# Incomplete Preferences and Equilibrium in Contingent Markets

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**Abstract**

This paper shows that betting or speculative trading between agents with incomplete preferences is likely to occur if agents have access to convex choice sets. This contrasts sharply with endowment-economy models where preference incompleteness often hinders either betting, speculative trading, or mutually beneficial insurance arrangements. Our results imply that decision-makers with identical tastes and identical feasible sets will potentially gain from speculative trade for generic status-quo allocations. We also develop a framework for endogenizing the status-quo allocations of decision-makers which are treated exogenously in the existing literature. Finally, we provide a tractable differential representation of status-quo allocations, equilibria, and conditions where speculative trade may or may not emerge.

**Keywords:** incomplete preferences, status quo, no-trade price zone

# 1 Introduction

Knight (2005) famously argued that profit arose as the economic return for bearing *uncertainty* rather than *risk*. He distinguished between the two by classifying risk as involving randomness "susceptible of measurement" by theoretical deduction, observed historical experience, or statistical calculation. Uncertainty is thus determined residually as randomness not "susceptible of measurement". Much later, Ellsberg (1961) introduced the notion of *ambiguity* to differentiate different forms of randomness. Ambiguity is defined, somewhat amorphously, as "...a quality depending upon the amount, type, reliability, and 'unanimity' of information ...giving rise to one's degree of 'confidence' in an estimate of relative likelihoods" (Ellsberg 1961, p. 657).

Knight (2005) clearly believed that individuals made subjective probability judgments. In fact, he discriminated between three different types of probability assessment: *a priori* probability assessment, statistical probability assessment, and estimates (Knight 2005, pp. 224-5). The first two characterize risk and the last uncertainty. However, Ellsberg's (1961) urn examples, and their later empirical validation, showed convincingly that many individuals prefer gambles with known odds to gambles with unknown odds. Such behavior is inconsistent with Savage's (1954) axioms and, thus, with judgments based on unique subjective probabilities.

The recognition that ambiguity concerns can yield preferences not describable by unique subjective probabilities naturally suggests that ambiguity might play an important role in market outcomes. A series of papers on equilibrium exchange have studied those potential effects. These efforts can be naturally segregated into two groups. Each is distinguished by the modified version of Savage's axioms it adopts to accommodate the potential effects of ambiguity. One group follows the trail initially blazed by Schmeidler (1989) and Gilboa and Schmeidler (1989) and concentrates on the market consequences of relaxing independence (Savage's Sure-thing Principle). The Choquet Expected Utility (CEU) and maximin Expected Utility (MEU) models have represented a particular focus of attention (Billot, Chateauneuf, Gilboa, and Tallon 2000; Chateauneuf, Dana, and Tallon 2000; Dana 2004; Trojani and Vanini 2004; Cao, Wang, and Zhang 2005; Kajii and Ui 2006; Rigotti, Shannon, and Strzalecki 2008; Easley and O'Hara 2009; Cao, Han, Hirshleifer, and Zhang 2011; Ui 2011; Rigotti and Shannon 2012).

Another follows Aumann (1962) and Bewley (1986, 2002) and studies the market consequences of relaxing complete ordering (Rigotti and Shannon 2005; Easley and O'Hara 2010; Chambers 2014). Our focus lies in that setting. Decision-makers with incomplete preferences "may find themselves unable to express preferences for one alternative over another or to choose between alternatives in a coherent manner " (Galaabaatar and Karni, 2013, p. 255). Relaxing complete ordering in a Savage setting yields preference maps characterized by multiple supporting hyperplanes. When coupled with *status-quo maintenance* (Bewley, 2002; Mandler, 2005), the resulting "kinks" can result in behavioral inertia that may appear anomalous.

For example, one naturally expects two individuals with identical expected-utility preferences but asymmetric allocations of state-contingent commodities to engage in betting or speculative trading, as characterized by trade in state-contingent consumption, as a way of providing mutual insurance. The intuitive argument is straightforward. Sharing common preferences but not the same allocation, each individual's marginal rate of substitution between state-contingent incomes would differ. Having different marginal rates of substitution, the individuals could gain by making side bets.

However, as is now well-known, relaxing complete ordering in a Savage setting yields preferences which can contradict this intuition and result in a situation where no bets would occur between identical individuals with asymmetric allocations of state-contingent commodities (Bewley, 2002; Rigotti and Shannon, 2005). Closely associated with this lack of betting is the occurrence of equilibrium price indeterminacies in the presence of an Arrow-Debreu-McKenzie economy with complete contingent markets (Rigotti and Shannon, 2005).

In the main, such "inertial results" have been obtained in an Arrow-Debreu-McKenzie endowment-economy setting where individual status-quo allocations are determined exogenously in the form of fixed endowments of state-contingent income vectors. Consequently, the set of autarkic state-contingent income vectors available to each decision-maker, also called her feasible or budget set, has the shape of a rectangular prism that rules out substitution between income in different states of nature. Thus, there are no possibilities for decision-makers to effect substitution between income in different states under autarky.

A number of environments exist, however, where decision-makers may possess a range of autarkic state-contingent income levels. Consider, for example, the case of isolated farmers all endowed with an identical natural resource base and who all operate a stochastic technology. How each marshals his or her productive resources in preparing for different possible states of nature determines the resulting stochastic product mix. One farmer may use the same resources to develop irrigation opportunities while another may not. If drought occurs that differential preparation will lead to differential production outcomes. And thus, it seems natural to think of these isolated farmers operating under autarky not with singleton bundles of state-contingent income but with *sets of feasible state-contingent income vectors.* Where in that feasible set the farmer chooses to operate will depend critically upon her beliefs about the relative likelihood of the different states of states of nature and her attitudes towards consumption in those states.

As another example, suppose that under autarky decision-makers have access to the same or different stochastic markets, but that their access is subject to frictions that can vary across individuals. Their autarkic choices will depend not only on market prices but also upon the extent of frictions and their preferences toward stochastic incomes.

When it comes to state-contingent exchange, the two decision environments differ importantly. If endowments are fixed, state-contingent exchange arises as a result of differences in either fixed endowments or differences in preferences. Kinkiness of the decision-maker's preference contour at the autarkic point, which is automatically identified with the endowment point, yields reluctance to bet for a range of priors and can create equilibrium price indeterminacies in an exchange setting (Rigotti and Shannon 2005; Easley and O'Hara 2010).

When means exist to transform income in one state of nature into income in another, that preference kinkiness has different implications. It ensures that a continuum of subjective probability structures is consistent with equilibrium for that allocation. When that continuum of subjective probability structures is confronted by a feasible set that permits substitution across income in different states, rational behavior requires matching those subjective probability structures with "marginal rates of transformation". If the feasible set supporting autarky is strictly convex, each marginal rate of transformation maps into an unique efficient point implying that a continuum of rational autarkic choices potentially exists. Even though these potential choices may be highly disparate, none may be strictly preferred to the others when preferences are incomplete.

So, the autarkic situations differ. In an endowment setting, the autarkic allocation is trivially determinant, but equilibrium supporting prices (priors) may not be. In the case of a feasible set, the autarkic choice may be indeterminant, but when the feasible set is strictly convex each potential autarkic allocation now has an unique supporting prior.

This difference creates opportunities for mutually beneficial exchange of state-contingent securities that would not exist in an endowment economy setting. This leads to a number of different results about the nature of the resulting state-contingent equilibrium. Particularly notable is the result that *individuals with identical feasible sets and identical binary preference orders can potentially gain from betting.* This contrasts strongly with standing results that emerge from state-contingent exchange in the presence of either complete preferences (for example, Magill and Quinzii 1996) or incomplete preferences. This finding cannot be generated by general equilibrium models where preferences are incomplete and endowments are fixed (for example, Rigotti and Shannon 2005).

The rest of the paper is organized as follows. We specify an economy where $N$ individuals have identical incomplete preferences over state-contingent income and general convex feasible sets. We then turn to defining the set of a decision-maker's status quo allocations and providing its differential characterization. It is shown that under relatively weak conditions the set is a continuum, in contrast to a singleton in an endowment economy framework. We subsequently characterize the equilibria under state-contingent exchange and compare how the addition of substitution possibilities affects Pareto optimality of the no-betting equilibrium, juxtaposing our framework to the corresponding endowment economy setup.

# 2 The Model

There are $N$ individuals. Uncertainty is represented by a finite state space, $S$, and states are indexed with a slight abuse of notation by $(1, 2, ..., S)$. $\Delta \subset \mathbb{R}_+^S$ represents the unit simplex. $X \subset \mathbb{R}^S$ denotes the constant acts (elements of $\mathbb{R}^S$ taking the same value in each state), and we write $x \in X$ to denote the constant act taking the same real value, $x$, in each state of Nature.

## 2.1 Preference Axioms and a Cardinal Representation

To focus the analysis, we assume that all $N$ individuals share common preferences that can be represented by a binary relation defined on $\mathbb{R}^S$ and denoted by $\succ$ where $y \succ q$ is to be read as $y \in \mathbb{R}^S$ is strictly preferred to $q \in \mathbb{R}^S$. The extension of our analysis to the case of non-identical individuals is straightforward, but only serves to complicate the notation. The *preference correspondence* $P : \mathbb{R}^S \rightrightarrows \mathbb{R}^S$, associated with the upper contour set of $\succ$, also called the strictly better than set, is defined as:

$$P(q) = \left\{ y \in \mathbb{R}^S : y \succ q \right\}.$$

The closure of $P(q)$ in the Euclidean metric is denoted by $\bar{P}(q)$.

We impose the following axioms on $\succ$:

(**A.1**) (Irreflexivity): $\nexists q \in \mathbb{R}^S$ such that $q \succ q$.

(**A.2**) (Transitivity): For all $p, q, r \in \mathbb{R}^S$, $p \succ q$ and $q \succ r$ implies $p \succ r$.

(**A.3**) (Continuity): For all $q \in \mathbb{R}^S$, $P(q)$ is open.

(**A.4**) (Monotonicity): For all $q \in \mathbb{R}^S$ and all $r \in \mathbb{R}_+^S \setminus \{0\}$, $q + r \succ q$.[1]

(**A.5**) (Independence): $p \succ q \iff \alpha p + (1 - \alpha) r \succ \alpha q + (1 - \alpha) r$ for all $\alpha \in (0, 1]$ and all $r \in \mathbb{R}^S$.

[1] Note that monotonicity of preferences implies completeness over constant acts.

Bewley (2002) provides an axiomatization of the incomplete expected utility model using these conditions.[2] Instead of relying on Bewley's representation, we analyze decision-maker behavior using a cardinal preference representation (in units of the riskless asset) traceable to Blackorby and Donaldson (1980). The use of this cardinalization, and the analogous shortage function developed below, is motivated by the fact that it permits the use of simple variational, and in particular differential, concepts in characterizing our various equilibria. The *translation function* for $\succ$, $T : \mathbb{R}^S \times \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined:

$$T(q; r) \equiv \sup \{ x \in \mathbb{R} : q - x \succ r, \quad x \in X \},$$

if there exists constant act $x \in X$ such that $q - x \succ r$ and $-\infty$ otherwise.[3] $T(q; r)$ represents the largest translation of $q$ in the direction $-1 \in X$ (the riskless asset) that remains strictly preferred to $r$. Intuitively, it can be thought of as an an individual's willingness to pay, as measured in the units of $1 \in X$, to move from $r$ to $q$. For $T(q; r)$ concave in $q$, its subdifferential with respect to $q$ at $(q; r) \in \mathbb{R}^S \times \mathbb{R}^S$ is defined by the compact, convex set:[4]

$$\bar{\partial} T(q; r) \equiv \left\{ \pi \in \mathbb{R}^S : \pi' (h - q) \geq T(h; r) - T(q; r) \text{ for all } h \in \mathbb{R}^S \right\}.$$

Figure 1 illustrates. The strictly better than set for $r$, $P(r)$, is given by the area above the kinked indifference curve $KK'$. The value of the translation function $T(q; r)$ is given by the length of the segment connecting points $A$ and $q$, and $\bar{\partial} T(r; r)$ can be visualized as the continuum of normals to the hyperplanes supporting $P(r)$ at $r$.

A preliminary lemma establishes useful properties of $T(q; r)$:

**Lemma 1** *If $\succ$ satisfies (A.1)-(A.5), then:*

1. $T(q; r) > 0 \Leftrightarrow q \in P(r)$; $T(q; q) = 0$ *for all* $q \in \mathbb{R}^S$;
2. $T(q + x; r) = T(q; r) + x$ *for all* $x \in X$ *(translatability)*;
3. $T(q; r)$ *is nondecreasing concave in* $q$;
4. *if* $\bar{\partial} T(q; r) \neq \emptyset$, $\bar{\partial} T(q; r) \subset \Delta$ *for all* $q, r \in \mathbb{R}^S$;
5. $T(\alpha q + y; \alpha r + y) = \alpha T(q - r; 0)$ *for all* $\alpha > 0$ *and all* $q, y, r \in \mathbb{R}^S$; *and*
6. $\bar{\partial} T(q; q) = \bar{\partial} T(0; 0)$ *for all* $q \in \mathbb{R}^S$.

[2] Ghirardato et al. (2003) provide an axiomatization of Bewley's model in a Savage setting. See also Galaabaatar and Karni (2013).

[3] The first $x$ in the definition of $T^n(q; r)$ stands for a scalar while the other two $x$'s stand for constant acts assuming that same $x$ value in all states. The same applies to the definitions of $\sigma^n(z)$ and $\Sigma(z)$ introduced later in the paper.

[4] Here and in the rest of the paper, $x'y$ denotes $\sum_{s \in S} x_s y_s$.

**Proof.** Proofs are in an appendix. ■

By Lemma 1.1, $T(q;r)$ characterizes $\succ$ and $P(r)$. Indeed, because it is a real-valued function of the "random variables" $q$ and $r$, it has a natural interpretation as a "sufficient statistic" for $P(r)$. Together the definition of $\bar{\partial}T(q;r)$ and Lemma 1 give

$$\begin{aligned}
\bar{\partial}T(q;q) &= \left\{\pi \in \Delta : \pi'(h-q) \geq T(h;q) - T(q;q) \text{ for all } h \in \mathbb{R}^S\right\} & (1)\\
&= \left\{\pi \in \Delta : \pi'(h-q-T(h-q;0)) \geq 0 \text{ for all } h-q-T(h-q;0) \in \mathbb{R}^S\right\}\\
&= \{\pi \in \Delta : \pi'(h-q) \geq 0 \text{ for all } h-q \in P(0)\}.
\end{aligned}$$

Lemma 1.3 and 1.5 imply that $T(q-y;0)$ is superlinear (positively homogeneous and concave) in $q-y$. Hence, it is the (lower) support function for the compact, convex set $\bar{\partial}T(0;0)$ (see, for example, Rockafellar 1970, Corollary 13.2.1 or Hiriart-Urruty and LeMarechal 2001, Remark D.1.2.3) and thus

$$T(q-y;0) = \inf\left\{\pi'(q-y) : \pi \in \bar{\partial}T(0;0)\right\}. \tag{2}$$

Upon noting that $\bar{\partial}T(0;0) \subset \Delta$, the preference representation in (2) is easily recognized as corresponding to Bewley's Theorem 1 (also see Rigotti and Shannon 2005, Corollary 1). In words, $q \succ y$ if and only if $\pi'(q-y) > 0$ for all $\pi \in \bar{\partial}T(0;0)$, and $\bar{\partial}T(0;0)$ is naturally interpretable as a set of "subjective beliefs" (Rigotti, Shannon, and Strzalecki 2008). In the following, $\bar{\partial}T(0;0)$ will be referred to as the *subjective beliefs structure or set.* When the subjective beliefs constitute a singleton set, $\succ$ is a complete order (Rigotti and Shannon 2005). And $P(r)$ corresponds to an open half space falling above the affine hyperplane passing through $r$ with normal given by the singleton prior $\left\{\bar{\partial}T(0;0)\right\} \in \Delta$. We refer to singleton subjective beliefs as the *risk case.*

The concave conjugate of $T(q;r)$, $T^* : \Delta \times \mathbb{R}^S \to \mathbb{R}$, gives the lowest expected value, using prior $\pi \in \Delta$, of any act $q \succ r$ :

$$\begin{aligned}
T^*(\pi;r) &= \inf_q \{\pi' q - T(q;r)\}\\
&= \inf_q \left\{\pi' q - \inf_\pi \left\{\hat{\pi}'(q-r) : \hat{\pi} \in \bar{\partial}T(0;0)\right\}\right\}\\
&= \inf_q \left\{\pi' r - \inf_\pi \left\{(\hat{\pi}-\pi)'(q-r) : \hat{\pi} \in \bar{\partial}T(0;0)\right\}\right\}\\
&= \begin{cases} \pi' r & \pi \in \bar{\partial}T(0;0) \\ -\infty & \text{otherwise} \end{cases}.
\end{aligned}$$

## 2.2 The Feasible Sets and a Cardinal Representation

The set of state-contingent income combinations available to individual $n$ is represented by a closed, nonempty, convex set, $Z^n \subset \mathbb{R}^S$ satisfying $z \in Z^n \Rightarrow z' \in Z^n$ for all $z' \leq z$ (free disposability). Some obvious special cases of the feasible set are the endowment economy where $Z^n = \{\hat{z} : \hat{z} \leq z^n\}$ for all $n$; an Arrow-Debreu-McKenzie complete market where for all $z$

$$Z^n = \{\hat{z} : \pi'\hat{z} \leq \pi' z^n\} \text{ for } \pi \in \Delta;$$

and strictly convex state-contingent production-possibility sets.

We use a construct similar to the translation function to characterize the set of feasible state-contingent income combinations $Z^n$. Luenberger (1994) defines the *shortage function*, $\sigma^n : \mathbb{R}^S \to \bar{\mathbb{R}}$, by:

$$\sigma^n(z) \equiv \min\{x \in \mathbb{R} : (z - x) \in Z^n, x \in X\},$$

if there exists $x \in X$ such that $z - x \in Z^n$ and $\infty$ otherwise. $\sigma^n(z)$ represents the smallest translation of $z$ in the direction $-1 \in X$ such that $(z - x)$ remains feasible. For $\sigma^n$ convex in $z$, its subdifferential[5] at $z$ is defined as

$$\underline{\partial}\sigma^n(z) \equiv \left\{\pi \in \mathbb{R}^S : \pi'(h - z) \leq \sigma^n(h) - \sigma^n(z) \text{ for all } h \in \mathbb{R}^S\right\}.$$

Because $\underline{\partial}\sigma^n(z)$ is derived from the physical opportunities to transform income in one state of nature into income in another as summarized by $Z^n$, we shall refer to $\underline{\partial}\sigma^n(z)$ as $n$'s *marginal rates of transformation* of state-contingent income at $z$.[6]

Parallel to Lemma 1, we have:

**Lemma 2** $\sigma^n(z)$ *satisfies:*

1. $\sigma^n$ *is nondecreasing in* $z$;
2. $z \in Z^n \Leftrightarrow \sigma^n(z) \leq 0$;
3. *for* $x \in X$, $\sigma^n(z + x) = \sigma^n(z) + x$ *(translatability); and*
4. $\sigma^n$ *is convex in* $z$, *and* $\underline{\partial}\sigma^n(z) \subset \Delta$ *if* $\underline{\partial}\sigma^n(z) \neq \emptyset$.

The convex conjugate of $\sigma^n(z)$ for prior $\pi \in \Delta$ gives maximal expected value attainable with feasible set $Z^n$ :

$$\sigma^{n*}(\pi) = \sup_z \{\pi' z - \sigma^n(z)\},$$

[5] Because $\sigma^n(z)$ is convex, its subdifferential is defined by reversing the inequality used to define the subdifferential for concave $T^n(h; q)$. Some authors refer to the latter as the superdifferential for $T^n(h; q)$. We follow the terminology established in Rockafellar (1970).

[6] In previous versions, we used the terminology risk-neutral probabilities in place of the marginal rate of transformation. This was intended to be evocative of Knight's distinction between uncertainty and risk. As a reviewer points out, other notions of risk-neutral probabilities already exist and we changed terminology to avoid unnecessary confusion.

## 2.3 The Choice Correspondence

We begin with some definitions and behavioral assumptions. We follow Masatlioglu and Ok (2005) and utilize choice correspondences to discuss the generic choice problems decision-makers face. Following their terminology, the pair $(A, x)$ is called a *choice problem* with *feasible set* $A$ and *status quo* $x \in A$. The symbol $\diamond$ is used to denote the absence of a status quo. Thus, a choice problem without a status quo is denoted as $(A, \diamond)$. A choice correspondence $c(A, x) \subseteq A$ denotes the set of alternatives that a decision-maker with status-quo $x$ may choose from the set $A$.

In the presence of a status quo, we require

$$c(A,x) = \begin{cases} \{q \in A : q \succ x \text{ and } P(q) \cap A = \emptyset\}, & \text{if } P(x) \cap A \neq \emptyset \\ x, & \text{otherwise} \end{cases}, \tag{3}$$

which is the set of alternatives that are strictly preferred to the status quo and not strictly worse than any alternative in $A$ if there are alternatives preferred to $x$. However, if there is no element of $A$ that $\succ$ dominates $x$, the choice remains $x$. Specification (3) is a behavioral assumption that encapsulates both Bewley's (2002) *Maximality Assumption* that a decision maker's actions be described by an undominated program and *Inertia Assumption* (or status-quo maintenance) that starting from a status-quo point $x$ a decision-maker only moves from $x$ if the move strictly dominates $x$.[7,8]

# 3 The Analysis

Although our model is formally timeless, our equilibrium analysis proceeds in two stages. In the first stage, isolated agents respond to their feasible sets $Z^n$ to determine their *individual autarkic* or *individual status quo* outcomes. These individual status quo outcomes are then treated as exogenous data to the equilibrium exchange problem that permits individuals to engage in exchange with others. This is equivalent to assuming that individuals derive their status quo outcomes naively without anticipating the possibility of trading with other individuals. The myopia assumption seems rather fitting for an interpretation of our model as a situation where the agents respond to the imperfections of existing markets by creating a

[7]Mandler (2005) demonstrates that status quo maintenance in "simple" choice environments, which include the setup considered in the present paper, is outcome rational in the sense that sequences of choices don't lead to dominated outcomes.

[8]Masatlioglu and Ok (2005), using a revealed-preference approach provide an axiomatic framework for characterizing rational choice with a status-quo bias. Ortoleva (2010) axiomatizes multi-prior preferences with a status quo bias taking as a primitive the preferences of an agent.

new one. In this case, it seems reasonable for agents to contemplate a "status quo" allocation that is independent of the new market.[9]

### 3.1 The Individual Status Quo

In an endowment (pure exchange) economy, where singleton endowments are treated as primitive, each individual's endowment is naturally interpreted as the individual's autarkic position. In our model, feasible sets, $Z^n$, are treated as primitives. As such that often leaves the individual's autarkic position undefined. We partially resolve that indeterminacy by assuming that the autarkic position will be determined endogenously according to $c(Z^n, \diamond)$.

More formally, we define the *individual status quo* (or interchangeably the status quo or autarkic position), in terms of the correspondence for $Z^n$ in the absence of a status quo. Thus, the *individual status quo* must belong to

$$c(Z^n, \diamond) = \{q^n \in Z^n : Z^n \cap P(q^n) = \emptyset\},$$

which corresponds to the set of undominated elements of $Z^n$.

Our definition of individual status quo has two motivations. First, it is consistent with Bewley's (2002) *Maximality Assumption* that the decision-maker's actions are determined by an undominated program. Second, as in many trade and bargaining models,[10] each individual's fallback option is given by what she can achieve on her own, that is, in autarky. In what follows, we frequently use the notational short-hand, $Q^n \equiv c(Z^n, \diamond)$.

This definition does not always identify an unique individual status quo. Sometimes, it may. For example, if $Z^n$ equals the singleton set $\{z^n\}$,

$$Q^n = \{z^n\}.$$

More generally, because subjective beliefs and objective beliefs can both encompass a range of priors, a range of status-quo points may exist. The convexity of $Z^n$ and $P(q)$ and our definition, however, ensure that any potential $Q^n$ satisfy standard separation theorems. These separation results, in turn, have particularly convenient manifestations in terms of those sets of subjective beliefs and marginal rates of transformation. We have as our first representation result:

[9] Although there are many additional environments where the myopia assumption makes sense, there are some where it is not realistic. An interesting and natural extension of the present analysis is a dynamic model wherein decision-makers choose their status quos in anticipation of exchange opportunities at a later period. We leave this as an important line of research to the future.

[10] These include trade models with ambiguity-sensitive decision-makers. These status-quo allocations can be rationalized within many environments. An example is a two-period model.

**Proposition 3** *$q^n \in Q^n$ if and only if*

$$q^n \in \arg\max_r \{T(r, q^n) - \sigma^n(r)\},$$

*and*

$$T(q^n, q^n) - \sigma^n(q_n) = 0.$$

At an operational level, Proposition 3 requires:

**Corollary 4** *$q^n \in Q^n$ with $q^n > 0$ if and only if*

$$\begin{aligned} \sigma^n(q^n) &= 0, \\ \bar{\partial}T(0;0) \cap \underline{\partial}\sigma^n(q^n) &\neq \emptyset. \end{aligned}$$

Corollary 4 establishes two important characteristics of a status quo. First, a status quo can exist only when the individual operates on the boundary of $Z^n$. This simply ensures that individuals never locate in autarky at outcomes that are dominated by other feasible outcomes.

Second, the existence of a status quo establishes an objective "bound" on the range of subjective beliefs that are consistent with status-quo behavior. And that objective bound is given by the requirement that subjective beliefs and marginal rates of substitution are *consistent for* $q^n$ in the sense that

$$\pi \in \bar{\partial}T(0;0) \cap \underline{\partial}\sigma^n(q^n), \tag{4}$$

so that there exists at least one $\pi$ that is common to both. Room exists for disagreement between subjective beliefs and marginal rates of transformation, but a status quo cannot be maintained if disagreement is complete between the two sets. Subjective beliefs and marginal rates of transformation must overlap.

In this context, it is important to recognize that subjective belief formation as encapsulated in A.1-A.5 is modelled as distinct from $Z^n$. This is consistent with the tradition in the decision-theoretic literature. Hence, both subjective beliefs and $Z^n$ are treated as independent primitives in our model. It is only at the stage of forming the individual status quo that subjective belief and marginal rates of transformation interact. One might usefully compare this interaction with the alternative axiomatic approach of de Finetti (1974) and later Walley (1990) where the notions of *prevision* and *coherence* are used to incorporate gamble-based information in the belief formation process. In our format, the formulation of the individual status quo serves a closely similar purpose, it confronts the individual with information on the states of the world as encapsulated in the feasible gambles $Z^n$, and it ensures that their subjective beliefs cohere with that objective information.

The consistency requirement, (4), can be usefully visualized with the aid of Figure 2. Suppose that $Z^n$ is strictly convex and that preferences are represented by the kinked indifference curve. That kinked indifference curve remains tangent to $Z^n$ for the range of marginal rates of transformation (visualized as the slope of the $Z^n$ curve) falling between the normals to the indifference-curve's "legs". One thus visualizes $Q^n$ as the continuum of points where the kinked indifference curve is tangent to $Z^n$'s outer boundary. The segment of $Z^n$'s boundary joining points $A$ and $B$ in Figure 2 illustrates. In that continuum, subjective beliefs and marginal rates of transformation overlap satisfying (4). But also note that the marginal rates of transformation at $A$ and $B$ are different, so that from an operational perspective an individual locating at point $A$ would appear distinct from an individual locating at point $B$ even if they shared a common $\succ$ and a common $Z^n$.

This visual intuition suggests that $Q^n$ may be a continuum, and not a singleton set, when preferences are incomplete. Thus, *a priori,* individual status quos need not be unique. And, in particular, if an individual with incomplete preferences is confronted with a strictly convex choice set, one expects to find a continuum of $q^n$ each of which is consistent with a unique, but distinct, marginal rate of transformation captured by $\underline{\partial}\sigma^n(q^n)$. This discussion is formalized as:

**Proposition 5** *If $Q^n$ is a non-singleton set with $\{q', q''\} \in Q^n$, then $q_\alpha - \sigma^n(q_\alpha) \in Q^n$ for all $\alpha \in (0,1)$ where*

$$q_\alpha = \alpha q' + (1-\alpha) q''.$$

Expanding upon Figure 2 helps illustrate the role that uncertainty plays in determining an individual status quo in different choice settings. Figure 3 distinguishes between two individual choice settings: a pure endowment associated with the rectangle $0ECE'$ and a strictly convex feasible set whose outer boundary is given by the smooth curve $ZZ'$. It also distinguishes between two preference classes. The *risk* class where the better than set for point $C$, $P(C)$, is the open half space above $RR'$. The *uncertainty* class applies when $P(C)$ is the area above the kinked indifference curve $KK'$.

Knight's essential argument was that economic profit was a reward for coping with uncertainty. Knight argues that if a probability can be determined by *"...a priori calculation..."* or *"...by the empirical method of applying statistics to actual instances"* markets should operate to eliminate profit (Knight 2005, pp. 214-215). But if those conditions are not met, uncertainty arises and with it the possibility of profit as a reward for coping with that uncertainty.

Although Knight's (2005) formal structure and terminology differ from ours, a similar intuition applies here. So, for example, suppose $RR'$ in Figure 3 could be associated with an objectively determined probability. In that case, which Knight would classify as *risk,* individuals with preferences matching $RR'$ strictly prefer points lying above $RR'$ to $C$.

Now suppose that objective probabilities do not exist, and individuals operate with a fixed endowment associated with the rectangle $0ECE'$ in Figure 3. Then an individual with a singleton subjective beliefs captured by $RR'$ would establish an individual status quo at $C$ and would accept gambles at odds favorable to those reflected by $RR'$. An individual with preferences reflected in $KK'$ establishes the same individual status quo, but he or she will avoid gambles that $RR'$ would accept. That no-gamble range of odds is determined by the individual's subjective belief structure as encapsulated in $\bar{\partial}T\left(0;0\right)$. In both cases, the status-quo choice is the same. The point $C$ is the unique individual status quo, but for $RR'$ only one prior supports that status quo while for $KK'$ a continuum of priors supports it. The individual status quo for $KK'$ is characterized by prior or price indeterminacy. And that price indeterminacy translates behaviorally into hesitancy to gamble or make state-contingent trades.

Now consider a decision setting characterized by a strictly convex feasible set as illustrated by $ZZ'$. When contrasted with $0ECE'$, one can envision $0ECE'$ as corresponding to a situation where $\underline{\partial}\sigma^n\left(C\right) = \Delta$. And while $ZZ'$ may not communicate information about the relative likelihood of the various states of Nature, it does convey *objective* knowledge about the rate at which income in one state of Nature can be transformed into income in other states. A rational individual will always use that objective knowledge in the decision process.

Individual $RR'$ again locates at $C$. She chooses $C$ by identifying a point in the feasible set where her subjective beliefs match a "marginal rate of transformation" dictated by $ZZ'$. For individual $KK'$, $C$ again represents an individual status quo. At $C$, the individual's preferences and feasible set agree on an unique assessment of the relative likelihood of the two states as given by $RR'$. But that unique assessment is made by $ZZ'$ and only applies locally. It is only one of many which the individual might consider. As $KK'$ is slid around $ZZ'$, it remains tangent to it for a continuum of points whose endpoints are illustrated by $A$ and $B$ in the figure. Each represents a potential individual status quo, each with unique odds determined by $ZZ'$. Thus, when compared to decision situation $0ECE'$, a different result emerges. Instead of an unique quantity with indeterminate odds, a continuum of quantity individual status quos emerges, each with unique odds. And the individual uses $ZZ'$ to process his or her disparate beliefs about the true state of Nature.

## 3.2 Exchange Equilibrium with Feasible Sets

The individual status quo identifies the individual's equilibrium behavior when the only opportunities for adjusting state-contingent consumption fall within $Z^n$. To determine how the possibility of speculative exchange alters individual behavior, we now allow individuals to make state-contingent trades with one another and then characterize the associated Paretian equilibrium. As we have noted in the very beginning of this section, we assume that individuals behave myopically when they choose $q^n \in Q^n$ and they treat $q^n$ as the status quo when contemplating the possibility of mutually beneficial exchanges.

In that setting, the aggregate state-contingent income allocation across individuals must fall within the Minkowski sum of the individuals' endowed feasible sets

$$\mathcal{Z} \equiv Z^1 + Z^2 + ... + Z^N = \left\{ \sum_{n=1}^{N} z^n : z^n \in Z^n, n = 1, 2, ..., N \right\}.$$

Because each $Z^n$ is convex, so too is $\mathcal{Z}$. The individual status quos $q^n \in Q^n$ obviously satisfy:

$$\sum_{n=1}^{N} q^n \in \mathcal{Z}.$$

### 3.2.1 Equilibrium Identification

The following definitions are standard:

**Definition 6** *An allocation vector $(y^1, y^2, ..., y^N)$ is feasible relative to $\mathcal{Z}$ if $\sum_{n=1}^{N} y^n \in \mathcal{Z}$.*

**Definition 7** *A feasible allocation vector $(y^1, y^2, ..., y^N)$ is Pareto optimal relative to $\mathcal{Z}$ if there is no other feasible allocation vector $(y^{01}, y^{02}, ..., y^{0N})$, such that $y^{0n} \succ y^n$ for all $n$.*

The definition of the *exchange equilibrium* is also standard. Let $q^n$ denote individual $n$'s status-quo position. Starting from her status quo $q^n$, individual $n$ chooses her *intermediate allocation* $z^n$ from the set of her feasible allocations $Z^n$ and trades with other $n-1$ individuals to arrive at her *final consumption*, which is denoted by $c^n$. We envision a pseudo-dynamic model with two periods. In the first period, the individuals cannot trade with each other. However, they have access to their respective $Z^n$. The positions $q^n$ $(n = 1, .., N)$ taken by these agents in the first period serve as status quos for the choices made in the second period which now involves possibilities of trade between all of these individuals. In the second period, individual $n$ chooses $z^n$ and $c^n$.

Returning to the two examples in the introduction, first, consider the case of $n$ farmers each having access to stochastic technologies. Under this scenario, the intermediate allocations correspond to the farmers' production choices $z^n$ from their production sets $Z^n$. After they have made these choices, the farmers get an opportunity to trade with each other in the second period which leads to their final consumption vectors $c^n$.

For our second example of autarkic market(s) with frictions, the intermediate allocations correspond to the individuals positions in these markets. After these choices have been made, these $n$ individuals realize an opportunity for mutually beneficial trades and create new markets[11] to take advantage of this opportunity.

The individual $n$'s choice correspondence under exchange can be written as

$$c\left(F^n(\pi), q^n\right),$$

where $F^n(\pi) \equiv \{c : \pi' c^n \leq \sigma^{n*}(\pi)\}$.

**Definition 8** *An exchange equilibrium, where individual $n$'s status quo is given by initial portfolio $q^n \in Z^n$, a probability vector $\pi \in \Delta$, a final-consumption vector $\left(c^1, c^2, ..., c^N\right)$, and an intermediate allocation vector $\left(z^1, z^2, ..., z^N\right)$ such that*

$$\begin{gathered} c^n \in c\left(F^n(\pi), q^n\right) \text{ for all } n, \\ z^n \in \underline{\partial}\sigma^{n*}(\pi) \text{ for all } n, \text{ and} \\ \sum_{n=1}^{N} c^n \leq \sum_{n=1}^{N} z^n. \end{gathered}$$

Adopting the equilibrium identification strategy of Luenberger (1994) as extended by Chambers (2014) to incomplete preference structures, we have:

**Proposition 9** *An allocation vector $\left(y^1, ...., y^N\right)$ is Pareto optimal relative to $\mathcal{Z}$ if and only if $\left(y^1, ...., y^N\right)$ is zero maximal for*

$$\max\left\{\sum_n T\left(q^n; y^n\right) - \sum_n \sigma^n\left(\hat{z}^n\right) : \sum_n q^n - \sum_n \hat{z}^n = 0\right\}.$$

Operationally, Proposition 9 translates the requirement for Pareto optimality into another overlapping belief criterion:[12]

[11] Although we assume that the new markets are frictionless, our findings hold for various scenarios where these markets have frictions.

[12] Bonnisseau (2003) proves existence and characterizes equilibria of exchange economies where preferences may be incomplete or intransitive. Rigotti and Shannon (2005) establish the following result, in a different notation, for the case of Bewley incomplete preferences and an endowment economy (where $Z$ is a prism). Chambers (2014) extends that result to convex incomplete preference structures with a convex feasible set. The result in the present paper extends Chambers (2014) to the case of multiple feasible sets.

**Corollary 10** *An interior allocation vector* $\left(y^1, ...., y^N\right)$ *is Pareto optimal relative to* $\mathcal{Z}$ *if and only if*

$$\left\{\cap_{n=1}^{N} \underline{\partial}\sigma^n\left(y^n\right)\right\} \cap \bar{\partial}T\left(0;0\right) \neq \emptyset$$

Proposition 9, the convex structure of our problem, Fenchel's duality theorem (Rockafellar 1970, Theorem 30.1) and Luenberger (1994) then ensure that a dual representation of a Pareto optimal allocation $\left(y^1, ...., y^N\right)$ requires $\pi$ to be zero minimal for

$$\max_{\hat{p}\in\Delta}\left\{\sum_n \sigma^{n^*}\left(\hat{p}\right) - \sum_n T^*\left(\hat{p}; y^n\left(\pi\right)\right)\right\},$$

where

$$y^n\left(\pi\right) \in \bar{\partial}T^*\left(\pi; y^n\left(\pi\right)\right).$$

It follows immediately from our assumptions on better-than sets and feasible sets that the first welfare and the second welfare theorems are satisfied. Thus, the conditions in Proposition 9 and Corollary 10 also characterize competitive equilibria for appropriately chosen prices.

### 3.2.2 Equilibrium with Identical Feasible Sets

The presence of ambiguity is a common explanation for a number of "inertial" market anomalies including among others inertia in trading, refusals to provide mutual insurance, bid-ask spreads, the status-quo bias, and the home-country bias (see, for example, Bewley 1986, 2002; Dow and Werlang 1992; Epstein and Wang 1994; Billot et al. 2000; Chateauneuf et al. 2000; Mandler 2004; Trojani and Vanini 2004; Cao, Wang, and Zhang 2005; Rigotti and Shannon 2005; Rigotti et al. 2008; Easley and O'Hara 2009; Bossaerts, Ghirardato, Guarnaschelli, and Zame 2010; Cao, Han, Hirshleifer, and Zhang 2011; Ui 2011). In the current setting, these inertial market anomalies translate into a failure to execute state-contingent exchanges in the form of speculative trades, "betting", or insurance.

Perhaps the clearest way to appreciate the import of Propositions 5 and 9 and Corollary 10 is to consider the polar case where all $N$ individuals share a common $Z^n$. The case of $N$ farmers possessing the same stochastic technology and natural resource endowments facing a common uncertain decision setting is one possible example. If we maintain our assumption of identical $\succ$ over these individuals, intuition might suggest that the status quo allocations would be Pareto optimal. After all, if identical farmers facing identical technologies accept trades away from their status-quo outcomes, it seemingly signals that *individuals are willing to bet against themselves.* In a pure exchange (endowment) economy, that is $Z^n = \{z\}$ for all $n$, this would never happen. In our setting, however, a different result can emerge.

More formally, we have:

**Definition 11** *A no-betting outcome consists of a final-consumption vector* $\left(c^1, ..., c^N\right)$ *and an intermediate allocation vector* $\left(z^1, ..., z^N\right)$ *such that* $c^n = z^n$ *for all* $n \in \{1, ..., N\}$.

Using this definition, we have:

**Proposition 12** *Suppose that all* $N$ *individuals share a common* $Z$. *Then*
*1. interior status quos* $\left(q^1, ..., q^N\right)$ *are Pareto optimal if and only if*

$$\left(\cap_{n=1}^{N} \underline{\partial}\sigma\left(q^n\right)\right) \cap \bar{\partial}T\left(0;0\right) \neq \emptyset.$$

*2. if the set* $\bar{\partial}T\left(0;0\right) \cap \left(\cup_{z\in B}\underline{\partial}\sigma\left(z\right)\right)$, *where* $B = \{z : \sigma^n\left(z\right) = 0\}$, *contains at least two elements, the set of possible status quo points has positive measure, but the subset for which no trade is Pareto optimal has measure zero.*

The second part of Proposition 12 establishes a reasonable expectation that $N$ farmers with identical technologies and identical $\succ$ would benefit from state-contingent trades, possibly in the form of options or futures contracts, away from their $Q^n$. By Corollary 10, an interior no-betting outcome $(c^1, ..., c^N)$ is Pareto optimal if and only if

$$\cap_{n=1}^{N}\left\{\bar{\partial}T\left(0;0\right) \cap \underline{\partial}\sigma\left(c^n\right)\right\} \neq \emptyset.$$

In the status quo, each individual's (common) preference structure and feasible set contain "overlapping" subjective beliefs and objective beliefs. Moreover, the overlap occurs for at least one prior common to all individuals. The convexity of $Z$ ensures, however, that objective belief structures are *cyclically monotone* in the sense that for all $\pi^n \in \underline{\partial}\sigma\left(q^n\right)$,

$$\left(\pi^2 - \pi^1\right)'\left(q^2 - q^1\right) + \left(\pi^3 - \pi^2\right)'\left(q^3 - q^2\right) + \cdots + \left(\pi^1 - \pi^K\right)'\left(q^1 - q^K\right) \leq 0,$$

with a strict inequality for $Z$ strictly convex (Rockafellar 1970). So, for example, when $Z$ is strictly convex, objective beliefs at $q^n$ are singleton sets, and their cyclically monotone structure ensures that

$$\underline{\partial}\sigma\left(q^n\right) = \underline{\partial}\sigma\left(q^{n'}\right),$$

if and only if $q^n = q^{n'}$. Thus, if all farmers share a common $\succ$ and a common strictly convex $Z$, the individual autarkic choices are Pareto optimal if and only if all autarkic choices are identical. (Formally, the condition in Proposition 12.2 that $\bar{\partial}T\left(0;0\right) \cap \left(\cup_{z\in B}\underline{\partial}\sigma\left(z\right)\right)$ contains at least two elements ensures that $Q$ is now a continuum rather than a singleton set.) But, unless serendipity or some force outside our model drives all farmers to make the same choice, potential benefits from speculative exchange exist at those individual status quos. It also follows from Proposition 12 and the ensuing arguments that the set of no-betting outcomes in which agents do not consume at their status quos has measure zero in the set of all Pareto optimal outcomes.

Figure 2 again demonstrates. Suppose that there are two identical individuals and one of them chooses point $A$ as her status quo while the other chooses point $B$. Because the marginal rates of substitution at $A$ and $B$ differ, the autarky positions cannot be Pareto optimal. Hence, if the individuals are given the opportunity to trade, they will depart from their autarkic positions and choose a common intermediate allocation vector (and common objective beliefs) as illustrated by $E$ in Figure 2. The first decision-maker can now make the state-contingent trade from point $E$ to point $A^T$ while the second will trade from point $E$ to point $B^T$. As a result, both decision-makers are made strictly better off than in autarky. But in the absence of the ability to trade, individuals located at either $A$ or $B$ would have no incentive to reallocate their choice from $Z$ to point $E$.[13]

These results appear paradoxical when viewed through the lens of an Arrow-Debreu-McKenzie economy derived in the presence of complete preferences and smooth feasible sets. In that familiar case, identical agents facing a common preference structure and common feasible sets would choose intermediate-production and final-consumption vectors identical to their individual status quos. And those status quos would typically offer no apparent gains from speculative exchange. As we have already noted, the same result emerges in an endowment economy setting.

Here a different dynamic emerges. First, identical agents can identify different individual status quos. And those individual status quo can admit the possibilities of gains from state-contingent exchange or betting. In short, the presence of preference incompleteness that is induced by the presence of uncertainty introduces the potential for *behavioral heterogeneity* across otherwise seemingly identical agents in their individual status-quo behavior. This behavioral heterogeneity takes the form of multiple singleton marginal rates of substitution supporting the common non-singleton subjective belief structure. It opens the potential for gains from betting or speculative trading that are then realized by the forces of trade homogenizing the behavioral heterogeneity by requiring a common singleton marginal rate of transformation to support the common non-singleton subjective probability structure.

[13] In the unlikely case when the boundaries of the set $Z$ and the strictly better than set $P(\cdot)$ are linear and parallel to each other, the two individuals may engage in betting but end up consuming at their status quos.

### 3.2.3 General Feasible Sets

Proposition 9 and Corollary 10 together imply conditions under which the individual status quos are Pareto optimal:

**Proposition 13** *Interior individual status quos* $\left(q^1, q^2, ..., q^N\right)$, *where* $q^n \in Q^n$ *for all* $n$, *are Pareto optimal relative to* $\mathcal{Z}$ *if and only if*

$$\cap_{n=1}^{N} \left\{\underline{\partial}\sigma^n\left(q^n\right) \cap \bar{\partial}T\left(0;0\right)\right\} \neq \emptyset. \tag{5}$$

As already discussed unless $Z^n$ is a rectangular prism, $Q^n$ is typically a continuum. This surely happens when $Z^n$ is strictly convex. In that case, each element of $Q^n$ possesses an unique objective probability that also falls in the individual's subjective belief set. In the general case, presumably each $n$ will have different $Q^n$. Thus, for individual status quos to be Pareto optimal, each individual, acting in isolation, must choose from their continuum $Q^n$ a particular $q^n$ supported by objective belief that is shared by all others. Even in simple cases where strict convexity of $Z^n$ is violated, the criteria in Proposition 13 seem unlikely to be satisfied. Thus, starting from $\left(q^1, q^2, ..., q^N\right)$, *it appears likely that possibilities may exist for mutually beneficial exchange even if beliefs overlap and* $Z^n$ is not strictly convex. This can easily be confirmed by the construction of simple parametric specifications (see Example 14 in the Appendix).

Proposition 13 relates to several issues raised in the literature on ambiguity and exchange. One such issue is whether in the presence of Knightian uncertainty insurance markets, which are predicated on the ability to price actuarially fair insurance contracts, might break down. Another is whether exchange economies whose aggregate endowments exhibit no uncertainty may have Pareto-optimal allocations that do not exhibit full insurance (Billot et al. 2000; Rigotti and Shannon 2005; Rigotti et al. 2008).

When allowing for non-identical $Z^n$, no reason exists to suggest the existence of $\left(z^1, ..., z^N\right)$ such that

$$\cap_{n=1}^{N}\underline{\partial}\sigma^n\left(z^n\right) \neq \emptyset \text{ and } \sum_{n=1}^{N} z^n \in X,$$

which is required if aggregate intermediate allocation is to be nonstochastic. And even if that were to occur, full-insurance equilibria would then only be Pareto optimal if the additional criterion is satisfied

$$\cap_{n=1}^{N} \left\{\underline{\partial}\sigma^n\left(q^n\right) \cap \bar{\partial}T\left(0;0\right)\right\} \neq \emptyset.$$

Thus, questioning whether full insurance allocations are Pareto-optimal may have little practical relevance in our setting because intermediate allocations exhibiting no aggregate uncertainty seem unlikely to occur.

The overarching message is clear. If the feasible choice setting communicates no information about possible exchange between income in different states, the only way to resolve uncertainty is through exchange. However, if all share a common nonsingleton subjective belief structure that exchange will not be Pareto optimal. In that case, equilibrium allocations are not associated with unique supporting probabilities. On the other hand, if objective information in the form of $Z^N$ exists about how to transform income in one state of nature into income in another, economic efficiency requires that information be incorporated both in individual status-quo formulation and in exchange. And if physical information permits one to make smooth (unique) probabilistic assessments of states, the resulting uniqueness will be communicated throughout the market.

Our findings in this and the preceding section have another interesting implication. The analysis of Figure 2 demonstrated that the two individuals with identical preferences choose safe intermediate allocations and then use them to speculate away from full insurance (certainty line). This tendency to bet rather than insure doesn't require identical preferences or identical feasible sets. As a result of incomplete preferences and convex feasible sets, there are generally gains from trade from betting (moving away from the certainty line).

# 4 Concluding Remarks

This paper shows that betting or speculative trading between agents with incomplete preferences is likely to occur if agents have access to convex choice sets. This contrasts sharply with endowment-economy models where preference incompleteness often hinders either betting, speculative trading, or mutually beneficial insurance arrangements. Our results imply that individuals with identical but incomplete $\succ$ and identical feasible sets can potentially gain from speculative trade. Were the preferences complete in this environment the identical individuals would never gain from speculative trade.

As a reviewer points out, the reason that this occurs in our setting is that preference completeness resolves an important analytic ambiguity that arises in the case of preference incompleteness. In the complete preference setting, the individual's $\succ$ and $Z$ and the assumption of rationality require an individual's behavior to be characterized by the maximization postulate. The behavioral mapping describing rational behavior can be a correspondence, but the individual would remain indifferent among all those options. Incompleteness, on the other hand, invalidates the use of the maximization postulate and requires that we rely upon the more general, but weaker, choice correspondence used here. It is the possibility of a continuum of undominated choices, consistent with rationality, that is the analytic keystone of our results. If, for example, one strengthened our behavioral axiom to include a specific criterion for selection among those undominated choices, our results may disappear. Needless to say, however, different criteria for selecting among undominated choices would result in different resolutions, and how those differences compare can only be determined by further research.

One also may wonder whether our main result would hold were we to consider an alternative to incomplete preferences that remained consistent with a multiple-prior representation. If the agents had a preference structure obtained by relaxing Savage's sure-thing principle (for example, maximin expected utility (Gilboa and Schmeidler 1989), Choquet expected utility (Schmeidler 1989), variational preferences (Maccheroni, Marinacci, Rustichini 2006), or smooth ambiguity (Klibanoff, Marinacci, and Mukerji 2005)) rather than the completeness axiom, the bet-creating effect would not materialize. In those cases, decision-makers with identical preferences and feasible sets would have identical status-quo points and would not benefit from speculative trade. In fact, Rigotti and Shannon (2012) have shown that variational preferences displaying ambiguity aversion generate equilibria in an endowment setting that cannot be distinguished from those that would arise in a subjective expected utility model. One expects similar results to emerge in the current setting. On the other hand, our results will hold for preferences which allow for kinks off the certainty line. An example is the set of reference dependent ambiguity sensitive preferences (Mihm 2014).

En route to characterizing equilibria of the economies considered in the paper and studying their properties, we have made two additional contributions. First, we have developed a framework to endogenize the status-quo allocations of decisionmakers. Our approach of identifying a status quo is very different from the standard treatment where an exogenously given status quo is the norm. Second, we have provided a tractable differential representation of status-quo allocations, equilibria, and conditions where speculative trade may or may not emerge.

# 5 Appendix : Proofs

**Proof of Lemma 1:** The demonstration for 1 through 4 is similar to Chambers (2014) and is presented here for completeness. 1) $\Rightarrow$ For $q \in P(y)$, (A.3) requires that there exists an $\varepsilon > 0$ such that $q + \varepsilon B \subset P(y)$ where $B$ is the Euclidean unit ball, and hence $q - \varepsilon \in P(y)$. $\Leftarrow$ If $T(q; y) > 0$, $\exists \varepsilon \in (0, T(q; y))$ such that $q - \varepsilon \succ y$. By monotonicity and transitivity, $q \succ y$. This establishes the first part. For the second part, (A.4) requires $y - \varepsilon \in P(y)$ for all $\varepsilon < 0$, but irreflexivity implies $y - 0 \notin P(y)$.

2) For $v \in X$, note

$$\begin{aligned} T(q+v; y) &= \sup\{x \in \mathbb{R} : q + v - x \succ y\} \\ &= \sup\{x + v - v \in \mathbb{R} : q + v - x \succ y\} \\ &= \sup\{x - v : q + v - x \succ y\} + v \\ &= T(q; y) + v. \end{aligned}$$

3) For $q' \in q + \mathbb{R}^S_+ \setminus \{0\}$, (A.4) requires

$$q' - T(q; y) \succ q - T(q; y) \in P(y),$$

which establishes the desired monotonicity. (A.5) requires

$$p \succ q \Longleftrightarrow \alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r \text{ for all } \alpha \in (0,1] \text{ and all } r \in \mathbb{R}^S.$$

Take arbitrary $p, q, y \in \mathbb{R}^S$ such that $p, q \in P(y)$. By (A.5)

$$\begin{aligned} \alpha p + (1-\alpha) q &\in P(\alpha r + (1-\alpha) q) \text{ for all } \alpha \in (0,1], \\ (1-\alpha) q + \alpha r &\in P((1-\alpha) y + \alpha y) = P(y) \text{ for all } \alpha \in [0,1). \end{aligned}$$

But then, by transitivity, $\alpha p + (1-\alpha) q \in P(y)$ for all $\alpha \in (0,1)$, which implies that $P(y)$ is a convex set for all $y \in \mathbb{R}^S$. By the definition of $T$, $q - T(q; y) \in P(y)$ and $h - T(h; y) \in P(y)$. Convexity of $P(y)$ implies convexity of $P(y)$, and thus $\lambda(q - T(q; y)) + (1-\lambda)(h - T(h; y))$ $= \lambda q + (1-\lambda) h - (\lambda T(q; y) + (1-\lambda) T(h; y)) \in P(y)$, $\lambda \in [0,1]$ which implies concavity.

4) $p \in \bar{\partial} T(q; y)$ with $\bar{\partial} T(q; y)$ nonempty requires

$$p'(h - q) \geq T(h; y) - T(q; y), \quad \forall h \in \mathbb{R}^S.$$

Taking $h = q + 1$ with $1 \in X$, translatability implies

$$p'1 \geq T(q+1; y) - T(q; y) = 1.$$

Symmetrically, taking $h = q - 1$, yields $p'1 \leq 1$. It remains to show that $\bar{\partial}T(q;y) \subset \mathbb{R}^S_+$. Take $h = q + \delta$, with $\delta \in \mathbb{R}^S_+ \backslash \{0\}$, then if $p \in \bar{\partial}T(q;y)$, $p'\delta \geq T(q+\delta;y) - T(q;y) \geq 0$ by the monotonicity established above. Because $\delta$ is arbitrary, $\bar{\partial}T(q;y) \subset \mathbb{R}^S_+$. And thus, because $p'1 = 1$ and $\bar{\partial}T(q;y) \subset \mathbb{R}^S_+$, $\bar{\partial}T(q;y) \subset \Delta$.

5) A.5 requires

$$p \succ q \Longleftrightarrow \alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r \text{ for all } \alpha \in (0,1] \text{ and all } r \in \mathbb{R}^S.$$

Take $r = 0$ and observe that for all $\alpha \in (0,1]$

$$\frac{p}{\alpha} \succ \frac{q}{\alpha} \Longleftrightarrow p \succ q \Longleftrightarrow \alpha p \succ \alpha q,$$

which, recognizing that $r$ is arbitrary, allows us to rewrite independence as

$$p \succ q \Longleftrightarrow \alpha p + r \succ \alpha q + r \text{ for all } \alpha > 0 \text{ and all } r. \tag{6}$$

By (6), for all $\alpha > 0$

$$\begin{aligned} T(\alpha q + r; \alpha y) &= \sup\{x : \alpha q + r - x \succ \alpha y + r\} \\ &= \sup\{x : \alpha(q-y) - x \succ 0\} \\ &= \alpha \sup\left\{\frac{x}{\alpha} : (q-y) - \frac{x}{a} \succ 0\right\} \\ &= \alpha T(q-y;0). \end{aligned}$$

6) Using this last result implies

$$\begin{aligned} \bar{\partial}T(q;q) &= \left\{\pi \in \mathbb{R}^S : \pi'(h-q) \geq T(h,q) - T(q,q) \text{ for all } h \in \mathbb{R}^S\right\} \\ &= \left\{\pi \in \mathbb{R}^S : \pi'(h-q) \geq T(h-q,0) - T(0,0) \text{ for all } h - q \in \mathbb{R}^S\right\} \\ &= \bar{\partial}T(0;0). \end{aligned}$$

. ■

**Proof of Lemma 2:** Closely follows the proof of Lemma 1. ■

**Proof of Proposition 3:** Our proof is similar to Luenberger (1994) and Chambers (2014). $\Rightarrow q^n \in Z^n$ requires $\sigma^n(q^n) \leq 0$. Suppose $\sigma^n(q^n) < 0$, then $q^n - \sigma^n(q^n) \in Z^n \cap P(q^n)$ by the definition of the shortage function and monotonicity of $\succ$ implying that $q^n$ is not a status quo. Hence, $\sigma^n(q^n) = 0$ for a status quo. Lemma 1 then ensures $T(q^n;q^n) = 0$ so that $\sigma^n(q^n) - T(q^n;q^n) = 0$ for the status quo. To prove zero maximality suppose that there exists some other $r$ such that

$$T(r;q^n) - \sigma^n(r) > 0.$$

Lemma 1 then ensures $T(r - \sigma^n(r); q^n) > 0$, but that contradicts the assumption that $q^n$ is status-quo because $r - \sigma^n(r) \in Z^n \cap P(q^n)$.

$\Leftarrow$ To prove sufficiency suppose that

$$0 = T(q^n; q^n) - \sigma^n(q^n) = \max\{T(r; q^n) - \sigma^n(r)\},$$

but that $q^n$ is not a status quo. By Lemma 1, $T(q^n; q^n) = 0$, and thus $\sigma^n(q^n) = 0$. Because $q^n$ is not a status quo, there must exist some $r$ such that $\sigma^n(r) \leq 0$ and $T(r; q^n) > 0$. If $\sigma^n(r) = 0$, we contradict zero-maximality, and the desired result follows. That leaves the case where $\sigma^n(r) < 0$. In this case, $r - \sigma^n(r) \in Z^n$ and by monotonicity $r < r - \sigma^n(r) \in P(q^n)$. Hence, by Lemma 1, $T(r - \sigma^n(r); q^n) = T(r; q^n) - \sigma^n(r) > 0$, which contradicts zero-maximality. ■

**Proof of Corollary 4:** The curvature properties of $T(\cdot, q^n)$ and $\sigma^n(\cdot)$ ensure that the maximization program in Proposition 3 is concave. It then follows from the standard results of convex analysis (Rockafellar 1970, Section 27) that

$$\begin{aligned} 0 &\in \bar{\partial} T(q^n, q^n) - \underline{\partial}\sigma^n(q^n) \\ &= \bar{\partial} T(0, 0) - \underline{\partial}\sigma^n(q^n) \end{aligned}$$

and $\sigma^n(q^n)$ must be equal to 0 for $q^n$ to be a status quo. ■

**Proof of Proposition 5:** Because $q', q'' \in Q^n$ requires $\sigma^n(q') = \sigma^n(q'') = 0$ and $Z^n$ is convex, we have for all $\alpha \in (0, 1)$

$$q_\alpha \equiv \alpha q' + (1 - \alpha) q'' \in Z^n,$$

$\sigma^n(q_\alpha) \leq 0$, $\sigma^n(q_\alpha - \sigma^n(q_\alpha)) = 0$, and $\underline{\partial}\sigma^n(q_\alpha - \sigma^n(q_\alpha)) = \underline{\partial}\sigma^n(q_\alpha)$ (this last equality follows from the translation property). By definition

$$\begin{aligned} \underline{\partial}\sigma^n(q') &= \left\{\pi \in \Delta : \pi'(h - q') \leq \sigma^n(h) - \sigma^n(q') \text{ for all } h \in \mathbb{R}^S\right\}, \\ \underline{\partial}\sigma^n(q'') &= \left\{\pi \in \Delta : \pi'(h - q'') \leq \sigma^n(h) - \sigma^n(q'') \text{ for all } h \in \mathbb{R}^S\right\}, \end{aligned} \tag{7}$$

and

$$\underline{\partial}\sigma^n(q_\alpha) = \left\{\pi \in \Delta : \pi'\left(h - \alpha q' - (1 - \alpha) q''\right) \leq \sigma^n(h) - \sigma^n(q_\alpha) \text{ for all } h \in \mathbb{R}^S\right\}.$$

By (7), $\pi \in (\alpha\underline{\partial}\sigma^n(q') + (1 - \alpha)\underline{\partial}\sigma^n(q''))$ must satisfy

$$\begin{aligned} \pi'(h - q') &\leq \sigma^n(h) - \alpha\sigma^n(q') - (1 - \alpha)\sigma^n(q'') \\ &\leq \sigma^n(h) - \sigma^n(q_\alpha) \text{ for all } h \in \mathbb{R}^S \end{aligned}$$

where the second inequality follows from the convexity of $\sigma(z)$, which requires for all $\alpha \in (0,1)$

$$\sigma^n(q_\alpha) \leq \alpha\sigma^n(q') + (1-\alpha)\sigma^n(q'').$$

This establishes that $(\alpha\underline{\partial}\sigma^n(q') + (1-\alpha)\underline{\partial}\sigma^n(q'')) \subset \underline{\partial}\sigma^n(q_\alpha) = \underline{\partial}\sigma^n(q_\alpha - \sigma^n(q_\alpha))$ for all $\alpha \in (0,1)$. ■

**Proof of Proposition 9:** See Luenberger (1994, 1995) and Chambers (2014). ■

**Proof of Proposition 12:** The first part of the proposition follows immediately from Corollary 4. Consider now the second part of the proposition. Under the condition that $\bar{\partial}T(0;0) \cap (\cup_{z\in B}\underline{\partial}\sigma(z))$ contains at least two points, there at least two hyperplanes that separate the boundary $B = \{z : \sigma^n(z) = 0\}$ of the set $Z$ and the better than set of the decision-maker. By strict convexity of the set $Z$, these two hyperplanes separate the two sets at two different points on the boundary $B$. Both of these points belong to the set of status quos $Q^n$. By Proposition 5, the continuum of points $q_\alpha - \sigma^n(q_\alpha) \in Q^n$ where $q_\alpha = \alpha q' + (1-\alpha) q''$ (for all $\alpha \in (0,1)$) also belongs to $Q^n$. Since the individuals choose their status quos independently, the probability that all of them will choose the the same point on the continuum is equal to zero. Otherwise, which has probability 1, they will choose different points and there will exist gains from speculative trade. ■

**Example 14** *Consider the two-person, two-state case where each individual is endowed with* $(\bar{z}_1, \bar{z}_2) > 0$ *where*

$$\frac{b^2}{1-b^2} < \frac{\bar{z}_1}{\bar{z}_2} < \frac{b^1}{1-b^1}$$

*and* $b^n \in (0,1)$ *for* $n = 1, 2$. *For individual* $n = 1, 2$

$$\sigma^n(z_1, z_2) = z_1 + b^n(z_2 - \bar{z}_2 + \bar{z}_1 - z_1)$$

*Preferences are given by*

$$T(q;r) = \min_\pi \{(1-\pi)(q_1 - r_1) + \pi(q_2 - r_2) : \pi \in [a,1] \subset [0,1]\}.$$

*Assume that*

$$\frac{b^n}{1-b^n} \in [a,1].$$

*for both* $n$. *In this instance,* $(\bar{z}_1, \bar{z}_2) \in Q^n$ *for both* $n$. *And for each* $n$, *so too is any* $(q_1^n, q_2^n)$ *lying on the hyperplane passing through* $(\bar{z}_1, \bar{z}_2)$ *with normal* $(1-b^n, b^n)$. *The no-betting outcome associated with* $q^n = (\bar{z}_1, \bar{z}_2)$, *however, is not Pareto optimal. For although,*

$$\sum_n \min_\pi \{(1-\pi)(\bar{z}_1 - \bar{z}_1) + \pi(\bar{z}_1 - \bar{z}_2) : \pi \in [a,1] \subset [0,1]\} - \max_n \left\{ b_1^n \left( \sum_n \bar{z}_1 - \bar{z}_1 \right) + b_2^n(\bar{z}_1 - \bar{z}_2) \right\} = 0,$$

*it is also true that the allocation,*

$$
\begin{aligned}
y_1^n &= \frac{\bar{z}_1 \left(1 - b^1\right) + b^1 \bar{z}_2}{2 \left(1 - b^1\right)} > \bar{z}_1, \\
y_2^n &= \frac{b^2 \bar{z}_2 + \left(1 - b^2\right) \bar{z}_1}{2 b^2} > \bar{z}_2,
\end{aligned}
$$

*which corresponds to individual 2 specializing in state 2 income and individual 1 specializing in state income, dominates* $q^n = (\bar{z}_1, \bar{z}_2)$ *and is feasible.*

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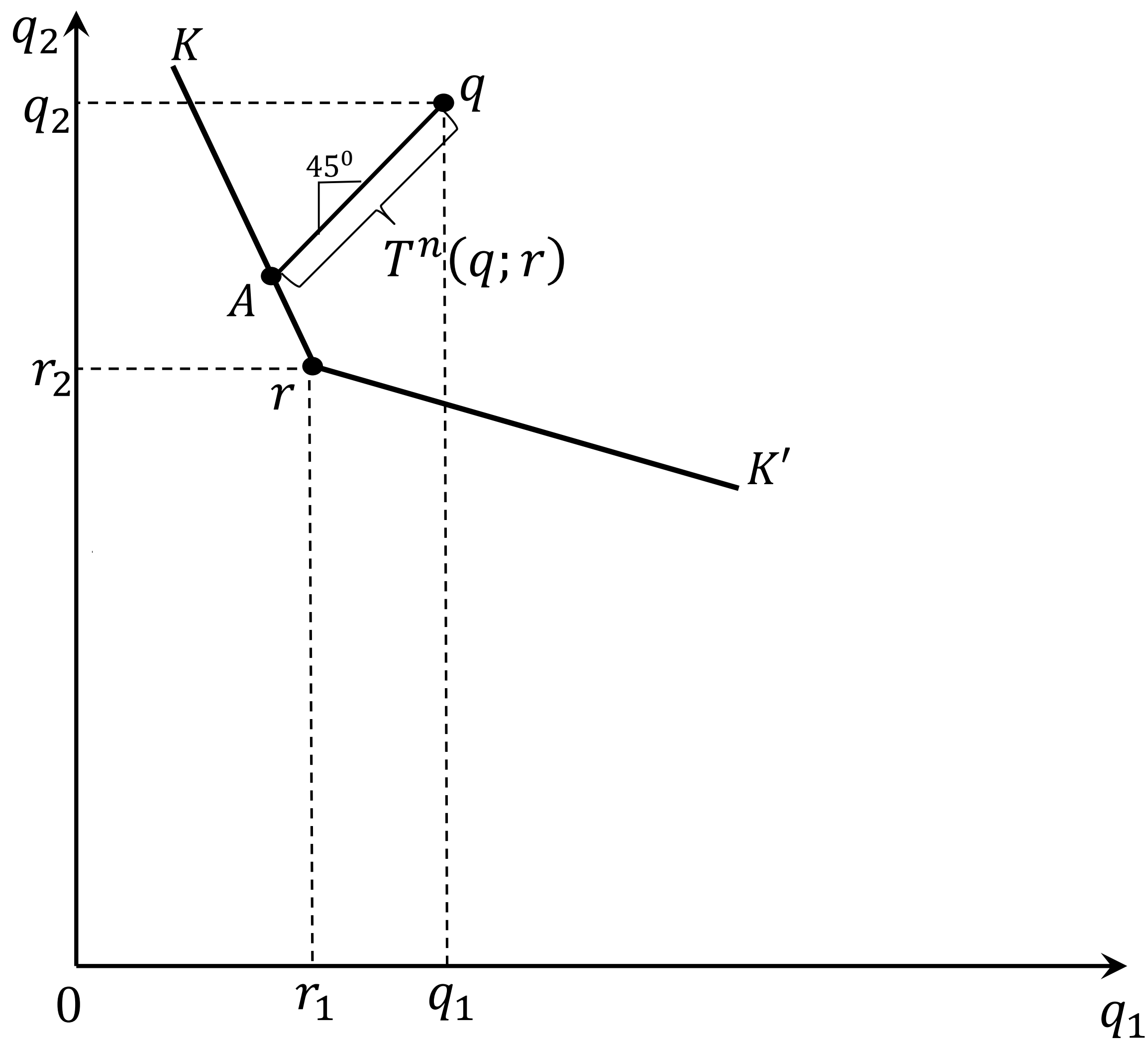


**Figure 1.** Strictly better than set and translation function

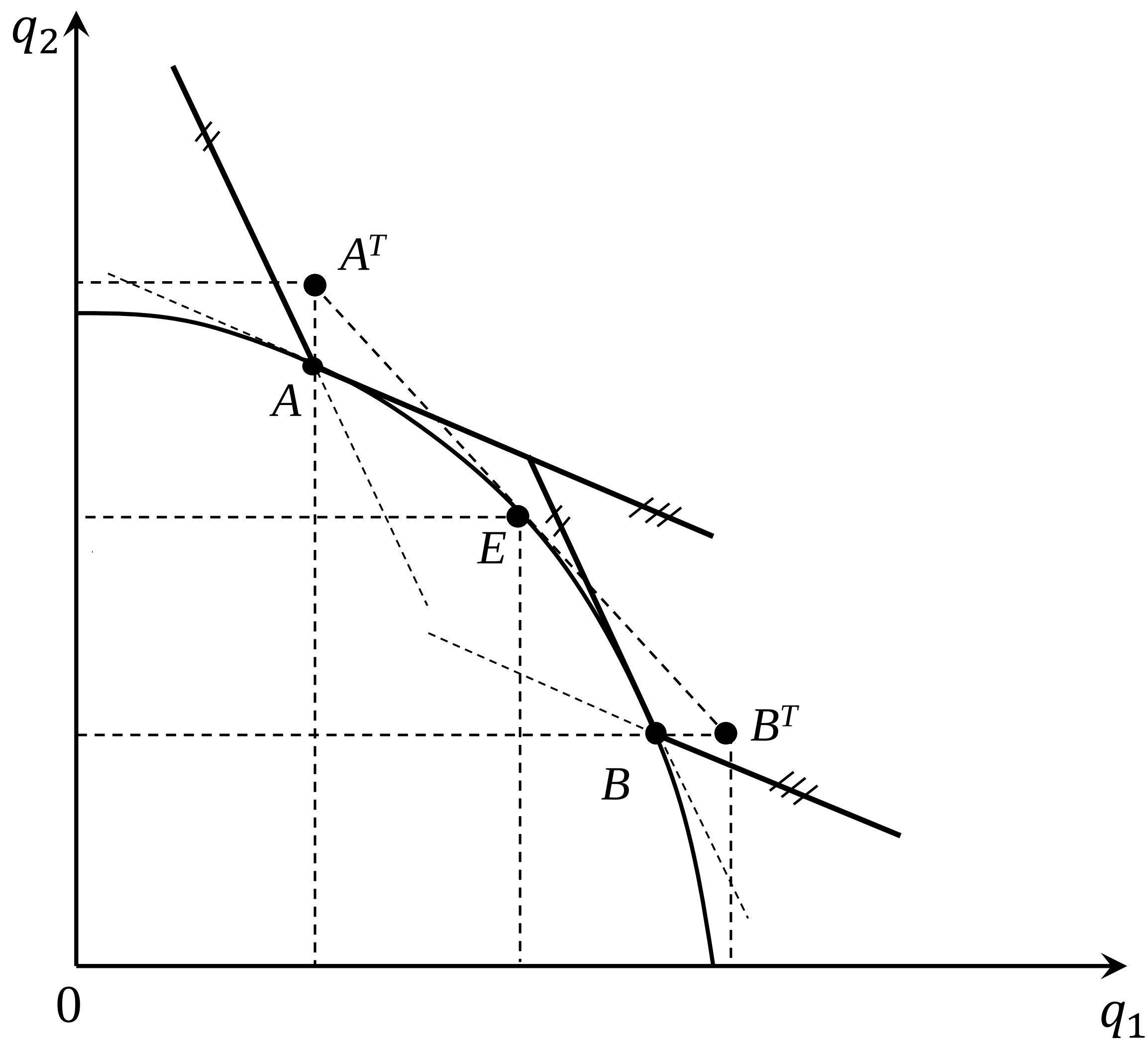


**Figure 2.** Gains from trade with identical preferences and technologies

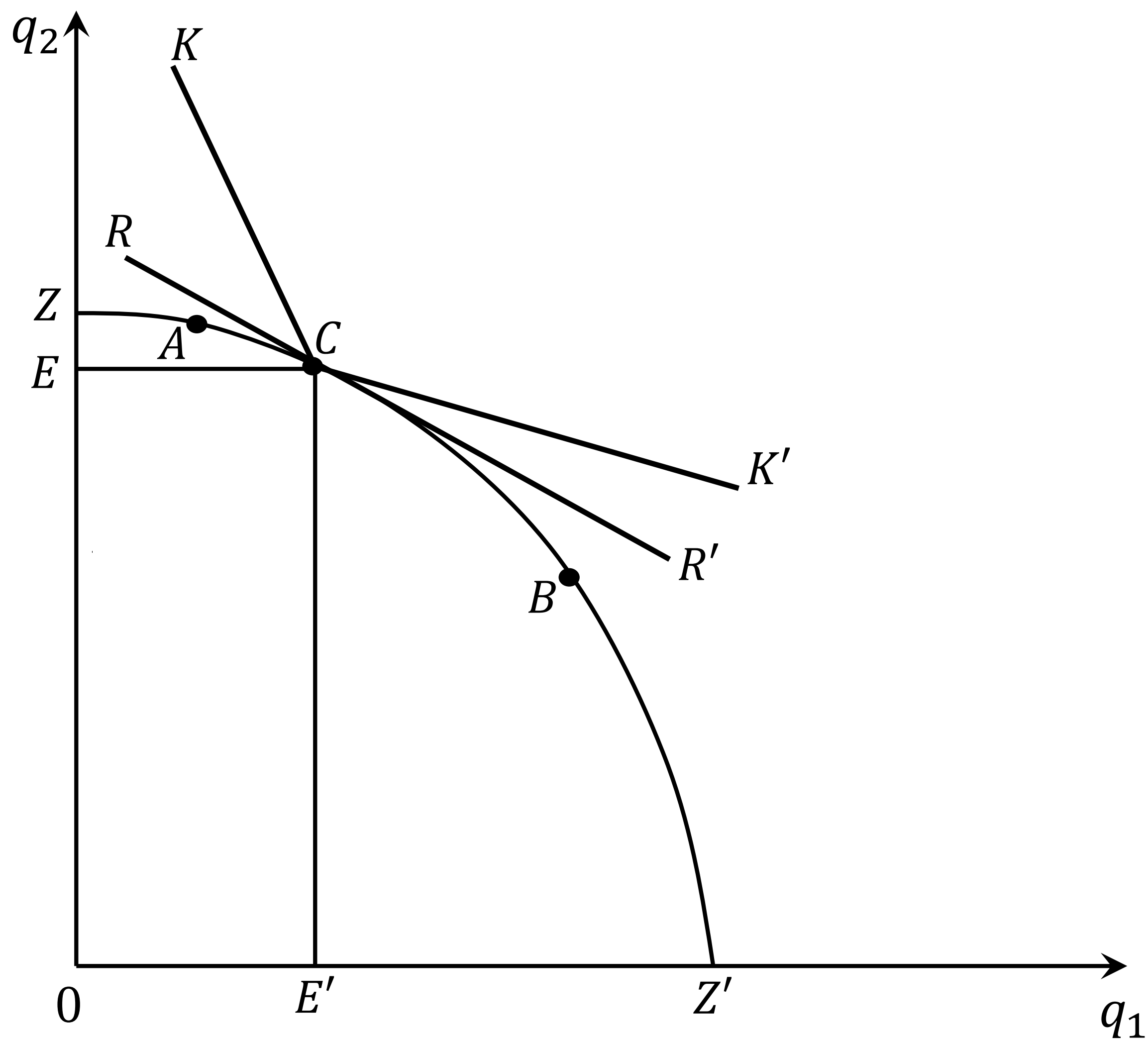


**Figure 3.** A comparison of feasible sets and preferences