Infinite dimensional finitely forcible graphon*

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Abstract

Graphons are analytic objects associated with convergent sequences of dense graphs. Finitely forcible graphons, i.e., those determined by finitely many subgraph densities, are of particular interest because of their relation to various problems in extremal combinatorics and theoretical computer science. Lovász and Szegedy conjectured that the topological space of typical vertices of a finitely forcible graphon always has finite dimension, which would have implications on the minimum number of parts in its weak ε-regular partition. We disprove the conjecture by constructing a finitely forcible graphon with the space of typical vertices that has infinite dimension.

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1 Introduction

Analytic objects associated with convergent sequences of combinatorial objects have recently attracted significant amount of attention. This line of research was initiated by the theory of limits of dense graphs [7–9, 33], followed by limits of sparse graphs [5,15], permutations [24,25], partial orders [27] and others. Analytic methods applied to such limit objects led to results in many areas of mathematics and computer science, in particular in extremal combinatorics [1–4, 19, 21–23, 28, 29, 38–42] and property testing [26, 36].

In this paper we are concerned with limits of dense graphs and in particular with those determined by finitely many subgraph densities. This phenomenon, which is known as finite forcibility, is closely related to quasirandomness of combinatorial objects, whose study was initiated by Chung, Graham and Wilson [12], Rödl [43] and Thomason [45,46]. In the setting of graph limits, large dense graphs are represented by analytic objects called graphons and the just mentioned results assert that every constant graphon is finitely forcible. This result was generalized by Lovász and Sós [31], see also [44], who proved that every step graphon, which is a multipartite graphon with uniform densities between and within its parts, is finitely forcible.

We are interested in the structure of the space of typical vertices of finitely forcible graph limits. We consider two spaces of typical vertices, which we formally define in Section 2. One is the space studied in [32] and is denoted by $T(W)$; informally speaking, $T(W)$ is formed by the neighbor functions ("rows" of a graphon W) with the L^1-topology. The other, which is denoted by $\overline{T}(W)$, is the space studied in [30, Chapter 13], where the L^1-metric is replaced by a finer metric. The structure of the space $\overline{T}(W)$ is closely related to weak ε-regular partitions of W [30, 35]; in particular, if $\overline{T}(W)$ has finite Minkowski dimension, then W has a weak ε-regular partition with a number of parts polynomial in ε^{-1}. We note that there are graphons W such that the minimum number of parts in a weak ε-regular partition of W is exponential in ε^{-2} [13]. In particular, graphons W such that the Minkowski dimension of $\overline{T}(W)$ is finite have simple structure from the regularity decomposition point of view.

Lovász and Szegedy [32, Conjecture 10], led by examples of finitely forcible graphons that were known at that time, conjectured that the space of typical vertices of a finitely forcible graphon always has finite dimension. We cite the conjecture verbatim.

Conjecture 1. *If W is a finitely forcible graphon, then $T(W)$ is finite dimensional. (We intentionally do not specify which notion of dimension is meant here—a result concerning any variant would be interesting.)*

In this paper we construct a graphon $W_\boxplus$, which we call a *hypercubical* graphon, such that $W_\boxplus$ is finitely forcible and both $T(W_\boxplus)$ and $\overline{T}(W_\boxplus)$ contain subspaces homeomorphic to $[0, 1]^\infty$.

Theorem 1. *The hypercubical graphon $W_\boxplus$ is finitely forcible and the topological spaces $T(W_\boxplus)$ and $\overline{T}(W_\boxplus)$ contain subspaces homeomorphic to $[0,1]^\infty$ equipped with the product topology.*

Looking at one of the motivations for studying the dimension of the spaces $T(W)$ and $\overline{T}(W)$, we remark that every weak ε-regular partition of $W_\boxplus$ has at least $2^{\Theta(\log^2 \varepsilon^{-1})}$ parts. We further discuss the existence of finitely forcible graphons with no weak ε-regular partition with a small number of parts in the concluding section.

The proof of Theorem 1 extends the methods from [18] and [37]. In particular, Norine [37] constructed finitely forcible graphons with the space of typical vertices of arbitrarily large (but finite) Lebesgue dimension. In his construction, both $T(W)$ and $\overline{T}(W)$ contain a subspace homeomorphic to $[0,1]^d$. One of the contributions of this paper is showing how the techniques from [18] and [37] can be refined to force a subspace homeomorphic to $[0,1]^\infty$, which turned out to be quite challenging. Another contribution of the paper is formalizing the methods used in [18] and [37], which are further used in the follow up papers [10, 11, 20].

We finish with giving a brief outline of the proof of Theorem 1 in informal terms. As in [18], the constructed hypercubical graphon $W_\boxplus$ has several parts (see Figure 1), which are determined by the degrees of the vertices that are contained in the parts. The parts $A_1, \dots, A_3$ serve to further partition the parts $B_1, \dots, B_5$ into infinitely many smaller parts; the part B_1 is split into parts $B_{1,d}$, $d \in \mathbb{N}$. The structure between the parts A_1 and A_0 plays the role of identifying the first of the smaller parts and the structure between A_1 and A_3 links consecutive smaller parts. The part C serves to introduce coordinate systems on the parts $A_0, \dots, A_3$ and $B_1, \dots, B_5$. The structure between the parts B_1 and B_2 provides a d-dimensional coordinate system on $B_{1,d}$, $d \in \mathbb{N}$, and is used to arrange that $B_{1,d}$ induces a subspace homeomorphic to $[0,1]^d$. The d-dimensional structure of the parts $B_{1,d}$ is forced in an iterative (induction like) way, increasing the dimension by one at each step. The proof is concluded by forcing the parts $B_{1,d}$ to be "projections" of the part D; in this way, we arrange that the subspace associated with the part D is homeomorphic to $[0,1]^\infty$.

2 Definitions

In this section we present the notation that we use throughout the paper; this includes the notions from the theory of graph limits, which originated in [7–9, 33].

A *graph* is a pair (V, E) where $E \subseteq \binom{V}{2}$. The elements of V are called *vertices* and the elements of E are called *edges*. All graphs considered in this paper are simple, i.e., without loops and parallel edges. The *order* of a graph G is the number of its vertices and is denoted by $|G|$. We use $\mathbb{N}^*$ for $\mathbb{N} \cup \{\infty\}$ and $[k]$ for $\{1, \dots, k\}$.

The *density* of a graph H in a graph G, which is denoted by $d(H, G)$, is the probability that a random set of $|H|$ distinct vertices of G induce a subgraph isomorphic to H. If $|H| > |G|$, we define $d(H, G)$ to be zero. A sequence of graphs $(G_i)_{i\in\mathbb{N}}$ is *convergent* if the sequence $(d(H, G_i))_{i\in\mathbb{N}}$ converges for every graph H. In general, we will consider sequences of graphs with their orders tending to infinity.

Convergent sequences of graphs can be associated with an analytic limit object, which we will now introduce. A *graphon* W is a symmetric measurable function from $[0, 1]^2$ to $[0, 1]$. Here, *symmetric* stands for the property that $W(x, y) = W(y, x)$ for every $x, y \in [0, 1]$. Very imprecisely speaking, one can think of a graphon as of a continuous version of the adjacency matrix of a graph. Mimicking the terminology for graphs, we refer to a graphon W restricted to $S \times T$, where S and T are two measurable subsets of $[0, 1]$, as to a *subgraphon* of W induced by $S \times T$.

We next link graphons to convergent sequences of graphs. A *W-random graph* of order k is obtained by sampling uniformly and independently k random points $x_1, \ldots, x_k \in [0, 1]$, which are associated with the vertices, and by joining the vertices corresponding to x_i and x_j by an edge with probability $W(x_i, x_j)$. Because of this connection, we refer to the points of $[0, 1]$ as to the *vertices* of W. The *density* of a graph H in a graphon W is the probability that the W-random graph of order $|H|$ is isomorphic to H. The definition of a W-random graph yields the following:

$$d(H, W) = \frac{|H|!}{|\mathrm{Aut}(H)|} \int_{[0,1]^{|H|}} \prod_{(i,j)\in E(H)} W(x_i, x_j) \prod_{(i,j)\notin E(H)} (1 - W(x_i, x_j)) \, \mathrm{d}\lambda_{|H|} \; ,$$

where $\mathrm{Aut}(H)$ is the automorphism group of H. Our results do not depend on whether we work with Borel or Lebesgue measure on $[0, 1]^d$, and we have made a choice of working with the Borel measure throughout the paper, which is denoted by λ or by λ_d if we wish to emphasize the dimension of the support space. When we talk about the measure on $[0, 1]^{\mathbb{N}}$, we mean the product measure of the measures on $[0, 1]$.

One of the key results in the theory of graph limits asserts [33] that for every convergent sequence $(G_i)_{i\in\mathbb{N}}$ of graphs with increasing orders, there exists a graphon W, which is called the *limit* of the sequence, such that for every graph H,

$$d(H, W) = \lim_{i\to\infty} d(H, G_i) \; .$$

Conversely, if W is a graphon, then the sequence of W-random graphs with increasing orders converges with probability one and its limit is W.

Two graphons W_1 and W_2 are *weakly isomorphic* if $d(H, W_1) = d(H, W_2)$ for every graph H. If $\varphi : [0, 1] \to [0, 1]$ is a measure preserving map, then the graphon $W^\varphi(x, y) := W(\varphi(x), \varphi(y))$ is always weakly isomorphic to W. The opposite is true in the following sense [6]: if two graphons W_1 and W_2 are weakly isomorphic, then there exist measure preserving maps $\varphi_1 : [0,1] \to [0,1]$ and $\varphi_2 : [0,1] \to [0,1]$ such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere.

The *degree* $\deg^W x$ of a vertex $x \in [0,1]$ in a graphon W is defined as

$$\deg^W x = \int_{[0,1]} W(x,y)\mathrm{d}y \ .$$

Note that the degree is well-defined for almost every vertex of W. We omit the superscript W whenever the graphon is clear from context. Let A be a measurable non-null subset of $[0,1]$. The *relative degree* $\deg_A^W x$ of a vertex $x \in [0,1]$ with respect A is defined as

$$\deg_A^W x = \frac{\int_A W(x,y)\mathrm{d}y}{\lambda(A)} \ .$$

Fix a graphon W, $x, x' \in [0,1]$ and a measurable set $Y \subseteq [0,1]$. The set $N_Y(x)$ is the set of $y \in Y$ such that $W(x,y) > 0$ and

$$N_Y(x \setminus x') = \{y \in Y \mid W(x,y) > 0 \text{ and } W(x',y) < 1\}.$$

Informally speaking, $N_Y(x \setminus x')$ contains $y \in Y$ such that a vertex associated with y can be a neighbor of a vertex associated with x and a non-neighbor of a vertex associated with x' in a W-random graph. We note that, assuming that Y is measurable, $N_Y(x)$ and $N_Y(x \setminus x')$ are measurable for almost every x and almost every pair x and x', respectively.

As mentioned in the Introduction, the structure and the complexity of a graphon can be studied by analyzing a topological space associated with its typical vertices [34]. We now give the formal definitions of the two types of such spaces that we mentioned in the Introduction. For a graphon W and $x \in [0,1]$, define a function $f_x^W : [0,1] \to [0,1]$ to be

$$f_x^W(y) := W(x,y).$$

Since the function f_x^W belongs to $L^1([0,1])$ for almost every $x \in [0,1]$, the graphon W naturally defines a probability measure μ on $L^1([0,1])$. The space $T(W)$ is formed by the support of the measure μ equipped with the topology inherited from $L^1([0,1])$. A vertex x of the graphon W is called *typical* if $f_x^W \in T(W)$. Another topological space, which is denoted by $\overline{T}(W)$, can be defined using the notion of *similarity distance*. If f and g are two functions from $L^1([0,1])$, define

$$d_W(f,g) := \int_{[0,1]} \left| \int_{[0,1]} W(x,y)(f(y) - g(y))\mathrm{d}y \right| \mathrm{d}x \ .$$

Note that the similarity distance d_W depends on the graphon W. The space $\overline{T}(W)$ is formed by the closure (with respect to d_W) of the support of μ equipped with the topology given by the metric d_W. The structure of the space $\overline{T}(W)$ is related to weak regularity partitions of W; in particular, if the Minkowski dimension of $\overline{T}(W)$ is d, then W has a weak ε-regular partition with $O(\varepsilon^{-d})$ parts. We refer the reader to [30, Chapter 13] for further details.

2.1 Finite forcibility

A graphon W is *finitely forcible* if there exist graphs $H_1, \ldots, H_k$ such that every graphon W' satisfying $d(H_i, W) = d(H_i, W')$ for every $i \in [k]$ is weakly isomorphic to W. For example, the result of Diaconis, Holmes, and Janson [14] is equivalent to the statement that the half-graphon $W_\triangle(x, y)$, which is defined as $W_\triangle(x, y) = 1$, if $x + y \geq 1$, and $W_\triangle = 0$, otherwise, is finitely forcible. We refer the reader to [32] for further examples of finitely forcible graphons and to Section 6 for the discussion of some further results on finitely forcible graphons.

Following the framework from [18], when proving that a graphon is finitely forcible, we give a set of *constraints* that uniquely determines W rather than listing the finitely many graphs and their densities that uniquely determine W. A *constraint* is an equality between two density expressions, where a *density expression* is a formal real polynomial combination of graphs, i.e., a real number or a graph H are density expressions, and if D_1 and D_2 are two density expression, then the sum $D_1 + D_2$ and the product $D_1 \cdot D_2$ are also density expressions. A graphon W *satisfies* a constraint $D_1 = D_2$ if both D_1 and D_2 are equal when evaluated with each H substituted with $d(H, W)$. As it was observed in [18], if a graphon W is a unique (up to weak isomorphism) graphon that satisfies a finite set $\mathcal{C}$ of constraints, then the graphon W is finitely forcible. In particular, W is the unique (up to weak isomorphism) graphon with densities of graphs appearing in $\mathcal{C}$ equal to their densities in W.

In [18], it was also observed that a more general form of constraints, called *rooted constraints*, can be used to prove that a graphon is finitely forcible. A graph is rooted if it has m distinguished vertices labeled with numbers $1, \ldots, m$; these vertices are referred to as *roots* while the other vertices are *non-roots*. Two rooted graphs are *compatible* if the subgraphs induced by their roots are isomorphic through an isomorphism mapping the roots with the same label to each other. Similarly, two rooted graphs are isomorphic if there exists an isomorphism mapping the i-th root of one of them to the i-th root of the other; in particular, if two rooted graphs are isomorphic, then they are compatible.

A *rooted density expression* is a formal real polynomial combination of compatible rooted graphs. We next describe how constraints formed by rooted expressions are interpreted. Consider a graphon W and a rooted graph H with m roots, and let H_0 be the graph induced by the m roots of H. We define the auxiliary function $c_H : [0, 1]^m \to [0, 1]$; the value of $c_H(x_1, \ldots, x_m)$ is equal to the probability that a W-random graph is isomorphic to H conditioned on the m roots being associated with $x_1, \ldots, x_m$ (in this order), i.e.,

$$c_H(x_1,\dots,x_m) = \frac{(|H|-m)!}{|\mathrm{Aut}(H)|} \int\limits_{(x_{m+1},\dots,x_{|H|})\in[0,1]^{|H|-m}} \prod_{(i,j)\in E(H)} W(x_i,x_j) \prod_{(i,j)\notin E(H)} (1-W(x_i,x_j))\,\mathrm{d}\lambda_{|H|-m},$$

where $\mathrm{Aut}(H)$ is the group of automorphisms of H that preserves the roots, and the vertices of H are numbered in a way that the first m vertices are the roots (in the order that they have).

Let $D = D'$ be a constraint such that D and D' are compatible rooted density expressions with graphs containing m roots. For every graph H appearing in D and D', substitute the function c_H; both D and D' can now be viewed as functions c_D and c'_D from $[0,1]^m$ to $[0,1]$. We say that the graphon W *satisfies* the constraint $D = D'$ if the functions c_D and c'_D are equal almost everywhere. We comment that, on several occasions, we consider constraints containing a fraction of two rooted density expressions D/D'. A constraint containing such fractions should be understood as saying that both sides are multiplied by the denominators of all the fractions, e.g., $D_1/D'_1 = D_2/D'_2$ should be understood as $D_1 \cdot D'_2 = D_2 \cdot D'_1$. One of the results in [18] asserts that for every two compatible rooted density expressions D and D', there exist density expressions C and C' such that a graphon W satisfies $D = D'$ if and only if it satisfies $C = C'$.

A graphon W is *partitioned* if there exist $k \in \mathbb{N}$, positive reals $a_1,\dots,a_k$ summing to one and distinct reals $d_1,\dots,d_k$ between 0 and 1 such that the set of vertices of W with degree d_i has measure a_i; we write A_i for the set of vertices of degree d_i for $i \in [k]$ and refer to A_i as to a *part* of the graphon W.

A graph H is *decorated* if its vertices are labeled with parts $A_1,\dots,A_k$. The density of a decorated graph H in a graphon W is the probability that the W-random graph is the graph H conditioned on the event that all sampled vertices are in the parts corresponding to their labels. For example, if H is an edge with its two vertices labeled with parts A_1 and A_2, then the density of H in W is the density of edges between the parts A_1 and A_2, i.e.,

$$d(H,W) = \frac{1}{\lambda(A_1)\lambda(A_2)} \int_{A_1}\int_{A_2} W(x,y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

Similarly as in the case of non-decorated graphs, we can define rooted decorated graphs, rooted decorated density expressions and form constraints using such expressions. A constraint that uses (rooted or non-rooted) decorated graphs is referred to as *decorated*. One of the results from [18], which we state as Lemma 3, asserts that for every decorated constraint, there exists an equivalent ordinary constraint.

The structural properties of a graphon W that satisfies a given set of constraints can be analyzed in several different ways. The constraints of the form $D = 0$ where D is a single graph G can be understood as forbidding G as a subgraph in a W-random graph. Consequently, the induced removal lemma and other combinatorial arguments can be used to derive some structural properties of every graphon satisfying $D = 0$. However, it is also possible to derive properties of such graphons in an analytic way, which is the way that we will generally use in our exposition.

We next introduce the convention for depicting decorated constraints used throughout the paper; an example of the use of this convention can be found in Figure 4. The roots of decorated graphs will be depicted by squares and non-root vertices by circles; all vertices will be labeled by the names of the corresponding parts of a graphon. The full lines connecting vertices correspond to edges and dashed lines to non-edges. No connection between a pair of vertices represents that both edge or non-edge are allowed between the vertices, i.e., the corresponding density expression should be understood as the sum of the expressions containing the graph with and without such the edge (unless the edge is missing between two roots). For example, if three pairs of vertices are missing a connection, the density expression is the sum of all eight graphs that can be obtained by including or not including the edge between the three pairs. If the edge is missing between two roots, then the density constraint is required to hold both when the edge is included between the pair of root vertices in all graphs and when it is included in no graph. To avoid any possible ambiguity with interpretations of the drawings of rooted constraints, the positions of the roots of all graphs appearing in a rooted decorated density constraint will always be identical (see Figure 15 for an example).

We conclude this section by explicitly stating three lemmas that were proven in [18] and that we use further. The first lemma guarantees the existence of a set of constraints that force a graphon satisfying these constraints to be a partitioned graphon with a given partition and given degrees.

Lemma 2. *Let $k \in \mathbb{N}$, $a_1, \dots, a_k$ be positive real numbers summing to one and let $d_1, \dots, d_k$ be distinct reals between 0 and 1. There exists a finite set of constraints $\mathcal{C}$ such that a graphon W satisfies $\mathcal{C}$ if and only if W is a partitioned graphon with k parts such that the i-th part has measure a_i and its vertices have degree d_i.*

The following lemma says that decorated constraints have the same expressing power as non-decorated constraints.

Lemma 3. *Let $k \in \mathbb{N}$, let $a_1, \dots, a_k$ be positive real numbers summing to one, and let $d_1, \dots, d_k$ be distinct reals between zero and one. Further, let D_1 and D_2 be two compatible rooted decorated density expressions with decorations $A_1, \dots, A_k$. There exist an ordinary density expression D, i.e., D has no roots and no decorations, such that every partitioned graphon W with k parts formed by vertices of degree d_i and measure a_i each satisfies $D_1 = D_2$ if and only if it satisfies $D = 0$.*

We remark that our definition of interpreting decorated density expressions differ from the definition given in [18]. However, the difference results only in a constant multiplicative factor depending on the measures of the parts of a graphon; in particular, Lemma 3 also holds with the definition of decorated constraints that we use.

part	$A_0^{\boxplus}$	$A_1^{\boxplus}$	$A_2^{\boxplus}$	$A_3^{\boxplus}$	$B_1^{\boxplus}$	$B_2^{\boxplus}$	$B_3^{\boxplus}$	$B_4^{\boxplus}$	$B_5^{\boxplus}$	$C^{\boxplus}$	$D^{\boxplus}$	$E_1^{\boxplus}$	$E_2^{\boxplus}$	$F^{\boxplus}$
degree	$\frac{110}{270}$	$\frac{111}{270}$	$\frac{112}{270}$	$\frac{113}{270}$	$\frac{114}{270}$	$\frac{115}{270}$	$\frac{116}{270}$	$\frac{117}{270}$	$\frac{118}{270}$	$\frac{119}{270}$	$\frac{40}{270}$	e_1	e_2	$\frac{45}{270}$

Table 1: The degrees of the vertices in the parts of the graphon $W_{\boxplus}$.

The last lemma states that there exists a finite set of constraints guaranteeing that a partitioned graphon is constant between a specific pair of its parts.

Lemma 4. *For all $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ summing to one, distinct reals $d_1, \ldots, d_k$ between zero and one, $\ell, \ell' \leq k$, $l \neq l'$, and $p \in [0,1]$, there exists a finite set of constraints $\mathcal{C}$ such that every partitioned graphon W with k parts $A_1, \ldots, A_k$ such that the measure of A_i is a_i and all vertices of A_i have degrees d_i satisfies $\mathcal{C}$ if and only if $W(x,y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$.*

3 The hypercubical graphon

In this section, we define the graphon from Theorem 1; the graphon is denoted by $W_{\boxplus}$ and referred to as the *hypercubical graphon*. For convenience, we provide a sketch of the structure of the graphon $W_{\boxplus}$ in Figure 1.

The *hypercubical graphon* $W_{\boxplus}$ is a partitioned graphon with 14 parts, which are denoted by $A_0^{\boxplus}, \ldots, A_3^{\boxplus}, B_1^{\boxplus}, \ldots, B_5^{\boxplus}, C^{\boxplus}, D^{\boxplus}, E_1^{\boxplus}, E_2^{\boxplus}, F^{\boxplus}$. Each part has measure $1/27$ except for the parts $E_1^{\boxplus}$ and $E_2^{\boxplus}$ that have measure $11/27$ and $4/27$, respectively. The degrees of the vertices in the parts are listed in Table 1. We will not compute the exact values e_1 and e_2 of the degrees of vertices in $E_1^{\boxplus}$ and $E_2^{\boxplus}$, respectively; however, the definition of the graphon will imply that $e_1 \in (4.5/27, 10/27)$ and $e_2 < 1/27$. In particular, vertices in different parts have different degrees. The high level overview of the roles of individual parts of the hypercubical graphon can be found at the end of Section 1.

We describe the graphon $W_{\boxplus}$ as a collection of functions $W_{\boxplus}^{X\times Y}$ on products of the parts $X^{\boxplus}$ and $Y^{\boxplus}$. To simplify our exposition, we define these as functions from $[0,1]^2$ to $[0,1]$, assuming that we have a fixed measurable bijection η_X from each part $X^{\boxplus}$ to $[0,1]$ such that $\lambda\left(\eta_X^{-1}(S)\right) = \lambda(S)\lambda(X^{\boxplus})$ for every measurable set $S \subseteq [0,1]$. So, it holds $W_{\boxplus}(x,y) = W_{\boxplus}^{X\times Y}(\eta_X(x), \eta_Y(y))$ for $x \in X^{\boxplus}$ and $y \in Y^{\boxplus}$, i.e., the graphon $W_{\boxplus}$ consists of appropriately scaled functions $W_{\boxplus}^{X\times Y}$. Note that, unlike graphons, the functions $W_{\boxplus}^{X\times Y}$ need not to be symmetric if $X \neq Y$; however, these functions satisfy $W_{\boxplus}^{X\times Y}(x,y) = W_{\boxplus}^{Y\times X}(y,x)$.

We now introduce additional notation used in the definition of the graphon $W_{\boxplus}$ and in the proof. For $x \in [0,1)$, let $\langle x \rangle$ be such $k \in \mathbb{N}$ that $x \in [1-2^{-k+1}, 1-2^{-k})$ and let $\widehat{x} = (x - (1-2^{-k+1})) \cdot 2^k$. Informally speaking, we imagine $[0,1]$ as partitioned into consecutive intervals of measures $1/2$, $1/4$, etc., and $\langle x \rangle$ indicates the index of the interval that x belongs to and $\widehat{x}$ is the relative position of x within

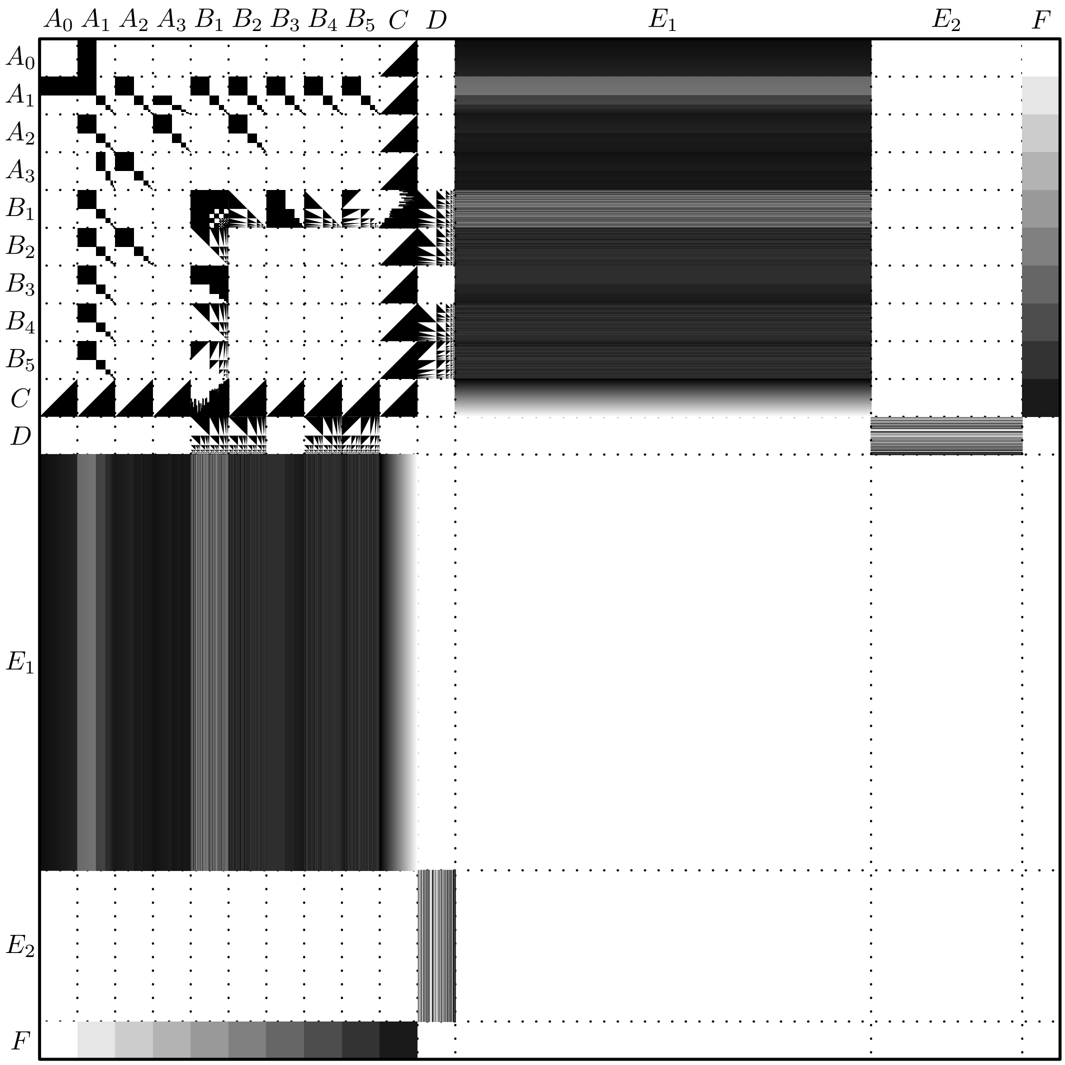

Figure 1: The hypercubical graphon. The origin of the coordinate system is in the top left corner; the values of the graphon are visualized using different shades of gray (with white being zero and black being one). The graphon between the parts X and Y, $X \in \{B_1^{\boxplus}, D^{\boxplus}\}$ and $Y \in \{B_1^{\boxplus}, B_2^{\boxplus}, B_4^{\boxplus}, B_5^{\boxplus}\}$, is drawn in an imprecise simplified way because of the complex structure. To simplify the picture, the parts are labeled by their names without the superscripts.

this interval. Observe that $x = 1 - 2^{1-\langle x\rangle} + \widehat{x}/2^{\langle x\rangle}$ for every $x \in [0,1)$. Using this notation, we define the *diagonal checker* function $\varkappa : [0,1]^2 \to [0,1]$ as follows (see Figure 2):

$$\varkappa(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle = \langle y\rangle \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to start with defining the structure between different parts of the graphon $W_\boxplus$.

$$W_\boxplus^{A_0\times A_1}(x,y) = \begin{cases} 1 & \text{for } (x,y) \in [0,1]\times[0,1/2], \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{A_1\times A_1} = W_\boxplus^{A_1\times A_2} = W_\boxplus^{A_1\times B_1} = W_\boxplus^{A_1\times B_2} = W_\boxplus^{A_1\times B_3} = W_\boxplus^{A_1\times B_4} = W_\boxplus^{A_1\times B_5} = W_\boxplus^{A_2\times A_3} = W_\boxplus^{A_2\times B_2} = \varkappa.$$

For $X \in \{A_0, \dots, A_3, B_2, \dots, B_5, C\}$, let:

$$W_\boxplus^{C\times X}(x,y) = \begin{cases} 1 & \text{for } x+y \geq 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The rest of the definition of the graphon $W_\boxplus$ depends on a collection of measure preserving functions, which we call a recipe. A *recipe* $\mathfrak{R}$ is a set of measure preserving maps r_n for $n \in \mathbb{N}^*$ such that $r_n : [0,1] \to [0,1]^n$. Recall that $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ and so we understand $[0,1]^\infty$ to be $[0,1]^\mathbb{N}$. An example of a recipe is a collection of maps that "zip" the standard binary representations of x_i, i.e., the digits of $r_n(x)$ on the positions congruent to i modulo n are determined by the digits of x_i, $i \in [n]$, and the digits of $r_\infty(x)$ on the positions

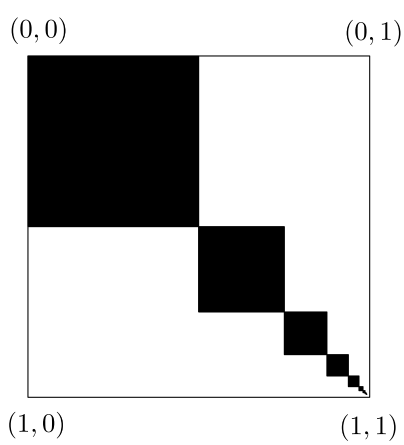

Figure 2: The diagonal checker function $\varkappa$.

congruent to 2^{i-1} modulo 2^i are determined by the digits of x_i, $i \in \mathbb{N}$. Observe that $\mathfrak{R} = \{r_n | n \in \mathbb{N}^*\}$ is a recipe if and only if

$$\lambda\left(\{x | \forall\, i \in [n]\ (r_n(x))_i \le z_i\}\right) = \prod_{i=1}^{n} z_i \text{ for every } (z_1, \dots z_n) \in [0,1]^n \tag{1}$$

for every $n \in \mathbb{N}$ and

$$\lambda\left(\{x | \forall\, i \in [k]\ (r_\infty(x))_i \le z_i\}\right) = \prod_{i=1}^{k} z_i \text{ for every } (z_1, \dots z_k) \in [0,1]^k \tag{2}$$

for every $k \in \mathbb{N}$, where $(x)_i$ is the i-th coordinate of $x \in [0,1]^n$, $n \in \mathbb{N}^*$. A recipe is *bijective* if all the maps r_n, $n \in \mathbb{N}^*$, are bijective.

For the rest of the definition of the graphon $W_\boxplus$, we fix a bijective recipe $\mathfrak{R}$. It can be shown that the definition of $W_\boxplus$ does not depend on this choice in the sense that the graphons defined for different choices of $\mathfrak{R}$ are weakly isomorphic (this statement stays true even if $\mathfrak{R}$ is a recipe that is not bijective).

$$W_\boxplus^{A_1\times A_3}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle = \langle y\rangle + 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{C\times B_1}(x,y) = \begin{cases} 1 & \text{for } (1 - 2^{1-\langle y\rangle}) + (r_{\langle y\rangle}(\widehat{y}))_1 \cdot 2^{-\langle y\rangle} + x \ge 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{B_1\times B_1}(x,y) = \begin{cases} 1 & \text{if } (r_{\langle x\rangle}(\widehat{x}))_k \le (r_{\langle y\rangle}(\widehat{y}))_k \text{ for every } k \le \min(\langle x\rangle, \langle y\rangle), \\ 1 & \text{if } (r_{\langle x\rangle}(\widehat{x}))_k \ge (r_{\langle y\rangle}(\widehat{y}))_k \text{ for every } k \le \min(\langle x\rangle, \langle y\rangle), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{B_1\times B_2}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \ge \langle y\rangle \text{ and } \widehat{y} \le (r_{\langle x\rangle}(\widehat{x}))_{\langle y\rangle}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{B_1\times B_3}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \ge \langle y\rangle, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_\boxplus^{B_1\times B_4}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \ge \langle y\rangle \text{ and } \widehat{y} \le \prod_{i=1}^{\langle y\rangle} (r_{\langle x\rangle}(\widehat{x}))_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{B_1\times B_5}(x,y) = \begin{cases} 1 & \text{if } \langle x\rangle \geq \langle y\rangle \text{ and } \widehat{y} \leq \prod_{i=1}^{\langle y\rangle}(1-(r_{\langle x\rangle}(\widehat{x}))_i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D\times B_1}(x,y) = \begin{cases} 1 & \text{if } r_\infty(\widehat{y})_k \leq (r_\infty(x))_k \text{ for every } \; k \leq \langle y\rangle, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D\times B_2}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \leq (r_\infty(x))_{\langle y\rangle}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D\times B_4}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \leq \prod_{i=1}^{\langle y\rangle}(r_\infty(x))_i, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

$$W_{\boxplus}^{D\times B_5}(x,y) = \begin{cases} 1 & \text{if } \widehat{y} \leq \prod_{i=1}^{\langle y\rangle}(1-(r_\infty(x))_i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For every $X \in \{A_0,\dots,A_3,B_1,\dots,B_5,C\}$, we set:

$$W_{\boxplus}^{E_1\times X}(x,y) = \; 1-1/11 \sum_{Y\in A_0^{\boxplus},\dots,A_3^{\boxplus},B_1^{\boxplus},\dots,B_5^{\boxplus},C^{\boxplus},D^{\boxplus}} \deg_Y y.$$

We further define

$$W_{\boxplus}^{E_2\times D}(x,y) = \; 1-1/4 \sum_{Y\in B_1^{\boxplus},B_2^{\boxplus},B_4^{\boxplus},B_5^{\boxplus}} \deg_Y y.$$

$W_{\boxplus}^{F\times A_1}(x,y) = 1/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times A_2}(x,y) = 2/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times A_3}(x,y) = 3/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times B_1}(x,y) = 4/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times B_2}(x,y) = 5/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times B_3}(x,y) = 6/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times B_4}(x,y) = 7/10$ for all $(x,y)\in[0,1]^2$,

$W_{\boxplus}^{F\times B_5}(x,y) = 8/10$ for all $(x,y)\in[0,1]^2$, and

$W_{\boxplus}^{F\times C}(x,y)=9/10$ for all $(x,y)\in[0,1]^2$.

If we have defined a function $W_{\boxplus}^{X\times Y}$, we set $W_{\boxplus}^{Y\times X}(x,y)=W_{\boxplus}^{X\times Y}(y,x)$. Finally, the graphon $W_{\boxplus}$ is equal to 0 between parts $X^{\boxplus}$ and $Y^{\boxplus}$ such that we have not defined a function $W_{\boxplus}^{X\times Y}$ or $W_{\boxplus}^{Y\times X}$. This completes the definition of the graphon $W_{\boxplus}$.

We now argue that $e_1\in(4.5/27,10/27)$ and $e_2<1/27$. Let $x\in E_1^{\boxplus}$. Since $N(x)$ is a subset of $A_0^{\boxplus}\cup\dots\cup A_3^{\boxplus}\cup B_1^{\boxplus}\cup\dots\cup B_5^{\boxplus}\cup C^{\boxplus}$, the measure of $N(x)$ is at most $10/27$. Since it does not hold that $W_{\boxplus}(x,y)=1$ for almost all $y\in N(x)$, we get that $e_1<10/27$. Observe that it holds for every $X\in\{A_0^{\boxplus},\dots,A_3^{\boxplus},B_2^{\boxplus},\dots,B_5^{\boxplus},C^{\boxplus}\}$ that $\deg_{E_1^{\boxplus}}(x)>1/2$ for every $x\in X$. It follows that $e_1>4.5/27$. Similarly, $N(x)$ is a subset of $D^{\boxplus}$ for every $x\in E_2^{\boxplus}$ and it does not hold that $W_{\boxplus}(x,y)=1$ for almost all $y\in N(x)$; this implies that $e_2<1/27$.

Before proceeding further, we introduce additional notation related to splitting parts A_i, $i\in\{1,2,3\}$, and B_j, $j\in\{1,\dots,5\}$, into smaller pieces. For $i\in\{1,2,3\}$, the set of vertices $x\in A_i^{\boxplus}$ with $\deg_{A_1^{\boxplus}}x=2^{-k}$ is denoted by $A_{i,k}^{\boxplus}$ and $A_{i,k}^{\boxplus}$ is called the *k-th level* of $A_i^{\boxplus}$. Similarly, $B_{j,k}^{\boxplus}$, $j\in\{1,\dots,5\}$, is the set of vertices $x\in B_j^{\boxplus}$ such that $\deg_{A_1^{\boxplus}}x=2^{-k}$. Note that measure of the k-th level $A_{i,k}^{\boxplus}$ is $2^{-k}/27$; the same holds for $B_{j,k}^{\boxplus}$.

3.1 Dimension of the space of typical vertices

We finish this section with showing that both $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ have infinite dimension.

Proposition 5. *Both $T(W_{\boxplus})$ and $\overline{T}(W_{\boxplus})$ contain a subspace homeomorphic to $[0,1]^\infty$.*

Proof. Observe that every vertex contained in $D^{\boxplus}$ is typical (both with respect to $T(W_{\boxplus})$ and with respect to $\overline{T}(W_{\boxplus})$) and define a map $h:D^{\boxplus}\to[0,1]^\infty$ as

$$h(x)=\left(\deg^{W_{\boxplus}}_{B_{2,i}^{\boxplus}}x\right)_{i\in\mathbb{N}}.$$

Because r_∞ is a bijection, h is a bijection between $D^{\boxplus}$ and $[0,1]^\infty$. We next show that h^{-1} is continuous when $D^{\boxplus}$ is equipped with the topology of the space $T(W_{\boxplus})$. To do so, we need to bound the L^1-distance of the functions $f_x^{W_{\boxplus}}$ and $f_{x'}^{W_{\boxplus}}$ in terms of $h(x)$ and $h(x')$ for all $x,x'\in D^{\boxplus}$, where $f_x^{W_{\boxplus}}(y):=W_{\boxplus}(x,y)$.

First note that

$$\deg^{W_{\boxplus}}_{B_{1,i}^{\boxplus}}x=\deg^{W_{\boxplus}}_{B_{4,i}^{\boxplus}}x=\prod_{k\in[i]}\deg^{W_{\boxplus}}_{B_{2,k}^{\boxplus}}x\quad\text{and}\quad\deg^{W_{\boxplus}}_{B_{5,i}^{\boxplus}}x=\prod_{k\in[i]}(1-\deg^{W_{\boxplus}}_{B_{2,k}^{\boxplus}}x)$$

for every $x \in D^{\boxplus}$. The value of $||f_x^{W_\boxplus} - f_{x'}^{W_\boxplus}||_1$ is the sum of the corresponding integrals over y from $B_1^{\boxplus}$, $B_2^{\boxplus}$, $B_4^{\boxplus}$, $B_5^{\boxplus}$ and $E_2^{\boxplus}$. The term corresponding to the integral over y from $B_2^{\boxplus}$ is equal to

$$\sum_{i=1}^{\infty} \lambda(B_{2,i}^{\boxplus}) \left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right| \ ,$$

the term corresponding to the integral over y from $B_4^{\boxplus}$ is equal to

$$\sum_{i=1}^{\infty} \lambda(B_{4,i}^{\boxplus}) \left|\prod_{k=1}^{i} \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \prod_{k=1}^{i} \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right| \ ,$$

and the term corresponding to the integral over y from $B_5^{\boxplus}$ is equal to

$$\sum_{i=1}^{\infty} \lambda(B_{5,i}^{\boxplus}) \left|\prod_{k=1}^{i} (1 - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x) - \prod_{k=1}^{i} (1 - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x')\right| \ .$$

The term corresponding to the integral over y from $B_1^{\boxplus}$ is at most

$$\sum_{i=1}^{\infty} \lambda(B_{1,i}^{\boxplus}) \sum_{k=1}^{i} \left|\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right| \ .$$

We next observe that

$$\left|\prod_{k=1}^{i} \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \prod_{k=1}^{i} \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right| \le \sum_{k=1}^{i} \left|\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right| \text{ and}$$

$$\left|\prod_{k=1}^{i} (1 - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x) - \prod_{k=1}^{i} (1 - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x')\right| \le \sum_{k=1}^{i} \left|\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|$$

for every $i \in \mathbb{N}$. Since it holds that $\lambda(B_{j,i}^{\boxplus}) = 2^{-i}/27$ for $j \in \{1,2,4,5\}$, we obtain that the sum of the terms corresponding to the integrals over y from $B_1^{\boxplus}$, $B_2^{\boxplus}$, $B_4^{\boxplus}$ and $B_5^{\boxplus}$ is at most

$$\frac{1}{27}\left(\sum_{i=1}^{\infty} 2^{-i} \left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right| + 3\sum_{i=1}^{\infty} 2^{-i} \sum_{k=1}^{i} \left|\deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,k}^{\boxplus}}^{W_\boxplus} x'\right|\right) \ ,$$

which is equal to

$$\frac{7}{27}\left(\sum_{i=1}^{\infty} 2^{-i} \left|\deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x - \deg_{B_{2,i}^{\boxplus}}^{W_\boxplus} x'\right|\right) \ .$$

Since the term corresponding to the integral over y from $E_2^{\boxplus}$ is at most the sum of the terms to the integrals over y from $B_1^{\boxplus}$, $B_2^{\boxplus}$, $B_4^{\boxplus}$ and $B_5^{\boxplus}$, we conclude that

$$||f_x^{W_{\boxplus}} - f_{x'}^{W_{\boxplus}}||_1 \leq \frac{14}{27} \left(\sum_{i=1}^{\infty} 2^{-i} \left| \deg_{B_{2,i}^{\boxplus}}^{W_{\boxplus}} x - \deg_{B_{2,i}^{\boxplus}}^{W_{\boxplus}} x' \right| \right) .$$

It follows that h^{-1} is a continuous map from $[0,1]^{\infty}$ to $D^{\boxplus}$. Since h^{-1} is a continuous injective map from a compact space to a Haussdorf space, it follows that h is a homeomorphism between $D^{\boxplus}$ with the topology given by $T(W_{\boxplus})$ and $[0,1]^{\infty}$. Since the identity map from $T(W_{\boxplus})$ to $\overline{T}(W_{\boxplus})$ is injective and continuous [34], it also follows that h is a homeomorphism between $D^{\boxplus}$ with the topology given by $\overline{T}(W_{\boxplus})$ and $[0,1]^{\infty}$. □

4 Constraints

This section and the next section are devoted to the proof of the following theorem, which together with Proposition 5 implies Theorem 1.

Theorem 6. *The hypercubical graphon $W_{\boxplus}$ is finitely forcible.*

In this section, we present the set $\mathcal{C}_{\boxplus}$ of the constraints that such that the graphon $W_{\boxplus}$ is the unique graphon satisfying $\mathcal{C}_{\boxplus}$. We only list the constraints contained in $\mathcal{C}_{\boxplus}$ and their analysis is postponed to the next section.

We present the constraints contained in the set $\mathcal{C}_{\boxplus}$ split into groups depending on the properties of a graphon that they force, and we informally describe these properties.

Partition constraints are the constraints given in the Lemma 2, which are satisfied by partitioned graphons with the same number of parts as $W_{\boxplus}$ and with the measures and the degrees of vertices of the parts as in $W_{\boxplus}$.

All the constraints that are presented in the rest are decorated constraints with vertices labelled by the parts $A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F$.

The zero constraints force that W equals 0 almost everywhere on

- $A_0 \times (A_0 \cup A_2 \cup A_3 \cup B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup D \cup E_2 \cup F)$,
- $A_1 \times (D \cup E_2)$,
- $A_2 \times (A_2 \cup B_1 \cup B_3 \cup B_4 \cup B_5 \cup D \cup E_2)$,
- $A_3 \times (A_3 \cup B_1 \cup \cdots \cup B_5 \cup D \cup E_2)$,
- $B_2 \times (B_2 \cup \cdots \cup B_5 \cup E_2)$,,
- $B_3 \times (B_3 \cup B_4 \cup B_5 \cup D \cup E_2)$,

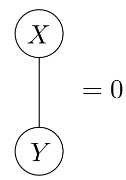

Figure 3: Constraint forcing zero edge density.

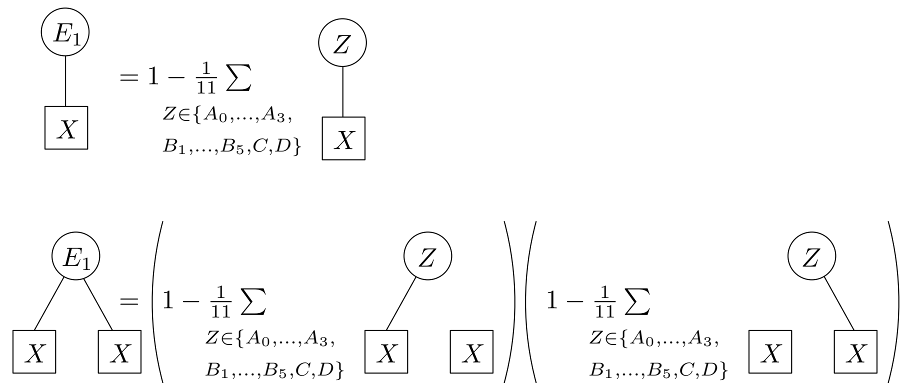

Figure 4: The degree unifying constraints contain the depicted constraints for all the choices of X in $\{A_0, \ldots, A_3, B_1, \ldots, B_5, C\}$.

- $B_4 \times (B_4 \cup B_5 \cup E_2)$,
- $B_5 \times (B_5 \cup E_2)$,
- $C \times (D \cup E_2)$,
- $D \times (D \cup E_1)$,
- $E_1 \times (E_1 \cup E_2 \cup F)$,
- $E_2 \times (E_2 \cup F)$, and
- $F \times F$.

The constraint forcing the zero edge density between parts X and Y is depicted in Figure 3.

The degree unifying constraints force that the relative degree of almost every vertex x from a part A_i, $i = 0, \ldots, 3$, a part B_j, $j = 1, \ldots, 5$ and the part C with respect to the complement of $E_2 \cup F$, i.e. $A_0 \cup \cdots \cup A_3 \cup B_1 \cup \cdots \cup B_5 \cup C \cup D \cup E_1$, is equal to $1/2$, and that $W(x, z)$ is constant for almost every such x when z ranges through the part E_1. These constraints also force that the degree of almost every vertex y from the part D is $4/27$ and $W(y, z)$ is constant for almost every such y when z ranges through the part E_2. The constraints are depicted in Figures 4 and 5.

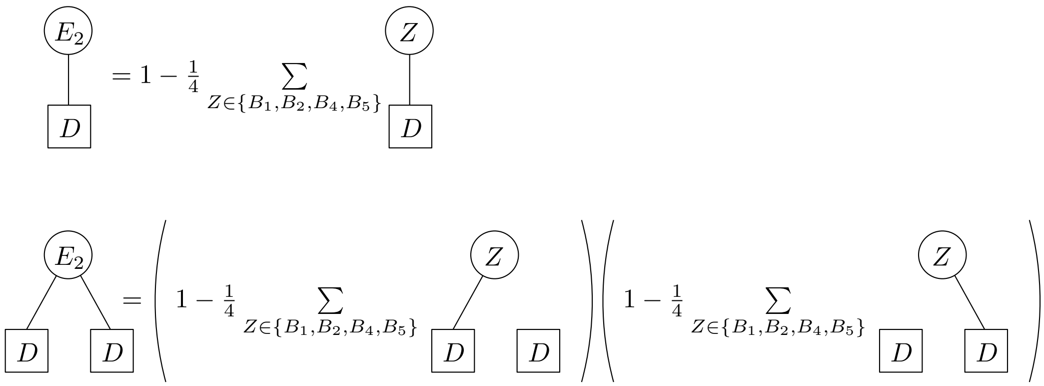

Figure 5: The degree unifying constraints for D.

Part	A_0	A_1	A_2	A_3	B_1	B_2	B_3	B_4	B_5	C
Density	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$	$\frac{5}{10}$	$\frac{6}{10}$	$\frac{7}{10}$	$\frac{8}{10}$	$\frac{9}{10}$

Table 2: Densities between the part F and the other parts.

The degree distinguishing constraints force that the graphon is constant between the part F and each of the parts $A_0, \ldots, A_3$, $B_1, \ldots, B_5$, C and D, and that this constant is equal to the value given in Table 2. The existence of finitely many such constraints follows from Lemma 4; Figure 6 contains an example of two constraints that can be used to force the graphon to be equal to $9/10$ between the parts C and F.

The triangular constraints force that the structure of the subgraphon induced by $C \times X$ is the same in $W_{\boxplus}$ for every $X \in A_0, \ldots, A_3, B_1, \ldots, B_5, C$, i.e., that the subgraphon induced by $C \times X$ is the half-graphon. Let H_i and d_i be the finitely many graphs and their densities that are satisfied by the half-graphon only; such a finite set of graphs exists since the half-graphon is finitely forcible [14, 32]. The structure of the subgraphon induced by $C \times C$ is forced by the constraints $H_i' = d_i$ where H_i' is the decorated graph obtained from H_i by labeling each vertex with C, and the structure of the subgraphon induced by $C \times X$ for $X \neq C$ is forced by the constraints

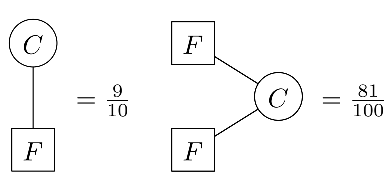

Figure 6: The degree distinguishing constraints for $F \times C$.

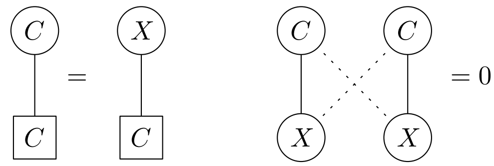

Figure 7: The triangular constraints include the depicted constraints for all the choices of X in $\{A_0, \ldots, A_3, B_1, \ldots, B_5\}$.

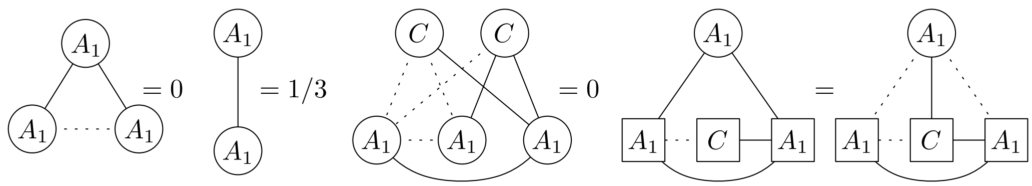

Figure 8: The main diagonal checker constraints.

depicted in Figure 7.

The main diagonal checker constraints force the diagonal checker structure of the subgraphon induced by $A_1 \times A_1$. They are depicted in Figure 8.

The complete bipartition constraints force, in particular, that the subgraphons induced by $A_1 \times A_2$, $A_1 \times A_3$, $A_1 \times B_1$, $\ldots$, $A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$ are unions of complete bipartite subgraphons. The constraints are given in Figure 9.

The auxiliary diagonal checker constraints determine the sizes of the sides of complete bipartite subgraphons in $A_1 \times A_2$, $A_1 \times A_3$, $A_1 \times B_1, \ldots, A_1 \times B_5$, $A_2 \times A_3$ and $A_2 \times B_2$. They are depicted in Figure 10.

The first level constraints force the structure of subgraphon induced by $A_0 \times A_1$ and they are depicted in Figure 11.

The stair constraints force the structure of subgraphon induced by $B_1 \times B_3$. They are depicted in Figure 12.

The coordinate constraints force some properties of the structure of the subgraphons induced by $B_1 \times (B_2 \cup B_4 \cup B_5)$ and $D \times (B_2 \cup B_4 \cup B_5)$. They can be found in Figure 13.

The initial coordinate constraint determines the relative degrees of vertices of B_1 in a subset of B_2. It is depicted in Figure 14.

The distribution constraints determine the relative degrees of vertices of B_2 in B_1 and D are depicted in Figure 15.

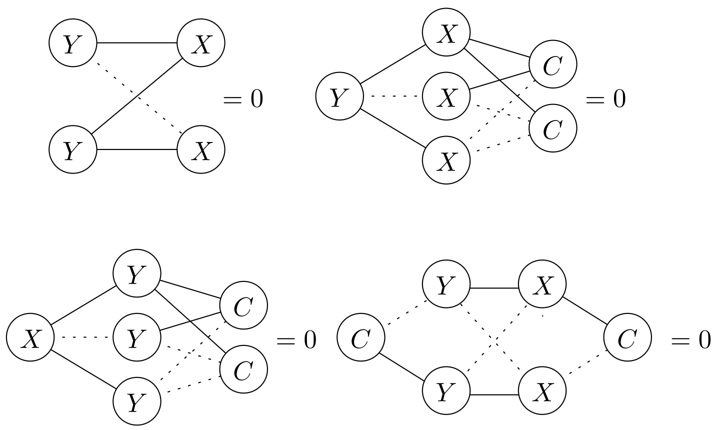

Figure 9: The complete bipartition constraints consist of the top two constraint for $(X,Y) \in \{(A_1,A_2),(A_1,A_3),(A_1,B_1),\ldots,(A_1,B_5),(A_2,A_3),(A_2,B_2)\}$ and the bottom two constraints for $(X,Y) \in \{(A_1,A_2),\ (A_1,A_3),\ (A_1,B_2),\ldots,$ $(A_1,B_5),\ (A_2,A_3),(A_2,B_2)\}$.

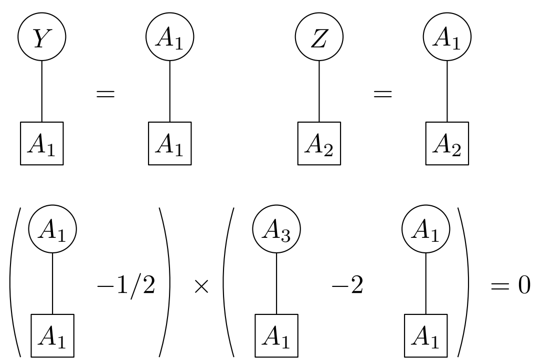

Figure 10: The auxiliary diagonal checker constraints consist of the depicted constraints, where Y in the first constrains attains all values in $\{A_2,B_1,\ldots,B_5\}$ and Z in the second constraint attains all values in $\{A_3,B_2\}$.

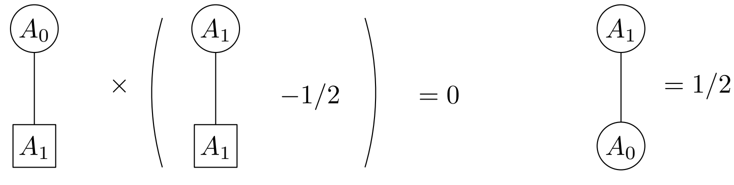

Figure 11: The first level constraints.

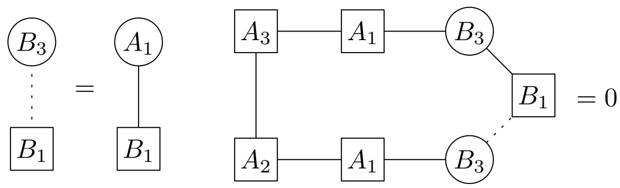

Figure 12: The stair constraints.

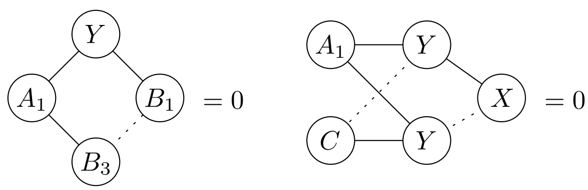

Figure 13: The coordinate constraints consist of the depicted constraints, where X and Y attain all values in $\{B_1, D\}$ and $\{B_2, B_4, B_5\}$, respectively.

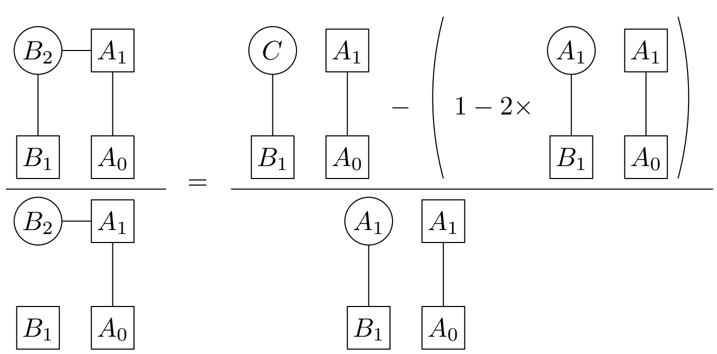

Figure 14: The initial coordinate constraint.

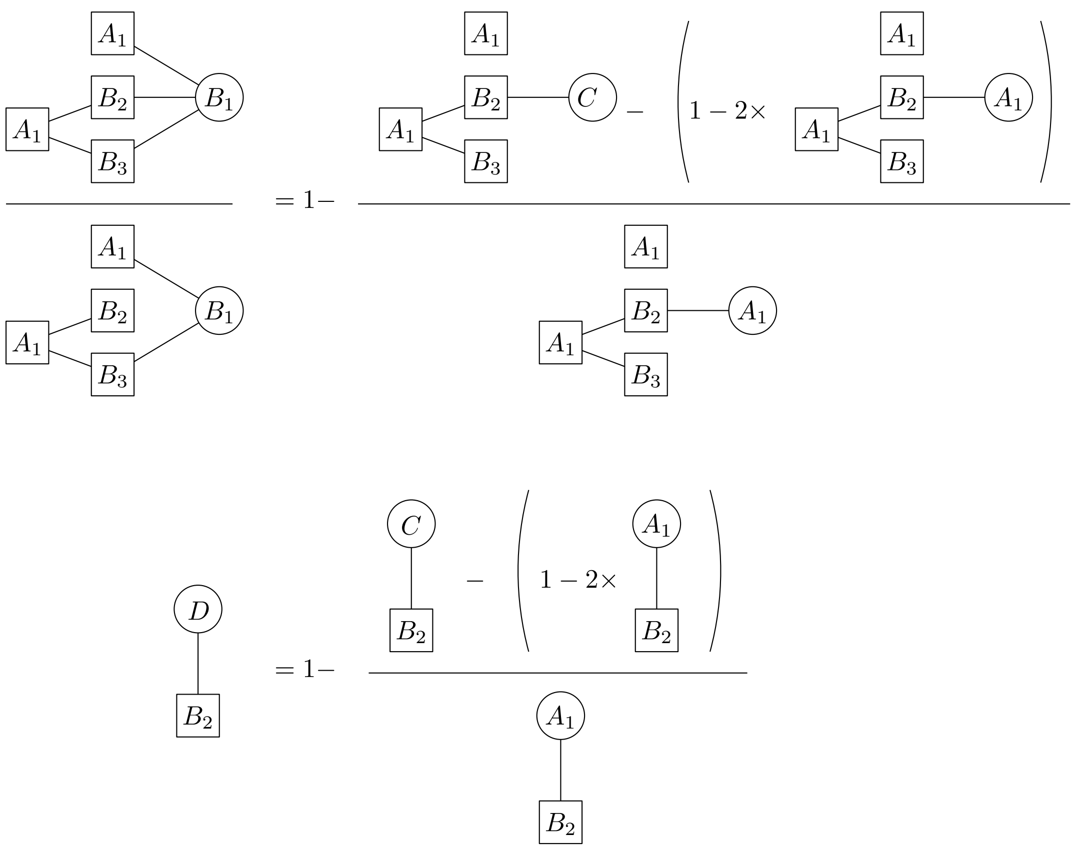

Figure 15: The distribution constraints.

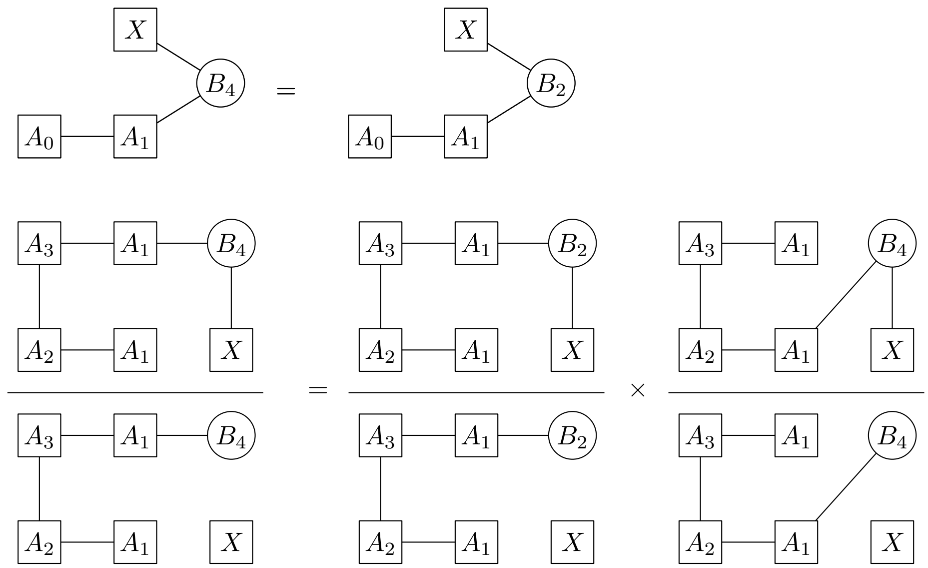

Figure 16: The product constraints forcing $B_1 \times B_4$ and $D \times B_4$ consist of the depicted constraints, where $X \in \{B_1, D\}$.

The product constraints force the structure of the subgraphons induced by $B_1 \times B_4$, $D \times B_4$, $B_1 \times B_5$ and $D \times B_5$. They are depicted in Figures 16, 17.

The projection constraints force the structure of the subgraphon induced by $B_1 \times B_1$. They are depicted in Figures 18 and 19.

The infinite constraints force the structure of the subgraphon iduces by $D \times B_1$ and $D \times B_2$. They are depicted in Figure 20.

This completes the list of the constraints that are contained in $\mathcal{C}_\boxplus$

5 Proof of Theorem 6

In this section, we prove Theorem 6. In particular, we will show that the hypercubical graphon $W_\boxplus$ is the unique (up to weak isomorphism) graphon satisfying set $\mathcal{C}_\boxplus$ of the constraints that we listed in Section 4.

Fix a bijective recipe $\mathfrak{R} = \{r_n | n \in \mathbb{N}^*\}$, which determines the graphon $W_\boxplus$. Suppose that W is a graphon that satisfies all constraints contained in $\mathcal{C}_\boxplus$. Our aim is to show that the graphons W and $W_\boxplus$ are weakly isomorphic. Since W satisfies the partition constraints, W is a partitioned graphon with parts of the same measure as those of $W_\boxplus$ and the vertices in the corresponding parts having the same degree as those in $W_\boxplus$. The parts of W are denoted by $A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F$ in such a way that the part X corresponds to the part $X^\boxplus$ of the graphon $W_\boxplus$. We will strictly use $A_0, \ldots, F$ in the context

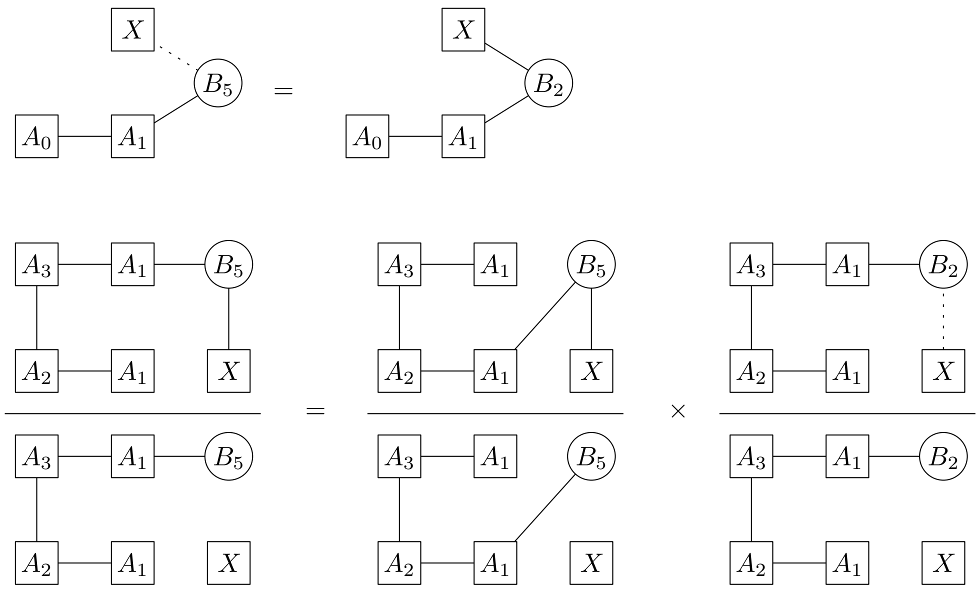

Figure 17: The product constraints forcing $B_1 \times B_5$ and $D \times B_5$ consist of the depicted constraints, where $X \in \{B_1, D\}$.

of the graphon W and $A_0^{\boxplus}, \dots, F^{\boxplus}$ in the context of the graphon $W_{\boxplus}$. In the analogy to $B_{1,n}^{\boxplus}$ and $B_{2,n}^{\boxplus}$, we define $B_{1,n}$ and $B_{2,n}$ to be the vertices of B_1 and B_2, respectively, that have relative degree 2^{-n} with respect to A_1 in W.

By the Monotone Reordering Theorem (see [30] for more details), there exist measure preserving maps $\psi_X : X \to X^{\boxplus}$ for $X = A_0, \dots, A_3,\ B_2, \dots, B_5,\ C,$ $E_1,\ E_2,\ F$ and non-decreasing functions $f_X : X^{\boxplus} \to [0,1]$ such that $f_X(\psi_X(x)) = \deg_C^W x$ for almost every $x \in X$. Note that we have not (yet) defined the functions ψ_{B_1} and ψ_D.

We now define a map $g_n : B_{1,n} \to [0,1]^n$ as

$$g_n(x) = \left(\deg_{B_{2,i}}^W(x)\right)_{i \in [n]}$$

for $x \in B_{1,n}$ and $g_\infty : D \to [0,1]^\infty$ as

$$g_\infty(x) = \left(\deg_{B_{2,i}}^W(x)\right)_{i \in \mathbb{N}}$$

for $x \in D$. Note that g_n is well-defined almost everywhere on $B_{1,n}$ and g_∞ almost everywhere on D. We next define a map $\psi_{B_1} : B_1 \to B_1^{\boxplus}$ as

$$\eta_{B_1}^{-1}\left(1 - \frac{1}{2^{n-1}} + \frac{r_n^{-1}(g_n(x))}{2^n}\right).$$

for $x \in B_{1,n}$, and we set $\psi_{B_1}(x)$ to be the same arbitrary vertex of $B_1^{\boxplus}$ for x that does not belong to any $B_{1,n}$, $n \in \mathbb{N}$. Similarly, we define

$$\psi_D(x) = \eta_D^{-1}(r_\infty^{-1}(g_\infty(x))) .$$

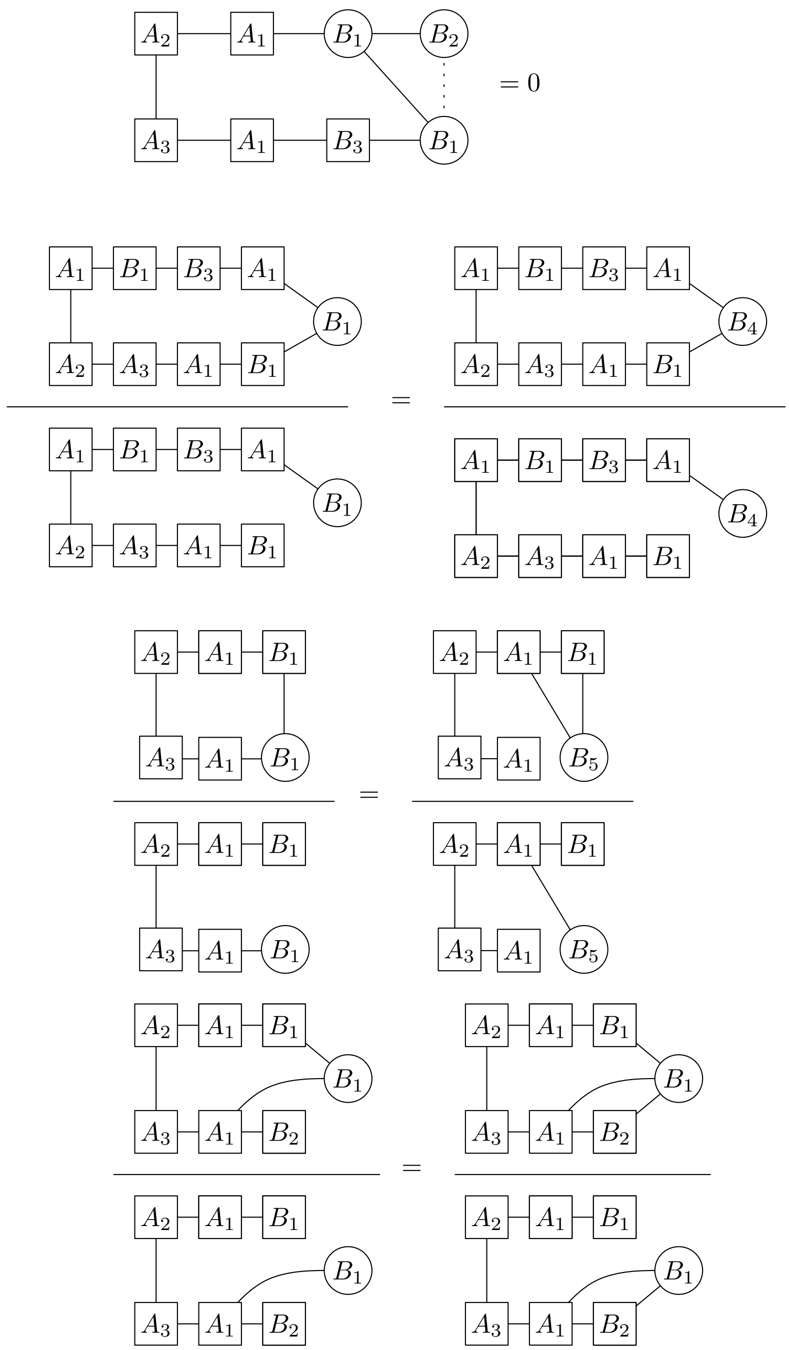

Figure 18: The first four projection constraints.

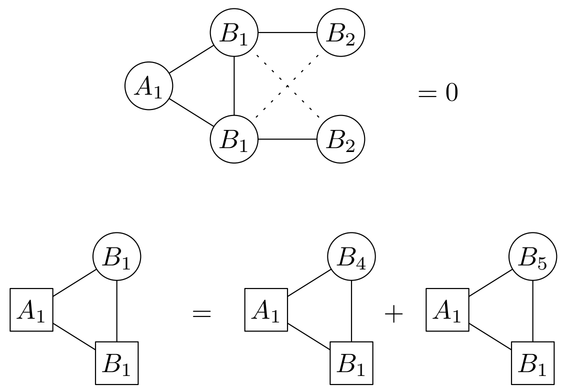

Figure 19: The last two projection constraints.

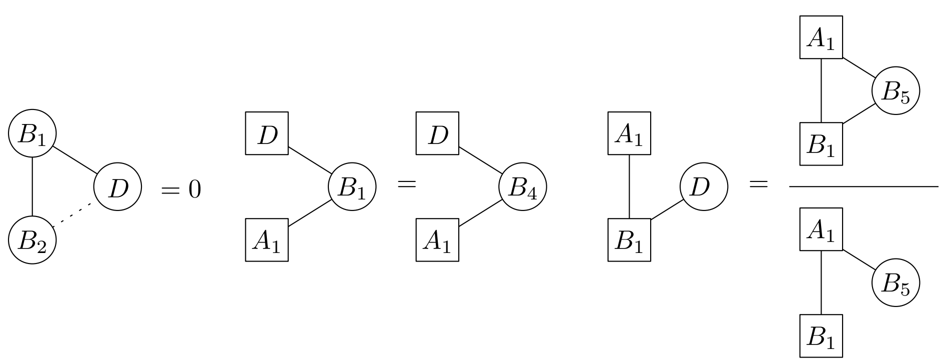

Figure 20: The infinite constraints.

Let ψ be the map from $[0,1]$ to $[0,1]$ equal to the map ψ_X on the part X for $X = A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F$.

In the rest of the section, we show that the graphons $W_{\boxplus}^{\psi}$ and W are equal almost everywhere and the map ψ is measure preserving. This would imply that the graphon $W_{\boxplus}$ is weakly isomorphic to W. Note at this point that the maps ψ_X for $X \neq B_1, D$, which form the map ψ, are measure preserving; so we only need to argue that ψ_{B_1} and ψ_D are measure preserving maps, which we will show in Subsections 5.11 and 5.12.

5.1 Zero and triangular tiles

The zero constraints guarantee that if $W_{\boxplus}$ is equal to zero almost everywhere on $X^{\boxplus} \times Y^{\boxplus}$ for $X, Y \in \{A_0, \ldots, A_3, B_1, \ldots, B_5, C, D, E_1, E_2, F\}$, then the graphon W is equal to zero almost everywhere on $X \times Y$. In particular, the graphons $W_{\boxplus}^{\psi}$ and W are equal almost everywhere on $X \times Y$.

The triangular constraints that correspond to those forcing the half-graphon guarantee that the subgraphon of W induced by $C \times C$ is weakly isomorphic to the half-graphon. The choice of ψ_C now implies that the graphons $W_{\boxplus}^{\psi}$ and W are equal almost everywhere on $C \times C$. We next analyze the constraints depicted in Figure 7. Fix $X \in \{A_0, \ldots, A_3, B_1, \ldots, B_5\}$. The first constraint in Figure 7 yields that $\deg_C^W(z) = \deg_X^W(z)$ for almost every $z \in C$. The second constraint yields that $N_C(x \setminus y)$ or $N_C(y \setminus x)$ or both has measure zero for almost every pair $x, y \in X$. This implies that the graphon W has values 0 and 1 almost everywhere on $X \times C$. The choice of ψ_X implies that W and $W_{\boxplus}^{\psi}$ are equal almost everywhere on $X \times C$ for $X \in \{A_0, \ldots, A_3, B_2, \ldots, B_5\}$. Note that we have not reached this conclusion for $X = B_1$ (because ψ_{B_1} is chosen differently) but we have still shown that the graphon W is equal to 0 or to 1 almost everywhere on $B_1 \times C$ and that the measure of the set containing $b \in B_1$ such that $\deg_C b \leq z$, $z \in [0,1]$, is equal to $z\lambda(B_1)$.

The subgraphon induced by $X \times C$ determines a preorder on the vertices of X according to their relative degrees in C. We often use this fact in our analysis. In this context, we write $x \prec_X y$ instead of $\deg_C x < \deg_C y$ for $x, y \in X$. We also extend this notation to subsets and write $Y \prec_X Z$ for subsets $Y, Z \subseteq X$ if $y \prec_X z$ for every $y \in Y$ and every $z \in Z$.

5.2 Forcing the structure on $A_1 \times A_1$

We now show that the main diagonal checker constraints, which are depicted in Figure 8, force that W and $W_{\boxplus}^{\psi}$ agree almost everywhere on $A_1 \times A_1$. Our line of arguments follows that in [18]; we sketch the arguments and refer the reader to [18] for a more detailed analysis.

The first constraint in Figure 8 implies that if x is a typical vertex of A_1 with respect to $T(W)$, then $W(x, y)$ is equal to 0 or 1 for almost every $y \in A_1$ and if x and x' are two typical vertices of A_1, then either $N_{A_1}(x)$ and $N_{A_1}(x')$ are equal up to a set of measure zero or they are disjoint up to a set of measure zero. Moreover, the measure of the pairs (x, x') such that $W(x, x') \neq 1$ and $N_{A_1}(x)$ and $N_{A_1}(x')$ are equal up to a set of measure zero is zero. Let $\mathcal{J}_{A_1}$ be the set of disjoint non-null measurable subsets of A_1 such that each $J \in \mathcal{J}_{A_1}$ is equal to $N_{A_1}(x)$ up to a set of measure zero for some typical vertex $x \in A_1$ and each $N_{A_1}(x)$ differs from a set contained in $\mathcal{J}_{A_1}$ on a set of measure zero. Our reasoning implies that, except for a subset of $A_1 \times A_1$ of measure zero, $W(x, y) = 1$ for $(x, y) \in A_1 \times A_1$ if and only if x and y belong to the same set $J \in \mathcal{J}_{A_1}$. Informally speaking, the graphon W on $A_1 \times A_1$ is a union of disjoint cliques on $J \in \mathcal{J}_{A_1}$. Observe that since the sets contained in $\mathcal{J}_{A_1}$ are non-null and disjoint, then $\mathcal{J}_{A_1}$ is countable.

The third constraint implies that for every set $J \in \mathcal{J}_{A_1}$, there exists a set $J' \subseteq A_1$ differing from J on a null set such that J' is an interval with respect to $\prec_{A_1}$, i.e., if $x, x' \in J'$ and $x \prec_{A_1} x'' \prec_{A_1} x'$, then $x'' \in J'$. Hence, we can assume without loss of generality that each $J \in \mathcal{J}_{A_1}$ is an interval with respect to $\prec_{A_1}$. The fourth constraint forces that it holds for almost every two vertices $x \prec_{A_1} x'$ from the same $J \in \mathcal{J}_{A_1}$ that

$$\begin{aligned} \lambda(J) &= \lambda\left(\{x'' \in A_1 \mid x \prec_{A_1} x'' \text{ and } x'' \notin J\}\right) \\ &= \lambda\left(\{x'' \in A_1 \mid x'' \notin J \text{ and } \exists z \in J\ z \prec_{A_1} x''\}\right) . \end{aligned}$$

Since the second constraint implies that $\sum\limits_{J \in \mathcal{J}_{A_1}} \lambda(J)^2 = \lambda(A_1)^2/3$, we obtain (see details of the analysis in [18]) that for every $J \in \mathcal{J}_{A_1}$, there exists $k \in \mathbb{N}$ such that J and the set

$$\{x \in A_1 \mid \deg_C x \in [1 - 2^{-k-1}, 1 - 2^{-k}]\}$$

differ on a set of measure zero. We conclude that W agrees with $W^{\psi}_{\boxplus}$ almost everywhere on $A_1 \times A_1$.

5.3 Forcing the structure of $A_0 \times A_1$

We now consider the first level constraints, which are depicted in Figure 11. The first constraint implies that $\deg_{A_0} y = 0$ or $\deg_{A_1} y = 1/2$ for almost every vertex in $y \in A_1$. In particular, $W(x, y) = 0$ for almost every $x \in A_0$ and $y \in A_1$ unless $\deg_{A_1} y = 1/2$. The second constraint forces that the density of W on $A_0 \times A_1$ is equal to $1/2$, which implies that $W(x, y) = 1$ for almost every $x \in A_0$ and $y \in A_1$ such that $\deg_{A_1} y = 1/2$. Therefore, W is equal to $W^{\psi}_{\boxplus}$ almost everywhere on $A_0 \times A_1$.

5.4 Forcing remaining diagonal checker subgraphons

We now use the bipartition constraints, which are depicted in Figure 9. Fix (X, Y) to be one of the pairs (A_1, A_2), (A_1, B_2), (A_1, B_3), (A_1, B_4), (A_1, B_5), (A_2, A_3) and

(A_2, B_2). Note that the list misses the pair (A_1, B_1), which is analyzed separately afterwards.

The first constraint in Figure 9 implies that there exist a set $\mathcal{J}_X$ formed by disjoint non-null subsets of X, a set $\mathcal{J}_Y$ formed by disjoint non-null subsets of Y, and a bijection $f : \mathcal{J}_X \to \mathcal{J}_Y$ such that except for a subset of $X \times Y$ of measure zero, it holds that $W(x, y) = 1$ for $(x, y) \in X \times Y$ iff there exist $J \in \mathcal{J}_X$ such that $x \in J$ and $y \in f(J)$, and $W(x, y) = 0$ elsewhere on $X \times Y$. Informally speaking, the graphon W on $X \times Y$ is a disjoint union of complete bipartite subgraphons between $J \in \mathcal{J}_X$ and $f(J) \in \mathcal{J}_Y$. We remark here that the set $\mathcal{J}_{A_1}$ can in principle differ from the set defined in Subsection 5.2, however, we will later argue that they actually coincide (in the sense that the elements of the set differ from each other on a set of measure zero).

Analogously to Subsection 5.2, the second constraint depicted in Figure 9 implies that each set contained in $\mathcal{J}_X$ differs from an interval with respect to $\preceq_X$ on a set of measure zero and the third constraint implies that each set contained in $\mathcal{J}_Y$ differs from an interval with respect to $\preceq_Y$ on a set of measure zero. Hence, we can assume without loss of generality that each set contained in $\mathcal{J}_X$ is an interval with respect to $\preceq_X$ and each set contained in $\mathcal{J}_Y$ is an interval with respect to $\preceq_Y$. Finally, the fourth constraint implies that the intervals are in the same order, i.e., if $J, J' \in \mathcal{J}_X$ satisfy that $J \preceq_X J'$, then $f(J) \preceq_Y f(J')$.

It remains to determine the measures of the sets contained in $\mathcal{J}_X$ and $\mathcal{J}_Y$. Recall that we have shown that W agrees with $W^{\psi}_{\boxplus}$ almost everywhere on $A_1 \times A_1$. We now split the argument depending on whether $X = A_1$ or $X = A_2$ and start with analyzing the case $X = A_1$. If $Y \neq A_3$, consider the first constraint depicted in Figure 10; this constraint implies that almost all the vertices of $X = A_1$ have the same relative degree with respect to A_1 as with respect to Y. Hence, almost every $x \in X$ belongs to some $J \in \mathcal{J}_X$ and for every $J \in \mathcal{J}_X$, it holds that $\lambda(f(J)) = \lambda(A_1) \deg_{A_1} x$ for almost every $x \in J$. Since f is a bijection and the sets contained in $\mathcal{J}_Y$ are disjoint, it follows that $\mathcal{J}_X$ coincides with the set $\mathcal{J}_{A_1}$ defined in Subsection 5.2 and $\lambda(f(J)) = \lambda(J)$ for every $J \in \mathcal{J}_X$. If $Y = A_3$, the third constraint in Figure 10 yields that almost all the vertices of $X = A_1$ have either the relative degree with respect to A_1 equal to $1/2$ or the relative degree with respect to Y double the relative degree with respect to A_1; this again implies that $\mathcal{J}_X$ coincides with the set $\mathcal{J}_{A_1}$ defined in Subsection 5.2 and $\lambda(f(J)) = 2\lambda(J)$ for every $J \in \mathcal{J}_X$ unless $\lambda(J) = \lambda(X)/2$. Since the elements of $\mathcal{J}_Y$ are disjoint intervals with respect to $\preceq_Y$ and the bijection f preserves their order, we conclude that for every $J \in \mathcal{J}_Y$, there exists $k \in \mathbb{N}$ such that J and the set

$$\{y \in Y \mid \deg_C y \in [1 - 2^{-k-1}, 1 - 2^{-k}]\}$$

differ on a set of measure zero (this holds both if $Y = A_3$ and if $Y \neq A_3$). It follows that W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $X \times Y$ if $X = A_1$, i.e, W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $A_1 \times A_2$, $A_1 \times A_3$ and $A_1 \times B_2, \ldots, A_1 \times B_5$.

We next finish the analysis of the case $X = A_2$; note that Y is either A_3 or B_2 in this case. The second constraint depicted in Figure 10 implies that almost all the vertices of $X = A_2$ have the same relative degree with respect to A_1 as with respect to Y. Hence, almost every $x \in X = A_2$ belongs to some $J \in \mathcal{J}_{A_2}$ and $\lambda(f(J)) = \lambda(A_1)\deg_{A_1} x$ for almost every $x \in J$. Since f is a bijection, we conclude (using the already analyzed structure on $A_2 \times A_1$) that $\lambda(f(J)) = \lambda(J)$ for every $J \in \mathcal{J}_{A_2}$. Hence, the set $\mathcal{J}_{A_2}$ coincides with the set $\mathcal{J}_{A_2}$ as defined in the case $(X,Y) = (A_1, A_2)$, and for every $J \in \mathcal{J}_Y$, there exists $k \in \mathbb{N}$ such that J and the set

$$\{y \in Y \mid \deg_{A_1} y \in [1 - 2^{-k-1}, 1 - 2^{-k}]\}$$

differ on a set of measure zero. It follows that W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $X \times Y$ if (X,Y) is (A_2, A_3) or (A_2, B_2).

In the previous analysis, we have omitted the case $(X,Y) = (A_1, B_1)$. As in the general case $X = A_1$ considered above, we derive that the top two constraints in Figure 9 imply that there exist a set $\mathcal{J}_X$ formed by disjoint non-null subsets of X that are intervals with respect to $\preceq_X$, a set $\mathcal{J}_Y$ formed by disjoint non-null subsets of Y, and a bijection $f : \mathcal{J}_X \to \mathcal{J}_Y$ such that except for a subset of $X \times Y$ of measure zero, it holds that $W(x,y) = 1$ for $(x,y) \in X \times Y$ iff there exist $J \in \mathcal{J}_X$ such that $x \in J$ and $y \in f(J)$. Note that we do not make any claims about the structure of the sets contained in $\mathcal{J}_Y$. The first constraint in Figure 10 implies that $\lambda(f(J)) = \lambda(J)$ for every $J \in \mathcal{J}_X$. It follows that $\mathcal{J}_X$ coincides with $\mathcal{J}_{A_1}$ defined in Subsection 5.2, almost every vertex $y \in Y$ belongs to $J \in \mathcal{J}_Y$ and the measure of $y \in Y$ with $\deg_{A_1} y = 2^{-k}$ is equal to 2^{-k}. In particular, the measure of $B_{1,k}$ is 2^{-k}. It follows that W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $A_1 \times B_1$.

Because each of the sets $\mathcal{J}_X$, $X \in \{A_1, \dots, A_3, B_1, \dots, B_5\}$, is the same in all the definitions (in the sense that its elements differ from each other on a set of measure zero) that we have given in this subsection and Subsection 5.2, we can just use $\mathcal{J}_X$ without referring to the particular place where the set was defined. We now split each part $X \in \{A_1, \dots, A_3, B_1, \dots, B_5\}$ into *levels* in the way analogous to that the parts of $W_{\boxplus}$ are split. For $k \in \mathbb{N}$, the k-th level $A_{i,k}$, $i \in \{1, 2\}$, of A_i is formed by $x \in A_i$ such that $\deg_{A_1} x = 2^{-k}$, the k-th level level $A_{3,k}$ of A_3 is formed by $x \in A_3$ such that $\deg_{A_2} x = 2^{-k}$, and the k-th level $B_{i,k}$, $i \in \{1, \dots, 5\}$, of B_i is formed by $x \in B_j$ such that $\deg_{A_1} x = 2^{-k}$. The levels of X coincide with sets contained $\mathcal{J}_X$ (up to a difference on a set of measure zero). Note that the measure of the level $A_{i,k}$ or $B_{i,k}$ is 2^{-k}. Also note that this coincides with our previous definition of $B_{1,k}$.

5.5 Using levels in density expressions

Many of the density expressions used in the following subsections use the level structure of the parts A_1 , A_2, A_3, $B_1, \dots, B_5$ of W in combination with the

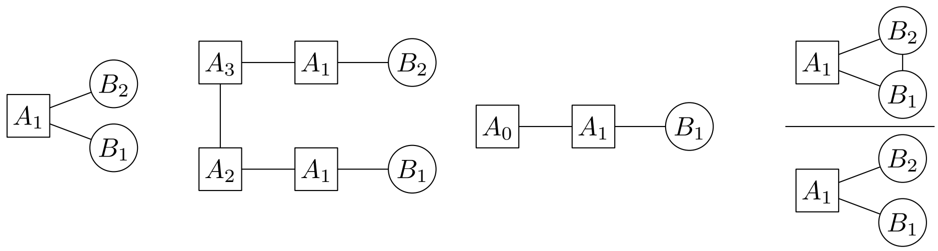

Figure 21: Density expressions specifying levels of vertices.

structure of W that we have already analyzed. Some examples of decorated graphs that we use are given in Figure 21. In the first decorated graph, all three vertices must belong to the same level (ignoring events with probability zero), i.e., if the root belongs to the k-th level of A_1, then the expression is equal (with respect to W) to 2^{-2k}, which is the product of the probabilities that a random vertex of B_1 belongs to $B_{1,k}$ and that a random vertex of B_2 belongs to $B_{2,k}$.

In the second decorated graph, if the root decorated with A_2 belongs to the k-th level of A_2, which is $A_{2,k}$, then its neighbors must belong to $A_{1,k}$ and $A_{3,k}$ and the remaining root to $A_{1,k+1}$. In such case, the expression is equal to 2^{-2k-1}, which is the product of the probabilities that a random vertex of B_1 belongs to $B_{1,k}$ and that a random vertex of B_2 belongs to $B_{2,k+1}$. In the third decorated graph, the root decorated with A_1 must belong to $A_{1,1}$ and the expression is equal to $1/2$, which is the probability that a random vertex of B_1 belongs to $B_{1,1}$.

The final expression is more complex. Suppose that the root belongs to $A_{1,k}$. The denominator is equal to 2^{-2k} as we have discussed earlier. The numerator is equal to 2^{-2k} multiplied by the density between $B_{1,k}$ and $B_{2,k}$, i.e., the whole expression is equal to the density of W between the $B_{1,k}$ and $B_{2,k}$.

5.6 Stair constraints

We now focus on the stair constraints, which are depicted in Figure 12. They are intended to force the desired structure on $B_1 \times B_3$. The first constraint in Figure 12 determines the relative degrees of vertices of B_1 in B_3, i.e., it enforces that $\deg_{B_3} x = 1 - 2^{-k}$ for almost every $x \in B_{1,k}$. The second constraint forces that the following holds for almost every vertex $x \in B_1$ and every $k \in \mathbb{N}$: if $\deg_{B_{3,k+1}} x > 0$, then $\deg_{B_{3,k}} x = 1$ (if $\deg_{B_{3,k}} x = 0$, then there exists a choice of the roots decorated with A_1, A_2, A_3 and A_1 such that the density expression is non-zero with x being the root decorated with B_1). Consequently, for almost every $x \in B_1$, there exists $k_0 \in \mathbb{N}$ such that $W(x, y) = 1$ for almost every $y \in B_{3,k}$, $k < k_0$ and $W(x, y) = 0$ for almost every $y \in B_{3,k}$, $k > k_0$. However, it is possible that $\deg_{B_3} x = 1 - 2^{-k}$ for almost every $x \in B_{1,k}$ only if it holds that for almost every $x \in B_1$, k_0 is equal to the level of x and $W(x, y) = 1$ for almost every $y \in B_{3,k_0}$. It follows that W agrees with $W^{\psi}_{\boxplus}$ almost everywhere on $B_1 \times B_3$.

5.7 Coordinate constraints

The coordinate constraints from Figure 13 force basic structure between the parts B_1 and D on one side and the parts B_2, B_4 and B_5 on the other side. Fix Y to be one of the parts B_2, B_4 and B_5. The first constraint depicted in Figure 13 implies that almost every vertex b of B_1 with non-zero relative degree with respect to Y_k has relative degree one with respect to $B_{3,k}$. Hence, almost every $b \in B_{1,k}$ satisfies that $W(b,y) > 0$ only if y belongs to $Y_{k'}$ with $k' \leq k$ except for a set of measure zero; in particular, $W(b,y) = 0$ for almost every $b \in B_{1,k}$ and $y \in Y_{k'}$, $k' > k$.

In addition to Y, fix X to be either B_1 or D. The second constraint implies that $N_X(y \setminus y')$ has measure zero for every k and almost every two $y, y' \in Y_k$ such that $y \prec_Y y'$; consequently, W is equal to 0 or 1 almost everywhere on $X \times Y$. It follows that for almost every $x \in X$ and every k, there exists $y_0 \in Y_k$ such that $W(x,y) = 1$ for almost every $y \in Y_k$ with $y \preceq_Y y_0$ and $W(x,y) = 0$ for almost every $y \in Y_k$ with $y_0 \preceq_Y y$. In particular, the definition of ψ on B_1 and D now yields that $W = W^{\psi}_{\boxplus}$ almost everywhere on $B_1 \times B_2$ and $D \times B_2$.

5.8 Initial coordinate constraint

We now consider the initial coordinate constraint, which can be found in Figure 14. The decorated graphs appearing in the constraint are evaluated to the following quantities when b is the root decorated with B_1:

$$\deg_{B_{2,1}} b = \frac{\deg_C b - (1 - 2\deg_{A_1} b)}{\deg_{A_1} b} \ .$$

Consider now $b \in B_{1,k}$. Unless b belongs to an exceptional set of measure zero, the right hand side belongs to the interval $[0,1]$ only if $\deg_C b$ belongs to the interval $[1 - 2^{-k+1}, 1 - 2^{-k}]$. This implies that W agrees with $W^{\psi}_{\boxplus}$ almost everywhere on $B_1 \times C$.

The results of Subsection 5.1 imply that the measure of $b \in B_1$ with $\deg_C b \leq z$ for $z \in [0,1]$ is equal to $z\lambda(B_1)$. Hence, it follows for every $z \in [0,1]$ and every $k \in \mathbb{N}$ that

$$\lambda\left(\{b \in B_{1,k} \mid (g_k(b))_1 \leq z\}\right) = z \cdot \lambda\left(B_{1,k}\right) \ . \tag{3}$$

5.9 Distribution constraints

The equality (3) can be interpreted as saying that the first coordinate of each g_k is uniformly distributed. We now argue that the same holds for the remaining coordinates of g_k, $k \in \mathbb{N}$, and all the coordinates of g_∞.

The decorated graphs appearing in the first constraint in Figure 15 are evaluated to the following quantities for every $k \in \mathbb{N}$ and almost every $b \in B_{2,k'}$,

$k' \leq k$:

$$\deg_{B_{1,k}} b = 1 - \frac{\deg_C b - (1 - 2\deg_{A_1} b)}{\deg_{A_1} b} \ .$$

Almost every $b \in B_{2,k'}$ satisfies that $\deg_{A_1} b = 2^{-k'}$; so, we get that

$$\deg_{B_{1,k}} b = 1 - 2^{k'} \left(\deg_C b - \left(1 - 2^{-(k'-1)}\right)\right) \ .$$

Informally speaking, the relative degree of almost every $b \in B_{2,k'}$ decreases linearly from 1 to 0 with its position within $B_{2,k'}$ given by $\prec_{B_2}$. This and the analysis of the structure between the parts B_1 and B_2 in Subsection 5.7 imply that

$$\lambda\left(\{b \in B_{1,k} \mid (g_k(b))_{k'} \leq z\}\right) = z \cdot \lambda\left(B_{1,k}\right) \tag{4}$$

for every $k \in \mathbb{N}$, every $k' \in [k]$ and every $z \in [0,1]$.

The second constraint depicted in Figure 15 implies the analogous statement for the structure between D and B_2. In particular, it holds that

$$\lambda\left(\{d \in D \mid (g_\infty(d))_{k'} \leq z\}\right) = z \cdot \lambda(D) \tag{5}$$

for every $k' \in \mathbb{N}$ and $z \in [0,1]$.

5.10 Product constraints

We now analyze the product constraints, which are depicted in Figures 16 and 17. Fix (X,Y) to be one of the pairs (B_1,B_4), (B_1,B_5), (D,B_4) and (D,B_5). The results on the structure of the graphon W between X and Y from Subsection 5.7 imply that if we show that $\deg_{Y_{k'}} x = \deg_{Y^{\boxplus}_{k'}} \psi(x)$ for every $x \in X$ and $k' \in \mathbb{N}$, then W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $X \times Y$.

Suppose that $(X,Y) = (B_1,B_4)$. The first constraint depicted in Figure 16 implies that $\deg_{B_{4,1}} b = \deg_{B_{2,1}} b$, i.e., $\deg_{B_{4,1}} b = (g_k(b))_1$, for almost every $b \in B_{1,k}$. The second constraint yields that $\deg_{B_{4,i}} b = \deg_{B_{2,i}} b \cdot \deg_{B_{4,i-1}} b$ for almost every $b \in B_{1,k}$; if $i > k$, then $\deg_{B_{2,i}} b$ is equal to zero and so is $\deg_{B_{4,i}} b$ for such b. If $i \leq k$, we obtain that it holds for almost every $b \in B_{1,k}$ that

$$\deg_{B_{4,i}} b = (g_k(b))_i \cdot \deg_{B_{4,i-1}} b = \prod_{i'=1}^{i} (g_k(b))_{i'} \ .$$

Hence, the graphons W and $W^{\psi}_{\boxplus}$ are equal almost everywhere on $X \times Y = B_1 \times B_4$. The remaining three choices of $X \times Y$ are analyzed in the completely analogous way.

5.11 Projection constraints

This subsection forms the core of our argument. We show that the mapping ψ_{B_1} is measure preserving; this will be implied by proving the following identity for every $k \in \mathbb{N}$.

$$\lambda\left(\{b \in B_{1,k} \mid (g_k(b))_i \leq z_i \ \forall i \in [k]\}\right) = \lambda(B_{1,k}) \prod_{i\in[k]} z_i \ \text{ for all } z \in [0,1]^k. \tag{6}$$

Note that if (6) holds, then $g_k(B_{1,k} \setminus Z)$, i.e., the image of g_k in $[0,1]^k$ after removing Z from the domain, is dense for every $Z \subseteq B_{1,k}$ of measure zero.

We prove (6) by induction on $k \in \mathbb{N}$. Note that (6) holds for $k = 1$ by (3). As a part of the induction argument, we will also show that W and $W^{\psi}_{\boxplus}$ are equal almost everywhere on $B_{1,k'} \times B_{1,k}$ if $k' < k$.

Fix integers k' and k such that $k' < k$ and assume using the induction that (6) holds for all smaller values of k. The first constraint depicted in Figure 18 yields that $N_{B_2}(b' \setminus b)$ has measure zero, i.e., $\deg_{B_{2,i}} b' \leq \deg_{B_{2,i}} b$, for almost every pair of vertices $b' \in B_{1,k'}$ and $b \in B_{1,k}$ such that $W(b', b) > 0$. In other words, the set

$$N_{B_{1,k'}}(b) \setminus \{b' \in B_{1,k'} \mid (g_{k'}(b'))_i \leq (g_k(b))_i \ \forall i \in [k']\}$$

has measure zero for almost every $b \in B_{1,k}$ and the set

$$N_{B_{1,k}}(b') \setminus \{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k'}(b'))_i \ \forall i \in [k']\}$$

has measure zero for almost every $b' \in B_{1,k'}$.

The second constraint in Figure 18 forces that

$$\deg_{B_{1,k'}} b = \deg_{B_{4,k'}} b$$

for almost every $b \in B_{1,k}$. Recall that we have shown in Subsection 5.10 that

$$\deg_{B_{4,k'}} b = \prod_{i\in[k']} (g_k(b))_i \tag{7}$$

for almost every $b \in B_{1,k}$. Since the equality (6) holds for k', it follows that the set $N_{B_{1,k'}}(b)$ and the set

$$\{b' \in B_{1,k'} \mid (g_{k'}(b'))_i \leq (g_k(b))_i \ \forall i \in [k']\}$$

differ on a set of measure zero and therefore the graphon W is equal to 1 almost everywhere on $N_{B_{1,k'}}(b)$ for almost every $b \in B_{1,k}$. Hence, W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $B_{1,k'} \times B_{1,k}$. It follows that $N_{B_{1,k}}(b')$ and the set

$$\{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k'}(b'))_i \ \forall i \in [k']\} \tag{8}$$

differ on a set of measure zero.

We now present the induction step for proving (6) by showing that it holds for k assuming that (6) holds for the previous value of k, i.e., $k-1$. The third constraint depicted in Figure 18 guarantees that

$$\deg_{B_{1,k}} b' = \deg_{B_{5,k-1}} b' \ ,$$

which yields that

$$\deg_{B_{1,k}} b' = \prod_{i \in [k-1]} (1 - (g_{k-1}(b'))_i)$$

for almost every $b' \in B_{1,k-1}$. Since $N_{B_{1,k}}(b')$ and the set (8) differ on a set of measure zero, we get that

$$\frac{\lambda(\{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k-1}(b'))_i \ \forall i \in [k-1]\})}{\lambda(B_{1,k})} = \prod_{i \in [k-1]} (1 - (g_{k-1}(b')_i)) \quad (9)$$

for almost every $b' \in B_{1,k-1}$.

The fourth constraint in Figure 18 implies that

$$\deg_{B_{1,k}} b' = \deg_{N_{B_{1,k}}(x)} b' \quad (10)$$

for almost every $b' \in B_{1,k-1}$ and almost every $x \in B_{2,k}$ (the vertex b' is the root labeled with B_1 and the vertex x is the root labeled with B_2 in the constraint); note that $W(x,y) = 1$ almost every $x \in B_{2,k}$ and for almost every $y \in N_{B_{1,k}}(x)$, by the structure of the graphon established in Subsection 5.7. The structure of the graphon established in Subsection 5.7 also implies the following: it holds for almost every $x \in B_{2,k}$ that the set $N_{B_{1,k}}(x)$ is the set of vertices $y \in B_{1,k}$ with $(g_k(y))_k \geq \zeta$ for some $\zeta \in [0,1]$, and it holds for almost every $\zeta \in [0,1]$ that there exists $x \in B_{2,k}$ such that the set $N_{B_{1,k}}(x)$ is the set of vertices $y \in B_{1,k}$ with $(g_k(y))_k \geq \zeta$. Hence, the equality (10) guarantees that

$$\begin{aligned} & \frac{\lambda(\{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k-1}(b'))_i \ \forall i \in [k-1]\})}{\lambda(B_{1,k})} \quad (11) \\ = \ & \frac{\lambda(\{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k-1}(b'))_i \ \forall i \in [k-1] \text{ and } (g_k(b))_k \geq \zeta\})}{\lambda(\{b \in B_{1,k} | (g_k(b))_k \geq \zeta\})} \end{aligned}$$

for almost every $b' \in B_{1,k-1}$ and almost every $\zeta \in [0,1]$. The equality (4) implies that

$$\lambda(\{b \in B_{1,k} | (g_k(b))_k \geq \zeta\}) = (1-\zeta)\lambda(B_{1,k})$$

for every $\zeta \in [0,1]$. This combined with (9) and (11) yields that

$$\frac{\lambda(\{b \in B_{1,k} \mid (g_k(b))_i \geq (g_{k-1}(b'))_i \ \forall i \in [k-1] \text{ and } (g_k(b))_k \geq \zeta\})}{(1-\zeta)\lambda(B_{1,k})}$$

$$= \prod_{i\in[k-1]} (1 - (g_{k-1}(b)_i))$$

for almost every $b' \in B_{1,k-1}$ and almost every $\zeta \in [0,1]$. Since the image of g_{k-1} is dense even after removing a set of measure zero from its domain, we conclude that

$$\frac{\lambda(\{b \in B_{1,k} \mid (g_k(b))_i \geq z_i \ \forall i \in [k]\})}{\lambda(B_{1,k})} = \prod_{i\in[k]} (1 - z_i)$$

for every $z \in [0,1]^k$. However, this is equivalent (by applying a straightforward manipulation using the principle of inclusion and exclusion) to (6) for k. Since g_k satisfies (6), the map $\psi_{B_{1,k}}$ is measure preserving. Consequently, ψ_{B_1} is a measure preserving map.

We have shown that the graphons W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $B_{1,k'} \times B_{1,k}$ for $k' \neq k$. It remains to analyze the structure of the graphon W on $B_{1,k} \times B_{1,k}$, $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. The first constraint in Figure 19 forces that $N_{B_2}(b' \setminus b)$ or $N_{B_2}(b \setminus b')$ has measure zero for almost all $b, b' \in B_{1,k}$ with $W(b,b') > 0$. Hence, $N_{B_{1,k}}(b)$ is a subset of the set

$$\{b' \in B_{1,k} \mid (g_k(b))_i \leq (g_k(b'))_i \ \forall i \in [k]\} \cup \{b' \in B_{1,k} \mid (g_k(b))_i \geq (g_k(b'))_i \ \forall i \in [k]\} \tag{12}$$

for almost every $b \in B_{1,k}$. Since ψ_{B_1} is a measure preserving map and the graphons W and $W^{\psi}_{\boxplus}$ are equal almost everywhere on $B_1 \times B_2$, $B_1 \times B_4$ and $B_1 \times B_5$, it follows that the measure of the set given in (12) is equal to

$$\lambda(B_{1,k}) \left(\prod_{i=1}^{k} \deg_{B_{2,i}} b + \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b) \right)$$
$$= \lambda(B_{1,k}) \left(\deg_{B_{4,k}} b + \deg_{B_{5,k}} b \right)$$

for almost every $b \in B_{1,k}$.

The second constraint in Figure 19 implies that $\deg_{B_{1,k}} b = \deg_{B_{4,k}} b + \deg_{B_{5,k}} b$ for almost every $b \in B_{1,k}$. Hence, it must hold that $N_{B_{1,k}}(b)$ is the set given in (12) and $W(b,b') = 1$ for almost every $b \in B_{1,k}$ and $b' \in N_{B_{1,k}}(b)$. We conclude that the graphons W and $W^{\psi}_{\boxplus}$ agree almost everywhere on $B_{1,k} \times B_{1,k}$ for every $k \in \mathbb{N}$.

5.12 Infinite constraints

In this subsection, we establish that the graphon W is equal to $W^{\psi}_{\boxplus}$ almost everywhere on $B_1 \times D$ by proving that the two graphons are equal almost everywhere on $B_{1,k} \times D$ for every $k \in \mathbb{N}$. We also establish that ψ_D is a measure preserving map by showing for every $k \in \mathbb{N}$ that

$$\lambda\left(\{d \in D \mid (g_\infty(d))_i \leq z_i \ \forall i \in [k]\}\right) = \lambda(D) \prod_{i\in[k]} z_i \ \text{ for all } z \in [0,1]^k \tag{13}$$

Fix $k \in \mathbb{N}$ for the rest of the subsection.

Let $d \in D$ and $b \in B_{1,k}$. We define

$$M^k_{B_1}(d) = \{b \in B_{1,k} \mid \forall\, i \in [k]\, \deg_{B_{2,i}} b \le \deg_{B_{2,i}} d\} \text{ and}$$

$$M^k_D(b) = \{d \in D \mid \forall\, i \in [k]\, \deg_{B_{2,i}} d \ge \deg_{B_{2,i}} b\}.$$

We obtain using (6) that

$$\lambda(M^k_{B_1}(d)) = \lambda(B_{1,k}) \cdot \prod_{i=1}^{k} \deg_{B_{2,i}} d = \lambda(B_{1,k}) \cdot \prod_{i=1}^{k} (g_\infty(d))_i$$

for almost every $d \in D$.

We now analyze the constraints depicted in Figure 20. The first constraint forces that $N_{B_2}(b \setminus d)$ has measure zero for almost every $b \in B_1$ and almost every $d \in D$ with $W(b, d) > 0$. It follows that the set $N_{B_{1,k}}(d)$ is a subset of $M^k_{B_1}(d)$ up to a set of measure zero for almost every $d \in D$ and the set $N_D(b)$ is a subset of $M^k_D(b)$ up to a set of measure zero for almost every $b \in B_{1,k}$.

The second constraint in Figure 20 implies that $\deg_{B_{1,k}} d = \deg_{B_{4,k}} d$ for almost every $d \in D$. We have shown in Subsection 5.10 that $\deg_{B_{4,k}} d = \prod_{i=1}^{k} \deg_{B_{2,i}} d$ for almost every $d \in D$. Therefore,

$$\lambda\left(N_{B_{1,k}}(d)\right) \ge \lambda(B_{1,k}) \cdot \deg_{B_{1,k}} d = \lambda(B_{1,k}) \prod_{i=1}^{k} \deg_{B_{2,i}} d = \lambda\left(M^k_{B_1}(d)\right)$$

for almost every $d \in D$. It follows that the sets $N_{B_{1,k}}(d)$ and $M^k_{B_1}(d)$ differ on a set of measure zero and $W(d, b) = 1$ for almost every $d \in D$ and almost every $b \in N_{B_{1,k}}(d)$. This determines the structure of W on $B_1 \times D$. In particular, it follows that W and $W^{\psi}_{\boxplus}$ are equal almost everywhere on $B_1 \times D$. Also note that the sets $N_D(b)$ and $M^k_D(b)$ differ on a set of measure zero for almost every $b \in B_{1,k}$.

We now show that g_∞ satisfies (13). The third constraint in Figure 20 implies that

$$\deg_D b = \deg_{B_{5,k}} b = \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b)$$

for almost every $b \in B_{1,k}$. Since $\deg_D b = \lambda(N_D(b))/\lambda(D)$ for almost every $b \in B_{1,k}$, we deduce that

$$\frac{\lambda(\{d \in D \mid \forall\, i \in [k]\ (g_\infty(d))_i \ge \deg_{B_{2,i}} b\})}{\lambda(D)} = \prod_{i=1}^{k} (1 - \deg_{B_{2,i}} b) = \prod_{i=1}^{k} (1 - (g_k(b))_i) \tag{14}$$

for almost every $b \in B_{1,k}$. Since the image of g_k is dense in $[0, 1]^k$ even after removing a set of measure zero from its domain, we conclude that g_∞ satisfies (13). It follows that ψ_D is measure preserving.

5.13 Structure involving the parts E_1, E_2 and F

Let $I = [0,1] \setminus (E_1 \cup E_2 \cup F)$. The degree unifying constraints, which are depicted in Figure 4, imply that for every $X \in \{A_0, \dots, A_3, B_1, \dots, B_5, C\}$ and almost every $x, x' \in X$:

$$\frac{1}{\lambda(E_1)} \int\limits_{E_1} W(x,z)\,\mathrm{d}z = (1 - \deg_I x) \text{ and}$$

$$\frac{1}{\lambda(E_1)} \int\limits_{E_1} W(x,z)W(x',z)\,\mathrm{d}z = (1 - \deg_I x)(1 - \deg_I x').$$

The reasoning given in [32, proof of Lemma 3.3] implies that the latter identity holds for almost every $x = x' \in X$, i.e., it holds that

$$\frac{1}{\lambda(E_1)} \int\limits_{E_1} (W(x,z))^2\,\mathrm{d}z = (1 - \deg_I x)^2$$

for almost every $x \in X$. The Cauchy-Schwarz inequality yields that $W(x,z) = 1 - \deg_I x$ for almost every $x \in X$ and $z \in E_1$. This implies that $\deg_{[0,1]\setminus(E_2\cup F)} x = 1/2$ for almost every $x \in I \setminus D$. Since the graphons $W^{\psi}_{\boxplus}$ and W agree almost everywhere on I^2, almost every $x \in I$ must have the same relative degree on I in both $W^{\psi}_{\boxplus}$ and W. It follows that $W^{\psi}_{\boxplus}$ and W agree almost everywhere on $I \times E_1$.

Similarly, the constraints depicted in Figure 5 imply that

$$\deg_{B_1\cup B_2\cup B_4\cup B_5\cup E_2} x = 1/2$$

for almost every $x \in D$, which implies that $W^{\psi}_{\boxplus}$ and W are equal almost everywhere on $D \times E_2$. Finally, the two degree distinguishing constraints yield that the graphon W on $X \times F$, for $X = A_1, \dots, A_3, B_1, \dots, B_5, C$, is constant and its density is the one given by Table 2. We conclude that the graphons $W^{\psi}_{\boxplus}$ and W are equal almost everywhere.

6 Conclusion

The method for establishing that a graphon is finitely forcible using decorated constraints, which originated in [18] and was further developed in this paper, turned out to be useful in several follow up results, which we now mention. First, Cooper et al. [10] addressed one of the motivations for Conjecture 1 and constructed a finitely forcible graphon W such that the number of parts in every weak ε-regular partition of W is at least $2^{\Omega\left(\varepsilon^{-2}/2^{5\log^*\varepsilon^{-2}}\right)}$ for an infinite sequence of ε tending to 0. This almost matches the general upper bound of $2^{\Theta(\log^2\varepsilon^{-1})}$ on the number of parts in weak ε-regular partitions [16]. It is worth noting that while for any $\varepsilon > 0$, there is a graphon W such that each weak ε-regular W has at least $2^{\Omega(\varepsilon^{-2})}$ parts, there is no graphon such that every weak ε-regular partition of W is at least $2^{\Omega(\varepsilon^{-2})}$ parts for an infinite sequence of ε tending to zero. The line of research on constructions of complex finitely forcible graph limits culminated with the result of Cooper et al. [11] that every graphon is a subgraphon of a finitely forcible graphon. This general result of Cooper et al. was also a key ingredient in the argument of Grzesik et al. in [20] for disproving a conjecture of Lovász that every extremal graph theory problem has a finitely forcible optimum, which was one of the most cited open problems on dense graph limits.

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References

[1] R. Baber: Turán densities of hypercubes, preprint available as arXiv:1201.3587.

[2] R. Baber and J. Talbot: A solution to the 2/3 conjecture, *SIAM J. Discrete Math.* **28** (2014), 756-766.

[3] R. Baber and J. Talbot: Hypergraphs do jump, *Combin. Probab. Comput.* **20** (2011), 161–171.

[4] J. Balogh, P. Hu, B. Lidický, and H. Liu: Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube, *European J. Combin.* **35** (2014), 75–85.

[5] B. Bollobás and O. Riordan: Sparse graphs: Metrics and random models, *Random Structures Algorithms* **39** (2011), 1–38.

[6] C. Borgs, J.T. Chayes, and L. Lovász: Moments of two-variable functions and the uniqueness of graph limits, *Geom. Funct. Anal.* **19** (2010), 1597–1619.

[7] C. Borgs, J. Chayes, L. Lovász, V.T. Sós, B. Szegedy, and K. Vesztergombi: Graph limits and parameter testing, in: *Proceedings of the 38rd Annual ACM*

Symposium on the Theory of Computing (STOC), ACM, New York, 2006, 261–270.

[8] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing, *Adv. Math.* **219** (2008), 1801–1851.

[9] C. Borgs, J.T. Chayes, L. Lovász, V.T. Sós, and K. Vesztergombi: Convergent sequences of dense graphs II. Multiway cuts and statistical physics, *Ann. of Math.* **176** (2012), 151–219.

[10] J.W. Cooper, T. Kaiser, D. Král', J.A. Noel: Weak regularity and finitely forcible graph limits, *Trans. Amer. Math. Soc.* **370** (2018), 3833–3864.

[11] J.W. Cooper, D. Král', T. Martins: Finitely forcible graph limits are universal, preprint available as arXiv:1701.03846.

[12] F.R.K. Chung, R.L. Graham, and R.M. Wilson: Quasi-random graphs, *Combinatorica* **9** (1989), 345–362.

[13] D. Conlon and J. Fox: Bounds for graph regularity and removal lemmas, *Geom. Funct. Anal.* **22** (2012), 1191–1256.

[14] P. Diaconis, S. Holmes, and S. Janson: Threshold graph limits and random threshold graphs, *Internet Math.* **5** (2009), 267–318.

[15] G. Elek: On limits of finite graphs, *Combinatorica* **27** (2007), 503–507.

[16] A. Frieze and R. Kannan: Quick approximation to matrices and applications, *Combinatorica* **19**, 175–220.

[17] R. Glebov, D. Král', and J. Volec: A problem of Erdős and Sós on 3-graphs, *Israel J. Math.* **211** (2016), 349–366.

[18] R. Glebov, D. Král', and J. Volec: Compactness and finite forcibility of graphons, preprint available as arXiv:1309.6695.

[19] A. Grzesik: On the maximum number of five-cycles in a triangle-free graph, *J. Combin. Theory Ser. B* **102** (2012), 1061–1066.

[20] A. Grzesik, D. Král', L.M. Lovász: Elusive extremal graphs, preprint available as arXiv:1807.01141.

[21] H. Hatami, J. Hladký, D. Král', S. Norine, and A. Razborov: Non-three-colorable common graphs exist, *Combin. Probab. Comput.* **21** (2012), 734–742.

[22] H. Hatami, J. Hladký, D. Král', S. Norine, and A. Razborov: On the number of pentagons in triangle-free graphs, *J. Combin. Theory Ser. A* **120** (2013), 722–732.

[23] J. Hladký, D. Král', and S. Norine: Counting flags in triangle-free digraphs, *Combinatorica* **37** (2017), 49–76.

[24] C. Hoppen, Y. Kohayakawa, C.G. Moreira, B. Ráth, and R.M. Sampaio: Limits of permutation sequences, *J. Combin. Theory Ser. B* 103 (2013), 93–113.

[25] C. Hoppen, Y. Kohayakawa, C.G. Moreira, and R.M. Sampaio: Limits of permutation sequences through permutation regularity, available as arXiv:1106.1663.

[26] C. Hoppen, Y. Kohayakawa, C.G. Moreira, and R.M. Sampaio: Testing permutation properties through subpermutations, *Theoret. Comput. Sci.* **412** (2011), 3555–3567.

[27] S. Janson: Poset limits and exchangeable random posets, *Combinatorica* **31** (2011), 529–563.

[28] D. Král', C.-H. Liu, J.-S. Sereni, P. Whalen, and Z. Yilma: A new bound for the 2/3 conjecture, *Combin. Probab. Comput.* **22** (2013), 384–393.

[29] D. Král', L. Mach, and J.-S. Sereni: A new lower bound based on Gromov's method of selecting heavily covered points, *Discrete Comput. Geom.* **48** (2012), 487–498.

[30] L. Lovász: **Large networks and graph limits**, AMS, Providence, RI, 2012.

[31] L. Lovász and V.T. Sós: Generalized quasirandom graphs, *J. Combin. Theory Ser. B* **98** (2008), 146–163.

[32] L. Lovász and B. Szegedy: Finitely forcible graphons, *J. Combin. Theory Ser. B* **101** (2011), 269–301.

[33] L. Lovász and B. Szegedy: Limits of dense graph sequences, *J. Combin. Theory Ser. B* **96** (2006), 933–957.

[34] L. Lovász and B. Szegedy: Regularity partitions and the topology of graphons, in: I. Bárány and J. Solymosi (eds.): An irregular mind. Szemerédi is 70, Bolyai Society Mathematical Studies **21**, Springer, 2010, 415–446.

[35] L. Lovász and B. Szegedy: Szemerédi's regularity lemma for the analyst, *Geom. Funct. Anal.* **17** (2007), 252–270.

[36] L. Lovász and B. Szegedy: Testing properties of graphs and functions, *Israel J. Math.* **178** (2010), 113–156.

[37] S. Norine: private communication.

[38] O. Pikhurko and A. Razborov: Asymptotic structure of graphs with the minimum number of triangles, *Combin. Probab. Comput.* **26** (2017), 138–160.

[39] O. Pikhurko and E.R. Vaughan: Minimum number of k-cliques in graphs with bounded independence number, *Combin. Probab. Comput.* **22** (2013), 910–934.

[40] A. Razborov: Flag algebras, *J. Symbolic Logic* **72** (2007), 1239–1282.

[41] A. Razborov: On 3-hypergraphs with forbidden 4-vertex configurations, *SIAM J. Discrete Math.* **24** (2010), 946–963.

[42] A. Razborov: On the minimal density of triangles in graphs, *Combin. Probab. Comput.* **17** (2008), 603–618.

[43] V. Rödl: On universality of graphs with uniformly distributed edges, *Discrete Math.* **59** (1986), 125–134.

[44] J. Spencer: Quasirandom multitype graphs, in: I. Bárány and J. Solymosi (eds.): An irregular mind. Szemerédi is 70, Bolyai Society Mathematical Studies **21**, Springer, 2010, 607–617.

[45] A. Thomason: Pseudo-random graphs, in: A. Barlotti, M. Biliotti, A. Cossu, G. Korchmaros, and G. Tallini, (eds.), North-Holland Mathematics Studies **144**, North-Holland, 1987, 307–331.

[46] A. Thomason: Random graphs, strongly regular graphs and pseudorandom graphs, in: C. Whitehead (ed.), Surveys in Combinatorics 1987, London Mathematical Society Lecture Note Series, **123**, Cambridge Univ. Press, Cambridge, 1987, 173–195.