Contributions to the Theory of Subnormal Subgroups and Factorized Groups

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Declaration

All results in Chapters 2,3 and 4 of this thesis are original unless otherwise explicitly stated.

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§1 Introduction

The purpose of this thesis is to investigate some problems concerning products of groups, that is given a group G with subgroups H and K we investigate what effect the structures of H and K have on the the structure of the product HK. In chapters 2 and 3 we consider the case where the product is itself a group and in chapter 4 we consider the case where the product need not be a group.

In this introductory chapter I would like to give a brief description of some of the results already proved in this area concerning groups which are factorised as a product of two of their subgroups.

Solubility in Factorised Groups

A question which has received much attention over the last few years is the following: given a group G=HK which is factorised as a product of subgroups H and K,what conditions on the structure of H and K are sufficient to ensure that G be soluble?

In 1955 Itô proved the following:-

Theorem (Itô[1]) Any group which can be expressed as a product of abelian subgroups is metabelian.

The proof of this theorem consists of a quite (elegantly) short commutator argument. The theorem is particularly remarkable as it imposes no conditions (such as finiteness) on the whole group.

A few years later, in 1961, it was proved in Kegel that:-

Theorem (Kegel[1] (and Wielandt[1])) Any finite group which can be expressed as a product of nilpotent subgroups is soluble.

Since that time some further generalizations of the above have been made, for example it is shown in Kazarin[1] that:-

Theorem (Kazarin[1]) Any finite group which can be expressed as a product of two subgroups each containing a nilpotent subgroup of index at most two is soluble.

Kazarin also observes that the factorisation of $S(5)$ as $S(5)=S(4)\langle(12345)\rangle$ shows that the above would not be true if 'index at most 2' was replaced by 'index at most 3'.

An interesting question arising from the above work is: given a finite group $G=HK$ factorised as a product of nilpotent subgroups H and K, what effect do the the nilpotent classes of H and K have on the derived length of G? The following has been conjectured:-

Conjecture(*) Suppose $G=HK$ is a (finite) group which can be expressed as the product of two subgroups H and K where H is nilpotent of class c and K is nilpotent of class d. Then is the derived length of G at most $c+d$?

Of course Itô's Theorem shows that this conjecture is true in the case where $c=d=1$ (even if G is infinite) and a result in Hall and Higman[1] shows that the conjecture is true if the subgroups H and K have coprime orders. In general it is not known if the derived length of G can be bounded at all in terms of c and d.

A 1973 paper by Pennington shows the following two results:-

Theorem (Pennington[1]) Suppose $G=HK$ is a finite group with subgroups H and K, where H is nilpotent of class c and K is nilpotent of class d. If π is the set of primes dividing both the order of H and the order of K, then the $(c+d)^{th}$ term of the derived series of G is a nilpotent π-group.

Theorem (Pennington[1])Suppose $G=HK$ is a finite group which can be expressed as a product of nilpotent subgroups H and K. Then for any prime p the maximal normal p-subgroup of G, $O_p(G)$, can be expressed as the product

$$O_p(G)=(O_p(G)\cap H)(O_p(G)\cap K).$$

We see that, combining these two results, in the situation of the first theorem we must have that:-

$$d(G) \leq c+d+\max_{p\in\pi}\{d((O_p(G)\cap H)(O_p(G)\cap K))\}$$
($d(G)$:=derived length of G),

and so the problem of finding a bound (though not the bound of the conjecture) is reduced to the case where G is a p-group.

Some progress has been made in bounding the derived length of a group that is the the product of an abelian subgroup and a nilpotent subgroup in terms of the order of the derived subgroup of the nilpotent subgroup. The best result of this kind is the following proved by Zaitsev:-

Theorem (Zaitsev[1]) Let $G=HK$ be a group with H abelian and K

nilpotent with finite derived subgroup K′ of order $p_1^{a_1}...p_n^{a_n}$ where $p_1,...,p_n$ are distinct primes. Then G is soluble with derived length at most $2+3(a_1+...+a_n)$.

Subnormality In Factorized Groups

Given a group G=HK which can be expressed as a product of subgroups H and K and which contains another subgroup X an interesting question to ask is: what 'subnormality relations' between the subgroups H and X and the subgroups K and X will suffice to enable us to deduce that X is subnormal in G ? The most striking result so far concerning this question is the following theorem proved by Wielandt (first proved by Maier for soluble subgroups):-

Theorem (Wielandt; Theorem 7.7 in Lennox and Stonehewer[1]) Suppose G=HK is a finite group that can be expressed as a product of subgroups H and K and suppose X is a subgroup of $H \cap K$ and that X is subnormal in both H and K. Then X is subnormal in G.

This theorem has also been extended to certain classes of infinite groups (e.g. periodic nilpotent-by-abelian-by-finite groups and soluble minimax groups) by Stonehewer (see Stonehewer[1]).

In the above theorem we require that the subgroup X be contained in both of the subgroups H and K. It is natural to try and generalize this result by taking away the requirement that X be contained in $H \cap K$ and imposing some new subnormality condition such as assuming that X is subnormal in $\langle X, X^l \rangle$ for every element l in $H \cup K$. To this end Maier and Sidki have proved the following result:-

Theorem (Maier and Sidki[1]) Let $G=HK$ be a finite soluble group with H and K subgroups and suppose that X is a p-subgroup of G such that X is subnormal in $\langle X,X^l\rangle$ for all l in $H\bigcup K$. Then X is subnormal in G.

In a similar vein Wielandt has proved the following:-

Theorem (Wielandt; Theorem 7.7.3 in Lennox and Stonehewer[1]) Suppose X is a subgroup of the finite group $G=HK$ where H and K are subgroups and suppose that $XX^a=X^aX$ is a group for every $a\in H\bigcup K$.

(a) If X is subnormal in XX^a for every $a\in H\bigcup K$, then X is subnormal in G.

(b) If X is nilpotent, then X is subnormal in G.

The Exponent of a Factorized Group

Another question concerning groups which can be expressed as products of two of their subgroups is what effect do the exponents of the 'factor' subgroups have upon the exponent of the whole group. The following result was proved by Howlett, refining work begun in Holt and Howlett[1].

Theorem (Howlett[1]) Suppose $G=AB$ is a group (finite or infinite) that can be expressed as the product of two abelian subgroups A and B where A has exponent e and B has exponent f. Then the exponent of G divides ef.

Sysak and Suchkov have constructed very clever examples which show that two natural conjectures that could be made are in fact untrue.

Theorem (Suchkov[1]) There exists a group G=HK which can be expressed as a product of two periodic subgroups H and K but which contains an element of infinite order.

Theorem (Sysak[1]) There exists a group G=HK which can be expressed as a product of (infinite) p-subgroups H and K (i.e. every element of H or K has order a power of p) but which contains an element of order q where q is a prime not equal to p.

Sysak also gives some conditions under which the exponent of the factorized group is 'better behaved' than in the above example.

Theorem (Sysak[1]) Let G=HK be a group with an ascending normal series with (locally nilpotent)-by-finite factors. Suppose that H and K are π-groups for some set of primes π and that either (i) H or K has an ascending normal series with almost abelian factors or (ii) $\pi(H)\cap\pi(K) = \emptyset$ (where $\pi(H)=\{p \mid p$ is prime and H has an element of order p$\}$. Then G is a π-group.

Rank, the Maximal and Minimal Conditions and Factorised Groups

A basic question about a group which can be expressed as a product of two of its subgroups is: if the two 'factor subgroups' have a certain property (such as having finite rank or satisfying the minimal /maximal condition) does the whole group also have this property. I would like in this section to give a brief list of some of the results which have been achieved so far.

Theorem (Lennox and Roseblade[1]) Suppose G=HK is a soluble group where H and K are polycyclic subgroups. Then G is polycyclic.

For the purposes of the next two results we will say that a group G has rank r if r is the smallest positive integer such that every finitely generated subgroup of G can be generated by r elements.

Theorem (Robinson[1]) Suppose G=HK is a soluble group, where H and K are subgroups of G of finite rank, at least one of which is nilpotent. Then G has finite rank.

Theorem (Chernikov[1]) Suppose G=HK is a locally finite group where H and K are subgroups of G of finite rank. Then G has finite rank.

Theorem (Chernikov[2]) Suppose G=HK is a locally stepped group where H and K are extremal subgroups. Then G is extremal. (A group is said to be locally stepped if each finitely generated subgroup contains a proper subgroup of finite index. A group is said to be extremal if it is an abelian-by-finite group satisfying the minimal condition.)

Factorizations of Simple Groups

Recently several papers have been published which use the classification of finite simple groups to study products of subgroups. One such paper, Arad and Fisman[1] gives a complete list of all the possible expressions of simple groups as products of subgroups of coprime order. There are remarkably few of them, in fact the only simple groups which can be expressed in this way are

(1) A_p for all primes $p \geq 5$ (this is the 'obvious' factorization where one of the factor subgroups is isomorphic to A_{p-1}).

(2) M_{11} and M_{23} (where one of the factor subgroups is isomorphic to M_{10} in the first case and M_{22} in the second).

(3) Certain projective special linear groups.

Also using this and the classification Arad an Fisman have given an affirmative answer to Szép's Conjecture.

Theorem (Arad and Fisman[2]) Suppose $G=HK$ is a finite group with H and K subgroups of G each having a non-trivial centre. Then G is not simple.

Finally there is the work of Liebeck, Praeger and Saxl who have classified the maximal factorizations of the simple groups. A factorization $G=HK$ by subgroups H and K is said to be maximal if both of H and K are maximal subgroups of G.

Theorem (Liebeck, Praeger and Saxl[1]) Let L be a finite simple group and G a group such that $L \leq G \leq \mathrm{Aut}L$. Suppose that $G=HK$ for some subgroups H and K, then the triple $(G;H,K)$ is explicitly known.

The Main Results of this Thesis

In Chapter 2 we study subnormality in groups that can be expressed as a product of their subgroups. The main results are the following:-

Theorem 2.3.5 Let $G=HK$ be a finite soluble group with H and K subgroups and suppose X is a subgroup of G for which there exists a positive integer n such that $[l,x_1,\ldots,x_n]\in X$ for all $l\in H\bigcup K$ and $x_1,\ldots,x_n\in X$. Then X is subnormal in G.

Theorem 2.4.2 Suppose $G=HK$ is a finite group with H and K subgroups. Let X be a subgroup of G such that X has a Sylow tower, X is subnormal in H and $X^kX=XX^k$ for every $k\in K$. Then X is subnormal in G.

In chapter 3 we investigate Conjecture($*$) (in the section on solubility above) and show that certain products of abelian and class two groups are soluble with derived length at most 3. The main results are:-

Theorem Let $G=A(XB)$ be a group with abelian subgroups A,X and B, where X normalizes B and BX is nilpotent of class at most 2. Then under the following conditions G has derived length at most 3:-

(i) G is a finite p-group for some odd prime p, X is cyclic, X normalizes A and AX has nilpotency class at most 2. (Corollary 3.2.2).

(ii) G is a finite 2-group, X has order 2 and A and B are elementary abelian 2-groups. (Corollary 3.4.11).

(iii) A and B are elementary 2-groups, X has order 2 and X normalizes A. (G can be infinite). (Corollary 3.3.2).

In Chapter 4 we investigate products of subgroups which are not themselves groups. We construct a group which shows that the obvious generalization of Wielandt's Theorem fails to be true (Example 4.2.1) and prove some easy results about the exponent a subgroup contained in a product of abelian subgroups.

Notation and Definitions

Let G be a group with subgroups H and K.

For any elements h and g we define the *conjugate* of h by g to be

$$h^g = g^{-1}hg.$$

We define the *commutator* $[h,g]$ to be

$$[h,g] = h^{-1}g^{-1}hg,$$

and we define the repeated commutator $[h,_n g]$ for any non-negative integer n inductively by

$$[h,_0 g]=h \quad \text{and } [h,_n g] = [[h,_{n-1} g],g].$$

Similarly, given any n elements $g_1,\ldots,g_n$ of G we define another element by

$$[g_1,g_2,\dots,g_n] = [[\dots[[g_1,g_2],g_3],\dots],g_n].$$

We can now define certain subgroups of G. We have

$$H^K = \langle h^k \mid \text{for all } h \in H,\ k \in K \rangle,$$

and in particular we call H^G the *normal closure* of H in G; it is the smallest normal subgroup of G containing H. Similarly, we define the subgroup [H,K] by

$$[H,K] = \langle [h,k] \mid \text{for every } h \in H,\ k \in K \rangle,$$

and we define the subgroup $[H,_nK]$ inductively by

$$[H,_1K] = [H,K] \quad \text{and} \quad [H,_nK] = [[H,_{n-1}K],K].$$

Also given any n subgroups $H_1,\dots,H_n$ of G, we define a new subgroup by

$$[H_1,H_2,\dots,H_n] = [[\dots[[H_1,H_2],H_3]\dots],H_n].$$

We define the derived series of G to be the series of subgroups $G^{(1)},G^{(2)},\dots$ inductively as follows :-

$$G^{(1)} = [G,G] \text{ and } G^{(n)} = [G^{(n-1)},G^{(n-1)}].$$

We sometimes write G′ for the *derived subgroup* $G^{(1)}$ and G″ for $G^{(2)}$ and so on. A group is called *soluble* if there exists a positive integer n for which $G^{(n)}=1$, and the smallest such integer is called the *derived length of G*, which we will denote by $d(G)$.

G is said to be nilpotent if there exists a positive integer n such that $[G,_n G] = 1$, and G will be said to be nilpotent of class c if c is the smallest positive integer such that $[G,_c G] = 1$.

H is said to be *subnormal* in G if there exists a series of subgroups $H_1,\ldots,H_n$ such that

$$H \triangleleft H_1 \triangleleft H_2 \triangleleft \ldots\ldots \triangleleft H_n = G$$

and the length of a shortest such series is called the *(subnormal) defect* of H in G.

List of Symbols

$H \leq G$	H is a subgroup of G
$H \triangleleft G$	H is a normal subgroup of G
G/N	the factor group of G by a normal subgroup N
H sn G	H is a subnormal subgroup of G
$H \triangleleft^n G$	H is a subnormal subgroup of G having defect at most n
$X \subset Y$	the set X is a subset of the set Y
$\langle X \rangle$	the subgroup generated by a subset X of G
$\lvert G \rvert$	the order of the group G
HK	the set $\{hk \mid h \in H, k \in K\}$
$\lvert G:H \rvert$	the index of H in G
$Z(G)$	the centre of G
$C_G(g)$	the centralizer of the element g in G
$C_G(H)$	the centralizer of the subgroup H in G

$N_G(H)$	the normalizer of H in G
$F(G)$	the Fitting subgroup of G (i.e. the largest normal subgroup of G that is nilpotent)
$O_\pi(G)$	the maximal normal π-subgroup of G for the set of primes π
$\mathbb{Z}_p$	the field containing p elements, where p is a prime

§2 Subnormalizers and Products

§2.1 Introduction

Given a subgroup H of a group G, the normalizer of H in G is defined to be

$$N_G(H) = \{g \in G \mid H \text{ is normal in } \langle H,g\rangle\},$$

and we know that H is normal in G if and only if $N_G(H) = G$.

In this chapter we look at some possible definitions (suggested by Wielandt) of an analagous set, namely the 'subnormalizer' of H, and investigate the properties of this set especially with regard to factorized groups.

The following theorem of Wielandt (Theorems 7.3.3 and 7.3.11 in Lennox and Stonehewer[1]) suggests several possible definitions of the subnormalizer.

Theorem 2.1.1(Wielandt) Suppose X is a subgroup of the finite group G. Then each of the following conditions is equivalent to X sn G.

(i) X sn $\langle X,g\rangle$ for all g in G;

(ii) X sn $\langle X,X^g\rangle$ for all g in G;

(iii) For all g in G and x in X there exists a positive integer n such that $[g,_n x] \in X$;

(iv) $g \in G$ and $g \in \langle X, X^g \rangle$ implies that $g \in X$.

Four possible subnormalizers of a subgroup X of a (finite) group G suggested by the above are:-

$S_G(X) = \{g \in G \mid X \text{ sn } \langle X, g \rangle\}$;

$S_G^{\,1}(X) = \{g \in G \mid X \text{ sn } \langle X, X^g \rangle\}$;

$S_G^{\,2}(X) = \{g \in G$ for which there exists a positive n for which $[g, x_1, x_2, \ldots, x_n] \in X$ for any elements $x_1, \ldots, x_n$ in $X\}$;

$S_G^{\,3}(X) = \{g \in G$ for which given any element x in X there exists a positive integer n such that $[g,_n x] \in X\}$.

<u>Note</u> $S_G(X) \subseteq S_G^1(X) \subseteq S_G^3(X) \subseteq S_G^2(X)$

The principal theorem regarding subnormalizers and products on which these investigations are based is the following theorem of Wielandt (Theorem 7.7.1 in Lennox and Stonehewer[1]).

<u>**Wielandt's Theorem**</u> Suppose G=HK is a finite group with H and K subgroups and suppose that X is a subgroup of G with X sn H and X sn K. Then X sn G.

In §2.2 we prove some easy results about subnormalizers in metabelian groups; then in §2.3 we prove a generalization of Wielandt's theorem to soluble groups. In §2.4 we look at what would happen if HK (as in Wielandts theorem above) were not necessarily a group. Finally in §2.5 we investigate subgroups of groups that permute with certain of their conjugates.

Before proceeding, we first state a result of Wielandt that is used several times in this chapter.

Theorem 2.1.2(Wielandt) Suppose $G=HK$ is a finite group with H and K subgroups and suppose p is a prime dividing the order of G. Then there exists Sylow p-subgroups H_p and K_p of H and K respectively such that H_pK_p is a Sylow p-subgroup of G.

§2.2 Metabelian Groups

In this section some easy results about metabelian groups are proved. Firstly it is shown that for various metabelian groups the "subnormalizers" $S_G(X)$, $S^1{}_G(X)$ and $S^2{}_G(X)$ suggested in 2.1 are actually groups. We note that in Casolo[1] a detailed characterization of the finite 'sn-groups' is given (an sn-group being a group for which $S_G(X)$ is a group for all subgroups X of G).

Proposition 2.2.1 Suppose G is a periodic metabelian group and X is a subgroup of G. Then $S^2{}_G(X)$ is a group.

Proof

$S^2{}_G(X) = \{g \in G$ such there exists $n=n(g)$ such that $[g,x_1,\dots,x_n] \in X$ for all $x_1,\dots,x_n \in X\}$,

and it follows that to prove the proposition it suffices to show that for any $h,k \in S^2{}_G(X)$ there exists an integer n such that $[hk,x_1,\dots,x_n] \in X$.

Let h and k be elements of $S^2{}_G(X)$ and choose positive integers m and p such that

$$[h,x_1,\dots,x_m] \in X \quad \text{for all } x_1,\dots,x_m \in X$$

and

$$[k,y_1,\dots,y_p] \in X \quad \text{for all } y_1,\dots,y_p \in X\ .$$

Let $x_1,\dots,x_{m+p}$ be arbitrary elements of X. Consider the element $[h,x_1]^k$. We have that

$$[[h,x_1]^k,x_2]=[[h,x_1],x_2{}^{k^{-1}}]^k=[[h,x_1],x_2]^k \quad \text{(since G is metabelian)},$$

and similarly

$$[[h,x_1]^k,x_2,\dots,x_m]=[h,x_1,\dots,x_m]^k=z^k=[k,z^{-1}]z \quad \text{for some } z \in X.$$

It follows that

$$[hk,x_1,\dots,x_{m+p}]=[[h,x_1]^k[k,x_1],x_2,\dots,x_{m+p}]$$

$$=[([k,z^{-1}]z[k,x_1, \ldots, x_m]), x_{m+1}, \ldots, x_{m+p}] \in X,$$

and hence $hk \in S^2{}_G(X)$ as required. □

__Corollary 2.2.2__ Suppose G is a finite metabelian group and X is a subgroup of G. Then $S^1{}_G(X)$ is a group.

__Proof__

$$S^1{}_G(X) = \{ g \in G \mid X \text{ sn } \langle X, X^g \rangle \},$$

and so to prove the corollary it suffices to show that if $h,k \in S^1{}_G(X)$ then X is subnormal in $\langle X, X^{hk} \rangle$.

Let $h,k \in S^1{}_G(X)$, then since $S^1{}_G(X) \leq S^2{}_G(X)$ and $X \leq S^2{}_G(X)$ it follows from Proposition 2.2.1 that

$$\langle X, X^{hk} \rangle \leq S^2{}_G(X),$$

and hence by Theorem 2.1.1(iii) it follows that X is subnormal in $\langle X, X^{hk} \rangle$ as required. □

It is easy to see that method of proof of the above corollary could be used to show that $S_G(X)$ is a group for any subgroup X of any *finite* metabelian group G. The final two results of this section show that $S_G(X)$ is a group for any subgroup X of *any* metabelian group G.

Proposition 2.2.3 Suppose $G=\langle H,K\rangle$ is a metabelian group with H and K subgroups of G, and suppose X is a subgroup of G such that $X \lhd^n H$ and $X \lhd^m K$. Then $X \lhd^{m+n} G$.

Proof Let $G=\langle H,K\rangle$ be as described in the theorem. We note first of all that

$$[G,X] = [H,X]^G[K,X]^G.$$

Also we note that since G is metabelian it follows that $[[H,X]^G,X]$ is normal in G and in fact in general we have that $[[H,X]^G,_sX]$ is normal in G for any s>0. Similarly $[[K,X]^G,_sX]$ is normal in G for any s>0. Hence we may deduce that for any s>0 we have that

$$[G,_sX] = [[H,X]^G,_{s-1}X]\ [[K,X]^G,_{s-1}X] \qquad (*).$$

Now $G = \langle H,K\rangle = H[H,K]K$ and so since G is metabelian it follows that

$$[H,X]^G = [H,X]^K.$$

Also we have that

$$[[h,x]^k,y]=[[h,x],[k^{-1},y^{-1}]y]^k = [[h,x],y]^k$$

$$\in [[H,X],X]^K = [[H,X],X]^G$$
for all $h\in H$, $k\in K$ and $x,y\in X$.

It follows that

$$[[H,X]^G,X] = [[H,X],X]^K$$

and clearly, using the above repeatedly, we will have that

$$[[H,X]^G{}_{,s-1}X] = [H,{}_sX]^K \qquad \text{for any } s>0.$$

In particular, if we take $s=n$ we find that

$$[H,X]^G{}_{,n-1}X] \leq X^K$$
and so, since $X \trianglelefteq^{m-1} X^K$, it follows that

$$[[H,X]^G{}_{,n+m-1}X] \leq X.$$

Similarly

$$[[K,X]^G{}_{,m+n-1}X] \leq X,$$

and so we can deduce from $(*)$ that

$$[G,{}_{m+n}X] \leq X,$$

and hence $X \lhd^{m+n} G$ as required. □

Using the observation made in §7.7 of Lennox and Stonehewer[1], we can deduce

Corollary 2.2.4 Suppose G is a metabelian group and X any subgroup of G. Then $S_G(X)$ is a group.

Proof

The set $S_G(X) = \{g \in G \mid X \text{ sn } \langle X,g \rangle\}$ is clearly closed under taking inverses and hence to show that it is in fact a group it suffices to show that it is closed under taking products.

Let h and k be any two elements of $S_G(X)$. Then X sn $\langle X,h \rangle$ and X sn $\langle X,k \rangle$ and hence using proposition 2.2.3 we can deduce that

$$X \text{ sn } \langle X,h,k \rangle \geq \langle X,hk \rangle.$$

Hence $hk \in S_G(X)$, which proves the corollary. □

Note Wielandt gives an example (to be found in §7.7 of Lennox and Stonehewer[1]) of a finite soluble group $G = \langle H,K \rangle$ of derived length 3 which contains a subgroup X such that $X \lhd H$ and X sn K but X is not subnormal in G (i.e. $S_G(X)$ is not a group).

§2.3 Soluble Groups

The aim of this section is to prove the following generalization of Wielandt's theorem (conjectured originally by Wielandt) for finite soluble groups:-

Suppose G=HK is a finite soluble group with H and K subgroups of G and suppose X is a subgroup of G with $H \cup K \subset S^2{}_G(X)$. Then X sn G.

This result was proved independently of the following earlier, slightly weaker result due to Carlo Casolo.

<u>**Theorem**</u> (Casolo) Let G=HK be a finite soluble group with H and K subgroups and suppose that X is a subgroup of G with $H \cup K \subset S^1{}_G(X)$. Then X is subnormal in G.

To prove this result we first show that it is true when X is a nilpotent group. In this case the proof is a slightly adapted version of the proof of the following theorem proved in Maier and Sidki[1].

<u>**Theorem**</u> (Maier and Sidki) Let G be a soluble finite group factorized as a product G=HK of two of its subgroups H and K. Let X be a subgroup of G of prime order such that $H \cup K \subset S^1{}_G(X)$. Then X sn G.

For completeness we give all of the adapted proof here.

First of all, we need to state two results, the first of which collects together some known facts about p'-groups of automorphisms of p-groups.

Lemma 2.3.1 Suppose Q is a p′-group of automorphisms of a p-group P. Then the following are true :-

(i) $P = C_P(Q)[P,Q]$,

(ii) $[P,Q] = [P,Q,Q]$,

(iii) if P is also abelian, then $P = C_P(Q) \times [P,Q]$.

Proof see Chapter 4 of Gorenstein[1]. □

The second result is a quite recent result due to Wolf.

Theorem 2.3.2 (Corollary 1.9 of Wolf[1]) Let V be an F[G]-module for a finite field F of order p and a p-soluble group G. Assume that $O_p(G)=1$, $C_G(V)=1$ and $P \in Syl_p(G)$. Then $|P| \leq |V|/2$ □

Lemma 2.3.3 Suppose G=HK is a finite soluble group with H and K subgroups of G and suppose X is a subgroup of G of prime order with $H \cup K \subset S^3{}_G(X)$. Then X sn G.

Proof (adapted from the proof in Maier and Sidki[1]).

Let G be a minimal counter example to the theorem. Let V be a minimal normal subgroup of G, then by the minimality of G we have that VXsn G and so since X is not subnormal in G it follows that that if the order of X=<x> (say) is p then V is an elementary abelian q-group for some prime q≠p.

Suppose V is not the only minimal normal subgroup of G. Let W be a minimal normal subgroup of G not equal to V. Since X is nilpotent, XV/V sn G/V and XW/W sn G/W, it follows that there exists a positive integer n such that

$$[G,_n X] \leq W \cap V = 1.$$

This implies that X is subnormal in G, contradicting our choice of G. Hence it follows that V is the unique minimal normal subgroup of G.

Now

$$\text{Fitting}(G)/\text{Frattini}(G) = \text{Fitting}(G/\text{Frattini}(G))$$

and it follows that we must have Frattini(G) = 1 (otherwise XFrattini(G) is subnormal in G, which would imply that X were contained in Fitting(G) (by the equality above), and hence that X were subnormal in G).

Since Frattini(G) = 1 there exists a maximal subgroup M of G which does not contain V. We must have that $G=MV$ and $M \cap V=1$ so we can deduce that $O_q(G) = (O_q(G) \cap M)\ V$, but $O_q(G) \cap M$ is normal in M and centralized by V and hence is normal in G. It follows that $O_q(G) \cap M = 1$ (as V is unique) and hence that

$$V = O_q(G).$$

<u>Claim</u> $V \cap H = V \cap K = 1$:-

Suppose, for example, that $V \cap H \neq 1$. Let $L = N_G(V \cap H)$; then $L \geq V, H$ and so using Dedekind's intersection lemma we have that

$$L = H(L \cap K).$$

Let $1 \neq v \in V \cap H$ then since $v \in H \subset S^3{}_G(X)$ it follows that for some positive integer n we must have

$$[v,_n x] = 1.$$

It follows from Lemma 2.3.1(iii) that $v = cw$ where $c \in C_V(X)$ and $w \in [V,X]$ and clearly we must have that

$$[w,_n x] = 1$$

so that $[w,_{n-1} x] \in C_V(X) \cap [V,X] = 1$ and applying this process repeatedly we find that $w = 1$ and so $v \in C_V(X)$. So we have shown that $V \cap H$ centralizes X and hence X is contained in L. If L were equal to G then it would follow that $V = V \cap H$ and hence X would be subnormal in VX, which is not the case. It follows that L is not equal to G and so using the minimality of G we can deduce that X is subnormal in L and hence is contained in $O_p(L)$ which is centralized by $V \leq O_q(L)$; this is a contradiction.

Hence we must have that $V \cap H = 1$ and similarly $V \cap K = 1$ also. This proves the claim.

By Theorem 2.1.2 there exists a Sylow q-subgroup Q of G which is factorized as

$$Q = (Q \cap H)(Q \cap K).$$

Now $V \leq Q$ and so any element v of V can be expressed in the form

$v = hk$ for some $h \in Q \cap H$, $k \in Q \cap K$.

Also, since $V \cap H = V \cap K = 1$, for any element $h \in Q \cap H$ there exists at most one element k in $Q \cap K$ such that $hk \in V$ (for suppose $k_1 \in K$, then if $hk_1 \in V$ then $k^{-1}k_1 = (hk)^{-1}hk_1 \in V \cap K = 1$ and so $k = k_1$). From this we can deduce that $|V| \leq |Q \cap H|$.

We now use Theorem 2.3.2 to derive a contradiction. For V is an F[G/V]-module for a finite field F of order q. Also $O_q(G/V) = 1$ and $QV/V \in Syl_q(G)$ and so by Theorem 2.3.2 we must have that

$$|QV/V| < |V|$$

and so

$$|V| \leq |Q \cap H| = |(Q \cap H)V/V| \leq |QV/V| < |V|$$

which is a contradiction. □

<u>Corollary 2.3.4</u> Let $G = HK$ be a finite soluble group with H and K subgroups and let X be a nilpotent subgroup of G with $H \cup K \subset S^3{}_G(X)$. Then X is subnormal in G.

<u>**Proof**</u> Let G be a minimal counter example to the theorem for which the order of X is also minimal. We note that for any subgroup Y of X we have that $H \cup K \subset S^3{}_G(Y)$, for X is nilpotent and hence for any elements $g \in H \cup K$ and $x \in X$ there exists a positive integer n such that

$[g,_n x]=1 \in Y$.

It follows, by the minimality of X, that every proper subgroup of X is subnormal in G and hence, if X is not to be subnormal in G, X must be a cyclic group of prime power order (otherwise we could find proper subgroups Y_1 and Y_2 of X such that $X=\langle Y_1, Y_2\rangle$ but then X would be subnormal in G by Wielandt's join criterion).

Now let Z be the subgroup of X that has prime order. By Lemma 2.3.3 $Z\neq X$ and so Z sn G, it follows that if p is the order of Z then $O_p(G)\neq 1$. But by the minimality of G, we have $O_p(G)X$ sn G and clearly

X sn $O_p(G)X$, and so X sn G which is a contradiction. □

We can now prove the main theorem of the section.

<u>**Theorem 2.3.5**</u> Let G=HK be a finite soluble group with H and K subgroups and suppose X is a subgroup of G with $H\bigcup K \subset S^2{}_G(X)$. Then X is subnormal in G.

<u>Proof</u> Let G be a minimal counter-example to the theorem for which the order of X is also minimal. Suppose V is a minimal normal subgroup of G so that VX is subnormal G and X is not subnormal in VX.

We first prove some statements about G.

(i) If F is the Fitting subgroup of G then $F=O_q(G)$ for some prime q and in particular V is a q-group.

<u>Proof</u> Let q and r be any primes for which $O_q(G)\neq 1$ and $O_r(G)\neq 1$;

we wish to show that $q=r$. Suppose not and let X_q and $X_{q'}$ be a Sylow q-subgroup of X and a Hall q'-subgroup of X respectively. Then we have that

$$\begin{aligned} O_q(G)X \cap O_{q'}(G)X &= (O_q(G)X_qX_{q'} \cap O_{q'}(G)X_{q'}X_q) \\ &= (O_q(G)X_q \cap O_{q'}(G)X_{q'}X_q)X_{q'} \\ &= (O_q(G)X_q \cap O_{q'}(G)X_{q'})X \\ &= X. \end{aligned}$$

But both $O_q(G)X$ and $O_{q'}(G)X$ are subnormal in G and it follows that for some positive integer n we have that

$$[G,_nX] \leq O_q(G)X \cap O_{q'}(G)X = X$$

and hence X is subnormal in G, which is a contradiction and so we must have that $q=r$ as required. □

(ii) Any proper subgroup of X containing X' ($\neq 1$) is subnormal in G and $X/X' \cong C_{p^m}$ for some prime p.

Proof First of all we note that $X' \neq 1$ by Corollary 2.3.4. Clearly for any proper subgroup Y of X containing X' we have that $H \cup K \subset S^2{}_G(Y)$ and so by the minimality of X, Y is subnormal in G. It follows that X/X' must be cyclic of prime power order (by the argument used in the proof of Corollary 2.3.4). □

Let X_p be a Sylow p-subgroup of X.

(iii) $p \neq q$.

Proof Suppose p=q. Let $X_{p'}$ be a Hall p'-subgroup of X. $X_{p'}$ is a subgroup of X' which is subnormal in G and so by Lemma 2.3.1(ii) we have that

$$[V,X_{p'}] = [V,X_{p'},X_{p'}] \leq V \cap X.$$

Also X_p is subnormal in VX_p and so we can find a positive integer s such that

$$[V,_s X_p]=1.$$

Now

$$[V,X] = [V,X_{p'}X_p] \leq (V \cap X)\,[V,X_p]\,,$$

and it is easy to see that in general that for any positive integer l we have that

$$[V,_l X] \leq (V \cap X)\,[V,_l X_p]$$

and so in particular

$$[V,_s X] \leq X$$

from which we deduce that X is subnormal in G. This is a contradiction and so we must have that p≠q as required.
□

(iv) $[V,_n X_p]$ is not contained in X for any positive integer n.
Proof Suppose $[V,_n X_p]$ is contained in X for some positive integer n; then since $p \neq q$ it follows from lemma 2.3.1(ii) that

$$[V,X_p] = [V,X_p,X_p] \leq V \cap X.$$

Now $X = X_p X'$ and X' sn VX so there exists a positive s for which

$$[V,_s X'] \leq X;$$

but

$$[V,X] = [V,X_p X'] \leq (V \cap X)\,[V,X'],$$

and in general it can be seen that

$$[V,_t X] \leq (V \cap X)\,[V,_{t-1} X'] \qquad \text{for any positive integer } t,$$

so that in particular

$$[V,_s X] \leq X,$$

from which we can deduce that X sn VX, which is a contradiction.
□

(v) $G = \langle H,V,X_p \rangle$.
Proof Suppose not, then if we let $M = \langle H,V,X_p \rangle$ then

$$M = H(M \cap K)$$

and clearly $H \cup (M \cap K) \subset S^2{}_G(M \cap X)$ and it follows from the minimality of G that

$$X_p \leq M \cap X \text{ sn } V(M \cap X),$$

and so there exists a positve integer n such that

$$[V,_n X_p] \leq X,$$

which contradicts (v). Hence $G = \langle H, V, X_p \rangle$ as required. □

X_p is a p-group of automorphisms of the the q-group F (since $C_G(F) \leq F$), and so by Lemma 2.3.1 we have that

$$F = C_F(X_p)\,[F,X_p] \quad \text{and } [F,X_p] = [F,X_p,X_p] \qquad (1).$$

If we let $D = F \cap X$ then since VX sn FX it follows that

$$[F,X_p] \leq F \cap VX = VD,$$

and hence

$$[F,X_p] = [F,X_p,X_p] \leq [V,X_p]\,[D,X_p] \leq [F,X_p] \quad (2).$$

If we think of $[V,X_p]$ as a $\mathbb{Z}_q X_p$-module and apply Maschke's

Theorem we can deduce that

$$[V,X_p] = W \oplus ([V,X_p] \cap D),$$

where W is a subgroup of V with $W^{X_p}=W$, $W \cap X=1$ and (using 2.3.1(iii)) $C_W(X_p)=1$. Using (1) and (2) we obtain

$$F = C_F(X_p)\ W\ D.$$

We note that by (iv), $W \neq 1$, and so in particular $C_{X_p}(W) \neq X_p$. Also $X' \cap X_p \leq X'$ sn G and so, using 2.3.1(ii), we have that

$$[W, X' \cap X_p] = [W, X' \cap X_p, X' \cap X_p] \leq W \cap X = 1$$

i.e. $X' \cap X_p \leq C_{X_p}(W)$. But $X_p/(X' \cap X_p) \simeq X/X' \simeq C_{p^m}$ and so it follows that there exists an element $x \in X_p$ for which $C_W(x)=1$. In particular, given an element $w \in W$, if there exists a positive integer r for which

$$[w,_r x] = 1,$$

then we must have that $w=1$.

Suppose f is an element of F for which there exists a positive integer r such that

$$[f,_r x] \in X$$

(for example given elements $h \in H$ and $d,d' \in D$, f could be any of $[h,d]$, $[d,h]$ $(= [h,d^{-1}]^{d})$ or $[h,d]^{d'}$), then we can write

$$f = cwd \quad \text{for some } c \in C_F(X_p) = C \text{ say}, w \in W \text{ and } d \in D.$$

Since V is in the centre of F we have that

$$[f,_r x] = [w,_r x]\, d' \in X \quad \text{for some } d' \in D$$

and it follows that $[w,_r x] \in W \cap X = 1$ and hence that $w=1$. So

$f \in CD$ (note that CD is not necessarily a group)
(3).

For any $h \in H$ and $d,d' \in D$ we have that

$$[h,d]^{d'^{-1}} \in CD$$

and in particular

$$d'[h,d] \in CD \qquad (4).$$

Given any element $b \in [H,D]$ it can be written in the form

$$b = [h_1,d_1] \ldots [h_t,d_t] \text{ for some } h_1,\ldots,h_t \in H \text{ and } d_1,\ldots,d_t \in D,$$

and it is clear that by using (3) and for (4) repeatedly we can show that $b \in CD$. Hence we have shown that $[H,D] \leq CD$ and hence that

$$D^H = [H,D]\,D \leq CD\,.$$

It follows from this that $[D^H, X_p] \leq D$ and so the group D^H is normalized by H, V_λ and X_p and hence is a normal subgroup of G.

Now D^H contains D and hence is not the the trivial group as the last (non-trivial) term of the derived series of X is an abelian subnormal subgroup of G and hence is contained in $F \cap X = D$. It follows that some minimal normal subgroup U of G is contained in D^H and hence in CD. It follows that

$$[U, X_p] \leq X$$

contradicting (iv) applied to U instead of V (the argument clearly applies to any minimal normal subgroup). This contradiction completes the proof. □

§2.4 Permuting Conjugates

In Wielandt[1] the following is proved:-

Suppose X *is a subgroup of the finite group* $G = AB$ *where* A *and* B *are subgroups, and suppose further that* $XX^g = X^gX$ *for all* $g \in A \cup B$. *If* P *is a normal Sylow subgroup of* X, *then* P *sn* G; *further, if* X *is nilpotent then* X *sn* G. (*)

In the same paper he conjectured that the above theorem would remain true even if X was not required to be nilpotent. The aim of this section is to prove the above conjecture for several special cases.

Theorem 2.4.1 Suppose $G=HK$ is a finite metabelian group with H and K subgroups, and suppose X is a subgroup of G such that $X^gX=XX^g$ for all $g\in H\bigcup K$. Then X is subnormal in G.

Proof Let G be a minimal counter example to the theorem for which the order of X is as small as possible. By (*) above we know that X is not nilpotent. Arguing as in the proof of 2.3.5(i) we can deduce that there exists a unique prime q such that $O_q(G)\neq 1$, in particular we have that $G'\leq O_q(G)$ and hence also that $O_q(G)$ is a Sylow q-subgroup of G.

Let Y be a Hall q'-subgroup of X and let V be a minimal normal subgroup of G, so that VX is subnormal in G. Since G is metabelian and V is contained in the centre of G' it follows that [V,Y] is a normal subgroup of G and hence that [V,Y]=V (if V centralized Y then clearly X would be subnormal in VX and hence in G which would be a contradiction). It follows from lemma 2.3.1(iii) that

$$C_V(Y)=1 \qquad (1).$$

If we let Q be the normal Sylow q-subgroup of X and let g be any element of $H\bigcup K$, then by Theorem 2.1.2 we have that $Q^gQ=QQ^g$ is the normal Sylow q-subgroup of X^gX and in particular if X_1 is any proper subgroup of X containing Q then

$$[X_1, X_1^{\,g}] \le QQ^g,$$

and so $X_1X_1^{\,g} = X_1^{\,g}X_1$. It follows from the minimality of X that X_1 is subnormal in G, and so using Wielandt's join criterion we can deduce that X has a unique maximal subgroup containing Q. Hence since X/Q is abelian it must in fact be cyclic, from which it follows that Y is cyclic of order p^n for some prime p. Let y be a generator of Y.

Again let g be any element of H or K and let $L = XX^g$, then since G is metabelian it follows that both X' and X'^g are normal in L. In L modulo $X'X'^g$, X and X^g both have unique Sylow p-subgroups, which must permute because of the theorem of Wielandt mentioned above. It follows that

$$(X'Y)(X'Y)^g = (X'Y)^g(X'Y).$$

If X did not equal $X'Y$ then it would follow from the minimality of X that $X'Y$ were subnormal in G; but that would imply that $X = QX'Y$ were subnormal in G and so we must have that $X = X'Y$ i.e.

$$Q = X' \qquad (1).$$

Regarding V as a $\mathbb{Z}_qY$-module and applying Maschke's Theorem we see that V must have a subgroup W such that

$$V = W(V \cap X), \text{where } W \cap X = 1 \text{ and } W^Y = W,$$

also $W \neq 1$ otherwise X would be subnormal in G. Now X is not subnormal in $WX = VX$ and so by Theorem 2.1.1(iv) it follows that there exists an element $w \in W$ such that $w \neq 1$ and

$$w \in \langle X, X^w \rangle = J;$$

also $w = hk$ for some elements h and k in H and K respectively.

<u>Claim</u> $\langle X, X^{hk} \rangle$ is contained in the product XX^kX^{hk} which is a group:-

Clearly it is enough to show that XX^kX^{hk} is a group. Let x_1 and x_2 be any elements of X, then

$$x_1^{k^{-1}} x_2^{h} = x_1[x_1,k^{-1}][h,x_2^{-1}]x_2 = x_1[h,x_2^{-1}][x_1,k^{-1}]x_2$$
$$\in X^h X^{k^{-1}} X \text{ (a set).}$$

It follows that $X^{k^{-1}}X^h \subset X^h X^{k^{-1}} X$ and hence that

$$XX^{hk} \subset X^{hk}XX^k = X^{hk}X^kX,$$

also noting that $X^{hk}X^k$ is a group we see that

$$XX^kX^{hk} = X^kXX^{hk} \subset X^kX^{hk}X,$$

and so we can deduce that XX^kX^{hk} is a group, proving the claim.

We have that $hk \in J=\langle X,X^{hk}\rangle$ and so using the claim it follows that we can find elements x_1,x_2 and x_3 of X such that

$$hk = x_1k^{-1}x_2kk^{-1}h^{-1}x_3hk$$

and hence we have that

$$h \in Xk^{-1}X$$

and in particular $hk \in XX^k$. It follows that

$$J \leq XX^k \qquad (2)$$

Suppose $h_1 \in H'$, then from (1) we have that X' is the Sylow q-subgroup of XX^{h_1} and hence

$$[h_1,y] \in XX^{h_1} \cap O_q(G) = X'.$$

It follows that H', and similarly K', are contained in the normalizer of X.

Let x be any element of X' and let H_q and K_q be the Sylow q-subgroups of H and K respectively . By the result of Wielandt mentioned above we have that H_qK_q is the Sylow q-subgroup of G, and so since X' is subnormal in G it follows that $X' \leq H_q K_q$ and so there exists elements h' and k' of H_q and K_q respectively such that $x=h'k'$. Now

$$[h^{-1},h'^{-1}] = [kw^{-1},k'x^{-1}] = [k,k']^{x^{-1}}[k,x^{-1}],$$

and so $[k,x^{-1}] \in \langle H',K',X\rangle \leq N_G(X)$ for any $x \in X'$.

By (2) we can find elements x and x' from X' such that $w=x[x',k]$ and it follows that $w \in N_G(X)$ so that in particular

$$[w,y] \in W \cap X = 1$$

but this implies that $w=1$ contrary to our assumption.

□

The other results of this section look at the case where X is assumed to have a Sylow tower. We recall that a group G is said to have a Sylow tower if it possesses a normal series

$$1=H_0 \leq H_1 \leq \ldots \leq H_n = G$$

such that each factor group H_i/H_{i-1} is a normal Sylow subgroup of G/H_{i-1} for $i=1,\ldots,n$.

We can now prove the following:-

<u>**Theorem 2.4.2**</u> Suppose $G=HK$ is a finite group with H and K subgroups. Let X be a subgroup of G such that X has a Sylow tower,

X is subnormal in H and $X^k X = X X^k$ for every $k \in K$. Then X is subnormal in G.

Proof Let G be a counter-example to the theorem of minimal order for which $|G:H|$ is also minimal. We can deduce the following:-

(i) (As in the proof of Wielandt's theorem) H must be a maximal subgroup of G.

For otherwise, if M is a maximal subgroup of G properly containing H then we have that $M = H(M \cap K)$ so that X is subnormal in M by the minimality of G and hence X is subnormal in $G = MK$ by the minimality of $|G:H|$.

(ii) $G = \langle K, X \rangle$.

For if not then $L = \langle K, X \rangle = (L \cap H)K$ and so by the minimality of G we have that X is subnormal in L, but then X is subnormal in $G = HL$ by Wielandt's Theorem which is a contradiction.

(iii) X is not nilpotent.

Suppose first of all that X is a p-group for some prime p. By (ii) we have that $X^G = X^K$ (since X^K is normalized by both X and K) but $X^K = \prod_{k \in K} X^k$ is a p-group and it follows that X is subnormal in X^G and hence in G which is a contradiction. Now suppose that X is nilpotent; then by theorem 2.1.2 all the (unique) Sylow subgroups of X satisfy the hypotheses of the Theorem and hence arguing as above they are subnormal in G and it follows from Wielandt's Join Theorem that X is subnormal in G which is a contradiction.

(iv) Let q be a prime for which X has a non-trivial normal q-subgroup Q, say (as X has a Sylow tower some such q exists).Then $O_q(G)\neq 1$.

For by Theorem 2.1.2 we have that $Q^kQ=QQ^k$ for all $k\in K$ and so by the argument used in (iii) above it follows that Q is subnormal in G and hence in particular $O_q(G)\neq 1$.

(v) Any Sylow q-subgroup of H is a Sylow q-subgroup of G.

Suppose not. By Theorem 2.1.2 we can find Sylow q-subgroups H_q and K_q of H and K respectively such that H_qK_q is a Sylow q-subgroup of G. It follows that if H_q is not a Sylow q-subgroup of G then there exists an element $k\in K\backslash H$ which normalizes H_q and hence which also normalizes $O_q(G)\cap H$ So $N_G(O_q(G)\cap H)\geq\langle H,k\rangle > H$, but H is a maximal subgroup of G and so $O_q(G)\cap H$ is normal in G. Now Q is a non-trivial subnormal subgroup of H and it follows that $O_q(G)\cap H$ is non-trivial and so by the minimality of G we must have that $(O_q(G)\cap H)X$ is subnormal in G, but X is subnormal in $(O_q(G)\cap H)X$ and so we derive the contradiction that X is subnormal in G.

From (v) we can know that H contains a Sylow q-subgroup of G and it follows that $O_q(G)$ is contained in H and so X is subnormal in $O_q(G)X$ but by (iii) we have that $O_q(G)\neq 1$ and so from the minimality of G we can deduce that $O_q(G)X$ is subnormal in G and so we have arrived at the contradiction that X is subnormal in G completing the proof of the theorem.

$\square$

Chapter 3

§3 GROUPS WHICH ARE PRODUCTS OF NILPOTENT SUBGROUPS

§3.1 Introduction

In 1961 it was proved in Kegel[1] that any finite group which can be written as a product of two nilpotent subgroups is soluble.Further it could be deduced from Hall and Higman[1] that if a finite group G can be written as a product of two subgroups H and K where H is nilpotent of class c,K is nilpotent of class d and the orders of H and K are coprime then G is soluble with derived length at most c+d. It has also been proved in Itō[1] that any group (finite or infinite) which is the product of two abelian subgroups is metabelian.In light of these two theorems the following has been conjectured:-

Conjecture 3.1.1 Any finite group G which is the product of two subgroups H and K, where H is nilpotent of class c and K is nilpotent of class d, is soluble of derived length at most c+d.

In Pennington[1] it was shown that if the above conjecture were true for finite p-groups then the following would be true for general finite groups:-

If G=HK is a finite group,where H is a nilpotent subgroup of

G of class c and K is a nilpotent subgroup of G of class d, then G is soluble of derived length at most 2(c+d).

So it is worthwhile investigating the conjecture in the case where G is a finite p-group.

In this chapter we continue work begun in Gold[1] and investigate p-groups of the form G=AXB where A is an abelian group, X is a cyclic group, B is an abelian group and XB is a nilpotent group of class 2. We show that various such groups have derived length three and hence satisfy the conjecture given above. In section 3.2 we consider such groups when G is finite and p is odd, in section 3.3 we assume that p is even, in section 3.4 we assume that p is even and that G is finite and finally in section 3.5 we look at some examples of these groups with properties of interest.

§3.2 Odd-order groups

This section is devoted to proving the following theorem:-

Theorem 3.2.1 Let G=AXB be a finite p-group where

1) p is odd,
2) A and B are abelian,
3) $A^X=A$ and $B^X=B$,
4) X=$\langle x\rangle$ is cyclic,
5) AX and BX are nilpotent of class at most 2.

Then $[[A,X],[B,X]]=1$.

Proof Suppose the theorem were false and let $G=AXB$ be a counter-example of minimal order. Let z be an element in the centre of G of order p, then by the minimality of G the theorem holds in

$$G/\langle z\rangle=((A\langle z\rangle)/\langle z\rangle)((X\langle z\rangle)/\langle z\rangle)((B\langle z\rangle)/\langle z\rangle)$$

and it follows that

$$[[A,X],[B,X]] \le \langle z\rangle \le Z(G).$$

Now clearly we can find elements $a\in A$ and $b\in B$ such that

$$[[a,x],[b,x]]=z.$$

We proceed to derive a contradiction.

i) $[[a,x],b],x]=z$:-

from the Hall-Witt identity we obtain $([x,y^{-1},z]^{y}[y,z^{-1},x]^{z}[z,x^{-1},y]^{x}=1)$

$$[[b^{-1},x^{-1}],[a,x]]^{x}\ [[a,x],b],x]^{b^{-1}}=1$$

(note that $[x,[x,a]]=1$ as AX has class 2 and similarly that

$$[b^{-1},x^{-1}]=[b,x]^{b^{-1}x^{-1}}=[b,x]).$$

It follows that

$$[[a,x],b],x] = [[a,x],[b,x]]^{xb} = z,$$

as required.

ii) $[[[b,a],x]^{a^{-1}},x]=1$:-

Let $[a,b]=a_1 x^{\alpha} b_1$ where $a_1 \in A, b_1 \in B$ and $\alpha \in \mathbb{N}$.
Let M_A be a maximal subgroup of G containing AX so that

$$M_A = M_A \cap BXA = (M_A \cap B)AX$$

by Dedekind's intersection lemma.
By the minimality of G we have

$$[[(M_A \cap B),X],[A,X]]=1.$$

Now $b_1 = x^{-\alpha} a_1^{-1}[a,b] \in M_A$ ($\geq G'$ since G is nilpotent) and it follows that

$$[[b_1^{-1},x],[A,X]]=1.$$

So

$$[[[b,a],x],x]=[[b_1^{-1}x^{-\alpha}a_1^{-1},x],x]$$

$$=[[b_1{}^{-1},x]^{a_1{}^{-1}}[a_1{}^{-1},x],x]$$

$$\sim[[b_1{}^{-1},x]^{a_1{}^{-1}},x] \qquad \text{(here $\sim$ means 'is conjugate to')}$$

$$\sim[[b_1{}^{-1},x],x^{a_1}]$$

$$=[[b_1{}^{-1},x],[x,a_1]]=1, \qquad (1)$$

and so

$$[[[b,a],x]^{a^{-1}},x]\sim[[[b,a],x],x^{a}]$$

$$=[[[b,a],x],x[x,a]]$$

$$=[[[b,a],x],[x,a]] \quad \text{(by (1))}$$

$$=[[b_1{}^{-1},x]^{a_1{}^{-1}}[a_1{}^{-1},x],[x,a]]$$

$$\sim[[b_1{}^{-1},x],[x,a]]=1,$$

as required.

iii) $[[[x,b^{-1}],a^{-1}],x]=z$:-

We note that $[x,b^{-1}]=[b,x]^{b^{-1}}=[b,x]$ and using the Hall-Witt identity we obtain

$$[[[b,x],a^{-1}],x]^{a}\,[[a,x^{-1}],[b,x]]^{x}=1,$$

and hence

$$[[[b,x],a^{-1}],x]=[[b,x],[x,a]] \qquad ([a,x^{-1}]=[x,a]^{x^{-1}}=[x,a])$$

$$=[[a,x],[b,x]]^{[x,a]}$$
$$=z\ ,$$

as required.

iv) $[[[x,b^{-1}],a^{-1}]^{b},x]=z$:-

$$[[x,b^{-1}],a^{-1}]^{b}=[[x,b^{-1}],a^{-b}]=[[x,b^{-1}],[b,a]a^{-1}]$$
$$=[[x,b^{-1}],a^{-1}]\ [[x,b^{-1}],[b,a]]^{a^{-1}}$$

and so by iii) to prove iv) it is enough to show that

$$[[[x,b^{-1}],[b,a]]^{a^{-1}},x]=1. \tag{2}$$

If we write $[a,b]=a_1x^{\alpha}b_1$ as in ii) then by a similar argument to that used in ii) we can show that $[[a_1,x],[B,X]]=1$ and so in particular,applying the Hall-Witt identity, we have

$$[[[x,b^{-1}],{a_1}^{-1}],x]=1. \tag{3}$$

Also, since $[[x,a],[b^{-1},x]]$ is central, applying Hall-Witt gives

$$[[[x,b^{-1}],{a_1}^{-1}],[x,a]]=1. \tag{4}$$

It follows that

$$
\begin{aligned}
[[[x,b^{-1}],[b,a]]^{a^{-1}},x] &= [[[x,b^{-1}],b_1{}^{-1}x^{-\alpha}a_1{}^{-1}]^{a^{-1}},x] \\
&= [[[x,b^{-1}],a_1{}^{-1}]^{a^{-1}},x] \\
&\sim [[[x,b^{-1}],a_1{}^{-1}],x^{a}] \\
&= [[[x,b^{-1}],a_1{}^{-1}],x[x,a]] \\
&= 1 \quad \text{(by (3) and (4)).}
\end{aligned}
$$

This proves (2) and hence we have shown that iv) holds.

v)The contradiction:-

From the Hall-Witt identity we obtain

$$[[a^{-1},x^{-1}],b]^{x}\ [[x,b^{-1}],a^{-1}]^{b}\ [[b,a],x]^{a^{-1}}=1,$$

and taking the commutator of the left-hand side with x we have

$[[[a^{-1},x^{-1}],b]^{x},x]\ [[[x,b^{-1}],a^{-1}]^{b},x]\ [[[b,a],x]^{a^{-1}},x]=1$ (each of the three commutators on the left-hand side is central)

and so by i),ii) and iv) (noting that $[a^{-1},x^{-1}]=[a,x]$) we have

$$z^2=1\ ,$$

which implies that $z=1$ (since z has odd order) which is a contradiction.
This proves the theorem.

□

Corollary 3.2.2 If $G=AXB$ is as in heorem 3.2.1 then

i) $X \triangleleft^2 G$

and ii) G has derived length at most 3.

Proof i) Let $g=ax^{\alpha}b$ be an element of G where $a\in A, b\in B$ and $\alpha\in \mathbb{N}$. Then

$$[g,x^{\beta}]=[a,x^{\beta}]^{b}[b,x^{\beta}].$$

It follows that $[G,X]$ is generated by elements of the form

$$[a,x^{\beta}]^{b}[b,x^{\beta}] \text{ where } a\in A;\ b\in B \text{ and } \beta\in \mathbb{N}$$

and each such element commutes with x. It follows that $[G,X,X]=1$ which proves i).

ii) We have shown that $[G,X,X]=1$ and it follows that $[G,X]$ is abelian, but modulo $[G,X]$, G is a product of two abelian subgroups (viz. A and XB) and hence by Itô's theorem $G'' \leq [G,X]$. It follows that G has derived length at most 3 as required. □

§3.3 2-Groups

In this section we show conjecture 3.1.1. holds for certain classes of 2-groups irrespective of whether they are finite are infinite.

Theorem 3.3.1 Let $G=AXB$ be a 2-group where

1) A and B are elementary abelian,
2) $A^X=A, B^X=B$,
3) $X=\langle x \rangle$ is a cyclic group of order 2,
(*)4) AX and BX are both nilpotent of class at most 2,
5) $AB \neq BA$.

Then $[[A,X],[B,X]]=1$.

Proof Suppose the theorem were false and let $G=AXB$ be a counter example.

Claim If a and b are any elements of A and B for which $[[a,x],[b,x]]\neq 1$ then $[a,b]\neq a'xb'$ for any $a'\in A, b'\in B$:-

Suppose that we have $[a,b]=a'xb'$ (*) for some $a'\in A, b'\in B$ then

$$abab=a'xb' \qquad (1)$$

and so

$$a^b=aa'xb'$$

and it follows (since a has order two) that

$$aa'xb'aa'xb'=1 \qquad (2)$$

* this hypothesis is implied by the others and hence is redundant.

and so

$$aa'xb'aa'b'xxb'xb'=1,$$

so that

$$[aa',b'x]=[b',x].$$

Now

$$[b',x]^{aa'}=[aa',b'x]^{aa'}=[b'x,aa']=[x,b']=[b',x],$$

so that

$$[aa',[b',x]]=1 . \qquad (3)$$

Similarly (1) also implies that

$$b^{a}=a'xb'b,$$

and so, since b has order 2, it follows that

$$a'xbb'a'xb'b=1$$

and so

$$a'xbb'a'b'b'bx=b'b'bxb'b'bx ,$$

i.e.

$$[a',b'bx]=[b'b,x],$$

and so we see that conjugation by a' $(=a'^{-1})$ inverts the involution $[b'b,x]$ i.e.

$$[a',[b'b,x]]=1\ . \tag{4}$$

From (1) we obtain

$$a'abab=xb'$$

and hence

$$a'aba=xb'b,$$

whence

$$baa'baa'=bxb'ba';$$

and so

$$[aa',b]=a'xb'[bb',x], \tag{5}$$

which is, of course, just (*) with aa' in place of a and $b'[bb',x]$ in place of b'. So making these substitutions for a and b' in (3), we obtain

$$[a,[b',x]]=1. \tag{6}$$

Also noting that $[b,a]=b'xa'$ we see that the above relations will hold if we interchange each a (or a') with each b (or b'). In particular from (5) we obtain

$$[a,bb']=a'[aa',x]xb', \tag{7}$$

and so as above we can substitute bb' for b and $a'[aa',x]$ for a' in (4) to obtain

$$[a'[aa',x],[b,x]]=1. \tag{8}$$

Now (3) and (6) together imply that

$$[a',[b',x]]=1, \tag{9}$$

and so using (4) we obtain

$$[a',[b,x]]=1. \tag{10}$$

(8) and (10) together give

$$[[aa',x],[b,x]]=1, \tag{11}$$

and, finally (10) (which implies $[[a',x],[b,x]]=1$) and (11), imply that

$[[a,x],[b,x]]=1$

a contradiction. This proves the claim.

G is a counter example to the theorem and so it follows that we can find certain fixed elements $a\in A, b\in B$ such that

$[[a,x],[b,x]]\neq 1.$

From the claim it follows that we can find elements $a_1\in A$ and $b_1\in B$ such that

$$ab=b_1a_1. \tag{12}$$

We now define two subsets, L_A and L_B, of G as follows:-

$L_A=\{\alpha\in A\text{: given any } \beta\in B,\ \alpha\beta=\beta'\alpha' \text{ for some } \alpha'\in A, \beta'\in B\},$

$L_B=\{\beta\in B\text{:given any } \alpha\in A,\ \beta\alpha=\alpha'\beta' \text{ for some } \alpha'\in A, \beta'\in B\}.$

There are two cases to consider:-

<u>Case (i)</u> $a\in L_A$ and $b\in L_B$:

Since $AB\neq BA$ we can find elements $\underline{a}$ and $\underline{b}$ such that

$\underline{a}\,\underline{b}=\underline{b}'x\underline{a}'$ for some $\underline{a}'\in A, \underline{b}'\in B$

and from the claim we have that

$$[[\underline{a},x],[\underline{b},x]]=1. \tag{13}$$

Now $a\in L_A$, and so

$$a\underline{b}'=\beta\alpha \text{ for some } \beta\in B,\alpha\in A.$$

and it follows that

$$a\underline{a}\,\underline{b}=\beta\alpha x\underline{a}'=\beta x[x,\alpha]\alpha\underline{a}',$$

and hence, from the claim, that

$$[[a\underline{a},x],[\underline{b},x]]=1. \tag{14}$$

Similarly, we can show that

$$\underline{a}\,\underline{b}b=b_2xa_2 \text{ for some } a_2\in A,b_2\in B,$$

and that

$$a\underline{a}\,\underline{b}b=b_3xa_3 \text{ for some } a_3\in A,b_3\in B.$$

It follows from the claim, that

$$[[\underline{a},x],[\underline{b}b,x]]=1 \tag{15}$$

and

$$[[a\underline{a},x],[b\underline{b},x]]=1. \qquad (16)$$

We can easily see from (13)-(16) that

$$[[a,x],[b,x]]=1$$

which is a contradiction.

<u>Case (ii)</u> either $a \notin L_A$ or $b \notin L_B$:

Without loss of generality we may assume that $b \notin L_B$. So from the definition of L_B it follows that there is an element $a_4 \in A$ such that

$$a_4{}^b = b'xa_4' \text{ for some } a_4' \in A, b' \in B. \qquad (17)$$

Now from (12) we have that

$$b^a = b_1a_1a, \qquad (18)$$

and (since b has order 2) it follows that

$$[b_1,a_1a]=b_1a_1ab_1a_1a=1,$$

and so

$$[b,a_1a]=[(b_1a_1a)^a,a_1a]=1.$$

Now (17) implies that

$$1=[a_4{}^b,a_1a]=[b'x,a_1a]=[b',a_1a]^x[x,a_1a]=[b',a]^x[b',a_1]^{ax}[x,a_1a],$$

and hence

$$[b',a_1]=[b',a][a_1a,x]. \tag{19}$$

Now either $[b',a_1]\in BA$ or $[b',a_1]\notin BA$ and we consider these two cases separately;-

(i) $[b',a_1]\in BA$:

In this case we can write

$$a_1b'=\beta\alpha \text{ for some } \alpha\in A, \beta\in B.$$

and so it follows that

$$(aa_4)^b=bb_1a_1b'xa_4'=b_1b\beta x[x,\alpha]\alpha a_4'. \tag{20}$$

From the claim, (17) and (20) we can deduce that

$$[[a_4,x],[b,x]]=[[aa_4,x],[b,x]]=1$$

and hence that

$$[[a,x],[b,x]]=1,$$

a contradiction.

(ii) $[b',a_1]\notin BA$:

In this case we can deduce from (19) that

$$b'^a=\beta x\alpha \text{ for some } \alpha\in A, \beta\in B,$$

so from the claim we have

$$[[a,x],[b',x]]=1. \qquad (21)$$

Now from (19) we can deduce that

$$[aa_1,b']=[aa_1,x]$$

and hence that

$$[aa_1,b'^a]=[aa_1,x] \quad (22)$$

and so

$$\begin{aligned}(bb')^a&=b_1aa_1b'^a\\&=b_1b'^a[b'^a,aa_1]aa_1\\&=b_1\beta x\alpha[x,aa_1]aa_1.\end{aligned}$$

From the claim we deduce that

$$[[a,x],[bb',x]]=1$$

and so from (21) we obtain

$$[[a,x],[b,x]]=1.$$

This is the final contradiction which completes the proof of the theorem. □

Corollary 3.3.2 Let $G=AXB$ be a 2-group where

1) A and B are elementary abelian
2) $A^X=A, B^X=B$
3) X is a cyclic group of order 2
(*) 4) AX and BX are both nilpotent of class at most 2.

Then G has derived length at most three.

Proof If $AB=BA$ is a group then by Itô's Theorem $M=AB$ is metabelian. Now $M \triangleleft G$ and G/M is clearly abelian so that $G' \leq M$ and the corollary follows.

If $AB \neq BA$ then by theorem 3.3.1 $[[A,X],[B,X]]=1$ and the result can be shown in this case by using the same proof as for Corollary 3.2.2. □

The example given below was constucted by Stonehewer and Busetto and it shows that Theorem 3.3.1 is not true if $AB=BA$.

(*) This hypothesis is implied by the others and hence is redundant.

<u>**Example 3.3.3**</u> We construct a group G, which is the wreath product of a dihedral group of order 8 and a cyclic group of order 2.

Let $G=(U_1 x U_2)] X$, where

$$U_1=\{u_1,v_1 : u_1^4=v_1^2=1, u_1^{v_1}=u_1^{-1}\}\simeq D_8,$$

$$U_2=\{u_2,v_2 : u_2^4=v_2^2=1, u_2^{v_2}=u_2^{-1}\}\simeq D_8,$$

$$X=\{x : x^2=1\}\simeq C_2,$$

and the action of x on $(U_1 x U_2)$ is given by the relations

$$u_1^x=u_2 \text{ and } v_1^x=v_2.$$

If we set

$$A=\{u_1^2,u_2^2,v_1,v_2\} \text{ and } B=\{u_1v_1,u_2v_2\};$$

then $(U_1 x U_2)=AB$. Also,

$G=AXB$,

$A^X=A, B^X=B$

A and B are elementary abelian,

AX and BX are nilpotent of class 2'

but $[[v_1,x],[u_1v_1,x]]=u_1^2u_2^2\neq 1$.

§3.4 Finite 2-Groups

The aim of this section is to show that Conjecture 3.1.1 holds for certain special types of finite 2-groups.

For all of this section we will be assuming the following hypotheses:-

(∗) $G=AXB$ is a finite 2-group such that :-

(i) A and B are elementary abelian 2-groups,

(ii) $X=\langle x\rangle$ is a cyclic group of order at most 2,

(iii) $B^X=B$,

(∗) (iv) BX is a nilpotent group of class at most 2.

We begin this section by giving brief proofs of three results from Gold[1] concerning groups satisfying (∗).

Lemma 3.4.1(Gold[1]) Let G be a group satisfying (∗) and let Z be the centre of G.Then either $Z\cap AB\neq 1$ or $[B,X]=1$.

Proof If $G=1$ then the lemma is trivially true, so we may assume that $G\neq 1$.

Suppose $Z\cap AB=1$.Since G is a non-trivial finite 2-group it follows that G must contain a non-trivial central element of the form axb for some $a\in A, b\in B$.Now

$$[A,xb]=[A,axb]=1,$$

(∗) This hypothesis is redundant

and hence $[b,x]=(bx)^2$ centralizes A,X and B and hence G. It follows that $[b,x]\in Z\cap AB$ and so by assumption $[b,x]=1$. Now $[G,\langle xb\rangle]$ is an normal subgroup of G and

$$[G,\langle xb\rangle]=[AXB,\langle xb\rangle]=[B,\langle xb\rangle]=[B,X],$$

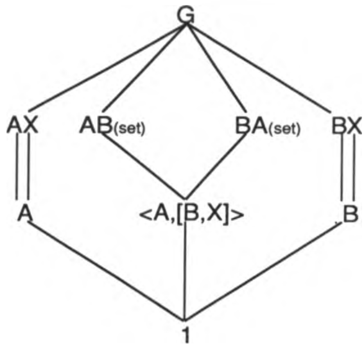
and so, if $[B,X]\neq 1$, then it must contain a non-trivial central element contrary to our supposition that $Z\cap AB=1$. Hence $[B,X]=1$ and the lemma is proved. □

Lemma 3.4.2(Gold[1]) Let G be any group satisfying (∗).Then $\langle A,[B,X]\rangle\leq AB\cap BA$.

Proof Suppose the lemma were false and let G be a counter-example of minimal order.
Clearly $[B,X]\neq 1$ and so by Lemma 3.4.1 it follows that AB contains a non-trivial element of the centre of G.Let z be such an element, then $z=ab=ba$ for some elements $a\in A$ and $b\in B$.Now $G/\langle z\rangle$ satisfies (∗) and so by the minimality of G it follows that the lemma is true in $G/\langle z\rangle$ and hence

$$\langle [B,X],A\rangle\leq AB\langle z\rangle\cap BA\langle z\rangle=AB\cap BA,$$

which proves the lemma. □



Corollary 3.4.3 $\langle [B,X],A\rangle$ is a metabelian group of exponent dividing 4 and $[A,[B,X]]$ is an elementary 2-group.

Proof Using Dedekind's intersection lemma and Lemma 3.4.2 we can write

$$\langle A,[B,X]\rangle=(\langle A,[B,X]\rangle\cap B)A,$$

and so $\langle A,[B,X]\rangle$ is a product of two elementary 2-groups. The result now follows from Theorem 1 of Holt and Howlett[1]. □

Lemma 3.4.4 Let $a\in A$ and $b\in B$ and suppose that $[a,b]=a'b'$ for some $a'\in A, b'\in B$. Then

(i) $[a,b]^2=1$,

(ii) $[a,b']=[a',b]=1$.

Proof We have that

$$abab=[a,b]=a'b',$$

and so,

$$a^b=aa'b'.$$

Since a has order two it follows that

$$1=(a^b)^2=aa'b'aa'b'=[aa',b'],$$

and so b' commutes with aa' and hence with $a=(aa'b')^b$, so

$$[a,b']=1,$$

and symmetrically

$$[a',b]=1,$$

proving (ii).

Also (by expansion)

$$[a,b]^2=[[b,a],a]=[b'a',a]=1,$$

and the lemma follows. □

Lemma 3.4.5 (For this lemma it is enough to assume that A and B are any abelian subgroups of any group G). Suppose $a,a'\in A$, $b,b'\in B$ and that $ab'=b_1{}'a_1$ and $ba'=a_1{}'b_1$ for some $a_1,a_1{}'\in A$, $b_1,b_1{}'\in B$. Then

$$[[a,b],[a'^{-1},b'^{-1}]]=1.$$

Proof The proof of this lemma is a simple adaption of the proof of Itô's Theorem. For

$$[a,b]^{a'b'}=[a_1,b_1],$$

and

$$[a,b]^{b'a'}=[a_1,b_1],$$

so that

$$[[a,b],[a'^{-1},b'^{-1}]]=([b,a]^{b'a'}[a,b]^{a'b'})^{a'^{-1}b'^{-1}}=1.$$ □

Lemma 3.4.6 Let $a\in A$ be such that $[a,x]=a'xb'$ for some $a'\in A,b'\in B$. Then

(i) $[a',x]=1$,

and

(ii) $[a,x]^2=[b',x]$.

<u>Proof</u> We have that

$$a'b'^x x=axax,$$

and so

$$a'b'^x=x^a,$$

whence

$$1=(x^a)^2=[a',b'^x]=[aaa',b'^x].$$

It follows that

$$[a,b'^x]=[aa',b'^x]. \qquad (1)$$

Also

$$xa=aa'b'^x,$$

and so

$$[x,a]=(xa)^2=[aa',b'^x]. \qquad (2)$$

Similarly,

$$aa'x=b'^{x}a,$$

and so

$$[x,aa']=(xaa')^2=[a,b'^{x}]. \qquad (3)$$

Now (1),(2) and (3) together imply that

$$[x,a]=[x,aa']$$

whence,

$$[x,a']=[x,aaa']=[x,aa'][x,a]^{aa'}=([aa',x][x,a])^{aa'}=1,$$

proving (i).

Also

$$[a,x]^2=([a,x]^2)^{a'}=([[x,a],x])^{a'}=([xb'^{x}a',x])^{a'}=[b'^{x},x]=[b',x]$$

proving (ii). □

<u>Lemma 3.4.7</u> Let $a\in A$ and $b\in B$ and suppose that $[a,b]=a'xb'$ for some $a'\in A$ and $b'\in B$. Then the following hold:-

(1) $[aa',[b',x]]=1$,

(2) $[aa',b']=[aa',x][b',x]$,

(3) $[b',[aa',x]]=[aa',x]^2$,

(4) $[a',[bb',x]]=1$,
(5) $[a',bb']=[a',x][bb',x]$,
(6) $[[a',x],bb']=[x,a']^2$.

<u>Proof</u> We have that

$$abab=a'xb',$$

and so

$$a^b=aa'xb',$$

and it follows that

$$1=(a^b)^2=aa'xb'aa'xb'=[aa',b'x][x,b'],$$

whence,

$$[aa',b'x]=[b',x].$$

Now

$$[b',x]=[x,b']=[b'x,aa']=[aa',b'x]^{aa'}=[b',x]^{aa'},$$

and (1) follows.
Also

$$[b',x]=[aa',b'x]=[aa',x][aa',b']^x=([x,aa'][aa',b'])^x,$$

and so

$$[aa',b']=[aa',x][b',x],$$

proving (2).
From (1) we have that

$$[x,aa'][b',x]=[b',x][x,aa']=[b',aa']=[aa',b']^{b'}\overset{\text{From (2)}}{=}[aa',x]^{b'}[b',x],$$

and so

$$[aa',x]^{b'}=[x,aa'],$$

and (3) follows.

We know that

$$b^a=a'xbb'$$

and so

$$1=(b^a)^2=a'xbb'a'xbb'=[a',bb'x][bb',x],$$

and it follows that

$$[a',bb'x]=[bb',x],$$

whence,

$$[bb',x]=[x,bb']=[a',bb'x]^{a'}=[bb',x]^{a'},$$

and (4) follows.
Also

$$[bb',x]=[a',bb'x]=[a',x][a',bb']^{x}=([x,a'][a',bb'])^{x},$$

and so

$$[a',bb']=[a',x][bb',x],$$

proving (5).
From (4) we have that

$$[x,a'][bb',x]=[bb',x][x,a']=[a',bb']^{bb'}=[a',x]^{bb'}[bb',x],$$

and so

$$[a',x]^{bb'}=[x,a'],$$

and (6) follows. □

Note We will use 7(1) to refer to Lemma 3.4.7(1) etc..

Corollary 3.4.8 Let $a\in A$ and $b\in B$ and suppose that $[a,b]=a'xb'$ for some $a'\in A$ and $b'\in B$. Then the following hold:-

(1) $[a,[b',x]]=1$,

(2) $[a,b'^x[b,x]]=[a,x][b',x]$,

(3) $[b'[b,x],[a,x]]=[a,x]^2$

(4) $[a',[b,x]]=1$,

(5) $[a',b^x b'^x]=[a',x][b',x][b,x]$,

(6) $[[a',x],b^x b'^x]=[x,a']^2$,

Proof

$$abab=a'xb'$$

and so

$$a'abaa'=xbb'a'$$

whence

$$[aa',b]=a'xb'^x[b,x].$$

(1),(2),(3),(5) and (6) now follow from the corresponding numbers in Lemma 3.4.7 on substituting aa' for a and $b'^x[b,x]$ for b',(4) follows from (1),7(1) and 7(4). □

Note We shall use 8(1) to refer to Corollary 3.4.8(1) etc..

Proposition 3.4.9 Let $a\in A$ and $b\in B$ and suppose that $[a,b]=a'xb'$ for some $a'\in A$ and $b'\in B$.

Then $[[a,x],[b,x]]=1$.

Proof

We shall consider three exhaustive cases:-

(i) $[aa',x] \in AB$,

(ii) $[a,x] \in AB$,

(iii) $[aa',x] \notin AB$ and $[a,x] \notin AB$.

<u>Case (i)</u> $[aa',x] \in AB$:-

In this case we can write $[aa',x]=\underline{a}\underline{b}$ for some $\underline{a} \in A$ and $\underline{b} \in B$. Now

$$abab=a'xb',$$

and so

$$babb'=aa'x,$$

whence,

$$bb'abb'a=b'aa'xa=b'x[x,aa']a'=b'x\underline{b}\underline{a}a',$$

i.e.

$$[a,bb']=a'\underline{a}x\underline{b}^{x}b'.$$

From 8(4) (with bb' in place of b etc.) we have that

$$[a'\underline{a},[bb',x]]=1,$$

and so from 7(4) we deduce that

$$[a,[bb',x]]=1,$$

and so

$$[[aa',x],[bb',x]]=[ab,[bb',x]]=1,$$

and we deduce from 7(1) that

$$[[aa',x],[b,x]]=1,$$

and finally, from 8(4) we deduce that

$$[[a,x],[b,x]]=1 \text{ as required.}$$

<u>Case (ii)</u> $[a,x]\in AB$:-

In this case we note that, since

$$abab=a'xb',$$

we have that

$$a'aba=xb'b,$$

and so

$$[b,aa']=[b,x]b'^{x}xa'$$

i.e.

$$[aa',b]=a'xb'^{x}[b,x].$$

It follows from case (i) (with aa' in place of a etc.) that $[[aa',x],[b,x]]=1$, and now 8(4) implies that

$$[[a,x],[b,x]]=1 \text{ as required.}$$

<u>Case (iii)</u> $[aa',x]\notin AB$ and $[a,x]\notin AB$:-

In this case we can write

$[a,x]=\bar{a}x\bar{b}$ for some $\bar{a}\in A$ and $\bar{b}\in B$

and

$[aa',x]=\bar{a}'x\bar{b}'$ for some $\bar{a}'\in A, \bar{b}'\in B$.

<u>Claim</u>:- In this case $[a',x]^2=1$.

First of all we note that by 3.4.6(i) we have

$$[\bar{a}',x]=1,$$

and so

$$1=[[aa',x],\bar{a}']=[\bar{b}',\bar{a}']. \qquad (1)$$

Now

$$aa'xaa'x=\bar{a}'x\bar{b}'$$

and so (inverting)

$$aa'x=\bar{b}'^{x}\bar{a}'aa'.$$

It follows that

$$[x,a]=[aa'x,a]=[\bar{b}'^{x}\bar{a}'aa',a]=[\bar{b}'^{x},a]^{\bar{a}'aa'}=[\bar{b}'^{x},a]^{aa'} \quad \text{(by (1))}$$
(2).

Also

$$[aa',x]=(aa'x)^{2}=[\bar{b}'^{x},\bar{a}'aa']=[\bar{b}'^{x},aa'],$$

and so

$$[aa',x]=[\bar{b}'^{x},a'][\bar{b}'^{x},a]^{a'}=[\bar{b}'^{x},a'][a,x] \qquad \text{(by (2))}.$$

It follows that

$$[\bar{b}'^{x},\bar{a}]=[aa',x]\,[x,a]=([aa',x]^{a}[a,x])^{a}=[a',x]^{a}, \qquad (3)$$

but by 3.4.6(i) $\bar{a}$ commutes with x and hence with $\bar{b}$ and $\bar{b}'$, and so

$$[aa',x]\,[x,a]=\bar{a}'x\bar{b}'\bar{b}x\bar{a}=\bar{a}'\bar{a}(\bar{b}'\bar{b})^{x}\in AB,$$

and it follows from 3.4.4 that
applied to $\bar{b}'$ and a

$$1=[[\bar{b}'^{x},\bar{a}],\bar{a}]=[[a',x]^{a},\bar{a}]=([x,\bar{a}]^{2})^{a},$$

which proves the claim.

We have that

$$abab=a'xb',$$

and so

$$ba=a'axbb'$$

hence

$$ab'=baa'xb,$$

so that

$$[a,b']=[aa',x]^{b},$$

and it follows that

$$[b',\bar{a}'][b',x]^{\bar{a}'}=[b',x\bar{a}']=[b',\bar{b}'x\bar{a}']=[b',[x,aa']]=[b',[b',a]^{b}]=([b',a]^{2})^{b}=$$
$$[aa',x]^{-2}=[\bar{b}',x]^{-1}=[\bar{b}',x] \text{ (by 3.4.6(ii)).}$$

We see from this that

$$[\bar{a}',b']=[\bar{b}',x]^{\bar{a}'}[b',x]=[\bar{b}',x]\,[b',x]\in AB,$$

and so by 3.4.4 we have that

$$[\bar{a}',[b',x]]=[\bar{a}',[\bar{b}',x]\,[b',x]]=1 \qquad (4)$$

Also, by the claim and 7(6), we know that

$$[[a',x],bb']=[x,a']^{2}=1. \qquad (5)$$

Now

$$abab=a'xb',$$

and so

$$bb'abb'a=b'aa'xa=b\bar{b}'^{x}\,\bar{a}'a',$$

i.e.

$$[a,bb']=\bar{a}'a'\bar{b}'\bar{b}'^{x}\in AB \qquad (6).$$

It follows from 3.4.4 that

$$[[a,bb'],bb']=1,$$

and so certainly

$$[[a,bb'],bb',x]=1.$$

Also we have by (5) and (6) that

$$[[a,bb'],x,bb']=[[a',x]^{b'\bar{b}'^{x}}[b'\bar{b}'^{x},x],bb']=1.$$

It follows, from the Hall-Witt identity, that

$$[[a,bb'],[bb',x]]=1,$$

and so from (6) we obtain

$$[\bar{a}'a',[bb',x]]=1,$$

whence, from 7(4), we have that

$$1=[\bar{a}',[b',x]\ [b,x]]=[\bar{a}',[b,x]]\ \text{(from (4))}=[\bar{b}'x\bar{a}',[b,x]],$$

and so

$$[[aa',x],[b,x]]=1 \; ,$$

and finally from 8(4) we deduce that

$$[[a,x],[b,x]]=1 \; ,$$ as required.

□

We now prove the main result of this section.

<u>**Thorem 3.4.10**</u> Let $G=AXB$ be a finite group such that :-

(i) A and B are elementary abelian 2-groups,

(ii) AB is not a group, (so that $G=\langle A,B\rangle$)

(iii) $X=\langle x\rangle$ is a cyclic group of order 2,

(iv) $B^X=B$,

(*) (v) BX is a nilpotent group of class at most 2.

Then $[[A,X],[B,X]]=1$.

<u>**Proof**</u> Suppose the theorem is not true and let $G=AXB$ be a counter example to the theorem of minimal order.

It is clear that $[B,X]\neq 1$ and so using Lemma 3.4.1 we may deduce that G has a central element, z say, of order 2 contained in AB. Since $\langle z\rangle \leq AB$ it follows that in the factor group $G/\langle z\rangle$ the subgroups $(A\langle z\rangle)/\langle z\rangle$ and $(B\langle z\rangle)/\langle z\rangle$ do not permute and also it is clear that the subgroup $(X\langle z\rangle)/\langle z\rangle)$ has order 2. So the group

(*) This hypothesis is redundant

$$G/\langle z\rangle=(A\langle z\rangle/\langle z\rangle)(X\langle z\rangle/\langle z\rangle)(B\langle z\rangle/\langle z\rangle)$$

satisfies the hypotheses of the theorem and hence, by the minimality of G, we must have that

$$[[A,X],[B,X]] \leq \langle z\rangle.$$

Let a_1 and b_1 be two elements of G for which $[[a_1,x],[b_1,x]]\neq 1$.

We note immediately that, by Proposition 3.4.9, we must have that

$$[a_1,b_1] = a_1'b_1' \text{ for some } a_1'\in A \text{ and } b_1'\in B. \quad (1)$$

We now define two subsets L_A and L_B, of A and B respectively, as follows:-

$$L_A = \{a\in A \text{ such that } aB\subseteq BA\},$$

$$L_B = \{b\in B \text{ such that } bA\subseteq AB\},$$

and note that both these sets are closed under taking products so that in fact both of these sets are subgroups (as G is finite); in particular if $b\in L_B$, then $b^{-1}\in L_B$ and it follows that $Ab\subseteq BA$.

Claim There exists an element $h\in B\setminus L_B$ such that $[[a_1,x],[h,x]]\neq 1$.

If $b_1 \notin L_B$ then the claim is certainly true so suppose that $b_1 \in L_B$. Since $AB \neq BA$ we can find elements $a \in A$ and $b \in B$ such that

$ab = b'xa'$ for some $a' \in A, b' \in B$.

Since $b_1 \in L_B$ we have that

$$abb_1 = b'xa'b_1 = b''xa'' \qquad \text{for some } a'' \in A, b'' \in B,$$

and in particular $bb_1 \notin L_B$. Now $[b_1,x] = [bb_1,x]\,[b,x]$ and so clearly either

$$[[a_1,x],[bb_1,x]] \neq 1$$

or

$$[[a_1,x],[b,x]] \neq 1,$$

and in either case the claim is true.

In light of the claim we may assume that $b_1 \notin L_B$. Either $a_1 \in L_A$ or $a_1 \notin L_A$; we consider the two cases separately.

<u>Case 1</u> $a_1 \in L_A$:-

First of all we note that $[B,X] \leq L_B$ from lemma 3.4.2 and so it follows from Lemma 3.4.5 that for any $b \in B$ $[a_1,[b,x]]$ is in the centre of $[A,B]=G'$. (*)

<u>Claim</u> $[G,b_1'] \le AB$ (recall $[a_1,b_1]=a_1'b_1'$).

Either $G=\langle B,X,a_1 \rangle$ or $G \neq \langle B,X,a_1 \rangle$.

If $G=\langle B,X,a_1 \rangle$, then we recall that from lemma 3.4.4 we have

$[a_1,b_1']=1$

and it follows that

$[G,b_1']=[x,b_1']^G \le AB$ by lemma 3.4.2.

If $G \neq \langle B,X,a_1 \rangle$ then let M be a maximal subgroup of G containing $\langle B,X,a_1 \rangle$. Using Dedekind's intersection lemma we can write

$$M=BX(M \cap A)$$

and so if B and $M \cap A$ do not permute then by the minimality of G we must have that

$$[[M \cap A,X],[B,X]]=1$$

and in particular

$$[[a_1,x],[b_1,x]]=1$$

which is a contradiction. Hence B and $M\cap A$ permute.

We note that $B(M\cap A)\lhd M$ as $|M:B(M\cap A)|\leq 2$. Let $J=(B(M\cap A))_G$. Suppose that a_m is an element of $A\cap M$ and let $g=abx^{\alpha}$ be an element of G,
where $a\in A, b\in B$ and α is 0 or 1 then

$$a_m{}^{g} = a_m{}^{bx^{\alpha}} \in (B(M\cap A))^{x^{\alpha}} = B(M\cap A).$$

It follows that $A\cap M \leq J \lhd G$. Now $a_1\in M\cap A$ and $a_1'=[a_1,b_1]b_1'\in M\cap A$ and so $b_1'=a_1'[a_1,b_1]\in J$. It follows that $[G,b_1']\leq J\leq AB$ and the claim is proved.

We know that $b_1\notin L_B$ and so we can find an element $a\in A$ such that

$$a^{b_1} = a'xb' \text{ for some } a'\in A\ ,\ b'\in B.$$

By Proposition 3.4.9

$$[[a,x],[b_1,x]]=1\ . \qquad (2)$$

From (1) we see that

$$(aa_1)^{b_1}=(a_1a)^{b_1} = a_1a_1'b_1'a'xb' = a_1a_1'a'[a',b_1']b_1'xb'$$

and so using the claim above it is easy to see that we can write

$$(aa_1)^{b_1} = a''xb'' \qquad \text{for some } a''\in A, b''\in B$$

and hence, from Proposition 3.4.9, it follows that

$$[[a,x]^{a_1}[a_1,x],[b_1,x]] = [[aa_1,x],[b_1,x]] = 1, \qquad (3)$$

But

$$[[a,x]^{a_1},[b_1,x]] = [[a,x],[b_1,x]^{a_1^{-1}}]^{a_1} = [[a,x],[[b_1,x],a_1]]^{a_1} \text{ (using(2))}$$
$$= 1 \quad \text{from } (*).$$

So (3) implies that $[[a_1,x],[b_1,x]] = 1$, a contradiction, which finishes the proof in Case 1.

Case 2 $a_1 \notin L_A$:-

Let M be a maximal subgroup of G containing BX.

Suppose that $a_1 \in M = BX(A\cap M)$, then since $a_1 \notin L_A$ it follows that B and $A\cap M$ don't permute and hence, because of the minimality of G, we have

$$[[A\cap M,X],[B,X]]=1,$$

and in particular,

$$[[a_1,x],[b_1,x]]=1$$

which is a contradiction.

So we may assume that $a_1 \notin M$ and hence that

$$G = M \,]\, \langle a_1 \rangle.$$

Claim $(M \cap A)\, B \neq B\, (M \cap A)$:-

We know that $b_1 \notin L_B$ so we can find an element $a \in A$ for which

$$ab_1 = b_1^{*} x\, a_1^{*} \qquad \text{for some } a_1^{*} \in A, b^{*} \in B.$$

Either $a \in M \cap A$ (which implies the claim from the above) or

$$a = a_m a_1 \qquad \text{for some } a_m \in M \cap A,$$

and in this case we have

$$a_m a_1 b_1 = b_1^{*} x\, a_1^{*},$$

and so

$$a_m b_1 [b_1, a_1] a_1 = b_1^{*} x\, a_1^{*},$$

whence (from (1))

$$a_m b_1 b_1' = b_1^{*} x\, a_1^{*} a_1 a_1',$$

so that the claim is also true if $a \notin A \cap M$.

The above claim together with the minimality of G implies that

$$[[M \cap A, X],[B, X]] = 1. \qquad (4)$$

We are assuming that $a_1 \notin L_A$, and so we can find an element $b \in B$ for which

$$a_1 b = b'xa' \qquad \text{for some } b' \in B, a' \in A.$$

By proposition 3.4.9 we have that

$$[[a_1, x],[b, x]] = 1, \qquad (5)$$

and also, since

$$b_1' b_1 b' x a' = b_1' b_1 a_1 b = b_1' [b_1, a_1] a_1 b_1 b = a_1' a_1 b_1 b,$$

we must (using proposition 3.4.9 again) have that

$$[[a_1' a_1, x],[b_1 b, x]] = 1,$$

and hence

$$[[a_1'a_1,x],[b_1,x]]^{[b,x]}=[[b,x],[a_1'a_1,x]].$$

Whence, conjugating by $[b,x]$, we obtain

$$[[a_1'a_1,x],[b_1,x]]=[[a_1'a_1,x],[b,x]]\,. \qquad (6).$$

<u>Claim 1</u> $[[a_1,x],[b_1,x]]=[[b_1,x],[b,x]^{a_1}]$:-

First we note that, since

$$[a_1,b_1]=a_1'b_1',$$

it follows that

$$a_1a_1'=b_1a_1b_1b_1';$$

and hence

$$[a_1a_1',x]=[b_1,x]^{a_1b_1b_1'}[a_1,x]^{b_1b_1'}[b_1b_1',x]. \qquad (7)$$

Also, by corollary 3.4.3,

$$[a_1,[b_1,x],[b_1,x]]=([b_1,x]a_1)^4=1. \qquad (8)$$

Combining (7) and (8) we obtain

$$
\begin{aligned}
[[a_1a_1',x],[b_1,x]] &= [[b_1,x]^{a_1b_1b_1'}[a_1,x]^{b_1b_1'}[b_1b_1',x],[b_1,x]] \\
&= [[a_1,x],[b_1,x]]^{b_1b_1'[b_1b_1',x]} \\
&= [[a_1,x],[b_1,x]] \quad (\in Z(G)),
\end{aligned}
$$

and so, using (6) and (7), we obtain

$$
\begin{aligned}
[[a_1,x],[b_1,x]] &= [[a_1a_1',x],[b,x]] \\
&= [[b_1,x]^{a_1b_1b_1'}[a_1,x]^{b_1b_1'}[b_1b_1',x],[b,x]] \\
\text{(using (5))} \quad &= [[b_1,x]^{a_1},[b,x]] \\
&= [[b_1,x],[b,x]^{a_1}] \quad \text{(since } [[a_1,x],[b_1,x]] \text{ is central)},
\end{aligned}
$$

proving claim 1.

<u>Claim 2</u> $[[b_1,x],[b,x]^{a_1}]=1$:-

Using the Hall-Witt identity we see that it is enough to show that

a) $[[b,x]^{a_1},x,b_1]=1$,

and

b) $[[b,x]^{a_1},b_1,x]=1$.

a) $[[b_1,x]^{a_1},x]=[[b_1,x],x^{a_1}]^{a_1}=[[b_1,x],[x,a_1]]^{a_1}\in Z(G)$, and so certainly

$$[[[b_1,x]^{a_1},x],b_1]=1 \qquad \text{as required.}$$

b) $[[b,x]^{a_1},b_1]=[[b,x],a_1,b_1]$,

and applying the Hall–Witt identity we obtain

$$\begin{aligned}[[b,x],a_1,b_1]&=[[b,x],[a_1,b_1]]^{b_1a_1}\\ &=[[b,x]^{a_1},[a_1,b_1]],\end{aligned}$$

but $[a_1,b_1]\in AB$, and so by lemmas 3.4.2 and 3.4.5 we have that

$$[[a_1,b_1],[a_1,[b,x]]]=1,$$

and it follows that

$$\begin{aligned}[[b,x]^{a_1},b_1]&=[[b,x],[a_1,b_1]]\\ &=[[b,x],a_1']\end{aligned}$$ $([a_1,b_1]=a_1'b_1'=b_1'a_1'$ by Lemma 3.4.4(i)).

Now $a_1'=[a_1,b_1]b_1'\in M\cap A$ (M contains B and G'), and so, by (4) we have that

$$[[a_1',x],[b,x]]=1.$$

Hence using this and the Hall-Witt identity we obtain

$$[[b,x]^{a_1},b_1,x]=[[b,x],a_1',x]\sim[[b,x],[a_1',x]]=1,$$

proving b) and claim 2.

Claims 1 and 2 together show that

$$[[a_1,x],[b_1,x]]=1,$$

which is a contradiction that proves the theorem in case 2 and hence completes the proof of the theorem. □

<u>Corollary 3.4.11</u> Any group G which satisfies the hypotheses of Theorem 3.4.10 has derived length at most 3

<u>Proof</u> First of all we note that if we set

$$L=[B,X]^G=[B,X]^A,$$

then the factor group G/L is metabelian as it is the product of two abelian groups. So in order to prove the corollary it is enough to show that L is abelian.

For any two elements $a\in A$ and $b\in B$ we have that

$$[[b,x]^a,x]=[[b,x],x^a]^a=[[b,x],[x,a]]^a=1 \quad \text{(by Theorem 3.4.10)}$$

and so

$$[[B,X]^G,X]=1, \qquad (1)$$

Now L′ is normally generated by elements of the form

$$[[b,x]^a,[b',x]] \qquad \text{where } b,b'\in B \text{ and } a\in A,$$

but from (1) we have that

$$[[b,x]^a,x,b']=1,$$

and

$$[[b,x]^a,b',x]\in[[B,X]^G,X]=1,$$

and so using the Hall–Witt identity, we can deduce that

$$[[b,x]^a,[b',x]]=1,$$

which proves that $L'=1$ and the corollary follows. □

§3.5 Examples

Chapter 3

In this section we present a few examples of groups which have similar [illegible] to those we have been examining in the previous sections.

All the groups are of the form $G=AXB$ where A, X and B are elementary abelian p-groups for some prime p. Also in each of the following groups we shall be assuming that X normalizes A and B, that both AX and BX are nilpotent of class at most 2 and that $A \cap B = A \cap X = B \cap X = 1$.

We will give a complete description of the groups in a list of (defining) properties as follows :-

(i) the prime p is ... ;

(ii) the (independent) generators of A, B and X are

$A=\langle a_1,\ldots,a_n\rangle$, $B=\langle b_1,\ldots,b_m\rangle$ and $X=\langle x,y\rangle$;

(iii) the action of X on A and B given by a list of the non-trivial commutators of each generator of A and B with each element of X;

(iv) a commutator table giving the commutators of each generator of A with each generator of B.

The verification, that each of the examples below is in fact a group with the required structure (i.e. that the presentation does not collapse), was done using Von Dyck's Theorem and is not included here, as it only involves routine calculations .

Example 3.5.1 A 2-group $G=AXB$ with $X \simeq C_2$ and $AB \neq BA$ which has derived length 3.

(i) $p=2$,

(ii) $A=\langle a_1, a_2, a_3 \rangle$, $B=\langle b_1, b_2, b_3 \rangle$ and $X=\langle x \rangle$,

(iii) $[a_1, x]=a_2$, $[b_1, x]=b_2$, (x commutes with the other generators),

(iv)

$[\,,\,]$	b_1	b_2	b_3
a_1	1	1	b_1
a_2	1	1	b_2
a_3	$a_2 b_2$	1	x

Note $G''=\langle b_2 \rangle$.

Example 3.5.2 A 2-group $G=AXB$ for which $X=\langle x,y \rangle \simeq C_2 \times C_2$, $AB \neq BA$ and $[[A,\langle x \rangle],[B,\langle y \rangle]] \neq 1$

(i) $p=2$,

(ii) $A=\langle a_1, \ldots, a_5 \rangle$, $B=\langle b_1, \ldots, b_8 \rangle$ and $X=\langle x,y \rangle$,

(iii) $[a_1,x]=a_2$, $[a_1,y]=a_4$, $[a_3,y]=a_5$, $[b_1,x]=b_2$, $[b_1,y]=b_4$, $[b_3,y]=b_8$ and $[b_6,y]=b_7$

(iv)

$[\ ,\ ]$	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
a_1	a_3xb_3	b_6	a_2b_6	a_5b_5	1	1	1	b_7
a_2	b_6	1	1	b_7	1	1	1	1
a_3	b_2b_6	1	b_6	b_8	b_7	1	1	1
a_4	b_5	b_7	1	1	1	1	1	1
a_5	b_8b_7	1	b_7	1	1	1	1	1

Note $[[a_1,y], [b_1,x]] = b_7 \neq 1$.

Example 3.5.3 A 2-group $G=AXB$ for which $X=\langle x,y\rangle \cong C_2 \times C_2$, $AB \neq BA$ and $[[A,\langle y\rangle],[B,\langle y\rangle]] \neq 1$.

(i) $p=2$.

(ii) $A=\langle a_1,\ldots,a_5\rangle$, $B=\langle b_1,\ldots,b_6\rangle$ and $X=\langle x,y\rangle$.

(iii) $[a_1,x]=a_2$, $[a_5,y]=a_4$, $[b_1,x]=b_2$, $[b_6,y]=b_2$ and $[b_5,y]=b_2$.

(iv)

[,]	b_1	b_2	b_3	b_4	b_5	b_6
a_1	1	1	b_1	1	1	1
a_2	1	1	b_2	1	1	1
a_3	a_2b_2	1	x	1	1	1
a_4	1	1	1	b_2	b_6	1
a_5	1	1	1	b_2b_6	1	1

<u>**Note**</u> $[[a_5,y],[b_5,y]]\neq 1$.

<u>**Example 3.5.4**</u> A p-group $G=AXB$ in which $X\cong C_p \times C_p$ and $[[A,X],[B,X]]\neq 1$.

(i) p is any prime.

(ii) $A=\langle a_1,a_2,a_3,a_4\rangle$, $B=\langle b_1,\ldots,b_{12}\rangle$ and $X=\langle x,y\rangle$.

(iii) $[a_1,x]=a_2$, $[a_1,y]=a_3$, $[b_1,x]=b_2$, $[b_1,y]=b_3$, $[b_5,x]=b_4$, $[b_7,x]=b_6$, $[b_8,y]=b_4$, $[b_9,y]=b_6$, $[b_{11},x]=b_{10}$ and $[b_{11},y]=b_{12}$.

(iv)

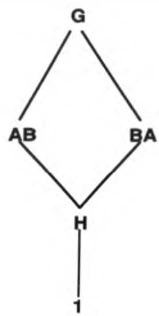
[,]	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}
a_1	1	b_8^{-1}	b_5	1	1	1	1	1	1	b_9	1	b_7^{-1}
a_2	b_8	1	b_4	1	1	1	1	1	1	1	b_9^{-1}	b_6^{-1}
a_3	b_5^{-1}	b_4^{-1}	1	1	1	1	1	1	1	b_6	b_7	1
a_4	b_{11}^{-1}	b_{10}^{-1}	b_{12}^{-1}	b_6	b_7	1	1	b_9	1	1	1	1

Note $[[b_1, x], [a_1, y]] = b_4^{-1} \neq 1$.

§4 Products of Subgroups which are not Groups

§4.1 Introduction

In this section we investigate the situation where we have a group G with three subgroups H, A and B such that H is contained in the product AB which need not be a group.



Given this situation it is natural to wonder about what effect the structures of A and B have upon the structure of H and whether or not theorems for products of subgroups which are groups will generalize to this situation (for example if A and B are nilpotent must H be soluble?).

The following result, due to Stonehewer and Busetto, generalizing Itô's Theorem gives some hope that other generalizations might be true.

Theorem (Stonehewer and Busetto) Suppose G is a group with subgroups A,B and H such that A and B are abelian and $H \leq AB$. Then H is metabelian.

The proof of this theorem is a generalized version of the proof of Itô's Theorem.

In this chapter we construct several examples of groups which show that certain theorems for factorized groups do not generalize in the above fashion and also we prove some elementary results concerning the exponent of a subgroup contained in the product of two abelian subgroups.

Notation Whenever we have a group G with subgroups A, B and X such that $X \leq AB$ we will use the following notation :-

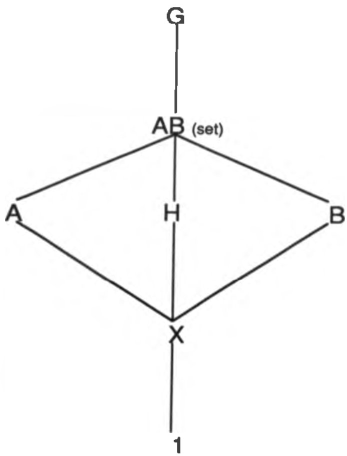
$$\pi_A(X) = \{a \in A \mid \exists\, b \in B \text{ such that } ab \in X\},$$

$$\pi_B(X) = \{b \in B \mid \exists\, a \in A \text{ such that } ab \in X\}.$$

§4.2 Wielandt's Theorem

In this section we consider the following situation in which we have a product of subgroups that is not a group:-

(∗) G is a finite group with subgroups A,B,H,X such that $X \leq H \subset AB$ (here AB is a set), X sn A and X sn B.



First of all we give an example to show that X need not be subnormal in H and then we give certain extra conditions which are sufficent to ensure that X is subnormal in G.

__Example 4.2.1__ We construct a group G satisfying (*) in which X is not subnormal in H.

Let K be the field of three elements and let W be the vector space over K consisting of all the 2-dimensional column vectors with coefficients in K. Let

$$u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and let A be the subgroup of GL(2,K) generated by the elements

$$a = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

then $A = \langle a, x \mid a^8 = x^2 = 1,\ a^x = a^3 \rangle$ is a group of order 16 (it is in fact a Sylow 2-subgroup of G(2,K)).

Let V=<u,v> be the additive group of W which we will write multiplicatively for convenience. Then A acts on V in a natural way i.e. u^a is the product of the 2x2 matrix a with the column vector u so we have

$$u^a = v,\ v^a = uv^{-1},\ u^x = u \text{ and } v^x = u^{-1}v^{-1}.$$

We take G to be the semidirect product V] A.

Let $b = u^{-1}a^{-1}u$ and define B to be the group $\langle b,x\rangle = A^{u}$ (since x commutes with u). We note the following

$a^{-1}b^{-1} = u^{-a}u = uv^{-1}$, $a^{-3}b^{-3} = u^{-a^{3}}\ u = u^{-1}v$ and $(uv^{-1})^{x} = u^{-1}v$.

Now let $h = uv^{-1}$ so that ,by the above, $\langle h\rangle \leq AB$ and let

$$H = \langle h,x \rangle \simeq S_3,$$

then clearly $H = \langle h\rangle\langle x\rangle \leq AB$.

Finally if we let $X = \langle x\rangle$ then we have that $X \leq H \leq AB$, X is subnormal in A and in B but X is not subnormal H. □

To end this section we give some conditions under which the subgroup X in situation (∗) (with G soluble as well) must be subnormal in H.

Proposition 4.2.4 Suppose G is a finite soluble group with subgroups A,B,H and X such that $X \leq A\cap B$, $X \leq H \leq AB$, X is cyclic with order a power of a prime and $[A,X,X] = [B,X,X] = 1$. Suppose also that at least one of the following conditions holds:-

(i) $A\cap B \leq X$

(ii) X is permutable in B (i.e. for any subgroup B_1 of B we have that XB_1 is a group),

(iii) $[y, b, b] = 1$ for all $y \in X$ and $b \in B$.

Then X sn H.

<u>**Proof**</u> Let G be a counter-example to the theorem of minimal order for which the order of H is also minimal.

Clearly $G = \langle A,B \rangle$. Let V be a minimal normal subgroup of G. By the minimality of G we must have that

$$(XV)/V \text{ sn } (HV)/V,$$

and it follows that $X(V \cap H)$ sn H and hence that X is not subnormal in $(V \cap H)X \leq AB$ and hence, by the the minimality of $|H|$ we must have that

$$H = (V \cap H)X.$$

Also it is clear, that if p^n is the order of X, where p is a prime, then V must be an elementary abelian q-group for some prime $q \neq p$. So, using Lemma 2.3.1, we can deduce that

$$(V \cap H) = [V \cap H, X] \times C_{V \cap H}(X),$$

and hence X is not subnormal in $[V \cap H, X]X$, so that, by the minimality of H, we must have that

$$V \cap H = [V \cap H, X],$$

whence we have that

$$C_{V \cap H}(X) = 1.$$

It is clear that we may assume that

$$A = \langle \pi_A(V \cap H) \rangle \ (\geq X) \text{ and } B = \langle \pi_B(V \cap H) \rangle,$$

and so, since both $\pi_B(V \cap H)$ and $\pi_A(V \cap H)$ are contained in $AV \cap BV$, it follows that

$$G = \langle A,B \rangle = AV = BV,$$

and hence

$$A \cap V = B \cap V = 1 \quad (\text{since } A \cap V \triangleleft G \text{ and clearly } V \neq V \cap A).$$

Suppose h is a non-identity element of $V \cap H$ then we can find elements a and b of A and B respectively such that $1 \neq h = ab$.
Let $X = \langle x \rangle$.

Claim There exists a positive integer r such that $[h,_r x] = h$:-

clearly the sequence

$$[h,_0 x] = h,\ [h,x],\ [h,_2 x],\ \ldots$$

repeats itself after finitely many steps. Let r be the least positive integer for which we can find an integer m with $0 \leq m < r$ such that

$$[h,_r x] = [h,_m x].$$

To prove the claim it is enough to show that m=0. Suppose not, so that m is a positive integer, then we have that

$$[[h,_{r-1} x],x] = [[h,_{m-1} x],x] = [\, x,([h,_{m-1} x])^{-1}]^{[h,_{m-1} x]},$$

and so, since V is abelian, we have that

$$1 = [[h,_{r-1} x],x]\,[([h,_{m-1} x])^{-1},x] = [\,[h,_{r-1} x]\,([h,_{m-1} x])^{-1},x]\,,$$

but $C_{V \cap H}(\langle x \rangle) = 1$ and so we must have that

$$[h,_{r-1} x] = [h,_{m-1} x]\,,$$

contradicting our choice of r. It follows that m=0 and so $[h,_r x] = h$ which proves the claim.

Now $[h,x] = [a,x]^b\,[b,x]$ which commutes with x^b, and so, since x^b also commutes with x, it follows that for any n>0

$$[[h,_n x],x^b] = 1\,,$$

and so using the claim we can deduce that

$$[h,x^b] = 1.$$

Now

$$[ba,x] = [h^{b^{-1}},x] = 1,$$

and so we have that

$$[b,x]^{a}[a,x] = 1,$$

and hence

$$[b,x] = [a^{-1},x].$$

<u>Claim</u> Each of the three conditions mentioned in the statement of the proposition now suffices to ensure that $[b^{-1},x] \in A$:-

(1) if condition (i) holds then we have that

$$[b,x] = [a^{-1},x] \in A \cap B \leq X,$$

and so

$$[b,x] = x^{\alpha} \text{ for some } \alpha > 0.$$

It follows that b, and hence b^{-1}, normalises X so that

$$[b^{-1},x] \in X \leq A.$$

(2) If condition (ii) holds then we have that

$$x^{b^{-1}} = b^{\alpha}x^{\beta} \quad \text{for some positive integers } \alpha \text{ and } \beta,$$

and so

$$[b^{-1},x] = [x,b]^{b^{-1}} = [x^{\beta},b] = [x,b]^{\beta} = [x,a^{-1}]^{\beta} \in A \text{ as required.}$$

(3) If condition (iii) holds then we have that

$$[b^{-1},x] = [x,b]^{b^{-1}} = [x,b] \in A \text{ finishing the proof of the claim.}$$

Finally we have that

$$[h,x] = [a,x]^{b}[b,x],$$

and so conjugating by b^{-1} and using the claim we have

$$[a,x][x,b^{-1}] \in V \cap A = 1,$$

and so $[h,x] = 1$, which implies that $h = 1$ giving us a contradiction that finishes the proof of the proposition. □

<u>**Corollary 4.2.3**</u> Suppose G is a finite soluble group with subgroups A,B,X,H such that $X \leq H \leq AB$, $X \leq A \cap B$, $[A,X,X]=[B,X,X]=1$ and $[y,b,b]=1$ for all $y \in X$ and $b \in B$. Then X sn

H.

Proof Since X is the join of a finite number of cyclic groups of prime power order each satisfying the hypotheses of Proposition 2.4.2 the corollary follows immediately from Proposition 2.4.2 and Wielandts Join Theorem (Satz 7 of Wielandt[2]) which states that in a finite group the join of any two subnormal subgroups is subnormal.

□

§4.3 Products of Cyclic Subgroups

It is well known that if a group can be expressed as a product of two cyclic subgroups then it is supersoluble. In this section we construct an example to show that a subgroup contained in a product of cyclic sugroups need not be supersoluble.

Example 4.3.1 (with S.E. Stonehewer) A finite soluble group $G=\langle A,B\rangle$ with A and B cyclic subgroups which has a subgroup X which is contained in the product AB but which is not supersoluble.

Let K be a field containing 16 elements which has minimal polynomial

$$r^4+r+1=0.$$

Let ø be a root of this polynomial.

Let V be the additive group of K (which we will write multiplicatively for convenience), so that V is an elementary abelian 2-group with basis u, v, w and x where

$u=1,\ v=\emptyset,\ w=\emptyset^2$ and $x=\emptyset^3$.

Let $A=\langle a\rangle$ be the multiplicative group of K i.e. a cyclic group of order 15 and let A act on V in the obvious fashion (i.e. '$u^a=u\emptyset$') so that

$$u^a=v,\ v^a=w,\ w^a=x,\ x^a=uv\ ('=\emptyset^4').$$

We define G to be the semi-direct product $G=V\rtimes A$.

We now define b to be $ua^{-1}u$ and let $B=\langle b\rangle$, a cyclic group of order 15, and note that, since V is a normal subgroup of G, we must have that

$$V\cap AB=\{a^ib^i \mid 0\le i\le 15\},$$

i.e. $V\cap AB$ contains every element of V except one (since the a^ib^i are clearly all distinct as $A\cap B=1$). By direct computation, we can show that

$$V\cap AB = V\setminus\{u\}.$$

Now we can show that

$$v^{a^5}=wx \text{ and } v^{a^{10}}=vwx$$

and hence the subgroup $\langle v,wx\rangle$ is normalized by a^5. It follows that if we set

$$X = \langle v, wx \rangle \rangle\!\!\!\lhd \langle a^5 \rangle,$$

Then X is contained in AB and furthermore X is clearly isomorphic to A_4 and is not supersoluble. □

The construction of the above example can be generalized to show that there is no bound on the rank* of a subgroup of a group contained in the product of two cyclic subgroups unlike the case where we consider a group that can be expressed as a product of product of cyclic subgroups (see for example Robinson[1]).

* Here we use the rank of a group G to mean the smallest positive integer r(G) such that every finitely generated subgroup of G can be generated by r(G) elements.

Example 4.3.2 (with S.E. Stonehewer) A finite soluble subgroup $G=\langle A,B\rangle$ with A and B cyclic subgroups which has a subgroup X which is contained in the product AB, and which has rank r, where r can be any positive integer.

Let K be a field of q^r elements for some prime q. Let V be the additive group of K and let $A=\langle a\rangle$ be a cyclic group of order q^r-1 which is acting on V in the manner described in example 4.3.1 . Set $G=V]A$ and let $b=u^{-1}au$ where u is any non-trivial element of V. If we let $B=\langle b\rangle$ then as in Example 4.3.1 we have that

$$V \cap AB = \left\{ a^i b^i \mid 0 \leq i \leq q^r - 1 \right\} = V \setminus \left\{ u \right\},$$

and so clearly AB must contain an elementary abelian q-subgroup of V of rank r-1, as required. □

§4.4 Exponents in Products of Subgroups

In Howlett[1] it is shown that if a group G can be expressed as a product of two abelian subgroups A and B, where A has exponent e and B has exponent f then the exponent of G divides ef. In this section we investigate the situation that arises when in a group G, we have two abelian subgroups A and B and look at what effect the exponents of A and B have on the exponent of any subgroup X contained in the product AB. Clearly, the result of Howlett mentioned above does not generalise in as strong a form, for if we consider the alternating group A_4 we see that

$\langle(12)(34)\rangle \leq \langle(134)\rangle\langle(132)\rangle$. In this section we present some elementary results regarding this situation in soluble groups.

Proposition 4.4.1 Let G be a finite soluble group with subgroups A, B and X such that A and B are abelian and have coprime orders and X is contained in the product AB. Then if A is π_1-group and B is a π_2-group then X is a $\pi_1 \cup \pi_2$ group.

Proof Suppose the proposition is not true and let G be a counter-example of minimal order and let V be a minimal normal subgroup of G. By the minimality of G we have that the proposition is true in

G/V and so XV/V is a $\pi_1 \bigcup \pi_2$-group. It follows that $X/(X \bigcap V)$ is a $\pi_1 \bigcup \pi_2$-group and so we must have that $V \bigcap AB \supset V \bigcap X \neq 1$ and moreover that V is a $(\pi_1 \bigcup \pi_2)'$-group. So there exists elements a and b from A and B respectively such that

$$1 \neq ab \in V \bigcap AB,$$

but $a \in A \bigcap VB = 1$ (as A and BV have coprime order) and $b \in B \bigcap AV = 1$, and so we have a contradiction. □

__Proposition 4.4.2__ Let G be a finite soluble group with subgroups A, B and X such that A and B are abelian and X is contained in the product AB. Suppose that the exponent of A is e and the exponent of B is f. If q is any prime dividing the order of X then $q \leq \min\{e, f\}$.

__Proof__ Suppose the result is false, and let G be a counter-example of minimal order. We may assume, without loss of generality, that $e \leq f$ and also that X is a cyclic group of order q. We may further assume that $G = \langle A, B \rangle$ and that $A = \langle \pi_A(X) \rangle$ and $B = \langle \pi_B(X) \rangle$.

Let V be a minimal normal subgroup of G, then as the result holds in G/V it follows that we must have that $X \leq V$ and so V must be an elementary abelian q-group and, in fact, it must also be the unique minimal normal subgroup of G.

Now clearly, $\pi_B(X) \leq AV$ and so it follows that $G = AV$ and hence the centralizer of V in A being a normal subgroup of G must be 1.

We can regard V as an irreducible $\mathbb{Z}_q A$-module upon which A acts faithfully. Applying Schur's Lemma, we can deduce that $\mathrm{End}_{\mathbb{Z}_q A}(V)$ is a division algebra. Since A is abelian, we have that $\mathbb{Z}_q A$ is contained in this algebra, and hence A is isomorphic to a subgroup of the multiplicative group of a field; i.e A is cyclic and hence has order exactly e.

So we have shown that $|X|>|A|$ but $A \cap X = B \cap X = 1$, and so for any element $a \in A$ there exists at most one element $b \in B$ for which $ab \in X$ (for if there were another element $b' \in B$ with $ab' \in X$ then we would have that $b^{-1}b' = (ab)^{-1}(ab') \in B \cap X = 1$ and hence $b' = b$). But $\bar{X} \subset \bar{A}\bar{B}$ and $B \cap X = 1$ and so it follows that $|X| \leq |A|$ and we have a contradiction. $\square$

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