TITLE Asymptotic formulae for closed orbits of hyperbolic flows

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Contents.

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Declaration.

The material in this thesis is original except where stated otherwise. Chapter 1 has been submitted to the Israel Journal of Mathematics. Chapter 2 has been submitted to Mathematische Zeitschrift.

Summary.

This thesis consists of three chapters and an appendix each with its own notation and references.

Chapter 0 is an introduction which sets out the definitions and results needed in the main part of the thesis.

In Chapter 1 we prove an analogue of Mertens' theorem of prime number theory for the closed orbits of an Axiom A flow (restricted to a non-trivial basic set). A similar result is established for the geodesic flow on a non-compact, finite area surface of constant negative curvature. Applying this result to the modular suface yields some asymptotic formulae concerning quadratic forms.

In Chapter 2 we give a new proof of an asymptotic formula for the number of closed orbits of a weak-mixing Axiom A flow subject to certain constaints due to S. Lalley. We extend this result to cover the case of finite group extensions and, for transitive Anosov flows, give an application to homology. We also discuss asymptotics for closed orbits in a fixed homology class, extending a result of A. Katsuda and T. Sunada.

The appendix, which is included for completeness, is an account of the proof of a technical result of A. Katsuda and T. Sunada which is used extensively in Chapter 2.

Chapter 0.
Introduction.

1. Axiom A flows.

The subject of this thesis is Axiom A flows and, more particularly, the distribution of their closed orbits. These flows (and the analogous Axiom A diffeomorphisms) were introduced by Smale in his influential 1967 paper [32]. They are C^1 flows on a compact C^∞ Riemannian manifold which when restricted to the *non-wandering set* – the set where the interesting dynamics takes place – satisfy a hyperbolicity condition.

We shall now be more precise. Let φ be a C^1 flow on the compact C^∞ Riemannian manifold M. A compact φ-invariant set Λ without fixed points is called *hyperbolic* if the tangent bundle of M resricted to Λ has a continuous splitting as the Whitney sum of three $D\varphi$-invariant sub-bundles:

$$T_\Lambda M = E + E^s + E^u$$

where E is the one-dimensional bundle tangent to the flow and E^s and E^u are, respectively, exponentially contracted and expanded by $D\varphi$, i.e. there exist positive constants C and λ such that

(a) $\| D\varphi_t(v) \| \leq C e^{-\lambda t} \| v \|, \qquad \text{for } v \in E^s,\ t \geq 0;$

(b) $\| D\varphi_{-t}(v) \| \leq C e^{-\lambda t} \| v \|, \qquad \text{for } v \in E^u,\ t \geq 0.$

A hyperbolic set Λ is called *basic* if

(i) the periodic orbits of φ restricted to Λ are dense in Λ;

(ii) Λ contains a dense φ-orbit;

(iii) there is an open neighbourhood U of Λ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(U).$$

The *non-wandering set* Ω of φ is defined by

$$\Omega = \{x \in M : \forall \text{ open } V \ni x\ \forall\, t_0 \geq 0\ \exists\, t > t_0 \text{ such that } \varphi_t(V) \cap V \neq \emptyset\}.$$

The flow φ satifies Axiom A if Ω can be written as the disjoint union of a finite number

of basic sets and hyperbolic fixed points. (A fixed point x of φ is hyperbolic if x is a hyperbolic fixed point of the time-one diffeomorphism $\varphi_1 : M \to M$. This means that $D\varphi_1 : T_xM \to T_xM$ has no eigenvalues of modulus 1.) In this thesis we shall be interested in Axiom A flows restricted to a basic set which is not trivial, i.e. one that does not consist of a single closed orbit.

Axiom A flows are a generalization of Anosov flows [4]. A flow $\varphi : M \to M$ is called Anosov if M is a hyperbolic set. Particularly interesting are the class of transitive Anosov flows, i.e. those for which $\Omega = M$. (There exist Anosov flows which are not transitive [10].) For transitive Anosov flows one can ask how the closed orbits of φ are related to the topology of M, for instance how they are distributed in $H_1(M,\mathbb{Z})$.

Prototypical examples of (transitive) Anosov flows are given by geodesic flows on the unit-tangent bundle of a manifold of (variable) negative sectional curvature. Such a flow φ is defined as follows. Let V be a manifold of negative sectional curvature and let T_1V be its unit-tangent bundle. For $(x,v) \in T_1V$ let $\gamma(t)$ be the unique unit-speed geodesic on V (parameterized by arc-length) such that $\gamma(0) = x$, $\dot\gamma(0) = v$ and define $\varphi_t(x,v) = (\gamma(t),\dot\gamma(t))$.

We now briefly discuss some of the ergodic properties of Axiom A flows. Throughout we assume that the flow φ is restricted to a non-trivial basic set Λ. The flow φ said to be topologically weak-mixing if there is no non-trivial solution to $F\circ\varphi_t = e^{iat}F$, $a > 0$, $F \in C(\Lambda)$. If this equation does have solutions then any a satisfying it is called an eigenfrequency for φ.

Associated with an Axiom A flow there is its topological entropy $h = h(\varphi)$. More generally given a real valued continuous function F defined on Λ there is its pressure P(F) which is defined in the following way. Fix a metric d on M. A set E is said to be (ε,T)-separated if for every pair x,y of elements of E with $x \neq y$ there exists $t \in [0,T]$ such that $d(\varphi_t x,\varphi_t y) > \varepsilon$. We define

$$Z_T(F,\varepsilon) = \sup\left\{ \sum_{x \in E} \exp \int_0^T F(\varphi_t x)\, dt \ : \ E \text{ is } (\varepsilon,T)\text{-separated} \right\},$$

$$P(F,\varepsilon) = \overline{\lim} \frac{1}{T} \log Z_T(F,\varepsilon)$$

and

$$P(F) = \lim_{\varepsilon \to 0} P(F,\varepsilon).$$

It turns out that this definition is independent of the choice of metric d. P(F) can also be characterized by the variational principle

$$P(F) = \sup_m h_m(\varphi) + \int F\, dm$$

where the supremum is taken over all φ-invariant probability measures m and $h_m(\varphi)$ is the measure theoretic entropy of φ with respect to m. If F is Hölder continuous there is a unique (ergodic) φ-invariant probability measure m_F, called the equilibrium state of F, such that

$$P(F) = h_{m_F}(\varphi) + \int F\, dm_F.$$

The topological entropy h is defined to be P(0) and the equilibrium state of 0, i.e. the unique φ-invariant probability measure m_0 such that $h_{m_0}(\varphi) = h$, is called the measure of maximal entropy for φ.

2. Closed orbits of Axiom A flows.

An Axiom A flow (restricted to a non-trivial basic set) has a countable number of closed orbits and we shall be concerned with asymptotic formulae related these closed orbits. We now give a brief survey of results in this area. Throughout τ will denote a generic closed orbit of least period $\lambda(\tau)$. Define

$$\pi(x) = \#\{\tau : \lambda(\tau) \le x\}.$$

For φ the geodesic flow on the unit-tangent bundle of an n-dimensional manifold

of *constant* negative sectional curvature Margulis announced [17] that

$$\pi(x) \sim \frac{e^{(n-1)x}}{(n-1)\,x}.$$

For such flows $n-1$ is equal to the topological entropy h. (Hejhal [13] gave an alternative proof of this result based on the Selberg trace formula and in fact obtained a far more precise asymptotic expression for $\pi(x)$.) According to the survey article of Alexeev and Jacobson [3], Margulis's dissertation [18] contained the more general result that for weak-mixing Anosov flows

$$\pi(x) \sim \frac{e^{hx}}{hx}.$$

Unfortunately, Margulis's proof of these results have never been published in English.

In [6] Bowen proved that for an Axiom A flow φ, weak-mixing or not, there exist positive constants A and B such that

$$A\,\frac{e^{hx}}{x} \le \pi(x) \le B\,\frac{e^{hx}}{x}$$

(for all x away from zero). He also conjectured that for a weak-mixing Axiom A flow the asymptotic formula of Margulis was true. This result was proved by Parry and Pollicott [25] and the also showed that for an Axiom A flow which is not weak-mixing with least positive eigenfrequency a,

$$\pi(x) \sim \frac{2\pi}{ax} \sum_{\frac{2\pi n}{a} \le x} e^{2\pi n h/a}.$$

These results are called prime orbit theorems and can be viewed as an analogue of the prime number theorem. The proof of Parry and Pollicott used the analytic properties of the so-called Ruelle zeta function (defined in analogy to the Riemann zeta function)

$$\zeta_\varphi(s) = \prod_\tau \left(1 - e^{-s\lambda(\tau)}\right)^{-1}.$$

This Euler product converges for Re $s > h$ and defines a function which is analytic and

non-zero in this half-plane. The key to the proof is extending $\zeta_\varphi(s)$ to a neighbourhood of Re s = h. If φ is weak-mixing then $\zeta_\varphi(s)$ is analytic in a neighbourhood of Re s ≥ h with the exception of a simple pole at s = h. If φ is not weak-mixing with least positive eigenfrequency a then $\zeta_\varphi(s)$ is analytic in a neighbourhood of Re s ≥ h with the exception of simple poles located at h+nia, n ∈ $\mathbb{Z}$. In the weak-mixing case we have that $\zeta_\varphi'(s)/\zeta_\varphi(s)$ is analytic in a neighbourhood of Re s ≥ h with the exception of of a simple pole with residue −1 at s = h. Writing

$$\frac{\zeta_\varphi'(s)}{\zeta_\varphi(s)} = -\int_0^\infty e^{-sx}\, dS(x)$$

where

$$S(x) = \sum_{n=1}^{[x/\lambda(\tau)]} \lambda(\tau)$$

and applying the Ikehara Tauberian theorem [15] one obtains $S(x) \sim e^{hx}$. The required result is then deduced by methods familiar from number theory. The proof in the non-weak-mixing case is more elementary.

The above results for $\pi(x)$ could be called temporal distribution results. One can also ask how closed orbits are distributed spatially. In this direction Bowen proved the following results [5], [6]. Let K ∈ C(A) and let

$$\lambda_K(\tau) = \int_0^{\lambda(\tau)} K(\varphi_t x)\, dt$$

(for any x ∈ τ). Also let m_0 be the measure of maximal entropy for φ. Then for any $\varepsilon > 0$

$$\frac{1}{\pi(x)} \sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \frac{\lambda_K(\tau)}{\lambda(\tau)} \to \int K\, dm_0 \qquad \text{as } x \to \infty$$

and

$$\frac{\sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \lambda_K(\tau)}{\sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \lambda(\tau)} \to \int K \, dm_0 \qquad \text{as } x \to \infty.$$

In [22] Parry gave an alternative proof of these results using a new zeta function and in [24] (cf. also [23]) went on to show that if G is a real valued Hölder continuous function defined on Λ with $P(G) \ge 0$ then for any $\varepsilon > 0$

$$\frac{\sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \lambda_K(\tau)\, e^{\lambda_G(\tau)}}{\sum_{x-\varepsilon<\lambda(\tau)\le x+\varepsilon} \lambda(\tau)\, e^{\lambda_G(\tau)}} \to \int K \, dm_G \qquad \text{as } x \to \infty$$

where m_G is the equilibrium state of G.

Another question one can ask is how closed orbits are distributed with respect to certain group extensions. The case of finite groups (or even compact Lie groups) was covered by Parry and Pollicott [26]. Let G be a finite group. Let φ be as above and let $\tilde{\varphi}$ be an Axiom A flow on the manifold $\tilde{M}$ restricted to the (non-trivial) basic set $\tilde{\Lambda}$. Suppose that G acts freely on $\tilde{M}$ by diffeomorphisms, commuting with $\tilde{\varphi}$, such that $M = \tilde{M}/G$, $\varphi = \tilde{\varphi}/G$ and $\Lambda = \tilde{\Lambda}/G$. A closed φ-orbit τ lifts to ℓ distinct closed $\tilde{\varphi}$-orbits $\tilde{\tau}_1,\ldots,\tilde{\tau}_\ell$, for some $\ell \mid |G|$. For each $\tilde{\tau}_i$, $i = 1,\ldots,\ell$, there exists a unique $\gamma(\tilde{\tau}_i) \in G$ such that, for any $x \in \tilde{\tau}_i$, $\gamma(\tilde{\tau}_i)x = \tilde{\varphi}_{\lambda(\tau)}x$. For any pair i,j, $\gamma(\tilde{\tau}_i)$ and $\gamma(\tilde{\tau}_j)$ are conjugate, thus each closed φ-orbit determines a unique conjugacy class in G. Let C be a fixed conjugacy class in G then

$$\#\{\tau : \lambda(\tau) \le x,\ \gamma(\tilde{\tau}) \in C\} \sim \frac{|C|}{|G|}\, \pi(x)$$

where $\tilde{\tau}$ is any lift of τ [26]. For groups with positive rank the situation is more complicated. Some results in this context are discussed in Chapter 2.

All the above results are obtained through the study of functions which generalize $\zeta_\varphi(s)$. Let $F : \Lambda \to \mathbb{C}$ be Hölder continuous and define

$$\zeta_\varphi(s,F) = \prod_\tau \left(1 - e^{-s\lambda(\tau) + \lambda_F(\tau)}\right)^{-1}$$

(whenever the product converges). If we set $F \equiv 0$, we recover the definition of $\zeta_\varphi(s)$. To study $\zeta_\varphi(s,F)$ we relate the Axiom A flow $\varphi : \Lambda \to \Lambda$ to a suspended flow, $\sigma^r : \Sigma^r \to \Sigma^r$, over a shift of finite type $\sigma : \Sigma_A \to \Sigma_A$ with a Hölder continuous ceiling function r. In the next section we describe shifts of finite type and their suspensions and then in Section 4 go on to discuss their relationship to Axiom A flows.

3. Shifts of finite type and suspended flows.

We begin by defining shifts of finite type (also sometimes called topological Markov chains). These were introduced in a mathematical context by Parry [21] (although they are closely related to the physicists notion of a one-dimensional lattice gas, see for example [29]). Let A be a $k \times k$ matrix of zeros and ones and suppose that A is aperiodic, i.e. that for some $n \geq 1$, A^n has all its entries positive. Let

$$\Sigma_A = \{ x \in \{1,\ldots,k\}^{\mathbb{Z}} : A(x_i,x_{i+1}) = 1 \ \forall\, i \in \mathbb{Z} \}$$

and

$$\Sigma_A^+ = \{ x \in \{1,\ldots,k\}^{\mathbb{N}\cup\{0\}} : A(x_i,x_{i+1}) = 1 \ \forall\, i \geq 0 \}.$$

Give $\{1,\ldots,k\}$ the discrete topology, $\{1,\ldots,k\}^{\mathbb{Z}}$ ($\{1,\ldots,k\}^{\mathbb{N}\cup\{0\}}$) the product topology and Σ_A (Σ_A^+) the subspace topology. This topology makes Σ_A (Σ_A^+) compact. The shift of finite type $\sigma : \Sigma_A \to \Sigma_A$ (one-sided shift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$) is defined by $(\sigma x)_i = x_{i+1}$ and is a homeomorphism (continuous map). The assumption that A is aperiodic is equivalent to σ being topologically mixing, i.e. that for every pair of non-empty open sets U,V there exists $n \geq 0$ such that $U \cap \sigma^n(V) \neq \emptyset$.

Define a metric d on Σ_A (Σ_A^+) by

$$d(x,y) = 2^{-\sup\{n\,:\,x_i = y_i \text{ for } |i| < n\}}$$

$$\left(d(x,y) = 2^{-\sup\{n\,:\,x_i = y_i \text{ for } 0 \leq i < n\}}\right).$$

Choose $\gamma > 0$ and let H_γ (H_γ^+) be the space of complex valued functions on Σ_A (Σ_A^+) Hölder continuous with respect to d with exponent γ. Thus for $f \in H_\gamma$ (H_γ^+) there exists a constant $C > 0$ such that for any $x,y \in \Sigma_A$ ($x,y \in \Sigma_A^+$)

$$|f(x) - f(y)| \leq C\, d(x,y)^\gamma.$$

Define $|f|_\gamma$ to be the infimum of all such C, i.e.

$$|f|_\gamma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)^\gamma}.$$

If we write $\|\,.\,\|_\gamma = |\,.\,|_\gamma + \|\,.\,\|_\infty$ (where $\|\,.\,\|_\infty$ is the uniform norm) then $\|\,.\,\|_\gamma$ is a norm on H_γ (H_γ^+) and with respect to this norm H_γ (H_γ^+) is a Banach space.

We can define the pressure of a real valued continuous function f, P(f), with respect to the shift in a similar way as we did for Axiom A flows. A set E is called (ε,n)-separated if for each pair $x,y \in E$ with $x \neq y$ there exists $i \in \{1,\ldots,n\}$ such that $d(\sigma^i x,\sigma^i y) > \varepsilon$. Define

$$Z_n(f,\varepsilon) = \sup\left\{ \sum_{x \in E} \exp f^n(x) \;:\; E \text{ is } (\varepsilon,n)\text{-separated} \right\}$$

(where $f^n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x)$),

$$P(f,\varepsilon) = \overline{\lim}\, \frac{1}{n} \log Z_n(f,\varepsilon)$$

and

$$P(f) = \lim_{\varepsilon \to 0} P(f,\varepsilon).$$

An equivalent definition is through the variational principle

$$P(f) = \sup_\mu h_\mu(\sigma) + \int f\, d\mu$$

where the supremum is taken over all σ-invariant probability measures μ and $h_\mu(\sigma)$ is the measure theoretic entropy of σ with respect to μ. If $f \in H_\gamma$ ($f \in H_\gamma^+$) then there is a unique (ergodic) σ-invariant probability measure μ_f, called the equilibrium state of f, such that

$$P(f) = h_{\mu_f}(\sigma) + \int f\, d\mu_f.$$

In the next section we shall show how to extend the definition of pressure to complex

valued functions.

Two functions $f,g \in C(\Sigma_A)$ ($f,g \in C(\Sigma_A^+)$) are said to be cohomologous if there exists $u \in C(\Sigma_A)$ ($u \in C(\Sigma_A^+)$) such that $f = g + u\circ\sigma - u$. Two functions in H_γ (H_γ^+) have the same equilibrium state if and only if their difference is cohomologous to a constant.

We now consider suspensions over Σ_A. Let r be strictly positive function in H_γ (for some $\gamma > 0$). Define the *r-suspension space*

$$\Sigma^r = \{ (x,t) : x \in \Sigma_A,\ 0 \le t \le r(x) \}$$

with $(x,r(x))$ identified with $(\sigma x,0)$. The metric d on Σ_A can be extended to give a metric on Σ^r. This construction is described in [9]. There is a natural flow σ^r on Σ^r, called the suspended flow, defined locally by $\sigma^r_t(x,s) = (x,s+t)$ and respecting indentifications.

As for Axiom A flows, σ^r is said to be topologically weak-mixing if there is no non-trivial solution to $F\circ\sigma^r_t = e^{iat} F$ with $a > 0$ and $F \in C(\Sigma^r)$. A necessary and sufficient condition for σ^r to be weak-mixing is for r not to be cohomologous to a function taking values in a discrete subgroup of the reals.

One can define pressure, P(F), of a continuous real valued function F with respect to σ^r in exactly the same way as for Axiom A flows and again we have the variational principle

$$P(F) = \sup_m h_m(\sigma^r) + \int F\, dm$$

where the supremum is taken over all σ^r-invariant probability measures m and $h_m(\sigma^r)$ is the measure theoretic entropy of σ^r with respect to m. Define a function f on Σ_A by

$$f(x) = \int_0^{r(x)} F(\sigma^r_t(x,0)\, dt \qquad (3.1).$$

If f is Hölder continuous then there is a unique (ergodic) σ^r-invariant probability m_F, called the equilibrium state of F such that

$$P(F) = h_{m_F}(\sigma^r) + \int F\, dm_F.$$

Every σ^r-invariant probability measure m is of the form $m = (\mu \times \ell)/\int r\, d\mu$ where μ is a σ-invariant probability measure and ℓ is Lebesgue measure on the line. Furthermore, if m is related to μ in this way then

$$h_m(\sigma^r) = \frac{h_\mu(\sigma)}{\int r\, d\mu}$$

(Abramov [1]). Thus

$$P(F) = \sup_\mu \frac{h_\mu(\sigma)}{\int r\, d\mu} + \frac{\int\int_0^{r(x)} F(\sigma_t^r(x,0))\, dt\, d\mu(x)}{\int r\, d\mu}$$

$$= \sup_\mu \frac{h_\mu(\sigma)}{\int r\, d\mu} + \frac{\int f\, d\mu}{\int r\, d\mu}$$

where the supremum is taken over all σ-invariant probability measures. Since m_F is the unique equilibrium state of F, if $m_F = (\mu^* \times \ell)/\int r\, d\mu^*$ then

$$P(F) = \frac{h_{\mu^*}(\sigma)}{\int r\, d\mu^*} + \frac{\int f\, d\mu^*}{\int r\, d\mu^*}$$

and if $\mu \neq \mu^*$ then

$$P(F) > \frac{h_\mu(\sigma)}{\int r\, d\mu} + \frac{\int f\, d\mu}{\int r\, d\mu}.$$

Rearranging these expressions we obtain

$$h_{\mu^*}(\sigma) + \int (-P(F)r + f)\, d\mu^* = 0$$

and for $\mu \neq \mu^*$

$$h_\mu(\sigma) + \int (-P(F)r + f)\, d\mu < 0.$$

Hence $\mu^* = \mu_{-P(F)r+f}$ and $P(-P(F)r + f) = 0$. Since r is strictly positive, $t \to P(-t\,r + f)$ ($t \in \mathbb{R}$) is strictly decreasing, so P(F) is the unique $c \in \mathbb{R}$ such that $P(-cr + f) = 0$.

4. Ruelle operators and zeta functions.

Define a function $\zeta : H_\gamma \to \mathbb{C}$ by

$$\zeta(g) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} \exp g^n(x)$$

where $\mathrm{Fix}_n = \{x \in \Sigma_A : \sigma^n x = x\}$. It will turn out that the study of the analytic properties of $\zeta_\varphi(s,F)$ can, after the introduction of symbolic dynamics, be reduced to that of $\zeta(g)$.

Remark. A map α from a domain $\Omega \subset \mathbb{C}$ to a complex Banach space X is said to be analytic if it is weakly analytic, i.e. if for every $u \in X^*$, $u \circ \alpha : \Omega \to \mathbb{C}$ is analytic. To say that $\zeta : D \to \mathbb{C}$ is analytic (where D is some domain in H_γ) is to say that for every domain $\Omega \subset \mathbb{C}$ and every analytic $\alpha : \Omega \to D$, $\zeta \circ \alpha : \Omega \to \mathbb{C}$ is analytic.

Our main tool in the study of $\zeta(g)$ will be the *Ruelle operator.* Before we define this operator we must first relate elements of H_γ to Hölder continuous functions defined on Σ_A^+. We do this by means of the following lemma due to Sinai.

Lemma 1. (Sinai [31].) If $g \in H_\gamma$ then one can find $u \in H_{\frac{1}{2}\gamma}$ such that the function $g' = g + u \circ \sigma - u$ depends only on the future co-ordinates $(x_i)_{i \geq 0}$. Hence g' can be identified with an element of $H_{\frac{1}{2}\gamma}^+$. Furthermore, it is possible to choose $u = u(g)$ so that the map $g \to g'$ is a bounded linear operator from $H_\gamma \to H_{\frac{1}{2}\gamma}^+$. Also g' is real valued if and only if g is real valued.

Now define the Ruelle operator $\mathcal{L}_{g'} : H_{\frac{1}{2}\gamma}^+ \to H_{\frac{1}{2}\gamma}^+$ by

$$\mathcal{L}_{g'}\psi(x) = \sum_{\sigma y = x} e^{g'(y)} \psi(y).$$

Note that the map $g' \to \mathcal{L}_{g'}$ is analytic and hence that the map $g \to \mathcal{L}_{g'}$ is analytic. We shall need some information about the spectrum of $\mathcal{L}_{g'}$. Write $\rho(\mathcal{L}_{g'})$ for the spectral radius of $\mathcal{L}_{g'}$, i.e.

$$\rho(\mathcal{L}_{g'}) = \sup\{|\lambda| : \lambda \in \mathrm{spectrum}(\mathcal{L}_{g'})\}.$$

We have the following propositions due to Ruelle and Pollicott.

Proposition 1. (Ruelle [29].) If $g' \in H_{1\gamma}^{+}$ is real valued then $e^{P(g)}$ is a simple eigenvalue of $\mathcal{L}_{g'} : H_{1\gamma}^{+} \to H_{1\gamma}^{+}$. Furthermore, the rest of the spectrum is contained in a disc of radius less than $e^{P(g)}$.

Proposition 2. (Pollicott [27].) For complex valued $g' \in H_{1\gamma}^{+}$ we have $\rho(\mathcal{L}_{g'}) \leq e^{P(\mathrm{Re}\, g)}$. If Im g is cohomologous to $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in [0,2\pi)$ then $e^{P(\mathrm{Re}\, g) + ic}$ is a simple eigenvalue of $\mathcal{L}_{g'}$ and the rest of the spectrum of $\mathcal{L}_{g'}$ is contained in a disc of radius less than $e^{P(\mathrm{Re}\, g)}$. If Im g is <u>not</u> cohomologous to $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in [0,2\pi)$ then $\rho(\mathcal{L}_{g'}) < e^{P(\mathrm{Re}\, g)}$.

Remark. If g' depends on only finitely many co-ordinates (we call such g' locally constant) then $\mathcal{L}_{g'}$ can be represented by a matrix. In this case Proposition 1 reduces to the Perron–Frobenius theorem and Proposition 2 reduces to Wielandt's theorem [11].

We can use this result to extend the definition of pressure to complex valued functions in H_γ (in a neighbourhood of the real valued functions). First we quote a lemma from perturbation theory.

Lemma 2. (Kato [14].) Let X be a complex Banach space and let B(X) denote the space of bounded linear operators on X. Let $\|\ \|$ be the operator norm on B(X). Suppose that $T_0 \in B(X)$ has a simple eigenvalue ρ_0 and that the rest of the spectrum of T_0 is contained in a disc of radius $r < |\rho_0|$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for $T \in B(X)$ with $\|T - T_0\| < \delta$, T has a simple eigenvalue $\rho_0(T)$ with $\rho_0(T_0) = \rho_0$ and

(i) the map $T \to \rho_0(T)$ is analytic

(ii) the rest of the spectrum of T is contained in a disc of radius $r + \varepsilon$.

In particular, provided we take ε sufficiently small, spectrum(T) $- \{\rho_0(T)\}$ is contained

in a disc of radius less than $|\rho_0(T)|$.

By Proposition 3, $\mathcal{L}_{g'}$ (for real valued g') satisfies the hypotheses on T_0 in Lemma 2. If h is a (complex valued) function in H_γ that is $\| \|_\gamma$-close to g then h' is $\| \|_{\frac{1}{2}\gamma}$-close to g' and so $\mathcal{L}_{h'}$ is $\| \|_{\frac{1}{2}\gamma}$-close to $\mathcal{L}_{g'}$ (where $\| \|_{\frac{1}{2}\gamma}$ also denotes the operator norm on $B(H^+_{\frac{1}{2}\gamma})$). Hence provided h is sufficiently $\| \|_\gamma$-close to g, $\mathcal{L}_{h'}$ has a simple eigenvalue $\rho_0(h)$ with $\rho_0(g) = e^{P(g)}$ and the rest of the spectrum of $\mathcal{L}_{h'}$ is contained in a disc of radius less than $|\rho_0(h)|$. Furthermore the map $h \to \rho_0(h)$ is analytic. We can now extend pressure analytically to complex valued functions in H_γ (in a neighbourhood of the real valued functions) by defining $P(h) = \log \rho_0(h)$. Note that P(h) is only defined modulo $2\pi i$, however we shall require that P(h) be real valued when h is real valued, in agreement with our original definition of the pressure of real valued functions. By Proposition 2 one sees that if $u \in C(\Sigma_A)$, $K \in C(\Sigma_A, 2\pi i\mathbb{Z})$ and c is a constant then $P(g+u\circ\sigma-u+K+c)=P(h)+c$ (modulo 2π).

We now relate $\mathcal{L}_{g'}$ to $\zeta(g)$ (for real valued g). First observe that, since g and g' are cohomologous,

$$\sum_{x\in \mathrm{Fix}_n} \exp g^n(x) = \sum_{x\in \mathrm{Fix}_n} \exp g'^n(x).$$

The sum on the R.H.S. above is reminiscent of the trace of $(\mathcal{L}_{g'})^n$ and in fact if g' is a locally constant function then, as we noted before, $\mathcal{L}_{g'}$ can be represented by a matrix and

$$\sum_{x\in \mathrm{Fix}_n} \exp g^n(x) = \sum_{x\in \mathrm{Fix}_n} \exp g'^n(x) = \mathrm{Trace}(\mathcal{L}_{g'}^n).$$

If this is the case and $\mathcal{L}_{g'}$ has eigenvalues $e^{P(g)}, \rho_1, \ldots, \rho_\ell$ with $|\rho_j| < e^{P(g)}$, $j = 1,\ldots,\ell$, then

$$\zeta(g) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Trace}(\mathcal{L}_{g'}^n) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(e^{nP(g)} + \rho_1^n + \ldots + \rho_\ell^n\right)$$

$$= \exp -\log\left((1-e^{P(g)})(1-\rho_1)\ldots(1-\rho_\ell)\right) = \frac{1}{\det(I - \mathcal{L}_{g'})}.$$

In particular,

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in \mathrm{Fix}_n} \exp g^n(x) = P(g) \qquad (4.1)$$

By approximating by locally constant functions, (4.1) holds for any real valued $g \in H_\gamma$ and so the series

$$Z(g) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x\in \mathrm{Fix}_n} \exp g^n(x)$$

defining $\zeta(g)$ converges whenever $P(g) < 0$. If we now suppose that g is complex valued then the inequality

$$\lim_{n\to\infty} \left| \sum_{x\in \mathrm{Fix}_n} \exp g^n(x) \right|^{1/n} \le \lim_{n\to\infty} \left(\sum_{x\in \mathrm{Fix}_n} \exp \mathrm{Re}\, g^n(x) \right)^{1/n}$$
$$= e^{P(\mathrm{Re}\, g)}$$

shows that $Z(g)$ converges whenever $P(\mathrm{Re}\, g) < 0$ and defines an analytic function there. Thus $\zeta(g) = \exp Z(g)$ is analytic and non-zero on $\{g \in H_\gamma : P(\mathrm{Re}\, g) < 0\}$.

We now extend $\zeta(g)$ to a neighbourhood of $\{g \in H_\gamma : P(\mathrm{Re}\, g) = 0\}$. First we have the following proposition., the proof of which again involves approximation by locally constant functions.

Proposition 3. (Parry [22], Pollicott [27].) Suppose $P(\mathrm{Re}\, g) = 0$ so that $\rho(\mathcal{L}_{g'}) \le 1$.

(i) If $\rho(\mathcal{L}_{g'}) < 1$, i.e. if Im g is <u>not</u> cohomologous to a function of the form $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in [0, 2\pi)$, then, for some $\varepsilon > 0$, $Z(h)$ converges absolutely on $\{h \in H_\gamma : \|h - g\|_\gamma < \varepsilon\}$.

(ii) If $\rho(\mathcal{L}_{g'}) = 1$, i.e. if Im g is cohomologous to a function of the form $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in [0, 2\pi]$, then, for some $\varepsilon > 0$,

$$Z_1(h) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{x \in \mathrm{Fix}_n} \exp h^n(x) - e^{P(h)} \right)$$

converges absolutely on $\{h \in H_\gamma : \|h - g\|_\gamma < \varepsilon\}$.

This leads to the following theorem.

Theorem 1. (Parry [22], Pollicott [27].) Suppose $P(\mathrm{Re}\, g) = 0$ so that $\rho(\mathcal{L}_{g'}) \le 1$.

(i) If $\rho(\mathcal{L}_{g'}) < 1$, i.e. if $\mathrm{Im}\, g$ is <u>not</u> cohomologous to a function of the form $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in [0, 2\pi)$, then, for some $\varepsilon > 0$, $\zeta(h) = \exp Z(h)$ is analytic and non-zero on $\{h \in H_\gamma : \|h - g\|_\gamma < \varepsilon\}$.

(ii) If $\rho(\mathcal{L}_{g'}) = 1$ but $\mathcal{L}_{g'}$ does <u>not</u> have 1 as an eigenvalue, i.e. if $\mathrm{Im}\, g$ is cohomologous to a function of the form $K + c$ with $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ and $c \in (0, 2\pi)$, then, for some $\varepsilon > 0$, $\zeta(h)$ has an analytic non-zero extension to $\{h \in H_\gamma : \|h - g\|_\gamma < \varepsilon\}$ by defining

$$\zeta(h) = \frac{\exp Z_1(h)}{1 - e^{P(h)}}.$$

(iii) If $\rho(\mathcal{L}_{g'}) = 1$ and $\mathcal{L}_{g'}$ does have 1 as an eigenvalue, i.e. $\mathrm{Im}\, g$ is cohomologous to a function in $C(\Sigma_A, 2\pi\mathbb{Z})$ then, for some $\varepsilon > 0$, $\zeta(h)$ has an analytic non-zero extension to $\{h \in H_\gamma : \|h - g\|_\gamma < \varepsilon\} - \{h : P(h) = 0\}$ by defining

$$\zeta(h) = \frac{\exp Z_1(h)}{1 - e^{P(h)}}.$$

5. Zeta functions for the suspended flow.

Let F be a continuous (complex-valued) function on Σ', let f be related to F by (3.1) and suppose that $f \in H_\gamma$. We can define a zeta function $\zeta_{\Sigma'}(s,F)$ in analogy to $\zeta_\varphi(s,F)$ by

$$\zeta_{\sigma^r}(s,F) = \prod_{\tau} \left(1 - e^{-s\lambda(\tau) + \lambda_F(\tau)}\right)^{-1}$$

(whenever the product converges) where the product is taken over all closed σ^r-orbits τ of least period $\lambda(\tau)$ and

$$\lambda_F(\tau) = \int_0^{\lambda(\tau)} F(\sigma_t^r z)\, dt, \qquad z \in \tau.$$

Every such τ corresponds to a closed σ-orbit, $\{x, \sigma x, \ldots, \sigma^{n-1}x\}$ say, (where n is the least period) and $\lambda(\tau) = r^n(x)$, $\lambda_F(\tau) = f^n(x)$. Let $\mathrm{Fix}_n = \{y \in \Sigma_A : \sigma^n y = y\}$ then $\sigma^i x \in \mathrm{Fix}_n$, $0 \le i \le n-1$, and if $n|m$ ($m = nk$ say) then $\sigma^i x \in \mathrm{Fix}_m$, $0 \le i \le n-1$, and $r^m(x) = kr^n(x)$, $f^m(x) = kf^n(x)$.

Hence we can rewrite $\zeta_{\sigma^r}(s,F)$ in terms of r and f as

$$\zeta_{\sigma^r}(s,F) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} \exp -sr^n(x) + f^n(x).$$

so that $\zeta_{\sigma^r}(s,F) = \zeta(-sr+f)$. First we observe that $\zeta(-sr+f)$ is analytic and non-zero whenever $P(-(\mathrm{Re}\, s)r + \mathrm{Re}\, f) < 0$. In view of the relationship $P(-P(\mathrm{Re}\, F)r + \mathrm{Re}\, f) = 0$ and the fact that the function $t \to P(-tr + \mathrm{Re}\, f)$ is strictly decreasing this gives that $\zeta_{\sigma^r}(s,F)$ is analytic and non-zero provided $\mathrm{Re}\, s > P(\mathrm{Re}\, F)$. Furthermore the behaviour of $\zeta_{\sigma^r}(s,F)$ for $\mathrm{Re}\, s = P(\mathrm{Re}\, F)$ is given by Theorem 1.

Remark. We have not specified a Banach space for the argument F of $\zeta_{\sigma^r}(s,F)$ to lie in, so we cannot talk about the analyticity of $F \to \zeta_{\sigma^r}(s,F)$. However, in all our applications F will take the form $F = z_1F_1 + \ldots + z_dF_d$, where $z_1, \ldots, z_d$ are complex numbers and we shall be interested in the analytic dependence of $\zeta_{\sigma^r}(s,F)$ on $z_1, \ldots, z_d$.

6. Symbolic dynamics for Axiom A flows.

Symbolic dynamics for hyperbolic systems has its origins in the work of

Hadamard [12] and Morse [19], [20]. More recent contributions were made by Adler and Weiss (for hyperbolic automorphisms of a 2-dimensional torus) [2], Ratner [28], Sinai [30] and, most importantly, by Bowen [7]. One can model an Axiom A flow by a suspended flow over a shift of finite type:

Proposition 4. (Bowen [7].) If φ is an Axiom A flow restricted to a non-trivial basic set then there exists a topologically mixing shift of finite type Σ_A, a strictly positive Hölder continuous function r and a Hölder continuous map $p : \Sigma^r \to \Lambda$ such that $p \circ \sigma^r_t = \varphi_t \circ p$. The map p is surjective, bounded-one and one-one on a residual set. Furthermore, if $F : \Lambda \to \mathbb{R}$ is Hölder continuous then f defined on Σ_A by

$$f(x) = \int_0^{r(x)} F \circ p(\sigma^r_t(x,0))\, dt$$

is Hölder continuous so that $F \circ p$ has a unique equilibrium state $m_{F \circ p}$. The measure $p^* m_{F \circ p}$ is equal to m_F, the equilibrium state of F. The map p is measure preserving with respect to $m_{F \circ p}$ and m_F and a.e. one-one with respect to $m_{F \circ p}$. The flow σ^r is called the principal suspension.

Remarks. (i) Since p is bounded-one, $P(F) = P(F \circ p)$. In particular the topological entropy $h(\varphi) = h(\sigma^r)$.

(ii) σ^r is weak-mixing if and only if φ is weak-mixing.

We shall briefly indicate how the map p is constructed. For each $x \in \Lambda$ one can define stable and unstable manifolds

$$W^s(x) = \{y \in \Lambda : d(\varphi_t x, \varphi_t y) \to 0 \text{ as } t \to +\infty\}$$

and

$$W^u(x) = \{y \in \Lambda : d(\varphi_t x, \varphi_t y) \to 0 \text{ as } t \to -\infty\}$$

respectively (where d is a metric on M induced by the Riemannian structure). One can choose co-dimension one sections T_i ($i = 1,\ldots,k$, say) transverse to the flow such that

for any $x \in \Lambda$ there exist sequences $t_j \uparrow \infty$ and $s_j \downarrow -\infty$ with the property that for every j

$$|t_j - t_{j+1}|, |s_j - s_{j+1}| < \sup\{\operatorname{diam} T_i : i = 1,\dots,k\}$$

and

$$\varphi_{t_j} x, \varphi_{s_j} x \in \bigcup_{i=1}^{k} T_i .$$

Let

$$P : \bigcup_{i=1}^{k} T_i \to \bigcup_{i=1}^{k} T_i$$

be defined by $P(y) = \varphi_{t^*(y)}(y)$ where

$$t^*(y) = \inf\{t > 0 : \varphi_t(y) \in \bigcup_{i=1}^{k} T_i\}.$$

It is possible to choose the sections with the following Markov property:

(i) $\operatorname{int} T_i \cap \operatorname{int} T_j = \emptyset$ for $i \neq j$ (where the interior is with respect to Λ)

(ii) if $y \in \operatorname{int} T_i \cap P^{-1}(\operatorname{int} T_j)$ then

$$P(T_i^u(y)) \supset T_j^u(P(y)) \text{ and } P^{-1}(T_j^s(P(y)) \supset T_i^s(y)$$

where, for example,

$$T_i^u(y) = T_i \cap W^u(y) \text{ and } T_i^s(y) = T_i \cap W^s(y).$$

Define a $k \times k$ matrix A by

$$A(i,j) = 1 \quad \text{if } \operatorname{int} T_i \cap P^{-1}(\operatorname{int} T_j) \neq \emptyset$$
$$= 0 \quad \text{otherwise.}$$

Given a point $y \in T_i$ associate with it the sequence $\bar{y} = \{\bar{y}_n\}_{n \in \mathbb{Z}}$ where $\bar{y}_0 = i$ and the φ-orbit of y intersects the sequence of sections $\{T_{\bar{y}_n}\}_{n \in \mathbb{Z}}$ as it flows forwards and backwards in time. If we write $y = p(\bar{y})$, this defines a surjective map

$$p : \Sigma_A \to \bigcup_{i=1}^{k} T_i .$$

Let $r = t^* \circ p : \Sigma_A \to \mathbb{R}$, then we can extend π to a map $\Sigma^r \to \Lambda$ by defining $p(\bar{y},t) = \varphi_t p(\bar{y})$.

Unfortunately, there is not a one-one correspondence between closed φ-orbits and closed σ^r-orbits. This discrepacy is caused by orbits which pass through the boundaries of the sections T_i. This means that the zeta functions for φ and σ^r will not be

equal. We can get around this difficulty by appealing to the following result of Bowen (itself a refinement of a similar result of Manning [16] for Axiom A diffeomorphisms).

Proposition 6. (Bowen [7]) In addition to the principal suspension, there exist a finite number of suspensions of shifts of finite type Σ^{r_i}, $i = 1,\ldots,p,\ldots,q$ (with each r_i Hölder continuous) and Hölder continuous maps $p_i : \Sigma^{r_i} \to \Lambda$ such that each p_i is bounded-one but not surjective. These suspensions have the property that if, for example, $\upsilon(\varphi,x)$ denotes the number of closed φ-orbits of least period x then

$$\upsilon(\varphi,x) = \upsilon(\sigma^{r},x) + \sum_{i=1}^{p} \upsilon(\sigma^{r_i},x) - \sum_{i=p+1}^{q} \upsilon(\sigma^{r_i},x).$$

Hence we have

$$\zeta_{\varphi}(s,F) = \zeta_{\sigma^{r}}(s,F\circ p) \prod_{i=1}^{p} \zeta_{\sigma^{r_i}}(s,F\circ p_i) \left(\prod_{i=p+1}^{q} \zeta_{\sigma^{r_i}}(s,F\circ p_i) \right)^{-1}.$$

From Section 5 each $\zeta_{\sigma^{r_i}}(s,F\circ p_i)$ is analytic and non-zero for $\mathrm{Re}\, s > P(\mathrm{Re}\, F\circ p_i)$. However, since each p_i is not surjective, $P(\mathrm{Re}\, F\circ p_i) < P(\mathrm{Re}\, F)$ and so

$$\frac{\zeta_{\varphi}(s,F)}{\zeta_{\sigma^{r}}(s,F)}$$

is analytic for $\mathrm{Re}\, s > P(\mathrm{Re}\, F) - \varepsilon$ (for some $\varepsilon > 0$). Hence the analytic behaviour $\zeta_{\varphi}(s,F)$ in a neighbourhood of $\mathrm{Re}\, s \geq P(\mathrm{Re}\, F)$ can be obtained from that of $\zeta_{\sigma^{r}}(s,F\circ p)$ and this in turn is given by Theorem 1.

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Chapter 1
An analogue of Mertens' theorem for closed orbits of Axiom A flows

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Abstract.
For an Axiom A flow restricted to a basic set we prove an analogue of Mertens' theorem of prime number theory. The result is also established for the geodesic flow on a non-compact, finite area surface constant negative curvature. Applying this to the modular surface yields some asymptotic formulae concerning quadratic forms.

0. Introduction.

In recent years a number of papers have pointed to similarities between the distributional properties of prime numbers and those of hyperbolic dynamical systems. In particular Parry and Pollicott [12] have proved an analogue of the prime number theorem for Axiom A flows. Precisely, they proved that for a (topologically) weak-mixing Axiom A flow, with closed orbits τ of least period $\lambda(\tau)$ and entropy h,

$$\text{card}\ \{\ \tau : N(\tau) \le x\ \} \sim x/\log x\ ,$$

where $N(\tau) = e^{h\lambda(\tau)}$ with a modified asymptotic formula for flows which are not weak-mixing.

This paper is motivated by Mertens' theorem of prime number theory ([8], pp. 349-353), which states that

$$\prod_{p\le x}(1 - \frac{1}{p}) \sim \frac{e^{-\gamma}}{\log x}$$

where the product is taken over all primes $p \le x$ and γ is Euler's constant. We prove

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that for an Axiom A flow φ

$$\prod_{N(\tau)\le x}\left(1-\frac{1}{N(\tau)}\right)\sim\frac{e^{-\gamma}}{\mathrm{Res}(\zeta_\varphi,1)\log x}$$

where ζ_φ is the Ruelle zeta function for φ. The proof, as one would expect, relies heavily on the symbolic dynamics of Bowen [5] and the thermodynamic formalism of Ruelle [15]. One should also note that the proof is elementary (just as the proof of Mertens' theorem is elementary) making no use of the deeper results about the analytic properties of ζ_φ obtained in [12] nor the associated Tauberian theorems.

We also establish a similar result in the case of the geodesic flow on a non-compact, finite area surface of constant negative curvature. To do this we make use of the prime geodesic theorem of Sarnak and Woo (cf. [16]). A classical example of such a surface is the modular surface and applying our theorem in this case, in conjunction with the relationship between closed geodesics on the modular surface and equivalence classes of quadratic forms, leads to some asymptotic formulae of a number theoretic character.

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1. Shifts of finite type and their suspensions.

Let A be an aperiodic $k\times k$ zero-one matrix (i.e. for some n, $A^n(i,j)>0$ for all $1\le i,j\le k$) and define

$$\Sigma_A=\left\{x\in\{1,\dots,k\}^{\mathbb{Z}}:A(x_n,x_{n+1})=1,\text{ for all }n\in\mathbb{Z}\right\}.$$

Give $\{1,\dots,k\}$ the discrete topology and Σ_A the product topology. With respect to this topology Σ_A is compact and zero-dimensional, with a basis for the topology being

given by finite unions of closed-open cylinders

$$[x_0,\dots,x_{n-1}]^m = \{ y : y_{i+m} = x_i,\ 0 \le i \le n-1 \}.$$

(We write $[x_0,\dots,x_{n-1}]$ for $[x_0,\dots,x_{n-1}]^0$.) The shift of finite type $\sigma:\Sigma_A \to \Sigma_A$ is defined by $(\sigma x)_i = x_{i+1}$ and is a homeomorphism with respect to the given topology.

For $f \in C(\Sigma_A)$ and $0 < \theta < 1$ we define

$$\mathrm{var}_n f = \sup\{ |f(x)-f(y)| : x,y \in \Sigma_A,\ x_i = y_i \text{ for } |i| \le n \}$$

$$\text{and } |f|_\theta = \sup(\mathrm{var}_n f/\theta^n).$$

The space $F_\theta = \{ f \in C(\Sigma_A) : |f|_\theta < \infty \}$ is a Banach space with respect to the norm

$$\|f\|_\theta = |f|_\infty + |f|_\theta$$

where $|\,.\,|_\infty$ is the uniform norm.

Define the pressure $P : C(\Sigma_A) \to \mathbb{R}$ by

$$P(f) = \sup\left\{ h_m(\sigma) + \int f\,dm : m \text{ a } \sigma\text{-invariant probability measure} \right\}$$

where $h_m(\sigma)$ denotes measure theoretic entropy. If $f \in F_\theta$, this supremum is attained for a unique, ergodic probability measure μ called the *equilibrium state* of f and μ has the following Gibbsian property. There exist positive constants C_1, C_2 such that for any $n \in \mathbb{N}$ and $x \in \mathrm{Fix}_n := \{ x : \sigma^n x = x \}$,

$$C_1\,\mu([x_0,\dots,x_{n-1}]) \le \exp(f^n(x) - nP(f)) \le C_2\,\mu([x_0,\dots,x_{n-1}]) \tag{1.1}$$

where $f^n(x) = f(x) + f(\sigma x) + \dots + f(\sigma^{n-1}x)$. (Bowen [6].)

If $f \in F_\theta$ is real and strictly positive, define the f suspension space $\Sigma_{A,f}$ to be

$$\left\{ (x,t) : x \in \Sigma_A \text{ and } 0 \le t \le f(x) \right\}$$

with $(x,f(x))$ and $(\sigma x,0)$ identified. Define a flow, σ^f, on this space by $\sigma^f_t(x,s) = (x,s+t)$, remembering identifications, i.e., a vertical flow under the graph of f. The entropy of σ^f is the entropy of σ^f_1. Using a result of Abramov [1], it is possible to show that this is the unique $h = h(\sigma^f)$ such that $P(-hf) = 0$ and that if μ is the unique equilibrium state of $-hf$ then the Lebesgue extension of μ is the unique measure of maximal entropy for σ^f. The flow σ^f is said to be topologically weak-mixing if there does not exist a non-

trivial solution to $F\sigma^f_t = e^{iat}F$ with $a > 0$ and $F \in C(\Sigma_{A,f})$. If σ^f is not weak-mixing such an a is called an eigenfrequency for σ^f.

Let τ denote a generic closed σ^f orbit and let $\lambda(\tau)$ be its least period. Define its norm $N(\tau)$ to be $e^{h\lambda(\tau)}$. We define the zeta function for σ^f by

$$\zeta_{\sigma^f}(s) = \prod_{\tau} \left(1 - N(\tau)^{-s}\right)^{-1} = \exp \sum_{\tau} \sum_{k=1}^{\infty} \frac{1}{k} N(\tau)^{-ks}.$$

Each such τ corresponds to n distinct elements of Fix_n (for some $n \in \mathbb{N}$), $\{\xi, \sigma\xi, \ldots, \sigma^{n-1}\xi\}$ say, and $\lambda(\tau) = f^n(\xi)$. Also, if nlm (m=ni say), these n elements are also in Fix_m and $f^m(\xi) = i\, f^n(\xi)$. Hence

$$\zeta_{\sigma^f}(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n} \exp -shf^n(\xi).$$

$\zeta_{\sigma^f}(s)$ is analytic and non-zero for $\mathrm{Re}\, s > 1$ and has a simple pole at $s = 1$ (Ruelle [15]). Furthermore, we have the following proposition.

Proposition 1. (Ruelle [15], Parry [11])

$$Z(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{\xi \in \mathrm{Fix}_n} \exp -shf^n(\xi) - e^{nP(-shf)} \right)$$

converges uniformly to a non-zero analytic function in a neighbourhood U of $s = 1$ and $\zeta_{\sigma^f}(s)$ can be analytically extended to $U - \{1\}$ by defining $\zeta_{\sigma^f}(s) = Z(s)/(1 - e^{P(-shf)})$.

We note that

$$\mathrm{Res}(\zeta_{\sigma^f}, 1) = \lim_{s \to 1} \frac{s-1}{1 - e^{P(-shf)}} Z(1) = -\left[\frac{de^{P(-shf)}}{ds}\right]^{-1}_{s=1} Z(1)$$

$$= \frac{1}{h\int f\,d\mu}\, Z(1) \tag{1.2}$$

where μ is the equilibrium state of $-hf$. This uses the fact that

$$\left[\frac{dP(-shf)}{ds}\right]_{s=1} = -h\int f\,d\mu$$

(Ruelle [15]).

2. Axiom A flows.

Let M be a compact Riemannian manifold and let φ be a C^1-flow on M. A compact φ-invariant set Λ containing no fixed points is said to be *hyperbolic* if the tangent bundle restricted to Λ can be written as the Whitney sum of three $D\varphi$-invariant continuous sub-bundles

$$T_\Lambda M = E + E^s + E^u$$

where E is the one-dimensional bundle tangent to the flow and there exist constants $C, \lambda > 0$ such that

(a) $\| D\varphi_t(v) \| \le C e^{-\lambda t} \| v \|$ for $v \in E^s$, $t \ge 0$

(b) $\| D\varphi_{-t}(v) \| \le C e^{-\lambda t} \| v \|$ for $v \in E^u$, $t \ge 0$.

A hyperbolic set Λ is said to be *basic* if

(i) the periodic orbits of φ restricted to Λ are dense in Λ

(ii) φ restricted to Λ is topologically transitive (i.e., Λ contains a dense orbit)

(iii) there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(U).$$

The non-wandering set Ω is defined by

$$\Omega = \{ x \in M : \forall \text{ open } V \ni x \ \exists \text{ a sequence } t_i \uparrow \infty \text{ with } \varphi_{t_i}(V) \cap V \neq \emptyset \}.$$

The flow satisfies *Axiom A* if Ω is a disjoint union of a finite number of basic sets and

hyperbolic fixed points. In what follows we will consider φ restricted to a basic set which is non-trivial (i.e., consists of more than one closed orbit).

Topological weak-mixing for φ is defined in the same way as for σ^f in the previous section. Bowen [3] has shown that either φ is not weak-mixing or φ is mixing with respect to the measure of maximal entropy.

As in the suspended flow case we define the zeta function for φ by

$$\prod_{\tau} \left(1 - N(\tau)^{-s}\right)^{-1}$$

where the product is taken over all closed φ-orbits τ of least period $\lambda(\tau)$ and $N(\tau)=e^{h(\varphi)\lambda(\tau)}$.

We can relate Axiom A flows to suspended flows by means of the following result due to Bowen [5].

Proposition 2. If φ is an Axiom A flow restricted to a (non-trivial) basic set Λ then there exists a suspension of a shift of finite type $\Sigma_{A,f}$ (where $f \in F_\theta$ for some $0 < \theta < 1$), and a Hölder continuous map $\pi : \Sigma_{A,f} \to \Lambda$, $\pi\sigma^f{}_t = \varphi_t\pi$, where π is surjective, finite-one, measure-preserving with respect to the measures of maximal entropy and one-one a.e. with respect to the measure of maximal entropy for σ^f.

We call σ^f the principal suspension. As a consequence of the above, φ is topologically weak-mixing if and only if σ^f is topologically weak-mixing. Clearly if σ^f is topologically weak-mixing then φ is topologically weak-mixing. On the other hand, if φ is topologically mixing then, by the comment above, it is mixing with respect to its measure of maximal entropy, hence σ^f is mixing with respect to its measure of maximal entropy, so it must be topologically weak-mixing. It also follows that $h(\varphi)=h(\sigma^f)$, since π is finite-one.

We wish to count the number of closed φ-orbits. We shall do this by means of the following proposition of Bowen [5] which is a refinement of work of Manning for the diffeomorphism case [10].

Proposition 3. (Bowen-Manning). In addition to the principal suspension, there exist suspensions of shifts of finite type, Σ_{A_i,f_i}, $f_i \in F_\theta$, $i=1,\ldots,p,\ldots,q$, Hölder continuous maps $\pi_i : \Sigma_{A_i,f_i} \to A$, $\pi_i\sigma^{f_i} = \varphi_i\pi_i$, where

(i) π_i is finite-one

(ii) π_i is not surjective

(iii) if $\upsilon(.,x)$ denotes the number of closed orbits of least period x then

$$\upsilon(\varphi,x) = \upsilon(\sigma^{f},x) + \sum_{i=1}^{p} \upsilon(\sigma^{f_i},x) - \sum_{i=p+1}^{q} \upsilon(\sigma^{f_i},x).$$

By (i) and (ii) we have that $h(\sigma^{f_i}) < h(\varphi)$, $i=1,\ldots,p,\ldots,q$ and from (iii) we obtain

$$\zeta_\varphi(s) = \zeta_{\sigma^f}(s) \frac{\prod_{i=1}^{p} \zeta_{\sigma^{f_i}}(hs/h(\sigma^{f_i}))}{\prod_{i=p+1}^{q} \zeta_{\sigma^{f_i}}(hs/h(\sigma^{f_i}))} \tag{2.1}$$

and

$$\mathrm{Res}(\zeta_\varphi,1) = \mathrm{Res}(\zeta_{\sigma^f},1) \frac{\prod_{i=1}^{p} \zeta_{\sigma^{f_i}}(h\ /h(\sigma^{f_i}))}{\prod_{i=p+1}^{q} \zeta_{\sigma^{f_i}}(h\ /h(\sigma^{f_i}))}. \tag{2.2}$$

Finally we remark that Parry and Pollicott [12] have shown that for a weak-mixing Axiom A flow

$$\pi(x) = \mathrm{card}\{\,\tau : N(\tau) \le x\,\} \sim \frac{x}{\log x}$$

and for an Axiom A flow that is not weak-mixing with least positive eigenfrequency a

$$\pi(x) \sim \frac{2\pi h/a}{\log x} \sum_{e^{2\pi nh/a} \le x} e^{2\pi nh/a}.$$

In either case there exist positive constants A, B such that

$$A \frac{x}{\log x} \le \pi(x) \le B \frac{x}{\log x} \qquad \text{for all large } x \tag{2.3}$$

(Bowen [4]).

3. The main theorem.

We will prove our result first for suspended flows and then use the results of the previous section to carry it over to Axiom A flows.

Let $\Sigma_{A,f}$, σ^f be as in section 1 and set $a = \inf\{f(\xi) : \xi \in \Sigma_A\}$, $b = \sup\{f(\xi) : \xi \in \Sigma_A\}$ and $y = (\log x)/h$. We begin by considering

$$K(x) = \sum_{N(\tau) \le x} \sum_{k=1}^{[y/\lambda(\tau)]} \frac{1}{k} N(\tau)^{-k} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi).$$

where Fix_n' denotes the set of $\xi \in \mathrm{Fix}_n$ with $f^n(\xi) \le y$.

First note that if $f^n(\xi) \le y$ for some $\xi \in \mathrm{Fix}_n$ then $na \le y$, so $n \le [\,y/a\,] + 1$ (here $[\ ]$ denotes integral part). Thus we have in fact

$$K(x) = \sum_{n=1}^{[y/a]+1} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi).$$

We split the range of summation into $1 \le n \le [\, y/\int f\, d\mu \,]$ and $[\, y/\int f\, d\mu \,] + 1 \le n \le [\, y/a \,]$ and note that if $f^n(\xi) > y$ for some $\xi \in \mathrm{Fix}_n$ then $nb > y$, so $n \ge [\, y/b \,] + 1$. Thus

$$K(x) = \sum_{n=1}^{[y/\int f\, d\mu]} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n} \exp -hf^n(\xi)$$

$$+ \sum_{n=[y/\int f\, d\mu]+1}^{[y/a]+1} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi) \quad - \sum_{n=[y/b]+1}^{[y/\int f\, d\mu]} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n''} \exp -hf^n(\xi)$$

(where Fix_n'' denotes the set of $\xi \in \mathrm{Fix}_n$ with $f^n(\xi) > y$)

$$= \sum_{n=1}^{[y/\int f\, d\mu]} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n} \exp -hf^n(\xi) \;+\; A(x) \;-\; B(x), \qquad \text{say.}$$

Now

$$\sum_{n=1}^{[y/\int f\, d\mu]} \frac{1}{n} = \log [y/\int f\, d\mu] + \gamma + o(1)$$

$$= \log \log x + \log \frac{1}{h\int f\, d\mu} + \gamma + o(1)$$

where γ is Euler's constant and Proposition 1 gives

$$\sum_{n=1}^{[y/\int f\, d\mu]} \frac{1}{n} \left\{ \sum_{\xi \in \mathrm{Fix}_n} \exp -hf^n(\xi) - 1 \right\} = \log Z(1) + o(1).$$

Combining this and (1.2) yields

$$K(x) = \log\log x + \gamma + \log \mathrm{Res}(\zeta_{\sigma'},1) + A(x) - B(x).$$

Our aim is now to show that $A(x) = o(1)$, $B(x) = o(1)$. We do this by means of the next two lemmas. Choose $0 < \varepsilon < \int f\,d\mu$, and write $A(x) = A_1(x) + A_2(x)$, $B(x) = B_1(x) + B_2(x)$, where

$$A_1(x) = \sum_{n=\lfloor y/(\int f\,d\mu - \varepsilon)\rfloor + 1}^{\lfloor y/a\rfloor + 1} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi),$$

$$A_2(x) = \sum_{n=\lfloor y/\int f\,d\mu\rfloor + 1}^{\lfloor y/(\int f\,d\mu - \varepsilon)\rfloor} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi),$$

$$B_1(x) = \sum_{n=\lfloor y/b\rfloor + 1}^{\lfloor y/(\int f\,d\mu + \varepsilon)\rfloor} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n''} \exp -hf^n(\xi),$$

and

$$B_2(x) = \sum_{n=\lfloor y/(\int f\,d\mu + \varepsilon)\rfloor + 1}^{\lfloor y/\int f\,d\mu\rfloor} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}_n''} \exp -hf^n(\xi).$$

By the ergodic theorem $f^n(\eta)/n \to \int f\,d\mu$ as $n \to \infty$ for μ-a.e. η, so we can choose N so large that for every $n \geq N$

$$\mu\left(\left\{\eta \in \Sigma_A : \left|\frac{f^n(\eta)}{n} - \int f\,d\mu\right| > \tfrac{1}{2}\varepsilon\right\}\right) < \varepsilon \tag{3.1}$$

and put

$$N' = \max\left(N, \left[\frac{2|f|_\theta}{\varepsilon(1-\theta)}\right] + 1\right).$$

Lemma 1. For $x \geq e^{N'b}$, i.e. for $y \geq N'b$,

$$A_1(x) < \left(\frac{\int f\,d\mu}{a} + \frac{2\int f\,d\mu}{N'b} - 1\right) C_2\,\varepsilon,$$

$$B_1(x) < \left(\frac{b}{\int f\,d\mu} - 1 + \frac{2}{N'}\right) C_2\,\varepsilon$$

where C_2 is defined by (1.1).

Proof. Suppose $[y/(\int f\,d\mu - \varepsilon)] + 1 \leq n \leq [y/a] + 1$ and $f^n(\xi) \leq y$, for some $\xi \in \mathrm{Fix}_n$. Then

$$\frac{f^n(\xi)}{n} < \int f\,d\mu - \varepsilon.$$

For n in this range we have

$$n \geq \frac{N'b}{\int f\,d\mu - \varepsilon} \geq N' > \frac{2|f|_\theta}{\varepsilon(1-\theta)},$$

so for every $\eta \in \{\xi_0, \xi_1, \ldots, \xi_{n-1}\}$,

$$\left|\frac{f^n(\eta)}{n} - \frac{f^n(\xi)}{n}\right| \leq \frac{|f|_\theta}{n}\left(1 + \theta + \ldots + \theta^{n-1}\right) \leq \frac{|f|_\theta}{n(1-\theta)} < \tfrac{1}{2}\varepsilon$$

and so

$$\frac{f^n(\eta)}{n} < \frac{f^n(\xi)}{n} + \tfrac{1}{2}\varepsilon < \int f\,d\mu - \tfrac{1}{2}\varepsilon.$$

Thus $\xi \in \mathrm{Fix}_{n,\varepsilon}$ where $\mathrm{Fix}_{n,\varepsilon}$ denotes the set of those $\xi \in \mathrm{Fix}_n$ such that for every $\eta \in \{\xi_0, \xi_1, \ldots, \xi_{n-1}\}$,

$$\left|\frac{f^n(\eta)}{n} - \int f\,d\mu\right| > \tfrac{1}{2}\varepsilon.$$

Hence for $[y/(\int f\,d\mu - \varepsilon)] + 1 \le n \le [y/a] + 1$,

$$\sum_{\xi \in \mathrm{Fix}_n'} \exp -hf^n(\xi) \le \sum_{\xi \in \mathrm{Fix}_{n,\varepsilon}} \exp -hf^n(\xi)$$

$$\le C_2\,\mu\Big(\bigcup_{\xi \in \mathrm{Fix}_{n,\varepsilon}} [\xi_0, \xi_1, \ldots, \xi_{n-1}]\Big) \qquad \text{by (1.1)}$$

$$< C_2\,\varepsilon \qquad \text{by (3.1).}$$

A similar argument gives that for $[y/b] + 1 \le n \le [y/(\int f\,d\mu + \varepsilon)]$,

$$\sum_{\xi \in \mathrm{Fix}_n''} \exp -hf^n(\xi) < C_2\,\varepsilon.$$

Thus

$$A_1(x) \le \left\{ \sum_{n=[y/(\int f\,d\mu - \varepsilon)]+1}^{[y/a]+1} \frac{1}{n} \right\} C_2\,\varepsilon$$

$$\le \frac{[y/a] - [y/(\int f\,d\mu - \varepsilon)] + 1}{[y/(\int f\,d\mu - \varepsilon)] + 1} C_2\,\varepsilon$$

$$\le \left\{ \frac{\int f\,d\mu - \varepsilon}{a} - 1 + \frac{2(\int f\,d\mu - \varepsilon)}{y} \right\} C_2\,\varepsilon \le \left\{ \frac{\int f\,d\mu}{a} - 1 + \frac{2\int f\,d\mu}{N^n b} \right\} C_2\,\varepsilon$$

and

$$B_1(x) \le \left\{ \sum_{n=[y/b]+1}^{[y/(\int f\,d\mu+\varepsilon)]} \frac{1}{n} \right\} C_2\varepsilon$$

$$\le \frac{[y/(\int f\,d\mu+\varepsilon)] - [y/b]}{[y/b]+1} C_2\varepsilon$$

$$\le \left\{ \frac{b}{\int f\,d\mu+\varepsilon} - 1 + \frac{2b}{y} \right\} C_2\varepsilon \le \left\{ \frac{b}{\int f\,d\mu} - 1 + \frac{2}{N'} \right\} C_2\varepsilon.$$

Lemma 2. For all $x > 1$

$$A_2(x) \le \left(\frac{\varepsilon}{\int f\,d\mu - \varepsilon} + \frac{2h\int f\,d\mu}{\log x} \right) C_2 B_2(x) \le \left(\frac{\varepsilon}{\int f\,d\mu} + \frac{2h\int f\,d\mu}{\log x} \right) C_2.$$

Proof. By (1.1), for every $n \ge 1$

$$\sum_{\xi \in \mathrm{Fix}_n} \exp -hf^{(n)}(\xi) \le C_2.$$

Hence

$$A_2(x) \le \left\{ \sum_{n=[y/\int f\,d\mu]+1}^{[y/(\int f\,d\mu-\varepsilon)]} \frac{1}{n} \right\} C_2$$

$$\le \frac{[y/(\int f\,d\mu-\varepsilon)] - [y/\int f\,d\mu]}{[y/\int f\,d\mu]+1} C_2$$

$$\le \left\{ \frac{\int f\,d\mu}{\int f\,d\mu-\varepsilon} - 1 + \frac{2h\int f\,d\mu}{\log x} \right\} C_2 = \left(\frac{\varepsilon}{\int f\,d\mu-\varepsilon} + \frac{2h\int f\,d\mu}{\log x} \right) C_2$$

and

$$B_2(x) \le \left\{ \sum_{n=[y/(\int f\,d\mu+\varepsilon)]+1}^{[y/\int f\,d\mu]} \frac{1}{n} \right\} C_2$$

$$\le \frac{[y/\int f\,d\mu] - [y/(\int f\,d\mu+\varepsilon)]}{[y/(\int f\,d\mu+\varepsilon)] + 1}\, C_2$$

$$\le \left\{ \frac{\int f\,d\mu+\varepsilon}{\int f\,d\mu} - 1 + \frac{2\int f\,d\mu}{y} \right\} C_2 = \left(\frac{\varepsilon}{\int f\,d\mu} + \frac{2h\int f\,d\mu}{\log x} \right) C_2 .$$

Since we may choose $\varepsilon > 0$ as small as we please, the two lemmas combine to give $A(x) = o(1)$, $B(x) = o(1)$, and so

$$K(x) = \log\log x + \gamma + \log \mathrm{Res}(\zeta_{\mathcal{O}_f}, 1) + o(1).$$

Now

$$\sum_{N(\tau)\le x} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) = \sum_{N(\tau)\le x} \sum_{k=1}^{\infty} \frac{1}{k} N(\tau)^{-k}$$

so

$$0 < \sum_{N(\tau)\le x} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) - K(x) = \sum_{N(\tau)\le x} \sum_{k=[y/\lambda(\tau)]+1}^{\infty} \frac{1}{k} N(\tau)^{-k}$$

$$\le \sum_{N(\tau)\le x} \frac{\log N(\tau)}{\log x} \sum_{k=2}^{\infty} \frac{1}{k} N(\tau)^{-k} = \frac{1}{\log x} \sum_{N(\tau)\le x} \frac{\log N(\tau)}{N(\tau)\,(N(\tau) - 1)}$$

and this last term tends to 0 as $x \to \infty$, since

$$\sum_{\tau} \frac{\log N(\tau)}{N(\tau)\,(N(\tau) - 1)}$$

converges. (To see this, write the sum as a Stieltjes integral with respect to $\pi(x)$,

partially integrate and apply (2.3).) Hence

$$\sum_{N(\tau)\le x} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) = \log\log x + \gamma + \log \mathrm{Res}(\zeta_{\sigma}, 1) + o(1).$$

Now let φ be an Axiom A flow and let $f, f_1, \ldots, f_p, \ldots, f_q$ be as in Propositions 2 and 3. By (2.1) and (2.2)

$$\sum_{\substack{N(\tau)\le x \\ \tau \text{ a } \varphi\text{-orbit}}} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) - \sum_{\substack{N(\tau)\le x \\ \tau \text{ a } \sigma\text{-orbit}}} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) \to \log \frac{\mathrm{Res}(\zeta_{\varphi}, 1)}{\mathrm{Res}(\zeta_{\sigma}, 1)}$$

as $x \to \infty$ and so for closed φ-orbits we also have

$$\sum_{N(\tau)\le x} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) = \log\log x + \gamma + \log \mathrm{Res}(\zeta_{\varphi}, 1) + o(1). \tag{3.2}$$

Now note

$$0 < \sum_{N(\tau)\le x} \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) - \sum_{N(\tau)\le x} \frac{1}{N(\tau)} = \sum_{N(\tau)\le x} \sum_{k=2}^{\infty} \frac{1}{k} N(\tau)^{-k}$$

$$\le \sum_{N(\tau)\le x} \frac{1}{2\, N(\tau)\, (N(\tau) - 1)}.$$

It is easy to see, by the same argument as above, that this last sum converges as $x \to \infty$ and, since

$$\sum_{N(\tau)\le x} \left\{ \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) - \frac{1}{N(\tau)} \right\}$$

is increasing, this ensures the convergence of

$$\sum_{\tau} \left\{ \log\left(\frac{1}{1 - N(\tau)^{-1}}\right) - \frac{1}{N(\tau)} \right\}. \tag{3.3}$$

As an immediate consequence of (3.2) and (3.3) we have the following theorem.

Theorem 1. For an Axiom A flow φ (restricted to a non-trivial basic set)

$$\prod_{N(\tau)\le x}\left(1-\frac{1}{N(\tau)}\right) \sim \frac{e^{-\gamma}}{\mathrm{Res}(\zeta_\varphi,1)\log x}$$

and

$$\sum_{N(\tau)\le x}\frac{1}{N(\tau)} = \log\log x + B + o(1)$$

where the constant B is given by

$$B = \gamma + \log \mathrm{Res}(\zeta_\varphi,1) - \sum_{\tau}\left\{\log\left(\frac{1}{1-N(\tau)^{-1}}\right) - \frac{1}{N(\tau)}\right\}.$$

Remarks. (i) Since this paper was first written, Mark Pollicott has pointed out to the author that, in the weak-mixing case, a considerably shorter but non-elementary proof of Theorem 1 is possible using a complex Tauberian theorem due to Agmon [2]. To apply this result one needs the additional information that if φ is weak-mixing then, apart from the simple pole at $s=1$, $\zeta_\varphi(s)$ is analytic and non-zero in a neighbourhood of $\mathrm{Re}\, s \ge 1$ [12]. An advantage of this approach is that it sharpens the error term in the expression for $\sum_{N(\tau)\le x} 1/N(\tau)$ from $o(1)$ to $o(1/\log x)$. This comment also applies to the results in the next two sections.

(ii) As we noted in section 2, for a weak-mixing Axiom A flow, $\pi(x) \sim x/\log x \sim \mathrm{li}\, x$ where

$$\mathrm{li}\, x = \int_2^\infty \frac{1}{\log t}\,dt.$$

It is interesting to have information about the error term in this theorem, i.e. $\psi(x) = \pi(x) - \mathrm{li}\, x$. It is known that for the geodesic flow on a manifold of constant negative curvature, $\psi(x) = O(x^\alpha)$ for some $0 < \alpha < 1$ [9]. On the other hand, for a

suspended flow with the suspending function depending on only finitely many coordinates this can never be the case [14]. Our result has some bearing on the average behaviour of $\psi(x)$. We have, using the fact that $\mathrm{li}\, x = x/\log x + O(x/(\log x)^2)$,

$$\sum_{N(\tau)\le x} \frac{1}{N(\tau)} = \int_2^x \frac{1}{t}\, d\pi(t) = \frac{\pi(x)}{x} + \int_2^x \frac{\pi(t)}{t^2}\, dt$$

$$= \int_2^x \frac{1}{t \log t}\, dt + \int_2^x \frac{\psi(t)}{t^2}\, dt + O\left(\frac{1}{\log x}\right) = \log\log x + \int_2^x \frac{\psi(t)}{t^2}\, dt + O\left(\frac{1}{\log x}\right)$$

(modulo a constant) and comparing this with our result reveals that

$$\int_2^\infty \frac{\psi(t)}{t^2}\, dt$$

converges.

(iii) The asymptotic formulae of Theorem 1 also hold for an Axiom A diffeomorphism φ. To see this consider the time-1 suspension flow. Then $N(\tau)$ has the same value whether τ is regarded as a closed orbit of the diffeomorphism or of the flow. The function $\zeta_\varphi(s)$ is defined by $\zeta_\varphi(s) = \zeta(e^{-hs})$ where $\zeta(z)$ is the usual Artin-Mazur zeta function for φ:

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \operatorname{card}\{ x : \varphi^n x = x \}.$$

We now turn our attention to the situation considered by Parry and Pollicott in [13]. Let $\tilde\varphi : \tilde M \to \tilde M$ be an Axiom A flow and let G be a finite group of diffeomorphisms which acts freely on $\tilde M$ and commutes with $\tilde\varphi$. This gives rise to a flow φ on the quotient manifold $M = \tilde M/G$, defined by $\varphi_t(Gx) = G(\tilde\varphi_t x)$. It can be shown that φ is also an Axiom A flow. We shall suppose that $\tilde\Lambda$ is a (non-trivial) basic set for $\tilde\varphi$, then $\Lambda = \tilde\Lambda/G$ is a basic set for φ. As usual, we shall consider $\tilde\varphi$, φ restricted to $\tilde\Lambda$, Λ

respectively.

For any closed φ-orbit τ, let $\tilde{\tau}_1,\ldots,\tilde{\tau}_n$ be the closed $\tilde{\varphi}$-orbits which lie above τ. Then $n \mid |G|$ and

$$\lambda(\tilde{\tau}_i) = \frac{|G|}{n}\lambda(\tau), \qquad i = 1,\ldots,n.$$

For each $\tilde{\tau}_i$ there exists a unique *Frobenius element* $[\tilde{\tau}_i] \in G$ such that $[\tilde{\tau}_i]\, x = \tilde{\varphi}_{\lambda(\tau)} x$ for every $x \in \tilde{\tau}_i$. If $g\,\tilde{\tau}_1 = \tilde{\tau}_j$, $g \in G$ then $[\tilde{\tau}_j] = g\,[\tilde{\tau}_i]\,g^{-1}$ so that the *Frobenius class*, the conjugacy class of $[\tilde{\tau}_i]$ depends only on τ.

Choose $g \in G$ and let $C = C(g)$ be its conjugacy class. Let R_χ be an irreducible representation of G with irreducible character χ, and define

$$L(s,\chi) = \prod_{\tau} \det\left(I - \frac{R_\chi(\tau)}{N(\tau)^s}\right)^{-1}$$

where $R_\chi(\tau) = R_\chi([\tilde{\tau}])$ for any $\tilde{\tau}$ lying above τ. This product converges for $\mathrm{Re}(s)>1$. Clearly, if χ_0 denotes the trivial character, $L(s,\chi_0) = \zeta_{\varphi}(s)$. On the other hand, if $\chi \neq \chi_0$, then $L(s,\chi)$ is analytic in a neighbourhood of $\mathrm{Re}(s)=1$ [13].

By the orthogonality relation for characters and (3.2)

$$\sum_{\substack{N(\tau)\le x\\ [\tilde{\tau}]\in C}} \log\left(1 - \frac{1}{N(\tau)}\right) = \sum_{\substack{N(\tau)\le x\\ [\tilde{\tau}]\in C}} \sum_{k=1}^{\infty} \frac{1}{k} N(\tau)^{-k}$$

$$= \frac{|C|}{|G|} \sum_{N(\tau)\le x} \sum_{k=1}^{\infty} \frac{1}{k} N(\tau)^{-k} + \frac{|C|}{|G|} \sum_{\chi\neq\chi_0} \chi(g^{-1}) \sum_{N(\tau)\le x} \sum_{k=1}^{\infty} \frac{\chi([\tilde{\tau}])}{k} N(\tau)^{-k}$$

$$= \frac{|C|}{|G|} \log\log x + \frac{|C|}{|G|}\gamma + \frac{|C|}{|G|} \log \mathrm{Res}(\zeta_{\varphi},1)$$

$$+ \frac{|C|}{|G|} \sum_{\chi\neq\chi_0} \chi(g^{-1}) \log L(1,\chi) + o(1).$$

Hence we have:

Corollary 1.

$$\prod_{\substack{N(\tau)\le x\\ [\tau]\in C}} \left(1 - \frac{1}{N(\tau)}\right) \sim \frac{e^{-(|C|/|G|)\gamma}}{A\,(\log x)^{|C|/|G|}}$$

where

$$A = \mathrm{Res}(\zeta_\varphi,1)^{|C|/|G|} \left\{ \prod_{\chi\neq\chi_0} e^{-\chi(g^{-1})}\, L(1,\chi) \right\}^{|C|/|G|}$$

and

$$\sum_{\substack{N(\tau)\le x\\ [\tau]\in C}} \frac{1}{N(\tau)} = \frac{|C|}{|G|} \log\log x + \text{constant} + o(1).$$

(This last formula is deduced in the same way as in Theorem 1.)

4. Non-compact quotients.

Let $\mathbb{H}^+$ denote the upper half plane $\{ x + iy \in \mathbb{C} : y > 0 \}$ equiped with the Poincaré metric $ds^2 = (dx^2 + dy^2)/y^2$. Let Γ be a discrete subgroup of $PSL(2,\mathbb{R})$, then Γ acts on $\mathbb{H}^+$ as linear fractional transformations, $z \to (az + b)/(cz + d)$, and these transformations are isometries of $\mathbb{H}^+$ with respect to the Poincaré metric. The surface S is the quotient space $\mathbb{H}^+/\Gamma$. With respect to the induced metric, this surface has curvature -1.

Let T_1S be the unit tangent bundle of S and let φ be the geodesic flow on T_1S, i.e. for $(x,v) \in T_1S$, $\varphi_t(x,v)$ is the point reached by starting at (x,v) and flowing for time t along the unique unit-speed geodesic through x in the direction v. Such flows have $h(\varphi) = 1$. There is an exact correspondence between closed geodesics on S and closed φ-orbits. We define the norm of a closed geodesic τ of length $\lambda(\tau)$ to be $N(\tau) = e^{\lambda(\tau)}$

and define

$$\zeta_\Gamma(s) = \prod_\tau (1 - N(\tau)^{-s})^{-1}.$$

If S is compact then φ satisfies Axiom A and the analysis of the previous section applies (with $\zeta_\Gamma(s) = \zeta_\varphi(s)$). We now consider the case where S is not compact but has finite area (with respect to the Riemann measure). It remains true in this situation that $\zeta_\Gamma(s)$ is analytic and non-zero for $\mathrm{Re}(s) > 1$ with a simple pole at $s=1$.

Let Δ be the Laplace-Beltrami operator on S, and let

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_k \le 3/16$$

be the discrete eigenvalues of $-\Delta$ in $[0,3/16]$. Write $\alpha_j = \frac{1}{2} + \sqrt{(1/4 - \lambda_j)}$, $j = 1,\ldots,k$, so that $\frac{1}{2} \le \alpha_j < 1$. We have the following result.

Proposition 4. (Sarnak and Woo, cf. [16].) Let $\pi(x)$ denote the number of closed geodesics τ on S with $N(\tau) \le x$, then

$$\pi(x) = \mathrm{li}\, x + \mathrm{li}\, x^{\alpha_1} + \ldots + \mathrm{li}\, x^{\alpha_k} + O(x^{3/4}(\log x)^2).$$

Since $\mathrm{li}\, x = x/\log x + O(x/(\log x)^2)$, we have

$$\pi(x) = x/\log x + O(x/(\log x)^2). \tag{4.1}$$

Theorem 2. For a non-compact, finite area surface of constant negative curvature $S = H^+/\Gamma$, we have

$$\sum_{N(\tau)\le x} \frac{1}{N(\tau)} = \log\log x + B + O\left(\frac{1}{\log x}\right)$$

B constant, and

$$\prod_{N(\tau)\le x}\left(1-\frac{1}{N(\tau)}\right)\sim\frac{e^{-\gamma}}{\mathrm{Res}(\zeta_{\Gamma},1)\log x}.$$

Proof. Let N_0 be the norm of the shortest geodesic on M. Then

$$\sum_{N(\tau)\le x}\frac{1}{N(\tau)}=\int_{N_0}^{x}\frac{1}{t}\,d\pi(t)=\int_{N_0}^{x}\frac{\pi(t)}{t^2}dt+\frac{\pi(x)}{x}$$

$$=\log\log x-\log\log N_0+\int_{N_0}^{x}\frac{\pi(t)-t/\log t}{t^2}\,dt+\frac{\pi(x)}{x}$$

By (4.1),

$$\int_{N_0}^{x}\frac{\pi(t)-t/\log t}{t^2}\,dt=\int_{N_0}^{\infty}\frac{\pi(t)-t/\log t}{t^2}\,dt+O\left(\frac{1}{\log x}\right)$$

and $\pi(x)/x=O(1/\log x)$, so we have

$$\sum_{N(\tau)\le x}\frac{1}{N(\tau)}=\log\log x+B+O\left(\frac{1}{\log x}\right)$$

where

$$B=\int_{N_0}^{\infty}\frac{\pi(t)-t/\log t}{t^2}\,dt-\log\log N_0.$$

By the same argument as in the previous section, the series

$$\sum_{\tau}\left\{\log\left(1-\frac{1}{N(\tau)}\right)+\frac{1}{N(\tau)}\right\}$$

is convergent, with sum F say. Hence

$$\log\prod_{N(\tau)\le x}\left(1-\frac{1}{N(\tau)}\right)=-\log\log x-B+F+o(1)$$

and so to deduce the result we only need to show that $F - B = -\gamma - \log \mathrm{Res}(\zeta_\Gamma, 1)$.

We shall use the fact that

$$\gamma = -\int_0^\infty e^{-u} \log u \, du.$$

If $\delta \geq 0$ it is easy to see that

$$0 < -\log(1 - N(\tau)^{-1-\delta}) - N(\tau)^{-1-\delta} \leq \tfrac{1}{2} N(\tau)^{-1} (N(\tau) - 1)^{-1}.$$

Hence the series

$$F(\delta) = \sum_\tau \left\{ \log\left(1 - N(\tau)^{-1-\delta}\right) + N(\tau)^{-1-\delta} \right\}$$

converges uniformly for all $\delta \geq 0$ and so $F(\delta) \to F(0) = F$ as $\delta \to 0$.

Now suppose $\delta > 0$. Then

$$F(\delta) = G(\delta) - \log \zeta_\Gamma(1+\delta)$$

where

$$G(\delta) = \sum_\tau N(\tau)^{-1-\delta}.$$

Write

$$L(x) = \sum_{N(\tau) \leq x} \frac{1}{N(\tau)} = \log\log x + B + E(x), \qquad E(x) = O\left(\frac{1}{\log x}\right).$$

Then

$$\sum_{N(\tau) \leq x} N(\tau)^{-1-\delta} = x^{-\delta} L(x) + \delta \int_{N_0}^{x} t^{-1-\delta} L(t) \, dt$$

If we let $x \to \infty$, $x^{-\delta} L(x) \to 0$, so

$$G(\delta) = \delta \int_{N_0}^{\infty} t^{-1-\delta} L(t) \, dt = \delta \int_{N_0}^{\infty} t^{-1-\delta} (\log\log t + B) \, dt + \delta \int_{N_0}^{\infty} t^{-1-\delta} E(t) \, dt.$$

Put $t = e^{u/\delta}$. Then

$$\delta \int_1^\infty t^{-1-\delta} \log\log t \, dt = \int_0^\infty e^{-u} \log(u/\delta)\, du = -\gamma - \log\delta$$

and

$$\delta \int_1^\infty t^{-1-\delta} dt = 1.$$

Hence

$$G(\delta) + \log\delta - B + \gamma = \delta \int_{N_0}^\infty t^{-1-\delta} E(t)\, dt - \delta \int_1^{N_0} t^{-1-\delta} (\log\log t + B)\, dt.$$

Now if $T = \exp(1/\sqrt{\delta})$,

$$\left| \delta \int_{N_0}^\infty t^{-1-\delta} E(t)\, dt \right| \le \text{const.}\ \delta \int_{N_0}^T t^{-1}\, dt + \frac{\text{const.}\ \delta}{\log T} \int_T^\infty t^{-1-\delta}\, dt$$

$$\le \text{const.}\ \delta \log T + \text{const.}\ \frac{T^{-\delta}}{\log T} \le \text{const.}\ \sqrt{\delta} + \text{const.}\ e^{-\sqrt{\delta}}\, \delta^{1/2} \to 0 \quad \text{as } \delta \to 0.$$

Also,

$$\left| \delta \int_1^{N_0} t^{-1-\delta} (\log\log t + B)\, dt \right| < \delta \int_1^{N_0} t^{-1} (|\log\log t| + |B|)\, dt \to 0 \quad \text{as } \delta \to 0.$$

Hence $G(\delta) + \log\delta \to B - \gamma$ as $\delta \to 0$, but $\log \zeta_\Gamma(1+\delta) + \log\delta \to \log \mathrm{Res}(\zeta_\Gamma, 1)$ as $\delta \to 0$, so $F(\delta) \to B - \gamma - \log \mathrm{Res}(\zeta_\Gamma, 1)$, i.e. $F = B - \gamma - \log \mathrm{Res}(\zeta_\Gamma, 1)$ and the proof is complete.

Remark. The above proof was inspired by [8], pp.349-353.

Now suppose that $W = \mathbb{H}^+/\Gamma(W)$ and $S = \mathbb{H}^+/\Gamma(S)$, where $\Gamma(W)$, $\Gamma(S)$ are discrete co-finite area subgroups of $PSL(2,\mathbb{R})$, with $\Gamma(W)$ a normal subgroup of $\Gamma(S)$ such that $G = \Gamma(S)/\Gamma(W)$ is finite. Then S is the quotient of W by the group G and a result analogous to Corollary 1 holds. As in the Axiom A case, lying above each closed geodesic τ in S there are a finite number of closed geodesics $\tilde{\tau}_1,\ldots,\tilde{\tau}_n$ in W. As before, each $\tilde{\tau}_i$ gives rise to a unique Frobenius element $[\tilde{\tau}_i] \in G$, the conjugacy class of which depends only on τ. If $n = |G|$ then we say that τ splits completely. This is the case if and only if $[\tilde{\tau}_i]$ is the identity, $1 \le i \le n$.

Once again let R_χ be an irreducible representation of G with irreducible character χ and define

$$L(s,\chi) = \prod_{\tau} \det\left(I - \frac{R_\chi([\tilde{\tau}])}{N(\tau)^s}\right)^{-1}$$

where $\tilde{\tau}$ lies over τ. If χ_0 denotes the trivial character, then $L(s,\chi) = \zeta_{\Gamma(S)}(s)$ and for $\chi \neq \chi_0$, $L(s,\chi)$ is analytic in a neighbourhood of $s = 1$. Applying the analysis of Section 3, we obtain the following.

Corollary 2. Let $g \in G$ and $C = C(g)$ be its conjugacy class, then

$$\prod_{\substack{N(\tau)\le x \\ [\tilde{\tau}]\in C}} \left(1 - \frac{1}{N(\tau)}\right) \sim \frac{e^{-(|C|/|G|)\gamma}}{A\,(\log x)^{|C|/|G|}}$$

where

$$A = \mathrm{Res}(\zeta_{\Gamma(S)},1)^{|C|/|G|} \left\{\prod_{\chi\neq\chi_0} e^{-\chi(g^{-1})} L(1,\chi)\right\}^{|C|/|G|}$$

and

$$\sum_{\substack{N(\mathfrak{c})\le x \\ [\mathfrak{c}]\in C}} \frac{1}{N(\mathfrak{c})} = \frac{|C|}{|G|} \log\log x + \text{constant} + o(1).$$

5. The modular surface and quadratic forms.

A classical example of a non-compact, finite area surface of constant negative curvature is the modular surface, $\mathbb{H}^+/\Gamma$ where $\Gamma = PSL(2,\mathbb{Z})$. (In fact its area is $2\pi^2/3$.) We shall now describe the elegant relationship between geodesics on the modular surface and quadratic forms (cf. Sarnak [16]).

We consider quadratic forms that are primitive and indefinite, for example

$$Q(x,y) = ax^2 + bxy + cy^2$$

where $(a,b,c)=1$ and the discriminant $d=b^2-4ac$ satifies $d > 0$, $d \equiv 0, 1 \pmod 4$ and is not a perfect square. Denote the set of such d by D. Two such forms Q, Q' are called equivalent if we can transform one to the other by a substitution

$$x' = \alpha x + \beta y$$
$$y' = \gamma x + \delta y$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\alpha\delta-\beta\gamma = 1$. This relation partitions forms into classes and it is clear that two forms from the same class have the same discriminant. Gauss showed that the number of classes with a given discriminant $d > 0$ is finite, this number is denoted by h(d) [7].

The substitutions which preserve Q are called the automorphs of Q. All the automorphs of Q may be written in terms of solutions of Pell's equation

$$t^2 - du^2 = 4, \qquad t, u \in \mathbb{Z}, \qquad (t,u) \neq (0,0)$$

by choosing

$$\alpha = \tfrac{1}{2}(t - bu) \qquad \beta = -cu$$

$$\gamma = au \qquad \delta = \tfrac{1}{2}(t + bu).$$

Let (t_0,u_0) be the solution for which $\varepsilon_d = \frac{1}{2}(t_0 + u_0\sqrt{d})$ is least, then all solutions (t,u) are generated by

$$\tfrac{1}{2}(t + u\sqrt{d}) = \varepsilon_d^n, \qquad n \in \mathbb{Z}.$$

In [7] Gauss noticed that

$$\sum_{\substack{d\in D\\ d\le x}} h(d) \log \varepsilon_d = \frac{\pi^2 x^{3/2}}{18\,\zeta(3)} + O(x \log x)$$

(here $\zeta(s)$ is the Riemann zeta function) and this was later proved by Siegel [18]. One would like to be able to separate the quantities $h(d)$ and $\log \varepsilon_d$ to obtain an asymptotic formula for $\sum_{d\in D, d\le x} h(d)$, but this appears to be difficult and remains an unsolved problem. However, using the next proposition, Sarnak obtained an asymptotic expression for $h(d)$ summed over the sets $D_x = \{ d \in D : \varepsilon_d \le x \}$ [16].

Proposition 5. There is a bijection between closed geodesics on the modular surface and equivalence classes of quadratic forms. Furthermore a closed geodesic corresponding to an equivalence class with discriminant d has length $2 \log \varepsilon_d$.

Applying this correspondence to Proposition 4 and using the fact that the Laplace–Beltrami operator on the modular surface has no eigenvalues in $(0,3/16]$, Sarnak obtained

$$\sum_{d\in D_x} h(d) = \frac{(i(x^2)}{} + O\left(x^{3/2} (\log x)^2\right),$$

Remark. The function $d \to \varepsilon_d$ seems fairly irregular and nothing much better than $\sqrt{d} \le \varepsilon_d \le e^d$ is known.

We now combine Proposition 5 with the results described in Theorem 2 to give two new asymptotic formulae involving $h(d)$ and ε_d.

Proposition 6.

$$\prod_{d \in D_x} \left(1 - \varepsilon_d^{-2}\right)^{h(d)} \sim \frac{e^{-\gamma}}{2\,\mathrm{Res}(\zeta_{\Gamma^+},1)\log x}$$

and

$$\sum_{d \in D_x} h(d)\,\varepsilon_d^{-2} = \log\log x + \text{constant} + O\left(\frac{1}{\log x}\right).$$

Let $p \ge 3$ be a prime and let $\Gamma(p)$ be the principal congruence subgroup of Γ of level p, i.e.

$$\Gamma(p) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod{p} \right\}.$$

The suface $\mathbb{H}^+/\Gamma(p)$ is a finite regular covering of $\mathbb{H}^+/\Gamma$ and the covering group $G = \Gamma/\Gamma(p) \cong PSL(2,\mathbb{Z}/p\mathbb{Z})$. In this situation a geodesic on $\mathbb{H}^+/\Gamma$ splits completely if and only if $d \in D_p = \{ d \in D : p \mid u_0 \}$ (where $\varepsilon_d = \frac{1}{2}(t_0 + u_0\sqrt{d})$). Applying Corollary 2 and noting that $|G| = (p^2 - 1)p/2$, we have:

Corollary 3.

$$\prod_{d\in D_{p,x}} \left(1 - \varepsilon_d^{-2}\right)^{h(d)} \sim \frac{e^{-2\gamma/p(p^2-1)}}{A\, 4^{1/p(p^2-1)} (\log x)^{2/p(p^2-1)}}$$

where $D_{p,x} = \{ d \in D_p : \varepsilon_d \le x \}$,

$$A = \mathrm{Res}(\zeta_{\Gamma'},1)^{2/p(p^2-1)} \left\{ \prod_{\chi\neq\chi_0} e^{-\dim \chi} L(1,\chi) \right\}^{2/p(p^2-1)}$$

(the product being taken over all non-trivial irreducible characters of $PSL(2,\mathbb{Z}/p\mathbb{Z})$) and

$$\sum_{d\in D_{p,x}} h(d)\, \varepsilon_d^{-2} = \frac{2}{p(p^2-2)} \log\log x + \text{constant} + o(1).$$

Remark. Similar results may be obtained for quadratic forms over the ring of integers in $\mathbb{Q}(\sqrt{-D})$ where $D \in \mathbb{Z}$ is positive and square free by means of the correspondence between equivalence classes of such forms and geodesics on certain arithmetic three manifolds (Sarnak [17]).

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Chapter 2
Prime orbit theorems with multi-dimensional constraints for Axiom A flows

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Abstract.
In this paper we give a new proof of an asymptotic formula for the number of closed orbits of an Axiom A flow subject to certain constraints due to S. P. Lalley. We consider finite group extensions and, for transitive Anosov flows, give an application to homology. We also discuss asymptotics for closed orbits in a fixed homology class, extending a result of Katsuda and Sunada.

0. Introduction.

In [17] Parry and Pollicott established a prime orbit theorem for weak-mixing Axiom A flows. They showed that if τ denotes a closed orbit of the flow with least period $\lambda(\tau)$ then $\#\{\tau : \lambda(\tau) \le x\} \sim e^{hx}/hx$, where h is the topological entropy of the flow. (They also obtained a modified formula for flows which are not weak-mixing.) Their proof was based on techniques from analytic number theory. A zeta function associated with the flow was analysed and the information thus obtained was translated into an asymptotic formula by means of a Tauberian theorem.

In [10] Lalley proved the following interesting asymptotic formula. Let $F_1,\ldots,F_d$ be real valued functions defined on a basic set of the flow satisfying suitable conditions and let $\lambda_{F_i}(\tau)$ denote the integral of F_i around τ. Then, for $\delta_i > 0$,

$$\#\{\tau : \lambda(\tau) \le x, |\lambda_{F_i}(\tau)| \le \delta_i, i = 1,\ldots,d\} \sim \text{const. } e^{\alpha x}/x^{\frac{1}{2}d+1}$$

with an explicit formula for the constant and where α is the supremum of the measure theoretic entropies of the flow taken over all those measures with respect to which each

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function F_i has zero integral. Lalley's proof makes no use of zeta functions or Tauberian theorems and indeed he comments that he does not believe that his result can be obtained in this way. The purpose of this paper is to show that infact such a proof is possible.

The main part of the proof uses work of Katsuda and Sunada on asymptotics for the number of closed orbits in a homology class for a weak-mixing transitive Anosov flow [9] (cf. also Pollicott [20], [21]). We use an L-function, which is analogous to one that they consider but with two main differences. Firstly, the terms in our L-function are weighted differently and secondly, the function Katsuda and Sunada work with is defined on $\mathbb{C} \times$ (a compact group) whereas ours is defined on $\mathbb{C} \times \mathbb{R}^d$. To overcome this problem we introduce a probability density on $\mathbb{R}^d$ whose Fourier transform has compact support and then, after obtaining an asymptotic formula, use approximation arguments (which are incidently more direct than the 'unsmoothing' arguments used [10], [11]) to deduce the desired result.

Although our approach differs from Lalley's it shares a common basis in its reliance on both Bowen's modelling theory for hyperbolic flows which reduces the problem to one of symbolic dynamics and Ruelle's thermodynamic ideas. Also it should be noted that many of the ideas in this paper were motivated by a study of [9], [10] and [17].

After we have obtained Lalley's result we go on to prove a version of the theorem in the context of finite group extensions of Axiom A flows. Restricting our attention to transitive Anosov flows, this yields an application to homology. In the last section of this paper we return to a situation closer to that originally studied by Katsuda and Sunada and give an extension of the results they obtained in [9].

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1. Shifts of finite type.

Let A be an aperiodic $k \times k$ zero-one matrix and let

$$\Sigma_A = \{ x \in \{1,\dots,k\}^{\mathbb{Z}} : A(x_i,x_{i+1}) = 1 \ \forall\, i \in \mathbb{Z} \},$$

$$\Sigma_A^+ = \{ x \in \{1,\dots,k\}^{\mathbb{N}\cup\{0\}} : A(x_i,x_{i+1}) = 1 \ \forall\, i \geq 0 \}.$$

Give $\{1,\dots,k\}$ the discrete topology and Σ_A (Σ_A^+) the topology it inherits as a subspace of $\{1,\dots,k\}^{\mathbb{Z}}$ ($\{1,\dots,k\}^{\mathbb{N}\cup\{0\}}$) equipped with the product topology. Define the shift of finite type $\sigma : \Sigma_A \to \Sigma_A$ (one-sided shift of finite type $\sigma : \Sigma_A^+ \to \Sigma_A^+$) by $(\sigma x)_i = x_{i+1}$. With respect to the given topology σ is a homeomorphism (continuous map). The assumption that A is aperiodic is equivalent to σ being topologically mixing (i.e. for every pair of non-empty open sets U and V there exists $n \geq 0$ such that $\sigma^n(U) \cap V \neq \emptyset$).

We can define a metric on Σ_A (Σ_A^+) by $d(x,y) = (\frac{1}{2})^n$ where

$$n = \sup\{m : x_i = y_i \text{ for } |i| < m\}$$

($n = \sup\{m : x_i = y_i \text{ for } i < m\}$). Choose $\gamma > 0$ and let H_γ (H_γ^+) be the space of Hölder continuous functions on Σ_A (Σ_A^+) with exponent γ, i.e. those functions f for which there exist a constant $C > 0$ such that

$$|f(x) - f(y)| \leq C\, d(x,y)^\gamma.$$

Let $|f|_\gamma$ be the smallest possible choice of C satisfying this inequality, i.e.

$$|f|_\gamma = \sup \frac{|f(x) - f(y)|}{d(x,y)^\gamma}.$$

With respect to the norm $\|\,.\,\|_\gamma = |\,.\,|_\gamma + \|\,.\,\|_\infty$, H_γ (H_γ^+) is a Banach space.

For a real valued $f \in C(\Sigma_A)$ ($f \in C(\Sigma_A^+)$) define the pressure of f to be

$$P(f) = \sup \{ h_m(\sigma) + \int f\, dm : m \text{ a } \sigma\text{-invariant probability measure} \}.$$

If $f \in H_\gamma$ ($f \in H_\gamma^+$) then there is a unique ergodic probability measure μ, called the *equilibrium state* of f such that $P(f) = h_\mu(\sigma) + \int f\, d\mu$. Two functions $f,g \in C(\Sigma_A)$ are said to be *cohomologous* if there exists $u \in C(\Sigma_A)$ such that $f = g + u\circ\sigma - u$. Two functions in H_γ (H_γ^+) have the same equilibrium state if and only if their difference is

cohomologous to a constant [5]. Pressure is a convex fuctional, i.e. for $f,g \in H_\gamma$ and $0 < \alpha < 1$, $P(\alpha f + (1-\alpha)g) \le \alpha P(f) + (1-\alpha)P(g)$ and this inequality is strict unless $f-g$ is cohomologous to a constant. For $t \in \mathbb{R}$ the function $t \to P(tg + f)$ is real-analytic and

$$\left[\frac{dP(tg+f)}{dt}\right]_{t=0} = \int g \, d\mu$$

where μ is the equilibrium state of f. (Ruelle [23].)

The definition of pressure can be extended to complex valued functions in the following way. Let $f \in H_\gamma$ be real valued, then there exists a function $u \in H_{\frac{1}{2}\gamma}$ such that the function $f' = f + u\circ\sigma - u$ depends only on the future coordinates $(x_i)_{i\ge 0}$ and which can hence be interpreted as a function defined on Σ_A^+. With this interpretation $f' \in H_{\frac{1}{2}\gamma}^+$ It is possible to choose $u = u(f)$ so that the map $f \to f'$ is a bounded linear operator from H_γ to $H_{\frac{1}{2}\gamma}^+$ ([25]). Define the Ruelle operator $\mathcal{L}_{f'} : H_{\frac{1}{2}\gamma}^+ \to H_{\frac{1}{2}\gamma}^+$ by

$$\mathcal{L}_{f'}\psi(x) = \sum_{\sigma y = x} e^{f'(y)} \psi(y).$$

Note that the map $f' \to \mathcal{L}_{f'}$ is analytic and hence that the map $f \to \mathcal{L}_{f'}$ is analytic. The number $e^{P(f)} = e^{P(f')}$ is a simple eigenvalue of $\mathcal{L}_{f'}$ and the rest of the spectrum of $\mathcal{L}_{f'}$ is contained in a disc of radius less than $e^{P(f)}$. Suppose g is $\|\ \|_\gamma$-close to f so that g' is $\|\ \|_\gamma$-close to f' then perturbation theory ensures that $\mathcal{L}_{g'}$ has a simple eigenvalue $\rho(g)$ with $\rho(f) = e^{P(f)}$ and the rest of the spectrum of $\mathcal{L}_{g'}$ is contained in a disc of radius less than $|\rho(g)|$. Futhermore the map $g \to \rho(g)$ is analytic. We define $e^{P(g)} = \rho(g)$. Thus pressure is defined (modulo $2\pi i$) and analytic on a neighbourhood of the real valued functions in H_γ It can also be shown that if g_1 is cohomologous to g and K is valued in $2\pi i\mathbb{Z}$ then $P(g_1 + K) = P(g)$. ([15], [19].)

We shall be interested in the function $\zeta : H_\gamma \to \mathbb{C}$ defined by

$$\zeta(f) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in Fix_n} \exp f^n(x)$$

whenever the above series converges. (Here $Fix_n = \{ x : \sigma^n x = x \}$ and

$f^n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1}x)$.) Convergence occurs whenever

$$\overline{\lim} \left| \sum_{x \in \mathrm{Fix}_n} \exp f^n(x) \right|^{1/n} \leq \overline{\lim} \left| \sum_{x \in \mathrm{Fix}_n} \exp \mathrm{Re}\, f^n(x) \right|^{1/n} < 1.$$

The latter limit is equal to $e^{P(\mathrm{Re}\, f)}$ and so the series defining $\zeta(f)$ converges whenever $P(\mathrm{Re}\, f) < 0$. In this domain $\zeta(f)$ is analytic and non-zero. ([23].)

We require some information about the behaviour of ζ in a neighbourhood of $\{ f : P(\mathrm{Re}\, f) = 0 \}$. This is provided by the next proposition.

Proposition 1. (Parry [15], Pollicott [19].) If $P(\mathrm{Re}\, f) = 0$ then ζ has a non-zero analytic continuation to a neighbourhood of f provided Im f is not cohomologous to a function valued in $2\pi\mathbb{Z}$. If Im f is cohomologous to a function valued in $2\pi\mathbb{Z}$ then $P(f) = 0$. In this case ζ has a non-zero analytic continuation to the set $\{ g : 0 < \| f-g \|_\gamma < \varepsilon,\ P(g) \neq 0 \}$ (for some $\varepsilon > 0$) given by $\zeta(g) = \varphi(g)/(1-e^{P(g)})$ where φ is a function which is non-zero and analytic on $\{ g : \| f-g \|_\gamma < \varepsilon \}$.

2. Suspended flows and L-functions.

Let $r \in H_\gamma$ (for some $\gamma > 0$) be real valued and strictly positive. Define the *r-suspension space*

$$\Sigma^r = \{ (x,t) : x \in \Sigma_A,\ 0 \leq t \leq r(x) \}$$

with $(x,r(x))$ and $(\sigma x,0)$ identified. We can use the metric d on Σ_A to construct a metric on Σ^r. This construction is described in [6]. Define the suspended flow σ^r on Σ^r by $\sigma^r_t(x,s) = (x,s+t)$, taking identifications into account. We shall suppose that σ^r is (topologically) weak-mixing which is to say that there is no non-trivial solution to $F \circ \sigma^r_t = e^{iat} F$ with $a > 0$ and $F \in C(\Sigma^r)$. σ^r is weak-mixing if and only if r is <u>not</u> cohomologous to a function valued in a discrete subgroup of $\mathbb{R}$.

If $F \in C(\Sigma^r)$ is real valued define the pressure

$$P(F) = \sup \{ h_m(\sigma^r) + \int F\, dm : m \text{ a } \sigma^r\text{-invariant probability measure} \}.$$

Those probability measures for which this supremum is attained are called equilibrium states of F. Define $f : \Sigma_A \to \mathbb{R}$ by

$$f(x) = \int_0^{r(x)} F(\sigma_t^r(x,0))\, dt.$$

If f is Hölder continuous then the equation $P(-c\, r + f) = 0$ is uniquely satisfied by $c = P(F)$ and furthermore if μ is the equilibrium state of $-P(F)\, r + f$ then $m = (\mu \times \ell)/\int r\, d\mu$, where ℓ is one-dimensional Lebesgue measure, is the unique equilibrium state of F. (This makes use of a result of Abramov [1]). $P(0) = h(\sigma^r)$, the topological entropy of σ^r, and the equilibrium state of 0 is called the measure of maximal entropy.

Two functions $F,G \in C(\Sigma^r)$ are said to be cohomologous if there exists a function $U \in C(\Sigma^r)$ such that for every $\varepsilon > 0$ and every $(x,t) \in \Sigma^r$

$$\int_0^{\varepsilon} F(\sigma_s^r(x,t))\, ds - \int_0^{\varepsilon} G(\sigma_s^r(x,t))\, ds = U(\sigma_\varepsilon^r(x,t)) - U(x,t).$$

As is the case for shifts, two functions have the same equilibrium state if and only if their difference is cohomologous to a constant.

Let $F_1,\dots,F_d \in C(\Sigma^r)$ be real valued functions such that the functions $f_1,\dots,f_d$ defined on Σ_A by

$$f_i(x) = \int_0^{r(x)} F_i(\sigma_t^r(x,0))\, dt, \qquad i = 1,\dots,d \tag{2.1}$$

are all Hölder continuous. We shall suppose that $F_1,\dots,F_d$ satisfy the following condition.

Condition (A). If $a_0,a_1,\dots,a_d$ are real numbers such that the flow T_t^G on $S^1 \times \Sigma^r$ defined by $T_t^G(e^{i\theta},(x,s)) = (e^{i\theta + iG^t(x,s)},\sigma^r{}_t(x,s))$ where $G = a_0 + a_1F_1 + \dots + a_dF_d$ and

$$G^t(x,s) = \int_0^t G(\sigma_\tau^r(x,s))\, d\tau,$$

is not topologically transitive then $a_0 = a_1 = \ldots = a_d = 0$.

Remark. Given continuous functions $G_1,\ldots,G_d$ and given $\varepsilon > 0$ there exist functions $F_1,\ldots,F_d$ satisfying (A) such that $\| G_i - F_i \|_\infty < \varepsilon$, $i = 1,\ldots,d$ (Lalley [11], Proposition 8).

One can show that (A) can be reformulated in terms of $f_1,\ldots,f_d$ to give

Condition (A'). If $a_0,a_1,\ldots,a_d$ are real numbers such that $a_0 r + a_1 f_1 + \ldots + a_d f_d$ is cohomologous to a function valued in $2\pi\mathbb{Z}$ then $a_0 = a_1 = \ldots = a_d = 0$.

Write $F = (F_1,\ldots,F_d)$ and $f = (f_1,\ldots,f_d)$. Note that in particular (A) implies that no non-trivial linear combination of $F_1,\ldots,F_d$ is cohomologous to a constant. This implies that for $w = (w_1,\ldots,w_d) \in \mathbb{R}^d$, the function $\beta_r(w) = P(\langle w,F\rangle)$ is strictly convex in each w_i, $i=1,\ldots,d$ (where $\langle w,F\rangle = w_1F_1 + \ldots + w_dF_d$).

We now extend $\beta_r(w)$ to complex values of the argument. Note that for every $w \in \mathbb{R}^d$, $P(-\beta_r(w)r + \langle w,f\rangle) = 0$. Since $P(-sr + \langle w+it,f\rangle)$ is analytic for $(s,w+it)$ in a neighbourhood of $\mathbb{R} \times \mathbb{R}^d$ in $\mathbb{C} \times \mathbb{C}^d$ and

$$\left[\frac{\partial P(-sr + \langle w,f\rangle)}{\partial s}\right]_{s=s_0} = -\int r\, d\nu < 0$$

(where ν is the equilibrium state of $-s_0 r + \langle w,f\rangle$) the implicit function theorem allows us to extend β_r to an analytic function on a neighbourhood of $\mathbb{R}^d$ in $\mathbb{C}^d$ by defining $\beta_r(w+it)$ by the equation $P(-\beta_r(w+it)r + \langle w+it,f\rangle) = 0$.

Lemma 1. For every $\xi \in \mathbb{R}^d$,

$$\nabla\beta_r(\xi) = \left(\int F_1\, dm_\xi, \ldots, \int F_d\, dm_\xi\right)$$

where m_ξ is the equilibrium state of $<\xi,F>$.

Proof. Since $P(-\beta_r(w)r + <w,f>)$ is identically zero, partial differentiation with respect to w_i yields

$$\frac{\partial P}{\partial \beta}\frac{\partial \beta_r}{\partial w_i} + \frac{\partial P}{\partial w_i} = 0. \qquad (2.2)$$

We have

$$\left[\frac{\partial P(-\beta r + <\xi,f>)}{\partial \beta}\right]_{\beta=\beta_r(\xi)} = -\int r\, d\mu_\xi$$

and

$$\left[\frac{\partial P(-\beta_r(\xi)r + <w,f>)}{\partial w_i}\right]_{w=\xi} = \int f_i\, d\mu_\xi$$

where μ_ξ is the equilibrium state of $-\beta_r(\xi) + <\xi,f>$. Hence

$$\left[\frac{\partial \beta_r(w)}{\partial w_i}\right]_{w=\xi} = \frac{\int f_i\, d\mu_\xi}{\int r\, d\mu_\xi} = \int F_i\, dm_\xi$$

and the lemma is proved.

In addition to (A) we shall also suppose that $F_1,\ldots,F_d$ satisfy the following condition.

Condition (B). The strictly convex function β_r has a finite minimum and from now on ξ will denote that minimum.

Clearly we then have $\nabla\beta_r(\xi) = 0$ and so, by Lemma 1, we can reformulate (B) as

Condition (B'). For some (necessarily unique) $\xi \in \mathbb{R}^d$

$$\int F_i\, dm_\xi = 0, \qquad i = 1,\ldots,d.$$

Since m_ξ is the equilibrium state of $<\xi,F>$

$$\beta_r(\xi) = h_{m_\xi}(\sigma^r) + \int <\xi,F> dm_\xi$$
$$= h_{m_\xi}(\sigma^r) > 0.$$

We shall write $\alpha = \beta_r(\xi)$. Note that we also have

$$\alpha = \sup\{ h_m(\sigma^r) : \int F_i \, dm = 0, \ i = 1,\ldots,d\}.$$

Remark. One can obtain an explicit expression for $\nabla^2\beta_r(\xi)$ in terms of $F_1,\ldots,F_d$. Substituting the information that $\nabla\beta_r(\xi) = 0$ into (2.2) we obtain

$$\left[\frac{\partial^2\beta_r(w)}{\partial w_i \, \partial w_j}\right]_{w=\xi} = \frac{1}{\int r \, d\mu_\xi}\left[\frac{\partial^2 P(-\beta_r(\xi)r + <w,f>)}{\partial w_i \, \partial w_j}\right]_{w=\xi}.$$

One can show ([23], Chapter 5, Exercise 5) that

$$\left[\frac{\partial^2 P(-\beta_r(\xi)r + <w,f>)}{\partial w_i \, \partial w_j}\right]_{w=\xi} = \sum_{n \in \mathbb{Z}} \int f_i(x) \, f_j(\sigma^n x) \, d\mu_\xi$$

and one can also show that

$$\frac{1}{\int r \, d\mu_\xi} \sum_{n \in \mathbb{Z}} \int f_i(x) \, f_j(\sigma^n x) \, d\mu_\xi(x)$$
$$= \lim_{T \to \infty} \frac{1}{T} \int \left(\int_0^T F_i(\sigma_t^r x) \, dt\right)\left(\int_0^T F_j(\sigma_t^r x) \, dt\right) dm_\xi(x)$$

(cf. [9]). Combining these equations gives an expression for $\nabla^2\beta_r(\xi)$.

We can use the above expression to help prove another consequence of (A) namely that $\nabla^2\beta_r(\xi)$ is positive definite. This assertion is equivalent to saying that for every $a \in \mathbb{R}^d - \{0\}$

$$<a\nabla^2\beta_r(\xi),a> = \sum_{i=1}^d\sum_{j=1}^d a_i \, a_j \left[\frac{\partial^2\beta_r(w)}{\partial w_i \, \partial w_j}\right]_{w=\xi} > 0.$$

By the above remark we have that

$$\langle a\nabla^2\beta_r(\xi),a\rangle = \lim_{T\to\infty}\frac{1}{T}\int\sum_{i=1}^{d}\sum_{j=1}^{d} a_i a_j\left(\int_0^T F_i(\sigma_t^r x)\,dt\right)\left(\int_0^T F_j(\sigma_t^r x)\,dt\right)dm_\xi.$$

If we write $G = a_1F_1 + \ldots + a_dF_d$ then this simplifies to

$$\langle a\nabla^2\beta_r(\xi),a\rangle = \lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T G(\sigma_t^r x)\,dt\right)^2 dm_\xi.$$

Now by a result of Ratner [22]

$$\lim_{T\to\infty}\frac{1}{T}\int\left(\int_0^T G(\sigma_t^r x)\,dt\right)^2 dm_\xi = 0$$

if and only if there exists $U \in L^2_{m_\xi}(\Sigma^r)$ such that

$$\lim_{\varepsilon\to 0}\frac{U(\sigma_\varepsilon^r x) - U(x)}{\varepsilon}$$

exists and is equal to $G(x)$ m_ξ-a.e.. This is equivalent ([22], Lemma 1.4) to the existence of $u \in L^2_{\mu_\xi}(\Sigma_A)$ such that

$$u(\sigma y) - u(y) = g(y) \qquad \mu_\xi\text{-a.e.}$$

where

$$g(y) = \int_0^{r(y)} G(\sigma_t^r(y,0))\,dt.$$

However it is well known that if such a function u exists then it can be chosen to be continuous (or even Hölder continuous) so that

$$u(\sigma y) - u(y) = g(y) \qquad \forall\, y \in \Sigma_A$$

[4]. By writing

$$U(y,t) = u(y) + \int_0^t G(\sigma_s^r(y,0))\,ds$$

one sees that this in turn is equivalent to the existence of $U \in C(\Sigma^r)$ such that for every

$\varepsilon > 0$ and every $x \in \Sigma$

$$U(\sigma_\varepsilon^r x) - U(x) = \int_0^\varepsilon G(\sigma_t^r x)\,dt$$

which is just to say that G is cohomologous to zero. However, since $a \neq 0$, (A) ensures that $G = a_1F_1 + \ldots + a_dF_d$ cannot be cohomologous to a constant and so we have

$$\langle a\nabla^2\beta_r(\xi), a\rangle > 0$$

as required.

Let τ be a closed σ^r-orbit with least period $\lambda(\tau)$ and let

$$\lambda_{F_i}(\tau) = \int_0^{\lambda(\tau)} F_i(\sigma_t^r x)\,dt, \text{ for any } x \in \tau,\ i = 1,\ldots,d.$$

Write $[\tau] = (\lambda_{F_1}(\tau),\ldots,\lambda_{F_d}(\tau)) \in \mathbb{R}^d$. For $s \in \mathbb{C}$, $t \in \mathbb{R}^d$, define

$$L_r(s,t) = \prod_\tau \left(1 - e^{-s\lambda(\tau) + \langle \xi + it, [\tau]\rangle}\right)^{-1}$$

$$= \exp \sum_\tau \sum_{n=1}^\infty \frac{1}{n} e^{n(-s\lambda(\tau) + \langle \xi + it, [\tau]\rangle)}$$

(where this converges).

Proposition 2.

(i) $L_r(s,t)$ is analytic and non-zero in the set $\{ s : \operatorname{Re} s > \alpha \} \times \mathbb{R}^d$.

(ii) For $t_0 \neq 0$, $L_r(s,t)$ is analytic and non-zero in a neighbourhood of $\{ s : \operatorname{Re} s \geq \alpha \} \times \{t_0\}$.

(iii) For each $t_0 \in \mathbb{R}^d$, $L_r(s,t)$ is analytic and non-zero in a neighbourhood of $\{ s : \operatorname{Re} s = \alpha,\ \operatorname{Im} s \neq 0 \} \times \{t_0\}$.

(iv) In a neighbourhood of $(\alpha,0)$, $L_r(s,t)$ takes the form $\psi(s,t)/(s-s(t))$ for some analytic and non-zero ψ.

Proof. Each closed σ^r-orbit τ corresponds to a closed σ-orbit, $\{x,\sigma x,\ldots,\sigma^{n-1}x\}$ say, and $\lambda(\tau) = r^n(x)$, $[\tau] = f^n(x)$. Hence, in the notation of Section 1, we have

$L_\gamma(s,t) = \zeta(-sr + \langle\xi+it,f\rangle)$. By the remarks preceeding Proposition 1, this is analytic and non-zero whenever $P(-(\mathrm{Re}\, s)r + \langle\xi,f\rangle) < 0$. Since $P(-\alpha r + \langle\xi,f\rangle) = 0$ and (because r is strictly positive) for $\sigma \in \mathbb{R}$ the function $\sigma \to P(-\sigma r + \langle\xi,f\rangle)$ is strictly decreasing, this will be the case whenever $\mathrm{Re}\, s > \alpha$. This proves (i).

Suppose $\mathrm{Re}\, s_0 = \alpha$ then Proposition 1 gives that $L_\gamma(s,t)$ is analytic and non-zero in a neighbourhood of (s_0,t_0) unless $-(\mathrm{Im}\, s_0)r + \langle t_0,f\rangle$ is cohomologous to a function valued in $2\pi\mathbb{Z}$. By (A') and the fact that σ^r is weak-mixing, this can only happen when $\mathrm{Im}\, s_0 = 0$, $t_0 = 0$. This proves (ii) and (iii).

In a neighbourhood of $(\alpha,0)$ we have, again by Proposition 1,

$$L_\gamma(s,t) = \frac{\varphi(s,t)}{1 - e^{P(-sr + \langle\xi+it,f\rangle)}}$$

for some analytic and non-zero φ. Since $P(-s(t)r + \langle\xi+it,f\rangle) = 0$, $L_\gamma(s,t)$ has a singularity at $(s(t),t)$. Since $e^{P(-sr + \langle\xi,f\rangle)}$ has a non-zero partial derivative with respect to s at $s = \alpha$, analyticity ensures that $e^{P(-sr + \langle\xi+it,f\rangle)}$ has a non-zero partial derivative with respect to s at $s = s(t)$. Hence $L_\gamma(s,t) = \psi(s,t)/(s-s(t))$ for some analytic and non-zero ψ and (iv) is true.

We now make some observations concerning s(t) which we shall need later.

Proposition 3.

(i) $\mathrm{Re}\, s(t)$ is an even function and $\mathrm{Im}\, s(t)$ is an odd function.

(ii) $\nabla \mathrm{Re}\, s(0) = 0$.

(iii) $\nabla \mathrm{Im}\, s(0) = 0$.

(iv) $\nabla^2 \mathrm{Re}\, s(0) = -\nabla^2 \beta_\gamma(\xi)$ and is thus negative definite.

(v) $\nabla^2 \mathrm{Im}\, s(0) = 0$.

Proof. Note that $L_\gamma(s,-t) = \overline{L_\gamma(\bar{s},t)}$. This implies that $s(t) = \overline{s(-t)}$. (i) follows immediately. It also implies that

$$\mathrm{Re}\, s(t) = \tfrac{1}{2}(s(t) + s(-t)) \tag{2.3}$$

and

$$\mathrm{Im}\, s(t) = -\tfrac{1}{2}i\,(s(t) - s(-t)) \tag{2.4}$$

This proves (ii) and (v). By (2.4) if we write $T_j = it_j$, $j = 1,\ldots,d$ we have

$$\left[\frac{\partial}{\partial t_j}\,\mathrm{Im}\, s(t)\right]_{t=0} = -i\left[\frac{\partial}{\partial t_j}\, s(t)\right]_{t=0} = \left[\frac{\partial}{\partial T_j}\, s(t)\right]_{t=0}$$

$$= \left[\frac{\partial}{\partial T_j}\,\beta_r(\xi+T)\right]_{T=0}$$

and so $\nabla \mathrm{Im}\, s(0) = \nabla\beta_r(\xi) = 0$. By (1) $\nabla^2 \mathrm{Re}\, s(0) = \nabla^2 s(0)$ but

$$\left[\frac{\partial^2 s(t)}{\partial t_j\,\partial t_k}\right]_{t=0} = -\left[\frac{\partial^2 s(t)}{\partial T_j\,\partial T_k}\right]_{t=0} = -\left[\frac{\partial^2 \beta_r(\xi+T)}{\partial T_j\,\partial T_k}\right]_{T=0}$$

so $\nabla^2 \mathrm{Re}\, s(0) = -\nabla^2\beta_r(\xi)$.

We shall now go on to consider, following [9], a certain higher logarithmic derivative (with respect to the first variable) of $L_r(s,t)$.

Let $\nu = [d/2]$ (here [ ] denotes integer part) and define

$$\eta_r(s,t) = \frac{\partial^{\nu+1}}{\partial s^{\nu+1}} \log L_r(s,t)$$

$$= \sum_{\tau} \sum_{n=1}^{\infty} n^{\nu} (-\lambda(\tau))^{\nu+1}\, e^{n(-s\lambda(\tau) + \langle\xi + it, \tau\rangle)}.$$

By Proposition 2, $\eta_r(s,t)$ is analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d$ except for a singularity at $(\alpha,0)$ and in a neighbourhood of $(\alpha,0)$,

$$\eta_r(s,t) = \frac{(-1)^{\nu+1}\,\nu!}{(s - s(t))^{\nu+1}} + \psi_1(s,t)$$

for some analytic function ψ_1.

Define

$$\eta_r^1(s,t) = \sum_\tau (-\lambda(\tau))^{\nu+1} e^{-s\lambda(\tau) + \langle \xi + it, f\tau \rangle}.$$

We shall show that $\eta_r(s,t) - \eta_r^1(s,t)$ is analytic on $\{ s : \mathrm{Re}\, s > (1-\varepsilon)\alpha \} \times \mathbb{R}^d$ for some $\varepsilon > 0$ and hence that $\eta_r^1(s,t)$ enjoys the properties of $\eta_r(s,t)$ described in the preceeding paragraph. We shall do this (cf. [17], Lemma 2) by showing that the series defining $\eta_r(s,t) - \eta_r^1(s,t)$, namely

$$\sum_\tau \sum_{n=2}^{\infty} n^\nu (-\lambda(\tau))^{\nu+1} e^{n(-s\lambda(\tau) + \langle \xi + it, f\tau \rangle)}$$

converges absolutely for $\mathrm{Re}\, s > (1-\varepsilon)\alpha$ and any $t \in \mathbb{R}^d$.

Recall first that $P(-\alpha r + \langle \xi, f \rangle) = 0$. It can be shown that this implies that $-\alpha r + \langle \xi, f \rangle$ is cohomologous to a strictly negative function, g say. We can choose $\varepsilon > 0$ so small that $g < -5\varepsilon\alpha r$ and hence, for any closed orbit τ,

$$-\alpha\lambda(\tau) + \langle \xi, f\tau \rangle < -5\varepsilon\alpha\lambda(\tau).$$

Thus

$$\sum_\tau \sum_{n=2}^{\infty} n^\nu \lambda(\tau)^{\nu+1} e^{n(-(1-\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle)}$$

$$\leq C \sum_\tau \sum_{n=2}^{\infty} \lambda(\tau)^{\nu+1} e^{n(-(1-2\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle)}$$

$$= C \sum_\tau \lambda(\tau)^{\nu+1} \frac{e^{2(-(1-2\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle)}}{1 - e^{-(1-2\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle}}$$

$$= C \sum_\tau \lambda(\tau)^{\nu+1} \frac{e^{-(1-2\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle}}{e^{(1-2\varepsilon)\alpha\lambda(\tau) - \langle \xi, f\tau \rangle} - 1}$$

$$\leq C' \sum_\tau \lambda(\tau)^{\nu+1} e^{-(1-2\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle} e^{-3\varepsilon\alpha\lambda(\tau)}$$

$$= C' \sum_\tau \lambda(\tau)^{\nu+1} e^{-(1+\varepsilon)\alpha\lambda(\tau) + \langle \xi, f\tau \rangle}$$

(for some positive constants C, C') and this last summation is (up to a factor $(-1)^{\nu+1}$)

part of $\eta_r(s,0)$ evaluated at $s = (1+\epsilon)\alpha$ and so it converges.

We now have

Proposition 4. The function $\eta_r^1(s,t)$ is analytic in a neighbourhood of $(\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d) - \{(\alpha,0)\}$ and in a neighbourhood of $(\alpha,0)$,

$$\eta_r^1(s,t) = \frac{(-1)^{\nu+1}\,\nu!}{(s - s(t))^{\nu+1}} + \psi_2(s,t)$$

for some analytic function ψ_2.

3. Axiom A flows.

We now turn our attention to differentiable flows satisfying Smale's Axiom A. Let M be a compact C^∞ Riemannian manifold and φ a C^1 flow on M. We shall be concerned with φ restricted to the non-wandering set Ω defined by

$$\Omega = \{ x \in M : \forall \text{ open } V \ni x\ \exists\, t_i \uparrow \infty \text{ such that } \varphi_{t_i}(V) \cap V \neq \emptyset \}.$$

A compact φ-invariant set Λ containing no fixed points is called basic if

(i) the tangent bundle of M restricted to Λ has a continuous splitting into three $D\varphi$-invariant sub-bundles

$$T_\Lambda M = E + E^s + E^u$$

where E is the one-dimensional bundle tangent to the flow and there exist constants $C, \lambda > 0$ such that

$$\| D\varphi_t(v) \| \leq C e^{-\lambda t} \| v \| \qquad \forall\, v \in E^s,\ t \geq 0$$

$$\| D\varphi_{-t}(v) \| \leq C e^{-\lambda t} \| v \| \qquad \forall\, v \in E^u,\ t \geq 0;$$

(ii) the intersection of Λ with the set of closed φ-orbits is dense in Λ;

(iii) the restriction of φ to Λ is topologically transitive;

(iv) there exists an open neighbourhood U of Λ such that

$$\bigcap_{-\infty<t<\infty} \varphi_t(U) = \Lambda.$$

The flow φ satisfies Axiom A if Ω is the union of a finite number of basic sets and

hyperbolic fixed points. Axiom A flows are a generalization of Anosov flows. A C^1 flow on M is Anosov if (i) holds with M replacing Λ. (The other conditions are not assumed.) Of particular interest are transitive Anosov flows, i.e. those for which $\Omega = M$. We shall always suppose that φ is restricted to a basic set and that this set is non-trivial, i.e. it consists of more than one closed orbit.

It is possible to model an Axiom A flow (restricted to a basic set) by a suspended flow over a shift of finite type.

Proposition 5. (Bowen [4].) If φ is an Axiom A flow restricted to a (non-trivial) basic set Λ then there exists a (topologically mixing) shift of finite type Σ_A, a strictly positive Hölder continuous function r and a Hölder continuous map $p : \Sigma^r \to \Lambda$ such that $p \circ \sigma^r_t = \varphi_t \circ p$. The map p is surjective and bounded-one. Furthermore, if $G : \Lambda \to \mathbb{R}$ is Hölder continuous then g defined on Σ_A by

$$g(x) = \int_0^{r(x)} G \circ p(\sigma^r_t(x,0))\, dt$$

is Hölder continuous so that $G \circ p$ has a unique equilibrium state m, say. The map p is measure preserving with respect to m and p^*m and one-one a.e. with respect to m. The measure p^*m is the unique equilibrium state of G.

Remarks. (i) Σ^r is called the principal suspension.

(ii) Pressure and equilibrium states are defined in the same way for φ as they were for σ^r and, since p is bounded-one, $P(G) = P(G \circ p)$. In particular the topological entropy $h(\varphi) = h(\sigma^r)$.

(iii) φ is weak-mixing if and only if σ^r is weak-mixixg.

Suppose now that φ is weak-mixing and that $F_1,\ldots,F_d$ are real valued, Hölder continuous functions defined on Λ and satisfying Condition (A) (with Λ and φ replacing Σ^r and σ^r in the statement of (A)). Then it is clear that $F_1 \circ p,\ldots,F_d \circ p$ also satisfy

(A). For $w \in \mathbb{R}^d$ define $\beta(w) = P(\langle w,F\rangle)$. By Remark (ii) we have $\beta(w) = \beta_r(w)$ (where β_r is defined with respect to $F_1 \circ p,\ldots,F_d \circ p$) and so β is strictly convex.

As before, we shall also suppose that $F_1,\ldots,F_d$ satisfy Condition (B), i.e. that the strictly convex function β has a finite minimum which we denote by ξ and we write $\alpha = \beta(\xi)$. Then

$$0 = \nabla\beta(\xi) = \nabla\beta_r(\xi) = (\int F_1 \circ p\, dm_\xi,\ldots,\int F_d \circ p\, dm_\xi)$$
$$= (\int F_1\, dp^* m_\xi, \ldots, \int F_d\, dp^* m_\xi),$$

so (B') is also satisfied. Since $p^* m_\xi$ is the unique equilibrium state of $\langle\xi,F\rangle$ the above shows that

$$\alpha = h_{p^* m_\xi}(\varphi) = \sup\{ h_m(\varphi) : \int F_i\, dm = 0,\ i = 1,\ldots,d\}.$$

We also have that

$$\nabla^2\beta(\xi) = \nabla^2\beta_r(\xi) = -\nabla^2 \mathrm{Re}\, s(0)$$

and the expression for $\nabla^2\beta_r(\xi)$ in terms of $F_1 \circ p,\ldots,F_d \circ p$ given in the last section leads to a corresponding expression for $\nabla^2\beta(\xi)$ in terms of $F_1,\ldots,F_d$.

We define a function η^1 analogous to $\eta_r{}^1$, i.e.

$$\eta^1(s,t) = \sum_\tau (-\lambda(\tau))^{\nu+1}\, e^{-s\lambda(\tau) + \langle\xi+it,[\tau]\rangle}$$

where the summation is taken over all closed φ-orbits τ of least period $\lambda(\tau)$ and $[\tau]$ denotes $(\lambda_{F_1}(\tau),\ldots,\lambda_{F_d}(\tau))$. We wish to relate $\eta^1(s,t)$ to $\eta_r{}^1(s,t)$. Unfortunately there is not a one-one correspondence between closed φ-orbits and closed σ^r-orbits. However we can overcome this discrepancy by means of the following proposition which is a refinement by Bowen of a result of Manning [14] for Axiom A diffeomorphisms.

Proposition 6. (Bowen [4].) In addition to the principal suspension, there exist a finite number of suspensions of shifts of finite type Σ^{r_i}, $i = 1,\ldots,p,\ldots,q$ (with each r_i Hölder continuous) and Holder continuous maps $p_i : \Sigma^{r_i} \to \Lambda$ such that each p_i is

bounded-one but <u>not</u> surjective. Moreover, if, for example, $\upsilon(\varphi,x)$ denotes the number of closed φ-orbits of least period x then

$$\upsilon(\varphi,x) = \upsilon(\sigma^r,x) + \sum_{i=1}^{p} \upsilon(\sigma^{r_i},x) - \sum_{i=p+1}^{q} \upsilon(\sigma^{r_i},x).$$

Hence we have

$$\eta^1(s,t) = \eta_r^1(s,t) + \sum_{i=1}^{p} \eta_{r_i}^1(s,t) - \sum_{i=p+1}^{q} \eta_{r_i}^1(s,t).$$

From Section 2, each $\eta_{r_i}{}^1(s,t)$ is analytic for Re $s > P(<\xi,F\circ p_i>)$. However, since each p_i is not surjective, $P(<\xi,F\circ p_i>) < P(<\xi,F>) = \alpha$, $i = 1,\ldots,q$ [23]. Thus each $\eta_{r_i}{}^1(s,t)$ is analytic in $\{s : \text{Re } s > \alpha - \varepsilon\} \times \mathbb{R}^d$ (for some $\varepsilon > 0$). Hence Proposition 4 remains true with η^1 replacing $\eta_r{}^1$ (and Proposition 3 with β replacing β_r).

4. Fourier analysis and Tauberian theorems.

Let $v : \mathbb{R}^d \to \mathbb{R}$ be a C^∞ function in $L^1(\mathbb{R}^d)$ and let k be a continuous probability density function on $\mathbb{R}^d$ such that the Fourier transform

$$\hat{k}(t) = \int_{\mathbb{R}^d} k(y)\, e^{i<t,y>}\, dy$$

is C^∞ in a neighbourhood of 0 and has compact support. Define

$$\tilde{\eta}(s) = \sum_{\tau} (-\lambda(\tau))^{\nu+1}\, e^{-s\lambda(\tau) + <\xi,[t]>}\, k*v([\tau]).$$

Clearly $\widehat{k*v}(t) = \hat{k}(t)\hat{v}(t)$ and the Fourier inversion formula gives

$$k*v(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}(-t)\, \hat{v}(-t)\, e^{i<t,y>}\, dt.$$

Substituting this into the definition of $\tilde{\eta}$ we obtain

$$\tilde{\eta}(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}(-t)\, \hat{v}(-t) \sum_{\tau} (-\lambda(\tau))^{\nu+1} e^{-s\lambda(\tau) + \langle \xi + it, \tau \rangle} dt$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}(-t)\, \hat{v}(-t)\, \eta^1(s,t)\, dt.$$

Using Proposition 4 and the fact that $\hat{k}$ has compact support we have that, apart from a singularity at $s = \alpha$, $\tilde{\eta}(s)$ is analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \}$.

We now shall examine the singularity of $\tilde{\eta}(s)$ at $s = \alpha$. If U is a neighbourhood of 0 in $\mathbb{R}^d$ we have

$$\tilde{\eta}(s) = \frac{1}{(2\pi)^d} \int_U \hat{k}(-t)\, \hat{v}(-t)\, \eta^1(s,t)\, dt$$

$$+ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d - U} \hat{k}(-t)\, \hat{v}(-t)\, \eta^1(s,t)\, dt$$

and the second of these integrals gives a function ψ_3 analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \}$. If U is sufficiently small and s is sufficiently close to α then

$$\tilde{\eta}(s) = \frac{1}{(2\pi)^d} \int_U \hat{k}(-t)\, \hat{v}(-t) \left\{ \frac{(-1)^{\nu+1}\, \nu!}{(s-s(t))^{\nu+1}} + \psi_2(s,t) \right\} dt + \psi_3(s)$$

$$= \frac{(-1)^{\nu+1}\, \nu!}{(2\pi)^d} \int_U \hat{k}(-t)\, \hat{v}(-t) \frac{1}{(s-s(t))^{\nu+1}}\, dt + \psi_4(s)$$

with $\psi_4(s)$ analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \}$.

We quote the following refinement of the Morse lemma from [9].

Lemma 2. Let $f(x)$ be a real valued even C^∞ function in a neighbourhood of 0 in $\mathbb{R}^d$. If 0 is a non-degenerate critical point for f then there exists a local coordinate system $y = (y_1, \ldots, y_d)$ in a neighbourhood of 0 such that $y(-x) = -y(x)$ (so that, in particular, $y(0) = 0$) and $f(y) = y_1^2 + \ldots + y_k^2 - y_{k+1}^2 - \ldots - y_n^2$, for some $0 \leq k \leq n$.

We apply this result to the even function Re $s(t) - \alpha$ to obtain local coordinates $\theta = (\theta_1,\ldots,\theta_d)$ around $0 \in \mathbb{R}^d$ such that $\theta(-t) = -\theta(t)$ and

$$\mathrm{Re}\, s(t) = \alpha - (\theta_1^2 + \ldots + \theta_d^2).$$

(The θ_i's all occur with negative sign because $\nabla^2 s(0)$ is negative definite.) Let $J(\theta)$ be the Jacobian determinant of this change of coordinates then

$$|J(0)| = \frac{2^{\frac{1}{2}d}}{(\det \nabla^2 \beta(\xi))^{\frac{1}{2}}}$$

(To see this first define a $d \times d$ matrix M by

$$M(i,j) = \left[\frac{\partial \theta_i(t)}{\partial t_j}\right]_{t=0}$$

Then

$$\nabla^2 \beta(\xi) = \nabla^2 \mathrm{Re}\, s(0)$$

$$= \left(2 \sum_{k=1}^{d} \left[\frac{\partial \theta_k(t)}{\partial t_i}\right]_{t=0} \left[\frac{\partial \theta_k(t)}{\partial t_j}\right]_{t=0}\right)_{i,j=1}^{d} = 2\, M^T M$$

and so

$$\det \nabla^2 \beta(\xi) = 2^d (\det M)^2.$$

However, $|\det M| = 1/|J(0)|$ giving the required result.)

Hence provided U is sufficiently small we may write (remembering that $\hat{k}(0) = 1$)

$$\hat{\eta}(s) = \frac{(-1)^{\nu+1}\, \nu!}{(2\pi)^d}\, |J(0)|\, \hat{v}(0) \int_{\theta(U)} \frac{1 + P(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu+1}}\, d\theta$$

$$+ \psi_4(s)$$

where $P(\theta)$ is defined by

$$|J(0)|\, \hat{v}(0)\, (1 + P(\theta)) = |J(\theta)|\, \hat{k}(-t(\theta))\, \hat{v}(-t(\theta))$$

(so that in particular $P(0) = 0$) and $Q(\theta) = \mathrm{Im}\, s(t(\theta))$ so that $Q(\theta)$ is an odd function and $Q(0) = 0$, $\nabla Q(0) = 0$, $\nabla^2 Q(0) = 0$. Without loss of generality, we may suppose that $\theta(U)$ takes the form $\{\, \theta : \theta_1^2 + \ldots + \theta_d^2 \le a^2 \,\}$ for some small $a > 0$.

Now this last integral is in precisely the form considered by Katsuda and Sunada

in [9] and if we write

$$C = \frac{1}{(2\pi)^{\frac{1}{2}d}\,(\det \nabla^2\beta(\xi))^{\frac{1}{2}}}$$

we may duplicate their analysis to obtain

Proposition 7.

(i) If d is even then

$$\lim_{\sigma \to \alpha}\left(\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}\,C\,\hat{\varphi}(0)}{\sigma+i\omega-\alpha}\right)$$

exists for almost every $\omega \in \mathbb{R}$ and is locally integrable and there exists a locally integrable function $h(\omega)$ such that

$$\left|\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}\,C\,\hat{\varphi}(0)}{\sigma+i\omega-\alpha}\right| \le h(\omega)$$

$(\sigma > \alpha)$.

(ii) If d is odd then

$$\lim_{\sigma \to \alpha}\left(\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}\,C\,\hat{\varphi}(0)\,\pi^{\frac{1}{2}}}{(\sigma+i\omega-\alpha)^{\frac{1}{2}}}\right)$$

exists for almost every $\omega \in \mathbb{R}$ and is an element of the the Sobolev space $W^{1,1}_{loc}(\mathbb{R})$ which consists of locally integrable functions with locally integrable first derivatives and there exists a locally integrable function $h(\omega)$ such that

$$\left|\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}\,C\,\hat{\varphi}(0)\,\pi^{\frac{1}{2}}}{(\sigma+i\omega-\alpha)^{\frac{1}{2}}}\right| \le h(\omega)$$

$(\sigma > \alpha)$.

Define

$$S_{k,\nu}(x) = \sum_{\lambda(\tau)\le x} \lambda(\tau)^{\frac{1}{2}d+1}\, e^{\langle \xi, \int t b\rangle}\, k*v([\tau])$$

and observe that if d is even then

$$\hat{\eta}(s) = (-1)^{\nu+1} \int_0^\infty e^{-sx}\, dS_{k,\nu}(x)$$

and if d is odd then

$$\hat{\eta}(s) = (-1)^{\nu+1} \int_0^\infty x^{-\frac{1}{2}} e^{-sx}\, dS_{k,\nu}(x).$$

We shall now convert the information we have about $\hat{\eta}(s)$ into an asymptotic formula for $S_{k,\nu}(x)$ by means of the following proposition.

Proposition 8.

(i) Let $\varphi(x)$ be an increasing function with $\varphi(0) = 0$ and let

$$f(s) = \int_0^\infty e^{-sx}\, d\varphi(x).$$

Suppose that f(s) is analytic in Re $s > \alpha > 0$ and that for some constant A,

$$\lim_{\varepsilon \to 0} \left(f(\alpha+\varepsilon+i\omega) - \frac{A}{\varepsilon+i\omega} \right)$$

exists for almost every $\omega \in \mathbb{R}$ and is locally integrable and there exists a locally integrable function $h(\omega)$ such that

$$\left| f(\alpha+\varepsilon+i\omega) - \frac{A}{\varepsilon+i\omega} \right| \leq h(\omega)$$

$(\varepsilon > 0)$. Then $\varphi(x) \sim A\, e^{\alpha x}$.

(ii) Let $\varphi(x)$ be an increasing function with $\varphi(x) = 0$ for $0 \leq x \leq \delta$, some $\delta > 0$ and let

$$f(s) = \int_0^\infty x^{-\frac{1}{2}} e^{-sx}\, d\varphi(x).$$

Suppose that f(s) is analytic in Re $s > \alpha > 0$ and that for some constant A,

$$\lim_{\varepsilon \to 0} \left(f(\alpha+\varepsilon+i\omega) - \frac{A}{(\varepsilon+i\omega)^{\frac{1}{2}}} \right)$$

exists for almost every $\omega \in \mathbb{R}$ and is in $W^{1,1}_{loc}(\mathbb{R})$ and there exists a locally integrable

function $h(\omega)$ such that

$$\left| f(\alpha+\varepsilon+i\omega) - \frac{A}{(\varepsilon+i\omega)^{\frac{1}{2}}} \right| \leq h(\omega)$$

$(\varepsilon > 0)$. Then $\varphi(x) \sim A\, \pi^{-\frac{1}{2}}\, e^{\alpha x}$.

Remark. (i) is (a weak version of) the Wiener–Ikehara Tauberian theorem (cf. for example [12]). A proof of (ii) is given in [9]. (A Tauberian theorem with the same type of singularity as in (ii) was proved by Delange [7] but with slightly stronger hypotheses.)

Applying this proposition to $\hat{\eta}(s)$, $S_{k,v}(x)$ in the two cases where d is even or odd enables us to conclude the next result.

Proposition 9.

$$S_{k,v}(x) \sim C\,\hat{v}(0)\, e^{\alpha x}.$$

5. Prime orbit asymptotics.

In this section we shall deduce an asymptotic formula for

$$\pi(x,F) = \#\{\, \tau : \lambda(\tau) \leq x,\ |\lambda_{F_i}(\tau)| \leq \delta_i,\ i = 1,\ldots,d \,\}$$

where $\delta_1,\ldots,\delta_d$ are fixed but arbitrary positive numbers. Write $\delta = (\delta_1,\ldots,\delta_d)$ and let ψ denote the indicator function of the box $B(\delta) = [-\delta_1,\delta_1] \times \ldots \times [-\delta_d,\delta_d]$. We shall first obtain an asymptotic expression for

$$S(x) = \sum_{\lambda(\tau)\leq x} \lambda(\tau)^{\frac{1}{2}d+1}\, \psi([\tau])$$

and then, using techniques familiar from number theory, deduce from this an asymptotic expression for $\pi(x,F)$.

Write $\psi'(y) = e^{-\langle \xi, y\rangle}\,\psi(y)$, so that

$$S(x) = \sum_{\lambda(\tau)\le x} \lambda(\tau)^{\frac{1}{2}d+1}\, e^{\langle \xi, \lfloor \tau \rfloor\rangle}\, \psi'([\tau]).$$

We shall estimate $S(x)$ by approximating ψ' above and below by functions of the form $k*v$ where k is a continuous probability density on $\mathbb{R}^d$ such that $\hat{k}$ is C^∞ in a neighbourhood of and has compact support and v is a C^∞ function in $L^1(\mathbb{R}^d)$ and then comparing with $S_{k,v}(x)$.

Choose a sequence of continuous probability densities k_N, $N = 1,2,\ldots$, such that each Fourier transform $\widehat{k_N}$ is C^∞ in a neighbourhood of 0 and has compact support and such that $\{k_N : N \ge 1\}$ is an approximate identity, i.e. such that, as $N \to \infty$, the sequence of probability measures determined by k_N converges weakly to the atomic measure giving unit mass to the origin. Also choose k_N in such a way that, for $\| y \|_\infty > \frac{1}{2}$, there exists a constant $\Delta > 0$ independent of N satisfying

$$| k_N(y) | \le \frac{\Delta}{\| y \|_\infty^2}, \qquad N = 1,2,\ldots \qquad (5.1).$$

An example of such a sequence of densities is given by defining

$$\widehat{k_N}(t) = \begin{cases} 1 - \dfrac{t_1^2 + \ldots + t_d^2}{N^2} & \text{if } t_1^2 + \ldots + t_d^2 \le N^2 \\ 0 & \text{otherwise} \end{cases}$$

and obtaining k_N by using the inverse Fourier transform.

Since the sequence of k_N's are an approximate identity for every $\eta \in (0,1)$ there exists $N_0(\eta) \ge 1$ such that for all $N \ge N_0(\eta)$

$$\int_{\| y \|_\infty \le \eta} k_N(y)\, dy > 1 - \eta \qquad (5.2).$$

Let $\varepsilon > 0$ be given. (For convenience we shall suppose that $\varepsilon < \| \psi' \|_\infty$.) Define

$$B_\varepsilon(\delta) = \left(\prod_{i=1}^{d} [-\delta_i-\varepsilon, \delta_i+\varepsilon] \right) - B(\delta).$$

Then

$$\mathrm{vol}(B_\varepsilon(\delta)) = 2^d \left(\prod_{i=1}^{d} (\delta_i+\varepsilon) - \prod_{i=1}^{d} \delta_i \right) \le M\varepsilon$$

for some constant $M > 0$.

We begin by approximating ψ' from above. First we choose a C^∞ function v_1 with the following properties. For $y \in B(\delta)$, $v_1(y) = \psi'(y) + \varepsilon$. For $y \in B_\varepsilon(\delta)$, $0 \le v_1(y) \le 2\| \psi' \|_\infty$. For $y \notin B(\delta) \cup B_\varepsilon(\delta)$, $v_1(y) = 0$. This function is clearly in $L^1(\mathbb{R}^d)$. Furthermore

$$\hat{v}_1(0) = \int_{B(\delta)} (\psi'(y) + \varepsilon)\, dy + \int_{B_\varepsilon(\delta)} v_1(y)\, dy$$

$$= \widehat{\psi'}(0) + \mathrm{vol}(B(\delta))\, \varepsilon + \int_{B_\varepsilon(\delta)} v_1(y)\, dy$$

$$\le \widehat{\psi'}(0) + \left\{ \mathrm{vol}(B(\delta)) + 2 \| \psi' \|_\infty M \right\} \varepsilon \qquad (5.3).$$

We now show that we can choose N large enough to ensure that

$$k_N * v_1(y) \ge \psi'(y) \qquad \forall\, y \in \mathbb{R}^d.$$

First note that if $y \notin B(\delta)$ then for any $N \ge 1$,

$$k_N * v_1(y) > 0 = \psi'(y)$$

so we only have to consider $y \in B(\delta)$. Now, for any $\eta \in (0,1)$,

$$k_N * v_1(y) = \int_{\mathbb{R}^d} k_N(x)\, v_1(y-x)\, dx$$

$$\ge \int_{\|x\|_\infty \le \eta} k_N(x)\, v_1(y-x)\, dx \ge \inf_{\|x\|_\infty \le \eta} v_1(y-x) \int_{\|x\|_\infty \le \eta} k_N(x)\, dx.$$

So provided $N \ge N_0(\eta)$ we have, by (5.2),

$$k_N * v_1(y) \ge \inf_{\|x\|_\infty \le \eta} v_1(y-x)\ (1 - \eta)$$

and we can choose $\eta > 0$ so small that for every $y \in B(\delta)$

$$\inf_{\|x\|_\infty \le \eta} v_1(y-x)\ (1-\eta) \ \ge\ v_1(y) - \varepsilon = \psi'(y).$$

Thus there exists $\eta > 0$ such that for $N \ge N_0(\eta)$

$$k_N * v_1(y) \ \ge\ \psi'(y) \qquad \forall\, y \in \mathbb{R}^d$$

and hence for every $x \ge 0$

$$e^{-\alpha x} S(x) \ \le\ e^{-\alpha x} S_{k_N, v}(x).$$

From this we obtain

$$\overline{\lim}\ e^{-\alpha x} S(x) \ \le\ \overline{\lim}\ e^{-\alpha x} S_{k_N, v}(x) = C\,\hat{v}_1(0)$$

$$\le\ C\left(\widehat{\psi'}(0) + \left\{\mathrm{vol}(B(\delta)) + 2\,\|\psi'\|_\infty M\right\}\varepsilon\right).$$

Since we can choose $\varepsilon > 0$ as small as we please this yields

$$\overline{\lim}\ e^{-\alpha x} S(x) \ \le\ C\,\widehat{\psi'}(0) \qquad (5.4).$$

Next we perform the slightly harder task of approximating ψ' from below. This time we choose a C^∞ function v_2 in such a way that for $y \in B(\delta)$, $v_2(y) < \psi'(y)$ but

$$\int_{B(\delta)} v_2(y)\,dy \ \ge\ \int_{B(\delta)} \psi'(y)\,dy - \varepsilon$$

while for $y \notin B(\delta)$,

$$v_2(y) = \frac{-\varepsilon}{\|y\|_\infty^{3/2}}.$$

It is clear that $v_2 \in L^1(\mathbb{R}^d)$. Furthermore

$$\hat{v}_2(0) = \int_{B(\delta)} v_2(y)\,dy - \varepsilon \int_{\mathbb{R}^d - B(\delta)} \|y\|_\infty^{-3/2}\,dy$$

$$\ge\ \widehat{\psi'}(0) - \left(1 + \int_{\mathbb{R}^d - B(\delta)} \|y\|_\infty^{-3/2}\,dy\right)\varepsilon \qquad (5.5).$$

We now show that we can choose N large enough to ensure that

$$k_N * v_2(y) \ \le\ \psi'(y) \qquad \forall\, y \in \mathbb{R}^d.$$

First suppose that $y \notin B(\delta) \cup B_{\frac{1}{2}}(\delta)$. This ensures that

$$\{x : \| x \|_\infty \le \tfrac{1}{2}\} \cap \{x : x - y \in B(\delta)\} = \emptyset \qquad (5.6).$$

Then for $N \ge N_0(\frac{1}{2})$

$$k_N * v_2(y) = \int_{\mathbb{R}^d} k_N(x)\, v_2(y-x)\, dx$$

$$\le \int_{y-x \in B(\delta)} k_N(x)\, v_2(y-x)\, dx + \int_{\| x \|_\infty \le \frac{1}{2}} k_N(x)\, v_2(y-x)\, dx$$

(since all the contributions to the integral that we have omitted in the latter expression are negative). In view of (5.1) and (5.6) this gives

$$k_N * v_2(y) \le \| v_2 \|_\infty \int_{y-x \in B(\delta)} \frac{\Delta}{\| x \|_\infty^2}\, dx - \varepsilon \int_{\| x \|_\infty \le \frac{1}{2}} \frac{k_N(x)}{\| y-x \|_\infty^{3/2}}\, dx$$

$$\le \| v_2 \|_\infty \Delta \operatorname{vol}(B(\delta)) \sup_{y-x \in B(\delta)} \| x \|_\infty^{-2}$$

$$- \varepsilon \int_{\| x \|_\infty \le \frac{1}{2}} k_N(x)\, dx \inf_{\| x \|_\infty \le \frac{1}{2}} \| y-x \|_\infty^{-3/2}$$

$$\le \| v_2 \|_\infty \Delta \operatorname{vol}(B(\delta)) \sup_{y-x \in B(\delta)} \| x \|_\infty^{-2} - \tfrac{1}{2} \varepsilon \inf_{\| x \|_\infty \le \frac{1}{2}} \| y-x \|_\infty^{-3/2}$$

(by (5.2)). Now it is possible to choose a compact set $K \subset \mathbb{R}^d$ (containing $B(\delta) \cup B_{\frac{1}{2}}(\delta)$) which is large enough to ensure that for $y \notin K$

$$\| v_2 \|_\infty \Delta \operatorname{vol}(B(\delta)) \sup_{y-x \in B(\delta)} \| x \|_\infty^{-2} - \tfrac{1}{2} \varepsilon \inf_{\| x \|_\infty \le \frac{1}{2}} \| y-x \|_\infty^{-3/2} < 0$$

so for every $y \notin K$ and $N \ge N_0(\frac{1}{2})$

$$k_N * v_2(y) < 0 = \psi'(y) \qquad (5.7).$$

We now have to deal with $y \in K$. Since $v_2(y) < \psi'(y)\ \forall\, y \in \mathbb{R}^d$ and since K is compact there exists $a > 0$ such that

$$v_2(y) \le \psi'(y) - a \qquad \forall\, y \in K.$$

Choose $\eta \in (0,1)$ then provided $N \ge N_0(\eta)$

$$k_N * v_2(y) = \int_{\mathbb{R}^d} k_N(x)\, v_2(y-x)\, dx$$

$$= \int_{\|x\|_\infty \le \eta} k_N(x)\, v_2(y-x)\, dx \;+\; \int_{\|x\|_\infty > \eta} k_N(x)\, v_2(y-x)\, dx$$

$$\le \max\left[\sup_{\|x\|_\infty \le \eta} v_2(y-x),\ (1-\eta) \sup_{\|x\|_\infty \le \eta} v_2(y-x) \right] + \eta \|v_2\|_\infty$$

(by (5.2) (which of the terms in square brackets is the larger depends on whether the supremum is positve or negative)). We can choose $\eta > 0$ sufficiently small to ensure that for every $y \in K$

$$\max\left[\sup_{\|x\|_\infty \le \eta} v_2(y-x),\ (1-\eta) \sup_{\|x\|_\infty \le \eta} v_2(y-x) \right] \le v_2(y) + \tfrac{1}{2}a$$

and that

$$\eta \|v_2\|_\infty \le \tfrac{1}{2}a.$$

Substituting this into our estimate for $k_N * v_2(y)$ shows that there exists $\eta \in (0,1)$ such that for $N \ge N_0(\eta)$ and for every $y \in K$

$$k_N * v_2(y) \le v_2(y) + a \le \psi'(y).$$

Combining this with (5.7) we have that for $N \ge \max(N_0(\tfrac{1}{2}), N_0(\eta))$,

$$k_N * v_2(y) \le \psi'(y) \qquad \forall\, y \in \mathbb{R}^d$$

and hence for every $x \ge 0$

$$e^{-\alpha x} S(x) \ge e^{-\alpha x} S_{k_N * v_2}(x).$$

From this we obtain

$$\underline{\lim}\, e^{-\alpha x} S(x) \ge \underline{\lim}\, e^{-\alpha x} S_{k_N * v_2}(x) = C\, \bar{v}_2(0)$$

$$\ge C\left(\widehat{\psi'}(0) - \left(1 + \int_{\mathbb{R}^d - B(\delta)} \|y\|_\infty^{-3/2}\, dy \right) \varepsilon \right)$$

(by (5.5)). Since we may choose $\varepsilon > 0$ as small as we please we have

$$\underline{\lim}\, e^{-\alpha x} S(x) \geqslant C\, \widehat{\psi'}(0).$$

Combining this with (5.4) and noting that $\widehat{\psi'}(0) = \bar{\psi}(i\xi)$ yields

Proposition 10.

$$S(x) \sim C\, \bar{\psi}(i\xi)\, e^{\alpha x}.$$

We shall now use Proposition 10 to deduce an asymptotic formula for $\pi(x,F)$. Clearly $S(x) \le \pi(x,F)\, x^{\frac{1}{2}d+1}$, so

$$\underline{\lim}\ \frac{x^{\frac{1}{2}d+1}\,\pi(x,F)}{e^{\alpha x}} \ \ge\ C\,\hat{\psi}(i\xi).$$

We now prove an asymptotic inequality in the other direction. Let $\sigma > 1$ and $x = \sigma y$, then

$$\pi(x,F) \ = \ \pi(y,F) \ + \sum_{y<\lambda(\tau)\le x} \psi(|\tau|)$$

$$\le\ \pi(y,F) \ + \sum_{y<\lambda(\tau)\le x} \frac{\lambda(\tau)^{\frac{1}{2}d+1}}{y^{\frac{1}{2}d+1}}\ \psi(|\tau|) \ \le\ \pi(y,F) \ + \frac{\sigma^{\frac{1}{2}d+1}}{x^{\frac{1}{2}d+1}}\ S(x).$$

Hence

$$\frac{x^{\frac{1}{2}d+1}\,\pi(x,F)}{e^{\alpha x}} \ \le\ \frac{(\sigma y)^{\frac{1}{2}d+1}\,\pi(y,F)}{e^{\alpha\sigma y}} \ + \ \sigma^{\frac{1}{2}d+1}\,\frac{S(x)}{e^{\alpha x}}$$

and so if the first term on the R.H.S. tends to zero as $y \to \infty$ then

$$\overline{\lim}\ \frac{x^{\frac{1}{2}d+1}\,\pi(x,F)}{e^{\alpha x}} \ \le\ \sigma^{\frac{1}{2}d+1}\,C\,\hat{\psi}(i\xi)$$

and hence, since $\sigma > 1$ is arbitrary,

$$\overline{\lim}\ \frac{x^{\frac{1}{2}d+1}\,\pi(x,F)}{e^{\alpha x}} \ \le\ C\,\hat{\psi}(i\xi).$$

To show that, for any $\sigma > 1$, $(\sigma y)^{\frac{1}{2}d+1}\,\pi(y,F)\,e^{-\alpha\sigma y} \to 0$ as $y \to \infty$ it is clearly enough to show that, for any $\sigma > 1$, $\pi(y,F)\,e^{-\alpha\sigma y}$ is bounded. Choose a C^∞ function v in $L^1(\mathbb{R}^d)$ and a continuous probability density k such that $\hat{k}$ is C^∞ in a neighbourhood of 0 and has compact support and such that $k*v(y) \ge \psi'(y)$ for every $y \in \mathbb{R}^d$. Then, if λ_0 denotes the smallest number that occurs as the length of a closed orbit,

$$\frac{\pi(y,F)}{e^{\alpha\sigma y}} \ \le\ \sum_{\lambda(\tau)\le y} e^{-\alpha\sigma\lambda(\tau)}\ \psi(|\tau|)$$

$$\leq \lambda_0^{-\nu-1} \sum_{\lambda(\tau)\leq y} \lambda(\tau)^{\nu+1} e^{-\alpha\sigma\lambda(\tau)} \psi([\tau])$$

$$\leq \lambda_0^{-\nu-1} \sum_{\lambda(\tau)\leq y} \lambda(\tau)^{\nu+1} e^{-\alpha\sigma\lambda(\tau)+<\xi,[\tau]>} k*v([\tau])$$

$$< \lambda_0^{-\nu-1} |\hat{\eta}(\alpha\sigma)| < \infty$$

(where $\hat{\eta}$ is defined with respect to v and k) as required.

Taken together our two asymptotic inequalities yield

Theorem 1. (Lalley [10].) Let φ be a weak-mixing Axiom A flow restricted to a (non-trivial) basic set A and suppose that $F_1,\ldots,F_d$ are real valued Hölder continuous functions defined on A satisfying conditions (A) and (B). Then

$$\pi(x,F) \sim \frac{\hat{\psi}(i\xi)}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}}} \frac{e^{\alpha x}}{x^{\frac{1}{2}d+1}}.$$

Remarks.(i) There is nothing special about the box $B(\delta)$ used in the definition of $\pi(x,F)$. Clearly the same proof will work for any box in $\mathbb{R}^d$ and by a simple approximationargument

$$\#\{\tau : \lambda(\tau) \leq x,\ [\tau] \in B\} \sim \frac{\int_B e^{-<\xi,y>}\, dy}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}}} \frac{e^{\alpha x}}{x^{\frac{1}{2}d+1}}$$

where B is any Borel set in $\mathbb{R}^d$ whose boundary has zero measure.

(ii) For $a = (a_1,\ldots,a_d)$ one can also ask about the behaviour as $x \to \infty$ of

$$\#\{\tau : \lambda(\tau) \leq x,\ |\lambda_{F_i}(\tau) - \lambda(\tau)a_i| \leq \delta_i,\ i = 1,\ldots,d\}$$

but this quantity is just $\pi(x,F')$ where $F_i' = F_i - a_i$. Furthermore, if $F_1,\ldots,F_d$ satisfy (A) then so do $F_1',\ldots,F_d'$ and $P(<w,F'>) = P(<w,F>) - <w,a>$. Thus $\nabla P(<w,F'>) = \nabla P(<w,F>) - a$ and $\nabla^2 P(<w,F'>) = \nabla^2 P(<w,F>)$. The condition (B) that $P(<w,F'>)$ has a finite minimum ξ, i.e. that $\nabla P(<\xi,F'>) = 0$, becomes, in terms of $F_1,\ldots,F_d$, that for

some $\xi \in \mathbb{R}^d$

$$\nabla P(<\xi,F>) = \left(\int F_1 \, dm_\xi, \ldots \int F_d \, dm_\xi\right) = a.$$

In this case the growth rate $\alpha = \sup\{h_m(\varphi) : \int F_i \, dm = a_i, \ i = 1,\ldots,d\}$.

(iii) If β does not have a finite minimum then

$$\overline{\lim} \frac{1}{x} \log \pi(x,F) \leq \inf_{w \in \mathbb{R}^d} \beta(w).$$

To see this fix $w \in \mathbb{R}^d$ and consider, for $s \in \mathbb{C}$, $t \in \mathbb{R}^d$,

$$\sum_\tau \lambda(\tau)^{\nu+1} \, e^{-s\lambda(\tau) + <w+it,[\tau]>}.$$

By the arguments of Section 2, this converges for $\operatorname{Re} s > \beta(w)$. Hence, for suitable k and v,

$$\sum_\tau \lambda(\tau)^{\nu+1} \, e^{-s\lambda(\tau) + <w,[\tau]>} \, k*v([\tau])$$

converges for $\operatorname{Re} s > \beta(w)$ and so, after writing the above summation as a Stieltjes integral, we obtain

$$\overline{\lim} \frac{1}{x} \log \sum_{\lambda(\tau) \leq x} e^{<w,[\tau]>} \, k*v([\tau]) \leq \beta(w).$$

From this one easily deduces that

$$\overline{\lim} \frac{1}{x} \log \pi(x,F) \leq \beta(w)$$

and since this holds for every $w \in \mathbb{R}^d$ the required result is proved.

In [2], [3] Bowen showed that the closed orbits of an Axiom A flow are equidistributed with respect to the measure of maximal entropy m_0 (cf. also [15]). More precisely, if $K \in C(\Lambda)$ then

$$\frac{\sum_{\lambda(\tau) \leq x} \lambda_K(\tau)}{\sum_{\lambda(\tau) \leq x} \lambda(\tau)} \to \int K \, dm_0 \qquad \text{as } x \to \infty \tag{5.8}$$

and

$$\frac{1}{\#\{\tau : \lambda(\tau) \le x\}} \sum_{\lambda(\tau)\le x} \frac{\lambda_K(\tau)}{\lambda(\tau)} \to \int K \, dm_0 \quad \text{as } x \to \infty \qquad (5.9).$$

Lalley used Theorem 1 to deduce a generalization of (5.8), (5.9), at least for weak-mixing flows, namely that *individual* closed orbits are almost surely equidistributed with respect to the measure of maximal entropy. In fact he proved a stronger result:

Corollary 1. (Lalley [10].) For $F_1,\dots,F_d$ as in Theorem 1 and for any $\varepsilon > 0$,

$$\frac{\#\left\{\tau : \lambda(\tau) \le x,\ |\lambda_{F_i}(\tau)| \le \delta_i,\ i=1,\dots,d,\ \left|\frac{\lambda_K(\tau)}{\lambda(\tau)} - \int K \, dm_\xi\right| < \varepsilon\right\}}{\#\{\tau : \lambda(\tau) \le x,\ |\lambda_{F_i}(\tau)| \le \delta_i,\ i=1,\dots,d\}} \to 1$$

as $x \to \infty$, where m_ξ is the equilibrium state of $\langle \xi, F\rangle$.

6. Finite group extensions.

In this section we turn our attention to the situation considered by Parry and Pollicott in [18]. Suppose that $\tilde\phi : \tilde M \to \tilde M$ is an Axiom A flow and that G is a finite group which acts freely on $\tilde M$ by diffeomorphism, the action commuting with $\tilde\phi$. Then $\tilde\phi$ induces a flow φ on the quotient $M = \tilde M/G$ defined by $\varphi_t(xG) = (\tilde\phi_t x)G$ and this flow also satisfies Axiom A. We shall consider $\tilde\phi$ restricted to a G-invariant basic set $\tilde\Lambda$ and φ restricted to the basic set $\Lambda = \tilde\Lambda/G$ which we suppose to be non-trivial. In this situation, $\tilde\phi$ is said to be a G-covering of φ.

If τ is a closed φ-orbit then it is covered by ℓ distinct closed $\tilde\phi$-orbits $\tilde\tau_i$, $i = 1,\dots,\ell$, for some $\ell \mid |G|$, each of length $(|G|/\ell)\lambda(\tau)$. For each $i = 1,\dots,\ell$ there exists a unique $\gamma(\tilde\tau_i) \in G$ such that for any $x \in \tilde\tau_i$, $\gamma(\tilde\tau_i)x = \tilde\phi_{\lambda(\tau)}x$. This is called the Frobenius element of $\tilde\tau_i$. If $\tilde\tau_j$ is another lift of τ then $\gamma(\tilde\tau_j)$ is conjugate to $\gamma(\tilde\tau_i)$. Thus each closed

φ-orbit τ gives rise to a unique conjugacy class, called the Frobenius class of τ, in G.

Suppose that φ (and hence $\tilde{\varphi}$) is topologically weak-mixing and that $F_1,\ldots,F_d$ satisfy the hypotheses of Theorem 1. Let C be the conjugacy class of some fixed $g \in G$. We shall derive an asymptotic formula for

$$\pi_C(x,F) = \#\{\, \tau : \lambda(\tau) \le x,\ |\lambda_{F_i}(\tau)| \le \delta_i,\ i = 1,\ldots,d,\ \gamma(\tilde{\tau}) \in C\}$$

where $\tilde{\tau}$ is any lift of τ.

Let χ be an irreducible character of G. Define

$$L(s,t,\chi) \;=\; \exp \sum_{\tau} \sum_{n=1}^{\infty} \frac{\chi(\gamma(\tilde{\tau})^n)}{n}\, e^{n(-s\lambda(\tau) + \langle \xi + it, f(\tau)\rangle)}$$

and

$$\eta^1(s,t,\chi) \;=\; \sum_{\tau} (-\lambda(\tau))^{\nu+1}\ \chi(\gamma(\tilde{\tau}))\ e^{-s\lambda(\tau) + \langle \xi + it, f(\tau)\rangle}$$

where, again, $\tilde{\tau}$ is any lift of τ. Since $\gamma(\tilde{\tau})$ is unique up to conjugacy, $L(s,t,\chi)$ and $\eta^1(s,t,\chi)$ are well defined. By comparison with $L(s,t)$ and $\eta^1(s,t)$ the series defining $L(s,t,\chi)$ and $\eta^1(s,t,\chi)$ converge absolutely for $\mathrm{Re}\, s > \alpha$, $t \in \mathbb{R}^d$ and so $L(s,t,\chi)$ is analytic and non-zero on $\{\, s : \mathrm{Re}\, s > \alpha \,\} \times \mathbb{R}^d$ and $\eta^1(s,t,\chi)$ is analytic on $\{\, s : \mathrm{Re}\, s > \alpha \,\} \times \mathbb{R}^d$. If χ_0 denotes the trivial character then, clearly, we have $\eta^1(s,t,\chi_0) = \eta^1(s,t)$. We shall show that if $\chi \ne \chi_0$ then $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $\{\, s : \mathrm{Re}\, s \ge \alpha \,\} \times \mathbb{R}^d$.

In view of Section 3 one can model φ by a weak-mixing suspended flow σ^r on Σ^r over a shift of finite type Σ_A. One can also, following [18], model $\tilde{\varphi}$ by a weak-mixing suspended flow $\tilde{\sigma}^{\tilde{r}}$ over a certain skew-product $\tilde{\sigma} : \Sigma_A \times G \to \Sigma_A \times G$. To see how this is done we first briefly recall how is related φ to the suspended flow σ^r. One can construct a family of (arbitrarily small) co-dimension one sections $\{T_1,\ldots,T_k\}$ on Λ transverse to φ such that $T = T_1 \cup \ldots \cup T_k$ is a global cross-section for φ, i.e. every point of Λ intersects T infinitely often as it flows backwards and forwards in time. Let $P : T \to T$ be the Poincaré map. Then one can define a map $p : \Sigma_A \to T$ by letting $p(x)$ be the (unique) point of Λ such that $P^n(p(x)) \in T_{x_n}$ for every $n \in \mathbb{Z}$. If $r : \Sigma_A \to \mathbb{R}$ is

defined by

$$r(x) = \inf\{t > 0 : \varphi_t(p(x)) \in T\}$$

then p can be extended to a map $p : \Sigma^r \to \Lambda$ by $p(x,t) = \varphi_t(p(x))$. (The map p is not one-one because of orbits which pass through the boundaries of the sections.) If $\pi_G : \widetilde{M} \to M$ is the covering map then $\widetilde{T} = \pi_G^{-1}T$ is a global cross section for $\tilde{\varphi}$ and, provided each T_i is sufficiently small, we can identify $\widetilde{T}$ with $T \times G$. Thus we can define $\tilde{p} : \Sigma_A \times G \to \widetilde{T}$ by $\tilde{p}(x,g) = (p(x),g)$. Define $\kappa : \Sigma_A \to G$ by the equation

$$\tilde{\varphi}_{r(x)}(\tilde{p}(x,g)) = \tilde{p}(\sigma x,\kappa(x)g)$$

then κ is a function of only two co-ordinates, i.e. $\kappa(x) = \kappa(x_0,x_1)$. Define a skew-product $\tilde{\sigma} : \Sigma_A \times G \to \Sigma_A \times G$ by $\tilde{\sigma}(x,g) = (\sigma x,\kappa(x)g)$ and a function $\tilde{r} : \Sigma_A \times G \to \mathbb{R}$ by $\tilde{r}(x,g) = r(x)$. Let $\widetilde{\Sigma}^{\tilde{r}}$ be the $\tilde{r}$-suspension space over $\Sigma_A \times G$ then one can extend $\tilde{p}$ to a map $\tilde{p} : \widetilde{\Sigma}^{\tilde{r}} \to \widetilde{\Lambda}$ by $\tilde{p}((x,g),t) = \tilde{\varphi}_t(\tilde{p}(x,g))$. If $x \in \mathrm{Fix}_n$ (and $x \notin \mathrm{Fix}_m$ for $m < n$) then $p(x,0)$ lies on a closed φ-orbit, τ say, and $\lambda(\tau) = r^n(x)$. Also $\tilde{p}((x,g),0)$ lies on a closed $\tilde{\varphi}$-orbit $\tilde{\tau}$ and we have that

$$\begin{aligned}\gamma(\tilde{\tau})\tilde{p}((x,g),0) &= \tilde{\varphi}_{\tilde{r}^n(x)}\tilde{p}((x,g),0)\\ &= \tilde{p}((x,g),\tilde{r}^n(x))\\ &= \tilde{p}((x,g),\tilde{r}^n(x,g))\\ &= \tilde{p}((\sigma^n x,\kappa_n(x)g),0)\end{aligned}$$

(where $\kappa_n(x) = \kappa(\sigma^{n-1}x)\ldots\kappa(\sigma x)\kappa(x)$)

$$= \tilde{p}((x,\kappa_n(x)g),0).$$

However $\gamma(\tilde{\tau})\tilde{p}((x,g),0) = \tilde{p}((x,\gamma(\tilde{\tau})g),0)$ so $\gamma(\tilde{\tau}) = \kappa_n(x)$.

We can define $\eta_r^1(s,t,\chi)$ in the same way as $\eta^1(s,t,\chi)$ except that the summation is taken over closed σ^r-orbits and, for a closed σ^r-orbit τ, $[\tau] = (\lambda_{F_1*p}(\tau),\ldots,\lambda_{F_d*p}(\tau))$. Then, in particular, $\eta^1(s,t,\chi) - \eta_r^1(s,t,\chi)$ is analytic on $\{ s : \mathrm{Re}\, s > \alpha - \varepsilon \} \times \mathbb{R}^d$ (for some $\varepsilon > 0$).

Define

$$L_r(s,t,\chi) = \exp \sum_{\tau} \sum_{n=1}^{\infty} \frac{\chi(\gamma(\tilde{\tau})^n)}{n} e^{n(-s\lambda(\tau) + \langle\xi+it,[\tau]\rangle)}.$$

Then we also have that $L(s,t,\chi)/L_r(s,t,\chi)$ is analytic and non-zero on a neighbourhood of $\{s : \mathrm{Re}\, s > \alpha - \varepsilon\} \times \mathbb{R}^d$ (for some $\varepsilon > 0$). In view of the above comments we may rewrite $L_r(s,t,\chi)$ in the form

$$L_r(s,t,\chi) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} \chi(\kappa_n(x))\, e^{-sr^n(x) + \langle \xi + it, f^n(x) \rangle}$$

with $f = (f_1,\ldots,f_d)$ and f_i related to $F_i \circ p$ by (2.1), $i = 1,\ldots,d$. Using the same argument as in Section 2 one can show that $\eta_r^1(s,t,\chi)$ (and hence $\eta^1(s,t,\chi)$) differs from $(\partial/\partial s)^{\nu+1} \log L_r(s,t,\chi)$ by a function analytic for $\mathrm{Re}\, s > \alpha - \varepsilon$ (some $\varepsilon > 0$). Hence to show that, for $\chi \neq \chi_0$, $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $\{s : \mathrm{Re}\, s \geq \alpha\} \times \mathbb{R}^d$ it is sufficient to show that $L_r(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\{s : \mathrm{Re}\, s \geq \alpha\} \times \mathbb{R}^d$.

First we show that it is enough to consider the case where G is cyclic. Suppose that H is a subgroup of G and that $g_1,\ldots,g_k$ are elements of G such that

$$G = \bigcup_{i=1}^{k} Hg_i.$$

If χ is a character of H then one can define the *induced character* χ^* of G by

$$\chi^*(g) = \sum_{g_i g g_i^{-1} \in H} \chi(g_i g g_i^{-1}).$$

A result of Frobenius states that for each non-trivial character χ of G there exist cyclic subgroups $H_1,\ldots,H_m$ of G and non-trivial characters $\chi_1,\ldots,\chi_m$ of $H_1,\ldots,H_m$ respectively such that χ is a rational combination of the induced characters $\chi_1^*,\ldots,\chi_m^*$ [24]. Thus there exist integers $n, n_1,\ldots,n_m$ such that

$$L(s,t,\chi)^n = \prod_{j=1}^{m} L(s,t,\chi_j^*)^{n_j}.$$

Furthermore, for each $j = 1,\ldots,m$, H_j acts freely on $\widetilde{M}$ by diffeomorphisms which commute with $\tilde{\varphi}$ and thus gives rise to an Axiom A flow $\varphi_j = \tilde{\varphi}/H_j$ on the quotient $M_j = \widetilde{M}/H_j$. If $\pi_j : M_j \to M$ is the natural projection we can define functions $L(s,t,\chi_j)$ with respect to φ_j by

$$L(s,t,\chi_j) = \exp \sum_{\tau} \sum_{n=1}^{\infty} \frac{\chi_j(\gamma_{H_j}(\tilde{\omega})^n)}{n} e^{n(-s\lambda(\omega) + \langle \xi + it, [\omega] \rangle)}$$

where the summation is taken over all closed φ_j-orbits ω, $\tilde{\omega}$ is any lift of ω to $\widetilde{M}$, $\gamma_{H_j}(\tilde{\omega}) \in H_j$ is the Frobenius element of $\tilde{\omega}$ with respect to ω (i.e. $\gamma_{H_j}(\tilde{\omega})$ is the unique element of H_j such that for any $x \in \tilde{\omega}$, $\gamma_{H_j}(\tilde{\omega})x = \tilde{\varphi}_{\lambda(\omega)}x$) and

$$[\omega] = (\lambda_{F_1 \circ \pi_j}(\omega), \ldots, \lambda_{F_d \circ \pi_j}(\omega)).$$

One can show ([18], Proposition 2) that

$$L(s,t,\chi_j) = L(s,t,\chi_j^*), \qquad j = 1,\ldots,m$$

and so

$$L(s,t,\chi)^n = \prod_{j=1}^{m} L(s,t,\chi_j)^{n_j}.$$

Thus $L(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$ provided each $L(s,t,\chi_j)$ is analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$.

We can model each φ_j by a suspended flow σ^{r_j} and define L-functions $L_{r_j}(s,t,\chi_j)$ in the obvious way. Then we will have that $L(s,t,\chi_j)/L_{r_j}(s,t,\chi_j)$ is analytic for $\operatorname{Re} s > \alpha - \varepsilon$ (for some $\varepsilon > 0$). Thus $L(s,t,\chi)$ and $L(s,t,\chi_j)$, $j = 1,\ldots,m$, are analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$ if and only if $L_r(s,t,\chi)$ and $L_{r_j}(s,t,\chi_j)$, $j = 1,\ldots,m$, are analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$, respectively. Hence to show that $L_r(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$ it is enough to show that each $L_{r_j}(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \mathbb{R}^d$, $j = 1,\ldots,m$ and we can suppose, without loss of generality, that G is cyclic.

Write $\chi(\kappa(x)) = \exp 2\pi i\theta(x)$ where we have chosen θ (as we may) to take values in $\{0, 1/|G|, \ldots, (|G|-1)/|G|\}$ and note that since G is abelian

$$\chi(\kappa_n(x)) = \prod_{i=0}^{n-1} \chi(\kappa(\sigma^i x)) = \exp 2\pi i\theta^n(x).$$

Hence, in the notation of Section 1, $L_\tau(s,t,\chi) = \zeta(-s\tau + \langle\xi+it,f\rangle + 2\pi i\theta)$. If $\mathrm{Re}\, s_0 = \alpha$ then, by Proposition 1, $L_\tau(s,t,\chi)$ is analytic and non-zero in a neighbourhood of (s_0,t_0) unless there exists $w \in C(\Sigma_A)$ and $K \in C(\Sigma_A, 2\pi\mathbb{Z})$ such that

$$\mathrm{Im}\, s_0 + \langle t_0,f\rangle + 2\pi\theta = K + w\circ\sigma - w,$$

i.e. such that

$$|G|\, \mathrm{Im}\, s_0 + |G| \langle t_0,f\rangle = -2|G|\pi\theta + |G|\, K + |G|\, w\circ\sigma - |G|\, w.$$

Since σ^τ is weak-mixing and f satisfies (A') this can only occur when $s_0=\alpha$ and $t_0=0$. In this case we have $H(\sigma x) = \chi(\kappa(x))\, H(x)$ (where $H = e^{iw}$) but, if we consider H' defined on $\Sigma_A \times G$ by $H'(x,g) = \chi^{-1}(g)\, H(x)$, this shows that $\tilde{\sigma}$ is not ergodic contradicting the assumption that $\tilde{\sigma}^\tau$ is weak-mixing. So we have shown that if $\chi \neq \chi_0$ then $L_\tau(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d$.

To summarize we have that $\eta^1(s,t,\chi_0) = \eta^1(s,t)$ but if $\chi \neq \chi_0$ then $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d$. Define

$$\eta_C(s,t) = \sum_{\substack{\tau \\ \gamma(\tau)\in C}} (-\lambda(\tau))^{\nu+1}\, e^{-s\lambda(\tau) + \langle\xi+it,\tau\rangle}.$$

By the orthogonality relation for characters

$$\sum_{\chi \text{ irreducible}} \chi(g^{-1})\, \eta^1(s,t,\chi) = \frac{|G|}{|C|}\, \eta_C(s,t)$$

(where g is an element in the conjugacy class C). Hence $(|G|/|C|)\eta_C(s,t) - \eta^1(s,t)$ is analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d$ and so $(|G|/|C|)\eta_C(s,t)$ is analytic in a neighbourhood of $(\{s : \mathrm{Re}\, s \geq \alpha\} \times \mathbb{R}^d) - \{(\alpha,0)\}$ and in a neighbourhood of $(\alpha,0)$

$$\eta_C(s,t) = \frac{|C|}{|G|}\, \frac{(-1)^{\nu+1}\, \nu!}{(s - s(t))^{\nu+1}}$$

(modulo an analytic function).

As in Section 2, let $v : \mathbb{R}^d \to \mathbb{R}$ be a C^∞ function in $L^1(\mathbb{R}^d)$ and let k be a continuous probabilty density function on $\mathbb{R}^d$ such that the Fourier transform $\hat{k}(t)$ is C^∞ in a neighbourhood of 0 and has compact support. Define $\tilde{\eta}_C(s)$ by restricting the sum defining $\tilde{\eta}(s)$ to those closed orbits τ with $\gamma(\tau) \in C$, i.e.

$$\hat{\eta}_C(s) = \sum_{\substack{\tau \\ \gamma(\tau)\in C}} (-\lambda(\tau))^{\nu+1}\, e^{-s\lambda(\tau) + \langle \xi, t\tau \rangle}\, k*v([\tau]).$$

Then

$$\hat{\eta}_C(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{k}(-t)\, \hat{v}(-t)\, \eta_C(s,t)\, dt$$

and we can repeat the analysis of Section 4 to obtain

Proposition 11.

(i) If d is even then

$$\lim_{\sigma \to \alpha} \left(\hat{\eta}_C(\sigma+i\omega) - \frac{|C|}{|G|} \frac{(-1)^{\nu+1}\, \hat{v}(0)}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}} (\sigma+i\omega-\alpha)} \right)$$

exists for almost every $\omega \in \mathbb{R}$ and is locally integrable and there exists a locally integrable function $h(\omega)$ such that

$$\left| \hat{\eta}_C(\sigma+i\omega) - \frac{|C|}{|G|} \frac{(-1)^{\nu+1}\, \hat{v}(0)}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}} (\sigma+i\omega-\alpha)} \right| \le h(\omega)$$

$(\sigma > \alpha)$.

(ii) If d is odd then

$$\lim_{\sigma \to \alpha} \left(\hat{\eta}_C(\sigma+i\omega) - \frac{|C|}{|G|} \frac{(-1)^{\nu+1}\, \hat{v}(0)\, \pi^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}} (\sigma+i\omega-\alpha)^{\frac{1}{2}}} \right)$$

exists for almost every $\omega \in \mathbb{R}$ and is an element of the the Sobolev space $W^{1,1}_{loc}(\mathbb{R})$ which consists of locally integrable functions with locally integrable first derivatives and there exists a locally integrable function $h(\omega)$ such that

$$\left| \hat{\eta}_C(\sigma+i\omega) - \frac{|C|}{|G|} \frac{(-1)^{\nu+1}\, \hat{v}(0)\, \pi^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}} (\sigma+i\omega-\alpha)^{\frac{1}{2}}} \right| \le h(\omega)$$

$(\sigma > \alpha)$.

It is now an easy matter to repeat the analysis of earlier sections to obtain the desired result:

Theorem 2. Let φ be a weak-mixing Axiom A flow restricted to a (non-trivial) basic set A and suppose that $F_1,\ldots,F_d$ are real valued Hölder continuous functions defined on A satisfying conditions (A) and (B). Also let $\tilde{\varphi}$ be a G-covering of φ. Then

$$\pi_C(x,F) \sim \frac{|C|}{|G|} \pi(x,F).$$

To end this section we consider an application to homology. In [18] Parry and Pollicott proved that the closed orbits of a weak-mixing transitive Anosov flow (considered as 1-cycles) on a manifold M generate $H_1(M,\mathbb{Z})$. We now prove a slightly stronger result.

Corollary 2. Suppose that φ, $F_1,\ldots,F_d$ satisfy the hypotheses of Theorem 1 but that in addition φ is a transitive Anosov flow. Then $\mathcal{O}_F = \{ \tau : |\lambda_{F_i}(\tau)| \le \delta_i,\ i = 1,\ldots,d \}$ generates $H_1(M,\mathbb{Z})$.

Proof. Let $\tilde{\tilde{M}}$ be the universal homology cover of M. This means that $H_1 = H_1(M,\mathbb{Z})$ acts freely on $\tilde{\tilde{M}}$ and $M = \tilde{\tilde{M}}/H_1$. Let H_0 be a subgroup of H_1 such that $G = H_1/H_0$ is finite and let $\tilde{M} = \tilde{\tilde{M}}/H_0$ so that $M = \tilde{M}/G$. In this situation, $\gamma(\tau)$, the Frobenius element of τ is well defined (because G is abelian) and is just the coset of H_0 in H_1 to which the homology class of τ belongs. Applying Theorem 2 we have

$$\#\{ \tau : \lambda(\tau) \le x,\ |\lambda_{F_i}(\tau)| \le \delta_i,\ i = 1,\ldots,d,\ \gamma(\tau) \in h + H_0 \} \sim \pi(x,F)/|G|$$

for any $h \in H_1$.

Suppose now that H_1 is not generated by $\mathcal{O}_F$. Then there exists H_0, a cofinite subgroup of H_1, such that H_0 contains the homology classes of all the elements of $\mathcal{O}_F$. This means that for any $\tau \in \mathcal{O}_F$, $\gamma(\tau)$ is the identity in $G = H_1/H_0$ which contradicts the

equidistribution result of the previous paragraph.

Remark. Except for the contribution from Theorem 2 the above proof is really just the same as that contained in Section 7 of [18]. It is included for completeness.

7. $\mathbb{Z}^d$-extensions and homology.

As we remarked in Section 2, a result analogous to Theorem 1 is valid for Hölder continuous functions $F_1,\ldots,F_d$ (defined on a basic set, Λ, of a weak-mixing Axiom A flow) such that the set of vectors $[\tau] = (\lambda_{F_1}(\tau),\ldots,\lambda_{F_d}(\tau))$ lie in a discrete subgroup of $\mathbb{R}^d$ which we shall suppose to be $\mathbb{Z}^d$. In fact the proof in this case, which we shall sketch, is substantially easier than what we have considered hitherto because $\widehat{\mathbb{Z}^d} = \mathbb{R}^d/\mathbb{Z}^d$ is compact, so there is no need to introduce the probability densities that we used in Section 4. In this situation we must replace (A) with the following two independence conditions:

Condition (A_1). If $a_1,\ldots,a_d$ are real numbers such that the function $a_1F_1 + \ldots + a_dF_d$ is cohomologous to a constant then $a_1 = \ldots = a_d = 0$;

and

Condition (A_2). If $a_0,a_1,\ldots,a_d$ are real numbers such that for the function $G = a_0 + a_1F_1 + \ldots + a_dF_d$, then the skew-product flow $T^G\colon S^1 \times \Lambda \to S^1 \times \Lambda$ is not topologically transitive if and only if $a_0 = 0$ and $a_1 = \ldots = a_d = 0 \pmod{2\pi}$.

In view of the results of Section 3 we can lift the problem to a (weak-mixing) suspended flow, σ^t which is related to φ by a semi-conjugacy $p : \Sigma^t \to \Lambda$ and for simplicity we shall only give the proof in this case. If $f_1,\ldots,f_d$ are related to $F_1\circ p,\ldots,F_d\circ p$ by (2.1) then (A_2) is equivalent to

Condition (A_2'). If $a_0, a_1, \ldots, a_d$ are real numbers then $a_0\tau + a_1 f_1 + \ldots + a_d f_d$ is cohomologous to a function valued in $2\pi\mathbb{Z}$ if and only if $a_0 = 0$ and $a_1 = \ldots = a_d = 0$ (mod 2π).

Remark. A particular consequence of (A_2), (A_2') is that the set of vectors $[\tau]$ lie in $\mathbb{Z}^d$ and in no proper subgroup of $\mathbb{Z}^d$.

The condition (A_1) ensures that the function $\beta_\tau(w) = P(\langle w, F\circ p\rangle)$ is strictly convex. We also suppose that $F_1, \ldots, F_d$ satisfy (B), i.e. that β_τ has a finite minimum and that ξ is that minimum. We write, as before, $\alpha = \beta_\tau(\xi)$. The condition (A_1) also ensures that $\nabla^2\beta_\tau(\xi)$ is positive definite.

Now for $s \in \mathbb{C}$, $t \in \mathbb{R}^d/\mathbb{Z}^d$ we define

$$L_\tau(s,t) = \prod_\tau \left(1 - e^{-s\lambda(\tau) + \langle \xi + 2\pi i t, [\tau]\rangle}\right)^{-1}$$

$$= \exp \sum_\tau \sum_{n=1}^{\infty} \frac{1}{n} e^{n(-s\lambda(\tau) + \langle \xi + 2\pi i t, [\tau]\rangle)}$$

(where this converges). The following analogue of Proposition 2 holds.

Proposition 12.

(i) $L_\tau(s,t)$ is analytic and non-zero in the set $\{ s : \mathrm{Re}\, s > \alpha \} \times \mathbb{R}^d/\mathbb{Z}^d$.

(ii) For $t_0 \neq 0$, $L_\tau(s,t)$ is analytic and non-zero in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} \times \{t_0\}$.

(iii) For each $t_0 \in \mathbb{R}^d/\mathbb{Z}^d$, $L_\tau(s,t)$ is analytic and non-zero in a neighbourhood of $\{ s : \mathrm{Re}\, s = \alpha, \mathrm{Im}\, s \neq 0 \} \times \{t_0\}$.

(iv) In a neighbourhood of $(\alpha, 0)$, $L_\tau(s,t)$ takes the form $\psi(s,t)/(s - s(t))$ for some analytic and non-zero ψ.

The proof is the same as for Proposition 2 except that after expressing $L_\tau(s,t)$ in

the form $\zeta(-sr + <\xi+2\pi it, f>)$ we apply (A_2') in the place of (A') to prove (ii) and (iii). The function $s(t)$ also enjoys the properties described in Proposition 3 (with the same proof).

As before, we shall define

$$\eta_r^1(s,t) = \sum_{\tau} (-\lambda(\tau))^{\nu+1} e^{-s\lambda(\tau) + <\xi+2\pi it, f\tau>}$$

(where $\nu = [d/2]$) and in exactly the same way as in Section 2 we can deduce

Proposition 13. The function $\eta_r^1(s,t)$ is analytic in a neighbourhood of $(\{ s : \mathrm{Re}\, s \geq \alpha \} \times \mathbb{R}^d/\mathbb{Z}^d) - \{(\alpha,0)\}$ and in a neighbourhood of $(\alpha,0)$,

$$\eta_r^1(s,t) = \frac{(-1)^{\nu+1}\nu!}{(s - s(t))^{\nu+1}} + \psi_1(s,t)$$

for some analytic function ψ_1.

We now consider, for a fixed $m \in \mathbb{Z}^d$,

$$\hat{\eta}(s) = \sum_{\substack{\tau \\ [\tau]=m}} (-\lambda(\tau))^{\nu+1} e^{-s\lambda(\tau) + <\xi, f\tau>}.$$

Elementary Fourier analysis gives

$$\hat{\eta}(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\mathbb{Z}^d} e^{-2\pi i<t,m>} \eta_r^1(s,t)\, dt.$$

By Propostion 13 (and since $\mathbb{R}^d/\mathbb{Z}^d$ is compact), $\hat{\eta}(s)$ is analytic in a neighbourhood of $\{ s : \mathrm{Re}\, s \geq \alpha \} - \{\alpha\}$ and if s is in a neighbourhood of α and U is a small neighbourhood of $0 \in \mathbb{R}^d/\mathbb{Z}^d$ then

$$\hat{\eta}(s) = \frac{1}{(2\pi)^d} \int_U e^{-2\pi i<t,m>} \left\{ \frac{(-1)^{\nu+1}\nu!}{(s - s(t))^{\nu+1}} + \psi_1(s,t) \right\} dt$$

$$+ \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d/\mathbb{Z}^d - U} e^{-2\pi i<t,m>} \eta_r^1(s,t)\, dt$$

$$= \frac{(-1)^{\nu+1}\nu!}{(2\pi)^d} \int_U e^{-2\pi i\langle t,m\rangle} \frac{1}{(s-s(t))^{\nu+1}}\, dt + \psi_2(s,t),$$

for some ψ_2 analytic in a neighbourhood of $(\alpha,0)$. After applying Lemma 2, the latter integral is, once again, in the form considered by Katsuda and Sunada [9] and so, writing

$$C = \frac{1}{(2\pi)^{\frac{1}{2}d}(\det \nabla^2\beta(\xi))^{\frac{1}{2}}},$$

we obtain

Proposition 14.

(i) If d is even then

$$\lim_{\sigma\to\alpha}\left(\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}C}{\sigma+i\omega-\alpha}\right)$$

exists for almost every $\omega \in \mathbb{R}$ and is locally integrable and there exists a locally integrable function $h(\omega)$ such that

$$\left|\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}C}{\sigma+i\omega-\alpha}\right| \le h(\omega)$$

$(\sigma > \alpha)$.

(ii) If d is odd then

$$\lim_{\sigma\to\alpha}\left(\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}C\pi^{\frac{1}{2}}}{(\sigma+i\omega-\alpha)^{\frac{1}{2}}}\right)$$

exists for almost every $\omega \in \mathbb{R}$ and is an element of the the Sobolev space $W^{1,1}_{loc}(\mathbb{R})$ which consists of locally integrable functions with locally integrable first derivatives and there exists a locally integrable function $h(\omega)$ such that

$$\left|\hat{\eta}(\sigma+i\omega) - \frac{(-1)^{\nu+1}C\pi^{\frac{1}{2}}}{(\sigma+i\omega-\alpha)^{\frac{1}{2}}}\right| \le h(\omega)$$

$(\sigma > \alpha)$.

If we write

$$S_m(x) = \sum_{\substack{\lambda(\tau)\le x \\ [\tau]=m}} \lambda(\tau)^{\frac{1}{2}d+1} e^{<\xi,m>}$$

then if d is even

$$\hat{\eta}(s) = (-1)^{\nu+1} \int_0^\infty e^{-sx}\, dS_m(x)$$

and if d is odd

$$\hat{\eta}(s) = (-1)^{\nu+1} \int_0^\infty x^{-\frac{1}{2}} e^{-sx}\, dS_m(x).$$

It now follows from Proposition 8 and Proposition 14 that

$$S_m(x) \sim C e^{\alpha x}$$

and hence that

$$\sum_{\substack{\lambda(\tau)\le x \\ [\tau]=m}} \lambda(\tau)^{\frac{1}{2}d+1} \sim C e^{-<\xi,m>} e^{\alpha x}.$$

From this it is easy to deduce

Theorem 3. Let φ be a weak-mixing Axiom A flow restricted to a (non-trivial) basic set A and suppose that $F_1,\ldots,F_d$ are real valued Holder continuous functions defined on A satisfying (A_1), (A_2) and (B). Then for every $m \in \mathbb{Z}^d$

$$\pi_m(x) \sim \frac{e^{-<\xi,m>}}{(2\pi)^{\frac{1}{2}d} (\det \nabla^2\beta(\xi))^{\frac{1}{2}}} \frac{e^{\alpha x}}{x^{\frac{1}{2}d+1}}.$$

We now go on to discuss asymptotics for closed orbits in fixed homology classes for transitive C^∞ Anosov flows and in doing so extend the results of Katsuda and

Sunada contained in [9].

Suppose that φ is a C^∞ weak-mixing transitive Anosov flow on a manifold M and that Z is the vector field generating the flow. The first homology group $H_1(M,\mathbb{Z})$ is isomorphic to $\mathbb{Z}^d \oplus G$ where the (finite) group G is the torsion subgroup. For a closed φ-orbit τ, $[\![\tau]\!]$ will denote its homology class, $[\tau]$ the projection of $[\![\tau]\!]$ onto $\mathbb{Z}^d$ and $\gamma(\tau)$ the projection of $[\![\tau]\!]$ onto G.

Let m be an ergodic φ-invariant probability measure on M. We define the winding cycle with respect to m, Φ_m, to be the linear map from the space of closed 1-forms on M to $\mathbb{R}$ given by

$$\Phi_m(\omega) = \int \langle\omega,Z\rangle \, dm.$$

It is clear that if ω is an exact 1-form then $\Phi_m(\omega) = 0$. Using the fact that $H^1(M,\mathbb{R}) \cong$ (closed 1-forms)/(exact 1-forms) we can regard Φ_m as a homology class in $H_1(M,\mathbb{R}) = \mathrm{Hom}(H^1(M,\mathbb{R}),\mathbb{R})$.

Note that there exist smooth closed 1-forms $\rho_1,\ldots,\rho_d$ such that

$$[\tau] = \left(\int_\tau \rho_1, \ldots, \int_\tau \rho_d\right)$$

for every closed orbit τ. Define $F_i = \langle\rho_i,Z\rangle$, $i = 1,\ldots,d$ so each function F_i is C^∞ and $[\tau] = (\lambda_{F_1}(\tau),\ldots,\lambda_{F_d}(\tau))$. Let $F = (F_1,\ldots,F_d)$ and, for $w \in \mathbb{R}^d$, let m_w be the equilibrium state of $\langle w,F\rangle$. Suppose that for some $\xi \in \mathbb{R}^d$, $\Phi_{m_\xi}(H^1(M,\mathbb{R})) = \{0\}$.

We begin by showing that the the functions $F_1,\ldots,F_d$ satisfy (A_1). Suppose that for $a = (a_1,\ldots,a_d) \in \mathbb{R}^d$, $\langle a,F\rangle$ is cohomologous to some constant, A say. Then

$$A = \int \langle a,F\rangle \, dm_\xi = a_1\Phi_{m_\xi}(\rho_1) + \ldots + a_d\Phi_{m_\xi}(\rho_d) = 0.$$

In particular this means that for every closed orbit τ, $\langle a,[\tau]\rangle = 0$. However, as we remarked in the last section, Parry and Pollicott have shown ([18]) that the set $\{[\![\tau]\!] : \tau$ a closed φ-orbit$\}$ generates $H_1(M,\mathbb{Z})$ and hence the set $\{[\tau] : \tau$ a closed φ-orbit$\}$ generates $\mathbb{Z}^d$. Hence $a = 0$ and (A_1) holds.

Since (A_1) is satisfied, $\beta(w) = P(\langle w,F\rangle)$ is strictly convex. We also have *without further assumption* that $\beta(w)$ has a minimum at $w = \xi$. This is because

$$\nabla\beta(\xi) = \left(\int F_1\, dm_\xi, \ldots, \int F_d\, dm_\xi\right) = \left(\Phi_{m_\xi}(\rho_1), \ldots, \Phi_{m_\xi}(\rho_d)\right) = 0.$$

(Note that since β is strictly convex this argument shows that ξ is the unique element of $\mathbb{R}^d$ such that $\Phi_{m_\xi}(H^1(M,\mathbb{R})) = \{0\}$.) We also have, again since (A_1) is satisfied, that $\nabla^2\beta(\xi)$ is positive definite.

We now define a function η^1 on $\mathbb{C} \times \widehat{H_1(M,\mathbb{Z})}$. Note first that every element of $\widehat{H_1(M,\mathbb{Z})}$, when evaluated at $[[\tau]]$ say, takes the form $e^{2\pi i<\lambda,f\tau>}\ \chi(\gamma(\tau))$, for some $t \in \mathbb{R}^d/\mathbb{Z}^d$, $\chi \in \widehat{G}$. For such t and χ define

$$\eta^1(s,t,\chi) \;=\; \sum_\tau\ (-\lambda(\tau))^{\nu+1}\ \chi(\gamma(\tau))\ e^{-s\lambda(\tau) + <\xi+2\pi i t,f\tau>}$$

(where this converges).

We already know how $\eta^1(s,t,\chi_0)$ behaves in a neighbourhood of $\{s : \mathrm{Re}\, s \geq \alpha\} \times \{0\}$ (where χ_0 is the trivial character in $\widehat{G}$). Also, by comparison with $\eta^1(s,0,\chi_0)$, $\eta^1(s,t,\chi)$ is analytic provided $\mathrm{Re}\, s > \alpha = \beta(\xi)$. We now show that, apart from $(t,\chi) = (0,\chi_0)$, $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $\mathrm{Re}\, s \geq \alpha$. As usual, we shall lift the problem to a suspended flow $\sigma^r : \Sigma^r \to \Sigma^r$ (over a shift of finite type $\sigma : \Sigma_A \to \Sigma_A$) which is semi-conjugated to φ by a map $p : \Sigma^r \to \Lambda$. If the closed σ^r-orbit passing through $x \in \mathrm{Fix}_n$ lies over the closed φ-orbit τ then, as in Section 6, there is a function $\kappa : \Sigma_A \to G$ such that $\gamma(\tau) = \kappa_n(x)$ and a function $\theta : \Sigma_A \to \mathbb{R}$ such that $\chi(\gamma(\tau)) = e^{2\pi i\theta^n(x)}$.

Using the results of Section 3 to show that $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $\mathrm{Re}\, s \geq \alpha$ it is enough to show that $L_\tau(s,t,\chi)$ is analytic and non-zero in a neighbourhood of $\mathrm{Re}\, s \geq \alpha$ where, in the notation of Section 1,

$$L_\tau(s,t,\chi) \;=\; \zeta(-sr + <\xi+2\pi i t,f> + 2\pi i\theta)$$

(here $f = (f_1,\ldots,f_d)$ and f_i is related to $F_i \circ p$ by (2.1), $i = 1,\ldots,d$). By Proposition 1, $L_\tau(s,t,\chi)$ is analytic and non-zero in a neighbourhood of (s,t,χ) with $\mathrm{Re}\, s = \alpha$ unless

$$-(\mathrm{Im}\, s)r + 2\pi<t,f> + 2\pi\theta$$

is cohomologous to a function valued in $2\pi\mathbb{Z}$. If $-(\mathrm{Im}\, s)r + 2\pi<t,f> + 2\pi\theta$ is cohomologous to such a function then for every $n \geq 1$ and every $x \in \mathrm{Fix}_n$ we have

$$\exp i(-(\mathrm{Im}\ s)r^n(x) + 2\pi\langle t, f^n(x)\rangle + 2\pi\theta^n(x)) = 1.$$

In terms of φ the condition becomes for every closed φ-orbit τ

$$e^{2\pi i\langle t, f\tau\rangle}\chi(\gamma(\tau)) = e^{i(\mathrm{Im}\ s)\lambda(\tau)} \tag{7.1}$$

which implies that for every τ

$$e^{2\pi i|G|\langle t, f\tau\rangle} = e^{2\pi i|G|(\mathrm{Im}\ s)\lambda(\tau)}.$$

We now apply the following lemma stated in [9].

Lemma 3. If H is a real-valued C^∞ function on M such that, for every closed φ-orbit τ, $\exp 2\pi i\lambda_H(\tau) = 1$ then there exists $u \in C^1(M)$ such that $|u| = 1$ and, for all $x \in M$ and all $\varepsilon > 0$,

$$u(\varphi_\varepsilon x) = u(x)\exp 2\pi i\int_0^\varepsilon H(\varphi_\ell x)\,d\ell.$$

This lemma is a multiplicative version of a result of Livsic [13] (cf. also Guillemin and Kazhdan, [8]). In our case we take $H = |G|\langle t, F\rangle - |G|(\mathrm{Im}\ s)$. From the lemma we obtain

$$\frac{1}{2\pi i}\frac{u'(x)}{u(x)} = |G|\langle t, F\rangle - |G|(\mathrm{Im}\ s) \tag{7.2}$$

where u' denotes the derivative of u with respect to φ, i.e.

$$u'(x) = \lim_{\varepsilon\to 0}\frac{u(\varphi_\varepsilon x) - u(x)}{\varepsilon}.$$

Locally, we can write u in the form $e^{2\pi i q}$, for some $q \in C^1(M)$ so that

$$\frac{1}{2\pi i}\frac{u'}{u} = q'.$$

We can associate a closed 1-form ω with u by defining (locally) $\omega = dq$. Then $\langle\omega, Z\rangle = \langle dq, Z\rangle = q'$ so we obtain (for all $x \in M$)

$$\frac{1}{2\pi i}\frac{u'(x)}{u(x)} = \langle\omega, Z\rangle(x).$$

Substituting this into (7.2) and integrating with respect to m_ξ yields

$$\Phi_{m_\xi}(\omega) = |G| \int \langle t,F\rangle \, dm_\xi - |G| (\operatorname{Im} s)$$

$$= |G| (t_1 \Phi_{m_\xi}(\rho_1) + \dots + t_d \Phi_{m_\xi}(\rho_d)) - |G| (\operatorname{Im} s).$$

Since Φ_{m_ξ} vanishes identically on the space of closed 1-forms, this shows that $\operatorname{Im} s = 0$. Substituting into (7.1) gives that for every closed φ-orbit τ

$$e^{2\pi i \langle t,\tau\rangle} \chi(\gamma(\tau)) = 1.$$

However, noting again that as τ runs over the set of closed φ-orbits, $[[\tau]]$ generates $H_1(M,\mathbb{Z})$, we must have that (t,χ) is the trivial character in $H_1(M,\mathbb{Z})$, i.e. $t = 0$ and $\chi = \chi_0$.

We have now shown that $\eta^1(s,t,\chi)$ is analytic in a neighbourhood of $(\{ s : \operatorname{Re} s > \alpha \} \times \mathbb{R}^d/\mathbb{Z}^d \times \hat{G}) - \{(\alpha,0,\chi_0)\}$. Combining this with our knowledge of $\eta^1(s,t,\chi_0)$ in a neighbourhood of $\{s : \operatorname{Re} s \geq \alpha\} \times \{0\}$ and applying the analysis of Section 6 and the first half of this section we obtain

Theorem 4. Let φ be a C^{∞}, transitve Anosov flow on a manifold M. Suppose that for some (necessarily unique) $\xi \in \mathbb{R}^d$, $\Phi_{m_\xi}(H^1(M,\mathbb{R})) = \{0\}$ and write $\alpha = \beta(\xi)$. Then, for each fixed homology class $(n,\gamma) \in \mathbb{Z}^d \oplus G$,

$$\#\{ \tau : \lambda(\tau) \leq x, \ [[\tau]] = (n,\gamma) \} \sim \frac{e^{-\langle \xi,n\rangle}}{(2\pi)^{\frac{1}{2}d} |G| (\det \nabla^2 \beta(\xi))^{\frac{1}{2}}} \frac{e^{\alpha x}}{x^{\frac{1}{2}d+1}}.$$

Remarks.(i) The results of Katsuda and Sunada [9] cover the case where $\xi = 0$. (In particular, this covers the case of geodesic flows on compact manifolds of negative curvature.) They also show that if $\Phi_{m_0}(H^1(M,\mathbb{R})) \neq \{0\}$ then for every $N \geq 1$

$$\frac{\#\{\tau : \lambda(\tau) \leq x, \ [[\tau]] = (n,\gamma)\}}{e^{hx}} = o(x^{-N}).$$

(ii) In the case of compact negatively curved surfaces S of genus g (so $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^{2g}$), Lalley [11] has asymptotics for closed geodesics in a homology class varying linearly with their length. Let $\rho_1,\dots,\rho_{2g}$ be closed 1-forms on S such that for each closed

geodesic τ

$$[\tau] = \left(\int_\tau \rho_1, \cdots, \int_\tau \rho_{2g}\right)$$

and define functions $F_1,\ldots,F_{2g}$ on the unit-tangent bundle T_1S by $F_i(x,v) = \langle\rho_i(x),v\rangle$, $i = 1,\ldots,2g$. Lalley shows that if for each $x > 0$ one assigns a homology class $m(x) \in \mathbb{Z}^{2g}$ in such a way that $m(x)/x \to a$ as $x \to \infty$ then, for a in a neighbourhood of $0 \in \mathbb{R}^{2g}$,

$$\#\{\tau : \lambda(\tau) \le x,\ [\tau] = m(x)\} \sim \frac{(\det \nabla^2\Gamma(a))^{\frac{1}{2}}}{(2\pi)^g\,(\langle\nabla\Gamma(a),a\rangle - \Gamma(a))}\ \frac{e^{-\Gamma(a)x}}{x^{g+1}}$$

where $-\Gamma(a) = \sup\{h_m(\varphi) : m\ \varphi\text{-invariant and} \int F_i\, dm = a_i,\ i = 1,\ldots,2g\}$ and φ is the geodesic flow on T_1S. This result does not, at present, seem to be accessible by the methods used in this paper.

(iii) One should note that if $F_1,\ldots,F_d$ are functions satisfying the hypotheses of Theorem 3 then one does not have an asymptotic formula of the form given in Theorem 4 for an arbitrary finite group extension. For instance, suppose $d = 1$ and let $G = \mathbb{Z}/2\mathbb{Z}$ with $\gamma(\tau) = 0$ if and only if $\lambda_{F_1}(\tau) \in 2\mathbb{Z}$. Then, for example, $\{\tau : [\tau] = 0, \gamma(\tau) = 1\} = \emptyset$.

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Appendix.

In this appendix we reproduce, for completeness, the analysis of Katsuda and Sunada [1] which we used in Section 4 of Chapter 2. We start with a technical lemma.

Lemma. Let a,b and μ be real numbers with both a and μ positive. If m and n are non-negative integers such that $m + n \geq 1$ then

$$\int_{-\mu}^{\mu} \frac{1}{(t^2 + a^2)^m ((t-b)^2 + a^2)^n} \, dt = O\left(a^{-2(m+n)+1}\right)$$

as $a \to 0$.

Proof.

$$\int_{-\mu}^{\mu} \frac{1}{(t^2 + a^2)^m ((t-b)^2 + a^2)^n} \, dt$$

$$= a^{-2(m+n)} \int_{-\mu}^{\mu} \left(\frac{t^2}{a^2} + 1\right)^{-m} \left(\frac{(t-b)^2}{a^2} + 1\right)^{-n} dt$$

$$= a^{-2(m+n)} \, I(a), \qquad \text{say.}$$

Put $x = t/a$, then

$$I(a) = \int_{-\mu/a}^{\mu/a} (x^2 + 1)^{-m} \left(\frac{(ax-b)^2}{a^2} + 1\right)^{-n} a \, dx$$

$$\leq a \max\left(\int_{-\mu/a}^{\mu/a} (x^2 + 1)^{-1} dx, \int_{-\mu/a}^{\mu/a} \left(\frac{(ax-b)^2}{a^2} + 1\right)^{-1} dx \right)$$

and these last integrals are clearly bounded independently of a.

Recall that we wished to analyse the integral

$$\int_U \frac{1 + P(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu+1}} \, d\theta \qquad (1)$$

where $U = \{ \theta : \theta_1^2 + \ldots + \theta_d^2 \le a^2 \}$ for some small $a > 0$, P is a C^∞ function on U such that $P(0) = 0$, Q is a C^∞ odd function on U such that $Q(0) = 0$, $\nabla Q(0) = 0$ and $\nabla^2 Q(0) = 0$ and $\nu = [d/2]$.

We first consider the case when d is even, so $\nu = \frac{1}{2}d$. Changing to spherical polar coordinates the integral (1) becomes

$$\int_{S^{d-1}} \int_0^a \frac{1 + P(r\omega)}{(s - \alpha + r^2 + iQ(r\omega))^{\nu+1}} \, r^{d-1} \, dr \, d\omega.$$

We rewrite the integral as

$$\Omega_{d-1} \int_0^a \frac{r^{d-1}}{(s - \alpha + r^2)^{\nu+1}} \, dr + \int_{S^{d-1}} \int_0^a \frac{R_0(r\omega)}{(s - \alpha + r^2)^{\nu+1}} \, dr \, d\omega$$

$$- \sum_{j=1}^{\nu+1} \int_{S^{d-1}} \int_0^a \frac{R_j(r\omega)}{(s - \alpha + r^2)^j (s - \alpha + r^2 + iQ(r\omega))^{\nu+1}} \, dr \, d\omega$$

$$= I(s) + I_0(s) - \sum_{j=1}^{\nu+1} I_j(s),$$

say, where

$$\Omega_{d-1} = \int_{S^{d-1}} d\omega,$$

$$R_0(r\omega) = r^{d-1} P(r\omega)$$

and

$$R_j(r\omega) = \binom{\nu+1}{j} r^{d-1} (1 + P(r\omega)) (iQ(r\omega))^j, \qquad j = 1, \ldots, \nu+1.$$

We have

$$| R_0(r\omega) | \le \text{const. } r^d.$$

$$|R_j(r\omega)| \le \text{const. } r^{d-1+3j}, \qquad j = 1,\ldots,\nu+1$$

(this uses the fact that Q and all its first and second order partial derivatives vanish at 0).

Now

$$I(s) = \frac{\Omega_{d-1}}{(s-\alpha)^{\nu+1}} \int_0^a r^{d-1} \left(1 + \frac{r^2}{s-\alpha}\right)^{-\nu-1} dr.$$

Substituting $x = r/(s-\alpha)^{\frac{1}{2}}$, this becomes

$$I(s) = \frac{\Omega_{d-1}}{s-\alpha} \int_0^{a/(s-\alpha)^{\frac{1}{2}}} \frac{x^{d-1}}{(1+x^2)^{\nu+1}} dx$$

$$= \frac{\Omega_{d-1}}{s-\alpha} \int_0^{b(s)} \sin^{d-1}\varphi \cos\varphi \, d\varphi$$

where $x = \tan\varphi$ and $b(s) = \tan^{-1}(a/(s-\alpha)^{\frac{1}{2}})$. Write

$$B(b) = \int_0^b \sin^{d-1}\varphi \cos\varphi \, d\varphi$$

then, in particular, B(b) is analytic in a neighbourhood of $b = \pi/2$. Thus, using the identity $\tan^{-1}x = \pi/2 - \tan^{-1}(1/x)$,

$$I(s) = \frac{\Omega_{d-1} B(\pi/2)}{s-\alpha} - \frac{\Omega_{d-1} B'(\pi/2)}{s-\alpha} \tan^{-1}\left(\frac{(s-\alpha)^{\frac{1}{2}}}{a}\right)$$

$$+ \text{ higher order terms}$$

$$= \frac{\Omega_{d-1}}{s-\alpha} B(\pi/2) - \frac{\Omega_{d-1} B'(\pi/2)}{(s-\alpha)^{\frac{1}{2}}} + \text{ higher order terms.}$$

Now $B(\pi/2) = 1/d$ and

$$\Omega_{d-1} = \frac{2\pi^{\frac{1}{2}d}}{(\frac{1}{2}d-1)!}$$

so

$$I(s) - \frac{\pi^{\frac{1}{2}d}}{\nu!} \frac{1}{s-\alpha}$$

converges to a locally integral function as $\mathrm{Re}\, s \to \alpha$ and, for $\sigma > \alpha$,

$$\left| I(\sigma+it) - \frac{\pi^{\frac{1}{2}d}}{\nu!} \frac{1}{\sigma+it-\alpha} \right| \le \text{const. } t^{-\frac{1}{2}}$$

giving the required bound by a locally integrable function.

We now show that the remaining terms converge to locally integrable functions as $\mathrm{Re}\, s \to \alpha$. Define

$$h_0(t) = \int_0^a \frac{r^d}{(t^2+r^4)^{(\nu+1)/2}}\, dr,$$

$$h_j(t) = \sup_{\omega \in S^{d-1}} \int_0^a \frac{r^{d-1+3j}}{(t^2+r^4)^{\frac{1}{2}j}\left((t+Q(r\omega))^2+r^4\right)^{\frac{1}{2}(\nu+1)}}\, dr, \qquad j = 1,\ldots,\nu+1.$$

Then, for $\sigma > \alpha$,

$$|I_0(\sigma+it)| \le \text{const. } h_0(t)$$

and

$$|I_j(\sigma+it)| \le \text{const. } h_j(t), \qquad j = 1,\ldots,\nu+1.$$

Let $\mu > 0$, then by the lemma,

$$\int_{-\mu}^{\mu} \frac{r^d}{(t^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dt = O(1),$$

$$\int_{-\mu}^{\mu} \frac{r^{d-1+3j}}{(t^2+r^4)^{\frac{1}{2}j}\left((t+Q(r\omega))^2+r^4\right)^{\frac{1}{2}(\nu+1)}}\, dt = O(r^{j-1}), \qquad j = 1,\ldots,\nu+1$$

(as $r \to 0$). Hence

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^d}{(t^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dt\, dr < \infty,$$

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^{d-1+3j}}{(t^2+r^4)^{\frac{1}{2}j}\left((t+Q(r\omega))^2+r^4\right)^{\frac{1}{2}(\nu+1)}}\, dt\, dr < \infty, \qquad j = 1,\ldots,\nu+1$$

(this last holding for every $\omega \in S^{d-1}$). Using Fubini's theorem and the fact that S^{d-1} is compact, this gives

$$\int_{-\mu}^{\mu} h_0(t)\ dt < \infty,$$

$$\int_{-\mu}^{\mu} h_j(t)\ dt < \infty, \qquad j = 1,\dots,\nu+1.$$

Hence $h_0, h_1, \dots, h_{\nu+1}$ are locally integrable functions which dominate $I_0, I_1, \dots, I_{\nu+1}$ respectively, as required.

We now turn to the case where d is odd, so that $\nu = (d-1)/2$. Rewrite (1) in the form

$$J_1(s) + J_2(s) + J_3(s) + J_4(s)$$

where

$$J_1(s) = \int_U \frac{1}{(s - \alpha + \theta_1^2 + \dots + \theta_d^2)^{\nu+1}}\, d\theta,$$

$$J_2(s) = \int_U \frac{\langle \nabla P(0), \theta \rangle}{(s - \alpha + \theta_1^2 + \dots + \theta_d^2)^{\nu+1}}\, d\theta,$$

$$J_3(s) = \int_U \frac{P(\theta) - \langle \nabla P(0), \theta \rangle}{(s - \alpha + \theta_1^2 + \dots + \theta_d^2)^{\nu+1}}\, d\theta,$$

$$J_4(s) = \int_U \left\{ \frac{1 + P(\theta)}{(s-\alpha+\theta_1^2+\dots+\theta_d^2+iQ(\theta))^{\nu+1}} - \frac{1 + P(\theta)}{(s-\alpha+\theta_1^2+\dots+\theta_d^2)^{\nu+1}} \right\} d\theta.$$

After changing to spherical polar coordinates,

$$J_1(s) = \Omega_{d-1} \int_0^a \frac{r^{d-1}}{(s - \alpha + r^2)^{\nu+1}}\, dr$$

$$= \frac{\Omega_{d-1}}{(s - \alpha)^{\nu+1}} \int_0^a r^{d-1} \left(1 + \frac{r^2}{s - \alpha}\right)^{-\nu-1} dr.$$

Substituting $x = r/(s-\alpha)^{\frac{1}{2}}$, we have

$$J_1(s) = \frac{\Omega_{d-1}}{(s-\alpha)^{\frac{1}{2}}} \int_0^{a/(s-\alpha)^{\frac{1}{2}}} \frac{x^{d-1}}{(1+x^2)^{\nu+1}}\, dx$$

$$= \frac{\Omega_{d-1}}{(s-\alpha)^{\frac{1}{2}}} \int_0^{b(s)} \sin^{d-1}\varphi \, d\varphi$$

where $x = \tan\varphi$ and $b(s) = \tan^{-1}(a/(s-\alpha)^{\frac{1}{2}})$. Write

$$D(b) = \int_0^{b} \sin^{d-1}\varphi \, d\varphi$$

then, in particular, $D(b)$ is analytic in a neighbourhood of $b = \pi/2$. Thus

$$J_1(s) = \frac{\Omega_{d-1}\, D(\pi/2)}{(s-\alpha)^{\frac{1}{2}}} + \frac{\Omega_{d-1}\, D'(\pi/2)}{(s-\alpha)^{\frac{1}{2}}} \tan^{-1}\left(\frac{(s-\alpha)^{\frac{1}{2}}}{a}\right)$$

$$+ \text{ higher order terms}$$

$$= \frac{\Omega_{d-1}\, D(\pi/2)}{(s-\alpha)^{\frac{1}{2}}} + \Omega_{d-1}\, D'(\pi/2) + \text{ higher order terms.}$$

Now

$$D(\pi/2) = \frac{\pi}{2}\, \frac{(2\nu)!}{(\nu!)^2\, 4^\nu}$$

and

$$\Omega_{d-1} = \frac{2\,\pi^{\nu+\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})} = \frac{2\,\pi^\nu\, \nu!\, 4^\nu}{(2\nu)!}$$

so

$$J_1(s) - \frac{\pi^{\frac{1}{2}d+\frac{1}{2}}}{\nu!}\, \frac{1}{(s-\alpha)^{\frac{1}{2}}}$$

converges to a locally integrable function with locally integrable first derivative as $\mathrm{Re}\, s \to \alpha$ and, for $\sigma > \alpha$,

$$\left| J_1(\sigma+it) - \frac{\pi^{\frac{1}{2}d+\frac{1}{2}}}{\nu!}\, \frac{1}{(\sigma+it-\alpha)^{\frac{1}{2}}} \right| \leq \text{ const.}$$

giving the required bound by an element of $W^{1,1}_{loc}$.

Now consider $J_2(s)$. The integral is taken over a ball centred at $0 \in \mathbb{R}^d$ and the integrand is an odd function so $J_2(s) = 0$.

We now consider $J_3(s)$. On changing to polar coordinates and noting that $P(\theta) - \langle \nabla P(0), \theta \rangle = O(r^2)$, we have, for $\sigma > \alpha$,

$$|J_3(\sigma+it)| \leq \text{const.} \int_0^a \frac{r^{d+1}}{\left((\sigma - \alpha + r^2)^2 + t^2\right)^{\frac{1}{2}(\nu+1)}}\, dr$$

$$\leq \text{const.} \int_0^a r^{d+1-2(\nu+1)}\, dr = \text{const. } a$$

so $J_3(\sigma+it)$ is dominated by a locally integrable function and

$$\lim_{\sigma \to \alpha} J_3(\sigma+it)$$

exists and is locally integrable. Also

$$\left| \frac{d}{dt} J_3(\sigma+it) \right| \leq \text{const.} \int_0^a \frac{r^{d+1}}{\left((\sigma - \alpha + r^2)^2 + t^2\right)^{\frac{1}{2}(\nu+2)}}\, dr$$

$$\leq \text{const.} \int_0^a \frac{r^{d+1}}{(r^4 + t^2)^{\frac{1}{2}(\nu+2)}}\, dr.$$

By the lemma, for $\mu > 0$,

$$\int_{-\mu}^{\mu} \frac{r^{d+1}}{(r^4 + t^2)^{\frac{1}{2}(\nu+2)}}\, dt = O(1),$$

so

$$\int_0^a \int_{-\mu}^{\mu} \frac{r^{d+1}}{(r^4 + t^2)^{\frac{1}{2}(\nu+2)}}\, dt\, dr < \infty.$$

Using Fubini's theorem, this implies that

$$\int_{-\mu}^{\mu} \int_0^a \frac{r^{d+1}}{(r^4 + t^2)^{\frac{1}{2}(\nu+2)}}\, dr\, dt < \infty,$$

so that

$$\lim_{\sigma \to \alpha} \frac{d}{dt} J_3(\sigma+it)$$

exists and is locally integrable. Since (in the sense of distributions)

$$\frac{d}{dt} \lim_{\sigma \to \alpha} J_3(\sigma+it) = \lim_{\sigma \to \alpha} \frac{d}{dt} J_3(\sigma+it),$$

we can conclude that as $\sigma \to \alpha$, $J_3(\sigma+it)$ converges to an element of $W^{1,1}_{loc}(\mathbb{R})$.

Finally we deal with $J_4(s)$. Write

$$J_4(s) = \sum_{j=0}^{\nu+1} K_j(s)$$

where

$$K_0(s) = \int_U \frac{T_0(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2)\,(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu+1}}\, d\theta,$$

$$K_j(s) = \int_U \frac{T_j(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2)^j\,(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu-1}}\, d\theta$$

$j = 1,\ldots,\nu+1$, and

$$T_0(\theta) = (\nu+1)\, i\, Q_3(\theta),$$

$$T_1(\theta) = (\nu+1)\, i\, (P(\theta)\, Q(\theta) + Q(\theta) - Q_3(\theta)),$$

$$T_j(\theta) = \binom{\nu+1}{j} (1 + P(\theta))\, (i\, Q(\theta))^j, \qquad j = 2,\ldots,\nu+1,$$

with $Q_3(\theta)$ denoting the terms of degree three in the Taylor expansion of $Q(\theta)$ around $\theta = 0$. We have

$$T_0(\theta) = O(r^3),$$

$$T_1(\theta) = O(r^4),$$

$$T_j(\theta) = O(r^{3j}), \qquad j = 2,\ldots,\nu+1.$$

Also write

$$K_0(s) = \sum_{j=0}^{\nu+1} L_j(s),$$

where

$$L_0(s) = \int_U \frac{T_0(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2)^{\nu+2}}\, d\theta,$$

$$L_1(s) = \int_U \frac{T_0(\theta)\,(T_0(\theta) + T_1(\theta))}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2)^2\,(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu+1}}\, d\theta,$$

$$L_j(s) = \int_U \frac{T_0(\theta)\,T_j(\theta)}{(s - \alpha + \theta_1^2 + \ldots + \theta_d^2)^2\,(s - \alpha + \theta_1^2 + \ldots + \theta_d^2 + iQ(\theta))^{\nu+1}}\, d\theta,$$

$j = 2,\ldots,\nu+1$.

Clearly $L_0(s) = 0$ (since the integral is over a ball centred at 0 and the integrand is an odd function). Also, after changing to polar coordinates, for $\sigma > \alpha$,

$$|K_1(\sigma+it)| \le \text{const.} \sup_{\omega\in S^{d-1}} \int_0^a \frac{r^{d+3}}{(t^2+r^4)^{\frac{1}{2}}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dr,$$

$$|K_j(\sigma+it)| \le \text{const.} \sup_{\omega\in S^{d-1}} \int_0^a \frac{r^{d-1+3j}}{(t^2+r^4)^{\frac{1}{2}j}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dr,$$

$j = 2,\ldots,\nu+1$,

$$|L_1(\sigma+it)| \le \text{const.} \sup_{\omega\in S^{d-1}} \int_0^a \frac{r^{d+6}}{(t^2+r^4)\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dr,$$

$$|L_j(\sigma+it)| \le \text{const.} \sup_{\omega\in S^{d-1}} \int_0^a \frac{r^{d+2+3j}}{(t^2+r^4)^{\frac{1}{2}(j+1)}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}}\, dr,$$

$j = 2,\ldots,\nu+1$.

We shall call the above integrands (suppressing their dependence on ω) $p_1(t,r)$, $p_j(t,r)$, $p_1'(t,r)$ and $p_j'(t,r)$ respectively. By the lemma we have, for $\mu > 0$,

$$\int_{-\mu}^{\mu} p_1(t,r)\ dt = O(r^2),$$

$$\int_{-\mu}^{\mu} p_j(t,r)\ dt = O(r^j), \qquad j = 2,\ldots,\nu+1,$$

$$\int_{-\mu}^{\mu} p_1'(t,r)\ dt = O(r^3),$$

$$\int_{-\mu}^{\mu} p_j'(t,r)\ dt = O(r^{j+1}), \qquad j = 2,\ldots,\nu+1$$

and so

$$\int_0^a \int_{-\mu}^{\mu} p_1(t,r)\ dt\, dr < \infty,$$

$$\int_0^a \int_{-\mu}^{\mu} p_j(t,r)\ dt\, dr < \infty, \qquad j = 2,\ldots,\nu+1,$$

$$\int_0^a \int_{-\mu}^{\mu} p_0'(t,r)\ dt\, dr < \infty,$$

$$\int_0^a \int_{-\mu}^{\mu} p_j'(t,r)\ dt\, dr < \infty, \qquad j = 2,\ldots,\nu+1.$$

Hence, using Fubini's theorem, each of $K_1(\sigma+it)$, $K_j(\sigma+it)$ $(j = 2,\ldots,\nu+1)$, $L_1(\sigma+it)$, $L_j(\sigma+it)$ $(j = 2,\ldots,\nu+1)$ converge to locally integrable functions as $\sigma \to \alpha$. Also

$$\left| \frac{d}{dt} K_j(\sigma+it) \right| \le \text{const.} \sup_{\omega \in S^{d-1}} \int_0^a q_j(t,r,\omega)\ dr,$$

$j = 1,\ldots,\nu+1$, and

$$\left| \frac{d}{dt} L_j(\sigma+it) \right| \le \text{const.} \sup_{\omega \in S^{d-1}} \int_0^a q_j'(t,r,\omega)\, dr,$$

$j = 1,\ldots,\nu+1$, where

$$q_1(t,r,\omega) = \frac{r^{d+3}}{(t^2+r^4)^{\frac{1}{2}}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}} \times \left\{ \frac{1}{(t^2+r^4)^{\frac{1}{2}}} + \frac{1}{((t+Q(r\omega))^2+r^4)^{\frac{1}{2}}} \right\},$$

$$q_j(t,r,\omega) = \frac{r^{d-1+3j}}{(t^2+r^4)^{\frac{1}{2}j}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}} \times \left\{ \frac{1}{(t^2+r^4)^{\frac{1}{2}}} + \frac{1}{((t+Q(r\omega))^2+r^4)^{\frac{1}{2}}} \right\},$$

$j = 2,\ldots,\nu+1$,

$$q_1'(t,r,\omega) = \frac{r^{d+6}}{(t^2+r^4)\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}} \times \left\{ \frac{1}{(t^2+r^4)^{\frac{1}{2}}} + \frac{1}{((t+Q(r\omega))^2+r^4)^{\frac{1}{2}}} \right\},$$

$$q_j'(t,r,\omega) = \frac{r^{d+2+3j}}{(t^2+r^4)^{\frac{1}{2}(j+1)}\,((t+Q(r\omega))^2+r^4)^{\frac{1}{2}(\nu+1)}} \times \left\{ \frac{1}{(t^2+r^4)^{\frac{1}{2}}} + \frac{1}{((t+Q(r\omega))^2+r^4)^{\frac{1}{2}}} \right\},$$

$j = 2,\ldots,\nu+1$.

By the lemma, for $\mu > 0$,

$$\int_{-\mu}^{\mu} q_1(t,r,\omega)\, dt = O(1),$$

$$\int_{-\mu}^{\mu} q_j(t,r,\omega)\ dt = O(r^{j-2}), \qquad j = 2,\ldots,\nu+1,$$

$$\int_{-\mu}^{\mu} q_1'(t,r,\omega)\ dt = O(r),$$

$$\int_{-\mu}^{\mu} q_j'(t,r,\omega)\ dt = O(r^{j-1}), \qquad j = 2,\ldots,\nu+1$$

and so

$$\int_0^a \int_{-\mu}^{\mu} q_j(t,r,\omega)\ dt\, dr < \infty, \qquad j = 1,\ldots,\nu+1,$$

$$\int_0^a \int_{-\mu}^{\mu} q_j'(t,r,\omega)\ dt\, dr < \infty, \qquad j = 1,\ldots,\nu+1.$$

Using Fubini's theorem, this gives that each of

$$\lim_{\sigma \to \alpha} \frac{d}{dt} K_j(\sigma+it), \qquad j = 1,\ldots,\nu+1$$

and

$$\lim_{\sigma \to \alpha} \frac{d}{dt} L_j(\sigma+it), \qquad j = 1,\ldots,\nu+1$$

exist and are locally integrable. However, in the sense of distributions,

$$\frac{d}{dt} \lim_{\sigma \to \alpha} K_j(\sigma+it) = \lim_{\sigma \to \alpha} \frac{d}{dt} K_j(\sigma+it), \qquad j = 1,\ldots,\nu+1$$

and

$$\frac{d}{dt} \lim_{\sigma \to \alpha} L_j(\sigma+it) = \lim_{\sigma \to \alpha} \frac{d}{dt} L_j(\sigma+it), \qquad j = 1,\ldots,\nu+1$$

so we have that, as $\sigma \to \alpha$, $K_j(\sigma+it)$, $j = 1,\ldots,\nu+1$, and $L_j(\sigma+it)$, $j = 1,\ldots,\nu+1$, converge to elements of $W^{1,1}_{loc}$.

Combining all the above results, we find we have proved Proposition of Chapter 2.

Reference.

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