see that $x_i^r = \boldsymbol{\lambda_i}\boldsymbol{\xi}^{\boldsymbol{r-a_i}}$. □

Lemma 3.8.2. *Let $\mathcal{I}$ be a monomial ideal defining a G-cluster. Then $A_{\mathcal{I}} = \sigma^\vee \cap M$, for some four-dimensional cone $\sigma \in \Sigma_{\text{G-Hilb}}$. Furthermore, every four-dimensional cone $\sigma \in \Sigma_{\text{G-Hilb}}$ corresponds to a monomial G-cluster $\mathcal{I}$ in this way.*

Proof. Let $\mathcal{I}$ be a monomial ideal defining a G-cluster, and Γ its associated G-graph. Then $\mathcal{I}$ is one of four types of the second part of Theorem 3.4.1 so we do a case-by-case analysis:

UP If the generators for $\mathcal{I}$ satisfy $\sum_{i=1}^4 a_i = 3r+1$ and $b_{ij} = a_i + a_j - r$, for all

distinct $i, j \in \{1,2,3,4\}$, then

$$\boldsymbol{\mu_{ij}} = \frac{(x_i x_j)^{b_{ij}}}{(x_k x_l)^{r-b_{ij}}} = \frac{(x_i x_j)^{a_i+a_j-r}}{(x_k x_l)^{2r-a_i-a_j}} = \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} \cdot \frac{x_j^{a_j}}{(x_i x_k x_l)^{r-a_j}} = \boldsymbol{\lambda_i \lambda_j},$$

$$\boldsymbol{\nu_{ijk}} = \frac{(x_j x_k x_l)^{r-a_l+1}}{x_l^{a_l-1}} = \frac{(x_i x_j)^{b_{ij}}}{(x_k x_l)^{r-b_{ij}}} \cdot \frac{x_k^{a_k}}{(x_i x_j x_l)^{r-a_k}} = \boldsymbol{\mu_{ij}\, \lambda_k} = \boldsymbol{\lambda_i\, \lambda_j\, \lambda_k}$$

$$\boldsymbol{\xi} = x_1 x_2 x_3 x_4 = \boldsymbol{\nu_{ijk}\, \lambda_l} = \boldsymbol{\lambda_1\, \lambda_2\, \lambda_3\, \lambda_4}.$$

Therefore $\underline{A_{\mathcal{I}} \text{ is generated by } \boldsymbol{\lambda_1, \lambda_2, \lambda_3, \lambda_4}}$. Now set $p = (p_1, p_2, p_3, p_4) \in \mathbb{Z}^4_{\geq 0}$ with $p_i = r - a_i$, for all $i \in \{1,2,3,4\}$. Then $\sum_{i=1}^4 p_i = r - 1$ so it defines an "up" tetrahedron of Definition 3.1.1:

$$\text{Cone}\,(U_p) = \text{Cone}\,(u_1^p, u_2^p, u_3^p, u_4^p)\,, \quad u_i^p = \tfrac{1}{r}\,(p + e_i)\,,\ i \in \{1,2,3,4\}\,.$$

This is a smooth cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$ with dual cone

$$\begin{aligned}
\text{Cone}(U_p)^\vee &= \text{Cone}\begin{pmatrix} (r-p_1, -r+a_1, -r+a_1, -r+a_1), \\ (-r+a_2, a_2, -r+a_2, -r+a_2), \\ (-r+a_3, -r+a_3, a_3, -r+a_3), \\ (-r+a_4, -r+a_4, -r+a_4, a_4) \end{pmatrix} \\
&= \text{Cone}\begin{pmatrix} (a_1, -r+a_1, -r+a_1, -r+a_1), \\ (-r+a_2, a_2, -r+a_2, -r+a_2), \\ (-r+a_3, -r+a_3, a_3, -r+a_3), \\ (-r+a_4, -r+a_4, -r+a_4, a_4) \end{pmatrix} \subset M \otimes_{\mathbb{Z}} \mathbb{R}.
\end{aligned}$$

Notice that the generators for the dual cone are exponents of $\boldsymbol{\lambda_1, \lambda_2, \lambda_3, \lambda_4}$ so $A_{\mathcal{I}} = \text{Cone}(U_p)^\vee \cap M$.

HWU $\mathcal{I}$ is generated by relations (3.4) with the exponents satisfying $\sum_{i=1}^4 a_i = 3r + 2$ and for a fixed $i \in \{1,2,3,4\}$ $\eta_{ij} = \eta_{ik} = \eta_{il} = 0$ and $\eta_{jk} = \eta_{kl} = \eta_{jl} = -1$, for $\{i,j,k,l\} = \{1,2,3,4\}$. We have

$$\boldsymbol{\nu_{ijk}} = \frac{(x_i x_j x_k)^{r-a_l+1}}{x_l^{a_l-1}} = \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} \cdot \frac{(x_j x_k)^{a_j+a_k-r-1}}{(x_i x_l)^{2r-a_j-a_k+1}} = \boldsymbol{\lambda_i\, \mu_{jk}}$$

$$\boldsymbol{\lambda_j} = \frac{x_j^{a_j}}{(x_i x_k x_l)^{r-a_j}} = \frac{(x_i x_j x_k)^{r-a_l+1}}{x_l^{a_l-1}} \cdot \frac{(x_j x_l)^{a_j+a_l-r-1}}{(x_i x_k)^{2r-a_j-a_l+1}} = \boldsymbol{\nu_{ijk}\, \mu_{jl}} = \boldsymbol{\lambda_i\, \mu_{jk}\, \mu_{jl}}$$

$$\boldsymbol{\mu_{ij}} = \frac{(x_i x_j)^{a_i+a_j-r}}{(x_k x_l)^{2r-a_i-a_j}} = \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} \cdot \frac{x_j^{a_j}}{(x_i x_k x_l)^{r-a_j}} = \boldsymbol{\lambda_i\, \lambda_j} = \boldsymbol{\lambda_i^2\, \mu_{jk}\, \mu_{jl}}$$

$$\boldsymbol{\nu_{jkl}} = \frac{(x_j x_k x_l)^{r-a_i+1}}{x_i^{a_i-1}} = \frac{x_j^{a_j}}{(x_i x_k x_l)^{r-a_j}} \cdot \frac{(x_k x_l)^{a_k+a_l-r-1}}{(x_i x_j)^{2r-a_k-a_l+1}} = \boldsymbol{\lambda_j\, \mu_{kl}} = \boldsymbol{\lambda_i\, \mu_{jk}\, \mu_{jl}\, \mu_{kl}}$$

$$\boldsymbol{\xi} = x_i x_j x_k x_l = \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} \cdot \frac{(x_j x_k x_l)^{r-a_i+1}}{x_i^{a_i-1}} = \boldsymbol{\lambda_i\, \nu_{jkl}} = \boldsymbol{\lambda_i^2\, \mu_{jk}\, \mu_{jl}\, \mu_{kl}}$$

Hence $\underline{A_{\mathcal{I}} \text{ is generated by } \boldsymbol{\lambda_i},\ \boldsymbol{\mu_{jk}},\ \boldsymbol{\mu_{kl}},\ \boldsymbol{\mu_{il}}.}$ Now define a point $p \in \mathbb{Z}^4_{\geq 0}$ by $(p_1, p_2, p_3, p_4) = (r - a_1, r - a_2, r - a_3, r - a_4)$. As $\sum p_i = r - 2$ this defines a "halfway up" tetrahedron

$$\sigma_{\mathrm{hwu},i} := \mathrm{Cone}\left(o^p_{jk}, o^p_{jl}, o^p_{kl}, m_p\right), \text{ where } o^p_{st} = \tfrac{1}{r}\left(p + e_s + e_t\right).$$

As this is a smooth cone, its dual cone $\sigma^\vee_{\mathrm{hwu},i}$ is generated by four elements – normals to its faces:

$$\begin{array}{rrrr}
-(p_j+p_k+1)f_i & +(p_i+p_l+1)f_j & +(p_i+p_l+1)f_k & -(p_j+p_k+1)f_l, \\
-(p_j+p_l+1)f_i & +(p_i+p_k+1)f_j & -(p_j+p_l+1)f_k & +(p_i+p_k+1)f_l, \\
-(p_k+p_l+1)f_i & -(p_k+p_l+1)f_j & +(p_i+p_j+1)f_k & +(p_i+p_j+1)f_l, \\
(p_j+p_k+p_l+2)f_i & -p_i f_j & -p_i f_k & -p_i f_l.
\end{array}$$

Here f_1, f_2, f_3, f_4 is the basis of overlattice $\mathbb{Z}^4$ of M. Once we translate the coefficients expressed in p_*'s back to a_*'s and use them as the exponents of

Laurent monomials, we see that the generators for $\sigma^{\vee}_{\mathrm{hwu},i}$ are

$$\frac{(x_jx_k)^{2r-a_i-a_l+1}}{(x_ix_l)^{2r-a_j-a_k+1}} = \frac{(x_jx_k)^{a_j+a_k-r-1}}{(x_ix_l)^{2r-a_j-a_k+1}} = \boldsymbol{\mu_{jk}},$$

$$\frac{(x_jx_l)^{2r-a_i-a_k+1}}{(x_ix_k)^{2r-a_j-a_l+1}} = \frac{(x_jx_l)^{a_j+a_l-r-1}}{(x_ix_k)^{2r-a_j-a_l+1}} = \boldsymbol{\mu_{jl}},$$

$$\frac{(x_kx_l)^{2r-a_i-a_j+1}}{(x_ix_j)^{2r-a_k-a_l+1}} = \frac{(x_kx_l)^{a_k+a_l-r-1}}{(x_ix_j)^{2r-a_k-a_l+1}} = \boldsymbol{\mu_{kl}},$$

$$\frac{x_i^{3r-a_j-a_k-a_l+2}}{(x_jx_kx_l)^{r-a_i}} = \frac{x_i^{a_i}}{(x_jx_kx_l)^{r-a_i}} = \boldsymbol{\lambda_i}.$$

We conclude that $A_{\mathcal{I}} = \sigma^{\vee}_{\mathrm{hwu},i} \cap M$.

HWD $\mathcal{I}$ is generated by relations (3.4) with the exponents satisfying $\sum_{i=1}^{4} a_i = 3r+2$ and for a fixed $i \in \{1,2,3,4\}$ $\eta_{ij} = \eta_{ik} = \eta_{il} = -1$ and $\eta_{jk} = \eta_{kl} = \eta_{jl} = 0$, for $\{i,j,k,l\} = \{1,2,3,4\}$. We have

$$\boldsymbol{\lambda_j} = \frac{x_j^{a_j}}{(x_ix_kx_l)^{r-a_j}} = \frac{(x_ix_j)^{a_i+a_j-r-1}}{(x_kx_l)^{2r-a_i-a_j+1}} \cdot \frac{(x_jx_kx_l)^{r-a_i+1}}{x_i^{a_i-1}} = \boldsymbol{\mu_{ij}\,\nu_{jkl}}$$

$$\boldsymbol{\nu_{ijk}} = \frac{(x_ix_jx_k)^{r-a_l+1}}{x_l^{a_l-1}} = \frac{x_j^{a_j}}{(x_ix_kx_l)^{r-a_j}} \cdot \frac{(x_ix_k)^{a_i+a_k-r-1}}{(x_jx_l)^{2r-a_i-a_k+1}} = \boldsymbol{\lambda_j\,\mu_{ik}} = \boldsymbol{\mu_{ij}\,\mu_{ik}\,\nu_{jkl}}$$

$$\boldsymbol{\mu_{jk}} = \frac{(x_jx_k)^{a_j+a_k-r}}{(x_ix_l)^{2r-a_j-a_k}} = \frac{(x_ix_jx_k)^{r-a_l+1}}{x_l^{a_l-1}} \cdot \frac{(x_jx_kx_l)^{r-a_i+1}}{x_i^{a_i-1}} = \boldsymbol{\nu_{ijk}\,\nu_{jkl}} = \boldsymbol{\mu_{ij}\,\mu_{ik}\,\nu_{jkl}^2}$$

$$\boldsymbol{\lambda_i} = \frac{x_i^{a_i}}{(x_jx_kx_l)^{r-a_i}} = \frac{(x_ix_l)^{a_i+a_l-r-1}}{(x_jx_k)^{2r-a_i-a_l+1}} \cdot \frac{(x_ix_jx_k)^{r-a_l+1}}{x_l^{a_l-1}} = \boldsymbol{\mu_{il}\,\nu_{ijk}} = \boldsymbol{\mu_{ij}\,\mu_{ik}\,\mu_{il}\,\nu_{jkl}}$$

$$\boldsymbol{\xi} = x_ix_jx_kx_l = \frac{x_i^{a_i}}{(x_jx_kx_l)^{r-a_i}} \cdot \frac{(x_jx_kx_l)^{r-a_i+1}}{x_i^{a_i-1}} = \boldsymbol{\lambda_i\,\nu_{jkl}} = \boldsymbol{\mu_{ij}\,\mu_{ik}\,\mu_{il}\,\nu_{jkl}^2}$$

It follows that $\underline{A_{\mathcal{I}} \text{ is generated by } \boldsymbol{\nu_{jkl}}, \boldsymbol{\mu_{ij}}, \boldsymbol{\mu_{ik}}, \boldsymbol{\mu_{il}}}$. As in the previous case, point $p \in \mathbb{Z}^4_{\geq 0}$ defined by $(p_1, p_2, p_3, p_4) = (r - a_1, r - a_2, r - a_3, r - a_4)$ satisfies $\sum_{i=1}^4 p_i = r - 2$ so it defines a "halfway down" tetrahedron

$$\sigma_{\mathrm{hwd},i} := \mathrm{Cone}\left(o^p_{ij}, o^p_{ik}, o^p_{il}, m_p\right).$$

Its dual cone is generated by four elements:

$$\begin{aligned}
(p_k + p_l + 1)f_i \quad &+(p_k + p_l + 1)f_j \quad -(p_i + p_j + 1)f_k \quad -(p_i + p_j + 1)f_l, \\
(p_j + p_l + 1)f_i \quad &-(p_i + p_k + 1)f_j \quad +(p_j + p_l + 1)f_k \quad -(p_i + p_k + 1)f_l, \\
(p_j + p_k + 1)f_i \quad &-(p_i + p_l + 1)f_j \quad -(p_i + p_l + 1)f_k \quad +(p_j + p_k + 1)f_l, \\
-(p_j + p_k + p_l - 1)f_i \quad &+(p_i + 1)f_j \quad +(p_i + 1)f_k \quad +(p_i + 1)f_l.
\end{aligned}$$

or equivalently

$$\frac{(x_i x_j)^{2r - a_k - a_l + 1}}{(x_k x_l)^{2r - a_i - a_j + 1}} = \frac{(x_i x_j)^{a_i + a_j - r - 1}}{(x_k x_l)^{2r - a_k - a_l + 1}} = \boldsymbol{\mu_{ij}},$$

$$\frac{(x_i x_k)^{2r - a_j - a_l + 1}}{(x_j x_l)^{2r - a_i - a_k + 1}} = \frac{(x_i x_k)^{a_i + a_k - r - 1}}{(x_j x_l)^{2r - a_i - a_k + 1}} = \boldsymbol{\mu_{ik}},$$

$$\frac{(x_i x_l)^{2r - a_j - a_k + 1}}{(x_j x_k)^{2r - a_i - a_l + 1}} = \frac{(x_i x_l)^{a_i + a_l - r - 1}}{(x_j x_k)^{2r - a_i - a_l + 1}} = \boldsymbol{\mu_{il}},$$

$$\frac{(x_j x_k x_l)^{r - a_i + 1}}{x_i^{3r - a_i - a_j - a_l + 1}} = \frac{(x_j x_k x_l)^{r - a_i + 1}}{x_i^{a_i - 1}} = \boldsymbol{\nu_{jkl}}.$$

As these are generators of semigroup $A_{\mathcal{I}}$, it follows that $A_{\mathcal{I}} = \sigma^\vee_{\mathrm{hwd},i} \cap M$.

DOWN The generators of ideal $\mathcal{I}$ satisfy $\sum_{i=1}^4 a_i = 3r + 3$ and $\eta_{ij} = -1$, for all distinct

$i, j \in \{1,2,3,4\}$. Then

$$\boldsymbol{\mu_{ij}} = \frac{(x_i x_j)^{a_i+a_j-r-1}}{(x_k x_l)^{2r-a_i-a_j+1}} = \frac{(x_i x_j x_k)^{r-a_l+1}}{x_l^{a_l-1}} \cdot \frac{(x_i x_j x_l)^{r-a_k+1}}{x_k^{a_k-1}} = \boldsymbol{\nu_{ijk}\,\nu_{ijl}}$$

$$\boldsymbol{\lambda_i} = \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} = \frac{(x_i x_j)^{a_i+a_j-r-1}}{(x_k x_l)^{2r-a_i-a_j+1}} \cdot \frac{(x_i x_k x_l)^{r-a_j+1}}{x_j^{a_j-1}} = \boldsymbol{\mu_{ij}\,\nu_{ikl}} = \boldsymbol{\nu_{ijk}\,\nu_{ijl}\,\nu_{ikl}}$$

$$\boldsymbol{\xi} = x_1 x_2 x_3 x_4 = \boldsymbol{\lambda_i\,\nu_{jkl}} = \boldsymbol{\nu_{ijk}\,\nu_{ijl}\,\nu_{ikl}\,\nu_{jkl}}$$

Therefore, semigroup $\underline{A_{\mathcal{I}} \text{ is generated by } \boldsymbol{\nu_{123}, \nu_{124}, \nu_{134}, \nu_{234}}}$. As before, the point $p \in \mathbb{Z}^4_{\geq 0}$ defined by $p_i = r - a_i$ for $i \in \{1,2,3,4\}$ has $\sum_{i=1}^4 p_i = r-3$ so it defines a "down" tetrahedron

$$\text{Cone}\,(D_p) = \text{Cone}\,(d_1^p, d_2^p, d_3^p, d_4^p)\,, \quad \text{where } d_i^p = \tfrac{1}{r}\,(p + (1,1,1,1) - e_i)\,.$$

Its dual cone is

$$\text{Cone}\,(D_p)^\vee = \text{Cone}\begin{pmatrix} (p_4-1, -p_4-1, -p_4-1, p_1+p_2+p_3+2) \\ (-p_3-1, -p_3-1, p_1+p_2+p_4+2, -p_3-1) \\ (-p_2-1, p_1+p_3+p_4+2, -p_2-1, -p_2-1) \\ (p_2+p_3+p_4+2, -p_1-1, -p_1-1, -p_1-1) \end{pmatrix}$$

$$= \text{Cone}\begin{pmatrix} (r-a_4+1, r-a_4+1, r-a_4+1, -a_4+1) \\ (r-a_3+1, r-a_3+1, -a_3+1, r-a_3+1) \\ (r-a_2+1, -a_2+1, r-a_2+1, r-a_2+1) \\ (-a_1+1, r-a_1+1, r-a_1+1, r-a_1+1) \end{pmatrix}$$

and it follows that $A_{\mathcal{I}} = \text{Cone}\,(D_p)^\vee \cap M$.

To show the second statement, notice that for $s = 1, 2, 3$, the sets

$$\left\{ (a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 \quad | \quad 1 \leq a_i \leq r, \ \sum_{i=1}^{4} a_i = 3r + s \right\}$$
$$\text{and} \quad \left\{ (p_1, p_2, p_3, p_4) \in \mathbb{Z}^4_{\geq 0} \quad | \quad \sum_{i=1}^{4} p_i = r - s \right\}$$

are in bijective correspondence by $p_i = r - a_i$, for all $i \in \{1, 2, 3, 4\}$. Therefore, every four-dimensional cone σ of $\Sigma_{\text{G-Hilb}}$ corresponds to a monomial G-cluster ideal $\mathcal{I}$ by $\sigma^\vee \cap M = A_{\mathcal{I}}$. □

For each monomial G-cluster $\mathcal{I}$, Nakamura [24, Definition 1.5] defined a cone $\sigma(\Gamma)$ such that $\sigma^\vee(\sigma) \cap M$ is the saturation of $A_{\mathcal{I}}$. As we have shown that for all the monomial G-clusters I, semigroup $A_{\mathcal{I}} = \sigma^\vee \cap M$ for some $\sigma \in \Sigma_{\text{G-Hilb}}$, we have $\sigma(\Gamma) = \sigma$, meaning especially that $\sigma(\Gamma)$ is non-empty. Following [8, Remark 4.13.] we conclude that $\mathcal{I} = \text{in}_{\text{w}}(I_M)$, for some $\text{w} \in \left(\mathbb{Q}^4_{\geq 0}\right)^*$, where

$$I_M = \left\langle x^u - x^{u'} \ | \ u, u \in \mathbb{Z}^4_{\geq 0}, u - u' \in M \right\rangle,$$

allowing us to use [8, Theorem 5.2.]:

Theorem 3.8.1 (Craw-Maclagan-Thomas)**.** $\text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$ *is covered by affine charts* $\text{Spec}\,\mathbb{C}\left[A_{\mathcal{I}}\right]$ *defined by the monomial ideals* $\mathcal{I} = \text{in}_{\text{w}}\left(I_M\right)$ *as* w *varies in* $\left(\mathbb{Q}^4_{\geq 0}\right)^*$.

We finish the section with explicit description of $\text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$.

Theorem 3.8.2. $\text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$ *is a smooth toric variety given by the fan* $\Sigma_{\text{G-Hilb}}$.

Proof. We have noted above that every monomial ideal $\mathcal{I}$ that defines a G-cluster is the initial ideal of I_M with respect to some $\text{w} \in \left(\mathbb{Q}^4_{\geq 0}\right)^*$. In consequence, Theorem 3.8.1 applied to our case says that the birational component Hilb^{G} is covered by affine charts $\text{Spec}\,\mathbb{C}\left[A_{\mathcal{I}}\right]$, when $\mathcal{I}$ runs through the set of all the monomial G-cluster ideals.

Now we just need to apply Lemma 3.8.2. For every monomial G-cluster ideal $\mathcal{I}$ there is a four-dimensional cone $\sigma \in \Sigma_{\text{G-Hilb}}$ such that

$$\operatorname{Spec} \mathbb{C}\left[A_{\mathcal{I}}\right] = \operatorname{Spec} \mathbb{C}\left[\sigma^{\vee} \cap M\right],$$

and every four-dimensional cone σ arises in this way from a monomial G-cluster. Thus $\operatorname{Hilb}^{\mathrm{G}}(\mathbb{C}^4)$ is covered by affine charts corresponding to cones of $\Sigma_{\text{G-Hilb}}$. $\square$

3.9 Proof of the main theorem

Finally, we show that $\text{G-Hilb}\left(\mathbb{C}^4\right) = \operatorname{Hilb}^{\mathrm{G}}\left(\mathbb{C}^4\right)$. We continue using the results of [8, Section 5]. Let Z be a G-cluster and $\mathcal{I}$ its defining ideal. Define set

$$\left\{u - u' \in M \;\mid\; u, u' \in \mathbb{Z}^4_{\geq 0},\ x^u \text{ is a minimal generator of } \mathcal{I},\ x^{u'} \notin \mathcal{I}\right\}.$$

Notice that this is equivalent to the set (3.17) from Lemma 3.8.1 and it consists of fifteen elements so we can write it as

$$U = \left\{u_t - u'_t \;\mid\; 1 \leq t \leq 15\right\}.$$

Let I_U be the kernel of the $\mathbb{C}$-algebra homomorphism $\varphi : \mathbb{C}\left[y_1, \ldots, y_{15}\right] \to \mathbb{C}\left[A_{\mathcal{I}}\right]$ sending y_i to $x^{u_i}/x^{u'_i}$. [8, Corollary 5.5.]

Proposition 3.9.1 (Craw-Maclagan-Thomas)**.** *The universal family above the chart* $\operatorname{Spec} \mathbb{C}\left[A_{\mathcal{I}}\right]$ *is given by*

$$F := \left\langle x^{u_i} - y_i x^{u'_i} \;\mid\; 1 \leq i \leq |U| \right\rangle + I_U$$

in the ring $\mathbb{C}\left[x_1, x_2, x_3, x_4\right]\left[y_1, \ldots y_{15}\right]$.

We now describe these families in more detail. Relabel the variables $y_1, \ldots, y_{15}$ to more natural names $\lambda_1, \ldots, \lambda_4,\ \mu_{12}, \mu_{13}, \ldots, \mu_{34},\ \nu_{123}, \ldots, \nu_{234}$ and ξ. order the

elements of U so that the map φ of $\mathbb{C}$-algebras acts by

$$\lambda_i \longmapsto \frac{x_i^{a_i}}{(x_j x_k x_l)^{r-a_i}} = \boldsymbol{\lambda_i}, \qquad \mu_{ij} \longmapsto \frac{(x_i x_j)^{b_{ij}}}{(x_k x_k)^{r-b_{ij}}} = \boldsymbol{\mu_{ij}},$$

$$\nu_{jkl} \longmapsto \frac{(x_j x_k x_l)^{r-a_i+1}}{x_i^{a_i-1}} = \boldsymbol{\nu_{jkl}} \qquad \xi \longmapsto x_1 x_2 x_3 x_4 = \boldsymbol{\xi}.$$

We can now use these new labels to rewrite family $F \subset \mathbb{C}[x_1, \ldots, x_4][\lambda_1, \ldots, \xi]$ as

$$\left\langle \begin{array}{lcl} x_i^{a_i} & - & \lambda_i \ (x_j x_k x_l)^{r-a_i}, \\ (x_i x_j)^{b_{ij}} & - & \mu_{ij} \ (x_k x_l)^{r-b_{ij}}, \\ (x_j x_k x_l)^{r-a_i+1} & - & \nu_{jkl} \quad x_l^{a_l-1}, \\ x_1 x_2 x_3 x_4 & - & \xi \end{array} \quad \text{for } \{i,j,k,l\} - \{1,2,3,4\} \right\rangle + I_U$$

The proof of Lemma 3.8.2 gives us the relations between elements of U for each of the four cases. We use this to describe the ideal I_U.

UP If $\mathcal{I}$ is a monomial G-cluster defined by relations (3.4) of type UP, for some $1 \leq a_1, a_2, a_3, a_4 \leq r$ satisfying $\sum_{i=1}^{4} a_i = 3r+1$, then

$$I_U = \langle \mu_{ij} - \lambda_i \lambda_j, \quad \nu_{ijk} - \lambda_i \lambda_j \lambda_k, \quad \xi - \lambda_1 \lambda_2 \lambda_3 \lambda_4 \rangle$$

so family F can be viewed as

$$\left\langle \begin{array}{lcl} x_i^{a_i} & - & \lambda_i \ \ (x_j x_k x_l)^{r-a_i}, \\ (x_i x_j)^{a_i+a_j-r} & - & \lambda_i \lambda_j \ \ (x_k x_l)^{2r-a_i-a_j}, \\ (x_i x_j x_k)^{r-a_l+1} & - & \lambda_i \lambda_j \lambda_k \quad x_l^{a_l-1}, \\ x_1 x_2 x_3 x_4 & - & \lambda_1 \lambda_2 \lambda_3 \lambda_4 \end{array} \right\rangle \subset \mathbb{C}[x_1, x_2, x_3, x_4][\lambda_1, \lambda_2, \lambda_3, \lambda_4]$$

HWU Fix $i \in \{1,2,3,4\}$ and let $\{j,k,l\} = \{1,2,3,4\} \setminus \{i\}$ and suppose $\mathcal{I}$ is generated by monomial relations of type HWU, for some $1 \leq a_1, a_2, a_3, a_4 \leq r$ such that

$\sum_{i=1}^{4} a_i = 3r+2$ and $b_{is} = a_i + a_s - r$, $b_{st} = a_s + a_t - r - 1$, $s,t \in \{j,k,l\}$. Then

$$I_U = \left\langle \begin{array}{llll} \lambda_j - \lambda_i\mu_{jk}\mu_{jl}, & \mu_{ij} - \lambda_i^2\mu_{jk}\mu_{jl}, & \nu_{ikl} - \lambda_i\mu_{kl}, & \\ \lambda_k - \lambda_i\mu_{jk}\mu_{kl}, & \mu_{ik} - \lambda_i^2\mu_{jk}\mu_{kl}, & \nu_{ijl} - \lambda_i\mu_{jl}, & \nu_{jkl} - \lambda_i\mu_{jk}\mu_{jl}\mu_{kl}, \\ \lambda_l - \lambda_i\mu_{jl}\mu_{kl}, & \mu_{il} - \lambda_i^2\mu_{jl}\mu_{kl}, & \nu_{ijk} - \lambda_i\mu_{kl}, & \xi - \lambda_i^2\mu_{jk}\mu_{jl}\mu_{kl} \end{array} \right\rangle$$

and, similarly to UP case, by removing the variables that are combinations of $\lambda_i, \mu_{jk}, \mu_{jl}, \mu_{kl}$, family of G-clusters F lies in $\mathbb{C}[x_1,x_2,x_3,x_4][\lambda_i, \mu_{jk}, \mu_{jl}, \mu_{kl}]$.

HWD Fix $i \in \{1,2,3,4\}$ and let $\{j,k,l\} = \{1,2,3,4\} \setminus \{i\}$ and suppose $\mathcal{I}$ is generated by monomial relations of type HWD. for some $1 \le a_1, a_2, a_3, a_4 \le r$ such that $\sum_{i=1}^{4} a_i = 3r+2$ and $b_{is} = a_i + a_s - r - 1$, $b_{st} = a_s + a_t - r$, $s,t \in \{j,k,l\}$. Then

$$I_U = \left\langle \begin{array}{llll} \lambda_j - \mu_{ij}\nu_{jkl}, & \mu_{kl} - \mu_{ik}\mu_{il}\nu_{jkl}^2, & \nu_{ikl} - \mu_{ik}\mu_{il}\nu_{jkl}, & \\ \lambda_k - \mu_{ik}\nu_{jkl}, & \mu_{jl} - \mu_{ij}\mu_{il}\nu_{jkl}^2, & \nu_{ijl} - \mu_{ij}\mu_{il}\nu_{jkl}, & \lambda_i - \mu_{ij}\mu_{ik}\mu_{il}\nu_{jkl}, \\ \lambda_l - \mu_{il}\nu_{jkl}, & \mu_{jk} - \mu_{ij}\mu_{ik}\nu_{jkl}^2, & \nu_{ijk} - \mu_{ij}\mu_{ik}\nu_{jkl}, & \xi - \mu_{ij}\mu_{ik}\mu_{il}\nu_{jkl}^2, \end{array} \right\rangle$$

and by removing the variables that are combinations of $\mu_{ij}, \mu_{ik}, \mu_{il}, \nu_{jkl}$, family of G-clusters F lies in $\mathbb{C}[x_1,x_2,x_3,x_4][\mu_{ij}, \mu_{ik}, \mu_{il}, \nu_{jkl}]$.

DOWN If $\mathcal{I}$ is a monomial G-cluster defined by relations (3.4) of type DOWN, for some $1 \le a_1, a_2, a_3, a_4 \le r$ satisfying $\sum_{i=1}^{4} a_i = 3r+3$, then

$$I_U = \langle \lambda_i - \nu_{ijk}\,\nu_{ijl}\,\nu_{ikl}, \quad \mu_{ij} - \nu_{ijk}\,\nu_{ijl}, \quad \xi - \nu_{123}\,\nu_{124}\,\nu_{134}\,\nu_{234} \rangle$$

so family F can be viewed as

$$\left\langle \begin{array}{lll} x_i^{a_i} & - & \nu_{ijk}\,\nu_{ijl}\,\nu_{ikl}\,(x_jx_kx_l)^{r-a_i}, \\ (x_ix_j)^{a_i+a_j-r-1} & - & \nu_{ijk}\,\nu_{ijl}\,(x_kx_l)^{2r-a_i-a_j+1}, \\ (x_jx_kx_l)^{r-a_i+1} & - & \nu_{jkl}\,x_i^{a_i-1}, \\ x_1x_2x_3x_4 & - & \nu_{123}\,\nu_{124}\,\nu_{134}\,\nu_{234} \end{array} \right\rangle$$

in the ring $\mathbb{C}[x_1,x_2,x_3,x_4][\nu_{123},\ \nu_{124},\ \nu_{134},\ \nu_{234}]$

Part two of Theorem 3.4.1 shows that every G-cluster Z, not necessarily generated by a monomial ideal, lies in one of the above families. Therefore $\text{G-Hilb}\left(\mathbb{C}^4\right) = \text{Hilb}^{\text{G}}\left(\mathbb{C}^4\right)$ and this, together with Theorem 3.8.2 proves Theorem 3.1.1.

Chapter 4

Further Questions

In this chapter we present several problems that arise from the results of this thesis. The first two sections give some background and basic terminology of the moduli spaces $\mathcal{M}_\theta$ of θ-stable G-constellations, and are mainly expository. Section 4.3 contains some worked examples. We show how one can "build" the G-constellations that are parametrised by the points of $Y_{[12\text{-}34]}$ from the known G-clusters, for $r = 2, 3$ and suggest conditions (see 4.1.1 (*iii*)) on the stability parameter θ in order to have $Y_{[12\text{-}34]} \cong \mathcal{M}_\theta$.

In Section 4.4, we discuss the McKay correspondence and the known results. In the worked examples, as a first step to proving some version of Conjecture 4.4.1, we mark the exceptional subvarieties of the [12-34]-crepant resolution by the characters of the group G as suggested in [27].

Conjecture 4.5.1 in Section 4.5 shows how one might use the existence of crepant resolutions for quotients studied in this thesis to obtain existence of crepant resolutions for more general finite abelian subgroups of $\mathrm{SL}(4, \mathbb{C})$. In the final section, we show by an example that in dimension five, $\text{G-Hilb}\left(\mathbb{C}^5\right)$ is neither smooth nor crepant for $G = (\mathbb{Z}/2)^4$.

4.1 Are crepant resolutions isomorphic to $\mathcal{M}_\theta$?

We have shown in the previous chapter that the G-Hilbert scheme for the group $G = (\mathbb{Z}/r)^{\oplus 3}$ acting on $\mathbb{C}^4$ is not a crepant resolution of singularities. However, by doing a number of contractions $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ of discrepant exceptional divisors, we obtain a crepant resolution. As the G-Hilbert scheme can be viewed as a moduli space of θ-stable G-constellations for a specific stability parameter θ (see [19], [7]), a natural question is which varieties might we obtain by varying θ. More specifically, is it true that the special crepant resolutions of Chapter 2 (see Theorem 2.3.1) have an interpretation as $\mathcal{M}_\theta$ for some θ? Is the same true for all the crepant resolutions of $\mathbb{C}^4/G$?

The motivation for these questions also lies in the result of Craw and Ishii [6]. They proved that when $A \subset \mathrm{SL}\,(3, \mathbb{C})$ is a finite Abelian group, every projective crepant resolution $Y \to \mathbb{C}^3/G$ is isomorphic to the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations, for some stability condition θ.

We present three open questions, sorted by the highest generality first.

Conjecture 4.1.1. *Let $G \cong (\mathbb{Z}/r)^{\oplus 3}$ act on $\mathbb{C}^4$ by* (3.1)*. Then*

(i) Every projective crepant resolution Y is isomorphic to $\mathcal{M}_\theta$ for some generic stability condition θ.

(ii) If the special crepant resolution Y_φ (see 2.3.1) is projective, then $Y \cong \mathcal{M}_\theta$, for some stability condition θ.

(iii) The crepant resolution $Y_{[12\text{-}34]} \cong \mathcal{M}_\theta$, for some stability parameter θ. Furthermore, such a θ must satisfy $\theta_{0ii} < 0$ for all $0 \leq i \leq r-1$.

Parts (ii) and (iii) of the conjecture are true in the case $r = 2$ (see Section 8 of [5]). All three special crepant resolutions arise as moduli space of θ-stable McKay quiver representations, where θ lies in one of the three adjacent GIT chambers to the chamber C_0, where C_0 is a chamber defining G-Hilb $\left(\mathbb{C}^4\right)$.

Another interesting question is the GIT chamber structure for the space of

stability conditions Θ. This could be computed using the library `gitfan.lib` of the computer algebra system SINGULAR, see Section 4 of [1].

In the next section, after a very brief overview of the theory, we show on two worked examples how to obtain torus invariant G-constellations that correspond to the torus invariant points of $Y_{[12\text{-}34]}$.

4.2 Moduli spaces of G-constellations

Let a finite abelian group $G \subset \mathrm{SL}(n, \mathbb{C})$ act on $\mathbb{C}^n$.

Definition 4.2.1. A **G-constellation** is a G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$, such that $H^0(\mathcal{F})$ is isomorphic to the regular representation $\mathbb{C}[G]$ as a $\mathbb{C}G$-module.

As the group G is abelian, the definition implies that every G-constellation is isomorphic to $\sum_{\rho \in \widehat{G}} \mathbb{C}\rho$ as a G-module. Let $R(G)$ be the representation ring of G. The set

$$\Theta = \{\theta \in \mathrm{Hom}_{\mathbb{Z}}(R(G), \mathbb{Q}) \ \mid \ \theta(\mathbb{C}[G]) = 0\}$$

is the space of stability parameters. In the case of our group $G \cong (\mathbb{Z}/r)^{\oplus 3}$, a stability parameter is a r^3-tuple of rational numbers $\theta = (\theta_{ijk})_{0 \leq i,j,k < r} \in \mathbb{Q}^{r^3}$ such that $\sum_{0 \leq i,j,k < r} \theta_{ijk} = 0$.

Definition 4.2.2. Let $\theta \in \Theta$ be a stability parameter. A G-constellation $\mathcal{F}$ is

- **θ-semistable** if $\theta(\mathcal{G}) \geq 0$, for every proper submodule $\mathcal{G} \subset \mathcal{F}$,
- **θ-stable** if $\theta(\mathcal{G}) > 0$, for every proper submodule $\mathcal{G} \subset \mathcal{F}$.

A stability parameter θ is said to be **generic** if θ-semistability implies θ-stability.

Now that we have the notion of stability, we define $\mathcal{M}_\theta$ to be a moduli space of θ-stable G-constellations. It is well known that the language of G-constellations is equivalent to the language of **McKay quiver representations**, and the spaces $\mathcal{M}_\theta$ can be defined as certain GIT-quotients. In this context, King [22] proved that

for a generic θ, $\mathcal{M}_\theta$ is a fine moduli space. Craw, Maclagan and Thomas [7] have shown that when θ is generic, the scheme $\mathcal{M}_\theta$ has a unique irreducible component Y_θ, known as the birational or coherent component, that contains the torus $(\mathbb{C}^*)^n/G$, for a group $G \subset \mathrm{SL}(n, \mathbb{C})$.

4.3 Examples of G-constellations corresponding to $Y_{[12\text{-}34]}$

Example 4.3.1. Let $r = 2$. In the fan of G-Hilb $\left(\mathbb{C}^4\right)$, there are four "up" tetrahedra, whose corresponding G-clusters have bases

$$\begin{aligned}
\Gamma_1 &= \{\mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{xy}, \mathbf{xz}, \mathbf{yz}, \mathbf{xyz}\},\\
\Gamma_2 &= \{\mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{xyt}, \mathbf{xy}, \mathbf{yt}, \mathbf{xt}, \mathbf{t}\},\\
\Gamma_3 &= \{\mathbf{1}, \mathbf{x}, \mathbf{xzt}, \mathbf{z}, \mathbf{zt}, \mathbf{xz}, \mathbf{xt}, \mathbf{t}\},\\
\Gamma_4 &= \{\mathbf{1}, \mathbf{yzt}, \mathbf{y}, \mathbf{z}, \mathbf{zt}, \mathbf{yt}, \mathbf{yz}, \mathbf{t}\}.
\end{aligned}$$

We want to keep these, as the "up" tetrahedra appear in the fan $\Sigma_{[12-34]}$ of the crepant resolution. To obtain the fan $\Sigma_{[12-34]}$ from the fan of the G-Hilb $\left(\mathbb{C}^4\right)$ we need to remove the four faces of cones in $\Sigma_{\text{G-Hilb}}$ marked by the ratio $x_1x_2 : x_3x_4$. Two cones joined by one such face are

$$\text{Cone}\begin{pmatrix} \frac{1}{2}(1,1,0,0), \\ \frac{1}{2}(1,0,1,0), \\ \frac{1}{2}(1,0,0,1), \\ \frac{1}{2}(1,1,1,1) \end{pmatrix} \quad \text{and} \quad \text{Cone}\begin{pmatrix} \frac{1}{2}(0,0,1,1), \\ \frac{1}{2}(1,0,1,0), \\ \frac{1}{2}(1,0,0,1), \\ \frac{1}{2}(1,1,1,1). \end{pmatrix}$$

The G-clusters parametrised by these two affine pieces have bases:

$$\{\mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{yz}, \mathbf{yt}, \mathbf{zt}\} \text{ and } \{\mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{yz}, \mathbf{yt}, \mathbf{xy}\}.$$

The only place where these two sets differ is the basis monomial for the eigenspace L_{011}, which is either xy or zt. We replace the basis element xy by a Laurent

monomial xy^{-1} to obtain the set (compare with definition of G-prebrick [20],[21]).

$$\Gamma_5 = \left\{ \mathbf{1}, \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{t}, \mathbf{yz}, \mathbf{yt}, \frac{\mathbf{x}}{\mathbf{y}} \right\}.$$

A G-constellation $\mathcal{F}$ whose $H^0(\mathcal{F})$ is based by the above monomials must have a multiplication defined by the $4 \cdot 2^3$ relations:

$$\begin{array}{llll}
x \cdot \mathbf{1} = \alpha_1\, \mathbf{x}, & y \cdot \mathbf{1} = \beta_1\, \mathbf{y}, & z \cdot 1 = \gamma_1\, \mathbf{z}, & t \cdot 1 = \delta_1\, \mathbf{t}, \\
x \cdot \mathbf{x} = \alpha_2\, \mathbf{1}, & y \cdot \mathbf{x} = \beta_2\, \mathbf{xy}^{-1}, & z \cdot \mathbf{x} = \gamma_2\, \mathbf{yt}, & t \cdot \mathbf{x} = \delta_2\, \mathbf{yz}, \\
x \cdot \mathbf{y} = \alpha_3\, \mathbf{xy}^{-1}, & y \cdot \mathbf{y} = \beta_3\, \mathbf{1}, & z \cdot \mathbf{y} = \gamma_3\, \mathbf{yz}, & t \cdot \mathbf{y} = \delta_3\, \mathbf{yt}, \\
x \cdot \mathbf{z} = \alpha_4\, \mathbf{yt}, & y \cdot \mathbf{z} = \beta_4\, \mathbf{yz}, & z \cdot \mathbf{z} = \gamma_4\, \mathbf{1}, & t \cdot \mathbf{z} = \delta_4\, \mathbf{xy}^{-1}, \\
x \cdot \mathbf{t} = \alpha_5\, \mathbf{yz}, & y \cdot \mathbf{t} = \beta_5\, \mathbf{yt}, & z \cdot \mathbf{t} = \gamma_5\, \mathbf{xy}^{-1}, & t \cdot \mathbf{t} = \delta_5\, \mathbf{1}, \\
x \cdot \mathbf{yz} = \alpha_6\, \mathbf{t}, & y \cdot \mathbf{yz} = \beta_6\, \mathbf{z}, & z \cdot \mathbf{yz} = \gamma_6\, \mathbf{y}, & t \cdot \mathbf{yz} = \delta_6\, \mathbf{x}, \\
x \cdot \mathbf{yt} = \alpha_7\, \mathbf{z}, & y \cdot \mathbf{yt} = \beta_7\, \mathbf{t}, & z \cdot \mathbf{yt} = \gamma_7\, \mathbf{x}, & t \cdot \mathbf{yt} = \delta_7\, \mathbf{y}, \\
x \cdot \mathbf{xy}^{-1} = \alpha_8\, \mathbf{y}, & y \cdot \mathbf{xy}^{-1} = \beta_8\, \mathbf{x}, & z \cdot \mathbf{xy}^{-1} = \gamma_8\, \mathbf{t}, & t \cdot \mathbf{xy}^{-1} = \delta_8\, \mathbf{z},
\end{array}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{C}$. and it must satisy certain commutativity relations. For example,

$$\alpha_1\, \mathbf{x} = x \cdot \mathbf{1} = 1 \cdot \mathbf{x} = \mathbf{x}$$

implies that $\alpha_1 = 1$. Similarly,

$$\alpha_3\, \mathbf{xy}^{-1} = x \cdot \mathbf{y} = y \cdot \mathbf{x} = \beta_2\, \mathbf{xy}^{-1},$$

so $\alpha_3 = \beta_2$. If we continue manipulating the relations like this, we conclude that all the parameters can be expressed through the four parameters $\beta_2, \gamma_2, \delta_2$ and γ_7. Hence, all the G-constellations $\mathcal{F}$ with basis for $H^0(\mathcal{F})$ chosen as above, are

parametrised by an affine piece

$$\operatorname{Spec}\mathbb{C}\left[\frac{xz}{yt},\frac{xt}{yz},y^2,\frac{yzt}{x}\right] \quad\longleftrightarrow\quad \operatorname{Cone}\begin{pmatrix} \frac{1}{2}(1,1,0,0), \\ \frac{1}{2}(1,0,1,0), \\ \frac{1}{2}(1,0,0,1), \\ \frac{1}{2}(0,0,1,1), \end{pmatrix}$$

Similarly, we replace the pairs of G-clusters

$$\left.\begin{matrix}\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{xt},\mathbf{xy}\}\\ \{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{xt},\mathbf{zt}\}\end{matrix}\right\} \quad \text{by } \Gamma_6=\left\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{xt},\mathbf{yx}^{-1}\right\}$$

$$\left.\begin{matrix}\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xt},\mathbf{yt},\mathbf{xy}\}\\ \{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xt},\mathbf{yt},\mathbf{zt}\}\end{matrix}\right\} \quad \text{by } \Gamma_7=\left\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xt},\mathbf{yt},\mathbf{zt}^{-1}\right\}$$

$$\left.\begin{matrix}\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{yz},\mathbf{xy}\}\\ \{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{yz},\mathbf{zt}\}\end{matrix}\right\} \quad \text{by } \Gamma_8=\left\{\mathbf{1},\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{t},\mathbf{xz},\mathbf{yz},\mathbf{tz}^{-1}\right\}$$

and the three affine piece that parametrise such G-constellations correspond to the remaining three orange-slice cones in $\Sigma_{[12\text{-}34]}$. The G-constellations obtained from $\Gamma_1,\Gamma_2,\ldots,\Gamma_8$ are all θ-stable if the stability parameter $\theta=(\theta_{ijk})_{0\le i,j,k<2}$ satisfies

$$\begin{aligned} \theta_{ijk}>0, &\quad \text{for all } (i,j,k)\neq(000),(011) \\ \theta_{000},\ \theta_{011}<0, & \\ \theta_{011}+\theta_{111}>0, &\quad \theta_{011}+\theta_{010}>0, \\ \theta_{011}+\theta_{100}>0, &\quad \theta_{011}+\theta_{001}>0. \end{aligned}$$

Example 4.3.2. In a similar way, we list the G-constellations that correspond to points of $Y_{[12\text{-}34]}$, but now when $r=3$. Fix the octahedron O_P for $P=(1,0,0,0)$ and choose two antipodal points $\frac{1}{3}(2,1,0,0)$ and $\frac{1}{3}(1,0,1,1)$. As in the last example, we intend to go from eight "halfway up"/"halfway down" tetrahedra to four orange-slice tetrahedra, by removing the face cut out by $xy:(zt)^2$. One pair of cones from

$\Sigma_{\text{G-Hilb}}$ that needs to be "merged" is

$$\begin{aligned}&\operatorname{Cone}\left((3,1,1,1),(2,1,0,0),(2,0,1,0),(2,0,0,1)\right)\\&\operatorname{Cone}\left((3,1,1,1),(1,0,1,1),(2,0,1,0),(2,0,0,1)\right)\end{aligned}$$

The G-clusters corresponding to these two pieces are

$$\begin{aligned}\Gamma = \{&\mathbf{1},\mathbf{t},\mathbf{z},\mathbf{y},\mathbf{x},\mathbf{t^2},\mathbf{zt},\mathbf{yt},\mathbf{z^2},\mathbf{yz},\mathbf{y^2},\mathbf{zt^2},\mathbf{yt^2},\mathbf{z^2t},\mathbf{yzt},\mathbf{y^2t},\mathbf{yz^2},\mathbf{y^2z},\mathbf{z^2t^2},\\&\mathbf{yzt^2},\mathbf{y^2t^2},\mathbf{yz^2t},\mathbf{y^2zt},\mathbf{y^2z^2},\mathbf{yz^2t^2},\mathbf{y^2zt^2},\mathbf{y^2z^2t}\}\\\Gamma' = \{&\mathbf{1},\mathbf{t},\mathbf{z},\mathbf{y},\mathbf{x},\mathbf{t^2},\mathbf{zt},\mathbf{yt},\mathbf{z^2},\mathbf{yz},\mathbf{y^2},\mathbf{xy},\mathbf{zt^2},\mathbf{yt^2},\mathbf{z^2t},\mathbf{yzt},\mathbf{y^2t},\mathbf{yz^2},\mathbf{y^2z},\\&\mathbf{xy^2},\mathbf{yzt^2},\mathbf{y^2t^2},\mathbf{yz^2t},\mathbf{y^2zt},\mathbf{y^2z^2},\mathbf{y^2zt^2},\mathbf{y^2z^2t}\}\end{aligned}$$

The elements of these two G-clusters differ only in two places: the second one has monomials xy and xy^2, where the other one has z^2t^2 and yz^2t^2. We construct a set

$$\Gamma'' = \left(\Gamma \cap \Gamma'\right) \cup \left\{xy^{-2}, xy^{-1}\right\}.$$

By deforming this constellation, we get the affine variety corresponding to the orange-slice cone $\operatorname{Cone}\left((2,1,0,0),(2,0,1,0),(2,0,0,1),(1,0,1,1)\right)$.

We can repeat the same process for other three pairs of cones of G-Hilb $\left(\mathbb{C}^4\right)$ that lie within the same octahedron, and then again for the remaining three octahedra. All the G-constellation we obtain this way impose that θ_{0ii} must be negative.

4.4 Towards McKay correspondence

The notion of McKay correspondence says that whenever a finite group G acts on a variety M, the crepant resolutions of the quotient give information about the G-equivariant geometry of M. The idea started with observation of McKay [23] that there is a one-to-one correspondence between the nontrivial irreducible representations of $G \subset \mathrm{GL}(2,\mathbb{C})$ and the exceptional prime divisors of the minimal resolution

of the Kleinian singularity $\mathbb{C}^2/G$. Following this observation, Gonzales-Sprinberg and Verdier [17] introduced the the **tautological sheaves** associated to each non-trivial irreducible representation of G whose first Chern classes give the basis for integral cohomology of the resolution. To obtain the McKay correspondence in higher dimensions, Reid [27] suggested a recipe that generalises their construction to higher dimensions:

Conjecture 4.4.1 (Reid's second McKay conjecture)**.** *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite group and suppose that $Y = \text{G-Hilb}(\mathbb{C}^n)$ is a crepant resolution of the quotient $X := \mathbb{C}^n/G$. Then*

(i) the locally free Gonzalez-Sprinberg and Verdier sheaves $\mathcal{F}_i$ on Y form a $\mathbb{Z}$-basis of the K-theory of Y.

(ii) "Reid's recipe" leads to a $\mathbb{Z}$-basis of $H^*(Y, \mathbb{Z})$ for which the following bijection holds:

$$\{\textit{irreducible representations of } G\} \longleftrightarrow \textit{basis of } H^*(Y, \mathbb{Z}). \tag{4.1}$$

Part (i) of the conjecture was proven for abelian groups $G \in \mathrm{SL}(3, \mathbb{C})$ by Ito and Nakajima [19], and later Bridgeland, King and Reid [2] proved a stronger statement involving derived categories for all finite groups $G \subset \mathrm{SL}(3, \mathbb{C})$. The second part of the conjecture was proven by Craw [4] for abelian groups in dimension three.

However, not much is known in higher dimension. Can we apply Reid's recipe on the crepant resolution $Y = Y^{[12\text{-}34]} \to \mathbb{C}^4/G$, where $G \simeq (\mathbb{Z}/r)^{\oplus 3}$ acts by

$$\tfrac{1}{r}(1, -1, 0, 0) \oplus \tfrac{1}{r}(1, 0, -1, 0) \oplus \tfrac{1}{r}(1, 0, 0, -1)$$

to get the basis for $H^*(Y, \mathbb{Z})$? The following two examples are naive attempts to mark the compact exceptional subvarieties of the crepant resolution by group characters, following the method of [27] and [4]. In both cases each character appears once on the fan $\Sigma_{[12\text{-}34]}$.

Example 4.4.2. Suppose that $r = 2$ and choose the crepant resolution $Y_{[12\text{-}34]}$. The up tetrahedron with vertex e_1 has a single interior face that is cut out by the G-invariant ratio of monomials $x_1 : x_2x_3x_4$. As the two monomials defining this ratio are in the character space L_{111}, we mark this face by χ_{111}. The other three "up" tetrahedra are the same: they all have a single face that is interior to the fan $\Sigma_{[12-34]}$, cut out by a ratio $x_i : x_jx_kx_l$, where $i \in \{2,3,4\}$ and $\{j,k,l\} = \{1,2,3,4\} \setminus \{i\}$. As

$$\begin{aligned}
x_2,\, x_1x_3x_4 \in L_{100} \quad &\longrightarrow \quad \text{mark the corresponding face by } \chi_{100},\\
x_3,\, x_1x_2x_4 \in L_{010} \quad &\longrightarrow \quad \text{mark the corresponding face by } \chi_{010},\\
x_4,\, x_1x_2x_3 \in L_{001} \quad &\longrightarrow \quad \text{mark the corresponding face by } \chi_{001}.
\end{aligned}$$

There are two more internal faces in the fan of the resolution: the remaining faces of the orange-slice tetrahedra: we mark the one cut out by the ratio $x_1x_3 : x_2x_4$ by χ_{101} and the one cut out by $x_1x_4 : x_2x_3$ we mark by χ_{110}. This way, all the compact exceptional curves ($\simeq \mathbb{P}^1$) are given a label.

There is only one internal edge in the fan $\Sigma_{[12\text{-}34]}$, namely the edge where the four orange-slice tetrahedra meet. We can see from the fan that this edge corresponds to a copy of $\mathbb{P}^1 \times \mathbb{P}^1$. The edge is an intersection of the faces already marked by χ_{101} and χ_{110} and we mark the edge by $\chi_{101} \otimes \chi_{110} = \chi_{011}$. Notice that we have marked all the compact exceptional strata with nontrivial characters of the group, and that every character appears exactly once, either marking a $\mathbb{P}^1$ or a $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 4.4.3. Suppose that $r = 3$ and the crepant resolution is $Y^{[12\text{-}34]} \to X$. Start with the 1-dimensional torus-invariant subspaces: the faces of the "up" and "down" tetrahedra are all cut out by the ratios: $x_i^2 : x_jx_kx_l$ and $x_i : (x_jx_kx_l)^2$. Mark

them with the corresponding characters

$$x_1, (x_2x_3x_4)^2 \in L_{222} \longrightarrow \chi_{222}, \qquad x_1^2, x_2x_3x_4 \in L_{111} \longrightarrow \chi_{111},$$
$$x_2, (x_1x_3x_4)^2 \in L_{200} \longrightarrow \chi_{200}, \qquad x_2^2, x_1x_3x_4 \in L_{100} \longrightarrow \chi_{100},$$
$$x_3, (x_1x_2x_4)^2 \in L_{020} \longrightarrow \chi_{020}, \qquad x_3^2, x_1x_2x_4 \in L_{010} \longrightarrow \chi_{010},$$
$$x_4, (x_1x_2x_3)^2 \in L_{002} \longrightarrow \chi_{002}, \qquad x_4^2, x_1x_2x_3 \in L_{001} \longrightarrow \chi_{001}.$$

There are four more compact exceptional curves: the ones determined by the faces of the orange-slice tetrahedra, marked by:

$$x_1x_3, (x_2x_4)^2 \in L_{202} \longrightarrow \chi_{202}, \qquad (x_1x_3)^2, x_2x_4 \in L_{101} \longrightarrow \chi_{101},$$
$$x_1x_4, (x_2x_3)^2 \in L_{220} \longrightarrow \chi_{220}, \qquad (x_1x_4)^2, x_2x_3 \in L_{110} \longrightarrow \chi_{110}.$$

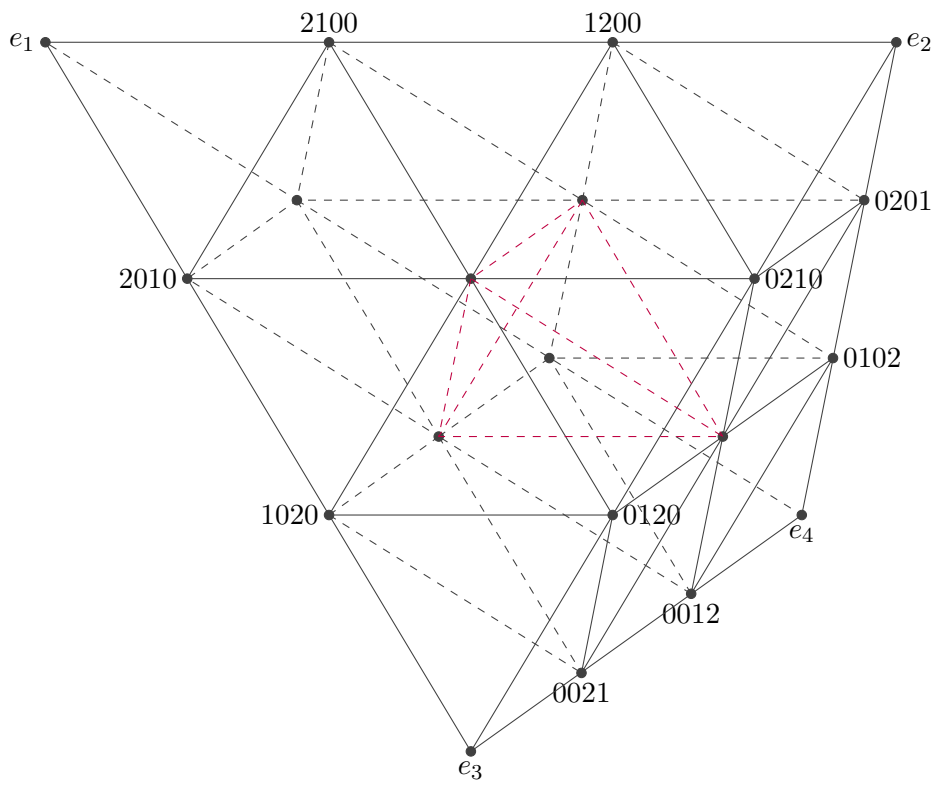


Figure 4.1: Partial subdivision of junior simplex for $r = 3$

This gives total of 12 curves. There are four edges that are intersections of the four neighbouring orange-slice cones, all of them are copies of $\mathbb{P}^1 \times \mathbb{P}^1$.

- The edge $\overline{(2,1,0,0),(1,0,1,1)}$ is an intersection of the faces cut out by the ratios $x_1x_3 : (x_2x_4)^2$ and $x_1x_4 : (x_2x_3)^2$. The monomials forming the first ratio belong to the eigenspace L_{101} and the monomials from the second ratio are in the eigenspace L_{110} so we mark the edge by the product of characters: $\chi_{110} \otimes \chi_{101} = \chi_{211}$.

- The edge $\overline{(1,2,0,0),(0,1,1,1)}$ is the intersection of the faces cut out by the ratios $(x_1x_3)^2 : x_2x_4$ and $(x_1x_4)^2 : x_2x_3$. Memebers of these two ratios belong to the eigenspaces L_{202} and L_{220} respectively, so we mark this copy of $\mathbb{P}^1 \times \mathbb{P}^1$ by $\chi_{220} \otimes \chi_{202} = \chi_{122}$.

- The edge $\overline{(1,1,1,0),(0,0,2,1)}$ is the intersection of the faces cut out by the ratios $x_1x_3 : (x_2x_4)^2$ and $(x_1x_4)^2 : x_2x_3$. We mark it by $\chi_{101} \otimes \chi_{220} = \chi_{021}$.

- The edge $\overline{(1,1,0,1),(0,0,1,2)}$ is the intersection of the faces cut out by the ratios $(x_1x_3)^2 : x_2x_4$ and $x_1x_4 : (x_2x_3)^2$ so mark the edge by: $\chi_{202} \otimes \chi_{110} = \chi_{012}$.

There are two more exceptional surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. They correspond to the edges that are lying in the direction of $\overline{e_1e_2}$ or $\overline{e_3e_4}$. Edge $\overline{(1,1,1,0),(1,1,0,1)}$ is at the intersection of two planes cut out by $x_1^2 : x_2x_3x_4$ and $x_2^2 : x_1x_3x_4$, so it is marked by $\chi_{222} \otimes \chi_{100} = \chi_{022}$. $\overline{(1,0,1,1),(0,1,1,1)}$ is at the intersection of planes cut out by $x_3^2 : x_1x_2x_4$ and $x_4^2 : x_1x_2x_3$ so is marked by $\chi_{010} \otimes \chi_{001} = \chi_{011}$.

Edge $\overline{(1,1,1,0),(1,0,0,1)}$ has three planes passing through it, cut out by ratios $x_1^2 : x_2x_3x_4$, $x_3^2 : x_1x_2x_4$ and $(x_2x_4)^2 : x_1x_3$. It corresponds to an exceptional del Pezzo surface of degree 6. To obtain two maps $\mathrm{dP}_6 \to \mathbb{P}^2$, we first realise dP_6 as the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^7_{\langle w_{000}, w_{001}, \ldots, w_{117} \rangle}$ given by the three ratios. The image of the map is

$$\big[(x_1x_2x_3x_4)^2 : x_1^3x_3^3 : (x_1x_2x_4)^3 : x_1^4x_2x_3x_4 :$$
$$(x_2x_3x_4)^3 : x_1x_2x_3^4x_4 : x_1x_2^4x_3x_4^4 : (x_1x_2x_3x_4)^2\big]$$

and the two maps from this exceptional surface to $\mathbb{P}^2$ are then given by the ratios

$$(w_{000} : w_{001} : w_{101}) = \left(x_1 (x_2x_4)^2 : x_1^2x_3 : x_2x_3^2x_4\right)$$

and

$$(w_{111} : w_{110} : w_{010}) = \left(x_1x_3^2 : x_3 (x_2x_4)^2 : x_1^2x_2x_4\right).$$

We mark the exceptional surface by the two characters determined by the elements of the ratios: χ_{212} and χ_{121}.

Similarly, the edge $\overline{(1,1,1,0),(0,1,1,1)}$ also defines a del Pezzo surface of degree 6 with three maps to $\mathbb{P}^1$ determined by the ratios $x_2^2 : x_1x_3x_4$, $x_3^2 : x_1x_2x_4$ and $(x_1x_4)^2 : x_2x_3$. The two maps to $\mathbb{P}^2$ are defined by ratios

$$\left(x_2 (x_1x_4)^2 : x_2^2x_3 : x_1x_3^2x_4\right) \quad \text{and} \quad \left(x_2x_3^2 : x_1^2x_3x_4^2 : x_1^2x_2x_4\right)$$

and we mark it by χ_{120} and χ_{210}.

The method for marking the remaining two edges $\overline{(1,1,0,1),(1,0,1,1)}$ and $\overline{(1,1,0,1),(0,1,1,1)}$ is the same as above two cases, and get marked by characters $\chi_{221} \oplus \chi_{112}$ and $\chi_{102} \oplus \chi_{201}$.

Again, every nontrivial character appears only once on the fan $\Sigma_{[12\text{-}34]}$: a single character marks a torus invariant $\mathbb{P}^1$ and invariant surface $\mathbb{P}^1 \times \mathbb{P}^1$, and two characters mark a torus invariant del Pezzo surface of degree 6.

In order to obtain such marking for $r \geq 4$, one first needs to understand the geometry of the exceptional divisors. As an example, when $r = 4$, there is a single internal lattice point $P := \frac{1}{4}(1,1,1,1)$, and its corresponding ray is surrounded by 24 four-dimensional cones in $\Sigma_{[12\text{-}34]}$. There are 18 curves, 9 $\mathbb{P}^1 \times \mathbb{P}^1$'s arising from meeting edge of orange-slice tetrahedra, 6 other $\mathbb{P}^1 \times \mathbb{P}^1$'s and 12 del Pezzo surfaces. One can check that they all have different characters. That is a total of $1 + 18 + 9 + 6 + 2 \cdot 12 = 58$ characters, which leaves the single divisor D_P marked by six characters:

$$\chi_{023}, \chi_{032}, \chi_{122}, \chi_{311}, \chi_{213}, \chi_{231}.$$

4.5 Inflation

Let $A \in \mathrm{SL}(4, \mathbb{C})$ be a finite Abelian group. Define the **r-th inflation** $\frac{1}{r}A$ of the group A as the group of diagonal matrices

$$\tfrac{1}{r}A = \{M \text{ diagonal } \mid M^r \in A\} \subset \mathrm{SL}(4, \mathbb{C}).$$

The results of this thesis motivate the conjecture

Conjecture 4.5.1. *If the quotient X of $\mathbb{C}^4$ by the group A has crepant resolution, then the quotient X_r of $\mathbb{C}^4$ by the group $\frac{1}{r}A$ also has a crepant resolution.*

4.6 Higher dimension

Same questions we have studied, or suggested to be studied, in the four dimensional case, may as well be posed for higher dimension. It may not be sensible to expect that one could easily generalize the the results made so far in smaller dimensions, but the symmetry inherent to the problem makes it intriguing.

Let $G = (\mathbb{Z}/r)^{\oplus(n-1)}$ be the maximal diagonal Abelian group of exponent r acting on the affine complex space $\mathbb{C}^n$, for integers $r \geq 2$ and $n \geq 4$, by

$$\alpha_i \mapsto \mathrm{diag}\left(\varepsilon, 1, \ldots, 1, \varepsilon^{-1}, 1, \ldots, 1\right), \tag{4.2}$$

with ε^{-1} being in the $(i+1)$-st place and $\varepsilon = e^{\frac{2\pi i}{r}}$, the primitive r-th root of unity. Set the coordinates of $\mathbb{C}^n$ to be $x_1, x_2, \ldots, x_n$. The affine quotient $X = \mathbb{C}^n/G$ is by definition equal to $\mathrm{Spec}\,\mathbb{C}[x_1, x_2, \ldots, x_n]^G$. Analogous to dimension four, the invariant ring is

$$\mathbb{C}[x_1^r, \ldots, x_n^r, x_1 x_2 \ldots x_n] \simeq \mathbb{C}[X_1, \ldots X_n, Y] / (X_1 X_2 \ldots X_n - Y^r),$$

so X can be embedded as a hypersurface in $\mathbb{C}^{n+1}$ and X is also toric: it is given by

a cone $\sigma = \mathrm{Cone}\,(e_1, e_2, \ldots, e_n)$ in the lattice

$$N = \mathbb{Z}^n_{\langle e_1,\ldots,e_n\rangle} + \tfrac{1}{r}\,(1,-1,0,\ldots,0) + \ldots + \tfrac{1}{r}\,(1,0,\ldots,0,-1)$$

Dais, Henk and Ziegler have shown in [10] that a crepant resolution of the singularity X exists in all dimensions. It would be interesting to know if the crepant resolutions have a moduli space interpretation. There is no reason to expect that G-Hilb $(\mathbb{C}^n)$ is a crepant resolution of singularities, but it might (or might not be at all) closely related to a crepant resolution. The worked example 4.6.2 shows that G-Hilb $(\mathbb{C}^n)$ does not have to be neither crepant nor smooth.

The relations generating a G-cluster have a very nice description 3.4 as seen in the statement of Theorem 3.4.1. A natural idea comes to mind of how one might generalise the part one of that theorem to higher dimensions:

Conjecture 4.6.1. *Let Z be a G-cluster. The generators of the ideal $\mathcal{I}_Z$ can be chosen as $2^n - 1$ equations:*

$$\left\{ \left(\prod_{i\in S} x_i\right)^{a_S} = \lambda_S \cdot \left(\prod_{j\notin S} x_j\right)^{r-a_S} \;\middle|\; \emptyset \neq S \subseteq \{1,2,\ldots,n\} \right\} \tag{4.3}$$

such that

- $a_S + a_{S^c} = r + 1$,
- $a_{S'} \leq a_S$ *whenever* $S \subseteq S'$ *and*
- $\pi = \lambda_S \lambda_{S^c}$ *for all the subsets* S, S' *of* $\{1,2,\ldots,n\}$.

It should be fairly easy to generalise Lemma 3.5.1 to show that the relations of the form 4.3, satisfying the first bulletpoint, do exist in the defining ideal $\mathcal{I}_Z$ of a G-cluster Z. However, existence of the relations satisfying all the three bulletpoints might require more work, and even then, this is only the first step in parametrising a G-cluster.

Example 4.6.2 (Dimension 5, $r = 2$.)**.** We use an adaptation of the algorithm from [25] (see Appendix 4.6) to compute the torus invariant G-clusters where the group $G \cong (\mathbb{Z}/2)^{\oplus 4}$ acts by (4.2). The computation shows that there are 81 affine pieces, 11 of which are singular.

Similar to the lower-dimensional cases, there are 5 smooth "corner" cones, that is cones that share a vertex with the junior simplex. For example, the cone with a ray through e_1 is

$$\mathrm{Cone}\begin{pmatrix} \frac{1}{2}(2, & 0, & 0, & 0, & 0) \\ \frac{1}{2}(1, & 1, & 0, & 0, & 0) \\ \frac{1}{2}(1, & 0, & 1, & 0, & 0) \\ \frac{1}{2}(1, & 0, & 0, & 1, & 0) \\ \frac{1}{2}(1, & 0, & 0, & 0, & 1) \end{pmatrix}$$

and the other four are obtained by permuting the columns in the array above. Once these five cones are cut off, we are left with a 4-dimensional polyhedron with $\binom{5}{2} = 10$ vertices $A_{ij} := \frac{1}{2}(e_i + e_j)$, where $i, j \in \{1, 2, 3, 4, 5\}$ are distinct.

There are five divisors of age 2 of the form $B_i := \frac{1}{2}\left(\sum_{j=1}^{5} e_j - e_i\right)$, and there are five age 3 divisors $C_i := \frac{1}{2}\left(\sum_{j=1}^{5} e_j + e_i\right)$, where $i \in \{1, \ldots, 5\}$.

In the rest of the section, the indices i, j, k, l, m are always distinct elements of $\{1, 2, 3, 4, 5\}$. The other smooth cones are

- The five cones obtained by permuting the indices of

$$\mathrm{Cone}\,(C_1, A_{12}, A_{13}, A_{14}, A_{15}) = \mathrm{Cone}\begin{pmatrix} \frac{1}{2}(2, & 1, & 1, & 1, & 1) \\ \frac{1}{2}(1, & 1, & 0, & 0, & 0) \\ \frac{1}{2}(1, & 0, & 1, & 0, & 0) \\ \frac{1}{2}(1, & 0, & 0, & 1, & 0) \\ \frac{1}{2}(1, & 0, & 0, & 0, & 1) \end{pmatrix}$$

- $\mathrm{Cone}\,(C_i, B_j, A_{ik}, A_{il}, A_{im})$. There are 20 such cones.

- $\mathrm{Cone}\,(C_i, B_j, B_k, A_{il}, A_{im})$. There are 30 such cones.
- $\mathrm{Cone}\,(B_i, B_j, A_{kl}, A_{km}, A_{lm})$. There are 10 such cones.

The remaining 11 cones are singular. 10 of them are of the form

$$S_{ij} := \mathrm{Cone}\,(C_i, C_j, B_k, B_l, B_m, A_{ij})\,.$$

For each of these singular cones the semigroup $M \cap S_{ij}^{\vee}$ has seven generators. When the cone is S_{12}, the generators are:

$$\frac{x_1x_2}{x_3x_4x_5}, \frac{x_1x_3x_4}{x_2x_5}, \frac{x_1x_3x_5}{x_2x_4}, \frac{x_1x_4x_5}{x_2x_3}, \frac{x_2x_3x_4}{x_1x_5}, \frac{x_2x_3x_5}{x_1x_4}, \frac{x_2x_4x_5}{x_1x_3}.$$

Finally, the central cone is

$$C = \mathrm{Cone}\,(C_1, C_2, C_3, C_4, C_5, B_1, B_2, B_3, B_4, B_5)\,.$$

The semigroup associated to the dual of the central cone is generated by 10 elements $(x_ix_jx_k)\,/\,(x_lx_m)$, for all distinct i, j, k, l, m.

The computation also shows that all the internal edges are cut out by the G-invariant ratios either of the form $x_i : x_jx_kx_lx_m$ or $x_ix_j : x_kx_lx_m$, for all the permutations $\{i, j, k, l, m\}$ of indices.

We can use the same computer algorithm to compute the torus-invariant G-clusters for $G \simeq (\mathbb{Z}/r)^{\oplus 4}$ when $r = 3, 4$. The number of G-invariant clusters is 471 when $r = 3$, and 1556 when $r = 4$. The table below shows that there is a certain pattern to the Euler number of the G-Hilbert scheme for the group action (4.2) in dimensions 2 to 4. Based on this, is it possible that the formula is similar in dimension 5?

n	$\chi\left(\text{G-Hilb}\left(\mathbb{C}^n\right)\right)$
2	$r = \binom{r}{1}$
3	$r^2 = \binom{r+1}{2} + \binom{r}{2}$
4	$\binom{r+2}{3} + 8\binom{r+1}{3} + \binom{r}{3}$
5	$\binom{r+3}{4} + \alpha\binom{r+2}{4} + \beta\binom{r+1}{4} + \binom{r}{4}$

By filling in the known values for $r = 2, 3, 4$ we get the folowing guess:

Conjecture 4.6.3. *Let the group $G \simeq (\mathbb{Z}/r)^{\oplus 4}$ act on $\mathbb{C}^5$ by* (4.2)*. Then the Euler number of* G-Hilb $\left(\mathbb{C}^5\right)$ *is*

$$\binom{r+3}{4} + 76\binom{r+2}{4} + 76\binom{r+1}{4} + \binom{r}{4}.$$

Appendix A

Code

Here we present the code used for computing all the torus invariant G-clusters, for the group $G = (\mathbb{Z}/r)^{\oplus n-1}$ acting on $\mathbb{C}^n$ by (4.2). The code is written in Sage [14]. At the end we show the basic usage of the class.

A.1 The `SymQuotSing` class

The main class is called `SymQuotSing` and it represents the quotient variety $\mathbb{C}^n/G$. An object of type `SymQuotSing` is initialized by two variables: the exponent of the group `r` and the dimension `dim` $= n$ of the affine space. During the initialization, a variable storing the order of the group is created, as well as the polynomial ring $\mathbb{C}[x_1, x_2, \ldots, x_n]$. For clarity, in dimensions up to five, the variables have names x, y, z, t, w instead of x_i. The code uses the packages `os.path` for accessing the file tree, `cPickle` for storing the computed data in the memory, and `random` that improves the output of `__relations` method, which have to be imported prior to running the script.

```
class SymQuotSing(object):
    def __init__(self, r, dim=4):
        self.r = r
        self.dim = dim
        self.ord = r**(dim-1)
        self.__L = ZZ**self.dim
```

```
        if self.dim == 2:
            self.__Q = PolynomialRing(QQ, 2, 'xy')
        elif(self.dim == 3):
            self.__Q = PolynomialRing(QQ, 3, 'xyz')
        elif(self.dim == 4):
            self.__Q = PolynomialRing(QQ, 4, 'xyzt')
        elif(self.dim == 5):
            self.__Q = PolynomialRing(QQ, 5, 'xyztw')
        else:
            self.__Q = PolynomialRing(QQ, self.dim, 'x')
        self.__Q.inject_variables()

        createDir('__EigSps')
        createDir('__ASets')
        createDir('__Relations')
        createDir('__AHilb')
```

All the methods that follow are defined within the SymQuotSing class. To improve efficiency, the class may internally store data for future usage without the need for recomputation. `__str__` creates a string that describes the created object, while `filename_str` creates a string used for storing the computed data.

```
    def __str__(self):
        return "Quotient of CC^"+str(self.dim)+" by the \
                 group (ZZ/"+str(self.r)+")^"+str(self.dim-1)

    def filename_str(self):
        return "sym-"+str(self.dim)+'-'+str(self.r)

    def __ZBasis(self):
        basis = []
        c = self.__L.basis()
        for i in range(self.dim):
            basis.append(self.r*c[i])
        return basis
```

The last method above, `__ZBasis`, creates a list of `dim` vectors that are basis of the sublattice $\mathbb{Z}^n \subset L$. All the lattice points are printed out as their r-th multiple, to avoid dealing with fraction $\frac{1}{r}$.

The private recursive method `__latptRec` computes a list of the lattice points in

$$L = \mathbb{Z}^n \oplus \frac{1}{r}(1,-1,0,\ldots,0) \oplus \frac{1}{r}(1,0,-1,\ldots,0) \oplus \frac{1}{r}(1,0,0,\ldots,-1)$$

that are contained in the junior simplex and stores the list into an empty list basket. With the public method `LatticePoints` we can return the data stored in `basket` without the need to state all the private arguments of `__latptRec` that is called internally.

```
    def __latptRec(self,current_vect, remaining_n, \
                                      remaining_r, basket):
        if remaining_n == 0 and remaining_r == 0:
            basket.append(current_vect)
            return
        if remaining_n < 0 or remaining_r <0:
            return

        for i in range(remaining_r+1):
            vs = current_vect + [i]
            self.__latptRec(vs, remaining_n - 1, \
                                remaining_r - i,  basket)
        return

    def LatticePts(self):     #, below = false):
        S = []
        self.__latptRec([], self.dim, self.r, S)

        S.sort();
        return S
```

The next method `weight` takes a monomial and returns the index of the eigenspace it belongs to. The argument `vect` can be either a monomial or an array of its exponents in lexicographical order. The eigenspaces $L_{a_1a_2a_3\ldots a_{n-1}}$ of the group action are labelled by the $n-1$ values $a_i \in \{0,1,\ldots,r-1\}$. The function first computes the $(n-1)$-tuple $a_1a_2\ldots a_{n_1}$ and in the next step treats it as an integer written in base r. The return value is the value of this integer in decimal base.

```
  def weight(self, vect):
     wt = 0
     if hasattr(vect, "exponents"):
        vect = vect.exponents()[0]
     mult = 1
     for i in range(self.dim-1,0,-1):
        wt += ((vect[0] - vect[i])%self.r)*mult
        mult *= self.r
     return wt
```

The following two methods are used to compute the minimal generators of each eigenspace, viewed as a module over the invariant ring. We run through all of the monomials dividing $(x_1x_2\dots x_n)^r$ and put them in eigenspaces they belong to. In `EigSp`, the method checks whether the list of eigenspaces has already been computed, that is if a file "`sym-dim-r__eigsps.p`" exists in the folder `__EigSps`. If yes, the data will just be read and returned. Otherwise, the private method `__eigspRecursion` is called, and the resulting list of lists of generators of the eigenspaces is stored in the previously mentioned file for future use.

```
 def __eigspRecursion(self, ind, currentExponents, EigSp):
    if ind == self.dim -1:
       monomial = 1
       for k in range(self.dim):
          monomial *= self.__Q.gen(k)**currentExponents[k]
       eig = self.weight(monomial)
       survived = true
       for i in range(len(EigSp[eig])):
          if EigSp[eig][i] <> [0]*self.dim and  \
             greaterThan(currentExponents, EigSp[eig][i]):
             survived = false
             break
       if survived:
          EigSp[eig].append(currentExponents)
       return

    for j in range(self.r):
       self.__eigspRecursion(ind+1, \
                              currentExponents + [j], EigSp)

```

```
def EigSp(self, exponents_only=false):
    fileName ='__EigSps/'+self.filename_str()+'__eigsps.p'
    if os.path.exists(filename):
        f = open(fileName, 'rb')
        EigSp = cPickle.load(f)
        f.close()

    else:
        EigSp = []
        for i in range(self.ord):
            EigSp.append([])
        for j in range(self.dim):
            t = [0]*self.dim
            t[j] = self.r
            EigSp[0].append(t)
         self.__eigspRecursion(-1, [], EigSp)
         f = open(fileName, 'wb')
         cPickle.dump(EigSp, f)
         f.close()
    if exponents_only:
        return EigSp
    for i in range(self.ord):
        for j in range(len(EigSp[i])):
            monom = 1
            for k in range(self.dim):
                monom *= self.__Q.gen(k)**EigSp[i][j][k]
            EigSp[i][j] = monom
    return EigSp
```

The key tree-traversal algorithm, producing all the monomial ideals in $\mathbb{C}[x_1, \ldots, x_n]$ that define a G-cluster is contained in the method `ASets`. It is a slight modification of the method of the same name in the Magma code written by Reid [25]. The main idea is that we pick a single monomial from the first nontrivial eigenspace (index 1 in the list of lists returned by `EigSp`), and add the remaining monomials from this eigenspace to the list of generators of an ideal `I`. We continue the same process with the remaining eigenspaces one by one and so on. The process either ends with exactly r^{n-1} chosen monomials, one from each eigenspace so the ideal `I` defines a G-cluster, or the ideal becomes "too big" and contains a whole following eigenspace, so it does not describe a G-cluster. In both cases, we backtrack one step, and choose

a different monomial from the previous eigenspace. As with the method `EigSp`, the data is stored into a file after being computed for the first time.

```
def ASets(self, exponents_only = false):
    fileName = '__ASets/'+self.filename_str()+'__a-sets.p'
    if os.path.exists():
        f = open(fileName, 'rb')
        asets = cPickle.load(f)
        f.close()
        for i in range(len(asets)):
            for j in range(len(asets[i])):
                monom = 1
                for k in range(self.dim):
                    monom *= self.__Q.gen(k)**asets[i][j][k]
                asets[i][j] = monom
        return asets

    EigSp = self.EigSp()
    I = []
    C = []
    M = []
    asets = []

    finished = false
    while not finished:

        S = exclude(EigSp[0],1)
        over = true
        max_i = -1
        for i in range(len(I)):
            S += exclude(EigSp[I[i]], C[i][M[i]])
            if len(C[i]) != M[i]+1:
                over = false
                max_i = i
        Id = self.__Q.ideal(S)
        Qbar = self.__Q.quotient_ring(Id)
        remaining = []
        exists_empty = false

        for i in range(1, self.ord):
            surviving_i = []
            for j in range(len(EigSp[i])):
                el = EigSp[i][j]
                if el not in Id:
```

```
                quot_el = Qbar.lift(Qbar.retract(el))
                surviving_i.append(quot_el)
            if len(surviving_i) != 1:
               remaining.append([i,surviving_i])
            if len(surviving_i) == 0:
               exists_empty = true

         if len(remaining) == 0:
            asets.append(Qbar.defining_ideal().\
                                    interreduced_basis())

         if len(remaining) != 0 and exists_empty == false:
            I.append(remaining[0][0])
            C.append(remaining[0][1])
            M.append(0)
         if len(remaining) == 0 or (len(remaining)!=0 and \
                                      exists_empty ==true):
            if over:
               finished = true
            else:
               broj = len(I) - max_i - 1
               remove_last(I, broj)
               remove_last(M, broj)
               remove_last(C, broj)
               M[max_i] += 1

      asets2 = []
      for i in range(len(asets)):
         asets2.append([])
         for j in range(len(asets[i])):
            asets2[i].append( asets[i][j].exponents()[0])
      f = open(fileName, 'wb')
      cPickle.dump(asets2, f)
      f.close()

      if exponents_only:
         return asets2

   return asets
```

The following method `EigenBases`, based on the corresponding `__eigenbases`, returns a list, each entry of which is a list of exactly `ord` monomials forming a basis for $\mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}_Z$ corresponding to the cluster Z defined by the ideal $\mathcal{I}_Z$

obtained from ASets.

```
def __eigenbasis(self, S):
    m = 1
    for i in range(self.dim):
        m *= self.__Q.gen(i)**self.r
    basis = Set( self.__Q.monomial_all_divisors(m) )
     for I in range(len(S)):
        opp = self.__Q.monomial_quotient(m, S[I])
        L = self.__Q.monomial_all_divisors(opp)
        basis -= Set(L)

    clus = [0]*self.ord

    for i in range(self.ord):
        mono = self.__Q.monomial_quotient(m, basis[i])
        ind = self.weight(mono)
        clus[ind] = mono
    return clus

def EigenBases(self):
    fileName = '__EigenBases/' + self.filename_str() + \
                                                \'__eigbas.p'
    if os.path.exists(fileName):
        f = open(fileName, 'rb')
        basket  = cPickle.load(f)
        f.close()

    else:
        basket = []
        A = self.ASets()
        for i in range(len(A)):
            basket.append( self.__eigenbasis(A[i]) )
        f = open(fileName, 'wb')
        cPickle.dump(basket, f)
        f.close()
     return basket
```

Once we obtain the bases of the vector space $\mathcal{O}_Z$, for a G-invariant cluster Z (using method EigenBases, we can deform the equations in $\mathcal{I}_Z$ and obtain an affine piece parametrising G-clusters with the origin being the torus invariant cluster Z. For each entry of the list EigenBases, method __relations returns a list of G-

invariant ratios of monomials that correspond to the coordinates of the affine piece. For example, when $r = 3$ and dimension $n = 4$, one of the ratios at index 1 looks like `[1,-1,-1,-1]` and this corresponds to the relation $x = \lambda yzt$ for some value of $\lambda \in \mathbb{C}$.

The method `__relations` creates a list of ratios in the following way. An element from the monomial ideal `X.ASets()[i]` is paired with an element from the basis of $\mathcal{O}_Z$ that lies in the same eigenspace. Once this is done, the function calls `__reduce_rels` to obtain a minimal set of relations, by removing the relations that are multiples of other relations from the list. To ensure the minimality, it randomly permutes the entries of the list containing the current relations, and runs the `__reduce_rels` again. Once no changes are made, the process stops. The function `Relations` simply iterates `__relations` over all the monomial ideals of G-clusters.

```
    def __reduce_rels(self, mat):
            M = matrix(mat)
        K = M.kernel().matrix().rows()
        indices = []

        for i in range(len(K)):
            npos = 0
            nneg = 0
            indp = indn = -1
            for j in range(len(mat)):
                if K[i][j] > 0:
                    npos += 1
                    indp = j
                else:
                    if K[i][j] < 0:
                        nneg += 1
                        indn = j
            if npos == 1  and K[i][indp] ==1 and nneg > 0:
                indices.append(indp)
            else:
                if npos > 0 and nneg == 1 and K[i][indn]==-1:
                    indices.append(indn)
        M = M.delete_rows(indices)
        return M.rows()
```

```
    def __relations(self, aset, basis):
        mat = []
        for i in range(len(aset)):
            bb = aset[i].exponents()[0]
            ind = self.weight(bb)
            cc = basis[ind].exponents()[0]
            #print bb, cc
            row = []
            for i in range(len(bb)):
                row.append(bb[i] - cc[i])
            mat.append( row )

        redmat = self.__reduce_rels(mat)
        if len(redmat) != self.dim:
            random.shuffle(redmat)
        while (redmat != mat):
            mat = redmat
            redmat = self.__reduce_rels(mat)
        for i in range(self.r):
            random.shuffle(redmat)
            mat = redmat
            redmat = self.__reduce_rels(mat)
        return mat

    def Relations(self):
        fileName = '__Relations/' + self.filename_str() + \
                                                    '__rels.p'
        if os.path.exists(fileName):
            f = open(fileName, 'rb')
            basket  = cPickle.load(f)
            f.close()

        else:
            basket = []
            A = self.ASets()
            B = self.EigenBases()
            for i in range(len(A)):
                basket.append( self.__relations(A[i], B[i]) )
            f = open(fileName, 'wb')
            cPickle.dump(basket, f)
            f.close()
        return basket
```

Finally, once the relations are computed, the private method `__affinepiece` checks whether there are exactly n generators. If this is true, the n-dimensional affine piece has exactly n coordinates so it must be a copy of $\mathbb{C}^n$. The method `_affinepiece` then takes the adjoint of the matrix which gives the vertices of the toric cone of the affine piece. As with the prior pair of private and public methods, the method `AHilbFan` iterates `__affinepiece` over all the computed G-clusters and returns a list of cones, where a cone is represented by a list of its vertices. In low dimensions, the data obtained from `AHilbFan` can be used to plot the fan, using the inbuilt Sage function `plot` or `plot3d`.

```
    def __affinepiece(self, mat):
        pts = []
        if len(mat) == self.dim:
            A = (1/self.r**(self.dim-2))*matrix(mat).adjoint()
            for i in range(self.dim):
                if A[0][i] < 0:
                    A *= -1
                    break
                if A[0][i] > 0:
                    break
            pts = A.columns()
        return pts

    def AHilbFan(self):
        fileName = '__AHilb/'+self.filename_str()+'__AHilb.p')
        if os.path.exists():
            f = open(fileName, 'rb')
            pieces  = cPickle.load(f)
            f.close()

        else:
            matrices = self.Relations()
            pieces = []
            for i in range(len(matrices)):
                pieces.append( self.__affinepiece(matrices[i]) )
            f = open(fileName, 'wb')
            cPickle.dump(pieces, f)
            f.close()
        return pieces
```

In addition to the class methods, there are several external methods we use. The function `createDir` takes a string as an argument and creates a directory with the name specified in the string.

```
def createDir(filename):
        try:
                if not os.path.exists(filename):
                        os.makedirs(filename)
        except OSError:
                print "Error: cannot create the folder"
```

The function `greaterThan` takes two vectors and returns `True` if every entry of the first vector is smaller or equal to the corresponding entry of the first vector. We use it to check whether a monomial divides another monomial in cases where monomials are represented by the list of their exponents.

```
def greaterThan(vector1, vector2):
        n = len(vector1)
        try:
                for i in range(n):
                        if (vector1[i] < vector2[i]):
                                return false
                return true
        except IndexError:
                print("Error: vectors of neq dimensions")
```

The final two functions, `exclude` and `remove_last` deal with lists. The first one `exclude` takes two arguments: a list and a potential element of a list. It returns a copy of the list, but without the element from the argument. Notice that it does not change the original list. The function `remove_last`, however, does change the list it takes as an argument, and simply removes the last `n` elements from it.

```
def exclude(lis, element):
        copyL = []
        for i in range(len(lis)):
                if lis[i] != element:
```

```
                        copyL.append(lis[i])
        return copyL

def remove_last(lis, n):
        for i in range(n):
                lis.pop()
```

A.2 Usage

Let a group $G = (\mathbb{Z}/2)^{\oplus 3}$ act on $\mathbb{C}^4$ by (4.2). To define an object of type `SymQuotSing` corresponding to this quotient variety, one needs to pass the value r and the dimension to the constructor:

```
\begin{minted}{python}
sage: X = SymQuotSing(2,4)
Defining x, y, z, t
sage: print X
Quotient of CC^4 by the group (ZZ/2)^3
```

If one later needs to check the dimension, the exponent r of the group, or its order, type:

```
sage: X.dim
4
sage: X.r
2
sage: X.ord
8
```

The method `LatticePts` lists all the lattice points contained in the junior simplex. The output `[0, 0, 1, 1]` refers to the point $\frac{1}{2}(0,0,1,1)$.

```
sage: X.LatticePts()
[[0, 0, 0, 2], [0, 0, 1, 1], [0, 0, 2, 0], [0, 1, 0, 1],
 [0, 1, 1, 0], [0, 2, 0, 0], [1, 0, 0, 1], [1, 0, 1, 0],
 [1, 1, 0, 0], [2, 0, 0, 0]]
```

The eigenspaces of the group action can be obtained by simply typing `X.EigSp()`. As we can see, the first entry, `X.EigSp()[0]`, consists of the generators of the ring of invariants, while the others are generators of the nontrivial eigenspaces over the ring of invariants.

```
sage: X.EigSp()
[[x^2, y^2, z^2, t^2, 1, x*y*z*t],
 [t, x*y*z],
 [z, x*y*t],
 [z*t, x*y],
 [y, x*z*t],
 [y*t, x*z],
 [y*z, x*t],
 [y*z*t, x]]
```

Running any of the commands `X.ASets()`, `X.EigenBases`, `X.Relations()` or `X.AHilb()` prints the long lists of the data. All of this four lists have the same length, determining the Euler number for the irreducible variety $\mathrm{Hilb}^{\mathrm{G}}(\mathbb{C}^n)$ that this program computes. Below we check how many affine pieces there are for the object `X` and we print the first three monomial ideals.

```
sage: len( X.ASets() )
27
sage: X.ASets()[0:3]
[[y^2, z^2, t^2, x],
 [y*z*t, x^2, x*y, y^2, x*z, z^2, x*t, t^2],
 [x^2, x*y, y^2, x*z, y*z, z^2, t^2]]
```

Finnaly, we show how to list all the data corresponding to a G-cluster: the generators of the ideal, the monomial basis, the relations and the vertices of the toric cone of the affine piece. Here we do so for the first two entries:

```
sage: for i in range(0,2):
....:     print i
....:     print "ideal: ", X.ASets()[i]
....:     print "basic monomials: ", X.EigenBases()[i]
....:     print "relations: ", X.Relations()[i]
```

```
....:     print "toric cone: ",  X.AHilb()[i]
....:     print " "
....:
0
ideal:  [y^2, z^2, t^2, x]
basic monomials:  [1, t, z, z*t, y, y*t, y*z, y*z*t]
relations: [(0, 0, 0, 2),
            (0, 2, 0, 0),
            (0, 0, 2, 0),
            (1, -1, -1, -1)]
toric cone: [(1, 0, 0, 1),
             (1, 1, 0, 0),
             (1, 0, 1, 0),
             (2, 0, 0, 0)]

1
ideal:  [y*z*t, x^2, x*y, y^2, x*z, z^2, x*t, t^2]
basic monomials:  [1, t, z, z*t, y, y*t, y*z, x]
relations:  [(1, -1, 1, -1),
             (-1, 1, 1, 1),
             (1, -1, -1, 1),
             (1, 1, -1, -1)]
toric cone:  [(1, 0, 1, 0),
              (1, 1, 1, 1),
              (1, 0, 0, 1),
              (1, 1, 0, 0)]
```

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