Dynamics of surface homeomorphisms: braid types and coexistence of periodic orbits

John Guaschi

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Nonlinear Systems Laboratory
Mathematics Institute
University of Warwick
Coventry
CV4 7AL
England

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Abstract

In this Thesis, we discuss the following general problem in dynamical systems: given a surface homeomorphism, and some information about its periodic orbits, what else can we deduce about its periodic orbit structure? Using the concept of the 'braid type' of a periodic orbit, its relation to Artin's braid group, and the Nielsen-Thurston classification of surface homeomorphisms, we examine problems pertaining to the coexistence of periodic orbits, in particular for homeomorphisms of the disc, annulus and 2-torus. We aim to elucidate the underlying geometry and topology in such systems. The main original results are the following:

- classification of braid types for periodic orbits of diffeomorphisms of genus one surfaces with topological entropy zero (Theorems 2.5 and 2.6).
- lower bounds on the size of the rotation sets of annulus homeomorphisms which possess certain periodic orbits or finite invariant sets (Theorems 3.17 and 3.19, Theorem 3.20).
- bounds on the size and shape of rotation sets of torus homeomorphisms possessing certain periodic orbits. (Theorems 3.24 and 3.25).
- the coexistence of periodic orbits in the disc, for periodic orbits of prime period (Theorem 4.2), of period 4 (Theorem 4.10), and for 3-point invariant sets (Theorem 4.11).
- the coexistence of periodic orbits in the annulus (Theorem 4.4), and of the sphere with a 4-point invariant set (Theorem 4.12).
- given a torus homeomorphism isotopic to the identity which possesses a fixed point, it is isotopic to the identity relative to that fixed point (Theorem 5.6).
- given a periodic orbit of a disc homeomorphism of period 3, the coexistence of a strongly linked fixed point (Theorem 5.10).
- given a periodic orbit of the annulus homeomorphism of pseudo-Anosov braid type, its rotation number lies in the interior of the rotation set (Theorem 6.1).
- amongst certain sets of braid types of the annulus and disc, the existence of minimal elements, which any other element dominates (Theorems 7.4 and 7.15).

Contents

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Declaration

Except where stated to the contrary, the results in this Thesis are the result of original work by the author. Chapter 2 was written with J. Llibre and R. S. MacKay, and is to appear in 'Publicacions Matemàtiques Universitat Autonoma de Barcelona'.

Chapter 0

Introduction

In the study of dynamical systems, we are interested in the following problem: given a map $f : M \to M$ of a topological space and a point $x \in M$, what do we know about the set

$$x, f(x), f^2(x), \ldots ?$$

We call this set the *forward orbit* of x under f. Some questions we may wish to ask are:

- is the forward orbit of x under f dense in M?
- Is x a periodic point of f i.e. does there exist $q \in \mathbf{N}$ such that $f^q(x) = x$? If x is periodic, we call the least such q the *period* of x.
- given the forward orbit of x, what can we say about other orbits of f?

Usually we do not know f explicitly. For example, given experimental data, we will only be able to follow the path of a few orbits – we cannot possibly expect to know everything about the system. Even if we do know f explicitly, it may be too difficult computationally to allow a thorough investigation of the system. Hence we are lead to ask: given a small amount of information about f, what properties can we deduce? For example, if f has a periodic orbit, what other periodic orbits coexist? This line of questioning was followed by Sharkovskii, who proved the following well-known Theorem.

Theorem 0.1 ([Sha, BGMY, CE]) *Define the following ordering $\succ$ on the natural numbers:*

$$3 \succ 5 \succ 7 \succ \ldots \succ 3.2 \succ 5.2 \succ \ldots \succ 3.2^2 \succ 5.2^2 \succ \ldots \succ 2^2 \succ 2 \succ 1.$$

If $f : \mathbf{R} \to \mathbf{R}$ is continuous and has a periodic orbit of period m, then it possesses a periodic orbit of period n for all $m \succ n$. Thus if f has a periodic orbit of period 3, it has periodic orbits of all periods.

We may refine further Sharkovskii's ordering of the natural numbers by considering the cyclic permutation induced by the action of f on the points of the periodic orbit rather than just the period (see for example, [Be, CE, MSS]).

Given a parametrized family of one-dimensional maps, Sharkovskii's Theorem generates an order on the parameter space by giving information on the order in which periodic orbits are created, for example, the existence of period-doubling cascades.

This provides an example of 'universal structure' in such families, in which we may also observe the phenomenon of 'self-similarity'. If we study the bifurcation diagram of the standard unimodal map $f_\mu : [0,1] \to [0,1]$, $f_\mu(x) = 4\mu x(1-x)$ for $0 \leq \mu \leq 1$, and magnify arbitrarily small parts of it, then these magnifications appear to be the same as the whole diagram [CE, TS]. For instance, it is possible to find infinitely many copies of the period-doubling cascade within the diagram.

Whilst the study of one-dimensional spaces continues to thrive, much attention has turned to the analogous problems in two dimensions. In particular, given a map f of a surface and information about its periodic orbits, what else can we say about f? In this case just the period of an orbit tells us nothing; for example consider a rigid rotation by $2\pi/3$ of the disc $\mathbf{D}^2$ about its centre. All points of this map have period 3, except for the centre which is a fixed point. It becomes necessary to know more about a periodic orbit to be able to make meaningful statements about the dynamics of f. One way of doing this is to specify a periodic orbit by the isotopy class of the map relative to the orbit, which is (up to conjugacy) what we shall call the *braid type* of a periodic orbit. As we shall see in Chapter 1, it is possible to define a partial ordering on the set of braid types. There we shall discuss in more detail the reasons for choosing the braid type as a specification. One motive for doing so is that we may utilize the Nielsen-Thurston classification of isotopy classes of surface homeomorphisms to furnish ourselves with a 'simplest' map in the isotopy class relative to the periodic orbit. According to this classification, a braid type is either simple ('periodic'), complicated ('pseudo-Anosov') or is made up of components of these two forms ('reducible'). Dynamically, the pseudo-Anosov isotopy classes are the most interesting, and we will concentrate our work on them. We examine several important properties of pseudo-Anosov homeomorphisms, including the persistence of their periodic orbits under isotopy, the existence of Markov partitions which provides symbolic dynamics, and the associated notion of an embedded branched 1-submanifold or 'train track', which enables us to understand something of the underlying topology of the homeomorphism.

In Chapter 2, we study the braid types of homeomorphisms of the 2-torus which are not pseudo-Anosov, and we shall see that this restriction places strong constraints on the dynamics.

In Chapter 3, we discuss the notions of *flow-equivalence* and *homology directions*, and relate them to homeomorphisms of the annulus and 2-torus. It turns out that flow-equivalence elucidates much of the inherent self-similarity in the partial order on braid types. As a consequence, we prove several results on the size and shape of the set of rotation numbers of annulus and torus homeomorphisms (the rotation number of a point is a measure of its average displacement). We study the *Weierstrass map*, which relates homeomorphisms of the 2-torus (which we understand) to those of the sphere S^2, enabling us to find lower bounds on the size of rotation sets for annulus homeomorphisms possessing 2-point invariant sets.

In Chapter 4, we turn our attention to homeomorphisms of the disc $\mathbf{D}^2$ and the sphere S^2, to prove several results analogous to those of Sharkovskii on the coexistence of periodic orbits, given a periodic orbit of a specific period. In particular, we show that any homeomorphism of the disc with a periodic orbit of period 4 whose braid type is pseudo-Anosov possesses periodic orbits of all periods, and we reach the same conclusion for any homeomorphism of the sphere with a 4-point invariant set whose braid type is pseudo-Anosov. We also study coexistence questions for homeomorphisms of the disc with a periodic orbit of prime period, and homeomorphisms of the annulus.

In Chapter 5, we discuss the notion of linking of periodic orbits of a homeomorphism, and the relation with the Lefschetz number of a map via algebraic representations such as the Burau representation, for which we exhibit two geometric interpretations. The Lefschetz number is a homotopy invariant which gives information about the fixed point structure. We give a number of examples of the use of such representations; in particular we show that given a homeomorphism of the 2-torus isotopic to the identity which has a fixed point, then it is isotopic to the identity relative to that fixed point. As another application, we answer in the affirmative the 'linking number' conjecture for the case of a periodic orbit of period 3, which asserts the existence of a fixed point which is strongly linked with the given periodic orbit.

In Chapter 6, we provide a generalisation of the Aubry-Mather Theorem by proving the existence of periodic orbits of homeomorphisms of the annulus whose braid type is periodic. Further, we show that if the braid type of a periodic orbit is pseudo-Anosov, then the corresponding rotation number lies in the interior of the rotation set. In order to do this, we discuss Nielsen fixed point theory and the Reidemeister trace which gives us an important homotopy invariant. We relate these ideas to the topological nature of pseudo-Anosov homeomorphisms.

In Chapter 7, we consider the set $B_{2,1}$ of pseudo-Anosov braid types of periodic orbits of the annulus of period 2, and show that there exist two 'minimal' elements, minimal in the sense that any element of $B_{2,1}$ dominates (at least) one of the two in the partial order. Using flow-equivalence, this has possible implications for periodic orbits of homeomorphisms of the annulus other than those of period 2. We also prove similar results for braid types in the set BT_3 of pseudo-Anosov braid types of periodic orbits of homeomorphisms of the disc of period 3. In this case, there exists a unique braid type which is dominated by all others; we conjecture analogous results for BT_n.

Finally, we give an extensive bibliography, containing many references not cited explicitly in the text, but which are related to the general development of the subjects discussed.

As well as proving the above results, we aim to give an overview of the techniques which have been developed in the dynamical study of surface homeomorphisms, with particular emphasis on the topological and geometric considerations involved, and their relation to algebraic structures.

Chapter 1

Thurston's classification of surface homeomorphisms and the braid type of a periodic orbit

In this Chapter, we recall the classification of Thurston of surface homeomorphisms up to isotopy, introduce the concept of the braid type of a periodic orbit of a surface homeomorphism, and relate it to Artin's braid group.

1.1 Notation

We start by fixing some notation which shall be used throughout. Let M be a compact, orientable, connected surface, with or without boundary. M may be characterized by its *genus* g (the number of torus summands), and the number of boundary components b (the number of discs removed). The *Euler characteristic* $\chi(M)$ of M is given by

$$\chi(M) = 2 - 2g - b.$$

There exists a Riemannian metric of constant curvature κ for M, which by a suitable choice of metric we can assume to be $+1$, 0 or -1 [Wo]. If $\chi(M) > 0$, e.g. the sphere, the metric is elliptic with $\kappa = +1$. If $\chi(M) = 0$ e.g. the 2-torus, the metric is Euclidean and $\kappa = 0$. If $\chi(M) < 0$, the Gauss-Bonnet Theorem [DC] implies that the metric is hyperbolic and $\kappa = -1$. It is the last case that we shall be interested in.

Let $Homeo^+(M)$ denote the space of orientation-preserving homeomorphisms of M, equipped with the topology of uniform convergence. We say that two homeomorphisms $f_1, f_2 \in Homeo^+(M)$ are *isotopic*, which we write $f_1 \simeq f_2$, if f_1 and f_2 may be joined by an arc in $Homeo^+(M)$. $\simeq$ defines an equivalence relation on $Homeo^+(M)$. Set

$$Aut(M) = Homeo^+(M)/\simeq . \tag{1.1}$$

Let $Isot(M)$ be the subset of $Homeo^+(M)$ of homeomorphisms isotopic to the identity. Then we can rewrite equation 1.1 as

$$Aut(M) = Homeo^+(M)/Isot(M).$$

For $f \in Homeo^+(M)$, let $[f]$ denote its *isotopy class* in $Aut(M)$. Let $A \subset M \setminus \partial M$ be a finite set, where ∂M denotes the boundary of M, then we introduce the following

notation:

$$Homeo^+(M, A) = \{f \in Homeo^+(M) : f(A) = A\}$$

and $Aut(M, A)$ is the group of isotopy classes in $Aut(M)$ relative to A.

1.2 The specification of a periodic orbit

1.2.1 Requirements for a specification

As outlined in the Introduction, the problem that we shall be interested in is to generalize Sharkovskii's theorem [Sha, Ste] to two dimensions. Let $f : M \to M$ be a homeomorphism of a surface M, such that f has a periodic orbit P of period n. What can we say about the set of periodic orbits of f, their structure, and the topological entropy of f?

The first question is what to choose as a specification of periodic orbits. In one dimension, one can choose the period of the orbit to deduce non-trivial information. One can gain more information by choosing the permutation induced by the periodic orbit, and this leads to a more refined ordering e.g. the MSS ordering in the unimodal case [MSS]. In two dimensions however, the period is not sufficient as a specification, as on the disc any period can come from a rigid rotation through an appropriate angle. Further, the concept of permutation is not well-defined. We require a generalization of the notion of permutation which describes the manner in which the map moves topologically the points of the orbit. In analogy with the one-dimensional case, we would like to choose a specification of periodic orbits satisfying the following conditions [Boy2, BoyF]:

1. given periodic orbits P,Q and a specification *spec* of periodic orbits, there exists an ordering on the set of specifications, such that $spec(P)$ dominates $spec(Q)$ in the ordering if and only if any map with a periodic orbit of type $spec(P)$ also has one of type $spec(Q)$.

2. given a periodic orbit P, and its specification $spec(P)$, there exists a 'simplest' map exhibiting P. In particular, for any periodic orbit Q of this simplest map with specification $spec(Q)$, there exists a periodic orbit of type $spec(Q)$ for any map which exhibits a periodic orbit of type $spec(P)$. So this simplest map has minimal structure with respect to the specification.

3. it is possible to calculate the structure of the ordering in case 1.

With such a structure, and a given family of maps, one can study how the family moves through the ordering, to suggest the bifurcations occurring.

1.2.2 The braid type of a periodic orbit

An idea, due essentially to Bowen [Bo], is to specify a periodic orbit $P = \{P_1, \ldots, P_n\} \subset \mathrm{Int}(M)$ of a homeomorphism f of a surface M by the isotopy class of f restricted to the surface minus the orbit, or equivalently, the isotopy class of the map relative to the orbit. We formalize this notion in the following manner (c.f. [Boy1, Boy3, LM1, GLM]). Define a surface M_P by deleting P from M and recompactifying by adding a circle, the circle of unit tangent vectors, at each end of $M \setminus P$. Let S_i be the circle replacing each P_i, and let $p : M_P \to M$ be the map which collapses each circle S_i to the point P_i of P.

If f is a C^1-diffeomorphism, then $f|_{M\setminus P}$ extends to a homeomorphism $f_P : M_P \to M_P$ by setting

$$f_P(x) = \frac{Df_{P_i}.x}{\|Df_{P_i}.x\|} \quad x \in S_i.$$

We call this process *blowing up* P. Given two diffeomorphisms f,g of M with periodic orbits P,$Q \subset \mathrm{Int}M$ respectively, we say that (P, f) and (Q, g) have the same *braid type* if there exists an orientation-preserving homeomorphism $h : M_P \to M_Q$ with hf_Ph^{-1} isotopic to g_Q. This defines an equivalence relation on such pairs, and we call the equivalence class of (P, f) its *braid type* $bt(P, f)$. This will serve as our characterization of periodic orbits. We remark that it was not necessary that P was a single periodic orbit; we could have taken P to be any finite union of periodic orbits.

We have assumed f to be C^1, but for the purposes of defining the braid type of a periodic orbit P of f, it suffices that f is differentiable at the points of P. We can in fact relax this condition by modifying slightly the definition of braid type. Given two homeomorphisms f,g of M with finite unions of periodic orbits $\mathcal{P}$,$\mathcal{Q} \subset \mathrm{Int}(M)$ respectively, we say that $(\mathcal{P}, f)$, $(\mathcal{Q}, g)$ have the same *braid type* if there exists an orientation-preserving homeomorphism $h : M \to M$ such that

1. $h(\mathcal{P}) = \mathcal{Q}$, and
2. hfh^{-1} is isotopic to g relative to $\mathcal{Q}$.

This defines an equivalence relation on such pairs, and we call the equivalence class of $(\mathcal{P}, f)$ its *braid type* $bt(\mathcal{P}, f)$. If f is a C^1-diffeomorphism, the two definitions coincide. This is a consequence of the following Theorem.

Theorem 1.1 *Suppose M is a compact, connected orientable surface, and suppose $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$ is an n-point set in $Int(M)$. Let M_Q be the compact surface obtained by blowing up the points of $\mathcal{Q}$. Then there exists a canonical isomorphism between $Aut(M_Q)$ and $Aut(M, \mathcal{Q})$.*

Proof

Let $p : M_Q \to M$ be the natural projection. p induces a morphism

$$p_* : Aut(M_Q) \to Aut(M, \mathcal{Q}).$$

The following Proposition affirms the surjectivity of p_*:

Proposition 1.2 ([Ko4]) *Let $f \in Homeo^+(M, \mathcal{Q})$. Then for all $\epsilon > 0$ sufficiently small, there exists $\delta > 0$ and $g \in Homeo^+(M, \mathcal{Q})$ isotopic to f relative to $\mathcal{Q}$, such that:*

1. *$f = g$ on $M \setminus \cup_{i=1}^n B(Q_i, \epsilon)$, where $B(Q_i, \epsilon)$ is the closed ball of radius ϵ centred on Q_i.*
2. *g is a C^∞-diffeomorphism on $\cup_{i=1}^n B(Q_i, \delta)$.*
3. *the isotopy between f and g fixes $M \setminus \cup_{i=1}^n B(Q_i, \epsilon)$.*

Proof

Pick $\epsilon > 0$ small enough so that the balls $B(Q_i, \epsilon)$ are pairwise disjoint. Let σ be the permutation induced by f on $\mathcal{Q}$, let $\mathcal{D}$ be a closed disc containing Q_1, satisfying

$$f^j(\mathcal{D}) \subset B(Q_{\sigma^j(1)}, \epsilon) \text{ for } 0 \leq j \leq n.$$

Choose $\delta > 0$ satisfying

$$\Delta_j \subset f^j(\mathcal{D}) \text{ for } 0 \leq j \leq n,$$

where $\Delta_j = B(\mathcal{Q}_{\sigma^j(1)}, \delta)$. Let T_j be a translation sending Δ_j onto Δ_{j+1}. Let

$$U_j = \text{Cl}(f^j(\mathcal{D})) \setminus \Delta_j \text{ for } j = 0, 1, \ldots, n,$$

where Cl denotes the closure. Define g on $(M \setminus \cup_{i=0}^{n-1} f^j(\mathcal{D})) \cup \bigcup_{j=0}^{n-1} \Delta_j$ as follows:

$$g = \begin{cases} T_j & \text{on } \Delta_j \text{ for } 0 \leq j \leq n-1 \\ f & \text{on } M \setminus \cup_{i=0}^{n-1} f^j(\mathcal{D}). \end{cases}$$

For each j, there exists an orientation-preserving homeomorphism which maps U_j to U_{j+1} and which coincides with g on ∂U_j, using the Alexander isotopy e.g. [Bi3]. So one may extend g to an orientation-preserving homeomorphism of M satisfying:

1. $f^{-1}g = Id$ on $M \setminus \cup_{i=0}^{n-1} f^j(\mathcal{D})$.
2. for each $j = 0, 1, \ldots, n-1$, $f^{-1}g|_{f^j(\mathcal{D})}$ is a homeomorphism of the disc $f^j(\mathcal{D})$ onto itself which is the identity on $\partial f^j(\mathcal{D})$ and which fixes the point $\mathcal{Q}_{\sigma^j(1)}$.

Thus $f^{-1}g|_{f^j(\mathcal{D})}$ is isotopic to the identity by an isotopy fixing $\mathcal{Q}_{\sigma^j(1)}$ and the complement of $f^j(\mathcal{D})$. □

Injectivity of p_* is a consequence of the following argument. Let $\tilde{f} \in Homeo^+(M_Q)$, and let $f \in Homeo^+(M, Q)$ be the homeomorphism induced by $\tilde{f}$. Pick a basepoint $\tilde{x}_0 \in \text{Int}(M_Q)$, and let $x_0 = p(\tilde{x}_0)$. Let $\mathsf{F} = \pi_1(M \setminus Q)$, and let $Aut_*\mathsf{F}$ be the group of all automorphisms of F which are induced by elements of $Homeo^+(M, Q)$. Let $Inn(\mathsf{F})$ denote the group of all inner automorphisms of F, then there exists a homomorphism

$$\lambda : Homeo^+(M, Q) \rightarrow Aut_*\mathsf{F}/Inn(\mathsf{F}).$$

But $Aut(M, Q) \cong Aut_*\mathsf{F}/Inn(\mathsf{F})$ [ZVC, Ep1]. So if f is isotopic to the identity (the basepoint is not necessarily fixed during the isotopy), the action induced by f on $\pi_1(M \setminus Q, x_0)$, an inner automorphism, is the same action induced by $\tilde{f}$ on $\pi_1(M_Q, \tilde{x}_0)$. It follows from Theorems of Whitehead, see [Sp], and Epstein [Ep1] that $\tilde{f}$ is isotopic to the identity. □

This ensures that the isotopy class of the map relative to the orbit is equivalent to the isotopy class of the map restricted to the surface minus the orbit. So we may (and shall) switch between these two viewpoints in any given situation.

In the case of the disc $\mathbf{D}^2$, it is convenient to work with a standard model $\mathbf{D}_n$ of the disc minus n points [Boy3, HT1]. Let $\mathbf{D}^2 = \{x \in \mathbf{R}^2 : \|x\| \leq 1\}$, and $x_i = (2i/(n+1) - 1, 0)$ for $i \in [n] = \{1, \ldots, n\}$. Then set

$$X = \{x_i : i \in [n]\}. \tag{1.2}$$

Let $P = \{P_1, \ldots, P_n\}$ be a periodic orbit in the interior of $\mathbf{D}^2$ of a C^1-diffeomorphism $f : \mathbf{D}^2 \rightarrow \mathbf{D}^2$. Let $\mathbf{D}_P, \mathbf{D}_n$ be the surfaces obtained by blowing up P and X respectively, and let f_P be the extension of f to $\mathbf{D}_P$. Let h be a homeomorphism from $\mathbf{D}_P$ to $\mathbf{D}_n$. Then f_P induces a homeomorphism $f_P^h = h \circ f_P \circ h^{-1}$ of $\mathbf{D}_n$ (see figure 1.1). A different choice of h replaces f_P^h by a homeomorphism which is conjugate to f_P^h. Thus the conjugacy class $< \alpha >$ of the isotopy class $\alpha = [f_P^h]$ of f_P^h in $Aut(\mathbf{D}_n)$ is independent of the choice of h.

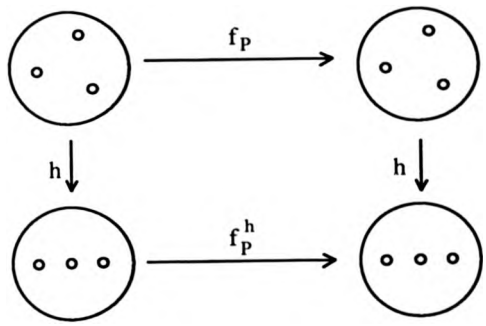

Figure 1.1: The homeomorphism induced by f_P

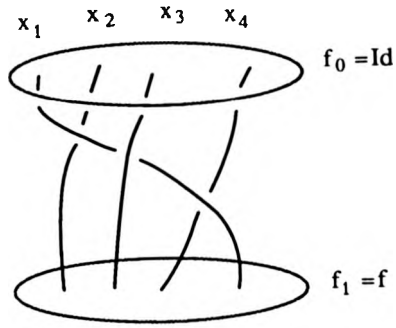

Figure 1.2: A geometric braid

1.3 Artin's braid group

In this Section, we shall motivate the choice of terminology 'braid type' defined in Section 1.2, by providing a link between $Aut(\mathbf{D}^2, X)$ and Artin's braid group B_n. We begin by recalling the definition of the latter [Bi3, Han].

1.3.1 Geometric braids

Let $f : (\mathbf{D}^2, X) \to (\mathbf{D}^2, X)$ be an orientation-preserving homeomorphism of the disc. Consider an isotopy $F : \mathbf{D}^2 \times I \to \mathbf{D}^2$, such that $f_0 = Id$, $f_1 = f$, where $f_t(x) = F(x,t)$. For each $i \in [n]$, let

$$\mathcal{A}_i = \cup_{t \in I}(f_t(x_i), t) \in \mathbf{D}^2 \times I$$

define the ith string. Then we call the set

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots \cup \mathcal{A}_n \tag{1.3}$$

a *geometric braid* (see figure 1.2). Note that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$. We define an equivalence relation $\sim$ on the set of geometric braids in the following way: $\mathcal{A} \sim \mathcal{A}'$ if there exists a continuous family of geometric braids $\mathcal{A}(s) = \mathcal{A}_1(s) \cup \ldots \cup \mathcal{A}_n(s)$, $s \in I$, such that the following hold:

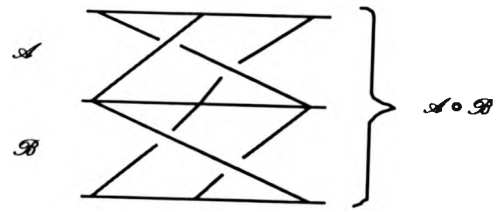

Figure 1.3: Composition of geometric braids

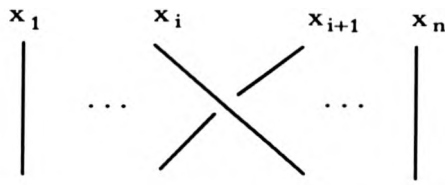

Figure 1.4: The generator σ_i of B_n

1. $\mathcal{A}(0) = \mathcal{A}$, $\mathcal{A}(1) = \mathcal{A}'$.
2. the endpoints of each strand are held fixed as s passes from 0 to 1.

The composition of geometric braids is defined by concatenation as shown in figure 1.3. If we substitute equivalent braids $\mathcal{A}'$, $\mathcal{B}'$ such that $\mathcal{A}' \sim \mathcal{A}$, $\mathcal{B}' \sim \mathcal{B}$, then $\mathcal{B} \circ \mathcal{A} \sim \mathcal{B}' \circ \mathcal{A}'$. Thus composition is well-defined on equivalence classes of geometric braids, and so provides a group structure on this set of classes. This group is B_n, *Artin's braid group* on n strings. We call an element of B_n a *braid*.

Theorem 1.3 (Artin [Ar, Bi3]) *The group B_n admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and defining relations*

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geq 2, \quad 1 \leq i,j \leq n-1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}, \quad 1 \leq i \leq n-2. \end{aligned} \tag{1.4}$$

The generator σ_i corresponds to the braid given in figure 1.4. The centre $Z(B_n)$ of B_n is the infinite cyclic subgroup generated by a full twist $\theta_n = (\sigma_1\sigma_2\ldots\sigma_{n-1})^n$.

1.3.2 Braid types and Artin's braid group

Given an element $f \in Homeo^+(\mathbf{D}^2, X)$ and an isotopy f_t from Id to f, it is clear from Section 1.3.1 that we may associate a braid with f. In fact, the following is true:

Theorem 1.4 ([Bi3]) $Aut(\mathbf{D}^2, X) \cong B_n/Z(B_n)$.

This correspondence can be understood by mapping each generator σ_i of B_n to a homeomorphism $m(\sigma_i)$ whose support is a small neighbourhood containing the points x_i and x_{i+1} of X, and which swaps the two points by a half-twist (see figure 1.5). The kernel of the mapping m is precisely $Z(B_n)$. So there exists a (canonical) bijection

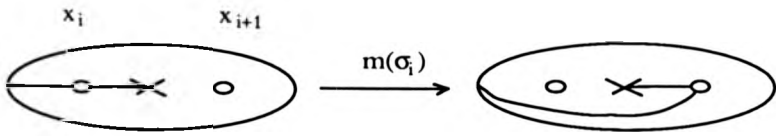

Figure 1.5: The homeomorphism $m(\sigma_i)$

between conjugacy classes of $Aut(\mathbf{D}^2, X)$ and of $B_n/Z(B_n)$, so we may identify the braid type $bt(P, f)$ of a periodic orbit P of period n by a conjugacy class in $B_n/Z(B_n)$. We shall say that a braid $\beta \in B_n$ *represents* the braid type of P if the conjugacy class of $\beta Z(B_n)$ is $bt(P, f)$. Hence we may express $bt(P, f)$ symbolically, in terms of the generators of B_n.

1.4 The partial order on braid types

In this Section, we describe the partial order on braid types [Boy3, HT1]. Let Σ_n denote the symmetric group on n elements i.e. the group of all permutations of the set $[n]$, and let $\pi_n : B_n \to \Sigma_n$ be the homomorphism defined by $\pi_n(\sigma_i) = (i, i+1)$. Since $\pi_n(\theta_n) = Id$, π_n induces a homomorphism $\overline{\pi}_n : B_n/Z(B_n) \to \Sigma_n$. We say a conjugacy class in $B_n/Z(B_n)$ is *cyclic* if its image under $\overline{\pi}_n$ is an n-cycle. Define

$$BT_n = \{\text{cyclic conjugacy classes in } B_n/Z(B_n)\}.$$

For $f \in Homeo^+(\mathbf{D}^2)$, define

$$bt(f) = \{bt(P, f) : P \text{ is a periodic orbit of } f\}.$$

Then

$$BT = \bigcup_{n \geq 1} BT_n = \bigcup_{f \in Homeo^+(\mathbf{D}^2)} bt(f). \tag{1.5}$$

We define an order relation on BT as follows:

$$\alpha \succ \beta \Leftrightarrow \text{ for all } f \in Homeo^+(\mathbf{D}^2), \alpha \in bt(f) \Rightarrow \beta \in bt(f). \tag{1.6}$$

Theorem 1.5 (Boyland [Boy3]) *The order relation $(BT, \succ)$ is a partial order.*

We give a sketch of the proof (a proof using similar ideas but in more detail may be found in Chapter 4). For $\succ$ to be a partial order, it must be

1. reflexive ($\beta \succ \beta$).
2. transitive ($\beta \succ \gamma, \gamma \succ \delta \Rightarrow \beta \succ \delta$).
3. antisymmetric ($\beta \succ \gamma, \gamma \succ \beta \Rightarrow \beta = \gamma$).

The first two properties follow from equation 1.6. The third follows from the fact that any two maps of the disc are isotopic, and a Theorem of Brunovskii [Bru] which

asserts the existence of an isotopy compelling two chosen braid types to disappear at different points during the isotopy. Hence $(BT, \succ)$ is also antisymmetric. □

The fundamental problem of the complete elucidation of this order is still open, but some partial results have been found. For example, Matsuoka [Ma2] has an algorithm to uncover the ordering in BT_3 (see Chapter 7 for more details), and Hall [HT1] has analysed braid types occurring in Smale's horseshoe.

As we have seen, the braid type of a periodic orbit is an isotopy class (up to conjugacy). We may gain a deeper insight into them by understanding isotopy classes on surfaces, which we shall now proceed to describe.

1.5 Nielsen-Thurston theory of surface homeomorphisms

In this Section, we describe Thurston's work on homeomorphisms of surfaces [T1, T2, CB, FLP, HT]. We motivate briefly this work by recalling the classification of toral automorphisms.

1.5.1 Classification of toral automorphisms

Consider the torus $\mathbf{T}^2$ as the quotient of $\mathbf{R}^2$ by the integer lattice $\mathbf{Z}^2$, equipped with a fixed orientation. Then $\pi_1(\mathbf{T}^2) \cong H_1(\mathbf{T}^2, \mathbf{Z}) \cong \mathbf{Z} \oplus \mathbf{Z}$. Given an element A of the special linear group $SL(2, \mathbf{Z})$, it preserves the lattice $\mathbf{Z}^2$, so there exists an induced homeomorphism $h_A : \mathbf{T}^2 \to \mathbf{T}^2$. Further, $(h_A)_* : \pi_1(\mathbf{T}^2) \to \pi_1(\mathbf{T}^2)$ has matrix A. On the other hand, two homeomorphisms of $\mathbf{T}^2$ are isotopic if and only if they induce the same isomorphism on $\pi_1(\mathbf{T}^2)$. Hence $Aut(\mathbf{T}^2) \cong SL(2, \mathbf{Z})$. The characteristic polynomial of A may be written as

$$\lambda^2 - (\mathrm{Trace}(A))\lambda + 1.$$

The eigenvalues are either [CB]:

1. complex i.e. Trace(A)=1, -1 or 0. Applying the Cayley-Hamilton Theorem leads to the conclusion that h_A has period 6,3 and 4 respectively. h_A is said to be *periodic*.

2. both ± 1. Then A has an integral eigenvector which projects to an invariant simple closed curve γ on $\mathbf{T}^2$, and which is homotopically non-trivial. Up to a change of coordinates,

$$A = \begin{pmatrix} \pm 1 & a \\ 0 & \pm 1 \end{pmatrix}$$

   which is a Dehn twist. This means that γ has a tubular neighbourhood U homeomorphic to an annulus $\mathbf{A}$, which we parametrize as $\{(r, \theta) : 0 \leq r \leq 1\}$. The *Dehn twist* in γ is the homeomorphism whose support is $\mathbf{A}$, and which is given by $(r, \theta) \mapsto (r, \theta + 2\pi r)$ on $\mathbf{A}$ (see figure 1.6). h_A is said to be *reducible*.

3. The eigenvalues of A are distinct real numbers λ, λ^{-1}, satisfying $|\lambda| > 1 > |\lambda^{-1}|$. h_A has infinite order, and leaves no simple closed curve invariant. The eigendirections induce two transverse, invariant foliations $\mathcal{F}^s$ (stable) and $\mathcal{F}^u$ (unstable) on $\mathbf{T}^2$ each parallel to the corresponding eigendirection, and transverse

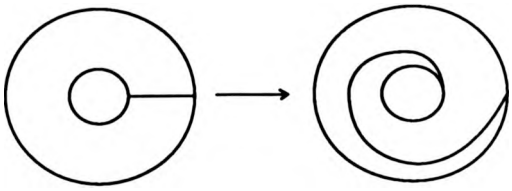

Figure 1.6: The Dehn twist

measures μ^s and μ^u such that

$$\begin{aligned} h_A(\mathcal{F}^s, \mu^s) &= (\mathcal{F}^s, \lambda^{-1}\mu^s) \\ h_A(\mathcal{F}^u, \mu^u) &= (\mathcal{F}^u, \lambda\mu^u), \end{aligned}$$

where $h_A(\mathcal{F}^s, \mu^s)$ denotes the image foliation $h_A(\mathcal{F}^s)$ (equal to $\mathcal{F}^s$) equipped with the image measure: the measure of an arc α transverse to the image foliation is the μ^s-measure of $h_A^{-1}(\alpha)$. So h_A contracts linearly in one direction and expands linearly in the other direction. h_A is said to be *Anosov*.

1.5.2 Classification of surface homeomorphisms

The notions expressed in Section 1.5.1 have been generalized by Thurston [T1] who has given a complete classification of homeomorphisms of hyperbolic surfaces, up to isotopy (in this vein, it is important to cite the work of Nielsen [Ni1, Ni2], who proved many of the statements of Thurston fifty years ago). The situation becomes more difficult, as the fundamental group is no longer Abelian, and one can find homeomorphisms in different isotopy classes whose action on homology is the same [T1, Pa1, Pa2].

We begin by describing the homeomorphisms analogous to those of Section 1.5.1. The theory gives a 'prime decomposition theorem'; isotopy classes of homeomorphisms are either simple (periodic), complicated (pseudo-Anosov), or have components of both types (reducible). One may find a more comprehensive exposition in [FLP, CB, Pe2]. We remark that if f is an orientation-preserving homeomorphism of a compact, orientable, connected surface M with $\chi(M) \geq 0$, then one of the following is true.

1. $g = 1$, $b = 0$, then $M = \mathbf{T}^2$, for which we have classified homeomorphisms up to isotopy in Section 1.5.1.

2. $g = 0$, $b = 0$, then $M = S^2$ and f is always isotopic to the identity.

3. $g = 0$, $b = 1$, then $M = \mathbf{D}^2$ and f is always isotopic to the identity.

4. $g = 0$, $b = 2$, then $M = \mathbf{A}$ and f is either isotopic to the identity (and preserves the boundary components), or is isotopic to the homeomorphism which swaps the boundary components given by

$$(r, \theta) \mapsto (-r, -\theta),$$

where we parametrize $\mathbf{A}$ as $[-1, 1] \times S^1$.

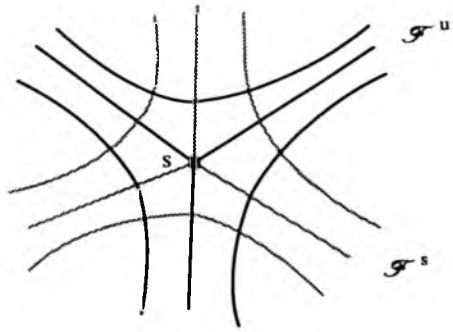

Figure 1.7: A three-pronged singularity

Let M be a compact, connected surface with Euler characteristic $\chi(M) < 0$, possibly with boundary, and let $f : M \to M$ be a homeomorphism. We consider the three cases in turn.

1. f is *periodic* if it is an isometry of a hyperbolic metric. As a consequence, $f^n = Id$ for some $n \in \mathbb{N}$ [Ni2].

2. f is a *pseudo-Anosov* homeomorphism if:

   (a) there exist two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ invariant under f having the same finite set of singularities in $\mathrm{Int}(M)$.

   (b) each singularity of $\mathcal{F}^s$ and $\mathcal{F}^u$ in $\mathrm{Int}(M)$ has at least three separatrices ('prongs'). A neighbourhood of a k-pronged singularity of a measured foliation may be obtained by gluing together k rectangles along edges e.g. see figure 1.7.

   (c) if $\partial M \neq \emptyset$, each component of ∂M is a cycle of leaves (i.e. a finite union of leaves and singular points) of $\mathcal{F}^s$ and $\mathcal{F}^u$, and contains at least one singularity of each foliation (see figure 1.8).

   (d) $\mathcal{F}^s$ and $\mathcal{F}^u$ are transverse in $\mathrm{Int}(M)$.

   (e) $\mathcal{F}^s$ and $\mathcal{F}^u$ admit transverse measures μ^s and μ^u respectively, and there exists a real number $\lambda > 1$ such that

   $$\begin{aligned} f(\mathcal{F}^s, \mu^s) &= (\mathcal{F}^s, \lambda^{-1}\mu^s) \\ f(\mathcal{F}^u, \mu^u) &= (\mathcal{F}^u, \lambda\mu^u), \end{aligned} \tag{1.7}$$

   where $f(\mathcal{F}^s, \mu^s)$ denotes the image foliation $f(\mathcal{F}^s)$ (equal to $\mathcal{F}^s$) equipped with the image measure. Thus f contracts on the leaves of $\mathcal{F}^s$ where distance is measured by μ^u, and expands on the leaves of $\mathcal{F}^u$ where distance is measured by μ^s. We call λ the *dilatation factor* of f.

Let $\mathcal{F}$ be one of the two invariant foliations, let P_s denote the number of prongs emanating from s, where s is a singularity of $\mathcal{F}$. If $s \in \partial M$, then arcs emanating from s along ∂M count as separatrices. For example both singularities s in

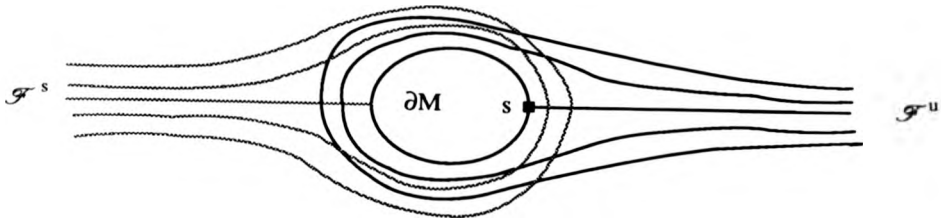

Figure 1.8: An example of the foliations at the boundary ∂M

figures 1.7 and 1.8 have $P_s = 3$ for $\mathcal{F}^u$. Then the Euler-Poincaré formula gives the following relation:

$$2\chi(M) = \sum_{s \in Sing\mathcal{F}} (2 - P_s). \tag{1.8}$$

This formula is a consequence of the Poincaré-Hopf Theorem [DC, CB] which relates the indices of (isolated) singularities of a line field (recall that a line field is locally the non-oriented version of a nowhere-zero vector field) to the Euler characteristic in the following manner:

$$\chi(M) = \sum I(s),$$

where $I(s)$ is the winding number of the line field obtained by counting the rotations of the tangent vector restricted to the boundary of a small disc containing s, and the sum is over all singularities of the line field defined by $\mathcal{F}$. We may verify that $I(s) = 1 - \frac{P_s}{2}$, and equation 1.8 follows.

3. f is *reducible* if there exists a family $\Gamma = \{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$ of simple closed curves such that

   (a) each Γ_i is *essential* i.e. it is neither null-homotopic nor boundary parallel.

   (b) $\Gamma_i \cap \Gamma_j = \emptyset$ if $i \neq j$.

   (c) $f(\Gamma) = \Gamma$.

Define a *generalized twist* to be a diffeomorphism of the open annulus $\mathbb{A}^\circ = S^1 \times (-1, 1)$ of the form

$$\theta' = \pm\theta + \omega(r), \quad r' = \pm r,$$

where $(\theta, r) \in S^1 \times (-1, 1), \omega : [-1, 1] \to S^1$ is C^1, and $\omega'(\pm 1) = 0$.

Theorem 1.6 ([T1]) *Let $\phi : M \to M$ be a homeomorphism, then ϕ is isotopic to f, for which exactly one of the following holds:*

1. *f is periodic.*

2. *f is pseudo-Anosov.*

3. *f is reducible. In this case, there exists an invariant tubular neighbourhood $U(\Gamma)$ of Γ, such that for each component $M_1, \ldots, M_k$ of $M \setminus U(\Gamma)$ and a (least) $n_i \in \mathbb{N}$, $f^{n_i}(M_i) = M_i$, and $f^{n_i}|_{M_i}$ satisfies 1 or 2. Further, if $U_1, \ldots, U_l$ are the components of $U(\Gamma)$ and $m_j \in \mathbb{N}$ is the (least) integer such that $f^{m_j}(U_j) = U_j$, then $f^{m_j}|_{U_j}$ is a generalized twist (not necessarily a Dehn twist).*

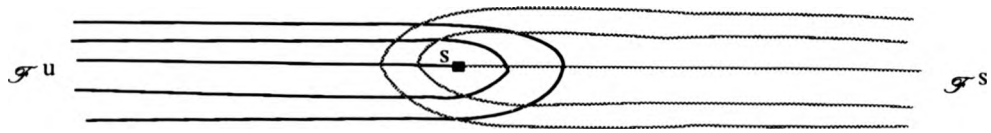

Figure 1.9: The local picture at a one-pronged singularity

We shall refer to the isotopy class $[\phi]$ as being periodic, pseudo-Anosov or reducible, and as *irreducible* if it is pseudo-Anosov or periodic. We shall also refer to the *Thurston type* of a homeomorphism, and call f the *Thurston canonical form* of $[\phi]$. If $[\phi]$ is periodic, then any complicated behaviour of ϕ can be isotoped away. However complicated behaviour of a pseudo-Anosov homeomorphism persists under isotopy, as we shall see later.

To conclude this section, we introduce a generalization of Handel ([H5], see also [FLP]) of pseudo-Anosov homeomorphisms which will be useful in what follows. Let $f : M \to M$ be a homeomorphism of a compact connected surface, and let K be a finite invariant set. Then f is *pseudo-Anosov relative to* K if it satisfies all the properties of a pseudo-Anosov homeomorphism except that the stable and unstable foliations are permitted to have one-pronged singularities at the points of K. Figure 1.9 illustrates the local picture at such a singularity. Equivalently, one can carry out the process of blowing up K, but as we have seen in Section 1.2.2 we may consider isotopy classes relative to a finite invariant set for the purposes of our investigation.

1.6 Properties of pseudo-Anosov homeomorphisms

In this Section, we review some of the properties of pseudo-Anosov homeomorphisms (see [FLP] for details). Let $g : M \to M$ be a homeomorphism of a compact orientable surface M and let K be a finite invariant set.

1. Let $o(x, g)$ be a periodic orbit of (least) period n. $o(x, g)$ is said to be *unremovable* if for any homeomorphism g_1 isotopic to $g = g_0$ (rel. K), there exists an isotopy (rel. K) $g_\mu, \mu \in I$, and a continuous arc $\gamma(\mu)$ with $\gamma(\mu) \cap K = \emptyset$, such that for all $\mu \in I$, $\gamma(\mu)$ is a period n point for g_μ and $\gamma(0) = x$ (see [AF, HT2] for more details).

   The following result is an extension of a Theorem of Asimov and Franks [AF].

   Theorem 1.7 (Boyland [Boy3], Hall [HT2]) *Let $g : M \to M$, K be as above, and suppose g is a pseudo-Anosov homeomorphism relative to K. Then all its periodic orbits not in ∂M are unremovable.*

   So periodic orbits of g persist under isotopy. We recall from Section 1.2.1 that this is the kind of behaviour that we are looking for in the 'simplest' map i.e. it has minimal structure with respect to periodic orbits. For the remainder of this Section, we take g to be pseudo-Anosov relative to K.

2. Global shadowing of pseudo-Anosov homeomorphisms

Theorem 1.8 (Handel [H3]) *Given f isotopic to g, there is a closed subset $Y \subset M$ and a continuous surjection $\pi : Y \to M$ that is homotopic to inclusion, such that $g \circ \pi = \pi \circ f|_Y$. Further, if f has the same topological entropy as g (it is at least as large), then f is semi-conjugate to g.*

Thus all the dynamics of g persist in its isotopy class. The topological entropy $h(f)$ of a homeomorphism f is a quantity which characterizes the dynamical complexity of f (see [Wa] for an exposition). If $h(f) = 0$, strong constraints are placed on the dynamics, in particular the periodic orbits of f, as we shall see in Chapter 2. For a pseudo-Anosov homeomorphism g, $h(g) = \log \lambda$, where λ is the dilatation factor of g. In particular, the topological entropy of a pseudo-Anosov homeomorphism is strictly positive.

3. For any $f \simeq g$, and for any essential curve C, $f^n(C) \not\simeq C$ for all $n \neq 0$ [Mi]. Thus there are no periodic points in the induced map on free homotopy classes.

4. g has a *Markov partition* by topological rectangles, i.e. there exists a finite collection of subsets $\mathcal{R} = \{R_1, \ldots, R_k\}$ of M such that:

   (a) for each R_i, $1 \leq i \leq k$, there exists an embedding $\phi_i : I \times I \to M$ whose image is R_i such that for all $t \in I$, $\cup_{s \in I} \phi_i(t,s)$ (respectively $\cup_{s \in I} \phi_i(s,t)$) is contained in a finite union of leaves and singularities of $\mathcal{F}^s$ (respectively $\mathcal{F}^u$), and in fact in one leaf if $t \in (0,1)$. We call the set $\cup_{s \in I} \phi_i(t,s)$ (respectively $\cup_{s \in I} \phi_i(s,t)$) an $\mathcal{F}^s$- (respectively $\mathcal{F}^u$-) fibre. A point $x \in R_i$ is contained in just one $\mathcal{F}^s$- (respectively $\mathcal{F}^u$-) fibre, which we denote by $\mathcal{F}^s(x, R_i)$ (respectively $\mathcal{F}^u(x, R_i)$).

   (b) $M = \cup_{i=1}^k R_i$.

   (c) Int $R_i \cap$ Int $R_j = \emptyset$ for $i \neq j$.

   (d) If $x \in$ Int R_i and $g(x) \in$ Int R_j then

   $$\begin{aligned} g(\mathcal{F}^s(x, R_i)) &\subset \mathcal{F}^s(g(x), R_j), \text{ and} \\ g^{-1}(\mathcal{F}^u(g(x), R_j)) &\subset \mathcal{F}^u(x, R_i). \end{aligned}$$

   Further we may choose $\mathcal{R}$ so that

   $$\begin{aligned} g(\mathcal{F}^u(x, R_i)) \cap R_j &= \mathcal{F}^u(g(x), R_j), \text{ and} \\ g^{-1}(\mathcal{F}^s(g(x), R_j)) \cap R_i &= \mathcal{F}^s(x, R_i). \end{aligned}$$

   So $g(R_i)$ crosses R_j just once.

Given $\mathcal{R}$, we construct the *subshift of finite type* as follows. Let A be the $k \times k$ matrix given by

$$\begin{aligned} a_{ij} &= 1 \text{ if } g(\text{Int } R_i) \cap \text{ Int } R_j \neq \emptyset \\ &= 0 \text{ otherwise.} \end{aligned}$$

Let

$$\Sigma_A = \{\mathbf{c} : \mathbf{c} \in [k]^{\mathbb{Z}} \text{ and } a_{c_i c_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}$$

be the *transition matrix* for $\mathcal{R}$, where $[k] = \{1, \ldots, k\}$. Let $\sigma_A : \Sigma_A \to \Sigma_A$ be the shift map, defined by $\sigma_A(c_i) = c_{i+1}$ for all $i \in \mathbb{Z}$, where $\mathbf{c} = (c_j)_{j \in \mathbb{Z}} \in \Sigma_A$. $\sigma_A : \Sigma_A \to \Sigma_A$ defines the subshift of finite type, which is semi-conjugate to g [FLP].

Figure 1.10: Examples of n-gons

Figure 1.11: structure of the vertex of a train track

5. g is topologically transitive [FLP] i.e. there exists a dense orbit.

6. Periodic points of g are dense [FLP], and there are a finite number of periodic points of each period.

7. Suppose g is pseudo-Anosov relative to a periodic orbit K. Since there are an infinite number of periodic orbits of g which persist under isotopy, then $bt(K, g)$ dominates an infinite number of other braid types in the partial order.

8. There exists an invariant train track for g. A train track is a branched 1-submanifold which is essentially a retract of the stable foliation. Its behaviour under the induced action of g is exactly that of a related Markov partition, so knowing the train track and the induced action of g on it, one may reconstruct the dynamics of g. We give a brief exposition of the theory of train tracks in the next Section.

1.7 Train tracks and periodic orbits

We start by recalling some basic definitions and facts about train tracks, more details of which may be found in [T3, HP, Pa1, Pe1, PP1, BH1, BH2]. Let M be a compact, connected, oriented surface of Euler characteristic $\chi(M) < 0$. We define an *n-gon* to be a disc with n discontinuities in the tangent space of the boundary. Figure 1.10 gives examples of a nullgon, a monogon, a bigon and a trigon. A *train track* τ on M is a branched 1-submanifold (i.e. a differentiable graph embedded in M such that the branches are tangent at the vertices), such that no component of $M \setminus \tau$ is a smooth annulus nor a nullgon, a monogon or a bigon. At each vertex of the train track, the branches partition into two disjoint sets depending on the tangency of the branch at the vertex. We label (arbitrarily) one set incoming and the other outgoing (see figure 1.11).

A *measured train track* (τ, μ) is a train track τ carrying a positive measure $\mu(b_i)$ on each branch b_i, such at each vertex v_k, a *switch condition* is satisfied (see figure 1.12). This means that at each vertex, the sum of the measures of the incoming branches equals the sum of the measures of the outgoing branches. A measured train track (τ, μ) defines a measured foliation, and conversely. For given a train track on M with branches $b_1, \ldots, b_n$, define $E(\tau) \subset \mathbf{R}^n$ to be the convex cone of positive measures μ

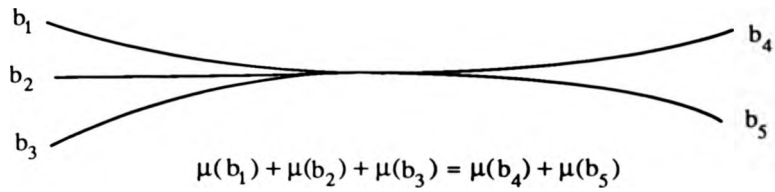

Figure 1.12: The switch condition at a vertex of a train track

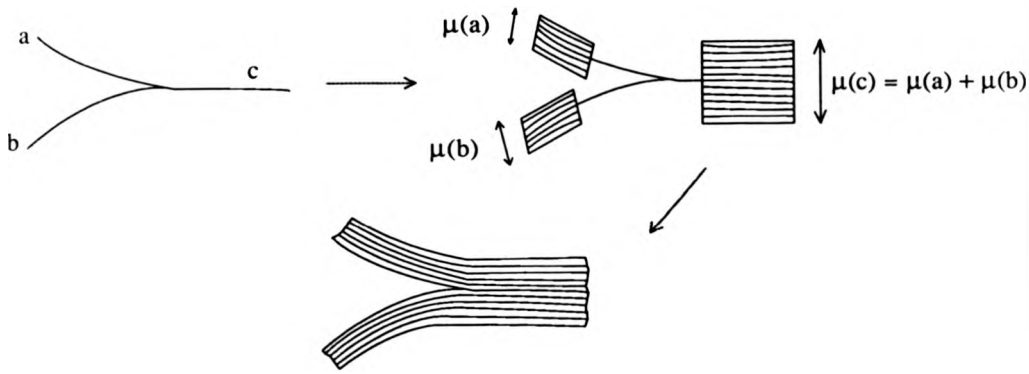

Figure 1.13: Construction of a partial measured foliation of M from a measured train track

such that (τ, μ) is a measured train track. Let $\mathcal{MF}(M)$ be the space of equivalence classes of measured foliations on M, with equivalence relation generated by isotopy and Whitehead moves (see [FLP]).

There exists a map $\varphi : E(\tau) \to \mathcal{MF}(M)$, which is a homeomorphism onto its image [Pa1]. φ is defined in the following way. Given a train track τ and a positive measure $\mu \in E(\tau)$, replace each branch b_i of τ by a rectangle of width $\mu(b_i)$ foliated by arcs parallel to b_i. The switch conditions guarantee that the rectangles may be glued together to provide a partial measured foliation $\mathfrak{F}$ of M (see figure 1.13). By identifying the edges of the complement in M of $\mathfrak{F}$, we obtain a measured foliation $\mathcal{F}$ of M whose equivalence class depends only on (τ, μ) (see figure 1.14). Distinct measures on τ define distinct classes of measured foliations, but there are distinct train tracks corresponding to the same class. By defining an equivalence relation on measured train tracks, there exists a one-to-one correspondence between classes of measured train tracks and classes of measured foliations of M [HP]. The equivalence relation on measured train tracks is generated by the following three moves:

1. isotopy.
2. shifting (letters indicate measures), see figure 1.15.
3. splitting, see figure 1.16.

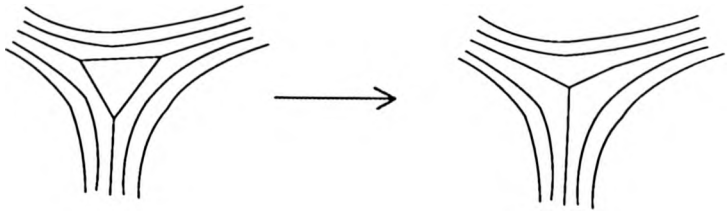

Figure 1.14: Identification of the edges of the complement in M of $\mathcal{F}$

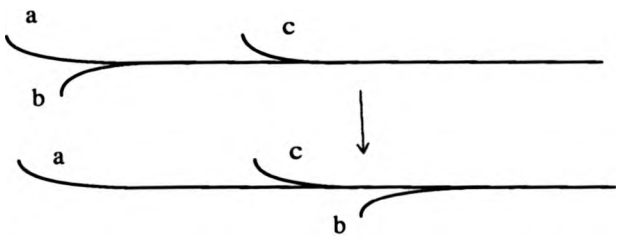

Figure 1.15: The shift move on train tracks

The inverse of splitting is called *collapsing*. Given a train track τ, there exists a fibred neighbourhood $N(\tau)$ of τ, with a retraction $N(\tau) \to \tau$, and whose fibres, or *ties*, are given by this retraction. A *singular tie* is a tie which meets a cusp of $N(\tau)$ (see figure 1.17). Identifying the complementary regions of $M \setminus N(\tau)$ gives a measured foliation $\mathcal{F}$ of M, and we say that $\mathcal{F}$ is *carried* by τ. We say that τ is *suited* to $\mathcal{F}$ if $\mathcal{F}$ can be represented by a partial measured foliation $\mathfrak{F}$ of support $N(\tau)$ which is transverse to the ties and has no leaves connecting cusps of $N(\tau)$. If σ is another train track on M, we say that σ is *carried* by τ if σ is isotopic to a train track which is contained in $N(\tau)$ and is transverse to the ties. Let $f : M \to M$ be a homeomorphism, then τ is *invariant* if the train track $f(\tau)$ is carried by τ.

Suppose τ is an invariant train track for f with branches $b_1, \ldots, b_n$. For each branch

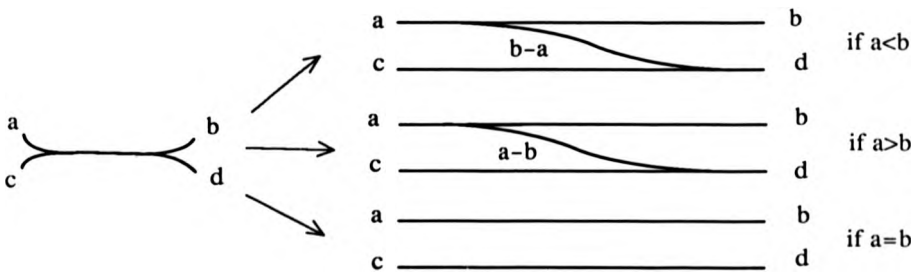

Figure 1.16: The split move on train tracks

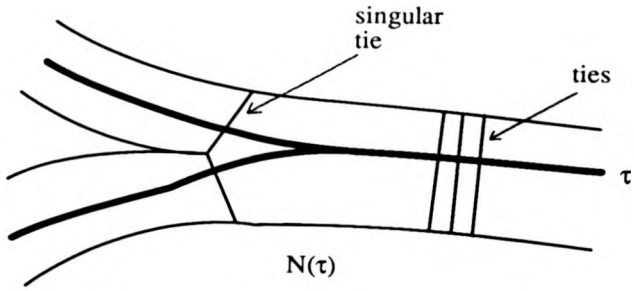

Figure 1.17: The fibred neighbourhood $N(\tau)$ of the train track τ

b_i, choose a point y_i in the interior of b_i. Let t_i be the tie which retracts to y_i. Let $\sigma(\tau)$ be the isotopic image of $f(\tau)$ carried by τ, then we define the *incidence matrix* $M(\tau)$ relative to $\sigma(\tau)$ to be

$$(M(\tau))_{ij} = \text{ Card } \{t_i \cap (\sigma(b_j))\}, \quad 1 \leq i, j \leq n. \tag{1.9}$$

If $(M(\tau))_{ij} \neq 0$, we say that the branch b_j *covers* the branch b_i.

Theorem 1.9 ([PP1]) *If $f : M \to M$ is a pseudo-Anosov homeomorphism and $\mathcal{F}^u \in \mathcal{MF}(M)$ is the class of its unstable foliation, then there exists an invariant train track τ on M suited to $\mathcal{F}^u$. Further, the incidence matrix $M(\tau)$ is Perron-Frobenius.*

$M(\tau)$ is *Perron-Frobenius* if there is a positive integer N such that each entry of M^N is strictly positive. This implies [Gan]:

1. It has a multiplicity one eigenvalue $\lambda_\tau > 1$, strictly larger than the modulus of all the other eigenvalues. Note that $\lambda_\tau > 1$ is the dilatation factor of the pseudo-Anosov homeomorphism f [Lo1] .

2. The corresponding eigenvector V_τ is positive i.e. each entry of V_τ is strictly positive.

So given a pseudo-Anosov map, there exists an invariant train track τ with its fibred neighbourhood $N(\tau)$. For each branch b_i of τ one may define a rectangle R_i in $N(\tau)$ bounded by the segments of the boundary $\partial N(\tau)$ which retract to b_i, and the singular ties of the cusps which correspond to the endpoints of b_i.

Proposition 1.10 ([Lo1]) *For a given pseudo-Anosov map $f : M \to M$ and its invariant train track, the collection $\mathcal{R}$ of rectangles defined above is a partial Markov partition in a fixed fibred neighbourhood $N(\tau)$. The transition matrix of this Markov partition is exactly the incidence matrix $M(\tau)$ of the train track.*

Let $f : M \to M$ be a pseudo-Anosov homeomorphism, let τ be an invariant train track for f, and let $(\mathcal{F}^s, \mathcal{F}^u)$ be the pair of invariant foliations. They have the same finite set of singularities in the interior of M, and on ∂M the singularities of $\mathcal{F}^s$ and $\mathcal{F}^u$ alternate. Card (Sing($\mathcal{F}^s$)) = Card (Sing($\mathcal{F}^u$)), and this quantity is bounded above

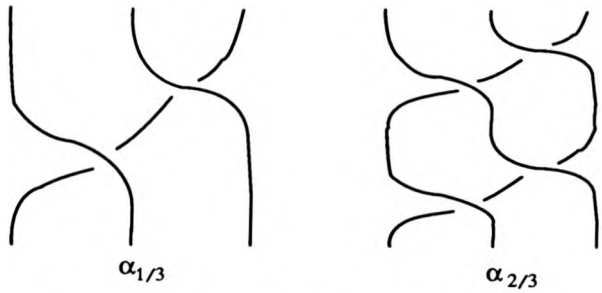

Figure 1.18: The periodic elements of BT_3

by $-2\chi(M)$ (this follows from the Euler-Poincaré formula). So the number of branches and the number of switches of τ are bounded and depend explicitly on $\chi(M)$ [PP2]. Recently, Bestvina and Handel [BH1, BH2] have found an algorithm to determine the train track of a pseudo-Anosov homeomorphism.

1.8 Irreducible and reducible braid types

In this Section, we combine the main ideas in this Chapter, viz. the braid type of a periodic orbit, the set BT of braid types, and the Nielsen-Thurston classification of surface homeomorphisms. The latter provides a decomposition for BT, and implies that we only need to know the partial order on a subset IBT of BT.

1.8.1 Thurston types of braid types

Let $f \in Homeo^+(\mathbf{D}^2)$, X be a periodic orbit of f of period n as in equation 1.2, and let $bt(X, f) = \beta \in BT_n$. Then we say that β is *pseudo-Anosov* (respectively *periodic*, *reducible*, *irreducible*) if the isotopy class of f rel. X is pseudo-Anosov (respectively periodic, reducible, irreducible). This is well-defined because Thurston type is an invariant of conjugacy. Let IBT_n be the subset of BT_n of irreducible braid types, and let $IBT = \cup_{n\geq 1} IBT_n$. Let $f_\beta \in Homeo^+(\mathbf{D}^2, X)$ be the Thurston representative in the isotopy class rel. X i.e. f_β is either pseudo-Anosov, periodic or reducible relative to X.

Suppose β is periodic, then there exists a (least) $p \in \mathbf{N}$ such that $f_\beta^p = Id$, and $p = n$. From a Theorem of Brouwer [Bro] (see also [Ei, Kel]), f_β is conjugate to a rotation through an angle $2k\pi/n$, for some k, $0 < k < n$. Hence β is conjugate to (up to an element of $Z(B_n)$)

$$\alpha_{k/n} = (\sigma_1\sigma_2 \ldots \sigma_{n-1})^k.$$

e.g. There are two periodic elements of BT_3 (see figure 1.18). We remark that there exists an algorithm [BGN] to determine the Thurston type of an element of BT.

1.8.2 Decomposition of reducible braid types

As we have seen in Section 1.8.1, we can find a simple representative in B_n of a periodic braid type. It is beneficial to find also a simple representative of a reducible braid type,

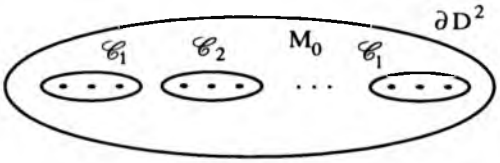

Figure 1.19: The arrangement of the discs $\mathcal{D}_1, \ldots, \mathcal{D}_l$ in $\mathbf{D}^2$

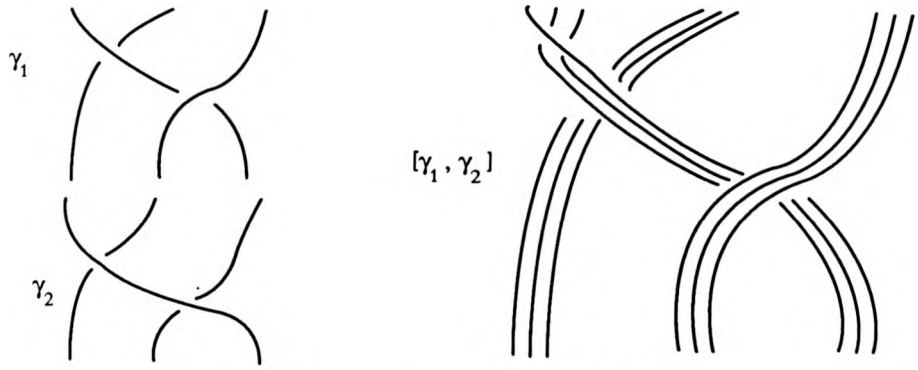

Figure 1.20: Decomposition of a reducible braid

for example, as some form of product, from which we may deduce the irreducible components, and how they relate to the partial order on BT.

So suppose f, X, β and f_β are as defined in Section 1.8.1, and suppose f_β is reducible relative to X. Let $M_0, M_1, \ldots, M_k$ be the decomposition components invariant under some iterate of f_β (see Theorem 1.6), and let M_0 be the component containing $\partial\mathbf{D}^2$. Since X is a periodic orbit, $M_0 \cap X = \emptyset$ and $f_\beta(M_0) = M_0$. Let $\partial M_0 = \{\partial\mathbf{D}^2, \mathcal{C}_1, \ldots, \mathcal{C}_l\}$, and let $\mathcal{D}_1, \ldots, \mathcal{D}_l$ be the discs bounded by the curves $\mathcal{C}_1, \ldots, \mathcal{C}_l$ respectively. As f_β permutes these discs, each $\mathcal{D}_i$ contains n/l points of X, and by conjugation if necessary, we may assume that $\mathcal{D}_1, \ldots, \mathcal{D}_l$ are as arranged in figure 1.19.

Let $\gamma_1 \in BT_l$ be the braid type of $[f_\beta|_{M_0}]$ in $Aut(\mathbf{D}^2 \setminus \cup_{j=1}^l \mathcal{D}_j)$ and $\beta' \in BT_{n/l}$ be the braid type of $[f_\beta^l|_{\mathcal{D}_j}]$ in $Aut(\mathbf{D}^2, X \cap \mathcal{D}_j)$ (note that $f_\beta^l|_{\mathcal{D}_i}$ and $f_\beta^l|_{\mathcal{D}_j}$ are conjugate for $1 \leq i, j \leq l$); both γ_1 and β' are uniquely defined. Write

$$\beta = [\gamma_1, \beta'],$$

where γ_1 is an irreducible braid (this follows from Theorem 1.6). For example, see figure 1.20.

This process is inductive, and we continue until $\beta = [\gamma_1, \ldots, \gamma_p]$, where each $\gamma_i \in BT_{n_i}$ is an irreducible braid type. It is clear from the construction that $n = n_1 \ldots n_p$. Since at each stage the braid types are uniquely determined, we recover the following Theorem.

Theorem 1.11 ([BoyF, Lo1]) *Each braid type $\beta \in BT$ admits a unique decomposition*

$$\beta = [\gamma_1, \ldots, \gamma_p]$$

where γ_i is an irreducible braid type. Further, if n is prime, then $BT_n = IBT_n$. □

This decomposition preserves the partial order, more precisely:

Theorem 1.12 ([BoyF, Lo1]) *Let $\gamma_1, \ldots, \gamma_p, \beta, \sigma \in IBT$, then*

1. $[\gamma_1, \ldots, \gamma_p] \succ [\gamma_1, \ldots, \gamma_{p-1}]$.
2. $[\gamma_1, \ldots, \gamma_p, \beta] \succ [\gamma_1, \ldots, \gamma_p, \sigma]$ *if and only if* $\beta \succ \sigma$.

So in BT, there is a certain self-similarity, the structure of which is determined by elements of IBT. Hence it is reasonable to concentrate on the study of irreducible braid types.

1.9 Concluding remarks

We conclude this Chapter with a few remarks concerning the choice of braid type as a specification for periodic orbits of homeomorphisms of the disc. Returning to the requirements of Section 1.2.1, we see that the first condition is satisfied by our definition of braid type and the partial order $\succ$ on BT. The second condition is satisfied by choosing the Thurston representative in the isotopy class; as we have just seen, one may recover the partial order on BT from that on IBT. Hence the 'simplest maps' are either pseudo-Anosov or periodic. Since the dynamics of periodic braid types (of the disc) are well-understood, one is naturally interested in the pseudo-Anosov case whose dynamics we can comprehend by studying the train track or Markov partition. As has been noted, given a braid type, there exists an algorithm to deduce the corresponding train track. However we would like to discover more general information, for instance, given *any* periodic orbit of some period whose braid type is pseudo-Anosov, what do we know about the dynamics? To deduce such information, it becomes necessary to consider other methods, for instance to look for other topological invariants, or to study homological or other homotopy information. We shall elaborate on these ideas in subsequent chapters.

Chapter 2

A classification of braid types for periodic orbits of diffeomorphisms of surfaces of genus one with topological entropy zero

We classify the braid types that can occur for finite unions of periodic orbits of diffeomorphisms of surfaces of genus one with zero topological entropy.

2.1 Introduction

In this Chapter we classify the braid types that can occur for finite unions of periodic orbits of diffeomorphisms of surfaces of genus one with zero topological entropy. This extends the analysis from the case of genus zero [LM1, GST1]. One case of interest to us is that of diffeomorphisms of the torus isotopic to the identity. This is relevant to the behaviour of three coupled oscillators, for example. A good picture of their dynamics is developing [AGK, KMG, LM2, MZ, H6, Fr7, BGKM1, BGKM2]. We hope that our results will help solve the intriguing problem of understanding the boundary of zero topological entropy in the space of C^1-diffeomorphisms of the torus. The work in this Chapter was done jointly with J. Llibre and R. S. MacKay [GLM], and is to appear in the Publicacions Matemàtiques Universitat Autonoma de Barcelona.

The plan of this Chapter is as follows. In Section 2.2 we describe three classes of diffeomorphisms with which we shall use to build up representatives of all isotopy classes with zero entropy. In Section 2.3, we classify finite order homeomorphisms of the torus, which is an important preliminary result. Then in Section 2.4, we use Nielsen-Thurston theory to isotope the homeomorphism to a standard form, and analyse the possibilities. Our main result is Theorem 2.6, but because it is rather long to state and requires notions introduced in Section 2.3, we leave its statement until Section 2.4. In Section 2.5 we rederive results of [H1, H6, LM2] as corollaries of those of Section 2.4.

2.2 Classes of diffeomorphisms

We say that two homeomorphisms $f : M \to M$, $g : N \to N$ of orientable surfaces M, N equipped with fixed orientations are *orientation-preserving conjugate*, written $f \approx g$, if there exists an orientation-preserving homeomorphism $k : M \to N$ such that $kf = gk$. We recall the definitions of the classes of diffeomorphisms which we call disc trees, reversing disc trees and reversing annulus trees from [LM1]. Suppose $f : M \to M$ is a homeomorphism of an orientable surface. A set S has *f-period* $n \in \mathbb{N}$ if $f^n S = S$ and n is the least such integer. If $\partial M \neq \emptyset$, let $N_f : \mathbb{N} \to \mathbb{Z}_+$ be the function giving the number $N_f(p)$ of boundary components of M of f-period p. Define L_i to be the following set for $i \in \mathbb{N}$, given by

- $L_1 = \{0\}$.
- for $m > 1$, let $L_m = \{l \in \{1, \ldots, m-1\} : l \text{ and } m \text{ are coprime}\}$.

We now consider canonical classes of diffeomorphisms. For $k \in \mathbb{Z}_+$, let A_k be the sphere S^2 with k equal-size discs equally spaced around the equator E removed, B_k be the disc $\mathbb{D}^2$ with k equal-size discs equally spaced around the circle E of radius $1/2$ removed, and C_k be the annulus $\mathbb{A}$ with k equal-size discs equally spaced around its mid-circle E removed.

Define K to be the set consisting of the following diffeomorphisms (see figure 2.1):

- $$a_{m,i,l} : A_{im} \to A_{im}$$

  for $m \geq 1$, $i \geq 1$, $l \in L_m$, is the rigid rotation of A_{im} by angle $2\pi l/m$ about the polar axis. Note that $a_{1,i,0}$ is the identity on A_i.

- $$b_{m,i,l} : B_{im} \to B_{im}$$

  for $m \geq 2$, $i \geq 1$, $l \in L_m$, is the rigid rotation of B_{im} by angle $2\pi l/m$.

- $$c_{m,i,l} : C_{im} \to C_{im}$$

  for $m \geq 2$, $i \geq 1$, $l \in L_m$, is the rigid rotation of C_{im} by angle $2\pi l/m$.

Note that distinct diffeomorphisms in K have different braid types.

We say that two circles $C_0, C_1 \subset M$ are *connected by an annulus* $\mathbb{A}^\circ \subset M$ if

$$\partial \mathbb{A}^\circ = C_0 \cup C_1.$$

Now we glue these diffeomorphisms together to obtain the class we call disc trees, defined recursively.

1. A *disc tree* (of *size* $n \in \mathbb{N}$) is a diffeomorphism $d : X \to X$ of a disc with holes such that d is one of $a_{1,i,0}$, $i = 1, 2$, or:

   (a) X has a boundary component $\partial_0(X)$ of period 1 called the *outer boundary* of X.

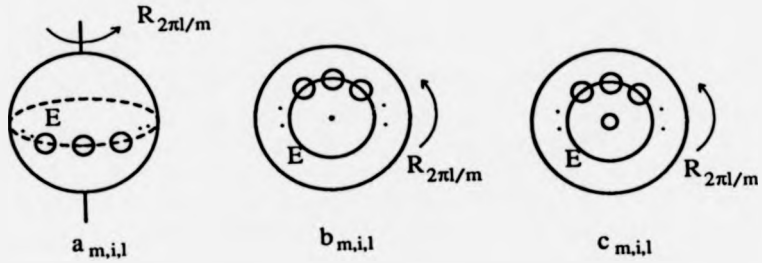

Figure 2.1: The standard diffeomorphisms $a_{m,i,l}$, $b_{m,i,l}$ and $c_{m,i,l}$

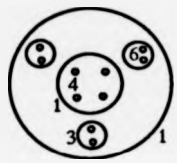

Figure 2.2: Example of a disc tree. The numbers indicate the f-periods of the circles.

(b) X has a decomposition as a disjoint union of n subsets S_j homeomorphic to sets of the form A_k,B_k,C_k, $k \in \mathbb{Z}_+$, and some number of subsets homeomorphic to $\mathbb{A}^\circ$.

(c) one of the subsets S_j, S_0 say, has $\partial_0(X)$ as a boundary component and $d|_{S_0}$ is orientation-preserving conjugate to one of $a_{1,i,0}$ ($i \geq 3$), $b_{m,i,l}$, or $c_{m,i,l}$.

(d) given any remaining orbit of boundary components of S_0, let p be its period (equal to 1 or m), and denote it by ∂_j, $j = 1, \ldots, p$. Then either ∂_j belongs to ∂X for all j, or there exists a disc tree $d' : X' \to X'$ and a generalized twist $g : \mathbb{A}^\circ \to \mathbb{A}^\circ$ such that each ∂_j is connected to the outer boundary of a copy X'_j of X' via an annulus $\mathbb{A}^\circ_j$, $d^p|_{X'_j} \approx d'$ and $d^p|_{\mathbb{A}^\circ_j} \approx g$.

(e) the sum of the sizes of d' over boundary components of S_0 is $n - 1$.

An example of a disc tree is given in figure 2.2.

2. Let $r_{i,j}$, $i,j \geq 0$ be reflection in a vertical diameter D of $R_{i,j}$, a disc minus i holes centred along D, with j holes in the left half-disc, and their j reflections in the right half-disc (see figure 2.3). A *reversing disc tree* (of size $n \in \mathbb{N}$) is a diffeomorphism $r : X \to X$ of a disc with holes such that

(a) X has n disjoint invariant subsets S_k homeomorphic to $R_{i,j}$ on which $r \approx r_{i,j}$, depending on k.

(b) each period 1 boundary component of S_k either lies in ∂X or is connected via an annulus on which r is orientation-preserving conjugate to a generalized twist, to another $S_{k'}$.

(c) given a period 2 orbit ∂_0,∂_1 of boundary components of S_k, either ∂_0,$\partial_1 \subset \partial X$ or there exists a disc tree $d : X' \to X'$ and a generalized twist $g : \mathbb{A}^\circ \to \mathbb{A}^\circ$ such

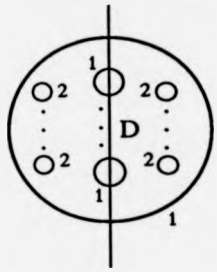

Figure 2.3: The standard diffeomorphism $r_{i,j}$

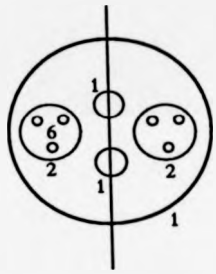

Figure 2.4: Example of a reversing disc tree

that each of ∂_j, $j = 0, 1$ is connected to the outer boundary of a copy X'_j of X' via an annulus A°_j, $r^2|_{X'_j} \approx d$ and $r^2|_{A^\circ_j} \approx g$.

An example of a reversing disc tree is given in figure 2.4.

3. Let $s_{m,i,j,l}$, $m \geq 2$, $l \in L_m$, be the diffeomorphism

$$\begin{aligned} r' &= -r \\ \theta' &= \theta + 2\pi l/m \end{aligned}$$

of $S_{m,i,j}$, the annulus $S^1 \times [-1, 1]$ with i orbits of the disc centred on $r = 0$ removed, all the removed discs being disjoint (see figure 2.5). A *reversing annulus tree* is a diffeomorphism $s : X \to X$ of an annulus with holes such that

(a) X has an invariant subset S on which s is conjugate to either a generalized twist or to $s_{m,i,j,l}$, for some m,i,j,l.

(b) for each orbit of boundary components ∂_j of S of even period p ($p = 2$ if m is odd and $p = 2$ or m if m is even) not contained in ∂X, there exists a disc tree $d : X' \to X'$ and a generalized twist g such that each ∂_j is connected to the outer boundary of a copy X'_j of X' via an annulus A°_j, and $s^p|_{X'_j} \approx d$, and $s^p|_{A^\circ_j} \approx g$.

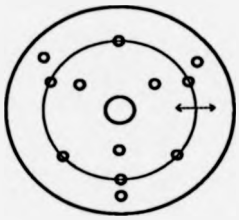

Figure 2.5: The standard diffeomorphism $s_{m,i,j,l}$

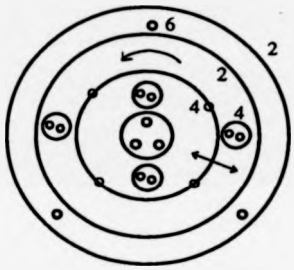

Figure 2.6: Example of a reversing annulus tree

(c) if m is odd, for each orbit of boundary components ∂_j of S of period m, not contained in ∂X, there exists a reversing disc tree $r : X' \to X'$ and a generalized twist g such that ∂_j is connected to the outer boundary of a copy X'_j of X' via an annulus A°_j, and $s^m|_{X'_j} \approx r$, $s^m|_{A^\circ_j} \approx g$.

An example of a reversing annulus tree is given in figure 2.6.

Note that any element of these three classes is a diffeomorphism and has zero topological entropy.

2.3 Classification of homeomorphisms of finite order of the torus

In this Section, we give a classification of finite order homeomorphisms of the torus up to orientation-preserving conjugacy. Let $T_x, x \in \mathbf{R}^2$, be translation by x on $\mathbf{R}^2$, i.e. $T_x(y) = y + x, y \in \mathbf{R}^2$. Let $R_\omega, \omega \in \mathbf{R}/2\pi\mathbf{Z}$ be rotation about the origin $\mathbf{0}$ on $\mathbf{R}^2$ by angle ω. Let r be the reflection $(x, y) \mapsto (x, -y)$ on $\mathbf{R}^2$. Define $\Gamma_\square$ to be the group generated by $T_{(1,0)}$ and $T_{(0,1)}$, $\Gamma_\triangle$ that generated by $T_{(1,0)}$ and $T_{(1/2,\sqrt{3}/2)}$. If $f : \mathbf{R}^2 \to \mathbf{R}^2$ commutes with the action of a group Γ on $\mathbf{R}^2$, we define $f/\Gamma : \mathbf{R}^2/\Gamma \to \mathbf{R}^2/\Gamma$ by identifying points of the same orbit under Γ. For $q \in \mathbf{N}$, let $\bar{\mathbf{Z}}_q$ be the quotient of $\mathbf{Z}$ under the equivalence relation generated by the relations $p \sim p + q$, $p \sim -p$.

Theorem 2.1 *If $f : \mathbf{T}^2 \to \mathbf{T}^2$ is a homeomorphism of finite order, then it is orientation-preserving conjugate to one of the following:*

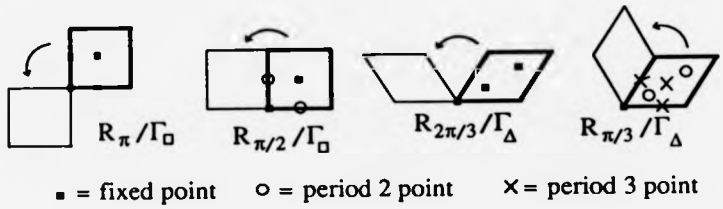

Figure 2.7: Finite order orientation-preserving diffeomorphisms of the torus with a fixed point

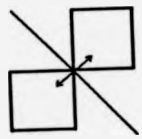

Figure 2.8: The orientation-reversing diffeomorphism $r \circ R_{\pi/2}/\Gamma_\square$

1. *$T_{(p,0)/q}/\Gamma_\square$ for some $q \in \mathbb{N}$, $p \in \mathbb{Z}_q$, with p,q having no common factor (order q).*
2. *$R_\omega/\Gamma_\square$, $\omega = \pm\pi/2, \pi$, or R_ω/Γ_Δ, $\omega = \pm\pi/3, \pm 2\pi/3$, (orders 4,2,6,3 respectively) (see figure 2.7).*
3. *$r \circ T_{(p,0)/q}/\Gamma_\square$, for some $q \in \mathbb{N}$, $p \in \mathbb{Z}_q$, with p,q having no common factor (order q if q is even, order 2q if q is odd).*
4. *$r \circ R_{\pi/2}/\Gamma_\square$ (order 2) (see figure 2.8).*

No two of the above are orientation-preserving conjugate.

The proof of Theorem 2.1 reduces to the classification of isometries of the torus, a well-studied problem (e.g. see [NS]). However, we have not found a compact treatment in the literature, nor one which considers the question of which isometries are equivalent up to orientation-preserving conjugacy. So we give a derivation in Section 2.6.

For case 2 of Theorem 2.1, it is known how many periodic orbits there are of smaller period than the order.

Theorem 2.2 ([Ep2])

- *The rotation of order 2 has 4 fixed points.*
- *The rotations of order 3 have 3 fixed points.*
- *The rotations of order 4 have 2 fixed points and one orbit of period 2.*
- *The rotations of order 6 have one fixed point, one orbit of period 2, and one orbit of period 3 (see figure 2.7).*

2.4 Results

We firstly recall some notions and results that we will need. We then go on to state and prove our results.

Let $f : M \to M$ be a homeomorphism of a surface of genus one, then its *completion* $g : \mathbb{T}^2 \to \mathbb{T}^2$, is defined by considering M to be $\mathbb{T}^2$ minus a disjoint union of equal size discs D_i, and extending $g|_{\partial D_i}$ radially into D_i [Ep2]. If $h(f) = 0$, then $h(g) = 0$. We say a simple closed curve on M is *rotational* if it is homotopically non-trivial on $\mathbb{T}^2$ after filling in the holes.

We recall the following two results.

Theorem 2.3 ([LM1]) *Let $f : X \to X$ be a homeomorphism of a surface of genus zero. If f is orientation-preserving, it has either a fixed point or an invariant boundary component. If f is orientation-reversing it has either a fixed point or an invariant boundary component, or an orbit of period 2 or a boundary component of period 2.*

So in both the orientation-preserving and orientation-reversing cases, if $\mathcal{P}$ is a finite union of periodic orbits for f, and the map $f_{\mathcal{P}}$ induced by blowing up $\mathcal{P}$ does not have a boundary component of period 1 (orientation-preserving) or period 1 or 2 (orientation-reversing), we can always append a fixed point or period 2 orbit to $\mathcal{P}$ in order to achieve this.

Theorem 2.4 ([LM1]) *Let $f : X \to X$ be a diffeomorphism of a surface of genus zero, with $h(f) = 0$, and let $\mathcal{P}$ be a finite union of periodic orbits of f.*

1. *If $\mathcal{P}$ contains a fixed point or f has an invariant boundary component, then*
   - *(a) if f is orientation-preserving, $bt(\mathcal{P}, f)$ has a representative which is a disc tree.*
   - *(b) if f is orientation-reversing, $bt(\mathcal{P}, f)$ has a representative which is a reversing disc tree.*
2. *If f is orientation-reversing, and $\mathcal{P}$ contains no fixed point and f has no invariant boundary component, but either $\mathcal{P}$ has a point of period 2 or f has a boundary component of period 2, then $bt(\mathcal{P}, f)$ has a representative which is a reversing annulus tree.*

Let $f : M \to M$ be a diffeomorphism of a surface of genus one with $h(f) = 0$, and let $\mathcal{P}$ be a finite union of periodic orbits for f. Let F be a Thurston canonical form for $f_{\mathcal{P}}$. Then from Theorem 1.6, F is either reducible, or periodic. First we consider the case that F has a rotational reducing curve.

Theorem 2.5 *Suppose $f : M \to M$ is a diffeomorphism of a surface of genus one with $h(f) = 0$, and $\mathcal{P}$ is a finite union of periodic orbits for f. Let F be a Thurston canonical form for $f_{\mathcal{P}}$, such that F has a rotational reducing curve Γ, of period p. Remove the tubular neighbourhood of Γ and its images, to obtain a disjoint union of punctured annuli A_i. Then:*

1. *if all A_i have period p, $bt(\mathcal{P}, f)$ has a representative which is p annuli, joined by generalized twists, permuted like a rotation, and if f is orientation-preserving or p is even there exists a disc tree $d : X' \to X'$ such that $F^p|_{A_i} \approx d$ or if f is orientation-reversing and p is odd there exists a reversing disc tree $d' : X' \to X'$ such that $F^p|_{A_i} \approx d'$.*

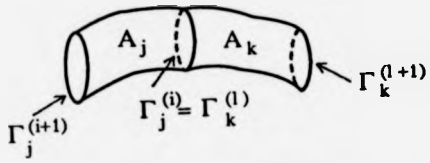

Figure 2.9: No two distinct A_k have a boundary component in common

2. *suppose some A_i has period $q \neq p$ (so $q = p/2$), then $p = 2$, and there are two such annuli A_1, A_2. If f is orientation-preserving, $bt(\mathcal{P}, f)$ has a representative which is two annuli, joined by generalized twists, and there exist disc trees $D_i : X' \to X'$, $i = 1, 2$ such that $F|_{A_i} \approx D_i$. If f is orientation-reversing, $bt(\mathcal{P}, f)$ has a representative which is two annuli, joined by generalized twists, and for each of A_1, A_2 there exist reversing disc trees $D_i' : X' \to X'$ such that $F|_{A_i} \approx D_i'$ or reversing annulus trees $S_i : X \to X$, such that $F|_{A_i} \approx S_i$, according as $F|_{A_i}$ has an invariant boundary component or fixed point, or not.*

Proof

Suppose Γ has period p. Define $\Lambda = \{F^k\Gamma : 0 \leq k < p\}$. Then F permutes the elements of Λ. If we remove the tubular neighbourhood of Γ and its images, we obtain a disjoint union of annuli A_i with holes, which are permuted. Then the A_i either have period p, or if p is even, some of the A_i may have period $p/2$ (i.e. the two boundaries of A_i in Λ are interchanged by $F^{p/2}$). This gives two cases:

1. if the A_i have period p, F permutes the A_i, and there are two subcases:

   (a) if f is orientation-preserving, or if f is orientation-reversing and p is even, then $F^p|_{A_i}$ is orientation-preserving, so by Theorem 2.4, there exists a disc tree $d : X' \to X'$ such that $F^p|_{A_i} \approx d$.

   (b) if f is orientation-reversing and p is odd, then $F^p|_{A_i}$ is orientation-reversing, so by Theorem 2.4, there exists a reversing disc tree $d' : X' \to X'$ such that $F^p|_{A_i} \approx d'$.

2. suppose p is even, and there exists some A_i (without loss of generality $i = 0$) such that A_i has period $p/2$. Put $q = p/2$. Then we claim that $q = 1$ and Γ has period 2.

   To prove the claim, suppose A_0 has period $q > 1$. Write $A_k = F^k(A_0), 1 \leq k < q$. Consider the situation on the torus; each A_k has two boundary components $\Gamma_k^{(1)}, \Gamma_k^{(2)} \in \Lambda$, say, for $0 \leq k < q$. Then for $j \neq k$, $\Gamma_j^{(i)} \neq \Gamma_k^{(l)}$, $i, l = 1, 2$, i.e. no two distinct A_k have a boundary component in common. For suppose $\Gamma_j^{(i)} = \Gamma_k^{(l)}$ for some $i, l \in \{1, 2\}$, $j \neq k$ (see figure 2.9). Since F^q interchanges $\Gamma_k^{(1)}$ and $\Gamma_k^{(2)}$ for each k,

   $$F^q(\Gamma_j^{(l)}) = \Gamma_j^{(i+1)} = F^q(\Gamma_k^{(i)}) = \Gamma_k^{(l+1)}$$

   (we take the value of the upper index of Γ up to mod2). Hence there are only two distinct elements of Λ, so $q = 1$, a contradiction.

   Since the A_k have no boundary components in common, there exist precisely q tubular regions $B_1, \ldots, B_q$, such that each B_i lies between two A_k, i.e. each B_i

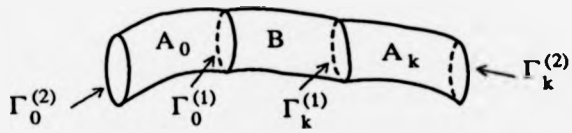

Figure 2.10: A tubular region B

has two distinct elements of Λ as its (rotational) boundary components. Consider one such element of $\{B_i\}_{i=1}^{q}$, B, say. Without loss of generality, it lies between A_0 and A_k, for some k, and has boundary components $\Gamma_0^{(1)}, \Gamma_k^{(1)} \in \Lambda$ (see figure 2.10). B is invariant under F^{2q}. Since Γ has period $2q$, B must have period either q or $2q$. Suppose it has period q, then F^q interchanges its boundary components, so $\Gamma_k^{(1)} = F^q(\Gamma_0^{(1)})$. But F^q interchanges the boundary components of A_0 so $F^q(\Gamma_0^{(1)}) = \Gamma_0^{(2)}$, therefore A_0 and A_k have a boundary component in common, which we have shown does not occur. Hence B has period $2q$.

However $F^k(A_0) = A_k$, for $k < q$, and considering $F^{k+q}(A_0) = A_k$, if necessary, then $F^k(\Gamma_0^{(1)}) = \Gamma_k^{(1)}$, and $F^k(\Gamma_0^{(2)}) = \Gamma_k^{(2)}$. But the elements of Λ are permuted like a rotation or rotation with reflection by F, since f is a diffeomorphism, hence $F^k(\Gamma_k^{(1)}) = \Gamma_0^{(1)}$, therefore $F^k(B) = B$, a contradiction, as B has period $2q$. This proves the claim.

So $q = 1$, and Γ has period 2, with $\Lambda = \{\Gamma, F(\Gamma)\}$. If we remove the tubular neighbourhoods of Γ and $F(\Gamma)$, we obtain two disjoint annuli A_0 and A_1. There are two subcases to consider:

(a) if f is orientation-preserving, then $F|_{A_i}$ $(i = 0, 1)$ is orientation-preserving, so by Theorem 2.4, $bt(\mathcal{P}, f)$ has a representative which is two annuli joined by generalized twists, such that $F|_{A_i}$ is a disc tree.

(b) if f is orientation-reversing, then $F|_{A_i}$ $(i = 0, 1)$ is orientation-reversing, so by Theorem 2.4, $bt(\mathcal{P}, f)$ has a representative which is two annuli joined by generalized twists, such that $F|_{A_i}$ is a reversing disc tree or a reversing annulus tree, according as f has an invariant boundary component or $\mathcal{P}$ contains a fixed point in A_i, or not. This completes the proof of Theorem 2.5. □

Next we consider the case where there is no rotational reducing curve. Either F is periodic, or has a non-rotational reducing curve C, in which case we may remove the decomposition components (of genus zero) of C and its images from M. So in both cases, we can find a unique decomposition component S of genus one, such that $F|_S$ is periodic. Let $G : \mathbb{T}^2 \to \mathbb{T}^2$ be the completion of $F|_S$. Then G is periodic, so is orientation-preserving conjugate to one of the cases of Theorem 2.1.

This leads to the following Theorem, which is our main result:

Theorem 2.6 *Suppose $F : M \to M$ is a Thurston canonical form for a homeomorphism of a surface of genus one, with unique decomposition component S of genus one. Let $G : \mathbb{T}^2 \to \mathbb{T}^2$ be the completion of $F|_S$. Then one of the following is true:*

1. *$G \approx T_{(p,0)/q}/\Gamma_\square$, and all boundary components of S have period q. Further, for each orbit $\partial_1, \ldots, \partial_q$ of boundary components of S not in ∂M, there exists a disc tree $d : X' \to X'$ such that the component X_i of $M \setminus S$ inside ∂_i is homeomorphic to X', and $F^q|_{X_i} \approx d$.*

2. *$G \approx R_\omega/\Gamma_\square$, $\omega = \pm\pi/2, \pi$ (order $k = 4, 2$), or $G \approx R_\omega/\Gamma_\triangle$, $\omega = \pm\pi/3, \pm 2\pi/3$ (order $k = 6, 3$).*

   - *If $k = 2$, then all boundary components of S have period 1 or 2, with at most 4 of period 1.*
   - *If $k = 3$, then all boundary components of S have period 1 or 3, with at most 3 of period 1.*
   - *If $k = 4$, then all boundary components of S have period 1, 2 or 4, with at most 2 of period 1, and 1 orbit of period 2.*
   - *If $k = 6$, then all boundary components of S have period 1, 2, 3 or 6 with at most 1 of period 1, 1 orbit of period 2, and 1 orbit of period 3.*

   In each case, for each orbit of boundary components of period p, $\partial_1, \ldots, \partial_p$, of S not in ∂M, there exists a disc tree $d : X' \to X'$ such that the component X_i of $M \setminus S$ inside ∂_i is homeomorphic to X', and $F^p|_{X_i} \approx d$.

3. (a) *$G \approx (r \circ T_{(p,0)/q})/\Gamma_\square$. If q is even, all boundary components have period q. If q is odd, all boundary components have period q or $2q$.*

   (b) *$G \approx (r \circ R_{\pi/2})/\Gamma_\square$. All boundary components of S have period either 1 or 2.*

In both parts of case 3, for each orbit of boundary components of period p, $\partial_1, \ldots, \partial_p$, of S not in ∂M, there exists a disc tree $d : X' \to X'$ such that the component X_i of $M \setminus S$ inside ∂_i is homeomorphic to X' and $F^p|_{X_i} \approx d$ if p is even, or there exists a reversing disc tree $d' : X' \to X'$ such that the component X_i' of $M \setminus S$ inside ∂_i is homeomorphic to X' and $F^p|_{X_i'} \approx d'$ if p is odd.

Proof

We apply Theorem 2.1 to the completion $G : \mathbf{T}^2 \to \mathbf{T}^2$ of $F|_S$.

1. $G \approx T_{(p,0)/q}/\Gamma_\square$, so all points of $\mathbf{T}^2$ have period q under G. Hence all boundary components of S have period q; if we consider the orbits $\partial_1, \ldots, \partial_q$ of those not in ∂M, the corresponding decomposition components X_i of $M \setminus S$ inside ∂_i are of genus zero, so we may apply Theorem 2.4. Thus there exists a disc tree $d : X' \to X'$ such that X_i is homeomorphic to X', and $F^q|_{X_i} \approx d$.

2. $G \approx R_\omega/\Gamma_\square, \omega = \pm\pi/2, \pi$: $k = 4, 2$, or $G \approx R_\omega/\Gamma_\square, \omega = \pm\pi/3, \pm 2\pi/3 : k = 6, 3$.

   The statements about the orbits of boundary components are an immediate consequence of applying Theorem 2.2 to the completion $G : \mathbf{T}^2 \to \mathbf{T}^2$. As in case 1, since G is orientation-preserving, we obtain disc trees in each orbit of boundary components of S not in ∂M.

3. (a) $G \approx (r \circ T_{(p,0)/q})/\Gamma_\square$.

   From the proof of Theorem 2.1, the boundary components of S have period q if q is even, or they have period q or $2q$ if q is odd.

(b) $G \approx (r \circ R_{\pi/2})/\Gamma_{\square}$.

Again from the proof of Theorem 2.1, the boundary components of S have period 1 or 2.

In both parts of case 3, each component of $\mathbf{T}^2 \setminus S$ inside a boundary component not in ∂M of even period r contains a disc tree since F^r is orientation-preserving, whilst those of odd period r' contain a reversing disc tree since $F^{r'}$ is orientation-reversing, by Theorem 2.4. □

2.5 Two corollaries

As a corollary of the results of Section 2.4 we obtain the genus one case of [H1]:

Theorem 2.7 *Let $f : M \to M$ be an orientation-reversing diffeomorphism of a surface M of genus one. If f has periodic orbits, or orbits of boundary components, with 3 distinct odd periods, then $h(f) > 0$.*

To derive this from the above, we require the genus zero result of [BlF, H1], also derived in [LM1]:

Theorem 2.8 *Let $f : X \to X$ be an orientation-reversing diffeomorphism of a surface of genus zero. If f has periodic orbits or orbits of boundary components with two distinct odd periods, then $h(f) > 0$.*

Proof of Theorem 2.7

Let $f : M \to M$ be an orientation-reversing diffeomorphism of a surface M of genus one. Let $\mathcal{P}$ be the union of three orbits of distinct odd period. Let F be a Thurston canonical form for $f_{\mathcal{P}}$. Suppose $h(f) = 0$. There are two cases:

1. suppose there is a rotational reducing curve, Γ, say, with period p. Remove its annular neighbourhood and its images, then we obtain a disjoint union of annuli $\mathbf{A}_i$. From Theorem 2.5, there are two possibilities. In the first case, the $\mathbf{A}_i$ have period p, so the $\mathbf{A}_i$ are permuted by F, and thus p divides the order of each periodic orbit, hence p is odd. So if we consider any $\mathbf{A}_i$, $F^p|_{\mathbf{A}_i}$ is orientation-reversing, since f is orientation-reversing. $\mathbf{A}_i$ must contain boundary components corresponding to the three orbits of distinct odd period. So by Theorem 2.8, $h(f) > 0$, a contradiction.

   The other possibility is when $p = 2$, and there are two invariant annuli $\mathbf{A}_0, \mathbf{A}_1$. Then one of $\mathbf{A}_0, \mathbf{A}_1$ must contain boundary components corresponding to at least two of the three orbits, and since $F|_{\mathbf{A}_i}$ $(i = 0, 1)$ is orientation-reversing, Theorem 2.8 implies that $h(f) > 0$, a contradiction.

2. suppose there is no rotational reducing curve. Then there exists a unique decomposition component S of genus one. Let $G : \mathbf{T}^2 \to \mathbf{T}^2$ be the completion of $F|_S$. Then we are in case 3 of Theorem 2.6. If $G \approx (r \circ T_{(p,0)/q})/\Gamma_{\square}$, all boundary components of S have period q or $2q$. If $G \approx (r \circ R_{\pi/2})/\Gamma_{\square}$, then all boundary components of S have period 1 or 2. In particular, all boundary components of S of odd period have the same period p, and the remaining boundary components corresponding to $\mathcal{P}$ must lie within decomposition components X_i of genus

zero whose outer boundaries have period p. Since $F^p|_{X_i}$ is orientation-reversing, and at least one of the X_i contains boundary components of odd order, not p, corresponding to the orbits of $\mathcal{P}$, then Theorem 2.8 implies that $h(f) > 0$, a contradiction. □

As a second corollary we will derive a result of [LM2, H6] for diffeomorphisms of the torus isotopic to the identity. Let $\mathbf{T}^2$ be the quotient $\mathbf{R}^2/\mathbf{Z}^2$, and let $\pi : \mathbf{R}^2 \to \mathbf{T}^2$ be the associated covering map. Given a continuous map $f : \mathbf{T}^2 \to \mathbf{T}^2$, there exists a continuous map $F : \mathbf{R}^2 \to \mathbf{R}^2$, a *lift* of f, such that $f \circ \pi = \pi \circ F$. Any two lifts F_1, F_2 of f differ by an element of $\mathbf{Z}^2$ i.e. there exists $n \in \mathbf{Z}^2$ such that

$$F_1(x) = F_2(x) + n \text{ for all } x \in \mathbf{R}^2 \tag{2.1}$$

Let Γ be the group of integer translations $\gamma_m : x \mapsto x + m$, $x \in \mathbf{R}^2$, $m \in \mathbf{Z}^2$. If f is homotopic to the identity, then

$$F\gamma = \gamma F \text{ for all } \gamma \in \Gamma. \tag{2.2}$$

Let $o(x)$ be a periodic orbit of f, of (least) period q. If f is homotopic to the identity, there exists $p \in \mathbf{Z}^2$ such that

$$F^q(\tilde{x}) = \tilde{x} + p \text{ for all } \tilde{x} \in \pi^{-1}(x),$$

and we say that x, and hence $o(x)$ has *F-rotation type* $\rho_F(x) = (p, q) \in \mathbf{Z}^2 \times \mathbf{N}$. If p and q have no common factor, we say that (p, q) is *primitive*. For any $x \in \mathbf{T}^2$ and $\tilde{x} \in \pi^{-1}(x)$, the *$F$-rotation vector* is defined to be

$$\rho(x, F) = \lim_{n \to \infty} \frac{F^n(\tilde{x}) - \tilde{x}}{n} \in \mathbf{R}^2$$

if the limit exists. This is independent of the choice of $\tilde{x}$. Thus a periodic point of rotation type (p, q) has rotation vector $p/q \in \mathbf{Q}^2$. Define the *F-rotation set $\rho(x, F)$ of* x to be the set of limit points of

$$\left\{ \frac{F^n(\tilde{x}) - \tilde{x}}{n} \right\}_{n \in \mathbf{N}},$$

where $\tilde{x} \in \pi^{-1}(x)$. For all x, $\rho(x, F)$ is compact and connected [LM2]. The *F-rotation set* is defined to be

$$\rho(F) = \{\rho \in \mathbf{R}^2 : f \text{ has an orbit of } F\text{-rotation vector } \rho\}.$$

If F_1, F_2 are two lifts satisfying equation 2.1, then

$$\rho_{F_1}(x) = \rho_{F_2}(x) + (n, 0)$$

for all periodic points $x \in \mathbf{T}^2$, and likewise for rotation vectors and rotation sets.

Theorem 2.9 *Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism of the torus isotopic to the identity, and suppose $h(f) = 0$. Then all rotation vectors associated with the periodic orbits of f are collinear.*

Proof

Suppose that f has a finite union of periodic orbits, with associated distinct rotation vectors p_i/q_i, $i = 1, \ldots, N$, not collinear. Then from [LM2], since f is homotopic to the identity, for each $i \in \{1, \ldots, N\}$ there exists a periodic orbit Q_i of primitive rotation type (p_i, q_i). Let $Q = \cup_{i=1}^{N} Q_i$, and let F be a Thurston canonical form for f_Q.

Suppose F is periodic, then using equation 2.2 and Theorem 2.1, we see that $F \approx T_{(p,0)/q}/\Gamma_\square$, for some $q \in \mathbb{N}$, $p \in \mathbb{Z}_q$, so all points of $\mathbb{T}^2$ are periodic with period q and rotation vector $(p, 0)/q$.

There are no non-rotational reducing curves for F. For suppose Γ were such a curve, then it must surround at least two holes, but they must come from the same orbit, so the rotation type of that orbit cannot be primitive, which is a contradiction.

Suppose there is a rotational reducing curve Γ for F. Then using equation 2.2, the permitted cases of Theorem 2.5 are the orientation-preserving ones. Remove the tubular neighbourhood of Γ and its images, to obtain a disjoint union of annuli $\mathbb{A}_i$ with holes. Let $G : \mathbb{T}^2 \to \mathbb{T}^2$ be the completion of F, and let $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of G. Suppose $\mathbb{A}_i$ has period q, then we may lift $\mathbb{A}_i$ to a compact $\tilde{\mathbb{A}}_i \subset \mathbb{R}^2$ such that $\tilde{G}^q(\tilde{\mathbb{A}}_i) = \tilde{\mathbb{A}}_i + p$ for some $p \in \mathbb{Z}^2$. Further, the rotation vector $\rho(x, \tilde{G}) = p/q$ for all $x \in \mathbb{A}_i$. From Theorem 2.5, we have two distinct cases.

1. Both Γ and $\mathbb{A}_i$ have period q, for $1 \leq i \leq q$, and F permutes the $\mathbb{A}_i$. Then each $\mathbb{A}_i$ contains at least one point of Q_j $(j = 1, \ldots, N)$, x_j, say, so from the above, $\rho(x_j, \tilde{G}) = p/q$ for $j = 1, \ldots, N$. But x_j has primitive rotation type (p_j, q_j), and so $\rho(x_j, \tilde{G}) = p_j/q_j$ and q divides q_j for each j. Hence $q = q_j$, $p = p_j$ for all j, so $N = 1$ – a contradiction.

   In this case, we can say more. There is exactly one point of the periodic orbit in each $\mathbb{A}_i$. If we consider F^q, $F^q(\mathbb{A}_i) = \mathbb{A}_i$. Then all rotation vectors lie on the line through p/q with slope corresponding to the homotopy class of the $\mathbb{A}_i$. In fact, in this case we can deduce more. There are 3 invariant boundary components in each $\mathbb{A}_i$. Since $h(f) = 0$, $F^q|_{\mathbb{A}_i}$ is periodic, and if we consider the genus zero completion of $F^q|_{\mathbb{A}_i}$ to S^2, then by a Theorem of Eilenberg and Kerekjarto [Ei, Ke1], $F^q|_{\mathbb{A}_i}$ is the identity.

2. Γ has period 2, and there are two invariant annuli $\mathbb{A}_0$,$\mathbb{A}_1$. Then we can lift each $\mathbb{A}_i$ to a compact $\tilde{\mathbb{A}}_i \in \mathbb{R}^2$ such that $\tilde{G}(\tilde{\mathbb{A}}_i) = \tilde{\mathbb{A}}_i + r_i$, for $i = 0, 1$, and some $r_i \in \mathbb{Z}^2$. Since the $\mathbb{A}_i$ are invariant, each Q_j must lie entirely in $\mathbb{A}_0$ or $\mathbb{A}_1$, so we may partition Q into two corresponding sets, $\mathcal{P}_0$ and $\mathcal{P}_1$, say, and in analogy with the analysis in case 1, if Q_j is contained in $\mathcal{P}_i$, then $q_j = 1$ and $p_j = r_i$, and all points of $\mathbb{A}_i$ have rotation vector r_i. From our hypotheses, this implies that $N = 2$, and so the rotation vectors of Q are collinear – a contradiction. This proves Theorem 2.9. □

2.6 Classification of isometries of the torus

In this Section, we provide a proof of Theorem 2.1. To do this, we require the following:

Theorem 2.10 ([Ep2]) *If f is a finite order homeomorphism of a compact orientable manifold M, then there exists a Riemannian metric $\mathcal{R}$ of constant curvature on M such that f is a diffeomorphism preserving $\mathcal{R}$.*

Let $\mathcal{E}$ be the Euclidean metric on $\mathbf{R}^n$, and let E^n be the group of isometries of $(\mathbf{R}^n, \mathcal{E})$. It consists of all transformations $y \mapsto Ay + x$, with $x \in \mathbf{R}^n$, $A \in O(n)$. In the case $n = 2$, E^2 is generated by T_x, R_ω and r, where $x \in \mathbf{R}^2$, $\omega \in \mathbf{R}/2\pi\mathbf{Z}$.

Theorem 2.11 (Killing, Hopf, see [Wo]) *Let $(M^n, \mathcal{R})$ be a Riemannian manifold of dimension $n \geq 2$ with metric $\mathcal{R}$. Then $(M^n, \mathcal{R})$ is complete, connected and of constant curvature $\kappa = 0$ if and only if it is isometric to a quotient $(\mathbf{R}^n, \mathcal{E})/\Gamma$, where Γ is a subgroup of E^n which acts freely and properly discontinuously.*

Theorem 2.12 ([Sr]) *Suppose $R_\omega \in SO(2)$, $R_\omega \neq Id$, commutes with Γ on $\mathbf{R}^2$, where Γ is a subgroup of E^2 with two generators, and which acts freely and properly discontinuously. Then $\omega = \pm\pi/3, \pm\pi/2, \pm 2\pi/3$, or π, and Γ is conjugate in the group generated by $SO(2)$ and isotropic scale changes to $\Gamma_\square$ if $\omega = \pm\pi/2$ or π, or $\Gamma_\triangle$ if $\omega = \pm\pi/3$ or $\pm 2\pi/3$.*

Proof of Theorem 2.1

Suppose f is a finite order homeomorphism of $\mathbf{T}^2$. Theorem 2.10 implies that there exists a Riemannian metric $\mathcal{R}$ of constant curvature on $\mathbf{T}^2$, such that f is a diffeomorphism and preserves $\mathcal{R}$. By the Gauss-Bonnet formula (e.g. [DC]), the curvature is zero. By Theorem 2.11, $(\mathbf{T}^2, \mathcal{R})$ is isometric to $(\mathbf{R}^2, \mathcal{E})/\Gamma$, for some $\Gamma < E^2$. Since $\mathbf{T}^2$ is compact, Γ must be a group generated by two translations T_{y_1}, T_{y_2}, with y_1, y_2 linearly independent, and without loss of generality, y_1 is along the x-axis (by rotating coordinates). To find the finite order isometries $f : (\mathbf{T}^2, \mathcal{R}) \to (\mathbf{T}^2, \mathcal{R})$, it suffices to find all the isometries $\bar{f} : (\mathbf{R}^2, \mathcal{E}) \to (\mathbf{R}^2, \mathcal{E})$, such that $\bar{f}\Gamma = \Gamma\bar{f}$, and $\bar{f}^m \in \Gamma$, for some m. If f is orientation-reversing, then $r \circ f$ is orientation-preserving and satisfies the same conditions. So let us take f to be orientation-preserving.

If $\bar{f}$ has no fixed point, then it is a translation T_x, $x \in \mathbf{R}^2$. If $\bar{f}^m \in \Gamma$ then x is a rational combination of y_1 and y_2. Hence $f \approx T_{p/q}/\Gamma_\square$, for some $p \in \mathbf{Z}^2$, $q \in \mathbf{N}$. However, many of these are orientation-preserving conjugate. Let $p/q = (p_1, p_2)/q$, which we assume is in lowest terms. Suppose p_1 and p_2 have no common factors. Consider the orbit of $T_{p/q}/\Gamma_\square$ under $SL(2, \mathbf{Z})$. Then for

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}),$$

$$A \begin{pmatrix} p_1/q \\ p_2/q \end{pmatrix} = \frac{(ap_1 + bp_2, cp_1 + dp_2)}{q}$$

and $(ap_1 + bp_2)/q$ and $(cp_1 + dp_2)/q$ have no common factors. For suppose $ap_1 + bp_2 = kn_1$, $cp_1 + dp_2 = kn_2$, for some integers n_1, n_2, k, $k \neq \pm 1$. Then

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = kA^{-1} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix},$$

but we assumed that p_1 and p_2 were coprime, a contradiction. Now p_1 and p_2 are coprime if and only if there exist integers a, b such that $ap_1 + bp_2 = 1$. Let $c = -p_2$ and $d = p_1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Conversely given coprime integers a,c we can find integers b,d satisfying $ad - bc = 1$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}).$$

Suppose p_1 and p_2 are not coprime but have highest common factor $\mathrm{hcf}(p_1, p_2) = k$. Then $(p_1, p_2)/q = k(m, n)/q$, where m and n are coprime. Hence the orbit of $T_{p/q}/\Gamma_\square$ under $SL(2, \mathbb{Z})$ is all $T_{p'/q'}/\Gamma_\square$, with $q' = q$ and $\mathrm{hcf}(p_1', p_2') = \mathrm{hcf}(p_1, p_2)$, where $p' = (p_1', p_2')$. Thus every element in this orbit is a translation by $k(a, b)/q$, with $k = \mathrm{hcf}(p_1, p_2)$, and $\mathrm{hcf}(a, b) = 1$. Thus we may choose a representative $T_{(p,0)/q}/\Gamma_\square$, with $p \in \mathbb{Z}_q$, and p,q having no common factors.

If $\bar{f}$ has a fixed point, we may assume by making a translation if necessary that $\mathbf{0}$ is fixed. So $\bar{f}(x) = Ax$, for some $A \in SO(2)$. The case of $\bar{f} = Id$ is included in case 1. Otherwise, by Theorem 2.12, Γ is conjugate to $\Gamma_\square$ or $\Gamma_\triangle$, and $\omega = \pm\pi/2, \pi$ in the case $\Gamma_\square$, $\pm 2\pi/3, \pm\pi/3$ in the case $\Gamma_\triangle$.

Thus when f is orientation-preserving, it is orientation-preserving conjugate to one of the cases 1 or 2. It can be seen that no two of these cases are orientation-preserving conjugate.

If f is orientation-reversing, then it is orientation-preserving conjugate to the composition of r with one of the cases 1 or 2. However, some of these are orientation-preserving conjugate to each other, so they are not distinct cases. There are two classes to consider.

1. $f \approx g = (r \circ T_{(p,0)/q})/\Gamma_\square$ (case 3). On $\mathbf{T}^2$, there are two curves invariant under g; they are $y = 0$ and $y = 1/2$. Under g, the x-coordinate has period q, and the y-coordinate has period 1 if $y = 0$ or $1/2$, and period 2 otherwise. So if q is even all points in $\mathbf{T}^2$ have period q. If q is odd, all points in $\mathbf{T}^2$ have period q or $2q$, according as they lie on an invariant curve or not. The invariant curves have rotation number $p/q \in \bar{\mathbf{T}}^1$, where $\bar{\mathbf{T}}^1$ stands for the set of equivalence classes of $\mathbf{R}$ under the equivalence relation generated by $x \sim x + 1$, $x \sim -x$. The rotation number is an invariant of orientation-preserving conjugacy. Hence the cases in part 3 are distinct.

2. $f \approx g = (r \circ R_\omega)/\Gamma_\square$, $\omega = \pi, \pm\pi/2$, or $f \approx g = (r \circ R_\omega)/\Gamma_\triangle$, $\omega = \pm\pi/3, \pm 2\pi/3$. Without loss of generality (by conjugation by $R_{-\omega}$), we take ω to be positive.

   - If $\omega = \pi$, then on $\mathbf{R}^2$ this corresponds to perpendicular reflection in $x = 0$, so by a change of coordinates, $f \approx r$, which is already included in case 3.
   - If $\omega = \pi/2$ (case 4), then on $\mathbf{R}^2$ this corresponds to perpendicular reflection in the curve $x + y = 0$. It is the only invariant curve, and is composed of fixed points.
   - If $\omega = \pi/3$, then on $\mathbf{R}^2$ this corresponds to perpendicular reflection in $x + \sqrt{3}y = 0$. On $\mathbf{T}^2$, there is one curve fixed pointwise by g, and g is a reflection in it.

- If $\omega = 2\pi/3$, then on $\mathbf{R}^2$ this corresponds to perpendicular reflection in $x + y/\sqrt{3} = 0$. On $\mathbf{T}^2$, $x + y/\sqrt{3} = 0$ is the only curve fixed pointwise by g, and g is a reflection in it.

Let $h_1 = r \circ R_{\pi/2}/\Gamma_\square$, $h_2 = r \circ R_{2\pi/3}/\Gamma_\triangle$. Then h_1 and h_2 are orientation-preserving conjugate. To see this, we need to find an orientation-preserving homeomorphism $k : \mathbf{R}^2/\Gamma_\square \to \mathbf{R}^2/\Gamma_\triangle$ such that $kh_1 = h_2k$, or equivalently an orientation-preserving homeomorphism $K : \mathbf{R}^2 \to \mathbf{R}^2$ such that $KH_1 = H_2K$, where $h_1 = H_1/\Gamma_\square$ and $h_2 = H_2/\Gamma_\triangle$, and also such that for each $\gamma_\square \in \Gamma_\square$ there exists a $\gamma_\triangle \in \Gamma_\triangle$ such that $K\gamma_\square = \gamma_\triangle K$ and vice versa. This is satisfied by

$$K = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

(with respect to the basis vectors of $\Gamma_\square$), which is a shear taking the generators of $\Gamma_\square$ to $\Gamma_\triangle$. Hence h_1 and h_2 are orientation-preserving conjugate. Similarly the case $r \circ R_{\pi/3}/\Gamma_\triangle$ is orientation-preserving conjugate to h_1, by taking

$$K = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

The cases h_1 and r are distinct because the case r has two invariant curves, whilst the case h_1 has only one invariant curve. $\square$

Chapter 3

Flow-equivalence and rotation sets

In this Chapter, we discuss the notions of flow-equivalence and homology directions, and relate them to the theory of cross-sections. We describe the connection between Anosov homeomorphisms of the 2-torus and pseudo-Anosov homeomorphisms of $\mathbf{D}_3$*, and prove several useful results about rotation sets of homeomorphisms of the annulus. Also, we discuss the structure of rotation sets of homeomorphisms of the 2-torus.*

3.1 Introduction

In Section 1.8.2, we exhibited some of the 'self-similarity' inherent in the structure of BT by expressing reducible braid types as products of irreducible braid types, and showing that the ordering in BT is determined by that in IBT. In this Chapter, we shall present another aspect of 'self-similarity' in IBT. Given a braid and one of its unremovable fixed points, one may transform it into another braid type with an unremovable fixed point using 'flow-equivalence'. This transformation is order-preserving, and provides a method by which one may compare different parts of the structure of IBT. In particular, one may transform periodic orbits of arbitrary period to one of small period which is easier to study. As an application of this, we shall give a proof of the following Theorem of Boyland:

Theorem 3.1 ([Boy3]) *Let* $f : \mathbf{A} \to \mathbf{A}$ *be an orientation-preserving homeomorphism of the annulus which preserves the boundary components, and let* $o(x)$ *be a* (p,q)*-periodic orbit (relative to some lift of* f*) such that* p *and* q *are coprime, and* $bt(o(x), f)$ *is pseudo-Anosov. Then the Farey interval of* p/q *is contained within the rotation set of* f*.*

We commence with a discussion on the geometry and homology of cross-sections to flows, in particular suspended flows of surface homeomorphisms. We describe the notion of the rotation number of a periodic orbit of a homeomorphism of the annulus, and relate it to the suspended flow. In this way, we are able to translate statements of Fried for flows into statements about rotation sets for homeomorphisms of the annulus. In particular, we use the idea of flow-equivalence to give a proof of Theorem 3.1.

We then describe the association between automorphisms of the 2-torus $\mathbf{T}^2$ and automorphisms of the 3-punctured disc, given by a 2-fold ramified covering of the 2-sphere S^2 by $\mathbf{T}^2$. An Anosov homeomorphism of $\mathbf{T}^2$ descends to a pseudo-Anosov

homeomorphism of S^2 minus 4 points, thus we can relate the dynamics of one to the other. Using this, we give a number of results about rotation sets for homeomorphisms of the annulus with a two-point invariant set, and using flow equivalence, we are able to give some results on the 'minimal size' of the rotation sets for homeomorphisms of the annulus with periodic orbits of certain rotation numbers. Many of these results will also be found to be useful in Chapter 4. Finally, we consider the case of the 2-torus, and give some possible 'smallest' rotation sets for homeomorphisms of $\mathbf{T}^2$.

3.2 Flow-equivalence and homology directions

3.2.1 The suspension construction and geometric braids

We begin by recalling the suspension construction. Suppose $f : N \to N$ is a homeomorphism of a compact metric space, then the *mapping torus* M_f is the quotient space

$$N \times \mathbf{R}/(f(x), s) \sim (x, s+1).$$

Denote the canonical projection of $\sim$ by π. The flow Φ_t on $N \times \mathbf{R}$ defined by

$$\Phi_t(x, s) = (x, s+t)$$

induces a flow ϕ_t on M_f, which is called the *suspension* of f. We remark that there exists a one-to-one correspondence between f-orbits of N and ϕ-orbits of M_f.

A *cross-section* to a flow ϕ on a compact 3-manifold M is a compact surface K transverse to the flow that meets every flowline, such that K is transverse to ∂M and $\partial K = K \cap \partial M$. Then for $k \in K$, the least time $t = t(k) > 0$ such that $\phi_t(k) \in K$ is called the *return time* of k, and the map $r : K \to K$ defined by $r(k) = \phi_{t(k)}(k)$ is the monodromy, or *return map* of K for ϕ. Hence $\pi(N \times \{0\})$ is a cross-section to ϕ_t with return time 1. Note that the corresponding return map and f are conjugate.

We now restrict ourselves to the case where $N = \mathbf{D}^2$. Given a periodic orbit P of period n of $f : \mathbf{D}^2 \to \mathbf{D}^2$, the suspension provides a canonical correspondence between the braid type $bt(P, f)$ and the conjugacy class $\beta Z(B_n)$, where $\beta \in B_n$ represents P. For given an isotopy f_t from Id to f, it defines a flow in the solid torus $\mathbf{T}$, where

$$\mathbf{T} = \mathbf{D}^2 \times \mathbf{R}/(x, s) \sim (x, s+1),$$

up to a change of coordinates, given by

$$\phi_t(\pi(x, 0)) = \pi(f_t(x), t).$$

See figure 3.1. Hence $f_t(P)$ is a geometric braid, which we denote as β, which we may consider as a closed braid $\hat{\beta}$ in $\mathbf{T}$. If we choose a different isotopy f'_t from Id to f, this defines a geometric braid β' and a closed braid $\hat{\beta}'$. Then $\hat{\beta}$ and $\hat{\beta}'$ are isotopic as simple closed curves in $\mathbf{T}$, up to full twists, as a suspension with the identity as return map is (up to isotopy) a full twist (see [Bi3]). This is true if and only if β and β' are conjugate in B_n, up to an element of $Z(B_n)$, which clarifies why we identify the conjugacy class of $\beta Z(B_n)$ with P in Section 1.3.2.

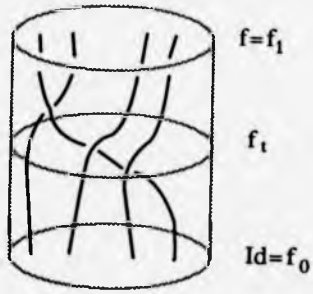

Figure 3.1: The suspended flow ϕ_t

3.2.2 Cross-sections and homology directions

Suppose ϕ is a flow on a compact 3-manifold M. We would like to decide when there exists a cross-section to ϕ, and if so, whether there exists a canonical or preferred cross-section. We can do this by studying invariants of the flow called homology directions, which we now proceed to do.

Suppose K is a cross-section to ϕ, then it has a preferred normal orientation induced by ϕ. It determines a class $u_K \in H^1(M;\mathbf{Z})$, for suppose γ is a simple closed oriented loop in M transverse to K, then

$$u_K(\gamma) = \sum_{x \in \gamma \cap K} \epsilon_x$$

where

$$\begin{aligned} \epsilon_x &= +1 \quad \text{if the orientation of } \gamma \text{ agrees with that of the flow at } x \\ &= -1 \quad \text{otherwise.} \end{aligned}$$

Then

Theorem 3.2 ([Fri1]) *Suppose K_0 and K_1 are cross-sections to ϕ. There exists an isotopy of M carrying K_0 to K_1 through cross-sections if and only if $u_{K_0} = u_{K_1}$.*

If this is the case, then the corresponding return maps are conjugate, and it is not necessary to distinguish between such cross-sections. So we are interested in the set

$$\{u_K \in H^1(M;\mathbf{Z}) : K \text{ is a cross-section to } \phi\}.$$

To this end, define the set D_M of *homology directions* of M as

$$D_M = H_1(M;\mathbf{R})/\mathbf{R}^+,$$

which we topologize as a sphere corresponding to the non-zero homology classes, plus an isolated point corresponding to zero. Let $p : H_1(M) \to D_M$ be the canonical projection map. So any closed loop γ in M determines a homology direction $p(\gamma) \in D_M$. Define $d \in D_M$ to be a *homology direction* [Fri1, Rh, Sn] for ϕ if there are points $m_i \to m$ and times $t_i \to \infty$ so that $\phi_{t_i} m_i \to m$, and $p((\phi_t m_i : 0 \le t \le t_i) \circ \text{short path}) \to d$ i.e. an accumulation point of almost closed orbits of ϕ. Thus the set D_ϕ of *homology directions for ϕ* is a compact nonempty subset of D_M. Then

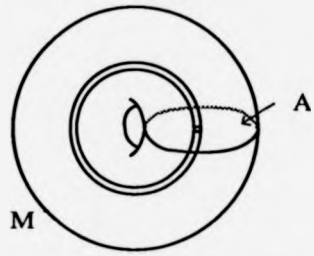

Figure 3.2: Suspended flow of the annulus

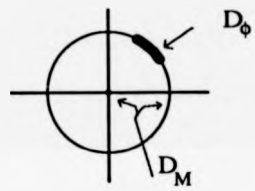

Figure 3.3: Homology directions of the annulus

Theorem 3.3 ([Fri1]) *The flow ϕ on the compact manifold M has a cross-section K dual to $u \in H^1(M;\mathbb{Z})$ if and only if $u(D_\phi) > 0$. Thus ϕ admits a cross-section if and only if D_ϕ lies in an open half-space of D_M.*

If there exists a cross-section K to ϕ such that the return map $r : K \to K$ has a Markov partition $\mathcal{R}$, then there exist a finite number of *minimal loops* i.e. permitted loops in the corresponding Markov graph such that no symbol occurs twice. Given the associated transition matrix A, we say that a sequence $s_0 \ldots_k$ is a minimal loop for $\mathcal{R}$ if $A_{s_i s_{i+1}} = 1$ for $i = 0, 1, \ldots, k-1$, $A_{s_k s_0} = 1$ and $s_i \neq s_j$ for $i \neq j$. To each minimal loop, there exists an associated set of periodic orbits of r. We denote the periodic orbits as $\gamma_i, i = 1, \ldots, k,$. We may describe the cross-sections to ϕ in the following manner.

Theorem 3.4 ([Fri1]) *For such a flow ϕ, a class $u \in H^1(M;\mathbb{Z})$ is dual to a cross-section L if and only if $u(\gamma_i) > 0, i = 1, \ldots, n$.*

We may regard the condition $u(D_\phi) > 0$ as determining a cone $\mathcal{C}(\phi) \subset H^1(M;\mathbb{R})$ for which lattice points correspond to cross-sections to ϕ. If $\mathcal{C}(\phi)$ is non-empty, then there exist infinitely many lattice points; under certain conditions (e.g. those of Theorem 3.4), $\mathcal{C}$ has finitely many flat, integrally defined sides. We may use this characterization to find 'simplest' cross-sections.

For example, suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus. Pick a suspended flow $\phi : M \to M$ of f, then $H_1(M;\mathbb{Z}) = \mathbb{Z}^2$ (see figure 3.2). Hence $D_M = S^1 \cup \{0\}$ (see figure 3.3). Since $f : \mathbb{A} \to \mathbb{A}$ is a return map, D_ϕ lies in the upper half plane. However, if we choose any hyperplane $\mathcal{P}$ of rational gradient passing through the origin such that D_ϕ lies entirely on one side of $\mathcal{P}$ and $\mathcal{P} \cap D_\phi = \emptyset$, then it corresponds to an element $u \in H^1(M;\mathbb{Z})$, and hence a cross-section to the flow (see figure 3.4).

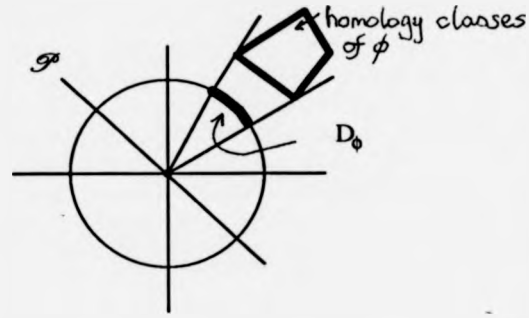

Figure 3.4: Cross-section to the suspended flow of the annulus

3.2.3 Flow-equivalence and pseudo-Anosov homeomorphisms

Suppose K,L are cross-sections to flows on compact 3-manifolds with return maps r_K,r_L respectively. We say that r_K and r_L are *flow-equivalent* if they have conjugate suspended flows. For instance, if K and L are (distinct) cross-sections to the same flow, then the return maps r_K and r_L are flow-equivalent. Thus in the example in Section 3.2.2, the return maps associated with the cross-sections corresponding to the hyperplanes $\mathcal{P}$ are flow-equivalent.

Suppose $\phi : M \to M$ is a flow on a compact 3-manifold, we say that ϕ is a *pseudo-Anosov flow* if there exists a cross-section for which the return map is pseudo-Anosov.

Theorem 3.5 (Fried, see [FLP]) *Any cross-section to a pseudo-Anosov flow will have pseudo-Anosov return map.*

This follows essentially from the fact that we may push forward the foliations and transverse measures of the cross-section for which the return map is pseudo-Anosov. For ϕ determines a system of local homeomorphisms between any two cross-sections and hence any local structure of one is translated to the other. For instance transversality of foliations, the local singularity structure and local expansion and contraction is preserved. It follows that one may push forward the Markov partition of the pseudo-Anosov homeomorphism, and that any other cross-section must have pseudo-Anosov return map.

Theorem 3.6 ([Fri1, Fri2]) *Suppose ϕ is a pseudo-Anosov flow, with cross-section K and return map $r : K \to K$. Let $\mathcal{R}$ be a Markov partition for r. For any closed orbit γ of ϕ, let $[\gamma]$ denote the corresponding homology class of $H_1(M;\mathbf{Z})$. Let Δ be the convex hull of*

$$\{[\gamma_i] : \gamma_i \text{ is a closed orbit corresponding to a minimal loop for } \mathcal{R}\} \subset H_1(M;\mathbf{R}).$$

If $0 \notin \Delta$, then $D_\phi = p(\Delta)$. Further, the homology classes $[\gamma_i]$ span $H_1(M;\mathbf{R})$.

We give a sketch of the proof. Suppose $\phi_t(m)$, $0 \le t \le T$, is a long, almost closed orbit, and let γ be the corresponding loop. Such an orbit corresponds to a loop in the graph associated with the Markov partition, and we may write it as a concatenation

of minimal loops. But the information given by homology just adds. So $[\gamma]$ is the corresponding sum of the $[\gamma_i]$. Taking longer and longer loops gives us the convex hull. Conversely, given any homology class in Δ, we may approximate it closely as a concatenation of minimal loops, and thus by an almost-closed orbit for ϕ. This outlines the first part. For the second part, we argue by contradiction. Suppose the classes $[\gamma_i]$ don't span $H_1(M;\mathbf{R})$, then there exists an integral class $u \in H^1(M;\mathbf{R})$ such that $u[\gamma_i] = 0$ for all $i = 1,\ldots,n$. u specifies a connected $\mathbf{Z}$-cover $\pi : \tilde{M} \to M$, and ϕ lifts to a flow $\tilde{\phi}$ of $\tilde{M}$. But any orbit of $\tilde{\phi}$ is uniformly bounded within $\tilde{M}$, for we may approximate its symbol sequence in $\mathcal{R}$ as a concatenation of minimal loops, on which u vanishes. This imposes that the orbit must be bounded uniformly for all time. By taking a large enough j-fold cover $M_j = \tilde{M}/j\mathbf{Z}$ of M, and its induced flow ϕ_j, which is also pseudo-Anosov, we find that since orbits are bounded uniformly in $\tilde{M}$, then there does not exist a transitive orbit in M_j, which contradicts ϕ_j being pseudo-Anosov. □

Hence the homology classes of periodic orbits span homology.

3.3 Rotation numbers of the annulus

In this Section, we recall some standard definitions and notions which we shall use throughout the rest of this Thesis. Suppose $f : \mathbf{A} \to \mathbf{A}$ is a homeomorphism of the annulus $\mathbf{A} = S^1 \times I$ isotopic to the identity. Let $\pi : \tilde{\mathbf{A}} \to \mathbf{A}$ be the *universal cover*, where $\tilde{\mathbf{A}} = \mathbf{R} \times I$, let $T : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ denote the deck transformation $T(x,y) = (x+1, y)$, and let $p_1 : \tilde{\mathbf{A}} \to \mathbf{R}$ denote projection onto the first factor. Let $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ be a lift of f. For any $x \in \mathbf{A}$ and $\tilde{x} \in \pi^{-1}(x)$, the *rotation number of x relative to F* is defined to be

$$\rho(x,F) = \lim_{n\to\infty} \frac{p_1(F^n(\tilde{x})) - p_1(\tilde{x})}{n}$$

if the limit exists. This is independent of the choice of lift $\tilde{x}$ of x. If $\rho(x,F)$ exists, then the fractional part of $\rho(x,F)$ is independent of the choice of lift F. The *rotation set of F* is defined to be

$$\rho(F) = \{\rho(x,F) : x \in A\}.$$

We say that a periodic orbit $o(x)$ of f is a *(p,q)-periodic orbit* of F or is of *F-rotation type* (p,q) if it has (least) period q, and $T^{-p}F^q(\tilde{x}) = \tilde{x}$ for some (and hence any) lift $\tilde{x}$ of x. Given $o(x)$, we may choose F appropriately, so that $0 \le p/q < 1$. Let $\partial^+\mathbf{A} = \{(x,y) \in \mathbf{A} : y = 1\}$ and $\partial^-\mathbf{A} = \{(x,y) \in \mathbf{A} : y = 0\}$, and define $\partial^\pm\tilde{\mathbf{A}} = \pi^{-1}(\partial^\pm\mathbf{A})$. One reason for studying the rotation set is the following Theorem:

Theorem 3.7 ([Fr3], see also [H5]) *Let $f : \mathbf{A} \to \mathbf{A}$ be a homeomorphism of the annulus isotopic to the identity, and let $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ be a lift of f. Then for each rational $p/q \in \rho(F)$, where p/q is in lowest terms, there exists a (p,q)-periodic orbit for F.*

The following Theorem is also important.

Theorem 3.8 ([H5]) *Let $f : \mathbf{A} \to \mathbf{A}$ be a homeomorphism of the annulus isotopic to the identity, and suppose f is pseudo-Anosov relative to some finite invariant set K. Let $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ be a lift of f, then*

1. *$\rho(F)$ is a closed interval.*

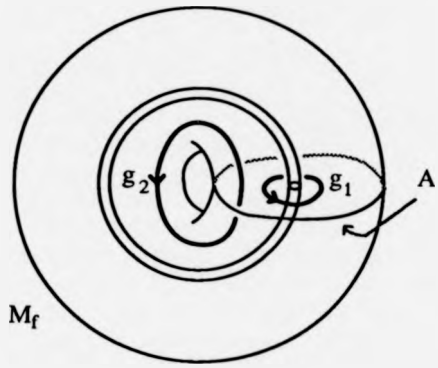

Figure 3.5: The generators of homology g_1 and g_2

2. *if $g \simeq f$ rel. K, and G is the lift of g equivariantly homotopic rel. $\pi^{-1}(K)$, then $\rho(G)$ is a closed set containing $\rho(F)$.*

We may generalize the notion of rotation number to homeomorphisms of the disc relative to an unremovable fixed point. Suppose $f \in Homeo^+(\mathbf{D}^2)$, and $x \in \mathrm{Fix}(f) \cap \mathrm{Int}(\mathbf{D}^2)$, then by blowing up x, we get an induced homeomorphism $f_{\{x\}} : \mathbf{A} \to \mathbf{A}$ of the annulus. Thus we may speak of *rotation numbers* of points $z \in \mathbf{D}^2 \setminus \{x\}$ *about x relative* to some lift $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ of $f_{\{x\}}$.

Given a homeomorphism of the annulus $f : \mathbf{A} \to \mathbf{A}$ isotopic to the identity, one may specify a choice of lift $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ by choosing a continuous path $\gamma : I \to \mathbf{A}$ satisfying $f(\gamma(0)) = \gamma(1)$, and picking the lift F which satisfies $F(\Gamma(0)) = \Gamma(1)$ for some (and hence any) lift Γ of γ. We shall refer to *rotation numbers and rotation sets relative to* γ. Using this idea, we shall indicate how to choose a lift which may be associated with a given suspension of f. Suppose f_t, $t \in I$ is an isotopy from $Id = f_0$ to $f = f_1$. It defines a flow ϕ_t in an analogous manner to the construction in Section 3.2.1. So the mapping torus $M_f = \mathbf{A} \times S^1$ is the quotient space

$$\mathbf{A} \times \mathbf{R}/(f(x), s) \sim (x, s+1),$$

where the canonical projection of $\sim$ is denoted by π. The flow

$$\begin{aligned} \Phi_t : \mathbf{A} \times \mathbf{R} &\to \mathbf{A} \times \mathbf{R} \\ \Phi_t(x, s) &= (x, s+t) \end{aligned}$$

induces the suspension ϕ_t of f on M_f, given by

$$\phi_t(\pi(x, 0)) = \pi(f_t(x), t).$$

Then $H_1(M_f; \mathbf{Z}) \cong \mathbf{Z}^2$, and we choose generators g_1 and g_2 of the first homology group as shown in figure 3.5. Any periodic orbit $\mathcal{P}$ of f defines a homology class

$$[\phi_t(\mathcal{P})] = pg_1 + qg_2 \in H_1(M_f; \mathbf{Z}),$$

where q is the f-period of $\mathcal{P}$. Let

$$\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_q\},$$

where

$$\begin{aligned} f(\mathcal{P}_i) &= \mathcal{P}_{i+1} \quad 1 \le i \le q-1 \\ f(\mathcal{P}_q) &= \mathcal{P}_1. \end{aligned}$$

By choosing γ to be the path $\gamma : I \to \mathbb{A}$ defined by

$$\gamma(t) = f_{q(t-\frac{i}{q})}(\mathcal{P}_{i+1}) \text{ for } \frac{i}{q} \le t \le \frac{i+1}{q},\ 0 \le i \le q-1 \tag{3.1}$$

it follows that the orbit $\mathcal{P}$ has rotation number p/q relative to γ; if we consider f^q, then $f^q(\mathcal{P}_1) = \mathcal{P}_1$, and we choose the lift F of f such that $F^q(\Gamma(0)) = \Gamma(1)$ where Γ is a lift of γ. Thus $F^q(\tilde{\mathcal{P}}_1) = T^p(\tilde{\mathcal{P}}_1)$, where $\tilde{\mathcal{P}}_1 \in \pi^{-1}(\mathcal{P}_1)$. So given a suspension ϕ_t, we may associate with a periodic orbit $\mathcal{P}$ a lift F of f. There is a one-to-one correspondence between orbits of ϕ_t in M_f and orbits of f in $\mathbb{A}$. In particular, if $\mathcal{Q}$ is a periodic orbit of f, then $\phi_t(\mathcal{Q})$ is a closed path lying in M_f. So it defines a homology class

$$[\phi_t(\mathcal{Q})] = rg_1 + sg_2 \in H_1(M_f, \mathbb{Z}),$$

for some integers r,s. It follows from the construction of F that if $x \in \mathcal{Q}$, then $F^s(\tilde{x}) = T^r(\tilde{x})$, where $\tilde{x} \in \pi^{-1}(x)$ i.e. x is an (r,s)-periodic orbit of F.

Given ϕ_t and the periodic orbit $\mathcal{P}$, then $\phi_t(\mathcal{P})$ defines a geometric braid $\beta \in B_{q+1}$. As we have seen, if we choose a different isotopy f'_t, this defines a suspension ϕ'_t, and $\phi'_t(\mathcal{P})$ defines a geometric braid β', such that β and β' are conjugate or differ by a multiple of the 'full-twist' braid θ_{q+1}. If β and β' are conjugate, then the corresponding lifts F and F' are equal, whilst if $\beta' = \theta^n_{q+1}\beta$ for some $n \in \mathbb{Z}$, then $F' = T^n F$, and so $F'^q(\tilde{\mathcal{P}}_1) = T^{p+nq}(\tilde{\mathcal{P}}_1)$. Thus given any lift F' of f, one may find an isotopy f'_t and a suspension ϕ'_t such that if $F'^q(\mathcal{P}_1) = T^{p+nq}(\mathcal{P}_1)$, and if $\gamma' : I \to \mathbb{A}$ is the path defined in equation 3.1 with f'_t replacing f_t, then $\mathcal{P}$ has rotation number $(p+nq)/q$ relative to γ', and

$$[\phi'_t(\mathcal{P})] = (p+nq)g_1 + qg_2,$$

where g_1 and g_2 are the canonical generators of $H_1(M_f, \mathbb{Z})$.

To sum up, we have shown that given a periodic orbit $\mathcal{P}$ of a homeomorphism $f : \mathbb{A} \to \mathbb{A}$ isotopic to the identity and any lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$, there exists an isotopy f_t between Id and f such that f_t is associated in a natural way with F, and vice versa i.e. an (r,s)-periodic orbit $\mathcal{Q}$ of F may be expressed as

$$[\phi_t(\mathcal{Q})] = rg_1 + sg_2 \in H_1(M_f, \mathbb{Z}),$$

where ϕ_t is the suspension defined by f_t. So there is a natural bijection between the rotation set $\rho(F)$ and the set of homology directions D_ϕ for ϕ, for both are sets of accumulation points of long, nearly-closed orbits. From now on, we shall make no distinction between the two. So using Theorem 3.6, we recover the following.

Proposition 3.9 *If in addition to the above conditions, f is pseudo-Anosov relative to $\mathcal{P}$, then the closed interval $\rho(F)$ has rational endpoints, for any lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$.* □

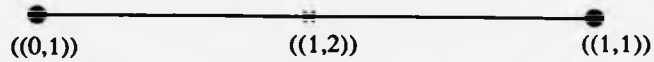

Figure 3.6: The level-0 Farey element and interval

3.4 Farey intervals and rotation sets

In this Section we define the Farey interval of a rational number, and then complete the proof of Theorem 3.1 using the notions in this Chapter.

Suppose p and q are coprime integers, such that $0 < p/q < 1$. Define

$$\frac{m_1}{n_1} = \max\{r/s < p/q, s < q, (r,s) = 1\}$$

$$\frac{m_2}{n_2} = \min\{r/s > p/q, s < q, (r,s) = 1\}$$

to be the *Farey neighbours* of p/q, where (r,s) denotes the highest common factor of r and s, and the *Farey interval* to be (see [HW] for more details)

$$I(p/q) = [m_1/n_1, m_2/n_2].$$

The Farey neighbours satisfy

$$\begin{aligned} pn_1 - qm_1 &= 1 \\ qm_2 - pn_2 &= 1 \end{aligned}$$

a consequence of which is

$$\frac{p}{q} = \frac{m_1 + m_2}{n_1 + n_2}.$$

A useful arrangement of rational numbers is the *Farey tree*, which we now describe. For more details, the reader may consult [KiO]. Given p,q two coprime integers, we write the rational $p/q = ((p,q))$. Define the 'Farey sum' $\oplus$ of two rationals to be the sum of their numerator and denominator:

$$((p_1,q_1)) \oplus ((p_2,q_2)) = ((p_1 + p_2, q_1 + q_2)).$$

We may interpret this sum as a sum of integer vectors, but it may also be considered as defining a 'mediant', for if $x < y$ are rationals, then $x < x \oplus y < y$. From this, it is possible to define a tree in the following way.

1. Start with $0/1 = ((0,1))$ and $1/1 = ((1,1))$ at either end of a line segment. We call this segment the level-0 Farey interval (see figure 3.6).

2. Applying the mediant operation $\oplus$ to the endpoints of the level-0 Farey interval we obtain $((1,2))$ at the midpoint. We call $1/2$ the level-0 Farey element.

3. The element $((1,2))$ divides the level-0 Farey interval into two subintervals, which we call the level-1 Farey intervals.

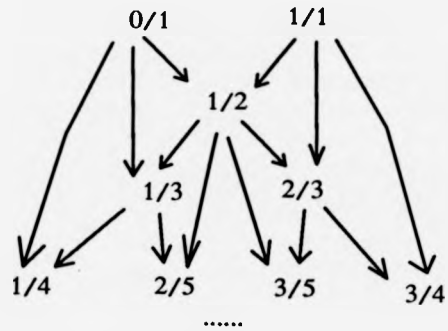

Figure 3.7: The Farey tree

4. Applying the mediant operation to the endpoints of the level-1 Farey intervals, we obtain two level-1 elements ((1,3)) and ((2,3)), and split the level-1 intervals at these elements to obtain the level-2 intervals. In this way, we obtain a tree as shown in figure 3.7, drawing arrows from the endpoints of each level-n interval to the corresponding level-n element.

5. Apply recursively steps 3 and 4 on all new Farey intervals.

Remarks

1. Each rational between 0 and 1 occurs precisely once in the tree.

2. The Farey neighbours of p/q are the endpoints of the corresponding interval on the same level (i.e. those rationals with arrows pointing to p/q in the tree). For example, $I(2/5) = [1/3, 1/2]$.

3. Each division of the level-n intervals using the level n element gives two level-$(n+1)$ intervals. Each level-n element has arrows pointing to two level-$(n+1)$ elements, we label the left arrow of the two with a '0' and the right arrow with a '1'. This generates a *binary tree*, which is a refinement of the Farey tree (see figure 3.8). So there exists a binary address for each rational p/q not equal to either 0/1,1/2 or 1/1 in the Farey tree, given by reading the path to p/q from 1/2 in the binary tree. For instance, we write the binary address of 2/5 as 01.

We now define two matrices

$$S_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

and make the following observations:

1. $S_0, S_1 \in SL(2, \mathbb{Z})$.

2. $S_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $S_0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

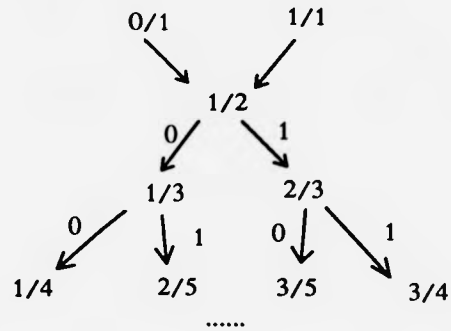

Figure 3.8: The binary tree obtained fron the Farey tree

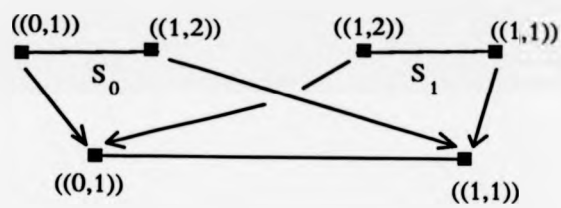

Figure 3.9: The action of S_0 and S_1 on the level-1 Farey intervals

3. $S_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $S_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus S_0 maps bijectively one level-1 Farey interval to the whole level-0 Farey interval, and S_1 does likewise with the other level-1 Farey interval (see figure 3.9), mapping endpoints to endpoints and interiors to interiors. By induction, it follows that there exists a matrix $A \in SL(2, \mathbb{Z})$ such that if p/q has binary address $I_0 I_1 \ldots I_N$ then [KiO]

1. $A = S_{I_N} \ldots S_{I_1} S_{I_0}$.
2. A maps bijectively the Farey interval $I(p/q)$ to $I(1/2)$.

To sum up,

Proposition 3.10 *Given a pair of coprime integers p,q, there exists a matrix $A \in SL(2, \mathbb{Z})$ such that $A \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and A maps $I(p/q)$ to $I(1/2)$ bijectively, mapping endpoints of $I(p/q)$ to those of $I(1/2)$.* □

The existence of such a matrix will prove to be useful later on in this Chapter.

We shall now give a proof of Theorem 3.1 using the concepts we have already discussed in this Chapter. Now suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism isotopic to the identity, which has a periodic orbit $o(x)$ of period q, such that there exists a lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of f for which $\rho(x, F) = p/q$, where $0 < p/q < 1$, and p and q are coprime. Suppose

further that the braid type $bt(o(x), f)$ is pseudo-Anosov, and let $g \in Homeo^+(\mathbb{A}, o(x))$ be the pseudo-Anosov representative in the isotopy class of f relative to $o(x)$. Let $G : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be the lift of g which is equivariantly homotopic to F. Theorem 3.8 implies that $\rho(G) \subset \rho(F)$ and that $\rho(G)$ is a closed interval, so it suffices to prove Theorem 3.1 for g. We give a variation of the argument given by Boyland in [Boy3].

We argue by contradiction. Assume $m_1/n_1 \notin \rho(G)$, the case $m_2/n_2 \notin \rho(G)$ is similar. Pick a suspension $\phi : M_g \to M_g$ of g, where M_g is the mapping torus, such that if we consider homology classes of orbits and choose standard generators g_1, g_2 of homology, then the loop γ corresponding to the orbit $\phi(x)$ has homology class $[\gamma] = (p, q)$ relative to the basis (g_1, g_2) i.e. as described in Section 3.3. It follows that for each homology class $[\gamma] = (r, s)$ of a closed orbit of ϕ relative to this basis, there exists a periodic orbit $o(y)$ of g such that y is an (r, s)-periodic point of G. Now we utilize the notion of flow-equivalence. Suppose we had chosen different generators g_1', g_2' as a basis for homology. Then it follows from linear algebra that

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = M \begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

where $M \in SL(2, \mathbb{Z})$. Suppose $[\gamma] = (p', q')$ relative to the basis (g_1', g_2'), then $(p', q') = (p, q).M^{-1}$. By choosing

$$M = \begin{pmatrix} -m_1 & -n_1 \\ p & q \end{pmatrix},$$

$(p', q') = (0, 1)$. Also, since $\rho(G)$ is a closed interval, then $r/s \in \rho(G)$ implies that $r/s > m_1/n_1$. Then $(r', s') = (r, s).M^{-1}$, and $s' = rn_1 - sm_1 > 0$. It follows from Theorem 3.3 that the change of basis corresponds to a change of cross-section, such that the (p, q)-orbit is transformed to a $(0, 1)$-orbit. Thus the return map to the new cross-section is a pseudo-Anosov homeomorphism of the annulus relative to a fixed point, by Theorem 3.5. But there do not exist pseudo-Anosov homeomorphisms of the 1-punctured annulus by the following (well-known) Proposition – a contradiction. Thus $\rho(G) \supset I(p/q)$. □

Proposition 3.11 *Suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus isotopic to the identity, and suppose x is a fixed point of f. Then $bt(x, f)$ is periodic i.e. f is isotopic to the identity relative to x.*

Proof

Consider the homeomorphism $F : \mathbb{D}_2 \to \mathbb{D}_2$ induced by blowing up x. There are no essential curves lying in $\mathbb{D}_2$, so $bt(x, f)$ cannot be reducible. Suppose it were pseudo-Anosov. From the Euler-Poincaré formula $\chi(\mathbb{D}_2) = -1$, so the number of singularities of each foliation is bounded above by 2. But each foliation of a pseudo-Anosov homeomorphism has at least one singularity on each boundary component, of which there are 3 – a contradiction. □

3.5 Ramified covering of the sphere

In this Section we describe a 2-fold ramified covering of the sphere S^2 by the torus $\mathbb{T}^2$, the *Weierstrass function* W. The action of $Aut(\mathbb{T}^2) \cong SL(2, \mathbb{Z})$ descends to a well-defined action on S^2, in particular, Anosov homeomorphisms of $\mathbb{T}^2$ descend to

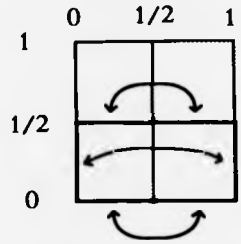

Figure 3.10: The fundamental domain for S^2

pseudo-Anosov homeomorphisms of S^2 (relative to 4 points). Since the Anosov homeomorphisms are well-understood, the correspondence provides an approach to understanding the associated pseudo-Anosov homeomorphisms. We now elucidate this relationship.

Consider the torus $\mathbf{T}^2$ as the quotient of $\mathbf{R}^2$ by the integer lattice $\mathbf{Z}^2$, by defining two translations $T_1, T_2 : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\begin{aligned} T_1 &: (x,y) \mapsto (x+1, y) \\ T_2 &: (x,y) \mapsto (x, y+1). \end{aligned}$$

Then $\pi : \mathbf{R}^2 \to \mathbf{R}^2 / < T_1, T_2 > \cong \mathbf{T}^2$ is the universal cover, and we choose a fundamental domain to be the unit square $\mathcal{D}$. Define $S : \mathbf{R}^2 \to \mathbf{R}^2$ to be the involution

$$S : (x,y) \mapsto (-x,-y).$$

Then the map $\bar{W} : \mathbf{R}^2 \to \mathbf{R}^2 / < T_1, T_2, S > \cong S^2$ induces a 2-fold ramified covering $W : \mathbf{T}^2 \to S^2$ of the sphere, defined by $W = \bar{W}\pi^{-1}$. Restricting $\bar{W}$ to $\mathcal{D}$ identifies points (x,y) with $(1-x, 1-y)$, thus the points of ramification occur at $(0,0)$, $(1/2,1/2)$, $(1/2,0)$, $(0,1/2)$ and their integer translates. We take the lower half of $\mathcal{D}$ as a fundamental domain $\mathcal{L}$ for S^2 (see figure 3.10). W is the *Weierstrass function* [Ah, Han].

Consider the action of a matrix $A \in SL(2,\mathbf{Z})$ on $\mathbf{T}^2$. S is in the centre of $SL(2,\mathbf{Z})$, so A descends to a well-defined homeomorphism h_A of S^2 which fixes the point $x_\infty = \bar{W}(0,0)$ on S^2, and permutes the three points $x_1 = \bar{W}(1/2,0)$, $x_2 = \bar{W}(1/2,1/2)$ and $x_3 = \bar{W}(0,1/2)$. By blowing up x_∞ to a circle S_∞, we may consider h_A as a homeomorphism of the disc, on which $SL(2,\mathbf{Z})$ acts, permuting the three points which we shall also denote by x_1,x_2 and x_3. In fact, there exists an isomorphism $PSL(2,\mathbf{Z}) \to B_3/Z(B_3)$. The matrices

$$\begin{aligned} a_1 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ a_2 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \end{aligned} \tag{3.2}$$

generate $SL(2,\mathbf{Z})$ and correspond to the generators σ_1 and σ_2^{-1} of B_3. This is because a_1 and a_2 correspond to shears in $\mathbf{R}^2$ (see figure 3.11). It isn't hard to see that the action of a_1 and a_2 induce Dehn twists on $\mathbf{T}^2$, along $y = 1/2$ and $x = 1/2$ respectively, and

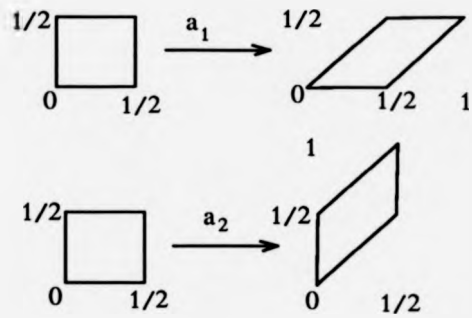

Figure 3.11: The shears a_1 and a_2

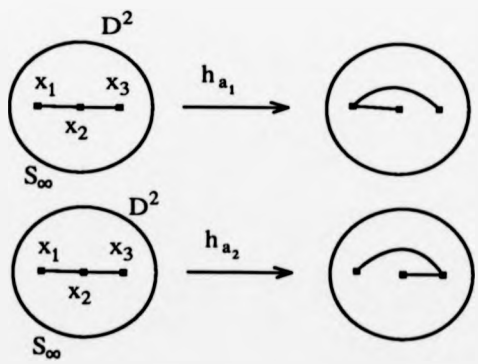

Figure 3.12: The induced action of a_1 and a_2 on $\mathbf{D}^2$

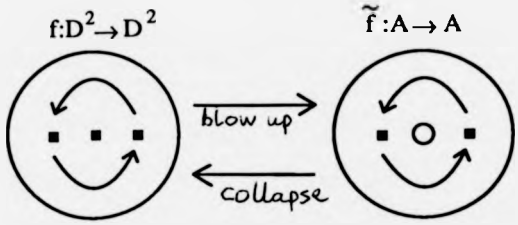

Figure 3.13: Defining a homeomorphism of the annulus from one of the disc, and vice-versa

induce the homeomorphisms $m(\sigma_1)$ and $m(\sigma_2^{-1})$ of Section 1.3.2 on S^2 (see figure 3.12). So the matrices a_1 and a_2 descend to give exactly the action of σ_1 and σ_2^{-1} on B_3, and conversely. Hence there exists an isomorphism between conjugacy classes in $PSL(2,\mathbb{Z})$ and braid types in $\mathbb{D}_3$, and so we may relate the dynamics of one to the other.

Suppose $A \in SL(2,\mathbb{Z})$, such that $|\text{Trace}(A)| > 2$, then A descends to an Anosov homeomorphism of $\mathbb{T}^2$. Further, its stable and unstable foliations descend to stable and unstable foliations on S^2, and there are one-pronged singularities at the ramification points; local expansion, contraction and transversality are preserved, and hence h_A : $\mathbb{D}^2 \to \mathbb{D}^2$ is a pseudo-Anosov homeomorphism relative to the 3 ramification points in $\text{Int}(\mathbb{D}^2)$. Conversely, for each such h_A, we can find $A \in PSL(2,\mathbb{Z})$ with $|\text{Trace}(A)| > 2$. Since the matrix corresponding to S is exactly $-I$ which corresponds to θ_3, the centre of B_3, we can ensure that h_A may be represented by a matrix $B \in SL(2,\mathbb{Z})$ with $\text{Trace}(B) > 2$. Further, there exist generators of $\mathbb{Z}^2$ with respect to which it is represented by a matrix whose elements are positive [Man].

3.6 Periodic orbits of period 2 of homeomorphisms of the annulus

3.6.1 Relative rotation numbers of periodic orbits

Suppose $f \in Homeo^+(\mathbb{D}^2, X)$, where $X = \{x_1, x_2, x_3\}$ is the set of points as defined in equation 1.2 with $n = 3$, and suppose f fixes x_2, swaps x_1 and x_3, and is pseudo-Anosov relative to X. By blowing up x_2, there exists an induced homeomorphism of the annulus $\tilde{f} : \mathbb{A} \to \mathbb{A}$, which is pseudo-Anosov relative to $\{x_1, x_3\}$. Conversely, given an orientation-preserving homeomorphism of the annulus isotopic to the identity, we may collapse the inner boundary of $\mathbb{A}$ to get a homeomorphism of the disc (see figure 3.13).

As we have seen in Section 3.5, we may find an element $A \in PSL(2,\mathbb{Z})$ which descends to f such that $\text{Trace}(A) > 2$ and the entries of A are positive. By a conjugacy if necessary, we may assume that the action of A restricted to $\mathcal{D}$ fixes $(1/2, 1/2)$ and swaps $(0, 1/2)$ and $(1/2, 0)$ (see figure 3.14). Thus the corresponding homeomorphism h_A fixes x_2 and x_∞, and swaps x_1 and x_3 (see figure 3.15). For A to have such an action, there are certain constraints. One may verify that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

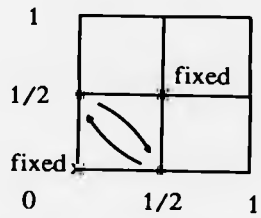

Figure 3.14: The action of A on $\mathbf{T}^2$

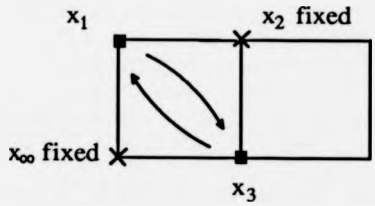

Figure 3.15: The action of A on S^2

then a and d are even, and b and c are odd.

So given A, how do we find the fixed points of f? There are two types of fixed points. Firstly, there are the fixed points of the induced homeomorphism of $\mathbf{T}^2$, and secondly, there are period 2 $\mathbf{Z}_2$-symmetric points which descend to fixed points on S^2.

So we look for points $\vec{x} \in \mathcal{L}$ satisfying either

1. $A\vec{x} = \vec{x} + \vec{m}, \quad \vec{m} \in \mathbf{Z}^2$.
2. $A\vec{x} = -\vec{x} + \vec{n}, \quad \vec{n} \in \mathbf{Z}^2$.

As is well-known (see [J]), the number of fixed points satisfying the first equation is exactly $|\det(A - I)|$, and the second is $|\det(A + I)|$. Since $\mathcal{D}$ is a 2-fold covering of $\mathcal{L}$, we have to divide each by 2, then there are two fixed points in common viz. $(0,0)$ and $(1/2, 1/2)$. So

$$\begin{aligned} \mathrm{Card}(\mathrm{Fix}(f)) &= \frac{|\det(A-I)|}{2} + \frac{|\det(A+I)|}{2} \\ &= \frac{\mathrm{Trace}(A) - 2}{2} + \frac{\mathrm{Trace}(A) + 2}{2} \\ &= \mathrm{Trace}(A). \end{aligned} \tag{3.3}$$

Two of the fixed points correspond to x_2 and x_∞. Let

$$\Delta = \{\vec{x} \in \mathcal{L} : \tilde{W}(\vec{x}) \in \mathrm{Fix}(f) \setminus \{x_2, x_\infty\}\}.$$

Given $\vec{x} \in \Delta$, we need to be able to keep track of its rotation number on the annulus $\mathbf{A}$ formed by blowing up x_2 and x_∞ on S^2, relative to some lift of $\tilde{f}$. One way of doing this is the following. Let $\mathcal{C}$ be a cut line on $\mathbf{A}$ as shown in figure 3.16 with a given orientation, such that the diagonal

$$\nabla = \{(t,t) : 0 \le t \le 1/2\} \subset \mathcal{L}$$

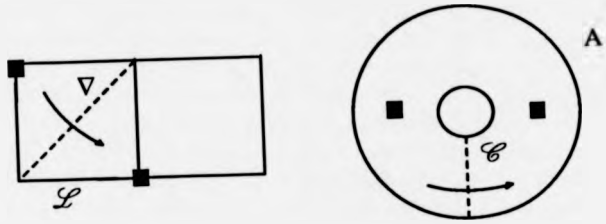

Figure 3.16: The cut line C on $\mathbf{A}$ with a chosen orientation

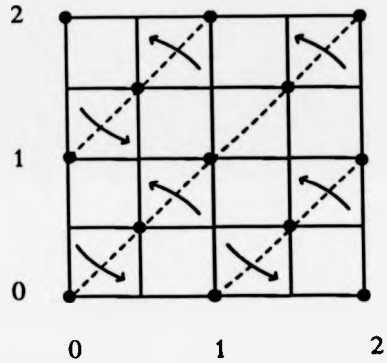

Figure 3.17: The plane $\mathbf{R}^2$ covered by copies of $\mathcal{L}$

descends to C, with the orientation shown. Glueing together $\mathbf{Z}$ copies of this cut annulus gives a canonical way of producing the covering space $\tilde{\mathbf{A}}$. Suppose α is an oriented path in $\mathbf{A}$, such that α and C are transverse at each point of $\alpha \cap C$, and such that this latter set has finite cardinality. Then we define the *winding number* $w(\alpha)$ of α to be the number of intersections of α with C counted algebraically i.e. count $+1$ for each intersection where the orientations of C and α agree, and -1 otherwise.

Suppose the endpoints of α are fixed points, then by comparing $w(\alpha)$ and $w(\tilde{f}(\alpha))$, one may compare their relative rates of rotation since the arcs lift in the obvious way to $\tilde{\mathbf{A}}$. Thus if we know the rotation number of one fixed point relative to some lift $\tilde{F} : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ of $\tilde{f}$, then we may calculate the rotation number of the other fixed point relative to $\tilde{F}$.

By covering $\mathbf{R}^2$ with copies of $\mathcal{L}$ with its oriented line segment ∇, we recover the situation in figure 3.17. Let $\tilde{\nabla}$ be the union of ∇ and all its translates under $< T_1, T_2, S >$. We may define a *winding number* $w'(\tilde{\beta})$ of an oriented path $\tilde{\beta}$ in the same way as on $\mathbf{A}$. We need $\tilde{\beta}$ to satisfy the following conditions.

1. $\tilde{\beta} \cap \mathbf{Z}^2 = \emptyset$.
2. $\tilde{\beta} \cap (\mathbf{Z}^2 + (1/2, 1/2)) = \emptyset$.
3. $\tilde{\beta}$ meets $\tilde{\nabla}$ transversally.

4. Card $(\tilde{\beta} \cap \tilde{\nabla})$ is finite.

Suppose $\tilde{\beta}$ descends to a path β on $\mathbf{A}$, then it is clear that $w'(\tilde{\beta}) = w(\beta)$, and so one may measure relative rotation numbers of points in $\mathrm{Fix}(\tilde{f})$ by studying the action of A on elements of Δ.

3.6.2 Calculations using the Weierstrass map

In this Section, we shall use the machinery described in Section 3.6.1 to give a proof of Theorem 3.1 in the case where $q = 2$. We then prove some more results for homeomorphisms of $\mathbf{D}_3$ which we use to determine the size of rotation sets of homeomorphisms of the annulus with periodic orbits of certain rotation numbers using flow-equivalence.

Suppose $A \in SL(2, \mathbf{Z})$ is as in Section 3.6.1. Then we have the following Proposition:

Proposition 3.12 *Trace*$(A) \equiv 0 \bmod 4$ *if and only if the rotation number of* $o(x_1)$ *relative to some lift of* f *is* $1/2$.

Proof

First note that Trace(A) is either congruent to 0 or 2 mod 4. As we have seen, there exist fixed points satisfying

$$A\vec{x} = \vec{x} + \begin{pmatrix} m \\ n \end{pmatrix} \quad m, n \in \mathbf{Z}.$$

Therefore

$$\vec{x} = \frac{1}{2-T} \begin{pmatrix} d-1 & -b \\ -c & a-1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} \text{ where } T = \text{ Trace}(A).$$

Choose $(m, n) = (1, 0)$, then

$$\vec{x} = \frac{1}{T-2} \begin{pmatrix} 1-d \\ c \end{pmatrix}$$

and

$$A^2\vec{x} = \vec{x} + (1, 0) + (a, c).$$

Also

$$A^2(1/2, 0) = ((aT-1)/2, cT/2).$$

Consider the straight line segment $\tilde{\beta}$ from $(1/2, 0)$ to $\vec{x}$: by comparing it with $A^2\tilde{\beta}$, we may calculate the rotation number of $\vec{x}$ relative to $(1/2, 0)$. Note that the set $\tilde{\nabla}$ consists of the diagonal given by $x = y$ on $\mathbf{R}^2$, and all its translates under T_1. To determine this relative rotation number, it suffices to calculate the quantity $w'(A^2(\tilde{\beta})) - w'(\tilde{\beta})$. Up to mod 2, this is given by the difference δ in the y- and x-coordinates of the vector

$$\begin{aligned} (A^2(\vec{x}) - A^2(1/2, 0)) \; &- \; (\vec{x} - (1/2, 0)) \\ &= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + (1 - \frac{T}{2}) \begin{pmatrix} a \\ c \end{pmatrix} \end{aligned}$$

which is

$$\delta = (c-a)(1-\frac{T}{2}) - 2.$$

Since a is even and c is odd, then $c-a$ is odd. Suppose $T \equiv 0 \bmod 4$, then δ is odd, but δ represents the translation in two iterates of a fixed point, relative to the period 2 orbit $o(x_1)$. Hence $o(x_1)$ could not have rotation number $0/2$ for some lift of $\tilde{f}$, since then we would need δ even (as we have iterated twice). So $o(x_1)$ must have rotation number $1/2$ for some lift of $\tilde{f}$. Similarly, $T \equiv 2 \bmod 4$ implies that $o(x_1)$ must have rotation number $0/2$ relative to some lift of $\tilde{f}$. □

Next, we verify Theorem 3.1 for the case where $q = 2$. More precisely, we prove:

Proposition 3.13 *Let $f : \mathbb{A} \to \mathbb{A}$ be an orientation-preserving homeomorphism of the annulus which preserves the boundary components, and let $o(x)$ be a $(1,2)$-periodic orbit (relative to some lift F of f) such that $bt(o(x), f)$ is pseudo-Anosov. Then the Farey interval $I(1/2) = [0,1]$ is contained within the rotation set of F.*

Proof

Let g be the pseudo-Anosov representative in the isotopy class of f relative to $o(x)$, let G be the lift of g such that $\rho(o(x), G) = 1/2$, and let $A \in SL(2,\mathbb{Z})$ be the associated toral automorphism. The idea is the following. Suppose α is an oriented arc joining $(1/2, 0)$ to a fixed point $\bar{x} \in \Delta$, then reflecting α in the point $(1/4, 1/4)$ on $\mathbb{R}^2$ gives an oriented arc α' which joins $(0, 1/2)$ to a point $\bar{y}$ which is also an element of Δ. The arcs α,α' are parallel but carry opposite orientation. Thus $A^2(\alpha)$ and $A^2(\alpha')$ are parallel, but since their orientations differ, $w'(A^2(\alpha)) = -w'(A^2(\alpha'))$. Since x_1 has rotation number $1/2$ relative to G, $w'(A^2(\alpha))$ must be odd, since $\bar{x}$ is a fixed point (as w' measures the rate of rotation of $\bar{x}$ relative to $(1/2, 0)$). In particular, $\bar{W}(\bar{x})$ must have rotation number

$$\frac{1 + w'(A^2(\alpha))}{2}$$

relative to G, and $\bar{W}(\bar{y})$ must have rotation number

$$\frac{1 + w'(A^2(\alpha'))}{2} = \frac{1 - w'(A^2(\alpha))}{2}.$$

Since $w'(A^2(\alpha)) \neq 0$, then there exist two fixed points, one whose rotation number is at least $1/1$, and the other is at most $0/1$. Since $\rho(G)$ is a closed interval,

$$\rho(G) \supset [0/1, 1/1] = I(1/2),$$

and Theorem 3.8 implies that $\rho(F) \supset \rho(G)$. All that remains now is to check that we can find suitable points $\bar{x}, \bar{y} \in \Delta$. Consider the points in Δ which project to period 2 $\mathbb{Z}_2$-symmetric points on $\mathbb{T}^2$. Such points satisfy

$$A\bar{x} = -\bar{x} + \begin{pmatrix} m \\ n \end{pmatrix} \quad m, n \in \mathbb{Z}. \tag{3.4}$$

Therefore

$$\bar{x} = \frac{1}{T+2} \begin{pmatrix} d+1 & -b \\ -c & a+1 \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Put $(m,n) = (1,0)$, then write

$$\vec{x}_1 = \frac{1}{T+2}\begin{pmatrix} 1+d \\ -c \end{pmatrix}$$

and similarly for $(m,n) = (0,1)$,

$$\vec{x}_2 = \frac{1}{T+2}\begin{pmatrix} -b \\ a+1 \end{pmatrix}.$$

Lemma 3.14 *Consider A as a map restricted to the domain $\mathcal{L}$. Then $\vec{x}_1, \vec{x}_2 \neq 0$, and at least one of $\vec{x}_1$ and $\vec{x}_2$ is not equal to $(1/2, 1/2)$.*

Proof
Since $a, d > 0$, $0 < a+1, d+1 < T+2$ so

$$0 < \frac{a+1}{T+2}, \frac{d+1}{T+2} < 1.$$

So $\vec{x}_1, \vec{x}_2 \neq 0$. Suppose $\vec{x}_1, \vec{x}_2$ are translates of $(1/2, 1/2)$ under $< T_1, T_2 >$. Then $(d+1)/(T+2) = 1/2$, so $a = d$ and $T+2 = 2(1+a)$. Further,

$$\frac{c}{T+2} = \frac{e}{2}, \quad \frac{b}{T+2} = \frac{f}{2}$$

for some $e, f \in \mathbb{N}$. Hence

$$A = \begin{pmatrix} a & f(1+a) \\ e(1+a) & a \end{pmatrix}$$

and $\det A = 1$ implies that

$$ef = \frac{a-1}{a+1}$$

and there are no integer solutions to the right-hand side with a strictly positive and even – a contradiction. □

Lemma 3.15 *Consider A as a map restricted to the domain $\mathcal{L}$, then at least one of $\vec{x}_1$ and $\vec{x}_2$ is not equal to $(3/4, 1/4)$ or $(1/4, 1/4)$.*

Proof
There are two cases to consider.

1. Suppose $a = d$, then $(d+1)/(T+2) = 1/2$ and so $\vec{x}_1, \vec{x}_2 \neq (3/4, 1/4)$ or $(1/4, 1/4)$.
2. Suppose $a \neq d$, and both are equal to either $(3/4, 1/4)$ or $(1/4, 1/4)$. Then

$$\frac{a+1}{T+2} = \frac{1}{4} \Leftrightarrow d+1 = 3(a+1) \Leftrightarrow \frac{d+1}{T+2} = \frac{3}{4}.$$

So

$$\begin{aligned} \frac{c}{T+2} &= \frac{c}{4(a+1)} = \frac{e}{4} \\ \frac{b}{T+2} &= \frac{b}{4(a+1)} = \frac{f}{4} \end{aligned}$$

where e, f are odd integers. Thus $d = 3a+2, c = e(a+1), b = f(a+1)$. $\mathrm{Det}(A) = 1$ implies that

$$ef = 3 - \frac{4}{a+1}$$

and there are no integer solutions to the right-hand side with a even and strictly positive – a contradiction. □

Lemma 3.16 *Let the point in Δ satisfying the conclusions of Lemmas 3.14 and 3.15 be $\vec{x}$, and let $\vec{y} = ((1/2, 1/2) - \vec{x})$. Then $\bar{W}(\vec{y}) \in Fix(g)$, and $\bar{W}(\vec{x}) \neq \bar{W}(\vec{y})$.*

Proof

$$\begin{aligned} A\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} - \vec{x}\right) &= \frac{1}{2}\begin{pmatrix} a+b \\ c+d \end{pmatrix} + \vec{x} - \begin{pmatrix} m \\ n \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} + \begin{pmatrix} m'-m \\ n'-n \end{pmatrix} - \vec{x} \end{aligned}$$

where

$$\frac{1}{2}(a+b) = \frac{1}{2} + m', \quad \frac{1}{2}(c+d) = \frac{1}{2} + n'.$$

Hence

$$A\vec{y} = -\vec{y} + (m'-m, n'-n)$$

where $m'-m$, $n'-n \in \mathbb{Z}$, and so $\bar{W}(\vec{y}) \in$ Fix(g). Since $\vec{x} \neq (3/4, 1/4)$ or $(1/4, 1/4)$, then the translate of $\vec{y}$ under $< T_1, T_2, S >$ is distinct from $\vec{x}$, and hence $\bar{W}(\vec{x}) \neq \bar{W}(\vec{y})$. □

One then makes the construction given on page 58, joining $(1/2, 0)$ to $\vec{x}$, and $(0, 1/2)$ to $\vec{y}$ with arcs α,α'. Thus the proof of Proposition 3.13 (the verification of Theorem 3.1 in the case $q = 2$) is completed. □

We now study the case in which the rotation type of the chosen period 2 orbit is $(0, 2)$ relative to some lift F of f. We have the following result.

Theorem 3.17 *Suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus isotopic to the identity, and suppose $o(x)$ is a period 2 orbit of f, such that $\rho(o(x), F) = 0$ for some lift $F : \bar{\mathbb{A}} \to \bar{\mathbb{A}}$ of f. Suppose further that $bt(o(x), f)$ is pseudo-Anosov. Then $\rho(F) \supset [-1, 1]$.*

Proof

Let g be the pseudo-Anosov homeomorphism in the isotopy class of f relative to $o(x)$, and let G be the lift of g such that $\rho(o(x), G) = 0$. Since $\rho(F) \supset \rho(G)$ by Theorem 3.8, it suffices to prove the Theorem for g. Let $A \in SL(2, \mathbb{Z})$ be a matrix whose action on $\mathbb{T}^2$ descends to that of g, as given in Section 3.6.1. From Proposition 3.12, Trace$(A) \equiv 2 \bmod 4$. Fixed points satisfy

$$A\vec{x} = \vec{x} + \vec{m}, \quad \vec{m} \in \mathbb{Z}^2 \tag{3.5}$$

so

$$\vec{x} = \frac{1}{T-2}\begin{pmatrix} 1-d & b \\ c & 1-a \end{pmatrix}\vec{m}$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

Lemma 3.18 *With A as above considered as a map restricted to $\mathcal{L}$, one of the following is true:*

1. $a + b \equiv 1 \bmod 4$, *then* $\vec{x} = (1/4, 1/4)$ *satisfies equation 3.5 for some* $m \in \mathbb{Z}^2$.
2. $a + b \equiv 3 \bmod 4$, *then* $\vec{x} = (3/4, 1/4)$ *satisfies equation 3.5 for some* $m \in \mathbb{Z}^2$.

Proof

Since a and d are even and b and c are odd, then $a + b \equiv 1$ or $3 \bmod 4$. As $T \equiv 2 \bmod 4$ and $ad - bc = 1$, it follows that

$$\begin{aligned} c + d &\equiv 1 \bmod 4 \text{ if } a + b \equiv 1 \bmod 4 \\ c + d &\equiv 3 \bmod 4 \text{ if } a + b \equiv 3 \bmod 4. \end{aligned}$$

So there are two cases to consider.

1. $a + b - 1$ and $c + d - 1$ are congruent to $0 \bmod 4$, so

$$A \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix} + \vec{m}_1$$

where

$$\vec{m}_1 = \begin{pmatrix} (a + b - 1)/4 \\ (c + d - 1)/4 \end{pmatrix} \in \mathbb{Z}^2.$$

2. $a + b$ is congruent to $3 \bmod 4$, so $b - a \equiv 3 \bmod 4$, and $d - c \equiv 1 \bmod 4$. Hence

$$A \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix} + \vec{m}_2$$

where

$$\vec{m}_2 = \begin{pmatrix} (b - a + 1)/4 \\ (d - c - 1)/4 \end{pmatrix} \in \mathbb{Z}^2.$$

Thus $(3/4, 1/4)$ is a fixed point of A considered as a map restricted to $\mathcal{L}$. □

We shall consider first the case where $a + b \equiv 1 \bmod 4$. Let α be the oriented straight line segment from $\tilde{x}_1 = (0, 1/2)$ to $\tilde{x}_3 = (1/2, 0)$. Then $w'(\alpha) = +1$. Since $x_1 = \tilde{W}(\tilde{x}_1)$ and $x_3 = \tilde{W}(\tilde{x}_3)$ are the two points of the period 2 orbit of g, it follows that $\rho(x_1, G^2) = \rho(x_3, G^2)$, and hence $w'(A^2(\alpha)) = w'(\alpha)$. But α consists of two subsegments, of which the midpoint $(1/4, 1/4)$ is a fixed point of A by Lemma 3.18 (considered as a map restricted to $\mathcal{L}$). It follows by looking at the behaviour of $A^2(\alpha)$ that $\tilde{W}(1/4, 1/4)$ must have rotation number $\rho(\tilde{W}(1/4, 1/4), G) = 0$ (since the action on the two subsegments is symmetric). This gives us a fixed point with respect to which we can measure the rotation numbers of other fixed points.

So suppose $\vec{y} \in \Delta$, then by linearity, it follows that $\vec{y}' = \vec{y} - (1/4, 1/4) \in \Delta$. Using an argument similar to that of the proof of Proposition 3.13 on page 58, one sees that

$\bar{W}(\vec{y})$ and $\bar{W}(\vec{y}')$ have equal and opposite rotation numbers, so it suffices to show that there exists a point $\bar{W}(\vec{y})$ with non-zero rotation number (as then $\bar{W}(\vec{y})$ and $\bar{W}(\vec{y}')$ must be distinct).

Suppose $\vec{y} \in \mathbf{R}^2$ satisfies

$$\vec{y} = \frac{1}{T-2}\begin{pmatrix} 1-d & b \\ c & 1-a \end{pmatrix}\vec{m}, \text{ for some } \vec{m} \in \mathbf{Z}^2.$$

Then $\vec{y} \in \Delta$. Let

$$\vec{m} = \begin{pmatrix} (a+b-1)/4+1 \\ (c+d-1)/4 \end{pmatrix}$$

then

$$\vec{y} = \begin{pmatrix} \frac{1}{4} + \frac{1-d}{T-2} \\ \frac{1}{4} + \frac{c}{T-2} \end{pmatrix}.$$

So

$$\{A\vec{y} - A\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}\} - \{\vec{y} - \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and the differences between the y- and x-coordinates of this gives the rotation number of $\bar{W}(\vec{y})$ relative to $\bar{W}(1/4, 1/4)$ up to mod2. Hence $\bar{W}(\vec{y})$ has non-zero rotation number relative to $\bar{W}(1/4, 1/4)$, and thus has non-zero rotation number relative to G.

Let

$$\vec{z} = \begin{pmatrix} 1/4 + b/(T-2) \\ 1/4 + (1-a)/(T-2) \end{pmatrix},$$

then $A\vec{z} = \vec{z} + \vec{n}$, where

$$\vec{n} = \begin{pmatrix} (a+b-1)/4 \\ (c+d-1)/4+1 \end{pmatrix}$$

and

$$\{A\vec{z} - A\begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}\} - \{\vec{z} - \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}\} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Similarly, $\bar{W}(\vec{z})$ has non-zero rotation number relative to G. Now we just have to check that one of $\bar{W}(\vec{y})$ and $\bar{W}(\vec{z})$ is not an element of $\{\bar{W}(0,0), \bar{W}(1/2,1/2), \bar{W}(1/4,1/4)\}$. Since their rotation numbers are non-zero relative to G, neither can be $\bar{W}(1/4,1/4)$. Suppose both are in $\{\bar{W}(0,0), \bar{W}(1/2,1/2)\}$, then it follows that $b/(T-2)$, $(1-a)/(T-2)$, $(1-d)/(T-2)$, $c/(T-2)$ are all odd multiples of $1/4$. Since $0 < a-1, d-1 < T-2$, one of $(a-1)/(T-2)$ or $(d-1)/(T-2)$ is equal to $1/4$, and the other is equal to $3/4$. Suppose $(d-1)/(T-2) = 1/4$ (the other case is exactly the same), then $a = 3d-2$. Also $c/(T-2) = k/4$, $b/(T-2) = l/4$, for $l,k \in \mathbf{Z}$. Thus $c = k(d-1)$ and $b = l(d-1)$. $\mathrm{Det}(A) = 1$ implies that $kl = 3 + 4/(d-1)$, and the only solutions satisfying all the conditions is when $d = 2$: then $a = 4$, and

$$A = \begin{pmatrix} 4 & 1 \\ 7 & 2 \end{pmatrix}.$$

In this case, we can find the corresponding braid type and check that it actually has fixed points with the required rotation numbers, using a representation of B_3 into the fundamental group of D_3 (see Proposition 5.3). Otherwise, one of $\bar{W}(\vec{y})$ and $\bar{W}(\vec{z})$ is not

an element of $\{\bar{W}(0,0), \bar{W}(1/2,1/2)\}$, and by taking the point $\bar{y}$ or $\bar{z}$ as appropriate, and the corresponding point $(1/4,1/4)-\bar{y}$ or $(1/4,1/4)-\bar{z}$, we have constructed two fixed points with non-zero, equal and opposite rotation numbers relative to G. Since $\rho(G)$ is a closed interval, $\rho(G) \supset [-1,1]$.

In the case where $a+b \equiv 3 \bmod 4$, we repeat this process with the following modifications. Let α be the oriented line segment from $\bar{x}_1 = (0,1/2)$ to $(-1/2,0)$ (note that $\bar{W}(-1/2,0) = x_3$). Then $w'(\alpha) = 0$, and so we see that $\rho(\bar{W}(-1/4,1/4),G) = 0$. Let

$$\vec{m} = \begin{pmatrix} (b-a+1)/4+1 \\ (d-c+1)/4 \end{pmatrix},$$

so

$$\vec{y} = \begin{pmatrix} \frac{-1}{4} + \frac{1-d}{T-2} \\ \frac{1}{4} + \frac{c}{T-2} \end{pmatrix},$$

and

$$\{A\vec{y} - A\begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}\} - \{\vec{y} - \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}\} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let

$$\vec{n} = \begin{pmatrix} (b-a+1)/4 \\ (d-c+1)/4+1 \end{pmatrix},$$

so

$$\vec{z} = \begin{pmatrix} \frac{-1}{4} + \frac{b}{T-2} \\ \frac{1}{4} + \frac{1-a}{T-2} \end{pmatrix},$$

thus

$$\{A\vec{z} - A\begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}\} - \{\vec{z} - \begin{pmatrix} -1/4 \\ 1/4 \end{pmatrix}\} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We complete the calculation as above, finding that the only allowed solution where $\bar{W}(\bar{y})$ and $\bar{W}(\bar{z})$ are both elements of $\{\bar{W}(0,0), \bar{W}(1/2,1/2), \bar{W}(1/4,1/4)\}$ is when

$$A = \begin{pmatrix} 4 & 7 \\ 1 & 2 \end{pmatrix}$$

which is also dealt with by Proposition 5.3. Otherwise by taking the point $\bar{y}$ or $\bar{z}$ as appropriate, and the corresponding point $(-1/4,1/4)-\bar{y}$ or $(-1/4,1/4)-\bar{z}$, $\rho(G) \supset [-1,1]$, and the proof of Theorem 3.17 is complete. □

We now consider the case where the points $x_1, x_3 \in \mathbb{A}$ are fixed by the homeomorphism.

Theorem 3.19 *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus isotopic to the identity, and suppose x_1, x_3 are two fixed points of f, such that $bt(\{x_1,x_3\},f)$ is pseudo-Anosov, and such that for some lift $F : \bar{\mathbb{A}} \to \bar{\mathbb{A}}$ of f, $\rho(x_1,F) = \rho(x_3,F) = 0$. Then $\rho(F) \supset [-1,1]$.*

Proof

Let g be the pseudo-Anosov homeomorphism in the isotopy class of f relative to $\{x_1,x_3\}$, and let G be the lift of g such that $\rho(x_1,G) = \rho(x_3,G) = 0$. Since $\rho(F) \supset \rho(G)$ by Theorem 3.8, it suffices to prove the Theorem for g. Let $A \in SL(2,\mathbb{Z})$ be the matrix

defined in Section 3.6.1 which descends to g except that now x_1 and x_3 are fixed points. So the restriction of A to $\mathcal{D}$ fixes $\vec{x}_1 = (0, 1/2)$, $\vec{x}_2 = (1/2, 1/2)$ and $\vec{x}_3 = (1/2, 0)$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

it follows that a and d are odd, and b and c are even. Then we argue in the usual manner. Suppose $\vec{x}$ satisfies

$$A\vec{x} = \vec{x} + \vec{m}, \quad \vec{m} \in \mathbb{Z}^2,$$

then $\vec{x} \in \Delta$. Consider the oriented straight line segments α from $\vec{x}_1$ to $\vec{x}$, and α' from $\vec{x}_3$ to $\vec{y} = (1/4, 1/4) - \vec{x}$. Then α and α' are parallel and of equal length, but carry opposite orientation. Then the winding numbers $w'(A^i(\alpha))$ and $w'(A^i(\alpha'))$ are equal and opposite for all $i \in \mathbb{Z}$. So if $w'(A(\alpha)) - w'(\alpha)$ is non-zero, then $\tilde{W}(\vec{x})$ has non-zero rotation number relative to x_1, and hence relative to G. Since x_3 has the same rotation number as x_1 relative to G, and $A^i(\alpha)$ and $A^i(\alpha')$ carry opposite orientations, it follows that $\tilde{W}(\vec{x})$ and $\tilde{W}(\vec{y})$ have equal and opposite rotation numbers relative to G, and hence $\rho(G) \supset [-1, 1]$. Thus it just remains to exhibit a suitable point $\vec{x}$.

Let

$$\vec{z}_1 = \left(\frac{1}{2} + \frac{b}{T-2}, \frac{1-a}{T-2}\right),$$

then

$$A\vec{z}_1 = \vec{z}_1 + \frac{1}{2}\begin{pmatrix} a-1 \\ c+2 \end{pmatrix},$$

and let

$$\vec{z}_2 = \left(\frac{1-d}{T-2}, \frac{1}{2} + \frac{c}{T-2}\right),$$

then

$$A\vec{z}_2 = \vec{z}_2 + \frac{1}{2}\begin{pmatrix} b+2 \\ d-1 \end{pmatrix}.$$

One may verify that

$$(A\vec{z}_1 - A\vec{x}_1) - (\vec{z}_1 - \vec{x}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.6}$$

and

$$(A\vec{z}_2 - A\vec{x}_3) - (\vec{z}_2 - \vec{x}_3) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.7}$$

As we have seen, the difference in the y- and x-coordinates of equations 3.6 and 3.7 gives the rotation number mod 2 of $\tilde{W}(\vec{z}_1)$ relative to x_1, and $\tilde{W}(\vec{z}_2)$ relative to x_3 respectively. We then argue as in Lemma 3.14 and Lemma 3.15 to show that if we consider A as a map restricted to the domain $\mathcal{L}$, then at least one of $\vec{z}_1$ and $\vec{z}_2$ isn't in the set $\{(0,0), (1/2, 1/2), (3/4, 1/4), (1/4, 1/4)\}$, and we choose such a point to be our representative $\vec{x}$. □

As an immediate application of our analysis, we prove the following Theorem.

Theorem 3.20 *Suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus isotopic to the identity, with a periodic orbit $o(x)$ of period $2q$, where $q \geq 2$, such that $bt(o(x), f)$ is pseudo-Anosov, and such that there exists a lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ for which $o(x)$ is a $(2p, 2q)$-periodic orbit of F, where $0 < p/q < 1$, and p and q are coprime. Let the Farey interval be $I(p/q) = [k_1/l_1, k_2/l_2]$, then $\rho(F)$ contains one of the sets $[(k_1 + p)/(l_1 + q), k_2/l_2]$ or $[k_1/l_1, (k_2 + p)/(l_2 + q)]$.*

Proof

Let $g \in Homeo^+(\mathbb{A}, o(x))$ be the pseudo-Anosov representative in the isotopy class of f relative to $o(x)$. Choose the lift G of g which is equivariantly homotopic to F. Since $\rho(G) \subset \rho(F)$ by Theorem 3.8, it suffices to prove the Theorem for G. Suppose $\rho(G)$ is the closed interval $[r/s, t/u]$, and suppose $r/s > k_1/l_1$ i.e. $rl_1 - k_1 s > 0$. Take a suspension ϕ of g such that ϕ is associated with G in the manner described in Section 3.3; if g_1 and $g_2 \in H_1(\mathbb{A} \times S^1; \mathbb{Z})$ are the standard generators of homology given there, then we may choose different generators g_1' and g_2', where

$$\begin{pmatrix} g_1' \\ g_2' \end{pmatrix} = M \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

with

$$M = \begin{pmatrix} -k_1 & -l_1 \\ p & q \end{pmatrix} \in SL(2, \mathbb{Z}).$$

If $m/n \in \rho(G)$, then $m/n > k_1/l_1$, and if $(m', n') = (m, n).M^{-1}$, then $n' = ml_1 - nk_1 > 0$. Theorem 3.3 implies that the change of basis corresponds to a new cross-section, such that the $(2p, 2q)$-periodic orbit is transformed to a $(0, 2)$-periodic orbit. Hence the return map to the new cross-section is a homeomorphism $h : \mathbb{A} \to \mathbb{A}$ which is pseudo-Anosov relative to a period 2 orbit, whose rotation number is zero relative to the lift $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ (associated with the suspension ϕ). But Theorem 3.17 states that $\rho(H) \supset [-1, 1]$. Since $(-1, 1).M = (k_1 + p, l_1 + q)$ and $(1, 1).M = (-k_1 + p, -l_1 + q) = (k_2, l_2)$, then $\rho(G) \supset [(k_1 + p)/(l_1 + q), k_2/l_2]$. The case where $t/u < k_2/l_2$ is similar, with

$$M = \begin{pmatrix} k_2 & l_2 \\ p & q \end{pmatrix}$$

yielding the result that $\rho(G) \supset [k_1/l_1, (p + k_2)/(q + l_2)]$. □

3.7 Flow-equivalence and the 2-torus

3.7.1 Introduction

As we have seen in Section 3.4, we may use number-theoretic properties to give a lower bound on the size of the rotation set for a homeomorphism of the annulus isotopic to the identity which has a periodic orbit of pseudo-Anosov braid type, and which is a (p, q)-periodic orbit of some lift, where p and q are coprime. The reason this is possible is because if there were no lower bound on the size of the rotation set, we could flow-equivalence it to a homeomorphism of the annulus which is pseudo-Anosov relative to a single fixed point, which gives a contradiction. In the case of the 2-torus we may gain some insight into rotation sets of homeomorphisms using flow-equivalence,

since there exist no homeomorphisms of $\mathbf{T}^2$ isotopic to the identity which are pseudo-Anosov relative to a single fixed point (we shall see this later in Theorem 5.6). These techniques do not give as much information for the 2-torus as they do for the annulus, for instance, there is no unique 'minimal' rotation set, but we can get a reasonable idea of the structure of the rotation sets.

Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism of the 2-torus isotopic to the identity and let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of f. Suppose $o(x)$ is a periodic orbit of f such that $bt(o(x), f)$ is pseudo-Anosov. In analogy with the case of the annulus, we may ask the following questions.

1. Is $\rho(o(x), F) \in \text{Int } \rho(F)$?
2. What is the analogue of the Farey interval, and does it provide a 'smallest' rotation set?
3. Let $\mathcal{M}_{p/q}$ be the set of lifts of homeomorphisms of $\mathbf{T}^2$ which possess a periodic orbit whose braid type is pseudo-Anosov and whose rotation type is (p, q) relative to that lift. Then does the set $\cap \rho(G)$ where the intersection runs over $G \in \mathcal{M}_{p/q}$ have a non-empty interior?

In this and the following Section, we shall discuss these questions, and provide some possible answers to them.

3.7.2 Reformulation of flow-equivalence

In order to make progress with the problems raised above, we shall reformulate slightly the ideas contained in the proof of Theorem 3.1 in Section 3.4. Suppose $f : \mathbf{A} \to \mathbf{A}$ is a homeomorphism of the annulus isotopic to the identity, let $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ be a lift of f, and let $o(x)$ be a periodic orbit of F of rotation type (p, q) such that p and q are coprime, and f is pseudo-Anosov relative to $o(x)$. Let $\phi : M_f \to M_f$ be the suspended flow such that ϕ is associated with the lift F as in Section 3.3. As we noted in Section 3.2.2, the set $p(D_\phi)$ of projected homology directions is a subset of S^1 contained in the upper half plane. If $\mathcal{P}$ is any hyperplane in $\mathbf{R}^2$ of rational gradient passing through the origin such that $p(D_\phi)$ lies entirely on one side of $\mathcal{P}$, then $\mathcal{P}$ corresponds to a different choice of cross-section to the flow ϕ. Let $u_{\mathcal{P}} \in H^1(M_f; \mathbf{Z})$ be the class dual to $\mathcal{P}$, and recall that $u_{\mathcal{P}}(\phi(o(x)))$ is the (algebraic) cardinality of $\phi(o(x)) \cap \mathcal{P}$. By taking standard bases for $H_1(M_f; \mathbf{Z})$ and $H^1(M_f; \mathbf{Z})$, we find that $[\phi(o(x))] = \begin{pmatrix} p \\ q \end{pmatrix}$ and $u_{\mathcal{P}} = (m, n)$, where m,n are coprime integers, and $\mathcal{P}$ has gradient $-n/m \in \mathbf{Q}$. Given a closed loop γ in M, let the corresponding homology direction $[\gamma] = \begin{pmatrix} r \\ s \end{pmatrix} \in H_1(M_f, \mathbf{Z})$. Then

$$u_{\mathcal{P}}[\gamma] = (m, n).\begin{pmatrix} r \\ s \end{pmatrix} = mr + ns$$

is the (minimum) cardinality of the set $\gamma \cap \mathcal{P}$. In particular, if we consider the set

$$\mathcal{C}_0 = \{u_{\mathcal{Q}} : u_{\mathcal{Q}} \in H^1(M; \mathbf{Z}), u_{\mathcal{Q}}.[\phi(o(x))] = 1\}$$

then Proposition 3.11 implies that $\rho(F)$ must meet all hyperplanes $\mathcal{Q}$ such that $u_{\mathcal{Q}} \in \mathcal{C}_0$ is dual to $\mathcal{Q}$. Since $\rho(F)$ is an interval, we see as in Section 3.4 that $\rho(F) \supset I(p/q)$. More exactly,

Proposition 3.21 *Suppose $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus isotopic to the identity, let $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of f, and suppose $o(x)$ is a periodic orbit of F-rotation type (p,q), where p,q are relatively prime integers, such that $bt(o(x), f)$ is pseudo-Anosov. Let*

$$\mathcal{L}_{(p,q)} = \{(m,n) \in \mathbb{Z}^2 : mp + nq = 1\}. \tag{3.8}$$

Then $-n/m \in \rho(F)$ for all $(m,n) \in \mathcal{L}_{(p,q)}$. □

Thus for any $(m,n) \in \mathbb{Z}^2$ satisfying

$$(m,n).\begin{pmatrix} p \\ q \end{pmatrix} = 1, \tag{3.9}$$

we know that $\rho(F)$ intersects the point $\alpha = -n/m$. The set $\mathcal{L}_{(p,q)}$ determines a set of hyperplanes which meet $\rho(F)$.

For any coprime pair (p,q), we can relate $\mathcal{L}_{(p,q)}$ to $\mathcal{L}_{(1,2)}$. First we fix $A \in SL(2,\mathbb{Z})$ as in Proposition 3.10, so that $A\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Take $(m,n) \in \mathbb{Z}^2$ satisfying equation 3.9, then

$$(m,n).A^{-1}.\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1,$$

and thus

$$(m',n').\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1,$$

where

$$(m',n') = (m,n).A^{-1} \in \mathbb{Z}^2. \tag{3.10}$$

The set $\mathcal{L}_{(p,q)}$ consists of all rationals $-n/m$. All pairs (m,n) satisfying equation 3.9 lie in $I(p/q)$, and since A maps $I(p/q)$ to $I(1/2)$ bijectively, there exists a bijection between pairs (m,n) satisfying equation 3.9 and pairs (m',n') satisfying equation 3.10. Hence the set $\mathcal{L}_{(1,2)}$ consists of all rationals $-n'/m'$, which we know in terms of $\mathcal{L}_{(p,q)}$ from equation 3.10. We sum this up as:

Theorem 3.22 *Let p,q be coprime integers and let $\mathcal{L}_{(p,q)}$ be the set defined by equation 3.8. Let $A \in SL(2,\mathbb{Z})$ be a matrix providing a bijection between $I(p/q)$ and $I(1/2)$ as above. Then*

$$\mathcal{L}_{(1,2)} = \mathcal{L}_{(p,q)}.A^{-1}.$$

□

This Theorem is important because it may be generalized to higher dimensions, where direct calculation of the hyperplanes which rotation sets must intersect is more difficult. We shall see presently how we may calculate these hyperplanes via a specific example.

3.8 Rotation sets for homeomorphisms of $\mathbb{T}^2$

3.8.1 Rotation sets and isotopy classes

In this Section, we shall use similar arguments to those described in Section 3.7.2 to calculate the analogue of the set $\mathcal{L}_{(p,q)}$ in the case of $Isot(\mathbb{T}^2)$. Given $(p,q,r) \in \mathbb{Z}^2 \times \mathbb{N}$,

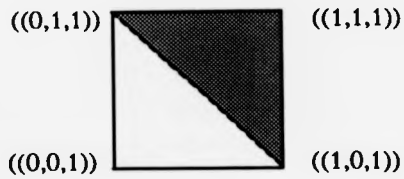

Figure 3.18: The level-0 Farey square

where (p, q, r) is primitive, we define an analogous set $\mathcal{L}_{(p,q,r)}$ and show how to relate it to the set $\mathcal{L}_{(1,1,2)}$. It is possible to determine the latter set, and so deduce the structure of $\mathcal{L}_{(p,q,r)}$.

Let $f : \mathbf{T}^2 \to \mathbf{T}^2$ be a homeomorphism of the 2-torus isotopic to the identity, and let $F : \mathbf{R}^2 \to \mathbf{R}^2$ be a lift of f. Suppose $o(x)$ is a periodic orbit of F-rotation type $(p, q) \in \mathbf{Z}^2 \times \mathbf{N}$, where (p, q) is primitive (see Section 2.5 for the definitions of these terms). Further, suppose f is pseudo-Anosov relative to $o(x)$. If $g \simeq f$ rel. $o(x)$, and G is the lift of g such that $o(x)$ is of G-rotation type (p, q), then by global shadowing for f, $\rho(G) \supset \rho(F)$. Since we are looking for 'smallest' rotation sets, it is sensible to assume that our homeomorphism is pseudo-Anosov relative to the chosen periodic orbit. It is a consequence of Theorem 3.6 that $\rho(F)$ is a convex set, and its extreme points are rational. In order to uncover the possible form of $\rho(F)$, we describe the analogue of the Farey interval.

3.8.2 The Farey square

The construction of the Farey square is a natural generalization of that of the Farey interval given in Section 3.4. We consider rational pairs $(p/r, q/r) \in \mathbf{Q}^2$ where (p, q, r) is primitive, which we associate with the triple $((p, q, r))$. We define a 'mediant' operation $\oplus$ as before, given by:

$$((p_1, q_1, r_1)) \oplus ((p_2, q_2, r_2)) = ((p_1 + p_2, q_1 + q_2, r_1 + r_2)).$$

Using this operation, we construct a lattice of rational pairs as follows (see [KiO] for further details).

1. We begin with the unit square whose corners are the triples $((0, 0, 1))$, $((1, 0, 1))$, $((1, 1, 1))$ and $((0, 1, 1))$, as shown in figure 3.18. We divide this square into two triangles by joining $((1, 0, 1))$ and $((0, 1, 1))$. We call these two triangles level-0 Farey triangles, and their union the level-0 Farey square. The rationals in the upper (respectively lower) Farey triangle are those (primitive) $((p, q, r))$ for which $p + q \geq r$ (respectively $p + q \leq r$) and $0 \leq p, q \leq r$. For ease of description, we shall just consider the upper level-0 Farey triangle (shaded in figure 3.18) in all that follows. We simply reflect the construction given below in the (common) hypotenuse of the triangles to recover the construction in the lower level-0 Farey triangle.

2. We apply the mediant operation on the vertices of the hypotenuse of the Farey triangle, putting $((1, 1, 2)) = ((1, 0, 1)) \oplus ((0, 1, 1))$ at the midpoint of the hypotenuse. The rational $((1, 1, 2))$ defines the level-0 Farey element of the lattice.

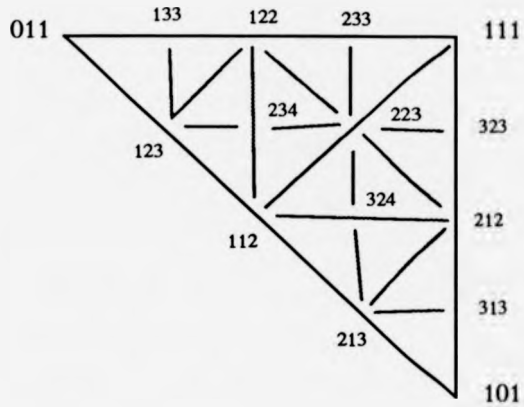

Figure 3.19: The Farey lattice up to level 3

3. Join $((1,1,1))$ to $((1,1,2))$ by a directed line segment; this divides the level-0 Farey triangle into two similar level-1 Farey triangles.

4. Carry out the mediant operation on the two hypotenuses of the level-1 Farey triangles to obtain $((2,1,2))$ and $((1,2,2))$ as level-1 elements, joining the level-0 Farey element to the level-1 element by a directed line segment.

5. Carry out steps 3 and 4 recursively to obtain successive levels of Farey elements and Farey triangles.

For example, the elements of the lattice up to level 3 are shown in figure 3.19.

Remarks

1. In this way, the whole lattice is produced.

2. Each pair of primitive rationals appears precisely once in the lattice, and those pairs which are not on the boundary of the level-0 Farey square i.e. those for which $0 < p, q < r$, can be reached by two different paths. For such pairs of rationals, there are two associated Farey triangles; if $((p,q,r))$ is a level-n Farey element, then there exists exactly two level-n Farey triangles whose hypotenuses contain $((p,q,r))$. We call their union the *Farey square* of $((p,q,r))$.

3. Each level-n triangle is the union of two level-$(n+1)$ triangles. We may generate a binary lattice, which we visualize in the following way. Given a Farey element, choose one path described by the above algorithm. We start from $((1,1,1))$ and follow the path to $((1,1,2))$. We assign a '0' if the path turns $3\pi/4$ to the right at $((1,1,2))$, and a '1' if the path turns $3\pi/4$ to the left. We continue this process, assigning a '0' or a '1' according to whether the path turns right or left, until we reach the given Farey element. For example, the binary address of $((3,3,4))$ is 0001 if we choose the path shown in figure 3.20. Since any Farey element $((p,q,r))$ satisfying $0 < p, q < r$ has two distinct paths leading to it, there are two distinct binary addresses associated with it.

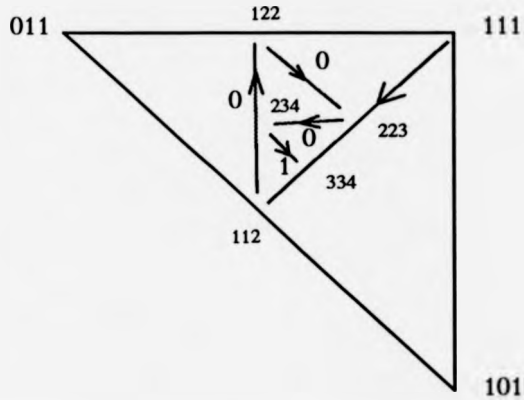

Figure 3.20: The path corresponding to the binary address 0001

Let

$$S_0 = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

then we may verify that:

1. $S_0, S_1 \in SL(3, \mathbb{Z})$.
2. $S_0 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $S_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $S_0 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.
3. $S_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $S_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $S_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Hence S_0 maps one level-1 Farey triangle bijectively to the level-0 Farey triangle, and S_1 does likewise with the other level-1 Farey triangle. Inductively, it follows that there exists a matrix $A \in SL(3, \mathbb{Z})$ such that if $((p, q, r))$ has binary address $I_0 I_1 \ldots I_N$ then [KiO]

1. $A = S_{I_N} \ldots S_{I_1} S_{I_0}$.
2. A maps bijectively the Farey triangle associated with the address $I_0 I_1 \ldots I_N$ to the level-0 Farey triangle.

Thus

Proposition 3.23 *Given $(p, q, r) \in \mathbb{Z}^2 \times \mathbb{N}$ a primitive triple and Δ one of the Farey triangles associated with the Farey element $((p, q, r))$, there exists a matrix $A \in SL(3, \mathbb{Z})$ such that $A \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Further A maps Δ to the level-0 Farey triangle bijectively, mapping corners to corners and edges to edges.* □

Given $(p,q,r) \in \mathbb{Z}^2 \times \mathbb{N}$ a primitive triple, define

$$\mathcal{L}_{(p,q,r)} = \{(l,m,n) \in \mathbb{Z}^3 : lp + mq + nr = 1\}.$$

Theorem 3.24 *Suppose $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism of the torus isotopic to the identity, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of f, and suppose $o(x)$ is a periodic orbit of F-rotation type (p,q,r) such that f is pseudo-Anosov relative to $o(x)$. Given any $(l,m,n) \in \mathcal{L}_{(p,q,r)}$, the rotation set $\rho(F)$ intersects the line*

$$lx_1 + mx_2 + n = 0 \tag{3.11}$$

in (x_1, x_2)-space.

Proof

Suppose not. Let $\phi : M_f \to M_f$ be the suspended flow associated with the lift F. The set $p(D_\phi)$ of projected homology directions is a subset of $S^2 \subset \mathbb{R}^3$ contained in the region $z > 0$. Let $\mathcal{P}$ be the hyperplane in $\mathbb{R}^3$ passing through the origin given by

$$(x,y,z) \in \mathcal{P} \Leftrightarrow lx + my + nz = 0.$$

Writing $x_1 = x/z$, $x_2 = y/z$, $\mathcal{P} \cap p(D_\phi) = \emptyset$ since we have assumed that $\rho(F)$ does not meet the line given by equation 3.11. As f is pseudo-Anosov relative to $o(x)$, $\rho(F)$ is a convex, connected set, and so $p(D_\phi)$ lies entirely on one side of $\mathcal{P}$. Theorem 3.3 implies that $\mathcal{P}$ corresponds to a new choice of crosss-section. Let $u_\mathcal{P} \in H^1(M_f;\mathbb{Z})$ be the class dual to $\mathcal{P}$. Taking standard bases for $H_1(M_f;\mathbb{Z})$ and $H^1(M_f;\mathbb{Z})$, we may write $u_\mathcal{P} = (l,m,n)$. But $u_\mathcal{P}(\phi(o(x))) = 1$ as $(l,m,n) \in \mathcal{L}_{(p,q,r)}$, so $\phi(o(x))$ intersects $\mathcal{P}$ precisely once. Let $g : \mathbb{T}^2 \to \mathbb{T}^2$ be the return map of $\mathcal{P}$ for ϕ. Theorem 3.5 implies that g is pseudo-Anosov relative to a single (fixed) point. However, g is isotopic to the identity, and as we shall see in Theorem 5.6, there are no homeomorphisms of the torus isotopic to the identity which are pseudo-Anosov relative to a single point. Hence we have reached a contradiction. □

So to find the possible structure of the rotation set $\rho(F)$, we must find the set $\mathcal{L}_{(p,q,r)}$. Choose $A \in SL(3,\mathbb{Z})$ as in Proposition 3.23, then $A\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. If $(l,m,n) \in \mathcal{L}_{(p,q,r)}$,

$$\begin{aligned} (l,m,n).\begin{pmatrix} p \\ q \\ r \end{pmatrix} &= 1 \\ \text{so } (l',m',n').\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} &= 1, \end{aligned}$$

where $(l',m',n') = (l,m,n).A^{-1} \in \mathcal{L}_{(1,1,2)}$. The converse is also true i.e. if $(l',m',n') \in \mathcal{L}_{(1,1,2)}$ then $(l,m,n) = (l',m',n').A \in \mathcal{L}_{(p,q,r)}$. So

Theorem 3.25 *Let $(p,q,r) \in \mathbb{Z}^2 \times \mathbb{N}$ be a primitive triple, and let $A \in SL(3,\mathbb{Z})$ be as defined in Proposition 3.23. Then $\mathcal{L}_{(1,1,2)} = \mathcal{L}_{(p,q,r)}.A^{-1}$.* □

Notice that this is a direct generalization of Theorem 3.22. So to find $\mathcal{L}_{(p,q,r)}$, it suffices to identify $\mathcal{L}_{(1,1,2)}$.

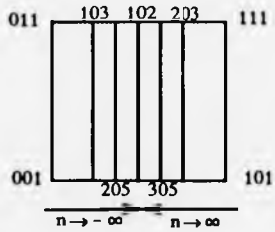

Figure 3.21: The set of lines $x_1 = -\frac{n}{l} = \frac{n}{2n-1}$.

3.8.3 The set $\mathcal{L}_{(1,1,2)}$

We now proceed with the classification of the set $\mathcal{L}_{(1,1,2)}$. More precisely, we would like to find the set of lines satisfying

$$lx_1 + mx_2 + n = 0 \tag{3.12}$$

in (x_1, x_2)-space such that $(l, m, n) \in \mathcal{L}_{(1,1,2)}$. Theorem 3.24 implies that if f is a homeomorphism of the torus isotopic to the identity possessing a periodic orbit of rotation type $(1, 1, 2)$ relative to some lift F of f, and if f is pseudo-Anosov relative to this periodic orbit, then $\rho(F)$ must meet all the lines given by equation 3.12. Suppose $(l, m, n) \in \mathcal{L}_{(1,1,2)}$, then

$$l + m + 2n = 1.$$

Combining this with equation 3.12,

$$lx_1 + (1 - l - 2n)x_2 + n = 0.$$

We consider two cases.

1. $m = 0$, then $l + 2n = 1$, and the pairs (l, n) are those as for $\mathcal{L}_{(1,2)}$, with

$$x_1 = -\frac{n}{l} = \frac{n}{2n-1}.$$

   So $\mathcal{L}_{(1,1,2)}$ contains the set $\{((1-2n), 0, n) : n \in \mathbb{Z}\}$, see figure 3.21.

2. $m \neq 0$, then

$$x_2 = \frac{l}{2n+l-1}x_1 + \frac{n}{2n+l-1}, \tag{3.13}$$

   and l and n parametrize this set of lines. Fix the parameter n. It may be verified easily that the point

$$x_1 = x_2 = \frac{n}{2n-1}$$

   satisfies equation 3.13 for each $l \in \mathbb{Z}$. For each such l, the point

$$(x_1, x_2) = (-n/l, 0)$$

   also satisfies equation 3.13, and so the line passes through the rationals $((n, n, 2n-1))$ and $((-n, 0, l))$ (or their lowest terms equivalent). Some of these lines are displayed in figure 3.22.

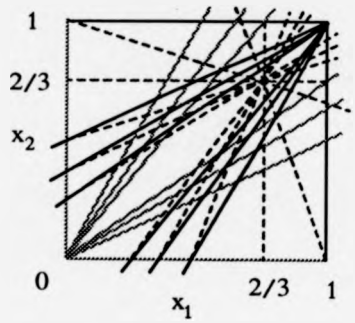

Figure 3.22: Some of the lines satisfying equation 3.13

Remarks

1. If we fix n, then as $l \to \pm\infty$, the gradient in equation 3.13 tends to 1. Thus the lines become closer to the diagonal $x_1 = x_2$ as $|l|$ increases.

2. Since figure 3.22 is symmetric about the diagonal $x_1 = x_2$, the structure of $\mathcal{L}_{(1,1,2)}$ is invariant under the action of the matrix

$$R = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z})$$

which rotates the level-0 Farey square through π about $((1,1,2))$, taking the upper level-0 Farey triangle to the lower one, and vice-versa.

3.8.4 The set $\mathcal{L}_{(p,q,r)}$

Let $(p,q,r) \in \mathbb{Z}^2 \times \mathbb{N}$ be a primitive triple such that $0 < p,q < r$, let Δ_0,Δ_1 be its associated Farey triangles, and let its (two) binary addresses be $I_0 I_1 \ldots I_N$ and $J_0 J_1 \ldots J_N$. Let Δ be the (upper) level-0 Farey triangle. Proposition 3.23 implies that there exist matrices A_0,$A_1 \in SL(3, \mathbb{Z})$ such that

1. $A_i \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ for $i = 0, 1$, and

2. A_i maps Δ_i to Δ bijectively for $i = 0, 1$.

We write $A_i \Delta_i = \Delta$ for $i = 0, 1$. Thus $\Delta_0 = A_0^{-1} A_1 \Delta_1$, the mapping is bijective between Δ_0 and Δ_1, and so the structure of $\mathcal{L}_{(p,q,r)}$ in Δ_1 is carried over to that of Δ_0, and vice versa. Since A_1^{-1} maps Δ to Δ_1, the structure of $\mathcal{L}_{(1,1,2)}$ is preserved when it is mapped to $\mathcal{L}_{(p,q,r)}$ – this follows from Theorem 3.25. So the set $\mathcal{L}_{(p,q,r)}$ has exactly the structure of figure 3.22, with the corners of the level-0 Farey triangles replaced with the corners of the relevant level-N Farey triangles. This provides another example of 'self-similarity' inherent within the structure of surface homeomorphisms.

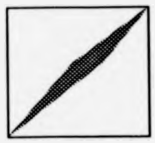

Figure 3.23: A possible rotation set allowed by the conclusions of Theorem 3.24

3.8.5 Conclusions about the structure of rotation sets

Because of the bijection between $\mathcal{L}_{(p,q,r)}$ and $\mathcal{L}_{(1,1,2)}$ on the relevant triangles, we shall just consider the case where $(p,q,r) = (1,1,2)$ of Section 3.8.3. We are not able to answer the questions posed in Section 3.7.1, but we are able to make some comments regarding them, on the basis of the results illustrated by figure 3.22.

1. It is perfectly possible that $\rho(o(x), F) \notin \text{Int } \rho(F)$ i.e. $\rho(o(x), F) \in \partial(\rho(F))$. For example this would be true if $\rho(F)$ were one of the level-0 Farey triangles. Such a triangle satisfies the constraints of Section 3.8.3.

2. It is possible that there is no 'smallest' rotation set. For instance, we can construct 'slivers' of rotation set lying along the diagonal $x_1 = x_2$ with arbitrarily small area e.g. in figure 3.23, the shaded area satisfies the conditions of Theorem 3.24.

3. So it is possible that the set $\cap\rho(G)$ defined in Section 3.7.1 may just consist of the point $\rho(o(x), F)$.

We reiterate that these are just possibilities; it may be that there exists further information contained within the dynamics of torus homeomorphisms which will answer the questions in the affirmative. However, on the basis of the work done here, we conjecture that they have negative answers. It is possible (using the techniques which will be touched upon in Section 5.3) to construct examples, and using their results have a better intuitive idea of the situation.

Chapter 4

Coexistence of periodic orbits in the disc

In this Chapter, we prove some results which are direct analogues of Sharkovskii's Theorem. In particular, we show that any homeomorphism of the disc with a periodic orbit of period 4 of pseudo-Anosov braid type has periodic orbits of all periods, and we reach the same conclusion for any homeomorphism of the sphere with a 4-point invariant set whose braid type is pseudo-Anosov. We also study coexistence questions for homeomorphisms of the disc with a periodic orbit of prime period, and homeomorphisms of the annulus, and their periodic orbits lying on the boundary.

4.1 Introduction

As we have already noted, given a continuous map of the interval, Sharkovskii's ordering of the natural numbers provides an order for the existence of periodic orbits [Sha]. In particular, if such a map has a point of period 3, it has periodic orbits of all other periods, and positive topological entropy [BoF, LY]. Analogous results in dimension two are no longer true: rotation by $2\pi/3$ in the disc $\mathbf{D}^2$ has only orbits of 1 and 3, with zero topological entropy. This example shows that the period alone is not sufficient to deduce information about the structure of periodic orbits of a homeomorphism f of a surface. Utilizing the notions discussed in previous Chapters, several results have appeared, among these, Theorems of Blanchard and Franks [BlF] for orientation-reversing homeomorphisms of the sphere, Handel [H2] for orientation-reversing homeomorphisms of compact orientable surfaces, and Katok [Ka] for the situation where f is $C^{1+\epsilon}$. More recently, a characterization of all braid types of genus zero surfaces with zero topological entropy has been made [LM1, GST1], and this has been extended to genus one [GLM] (see Chapter 2). With a Sharkovskii-type result in mind, Gambaudo, van Strien and Tresser have given an example of a braid type of period 3 of the disc which implies all other periods and positive topological entropy [GST2]. Kolev has shown that this is a general result, in fact:

Theorem 4.1 ([Ko3]) *If f is an orientation-preserving homeomorphism of the disc which has a periodic point z of period 3, then either f is isotopic to a homeomorphism g which is conjugate to rotation by $2\pi/3$ or $4\pi/3$, or f has periodic points of least period n for each $n \in \mathbf{N}$.*

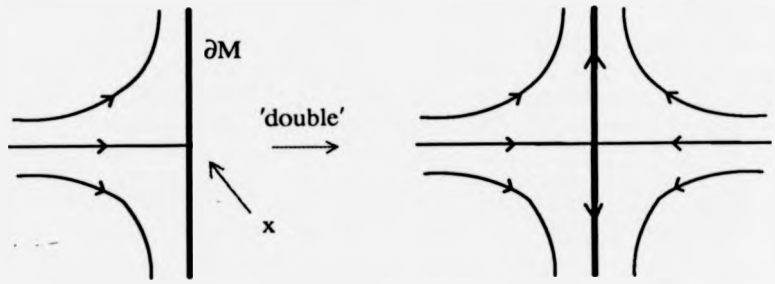

Figure 4.1: The 'double' of M close to x

In this Chapter, we extend the ideas of Kolev, and use Theorem 3.1 to prove the following Theorem:

Theorem 4.2 *Suppose $f \in Homeo^+(\mathbf{D}^2)$, and suppose it has a periodic point of prime period $n > 3$, then either f is isotopic to a homeomorphism g which is conjugate to rotation by $2\pi k/n$ for some $k \in \{1, \ldots, n-1\}$, or the set $Per(f)$ of periods of f contains the set*

$$\cap_{k=0}^{n-1} P_{k/n} \cup \{1\}$$

where

$$P_{0/n} = \{1, n-2, n-1, n, n+1, \ldots\},$$

$$P_{k/n} = \{l : l \in \mathbf{N} \text{ and there exists } j/l \in I(k/n) \text{ such that } j \text{ and } l \text{ are coprime}\}$$

for $1 \leq k < n$, and $I(k/n)$ is the Farey interval of k/n. Further $Per(f)$ is cofinite i.e. there exists a finite subset $S \subset \mathbf{N}$ such that $Per(f) = \mathbf{N} \setminus S$.

We show that a Theorem similar to Theorem 4.1 holds for period 4, and generalize Theorem 4.1 to the case where the set under consideration is an invariant 3-point set i.e. it is not necessarily a period 3 orbit. From this it follows (see Theorem 4.12) that any homeomorphism of the sphere which has an invariant 4-point set X such that $bt(X, f)$ is pseudo-Anosov has periodic orbits of all periods. We also discuss coexistence questions for homeomorphisms of the annulus and their periodic orbits lying on the boundary.

4.2 Notation and indices of fixed points

We shall use the standard notions defined in Section 3.3 for an orientation-preserving homeomorphism of the annulus. We will use Lefschetz fixed point theory [Br], which we now describe. First we make some definitions and new notation.

Let $f : M \to M$ be an orientation-preserving homeomorphism of a compact, connected surface M. Let x be an isolated fixed point of f^n. The *Lefschetz index* $Ind(x, f^n)$ is defined as follows (see [J] for further details). If $x \notin \partial M$, pick a small circle C_x centred on x such that $\mathrm{Fix}(f^n) \cap \mathcal{D}_x = \{x\}$, where $\mathcal{D}_x$ is the disc bounded by C_x. If $x \in \partial M$, then construct the 'double' of M close to x, for example, see figure 4.1. Pick a small

circle C_x as above, for the 'double'. Since the vector $z - f^n(z) \neq 0$ on C_x, define a direction field $\phi : C_x \to S^1$ by

$$\phi(z) = \frac{z - f^n(z)}{|z - f^n(z)|}.$$

Then we define

$$\begin{aligned} \mathrm{Ind}(x, f^n) &= \mathrm{degree}(\phi) \text{ if } x \notin \partial M \\ &= \frac{1}{2}\,\mathrm{degree}(\phi) \text{ if } x \in \partial M. \end{aligned}$$

This definition is independent of the radius of C_x, provided the radius is small enough. The *Lefschetz number* $L(f)$ of a homeomorphism f such that $\mathrm{Fix}(f)$ is finite, is the sum of the indices of the fixed points. Now suppose $f : \mathbf{D}^2 \to \mathbf{D}^2$ is an orientation-preserving homeomorphism of the disc, then $L(f) = 1$. Let $\mathcal{O}$ be a periodic orbit of period n of f, such that f is pseudo-Anosov relative to $\mathcal{O}$, and let $\mathcal{F}^s, \mathcal{F}^u$ be the associated stable and unstable foliations. Recall from Chapter 1 that:

1. all periodic orbits not on $\partial\mathbf{D}^2$ of f are unremovable.
2. that each foliation has a finite number of singularities with at least one on each boundary component and at each point of $\mathcal{O}$.
3. the singularities at the points of $\mathcal{O}$ are one-pronged singularities – this follows from the fact that each point of $\mathcal{O}$ has the same number of prongs, p, say. If $p > 1$ then blowing up the points of $\mathcal{O}$ and applying the Euler-Poincaré formula leads to a contradiction. All interior singularities have at least three prongs.
4. the singularities of $\mathcal{F}^s$ and $\mathcal{F}^u$ coincide in the interior of $\mathbf{D}^2$ and at the points of $\mathcal{O}$, and alternate on the boundary components.
5. the number of periodic orbits of each period is finite.
6. f has a dense orbit.

Then $Ind(x, f^n)$ is non-zero for all n and all $x \in \mathrm{Fix}(f^n)$, except at points of $\mathcal{O}$, where it is zero. It is strictly negative if x is on the boundary of $\mathbf{D}^2$, and less than or equal to one otherwise. We now give some facts, which are easily verifiable.

1. If x is a fixed point of f such that $Ind(x, f) = -1$, then $Ind(x, f^n) = -1$ for all n.
2. If $Ind(x, f) = +1$, there exist integers p, q such that

$$\begin{aligned} \mathrm{Ind}(x, f^k) &= +1 \quad k = 1, \ldots, q-1 \\ &= 1 - p \quad k = q \end{aligned} \tag{4.1}$$

   where there are exactly p stable leaves emanating from x, and the induced permutation of f on these leaves consists of p/q q-cycles.

Let $g : \mathbb{A} \to \mathbb{A}$ be an orientation-preserving homeomorphism of the annulus. Given a lift $G : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of g, we assign a rotation number relative to G of each boundary component of $\mathbb{A}$, which is well-defined and independent of choice of point on the component. Call these numbers $\rho(\partial_+, G), \rho(\partial_-, G)$ corresponding to the upper and lower components respectively.

Define an *ϵ-chain* for g from x to y to be a sequence of points $x_0 = x, x_1, \ldots, x_n = y$ in $\mathbb{A}$ such that $d(g(x_i), x_{i+1}) < \epsilon$ for $i = 0, 1, \ldots, n-1$. g is *chain transitive* if for any $x, y \in \mathbb{A}$ and $\epsilon > 0$, there exists an ϵ-chain from x to y. Then we have the following Theorem.

Theorem 4.3 ([Fr3]) *Let $g : \mathbb{A} \to \mathbb{A}$ be an orientation-preserving homeomorphism which preserves boundary components and is chain transitive. Choose a lift $G : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of g, then for every rational p/q (in lowest terms) between $\rho(\partial_+, G)$ and $\rho(\partial_-, G)$, G has a (p, q)-periodic point.*

Remark

- Suppose $f : \mathbb{D}^2 \to \mathbb{D}^2$ is an orientation-preserving homeomorphism of the disc, such that f is pseudo-Anosov relative to some periodic orbit $\mathcal{O}$. Since $f_{\mathcal{O}}$ has a dense orbit, it is chain transitive.

We now prove the following Theorem and a Corollary, which will be used several times in this Chapter.

Theorem 4.4 *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus isotopic to the identity, and suppose $o(x)$ is a periodic orbit of f such that f is pseudo-Anosov relative to $o(x)$. Then there exist periodic orbits lying in the interior of $\mathbb{A}$ such that their rotation numbers relative to a lift of f are those of the rotation numbers of the boundary.*

Proof

Assume that f is pseudo-Anosov relative to $o(x)$, and choose a lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of f. We shall prove the Theorem for the boundary component $\partial^+\mathbb{A}$ – the proof for $\partial^-\mathbb{A}$ is similar. Since $f|_{\partial^+\mathbb{A}}$ is a homeomorphism of S^1, it has a well-defined rotation number $\rho(\partial_+, F)$. Let $\mathcal{F}^u$ be the invariant unstable foliation, then $\mathcal{F}^u$ has at least one singularity on $\partial^+\mathbb{A}$, and such singularities are permuted by f. Hence $\rho(\partial_+, F)$ is rational. Further, if we write $\rho(\partial_+, F) = p_+/q_+$, where p_+ and q_+ are coprime, then any singularity of $\mathcal{F}^u$ on $\partial^+\mathbb{A}$ has (least) period q_+ (so q_+ divides the number of singularities on $\partial^+\mathbb{A}$).

Let $F_+ = T^{-p_+}F^{q_+}$. If we lift the foliations $(\mathcal{F}^s, \mathcal{F}^u)$ of $\mathbb{A}$ to foliations $(\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u)$ of $\tilde{\mathbb{A}}$, then F_+ fixes the set of singularities on $\partial^+\tilde{\mathbb{A}}$ pointwise. By choosing $m \in \mathbb{N}$ large enough, we may arrange that the m-fold covering $\pi_m : \tilde{\mathbb{A}} \to \mathbb{A}_m$, where $\mathbb{A}_m = \tilde{\mathbb{A}}/m\mathbb{Z}$, with the homeomorphism $F_{m,+}$ of $\mathbb{A}_m$ induced by F_+ satisfies

$$F_+(\tilde{y}) = \tilde{y} \Leftrightarrow F_{m,+}(\pi_m(\tilde{y})) = \pi_m(\tilde{y}) \text{ for } \tilde{y} \in \tilde{\mathbb{A}} \tag{4.2}$$

i.e. the lift of a fixed point of $F_{m,+}$ is a fixed point of F_+ (and vice versa). For any $\tilde{y} \in \tilde{\mathbb{A}}$ satisfying equation 4.2, the point $y = \pi(\tilde{y}) \in \mathbb{A}$ must be a (p_+, q_+)-periodic point of F. So to prove the Theorem, it suffices to show that $F_{m,+}$ has an interior fixed point. This follows from the fact that $F_{m,+}$ is pseudo-Anosov relative to the set $\pi_m(\pi^{-1}(o(x)))$ and has fixed points on its upper boundary: consider the homeomorphism $\overline{F_{m,+}} : \mathbb{D}^2 \to \mathbb{D}^2$

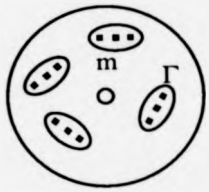

Figure 4.2: The reducing curve Γ and its iterates

induced by collapsing $\partial^- \mathsf{A}$ to a fixed point. The Lefschetz index of this point is at most $+1$. The Lefschetz index of a fixed point on the boundary is always strictly negative, and since $L(\overline{F_{m,+}}) = +1$, there must exist a fixed point of $F_{m,+}$ of strictly positive index in the interior of A_m. Thus the proof is complete. □

Although we have only defined the notion of 'braid type' for a finite invariant set $X \subset \text{Int } M$ for a surface M, we shall extend the definition slightly, in the case that $M = \mathsf{A}$ and X is a periodic orbit contained wholly in one boundary component, whence we shall say that the braid type $bt(X, f)$ is periodic.

Corollary 4.5 *Given $f : \mathsf{A} \to \mathsf{A}$ a homeomorphism of the annulus isotopic to the identity, with a periodic orbit $o(x)$ such that f is pseudo-Anosov relative to $o(x)$, then if $y \in Sing(\mathcal{F}^u) \cap \partial^+ \mathsf{A}$ and $z \in Sing(\mathcal{F}^u) \cap \partial^- \mathsf{A}$, then there exist periodic orbits $o(y')$, $o(z') \in Int\, \mathsf{A}$ such that $bt(o(y), f) = bt(o(y'), f)$ and $bt(o(z), f) = bt(o(z'), f)$.*

Proof

Let $o(x_+)$,$o(x_-) \subset \text{Int } \mathsf{A}$ be the (p_+, q_+)- and (p_-, q_-)-periodic orbits constructed in Theorem 4.4. Then $bt(o(x), f) \succ bt(o(x_\pm), f)$. We shall consider the orbit $o(x_+)$ – the case of $o(x_-)$ is similar. Since $f|_{\partial^+ \mathsf{A}}$ is a homeomorphism of S^1, it follows that the braid type $bt(o(y), f)$ is periodic. Suppose $bt(o(x_+), f)$ is periodic, then since $\rho(o(x_+), F) = \rho(o(y), F)$, it follows that $bt(o(x_+), f) = bt(o(y), f)$. Otherwise, suppose $bt(o(x_+), f)$ is not periodic. Since p_+ and q_+ are coprime, it cannot be reducible by the following Lemma.

Lemma 4.6 *Suppose $f : \mathsf{A} \to \mathsf{A}$ is a homeomorphism of the annulus isotopic to the identity possessing a periodic orbit $o(x)$, such that $o(x)$ is a (p, q)-periodic orbit of some lift F of f, where p and q are coprime. Then $bt(o(x), f)$ cannot be reducible.*

Proof

Suppose f is reducible relative to $o(x)$, then there exists a reducing curve Γ containing $j \geq 2$ points of $o(x)$, and some (least) $m \in \mathbb{N}$ such that $f^m(\Gamma) = \Gamma$. So each curve $f^s(\Gamma)$, $0 \leq s < m$, contains j points of $o(x)$, so j divides q and $m = q/j$ (see figure 4.2). There exists some $l \in \mathbb{Z}$ such that $F^m(\tilde{\Gamma}) = T^l(\tilde{\Gamma})$ for any $\tilde{\Gamma} \in \pi^{-1}(\Gamma)$. Thus $F^{mj}(\tilde{\Gamma}) = F^q(\tilde{\Gamma}) = T^{lj}(\tilde{\Gamma})$. So for any point $y \in o(x)$ contained in the region bounded by Γ, $F^q(\tilde{y}) = T^{lj}(\tilde{y})$ for any $\tilde{y} \in \pi^{-1}(y)$. But y is a (p, q)-periodic orbit of F, and thus $p = lj$. This contradicts p and q being coprime, and thus f cannot be reducible relative to $o(x)$. □

Thus $bt(o(x_+), f)$ must be pseudo-Anosov. We finish the proof by utilizing an argument of Boyland [Boy3], which follows from a Theorem of Brunovskii [Bru], to

show that there exists a periodic orbit $o(y') \subset \text{Int } \mathbb{A}$ of f such that $bt(o(x_+), f) \succ bt(o(y'), f) = bt(o(y), f)$. Let $g \simeq f$ rel. $o(x_+)$ be the homeomorphism such that g is pseudo-Anosov relative to $o(x_+)$. Let G be the lift of g such that $\rho(o(x_+), G) = p_+/q_+$. For all $x_i \in o(x_+)$, $Ind(x_i, g^{q_+})$ is zero, since each x_i is a 1-pronged singularity of each invariant foliation. Since p_+, q_+ are coprime and $q_+ \neq 1$ by Proposition 3.11 then Theorem 3.1 implies that $p_+/q_+ \in \text{Int}\rho(G)$, we may invoke the following Theorem of Franks:

Theorem 4.7 ([Fr6]) *Let $h : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the closed annulus homotopic to the identity and with every point non-wandering, and let $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h. If $0 \in$ Int $\rho(H)$, then h has at least two distinct fixed points, one of strictly positive Lefschetz index, the other of strictly negative Lefschetz index.*

So g has a periodic point u, such that u is a (p_+, q_+)-periodic point of G, and $Ind(u, g^{q_+}) = +1$. Let $\beta = bt(o(x_+), g)$, then there exists a diffeomorphism of the annulus with just one periodic orbit of braid type β; for let f_0, f_1 be Kupka-Smale maps with $\beta \notin bt(f_0)$ and $\beta \in bt(f_1)$. Brunovskii's Theorem states that given $N > 0$, then there exists a diffeotopy f_μ between f_0 and f_1, such that all orbits of periods less than N undergo only saddle-node and pitchfork bifurcations, and such bifurcations occur at a finite number of distinct values of μ. Take N to be larger than q_+, and let

$$\mu_0 = \inf\{\mu : \beta \in bt(f_\mu)\}.$$

Then there must be a bifurcation at μ_0. If it is a saddle-node bifurcation, the two orbits created as μ increases have the same braid type, but at $\mu = \mu_0$, f_{μ_0} has a single periodic orbit whose braid type is β. If the bifurcation were a pitchfork, the orbit that persists during the bifurcation existed for $\mu < \mu_0$, so its braid type cannot be β. Otherwise, the braid type β must come from the doubled orbit. But this would imply that the orbit is reducible – a contradiction.

So in particular, $\beta \neq bt(o(u), g)$ i.e. given a pseudo-Anosov braid type of rotation type (p_+, q_+), there exists another braid type $\gamma = bt(o(u), g)$ of the same rotation type, such that $\beta \succ \gamma, \beta \neq \gamma$. Let

$$\mathcal{D} = \{\alpha \ : \ \alpha \text{ is a braid type realized by a periodic orbit of } g \text{ of rotation type } (p_+, q_+), \beta \succ \alpha\}.$$

Since g is pseudo-Anosov, it has only finitely many periodic orbits of each period, so $\mathcal{D}$ is finite. Suppose $\delta \in \mathcal{D}$ is minimal i.e. $\xi \prec \delta \Rightarrow \xi = \delta$ for all $\xi \in \mathcal{D}$. Then ξ cannot be pseudo-Anosov by the above argument, so is periodic. Thus there exists a periodic orbit of g, and hence one, $o(y')$ of f, such that $bt(o(y'), f) = bt(o(y), f)$. Hence the proof of Corollary 4.5 is complete. □

4.3 Coexistence of periodic orbits for prime periods

In this Section, we shall prove Theorem 4.2. This is achieved essentially by blowing up a fixed point of the homeomorphism of the disc, considering the induced homeomorphism of the annulus, and using index theory and Theorem 4.3 to produce the required periodic orbits.

We now proceed with the proof of Theorem 4.2. Let $\mathcal{O}$ be the given periodic orbit of f of period n, where n is prime. Then the isotopy class of f relative to $\mathcal{O}$ is irreducible by Theorem 1.11, and so there exists a homeomorphism g isotopic to f rel. $\mathcal{O}$ with g either pseudo-Anosov or periodic relative to $\mathcal{O}$. We consider these two cases in turn.

1. g is periodic relative to $\mathcal{O}$. Thus $g^n = Id$. By a Theorem of Brouwer [Bro], g is conjugate to a rotation by $2k\pi/n$ for some $k = 1, \ldots, n-1$.

2. g is pseudo-Anosov relative to $\mathcal{O}$. Since $L(g) = 1$ and the indices of any fixed points on the boundary of the disc $\partial \mathbf{D}^2$ are negative, then g must have an interior fixed point x_0, with $Ind(x_0, g) = 1$. Denote the boundary of the disc by the circle S_∞. We now embed this map as a map $\bar{g}$ of the sphere S^2 in the natural way (stereographic projection). By collapsing down the boundary circle S_∞ to a point x_∞, we may extend $\bar{g}$ to a homeomorphism $G : S^2 \to S^2$, and we still have the invariant foliations induced by g, in particular there exists a dense orbit for G.

   The idea is the following [Ko3]. Take the set of fixed points F of G. By taking a pair (p_1, p_2) from $F \times F$, and blowing them up, we have an induced homeomorphism $G(p_1, p_2)$ of the annulus, in particular, we can define rotation numbers $\rho(p_1)$, $\rho(p_2)$ for the boundary relative to some lift of $G(p_1, p_2)$. This latter map is chain transitive, so by Theorem 4.3, for each p/q in lowest terms such that p/q is between $\rho(p_1)$ and $\rho(p_2)$, there exists a (p, q)-periodic orbit. This gives a rotation interval $\rho(p_1, p_2) =< \rho(p_1), \rho(p_2) >$, and we take the largest such interval over pairs (p_1, p_2) of $F \times F$, where $< a, b >$ equals $[a, b]$ if $a \leq b$ and $[b, a]$ if $a > b$. From this, we can deduce the periods that coexist, given the periodic orbit $\mathcal{O}$.

   We now formalize this. From the preceding discussion, G has (at least) two fixed points, x_0,x_∞, and $Ind(x_0, G) = 1$. Blow up these two points to give two boundary circles S_0,S_∞, and an induced homeomorphism $G(x_0, x_\infty)$ of the annulus. There are two distinct cases to consider.

   (a) There exists a lift $\tilde{G}(x_0, x_\infty)$ of $G(x_0, x_\infty)$ such that the rotation number $\rho(\mathcal{O}, \tilde{G}(x_0, x_\infty))$ of $\mathcal{O}$ with respect to $\tilde{G}(x_0, x_\infty)$ is zero.

   (b) There exists a lift $\tilde{G}(x_0, x_\infty)$ of $G(x_0, x_\infty)$ such that the rotation number $\rho(\mathcal{O}, \tilde{G}(x_0, x_\infty))$ of $\mathcal{O}$ with respect to $\tilde{G}(x_0, x_\infty)$ is k/n for $1 \leq k < n$.

   We consider these cases in turn.

   (a) In this case, we claim that g has at least three fixed points x_0,x_1,x_2, where $Ind(x_0, g) = Ind(x_1, g) = 1$ and $Ind(x_2, g) < 0$. This results from the following Lemma.

   Lemma 4.8 *Suppose $h : \mathbf{D}^2 \to \mathbf{D}^2$ is a homeomorphism of the disc with a periodic orbit $o(x)$ of period $n > 1$. Let x_0 be an interior fixed point of Lefschetz index $+1$, and let $\bar{h} : \mathbb{A} \to \mathbb{A}$ be the homeomorphism of the annulus induced by h by blowing up x_0. Suppose there exists a lift $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of $\bar{h}$ such that $\tilde{x} \in \pi^{-1}(x)$ is a $(0, n)$-periodic point of H. Then h has at least three fixed points.*

   Proof

Consider $H : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$. $\tilde{x}$ is a periodic point of H. It is a consequence of Brouwer's Lemma [Br1] (see also [Fa]) that H has an interior fixed point $\tilde{x}_1$ of (strictly) positive Lefschetz index (this follows easily from Brown's proof). $\tilde{x}_1$ projects to a fixed point of positive index of $\bar{h}$, and hence to one, x_1, of h. But $L(h) = 1$, is the sum of the indices of the fixed points. Hence there must exist at least one more fixed point x_2 of negative index to restore the Lefschetz index to $+1$ (note that x_2 could be on $\partial \mathbf{D}^2$). □

$Ind(x_2, g) = -1$ implies that $Ind(x_2, g^k) = -1$ for all $k \in \mathbf{N}$. The prongs of the foliation emanating from x_2 are fixed by g, so the rotation number on this boundary is an integer, K, say. Let the number of prongs emanating from x_0 and x_1 be p_0,p_1 respectively. Note that p_0 and p_1 are both at least two, and that g rotates the prongs at each of these points. So the rotation number corresponding to each of x_0 and x_1 is a (non-integer) rational whose denominator is a divisor of p_0 and p_1 respectively. We search for the maximum number of prongs each point can have, in order to minimize the rotation intervals of $G(x_0, x_2)$ and $G(x_1, x_2)$. From equation 1.8,

$$\sum_{s \in Sing\mathcal{F}} (2 - p_s) = 2(1 - n)$$

where p_s is the number of prongs at each singularity of one of the foliations $\mathcal{F}$. At least $-(1 + n)$ of this sum is accounted for by the singularities on S_∞ and at the points of $\mathcal{O}$. Thus there are at most the equivalent of (n-3) (interior) three-pronged singularities to complete this sum. In order to make the rotation intervals of $G(x_0, x_2)$ and $G(x_1, x_2)$ as small as possible, it is necessary to make the rotation numbers on the boundaries corresponding to x_0 and x_1 as close to K as possible. We do this by sharing the number of 'free' prongs between x_0 and x_1, so that p_0 and p_1 are $(n + 1)/2$ (n is odd). Consequently, one of the rotation intervals of $G(x_0, x_2)$ and $G(x_1, x_2)$ (or their inverses) must contain the set $[0, 2/(n + 1)]$ for some suitable lift. For $n > 3$ prime, such a set contains all numbers of the form

$$\left\{ \frac{1}{n - 2 + j} : j \in \mathbf{Z}_+ \right\},$$

and so there exist periodic orbits of all periods in $P_{0/n}$ by Theorem 4.3, where

$$P_{0/n} = \{1, n - 2, n - 1, n, n + 1, \ldots\}.$$

The existence of periodic orbits whose rotation numbers are those of the boundary components for the homeomorphisms $G(x_0, x_2)$ and $G(x_1, x_2)$ is guaranteed by Theorem 4.4.

(b) Since n is prime, we may invoke Theorem 3.1. So $\rho(\tilde{G}(x_0, x_\infty)) \supset I(k/n)$, the Farey interval, for some suitable lift $\tilde{G}(x_0, x_\infty)$. For all rationals $r/s \in I(k/n)$ such that r and s are coprime, there exists an (r, s)-periodic orbit, and thus a periodic orbit of period s. So $G(x_0, x_\infty)$ has periodic orbits of all periods in the set $P_{k/n}$, where

$$P_{k/n} = \{l : l \in \mathbf{N} \text{ and there exists } j/l \in I(k/n) \text{ such that } j \text{ and } l \text{ are coprime}\}.$$

Again, the existence of periodic orbits whose rotation numbers are those of the boundary components for the homeomorphism $G(x_0, x_\infty)$ is guaranteed by Theorem 4.4. But *a priori*, we do not know the value of k, hence $G(x_0, x_\infty)$ has periodic orbits of all periods in the set

$$\cap_{k=1}^{n-1} P_{k/n}.$$

Hence it follows that f has periodic orbits of all periods contained in the set

$$\cap_{k=0}^{n-1} P_{k/n} \cup \{1\}.$$

The fact that $Per(f)$ is cofinite is a consequence of the following Proposition.

Proposition 4.9 ([GaL]) *Let M be a compact, connected, oriented surface, possibly with boundary, and let $f : M \to M$ be a pseudo-Anosov homeomorphism, with dilatation factor $\lambda > 1$ and A the transition matrix for a Markov partition of f. Let $\mu \in \mathbf{R}$ be such that $2 < \mu < \lambda$ if $\lambda > 2$, and $1 < \mu < \lambda$ if $\lambda \leq 2$. Then the following statements are true.*

1. *There exists $N_0 \in \mathbf{N}$ such that Trace (A^{n+1})/ Trace $(A^n) > \mu$ for all $n \geq N_0$.*
2. *If $\mu > 2$, then f has periodic points of all periods greater than or equal to $2N_0$ with (perhaps) the exception of finitely many.*
3. *If $\mu < 2$, then f has periodic points of all periods greater than or equal to the maximum of $2N_0$ and $4(1 + |\log(\mu - 1)/\log\mu|)$ with (perhaps) the exception of finitely many.*

Hence the proof of Theorem 4.2 is completed. □

4.4 Coexistence of periodic orbits with one of period 4

In this Section, we prove the following Theorem.

Theorem 4.10 *Suppose $f \in Homeo^+(\mathbf{D}^2)$ and $o(x)$ is a periodic orbit of f of period 4. If $bt(o(x), f)$ is pseudo-Anosov, then f has periodic orbits of all periods.*

Proof

Suppose $g \simeq f$ rel. $o(x)$ is pseudo-Anosov relative to $o(x)$. Then since $\chi(\mathbf{D}_4) = -3$,

$$\sum_{s \in Sing\mathcal{F}^u} (2 - p_s) = -6,$$

where $\mathcal{F}^u$ is the invariant unstable foliation of g. The singularities at $o(x)$ account for -4 in this sum. To make up the remaining -2, there are two possibilities.

1. $\mathcal{F}^u$ has one 1-pronged singularity on $\partial\mathbf{D}^2$ and one 3-pronged singularity in $\mathrm{Int}(\mathbf{D}^2)$, or
2. $\mathcal{F}^u$ has two 1-pronged singularities on $\partial\mathbf{D}^2$.

We consider these cases in turn.

1. Let z be the 1-pronged singularity of $\mathcal{F}^u$ on ∂D^2, and let y be the 3-pronged singularity in $\mathrm{Int}(D^2)$. Let z' be the 1-pronged singularity of $\mathcal{F}^s$ on ∂D^2, then $Ind(z,g) = Ind(z',g) = -1/2$, and $Ind(y,g) \leq +1$. Since $L(g) = +1$, there exists an interior fixed point u (distinct from y) of g of positive index, and it cannot be a singularity of the foliations, so must be a regular point of the foliations. Thus g swaps the two prongs of $\mathcal{F}^u$ emanating from u. Let $\bar{g} : A \to A$ be the homeomorphism of the annulus induced by g by blowing up u. Fixing a lift of $\bar{g}$, the rotation numbers of the two boundaries are $\rho(\partial_+) = k$, $\rho(\partial_-) = (2l+1)/2$, $k, l \in \mathbb{Z}$, as the outer boundary $\partial^+ A$ contains the fixed point z, and the inner boundary $\partial^- A$ contains the period 2 orbit induced by the action of g on the prongs emanating from u. The corresponding rotation set contains the set $< k, (2l+1)/2 >$, so Theorem 4.3 implies that $\bar{g}$ has periodic orbits of all periods, and Theorem 4.4 implies that g does too.

2. $\mathcal{F}^u$ has two 1-pronged singularities on ∂D^2. Since $L(g) = 1$, there exists a fixed point z in the interior of D^2 which is a regular point of the foliation and whose index is $+1$. If g fixes the two singularities on ∂D^2, then we argue as in case 1 to show that g has points of all periods. Otherwise suppose g swaps them, let $\bar{g} : A \to A$ be the homeomorphism of the annulus induced by g by blowing up z. If the rotation numbers of the boundaries (relative to any lift of $\bar{g}$) are not equal, they differ by an integer, and we apply Theorem 4.3 to show that $\bar{g}$, and hence g, have periodic orbits of all periods. Otherwise suppose the rotation numbers of the boundaries of A are both $1/2$ relative to some lift $\tilde{G} : \tilde{A} \to \tilde{A}$. There are three subcases:

   (a) $\rho(o(x), \tilde{G}) = l/4$, where $l \equiv 0 \bmod 4$. Then $\rho(\tilde{G}) \supset [0/1, 1/2]$ or $\rho(\tilde{G}) \supset [1/2, 1/1]$, and we argue as in case 1 to deduce that g has points of all periods.

   (b) $\rho(o(x), \tilde{G}) = l/4$, where $l \equiv 1$ or $3 \bmod 4$. By considering $\bar{g}^{-1}$ if necessary, we may assume that $l \equiv 1 \bmod 4$. Then $\rho(\tilde{G}) \supset [1/4, 1/2]$ or $[-1/2, 1/4]$. But Theorem 3.1 implies that $\rho(\tilde{G}) \supset [0, 1/3]$. So $\rho(\tilde{G}) \supset [0, 1/2]$ or $[-1/2, 1/3]$, and hence g has periodic orbits of all periods.

   (c) $\rho(o(x), \tilde{G}) = l/4$, where $l \equiv 2 \bmod 4$. If $l \neq 2$, then $\rho(\tilde{G}) \supset [1/2, 3/2]$ or $[-1/2, 1/2]$, and we are done, using Theorem 4.3. So suppose $l = 2$. Then we use a flow-equivalence argument similar to the proof of Theorem 3.1 in Section 3.4. If $[0,1] \subset \rho(\tilde{G})$, then we are done. Otherwise, there are two subcases:

      i. $0 \notin \rho(\tilde{G})$. We make a suitable change of basis for the homology of the suspended flow $\phi : A \times S^1 \to A \times S^1$ defined so that the homology class $[\phi(x)] = (2,4)$ relative to the standard basis g_1, g_2 of $H_1(A \times S^1; \mathbb{Z})$. Let

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \in SL(2, \mathbb{Z})$$

      be the matrix associated with the change of basis. Note that $(0,2) = (2,4).M^{-1}$. For any $r/s \in \rho(\tilde{G})$, $r > 0$, so if we write $(r', s') = (r, s).M^{-1}$, then $s' = r > 0$, and Theorem 3.3 implies that the change of basis corresponds to a change of cross-section, such that the $(2,4)$-periodic orbit is transformed to a $(0,2)$-periodic orbit $o(y)$. The return

map $h : \mathbb{A} \to \mathbb{A}$ to the new cross-section is pseudo-Anosov relative to $o(y)$ by Theorem 3.5, and so Theorem 3.17 implies that $\rho(H) \supset [-1,1]$, where $H : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ is the lift of h such that $\rho(o(y), H) = 0$. Since $(-1,1).M = (1,3)$ and $(1,1).M = (1,1)$, then $\rho(\tilde{G}) \supset [1/3, 1]$, and by arguing as in case 1, we find that $\tilde{g}$, and hence g have periodic orbits of all periods.

ii. $1 \notin \rho(\tilde{G})$. We argue as in case 2(c)i, replacing M by

$$M' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

finding that $\rho(\tilde{G}) \supset [0, 2/3]$, and once again g has periodic orbits of all periods.

Theorem 4.4 guarantees that there exist periodic orbits in $\mathrm{Int}(\mathbb{A})$ of the same rotation type as the periodic orbits lying on $\partial\mathbb{A}$. Since periodic orbits of g are unremovable relative to $o(x)$, it follows that f has periodic orbits of all periods. □

4.5 Coexistence of periodic orbits in $\mathbb{D}_3$

In this Section, we prove the following generalization of Theorem 4.1.

Theorem 4.11 *Let $f : (\mathbb{D}^2, X) \to (\mathbb{D}^2, X)$ be an orientation-preserving homeomorphism, where $X = \{x_1, x_2, x_3\}$ is an invariant 3-point set. Then either f has periodic points of (least) period n for all $n \in \mathbb{N}$, or f is isotopic to a homeomorphism g such that one of the following is true:*

1. *g is conjugate either to rotation by π, $2\pi/3$ or $4\pi/3$, or is the identity.*
2. *g is reducible with two components C_1 and C_2; $g|_{C_1}$ is either the identity or conjugate to rotation by π, and $g|_{C_2}$ is the identity.*

Proof
We analyse the cases according to the permutation σ induced by f on X.

1. σ is a 3-cycle. Then X consists of a period 3 orbit, and Theorem 4.1 implies that either f has points of all periods, or f is isotopic to a homeomorphism g such that g is conjugate to rotation by $2\pi/3$ or $4\pi/3$.
2. σ is a 2-cycle. Then X consists of a period 2 orbit and a fixed point. Consider the Thurston type of the isotopy class of f relative to X. It is either
   (a) periodic: in this case, there exists $g \simeq f$ rel. X such that $g^2 = Id$. Hence g is conjugate to rotation by π.
   (b) reducible: so there exists $g \simeq f$ rel. X such that g is reducible. It must have two invariant components C_1 and C_2, such that the period 2 orbit is contained in C_1, and the fixed point and $\partial\mathbb{D}^2$ are contained in C_2 (see figure 4.3). Then Proposition 3.11 implies that the braid types $bt(X \cap C_i, g|_{C_i})$, $i = 1, 2$ are periodic, hence $g|_{C_1}$ is conjugate to rotation by π and $g|_{C_2}$ is the identity.

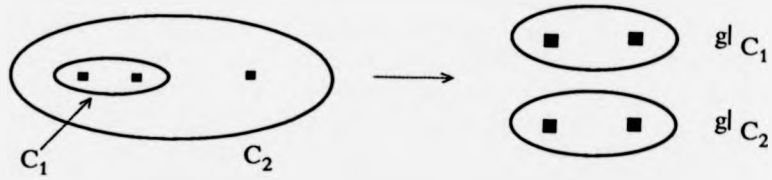

Figure 4.3: The reducible case for when σ is a 2-cycle

(c) pseudo-Anosov: then there exists $g \simeq f$ rel. X such that g is pseudo-Anosov relative to X. Let $\mathcal{F}^u$ be its unstable foliation, then the Euler-Poincaré formula implies that $\mathcal{F}^u$ has at most 4 singularities – in fact it has exactly 4, one at each point of X, and one on $\partial\mathbf{D}^2$. By renumbering X if necessary, suppose x_3 is fixed by g, then $Ind(x_3, g) = 0$ as it is a 1-pronged singularity. $L(g) = 1$ implies that g has a fixed point y of index $+1$, which must be a regular (interior) point of the foliations, as any fixed point on $\partial\mathbf{D}^2$ has strictly negative index. There is exactly one singularity of $\mathcal{F}^u$ on $\partial\mathbf{D}^2$, so it is a fixed point. Let $h : \mathbf{A} \to \mathbf{A}$ be the homeomorphism induced by g by blowing up y; fixing a lift of h, the rotation number of the outer boundary is k, for some $k \in \mathbf{Z}$, as it contains a fixed point, and that of the inner boundary is $(2l+1)/2$, for some $l \in \mathbf{Z}$, as it contains a period 2 orbit induced by the action of g on the prongs emanating from y. So the corresponding rotation set contains the set $< k, (2l+1)/2 >$. Thus Theorem 4.3 implies that h has periodic orbits of all periods, Theorem 4.4 guarantees that there exist periodic orbits in Int($\mathbf{A}$) of the same rotation type as the periodic orbits lying on $\partial\mathbf{A}$. Hence g has periodic orbits of all periods, and by unremovability of periodic orbits, so does f.

3. σ is the identity. Consider the Thurston type of the isotopy class of f relative to X. It is either

   (a) periodic: so there exists $g \simeq f$ rel. X such that $g = Id$.

   (b) reducible: so there exists $g \simeq f$ rel. X such that g is reducible. It must have two invariant components C_1 and C_2 such that two points of X lie in C_1. Thus $g|_{C_1} = Id$ and $g|_{C_2} = Id$.

   (c) pseudo-Anosov: then we argue exactly as in case 2c to conclude that g, and hence f has points of all periods. □

4.6 Coexistence of periodic orbits in S^2

In this Section, we combine the results of Theorems 4.10 and 4.11 to prove the following Theorem about homeomorphisms of S^2 with an invariant 4-point set.

Theorem 4.12 *Let $g : S^2 \to S^2$ be an orientation-preserving homeomorphism of the sphere, and let X be a g-invariant 4-point set. Then either g has periodic orbits of (least) period n for all $n \in \mathbf{N}$, or g is isotopic to a homeomorphism h such that one of the following is true:*

1. h is conjugate either to rotation by π, $2\pi/3$, $4\pi/3$, $\pi/2$ or $3\pi/2$, or is the identity.

2. h is reducible with two components C_1,C_2 and either:

(a) $h(C_i) = C_i$, $i = 1,2$ and $h|_{C_i}$ is either the identity, or conjugate to rotation by π, or

(b) $h(C_1) = C_2$, then $h^2|_{C_i}$, $i = 1,2$ is either the identity, or conjugate to rotation by π.

Proof
There are three cases to consider.

1. $X \cap Fix(g) \neq \emptyset$. Let $x_0 \in X \cap Fix(g)$, and let $\overline{g} : \mathbf{D}^2 \to \mathbf{D}^2$ be the orientation-preserving homeomorphism induced by g by blowing up x_0. Then $\overline{g}$ satisfies the conditions of Theorem 4.11 and it follows that either $Per(g) = \mathbf{N}$, or g is isotopic to a homeomorphism h such that either

   (a) h is conjugate to rotation by π, $2\pi/3$ or $4\pi/3$, or is the identity, or

   (b) h is reducible with two components C_1 and C_2; $h|_{C_1}$ is either the identity or conjugate to rotation by π, and $h|_{C_2}$ is the identity.

2. $X \cap Fix(g) = \emptyset$, and X is an orbit of period 4. Let h be the Thurston representative of g relative to X. It is either

   (a) periodic: then $h^4 = Id$, and so h is conjugate to rotation by $\pi/2$ or $3\pi/2$, or

   (b) reducible: then there are two decomposition components C_1 and C_2, and $h|_{C_1} = C_2$. So $h^2|_{C_i}$, $i = 1,2$ is conjugate to rotation by π, or

   (c) pseudo-Anosov: $L(h) = 2$ implies that h has a fixed point z_0 of (strictly) positive Lefschetz index. Let $\overline{h} : \mathbf{D}^2 \to \mathbf{D}^2$ be the orientation-preserving homeomorphism induced by h by blowing up z_0. Then $\overline{h}$ satisfies the conditions of Theorem 4.10, and it follows that h, and thus g has periodic points of all periods.

3. $X \cap Fix(g) = \emptyset$, and X consists of two period 2 orbits. Let h be the Thurston representative of g relative to X. It is either

   (a) reducible: then there are two decomposition components C_1, C_2 and either

      i. $h(C_i) = C_i$, $i = 1,2$, and $h^2|_{C_i} = Id$. So $h|_{C_i}$ is conjugate to rotation by π, or

      ii. $h(C_1) = C_2$, then $h^2|_{C_i}$, $i = 1,2$ is the identity, or

   (b) pseudo-Anosov. Let the two period 2 orbits contained in X be $o(x_1)$ and $o(x_2)$. Since $L(h) = 2$, there exists a fixed point z_0 of h of strictly positive Lefschetz index. Let $\overline{h} : \mathbf{D}^2 \to \mathbf{D}^2$ be the orientation-preserving homeomorphism induced by h by blowing up z_0. Consider the periodic orbit $o(x_1)$, then the braid type $bt(o(x_1), \overline{h})$ is periodic, so there exists a fixed point z_1 in $\mathrm{Int}(\mathbf{D}^2)$ such that if $f : \mathbf{A} \to \mathbf{A}$ is the orientation-preserving homeomorphism induced by $\overline{h}$ by blowing up z_1, then there exists a lift $F : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ of f such that $\rho(x_1, F) = 1/2$. There are two possibilities.

i. $\rho(x_1, F) \neq \rho(x_2, F)$, then $\rho(F)$ contains either $[0, 1/2]$ or $[1/2, 1]$, and it follows by Theorem 4.3 that f has periodic orbits of all periods. Theorem 4.4 guarantees that h does too, and so by unremovability of periodic orbits, so does g.

ii. $\rho(x_1, F) = \rho(x_2, F) = 1/2$. The proof is completed by the following Proposition, and Theorems 4.3 and 4.4.

Proposition 4.13 *Given the above case 3(b)ii, then $\rho(F)$ contains one of* $[0, 2/3]$ *or* $[1/3, 1]$.

Proof

The proof is similar to that of case 2(c)i of Theorem 4.10. There are two cases to consider.

1. Suppose $0 \notin \rho(F)$. Then we make a suitable change of basis for the homology of the suspended flow $\phi : \mathbb{A} \times S^1 \to \mathbb{A} \times S^1$ defined so that the homology class $[\phi(x_1)] = [\phi(x_2)] = (1, 2)$ relative to the standard basis g_1, g_2 of $H_1(\mathbb{A} \times S^1; \mathbb{Z})$. Let
$$M = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \in SL(2, \mathbb{Z})$$
be the matrix associated with the change of basis. Note that $(0, 1) = (1, 2).M^{-1}$. For any $r/s \in \rho(F)$, $r > 0$, so if we write $(r', s') = (r, s).M^{-1}$, then $s' = r > 0$, and Theorem 3.3 implies that the change of basis corresponds to a change of cross-section, such that the $(1, 2)$-periodic orbits are transformed to $(0, 1)$-periodic orbits y_1, y_2. The return map $r : \mathbb{A} \to \mathbb{A}$ to the new cross-section is pseudo-Anosov relative to $\{y_1, y_2\}$ by Theorem 3.5, and so Theorem 3.19 implies that $\rho(R) \supset [-1, 1]$, where $R : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ is the lift of r such that $\rho(y_1, R) = \rho(y_2, R) = 0$. Since $(-1, 1).M = (1, 3)$ and $(1, 1).M = (1, 1)$, then $\rho(F) \supset [1/3, 1]$, and by arguing as in case 1 of Theorem 4.10, we find that f has periodic orbits of all periods, and so Theorem 4.4 implies that $\overline{h}$ and hence h have periodic orbits of all periods.

2. Suppose $1 \notin \rho(F)$. We argue as in case 1 above, replacing M by
$$M' = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$
finding that $\rho(F) \supset [0, 2/3]$. □

Once again, f,$\overline{h}$ and h have periodic orbits of all periods, and since periodic orbits of h are unremovable relative to X, it follows that g has periodic orbits of all periods. Thus the proof of Theorem 4.12 is completed. □

Chapter 5

The Burau representation and linking

In this Chapter, we construct the Burau representation of B_n, and relate it to the Lefschetz number, and show two geometric interpretations of it. We use this relation with the Lefschetz number to give information about the number of fixed points of a homeomorphism, and their linking with a given periodic orbit. We discuss various notions of linking, and use the Burau representation to provide a partial solution to the 'linking number' conjecture.

5.1 Introduction

In this Chapter, we shall describe the Burau representation of B_n, which may be used to deduce homological fixed point and linking information. We instigate this discussion by defining the representation using Fox's free differential calculus, then exhibiting two geometric interpretations. It is then possible to relate the representation to the Lefschetz number, a quantity which determines a lower bound on the number of fixed points of a map, and tells us about the linking of such fixed points with a given braid type. We give examples which complete proofs of statements made earlier in this Thesis, including the non-existence of homeomorphisms of the torus isotopic to the identity which are pseudo-Anosov relative to a single fixed point. Finally, we discuss two notions of linking, with reference to the 'linking number' problem posed by John Franks, and resolve it in the case where the given braid type is in BT_3, using properties of the Burau representation.

5.2 Free differential calculus

We start by reviewing some simple facts about the free differential calculus of Fox [Fo, Bi3]. Let F_n be a free group on generators $x_1, \ldots, x_n$, and let $\mathbb{Z}\mathsf{F}_n$ be the ring of F_n with coefficients in $\mathbb{Z}$. The free derivative operators

$$\frac{\partial}{\partial x_i} : \mathbb{Z}\mathsf{F}_n \to \mathbb{Z}\mathsf{F}_n$$

have the properties

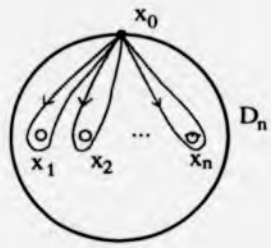

Figure 5.1: Identification of $\pi_1(\mathbf{D}_n, x_0)$ with F_n

1.

$$\frac{\partial}{\partial x_i}(g_1 + g_2) = \frac{\partial}{\partial x_i}(g_1) + \frac{\partial}{\partial x_i}(g_2) \quad g_1, g_2 \in \mathbb{Z}\mathsf{F}_n.$$

2.

$$\frac{\partial}{\partial x_i}(w_1 w_2) = \frac{\partial}{\partial x_i}(w_1) + w_1 \frac{\partial}{\partial x_i}(w_2) \quad w_1, w_2 \in \mathsf{F}_n.$$

3.

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij} \quad \frac{\partial}{\partial x_i}(1) = 0 \quad 1 \leq i \leq n.$$

Let $< t >$ be an infinite cyclic group, and define $\psi : \mathsf{F}_n \to < t >$ by $\psi(x_i) = t$, $i = 1, \ldots, n$. This extends to a map $\mathbb{Z}\mathsf{F}_n \to \Lambda$ which we shall also call ψ, where $\Lambda = \mathbb{Z}[t, t^{-1}]$ is the ring of integral Laurent polynomials in the indeterminate t. Identify F_n with the fundamental group $\pi_1(\mathbf{D}_n, x_0)$ where $x_0 \in \partial \mathbf{D}^2$ is a basepoint, as indicated in figure 5.1. Then

Theorem 5.1 ([Ar, Bi3]) *The braid group B_n has a faithful representation as a group of (right) automorphisms of F_n induced by the map $\mu : B_n \to Aut(\mathsf{F}_n)$ given by*

$$\begin{aligned} (\sigma_i)\mu : \quad x_i &\mapsto x_i x_{i+1} x_i^{-1} \\ x_{i+1} &\mapsto x_i \\ x_j &\mapsto x_j \text{ if } j \neq i, i+1. \end{aligned}$$

(see figure 5.2). Further, B_n is isomorphic to a subgroup of $Aut(\mathsf{F}_n)$ satisfying

$$\begin{aligned} (x_i)(\beta)\mu &= A_i x_{\delta_i} A_i^{-1} \quad 1 \leq i \leq n \\ (x_1 \ldots x_n)(\beta)\mu &= x_1 \ldots x_n \end{aligned}$$

where $(\delta_1, \ldots, \delta_n) \in \Sigma_n$ is a permutation, and $A_i \in \mathsf{F}_n$.

Remarks:

- $(\sigma_i)\mu$ is the isomorphism induced by the homeomorphism $m(\sigma_i)$ in Section 1.3.2.

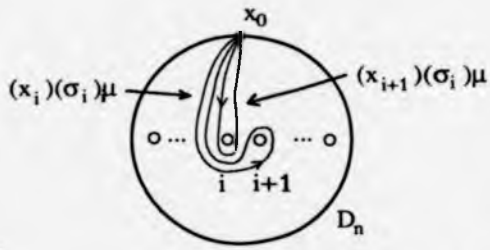

Figure 5.2: The homeomorphism $(\sigma_i)\mu$

- We shall not distinguish between the element $\beta \in B_n$ and the isomorphism $(\beta)\mu$ of F_n.

We define a homomorphism $R : B_n \to GL(n, \Lambda)$ by

$$(R(\beta))_{ij} = \psi\left(\frac{\partial}{\partial x_j}((x_i)\beta)\right) \quad 1 \le i, j \le n.$$

This defines the *full Burau representation.* By choosing generators $g_i = x_1 \ldots x_i$, $i = 1, \ldots, n$ for F_n, and defining $R' : B_n \to GL(n, \Lambda)$ by

$$(R'(\beta))_{ij} = \psi\left(\frac{\partial}{\partial g_j}((g_i)\beta)\right) \quad 1 \le i, j \le n$$

we find that the last row is always $(0, 0, \ldots, 1)$ (one may check this by calculating the matrices $R'(\sigma_i)$, $i = 1, \ldots, n-1$). So the last row and column may be deleted to obtain the *reduced Burau representation* $r : B_n \to GL(n-1, \Lambda)$ (see [Bi3] for more details). We find that

$$\begin{aligned}
r(\sigma_1) &= \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \\
r(\sigma_i) &= \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix} \quad \text{for } 1 \le i \le n-1 \\
r(\sigma_{n-1}) &= \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}.
\end{aligned}$$

It is faithful for $n = 3$, but it has been shown recently that in general [Md] it is not faithful.

5.3 Geometric interpretations and the Lefschetz number

There are two important geometric interpretations of the Burau representation. The first is as a signed linking transition matrix. We first define the linking number of a fixed point with a periodic orbit.

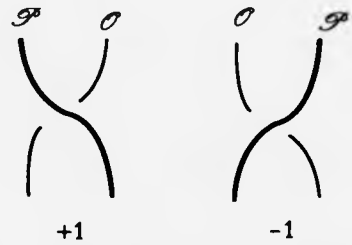

Figure 5.3: Convention for summing crossings of braids

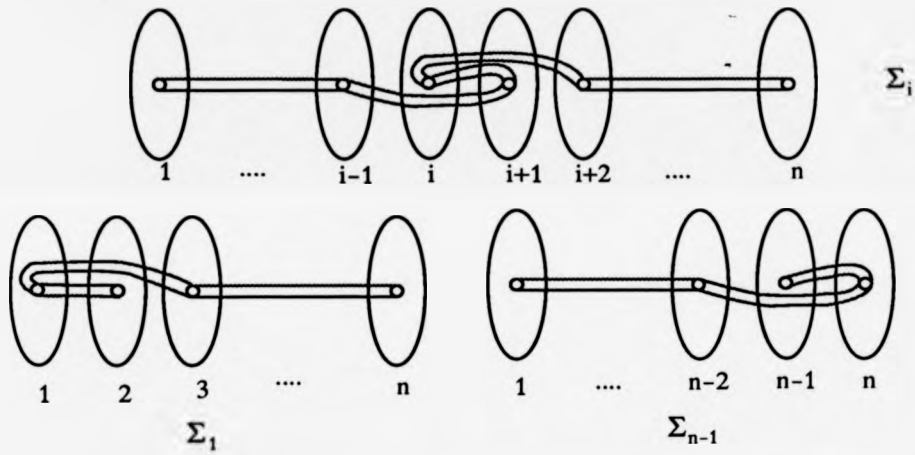

Figure 5.4: The disc $\mathbf{D}^2$ and the action of A_β

Let $f : \mathbf{D}^2 \to \mathbf{D}^2$ be an orientation-preserving homeomorphism, let $\mathcal{O}$ be a periodic orbit of period n, and let $\mathcal{P}$ be a fixed point of f. Pick an isotopy f_t from Id to f, and let $\phi_t : \mathbf{T} \to \mathbf{T}$ be the suspended flow in the solid torus as defined in Section 3.2.1. Then $f_t(\mathcal{P} \cup \mathcal{O})$ is a geometric braid β, say. We sum the number of crossings of the strands of $\mathcal{P}$ with those of $\mathcal{O}$ algebraically, with the convention in figure 5.3. We call this sum the *linking number* of $\mathcal{P}$ with $\mathcal{O}$, relative to the isotopy f_t. Recall that a different choice of isotopy f_t' defines another geometric braid β', such that β and β' are conjugate or differ by an element of $Z(B_{n+1})$. Conjugacy does not change the linking number of $\mathcal{P}$ with $\mathcal{O}$, whilst composition with $\theta_{n+1} \in Z(B_{n+1})$ changes it by n. Hence the *linking number (mod n) of $\mathcal{P}$ with $\mathcal{O}$* is independent of the choice of isotopy. We shall denote this quantity by $lk(\mathcal{P}, \mathcal{O})$.

Fix $\beta \in B_n$, then we consider an Axiom A homeomorphism A_β such that it has a periodic orbit $\mathcal{O}$ which represents β. We do this in the following manner ([Fr2]). Consider the disc $\mathbf{D}^2$ as being the union of n discs joined by $n-1$ boxes $B_1, \ldots, B_{n-1}$ as in figure 5.4. Let Σ_i be a map of $\mathbf{D}^2$ to itself, such that:

1. the n smaller discs are contracted.

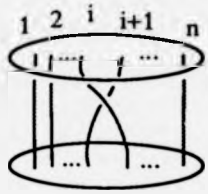

Figure 5.5: The geometric braid associated with ϕ_i is σ_i

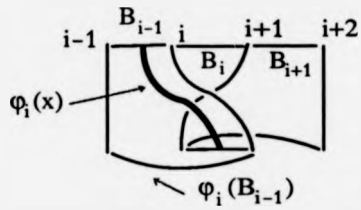

Figure 5.6: The linking of the boxes B_{i-1}, B_i and B_{i+1}

2. horizontal and vertical lines are preserved in the $n-1$ boxes; they are contracted vertically in a uniform manner, and if $j = i \pm 1$, Σ_i expands B_j horizontally in a uniform manner.

3. the ith and $(i+1)^{th}$ discs are swapped as in figure 5.4.

One may define maps Σ_i for $i = 1, \ldots, n-1$, then choose a suspension ϕ_i of each Σ_i such that the geometric braid associated with ϕ_i is exactly σ_i (see figure 5.5). Thus if $\beta = \sigma_{i_1} \ldots \sigma_{i_k}$, then $A_\beta = \Sigma_{i_1} \ldots \Sigma_{i_k}$, and the composition of the suspended flows $\phi_{i_1}, \ldots, \phi_{i_k}$ is associated with the geometric braid β. One may arrange that A_β is Axiom A ([Fr2]), so the non-wandering set is a union of a finite number of sources and sinks, and a Cantor set whose dynamics is given by a subshift of finite type. We may get an enhanced description by defining a signed linking matrix. Consider the braid σ_i (see figure 5.6). We study the manner in which the boxes cover each other. In particular, for orbit segments of ϕ_i, we multiply their representative of the subshift of finite type by factor of $+1$ (respectively -1) if the local orientation is preserved (respectively reversed), and a power of the indeterminate t, where the power is the crossing number of the orbit segment with the braid σ_i (relative to ϕ_i). The *crossing number* is the algebraic sum of crossings of the orbit segment with σ_i, with the convention given in figure 5.7. For example, in figure 5.6, with $x \in \mathrm{Int}(B_{i-1})$ and $\Sigma(x) \in \mathrm{Int}(B_i)$, then the crossing number of $\phi_i(x)$ with σ_i is $+1$. So, for σ_i, the subshift of finite type is

$$\begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}$$

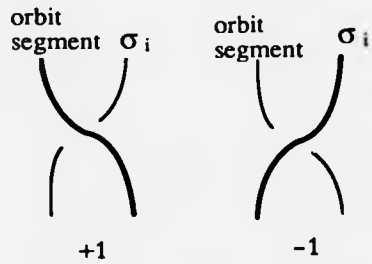

Figure 5.7: Convention for summing crossings of braids

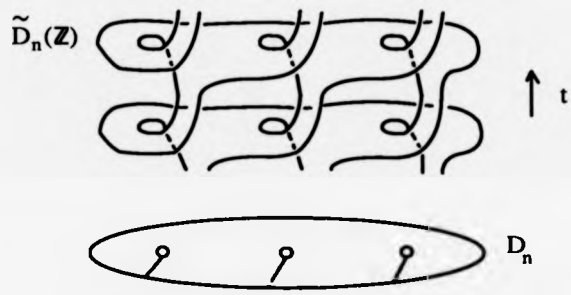

Figure 5.8: The infinite cyclic covering $\tilde{D}_n(\mathbf{Z})$ of $\mathbf{D}_n$

and the associated signed linking matrix is

$$\begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-i-2} \end{pmatrix}.$$

It may be verified that the signed linking matrix associated with the braid σ_i is exactly the matrix $r(\sigma_i)$, and that the signed linking matrix of the braid β is the composition $r(\sigma_{i_1})\dots r(\sigma_{i_k})$.

The second interpretation of the Burau representation is as the induced action on homology of a covering of $\mathbf{D}_n$. Consider the mapping $\psi : \mathbf{Z}\mathsf{F}_n \to \Lambda$; the normal subgroup $\ker\psi$ satisfies $(\ker\psi)\beta \subset \ker\psi$, so there exists a covering $\tilde{D}_n(\mathbf{Z})$ of $\mathbf{D}_n$ whose group of covering transformations is isomorphic to $\mathsf{F}_n/\ker\psi \cong < t >$. So $\tilde{D}_n(\mathbf{Z})$ is the infinite cyclic covering of $\mathbf{D}_n$ (see figure 5.8). Let G_n be the graph with one vertex x_0 and n oriented branches $e_1, \dots, e_n$ formed by retracting $\mathbf{D}_n$ onto its 1-skeleton, and let $\tilde{G}_n(\mathbf{Z})$ be its infinite cyclic covering (see figure 5.9).

Proposition 5.2 ([FH, Fri4]) *Let $w = w(x_1, \dots, x_n)$ be a word in the generators of F_n. Fix a lift $\tilde{x}_0$ of x_0, and lift the corresponding loop w with initial point $\tilde{x}_0$, then the*

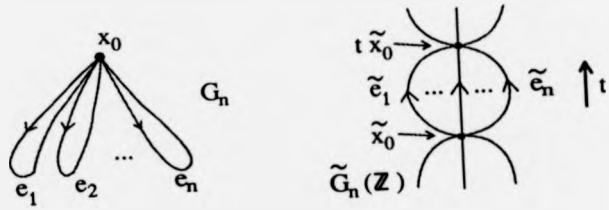

Figure 5.9: G_n and its infinite cyclic covering $\tilde{G}_n(\mathbf{Z})$

associated 1-chain in the cellular chain group $C_1(\tilde{G}_n(\mathbf{Z}))$ has the form

$$\gamma(w) = \psi\left(\frac{\partial w}{\partial x_1}\right)\tilde{e}_1 + \cdots + \psi\left(\frac{\partial w}{\partial x_n}\right)\tilde{e}_n,$$

where $\tilde{e}_1, \ldots, \tilde{e}_n$ correspond to lifts of $e_1, \ldots, e_n$ with basepoint $\tilde{x}_0$, and where the partial derivatives are taken with respect to the free differential calculus.

In particular, if $w = (x_i)\beta$, the induced action $F_1 : C_1(\tilde{G}_n(\mathbf{Z})) \to C_1(\tilde{G}_n(\mathbf{Z}))$ of β is

$$F_1(\tilde{e}_i) = \gamma((x_1)\beta) = \psi\left(\frac{\partial((x_i)\beta)}{\partial x_1}\right)\tilde{e}_1 + \cdots + \psi\left(\frac{\partial((x_i)\beta)}{\partial x_n}\right)\tilde{e}_n.$$

So the full Burau representation arises as the matrix of the induced action of β on 1-chains in $C_1(\tilde{G}_n(\mathbf{Z}))$.

Moreover, we have an equivariant version of the Lefschetz formula [FH, Fri4]

$$L_{<t>}(\beta) = \sum_{i \geq 0}(-1)^i \operatorname{Trace}(F_i) \tag{5.1}$$

where $F_i : C_i(\tilde{G}_n(\mathbf{Z})) \to C_i(\tilde{G}_n(\mathbf{Z}))$ are the induced maps on the cellular chain groups. It follows [FH] that

$$\begin{aligned} L_{<t>}(\beta) &= 1 - \operatorname{Trace} R(\beta) && (5.2)\\ &= -\operatorname{Trace} r(\beta) \in \Lambda. && (5.3) \end{aligned}$$

The *Lefschetz number* $L_{<t>}(\beta)$ [Br] is an isotopy invariant which gives fixed point information. So if $f \in Homeo^+(\mathbf{D}^2)$ has a periodic orbit $\mathcal{O}$ such that $bt(\mathcal{O}, f) = \beta$, and Trace $r(\beta)$ contains a term $a_i t^i$ where $a_i \neq 0$, then f has a fixed point $\mathcal{P}$ such that the linking number of $\mathcal{P}$ with $\mathcal{O}$ is i (relative to the suspension $\phi_{i_1} \ldots \phi_{i_k}$). Given a braid β, the Burau representation enables us to deduce fixed point and linking information as homology invariants.

Remark : one may replace the map ψ by the Abelianization operator, whence Λ is replaced by $\mathbf{Z}[a_1^{\pm 1}, \ldots, a_n^{\pm 1}]$. Defining analogous representations ('twisted') allows us to deduce more detailed information. This is obviously useful if one is considering surface homeomorphisms of, say, the annulus, where we would want to keep track of how fixed points link with a given periodic orbit and rotate around the annulus. The corresponding Lefschetz number is a special case of the more general Reidemeister trace, which we shall review in Chapter 6. As an example of this, we prove the assertion made in the proof of Theorem 3.17.

Proposition 5.3 *Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism of the annulus isotopic to the identity, let $o(x_1)$ be a period 2 orbit of f, such that $\rho(o(x_1), F) = 0$ relative to some lift $F : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of f, and such that f is pseudo-Anosov relative to $o(x_1)$. Suppose further that f is such that one of the induced Anosov diffeomorphisms of $\mathbb{T}^2$*

$$A = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}$$

descends to f in the manner described in Section 3.6.1. Then $\rho(F) \supset [-1, 1]$.

Proof

Consider the matrices in equation 3.2, then one may verify that

$$a_1 a_2 a_1^3 = \begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$$

and

$$a_2^3 a_1 a_2 = \begin{pmatrix} 2 & 1 \\ 7 & 4 \end{pmatrix}.$$

The calculation in each case is similar, so we shall just carry out the first one. Let $\beta = \theta_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^3$, then its associated Anosov diffeomorphism is

$$\begin{pmatrix} 2 & 7 \\ 1 & 4 \end{pmatrix}$$

(up to the involution $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S(x, y) = -(x, y)$). Let ϕ be the suspension of f such that $\phi(o(x_1))$ is the geometric braid β in the annulus, then as we have noted in Section 3.4, there is a bijection between the homology directions of ϕ and the rotation set $\rho(F)$. So it suffices to show that the action induced by β on $\mathbb{F}_3$ implies the existence of two fixed points, one with strictly positive linking number and one with strictly negative linking number about the annulus. Using the relations in Theorem 5.1 one may verify that

$$\begin{aligned} (x_1)\beta &= A_1 x_3 A_1^{-1} \\ (x_2)\beta &= A_2 x_2 A_2^{-1} \\ (x_3)\beta &= A_3 x_1 A_3^{-1}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= x_3^{-1} x_1^{-1} x_3 x_1 x_2 x_1^{-1} \\ A_2 &= x_3^{-1} x_1^{-1} x_3 x_1 \\ A_3 &= x_3^{-1} x_2^{-1} x_1^{-1} x_3^{-1}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial}{\partial x_1}((x_1)\beta) &= -x_3^{-1}x_1^{-1} + x_3^{-1}x_1^{-1}x_3 - \\ &\quad x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1} + x_3^{-1}x_1^{-1}x_3x_1x_2x_1^{-1}x_3 - \\ &\quad ((x_1)\beta).x_3^{-1}x_1^{-1}x_3 + ((x_1)\beta).x_3^{-1}x_1^{-1} \\ \frac{\partial}{\partial x_2}((x_2)\beta) &= x_3^{-1}x_1^{-1}x_3x_1 \\ \frac{\partial}{\partial x_3}((x_3)\beta) &= -x_3^{-1} - x_3^{-1}x_2^{-1}x_1^{-1}x_3^{-1} + \\ &\quad x_3^{-1}x_2^{-1}x_1^{-1}x_3^{-1}x_1 + ((x_3)\beta).x_3^{-1}. \end{aligned}$$

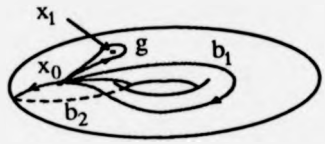

Figure 5.10: The elements g,b_1,b_2 of $\pi_1(M, x_0)$

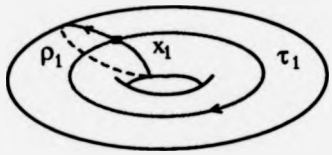

Figure 5.11: The generators τ_1,ρ_1 of $\pi_1(B(\mathsf{T}_1^2))$

Let ψ be the Abelianization operator, then the corresponding Lefschetz number is [FH]

$$\begin{aligned} L_{Ab}(\beta) &= 1 - \text{ Trace } \psi\left(\frac{\partial}{\partial x_j}((x_i)\beta)\right) \\ &= x_1^{-1}x_3^{-1} - x_1^{-1} + x_1^{-1}x_2 - x_1^{-1}x_2x_3 \\ &\quad + x_1^{-1}x_2^{-1}x_3^{-2} - x_2^{-1}x_3^{-2}. \end{aligned}$$

In particular, this implies the existence of fixed points which have strictly positive linking number about the annulus (the $x_1^{-1}x_2$ and $x_1^{-1}x_2x_3$ terms) and strictly negative linking number (the $x_1^{-1}x_2^{-1}x_3^{-2}$ and $x_2^{-1}x_3^{-2}$ terms), as x_2 is the loop corresponding to rotation about the annulus. This information is homotopy invariant, so f has fixed points of rotation type $(-1,1)$ and $(1,1)$, and the Proposition is proved. □

By using a similar approach, we may relate the group $Aut(M)$ to the braid group for an arbitrary manifold M [Bi1, Bi2]. One case that we are particularly interested in is that where $M = \mathsf{T}_1^2$, the 2-torus minus a single point x_1. Fix a basepoint $x_0 \in M$, then $\pi_1(M, x_0)$ is a free group of rank 2. Define elements g,b_1,b_2 as shown in figure 5.10. Then $g = b_1b_2b_1^{-1}b_2^{-1}$, and $\{b_1, b_2\}$ is a basis for $\pi_1(M, x_0)$. Let $\pi_1(B(\mathsf{T}_1^2))$ denote the braid group of T_1^2, then:

Proposition 5.4 ([Bi1, HuJ]) *$\pi_1(B(\mathsf{T}_1^2))$ is generated by the generators τ_1,ρ_1, where τ_1 and ρ_1 are as shown in figure 5.11.*

By inspection, one may verify that ρ_1 and τ_1 define automorphisms $\rho_1, \tau_1 : \mathsf{F}_2 \to \mathsf{F}_2$ whose action on $\{b_1, b_2\}$ is given by:

$$\rho_1 \ : \ \begin{cases} b_1 \mapsto b_2b_1b_2^{-1} \\ b_2 \mapsto b_2 \end{cases} \tag{5.4}$$

$$\tau_1 \ : \ \begin{cases} b_1 \mapsto b_1 \\ b_2 \mapsto b_1b_2b_1^{-1}. \end{cases} \tag{5.5}$$

Given a simple closed loop $\gamma \in \pi_1(\mathbf{T}_1^2, x_0)$, we may write it as a word in the generators $\{b_1, b_2\}$. Moreover, it defines a free homotopy class $[\gamma]$, which may also be written as a word $w([\gamma])$ in $\{b_1, b_2\}$. But $w([\gamma])$ is only defined up to conjugacy (i.e. up to an inner automorphism). So we may rewrite equations 5.4 and 5.5 in terms of their action on free homotopy classes:

$$\rho_1 \quad : \quad \begin{cases} [b_1] \mapsto [b_1] \\ [b_2] \mapsto [b_2] \end{cases}$$

$$\tau_1 \quad : \quad \begin{cases} [b_1] \mapsto [b_1] \\ [b_2] \mapsto [b_2] \end{cases}$$

i.e. both induce the identity on free homotopy classes. ρ_1 and τ_1 generate $\pi_1(B(\mathbf{T}_1^2))$, so if $f : \mathbf{T}^2 \to \mathbf{T}^2$ were a homeomorphism isotopic to the identity such that $bt(x_1, f)$ were pseudo-Anosov, then one could write $\beta \in bt(x_1, f)$ as a word in τ_1 and ρ_1. But β induces the identity on free homotopy classes – a contradiction of property 3 in Section 1.6, as both b_1 and b_2 are essential curves. So $bt(x_1, f)$ cannot be pseudo-Anosov. We sum this up as:

Proposition 5.5 *Suppose $f : \mathbf{T}^2 \to \mathbf{T}^2$ is a homeomorphism of the 2-torus isotopic to the identity, and suppose that x_1 is a fixed point of f. Then $bt(x_1, f)$ is not pseudo-Anosov.* □

We can say more. Suppose F is the Thurston canonical form for $f_{\{x_1\}}$, and $bt(x_1, f)$ is reducible. Clearly, there are no non-rotational reducing curves on $\mathbf{T}_1^2$. So suppose Γ is a rotational reducing curve for F. Γ must have period one, or else when we remove the tubular neighbourhood of Γ and its images, we obtain a disjoint union of at least 2 annuli, only one of which has holes, which implies that $\Gamma \simeq F^m(\Gamma)$ for some $1 \leq m < p$, which contradicts Γ having period $p > 1$. So removing Γ and its tubular neighbourhood, we obtain an annulus $\mathbf{A}_1$ with one hole. Proposition 3.11 implies that $F|_{\mathbf{A}_1}$ is periodic, and since it fixes all the boundary components $\partial\mathbf{A}_1$, $F|_{\mathbf{A}_1}$ is the identity. There can be no Dehn twist around Γ, since β induces the identity on free homotopy classes. Thus F is not reducible, and so must be periodic.

Suppose $bt(x_1, f)$ is periodic, then by Theorem 2.1, $F \approx T_{(p,0)/q}/\Gamma_{\square}$, for some $q \in \mathbf{N}$, $p \in \mathbf{Z}_q$, so all points of $\mathbf{T}^2$ are periodic with period q. But the completion of F has a fixed point, hence F is the identity. We summarize this as:

Theorem 5.6 *Suppose $f : \mathbf{T}^2 \to \mathbf{T}^2$ is a homeomorphism isotopic to the identity with a fixed point x, and let $F : \mathbf{T}_1^2 \to \mathbf{T}_1^2$ be the Thurston representative of $f_{\{x\}}$. Then F is the identity.* □

5.4 The linking number problem

In this Section, we discuss the following question posed by John Franks:

given $f \in Homeo^+(\mathbf{D}^2)$ and a periodic orbit $\mathcal{O} \subset Int(\mathbf{D}^2)$ of f, does there always exist a fixed point about which $\mathcal{O}$ has non-zero rotation number?

This problem is still unsolved, although some partial results are known. For instance, if $bt(\mathcal{O}, f)$ is periodic, there always exists such a fixed point; this follows from the fact that

$f \simeq g$ rel. $\mathcal{O}$, where g is conjugate to a non-trivial rotation (see Section 1.8.1). Thus g has a fixed point satisfying the conclusions of the question, and the rotation number of $\mathcal{O}$ about this fixed point does not change under isotopy rel. $\mathcal{O}$. For this reason, if $\mathcal{O}$ has period 2 the question is answered in the affirmative, since $bt(\mathcal{O}, f)$ is periodic in this case. We shall discuss various notions of linking between two periodic orbits, and how we can relate the geometric ideas to algebraic considerations, in particular the Burau representation of B_n. In this way, we are able to answer the question in the affirmative for the case when $\mathcal{O}$ has period 3.

5.4.1 Linking of a fixed point with a periodic orbit

We have already defined the linking number of a fixed point with a periodic orbit for a homeomorphism of the disc in Section 5.3, where the definition was made relative to a particular suspended flow. We may also think of the linking number without reference to the suspension. Consider the homeomorphism $f_{\mathcal{P}} : \mathbf{A} \to \mathbf{A}$ of the annulus induced by blowing up $\mathcal{P}$, and let $F : \bar{\mathbf{A}} \to \bar{\mathbf{A}}$ be a lift of $f_{\mathcal{P}}$. The rotation number of $\mathcal{O}$ about $\mathcal{P}$ relative to F is a rational l_F/n. If F' is another lift of $f_{\mathcal{P}}$, then l_F and $l_{F'}$ differ by a multiple of n. Then we define $lk(\mathcal{P}, \mathcal{O}) = l_F \bmod n$, and one may verify that it coincides with the definition given in Section 5.3. We say that $\mathcal{P}$ and $\mathcal{O}$ are *strongly linked* if $0 < lk(\mathcal{P}, \mathcal{O}) < n$. These two ways of characterizing the linking number are related to the discussion in Section 3.3.

We discuss another notion of linking of a fixed point $\mathcal{P}$ with a periodic orbit $\mathcal{O}$ (due to Gambaudo). We say that $\mathcal{P}$ and $\mathcal{O}$ are *not linked* if there exists a Jordan curve $\mathcal{C}$ in $\mathbf{D}^2$ bounding a disc $\mathcal{D}$ such that:

1. $\mathcal{O} \subset \mathrm{Int}(\mathcal{D})$
2. $\mathcal{P} \subset \mathbf{D}^2 \setminus \mathcal{D}$
3. $f(\mathcal{C}) \simeq \mathcal{C}$ in $\mathbf{D}^2 \setminus \{\mathcal{O} \cup \mathcal{P}\}$.

Otherwise we say that $\mathcal{P}$ and $\mathcal{O}$ are *linked* (see figure 5.12). If $\mathcal{P}$ and $\mathcal{O}$ are linked, then the suspended orbits are linked as knots in the solid torus. Gambaudo has shown

Theorem 5.7 ([Ga]) *Let f be an orientation-preserving embedding of $\mathbf{D}^2$, and let $\mathcal{O}$ be a periodic orbit of f, then there exists a fixed point $\mathcal{P}$ of f such that $\mathcal{P}$ and $\mathcal{O}$ are linked.*

Kolev has strengthened this to the case where $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a orientation-preserving homeomorphism of the plane [Ko3]. The affirmation of the question of Franks would express a stronger version of Theorem 5.7.

5.4.2 The Alexander polynomial and $n = 3$

We do not intend to discuss the Alexander polynomial in any detail, except to give a few brief properties, and its relation to the Burau representation. For a detailed exposition, one may consult [BZ], for example.

Suppose $\beta \in B_n$ is a braid. We say it is *positive* if it may be expressed as a word in positive powers of the σ_i.

Proposition 5.8 (Garside [Bi3]) *For all $\beta \in B_n$, there exists β_+ positive and $m \in \mathbf{Z}$ such that $\beta = \beta_+(\theta_n)^m$.*

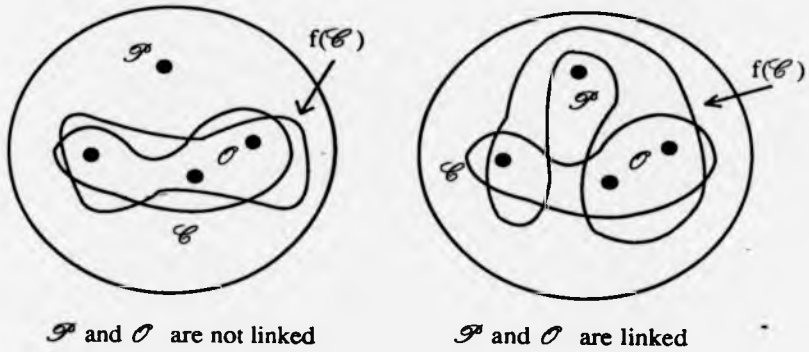

Figure 5.12: Examples of linking and non-linking

Suppose $f \in Homeo^+(\mathbf{D}^2, X)$ such that β represents $bt(X, f)$. By definition, β_+ also represents $bt(X, f)$ (β and β_+ correspond to different choices of suspension of f). Since they differ by full twists, the suspended orbits of fixed points of f have the same linking number (mod n) with β as with β_+. So dynamically, β_+ is as good a choice of representative for $bt(X, f)$ as β. This is reflected in the reduced Burau representation, one may verify that $r(\theta_n) = t^n I_{n-1}$. So without loss of generality, we shall assume that β is positive. Let $\bar{\beta}$ be the link associated with β. Then

Theorem 5.9 (Burau [Bi3]) *Suppose $\beta \in B_n$ is a (positive) braid word of length m involving all the generators $\sigma_1, \ldots, \sigma_{n-1}$. Then the Alexander polynomial $\nabla_\beta(t)$ of $\bar{\beta}$ has degree $m - n + 1$, and has leading coefficient 1. Moreover,*

$$(1 + t + \cdots + t^{n-1})\nabla_\beta(t) = \det(r(\beta) - Id). \tag{5.6}$$

Remarks

1. Write $D(t) = \det(r(\beta) - Id)$. Then it follows that $D(t)$ has degree m and leading coefficient 1.

2. For any n, $\det(r(\sigma_i)) = (-t)$, so $\det(r(\beta)) = (-t)^m$.

The *Alexander polynomial* has the following properties:

1. $\nabla_\beta(t) = \sum_{i=0}^{2r} a_i t^i$ for some r.

2. $a_{2r-i} = a_i$ for $i = 1, \ldots, 2r$.

3. $\nabla_\beta(0) = a_0 = a_{2r} = \pm 1$.

Equation 5.6 implies that $D(t)$ has corresponding properties. We deduce that n is odd if and only if m is even.

We now restrict to the case $n = 3$. Then

$$\begin{aligned} D(t) &= 1 - \text{Trace}\,(r(\beta)) + \det(r(\beta)) \\ &= 1 - \text{Trace}\,(r(\beta)) + t^m \end{aligned}$$

and Trace $(r(\beta))$ is a polynomial of degree strictly less than m (see Appendix to Section 3, [Bi3]).

Assume that there exist no fixed points $\mathcal{P}$ for f such that $lk(\mathcal{P}, X) \neq 0$. Then

$$\text{Trace } (r(\beta)) = \sum_{i=0}^{s} b_i t^{3i}$$

for some $s \geq 0$. From equation 5.2

$$L_{<t>}(f) = - \text{ Trace } r(\beta)$$

hence the Lefschetz number $L(f)$ of f is

$$L(f) = - \text{ Trace } r(\beta)|_{t=1}.$$

But $L(f) = 1$, thus

$$\sum_{i=0}^{s} b_i = -1.$$

Equation 5.6 implies that

$$(1 + t + t^2)(a_0 + a_1 t + \cdots + a_{m-3}t^{m-3} + a_{m-2}t^{m-2}) = \\ 1 - b_0 - b_1 t^3 - \cdots - b_s t^{3s} + t^m.$$

Comparing coefficients, and using the properties of $D(t)$, we find that $6|m$, $b_0 = 0$, $a_0 = 1$, $a_1 = -1$, $a_2 = 0$, and in general,

$$\begin{aligned} a_j &= 1 - b_1 - \cdots - b_{j/3} \quad &\text{if} \quad j \equiv 0 \bmod 3 \\ a_j &= -(1 - b_1 - \cdots - b_{(j-1)/3}) \quad &\text{if} \quad j \equiv 1 \bmod 3 \\ a_j &= 0 \quad &\text{if} \quad j \equiv 2 \bmod 3. \end{aligned}$$

Since $m \equiv 0 \bmod 6$, then $m - 2 \equiv 1 \bmod 3$, so $a_{m-2} = a_0 = -(1 - \sum_{i=0}^{s} b_i) = 1$. Therefore we conclude that $\sum_{i=0}^{s} b_i = 2$ – a contradiction. Hence Trace $(r(\beta))$ must contain a term $a_i t^i$ where $i \not\equiv 0 \bmod 3$ and $a_i \neq 0$, and thus f has a fixed point $\mathcal{P}$ such that $lk(\mathcal{P}, X) \neq 0$. We have shown:

Theorem 5.10 *Given $f \in Homeo^+(\mathbf{D}^2)$ and a periodic orbit $\mathcal{O} \subset Int(\mathbf{D}^2)$ of period 3 of f, there always exists a fixed point about which $\mathcal{O}$ is strongly linked.* □

So the linking number conjecture of Franks is true for the case when the period of the periodic orbit is 3. It is still open in all other cases. As a corollary of Theorem 5.10, we recover Theorem 4.1.

Proof of Theorem 4.1

Let g be the Thurston representative in the isotopy class of f relative to $o(z)$. g cannot be reducible relative to $o(z)$, from Theorem 1.11. Suppose g is periodic relative to $o(z)$, then by a Theorem of Brouwer [Bro] (see also [Ei, Kel]), g is conjugate to rotation through an angle of $2\pi/3$ or $4\pi/3$. Otherwise g is pseudo-Anosov relative to $o(z)$. Let y be the fixed point given by Theorem 5.10, and let $\bar{g}$ be the homeomorphism of the annulus $\bar{\mathbf{A}}$ induced by g by blowing up y. Since $o(z)$ has non-zero rotation number about y, there exists a lift $G : \tilde{\mathbf{A}} \to \tilde{\mathbf{A}}$ of $\bar{g}$ such that $\rho(o(z), G) = 1/3$ or $\rho(o(z), G) = 2/3$. As g is pseudo-Anosov relative to $o(z)$, then $\bar{g}$ is pseudo-Anosov relative to $o(z)$, and Theorem 3.1 implies that either $\rho(G) \supset I(1/3) = [0, 1/2]$ or $\rho(G) \supset I(2/3) = [1/2, 1]$. Theorem 3.7 and Theorem 4.4 guarantee that g has periodic orbits of all periods, and by unremovability of periodic orbits, so does f. □

Chapter 6

Rotation sets of homeomorphisms of the annulus

In this Chapter, we provide a generalization of the Aubry-Mather Theorem, by demonstrating the existence of periodic orbits whose braid type is periodic with specified rotation numbers for homeomorphisms of the annulus. Further, using topological methods and Nielsen-Reidemeister fixed point theory, we show that if the braid type of a periodic orbit is pseudo-Anosov, then the corresponding rotation number lies in the interior of the rotation set.

6.1 Introduction

Much is known about periodic orbits of homeomorphisms of the annulus, especially in the case where the homeomorphism is monotone twist and area-preserving [Boy4]. Then the Aubry-Mather Theorem implies the existence of monotone periodic orbits. It has been generalized to the case where the homeomorphism is just monotone twist by Hall [HG1], and where it is just area-preserving by Le Calvez [LC]. In [HG2], the former asked if the appropriate generalization of the Aubry-Mather Theorem was true without both conditions. Recently, Boyland [Boy3] has shown that if f is a homeomorphism of the annulus and p/q is a rational in lowest terms that is contained in the rotation set of f, then f has a periodic orbit of period q, rotation number p/q and whose braid type is periodic. As we have seen in Chapter 3, if f has a p/q-periodic orbit whose braid type is pseudo-Anosov then the Farey interval of p/q is contained in the rotation set of f. However this does not deal with the case where p and q have a common factor, since the proof relies on number-theoretic properties. In this Chapter we generalize this result to the case where p and q have a common factor, and give a proof which uses the topological nature of f in an explicit manner. The statement of the main Theorem is:

Theorem 6.1 *Suppose $f : A \to A$ is an orientation-preserving homeomorphism of the annulus which preserves the boundary components, with a periodic orbit $o(x)$, such that the rotation number of $o(x)$ relative to a lift $F : \tilde{A} \to \tilde{A}$ of f is p/q . If the braid type of $o(x)$ is pseudo-Anosov, then:*

1. *p/q is contained in the interior of the rotation set of F.*
2. *f has a periodic orbit whose braid type is periodic and whose rotation number is p/q relative to F.*

6.2 Markov partitions and periodic orbits

In this Section, we give the proof of part 1 of Theorem 6.1, although the proof of Proposition 6.4 is deferred until Section 6.3 after we have discussed Nielsen fixed point theory. We shall show how one may use Markov partitions associated with a pseudo-Anosov homeomorphism to approximate the forward orbit of a point by periodic orbits. We prove that if the chosen periodic orbit $o(x)$ is of pseudo-Anosov braid type, then on the universal cover of the annulus, there are orbits which advance faster (asymptotically) than $o(x)$ under iteration, and ones which advance slower. In this way, we show that the rotation number of $o(x)$ lies in the interior of the rotation set.

Proof of Theorem 6.1

Let $g : \mathbb{A} \to \mathbb{A}$ be the pseudo-Anosov homeomorphism in the isotopy class of f relative to $o(x)$. Pick the lift $G : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of g such that $\rho(o(x), G) = p/q$ i.e. G is equivariantly homotopic to F rel. $\pi^{-1}(o(x))$. Theorem 3.8 and Proposition 3.9 imply that $\rho(G)$ is a closed interval with rational endpoints contained in $\rho(F)$. So it suffices to prove the Theorem for g. As we have seen in Section 3.3, $\rho(G)$ is related to the set of homology directions D_ϕ of some suspension ϕ of g. By Theorem 3.6, the homology classes of closed orbits corresponding to minimal loops in a Markov partition span integral homology. It follows that the set $\rho(G)$ of rotation numbers has non-empty interior, i.e. Int $\rho(G) \neq \emptyset$.

We shall argue by contradiction. Suppose $p/q \notin$ Int $\rho(G)$ i.e. p/q is an endpoint of $\rho(G)$. By considering g^{-1} if necessary, we shall assume it is the lower endpoint. Let $H = T^{-p}G^q$, and let $h : \mathbb{A} \to \mathbb{A}$ be the projection of H. Then h is pseudo-Anosov relative to $o(x)$, and $H\tilde{x} = \tilde{x}$ for any $\tilde{x} \in \pi^{-1}(o(x))$. Thus $\rho(H) = [0, a]$, where $a > 0$.

Let $\tilde{o}(x) = \pi^{-1}(o(x))$. By blowing up the points of $o(x)$ to a set $\mathcal{C}$ of boundary circles, we obtain the induced pseudo-Anosov homeomorphism $h_{\mathcal{O}} : \mathbb{A}_{\mathcal{O}} \to \mathbb{A}_{\mathcal{O}}$, and its lift $H_{\mathcal{O}} : \tilde{\mathbb{A}}_{\mathcal{O}} \to \tilde{\mathbb{A}}_{\mathcal{O}}$. Let $\tilde{\mathcal{C}} = \pi^{-1}(\mathcal{C})$ be the set of boundary circles in $\tilde{\mathbb{A}}_{\mathcal{O}}$. Let $\pi_{\mathcal{C}} : \tilde{\mathbb{A}}_{\mathcal{O}} \to \mathbb{A}_{\mathcal{O}}$ denote the induced projection. There exists a Markov partition $\mathcal{R} = \cup_{i=1}^{N} R_i$ for $h_{\mathcal{O}}$, which we can lift to give a partition $\tilde{\mathcal{R}}$ for $H_{\mathcal{O}}$. By refining $\mathcal{R}$ if necessary, we may assume that the corresponding transition matrix is $0-1$ i.e. the intersection Int $R_i \cap$ Int $h_{\mathcal{O}}(R_j)$ consists of at most one connected component. We call such a partition *fine*. Choose a (connected) fundamental domain $\mathcal{D}$ of $\tilde{\mathbb{A}}_{\mathcal{O}}$, in such a way that it is the union of elements of $\pi_{\mathcal{C}}^{-1}(R_i)$ for $1 \leq i \leq N$ (so that the boundaries of $\mathcal{D}$ are also boundaries of elements of the partition $\tilde{\mathcal{R}}$). Let $R_i(0) = \mathrm{Cl}(\mathcal{D} \cap \text{ Int } \pi_{\mathcal{C}}^{-1}(R_i))$ for $i = 1, \ldots, N$, and let $R_i(j) = T^j R_i(0)$ for $j \in \mathbb{Z}$. (T is the usual integer translation). So the union $\cup_{i=1}^{N} R_i(j)$ partitions $T^j\mathcal{D}$.

Consider the set of minimal loops associated with $\mathcal{R}$ i.e. the set of loops in the graph associated with $\mathcal{R}$ such that no symbol is repeated. To such a loop $s(0)\ldots s(n)$, we may associate a unique translation, as there exists a point $x \in \mathbb{A}$ satisfying $h_{\mathcal{O}}^i(x) \in R_{s(i)}$ for $i = 0, 1, \ldots, n$. Take $\tilde{x} \in \pi_{\mathcal{C}}^{-1}(x) \cap \mathcal{D}$ then $\tilde{x} \in R_{s(0)}(0)$ and $H_{\mathcal{O}}^n(\tilde{x}) \in R_{s(n)}(l) = R_{s(0)}(l)$ for some $l \in \mathbb{Z}$. Since $\mathcal{R}$ is a fine partition, l is uniquely determined, and we say that the closed loop $s(0)\ldots s(n)$ has *translation* l. But each such loop has a periodic orbit of $h_{\mathcal{O}}$ associated to it whose rotation number is non-negative. Hence $l \geq 0$ for all minimal loops, and thus for all loops in $\mathcal{R}$.

Lemma 6.2 *For any point $z \in \tilde{\mathbb{A}}_{\mathcal{O}}$, there exists a uniform bound $M > 0$, such that for any $n \in \mathbb{N}$, $p_1(H_{\mathcal{O}}^n(z)) - p_1(z) \geq -M$.*

This says that any point $(x, y) \in \tilde{\mathbb{A}}_{\mathcal{O}}$ under iteration by $H_{\mathcal{O}}$ can only move a bounded

distance in the negative x-direction. Intuitively, this is because we may approximate its orbit by periodic orbits (as symbol sequences) whose translations are all non-negative, and any discrepancies in this approximation can always be bounded. We now make this precise.

Proof

Take any point $z \in \mathcal{D}$, and suppose it has forward symbol sequence $s(0)s(1)s(2)\ldots$ Consider the segment $s(0)\ldots s(n)$ for any n; choose i_1 as large as possible so that $s(i_1) = s(0)$, then i_2 as large as possible so that $s(i_2) = s(i_1+1)$ etc. to generate a sequence $0 \leq i_1 < i_2 < \ldots \leq i_K = n$. Note that $1 \leq K \leq N$. Then for $1 \leq j \leq K-1$, the sequence $s(i_j+1), \ldots, s(i_{j+1}-1), s(i_{j+1})$ corresponds to a closed loop in $\mathcal{R}$, and is associated with a periodic orbit of $h_{\mathcal{O}}$. So we have the following equality:

$$\begin{aligned} p_1(H^n_{\mathcal{O}}(z)) - p_1(z) &= \sum_{j=1}^{K-1} p_1(H^{i_{j+1}}_{\mathcal{O}}(z)) - p_1(H^{i_j}_{\mathcal{O}}(z)) \\ &= \sum_{j=1}^{K-1} p_1(H^{i_{j+1}}_{\mathcal{O}}(z)) - p_1(H^{i_j+1}_{\mathcal{O}}(z)) \\ &+ \sum_{j=1}^{K-1} p_1(H^{i_j+1}_{\mathcal{O}}(z)) - p_1(H^{i_j}_{\mathcal{O}}(z)). \end{aligned}$$

Since $s(i_j+1) = s(i_{j+1})$, each term in the first summation has an associated periodic orbit whose translation is non-negative. Let

$$L = \max_{1 \leq i \leq N} \max_{x,y \in R_i(0)} (p_1(x) - p_1(y)) > 0$$

be the maximum projected width of the Markov boxes. The second summation is

$$\sum_{j=1}^{K-1} p_1(H_{\mathcal{O}}(H^{i_j}_{\mathcal{O}}(z))) - p_1(H^{i_j}_{\mathcal{O}}(z)) \geq -S(K-1),$$

where

$$S = \max_{x \in \mathcal{D}} |p_1(x) - p_1(H_{\mathcal{O}}(x))|.$$

So

$$p_1(H^n_{\mathcal{O}}(z)) - p_1(z) \geq -(S+L)(K-1),$$

where S,L and $K \leq N$ are independent of z. Hence we can find an $M > 0$, satisfying the above inequality for all z. □

We now make some definitions. Let $\alpha : I \to \tilde{\mathbb{A}}_{\mathcal{O}}$ be a continuous path. Let $(-\alpha) : I \to \tilde{\mathbb{A}}_{\mathcal{O}}$ be the path defined by

$$(-\alpha)(t) = \alpha(1-t), \quad 0 \leq t \leq 1.$$

Let $\circ$ denote the *composition* of two arcs i.e. if $\gamma_1, \gamma_2 : I \to \mathbb{A}_{\mathcal{O}}$ are two arcs such that $\gamma_1(0) = \gamma_2(1)$, then we define

$$\begin{aligned} \gamma_1 \circ \gamma_2 &= \gamma_2(2t) \quad 0 \leq t \leq 1/2 \\ &= \gamma_1(2t-1) \quad 1/2 \leq t \leq 1. \end{aligned}$$

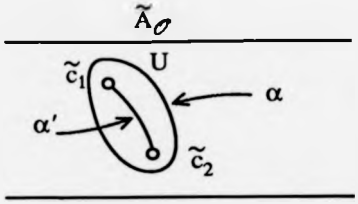

Figure 6.1: The essential loop α and its retract α'

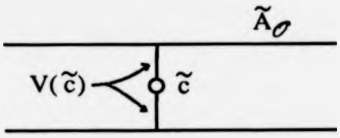

Figure 6.2: The vertical $V(\tilde{c})$ passing through $\tilde{c}$

Now take any two boundary circles $\tilde{c}_1, \tilde{c}_2 \in \tilde{\mathcal{C}} \cap \mathcal{D}$, and consider a simple loop $\alpha : I \to \mathcal{D}$, such that the compact domain U bounded by α satisfies $U \cap \tilde{\mathcal{C}} = \{\tilde{c}_1, \tilde{c}_2\}$. Thus α surrounds precisely $\tilde{c}_1$ and $\tilde{c}_2$, and so is essential (see figure 6.1). Let α' be a retract of α within U onto its skeleton i.e. α' has the form as in figure 6.1. Since α is essential, $H_{\mathcal{O}}(\alpha) \not\simeq \alpha$, so $H_{\mathcal{O}}(\alpha') \not\simeq \alpha'$. Thus if W is the compact domain bounded by the loop $H_{\mathcal{O}}(\alpha') \circ (-\alpha')$, then there exists $\tilde{c} \in \tilde{\mathcal{C}} \setminus \{\tilde{c}_1, \tilde{c}_2\}$ such that $\tilde{c} \subset W$.

Proposition 6.3 *With the above notation, suppose $\tilde{c} \subset T^l(\mathcal{D})$, where $l \geq 2M$, M as in Lemma 6.2. Then there exists a point y satisfying*

$$p_1(H_{\mathcal{O}}(y)) - p_1(y) \geq -M.$$

Proof

By an isotopy if necessary, choose α' so that it lies wholly within $\mathcal{D}$. Let $V(\tilde{c})$ be the 'vertical' passing through $\tilde{c}$ (see figure 6.2) i.e.

$$V(\tilde{c}) = \{u \in \tilde{A}_{\mathcal{O}} : p_1(u) = p_1(\tilde{c})\}.$$

By definition of M, $|p_1(\tilde{c}_1) - p_1(\tilde{c}_2)| < M$, so $p_1(\tilde{c}) - p_1(x) > M$ for all $x \in \alpha'$. Since $\alpha \cap V(\tilde{c}) = \emptyset$, then $H_{\mathcal{O}}(\alpha') \cap H_{\mathcal{O}}(V(\tilde{c})) = \emptyset$. But $H_{\mathcal{O}}$ fixes $\tilde{c}$, so $\alpha' \cap H_{\mathcal{O}}(V(\tilde{c})) \neq \emptyset$ (see figure 6.3). So there exists a point $y \in V(\tilde{c})$ such that $H_{\mathcal{O}}(y) \in \alpha'$ for which $p_1(y) - p_1(H_{\mathcal{O}}(y)) > M$. □

Proposition 6.4 *For $H_{\mathcal{O}}$ as in the Proof of Theorem 6.1, but not assuming $0 \notin$ Int $\rho(H_{\mathcal{O}})$, a circle $\tilde{c} \in \tilde{\mathcal{C}}$ satisfying the conditions of Proposition 6.3 for some iterate of $H_{\mathcal{O}}$ exists.*

To complete the proof of Proposition 6.4, we have to discuss the fixed point theory of Nielsen. Given Proposition 6.4, this implies part 1 of Theorem 6.1, because the conclusion of Proposition 6.3 contradicts Lemma 6.2.

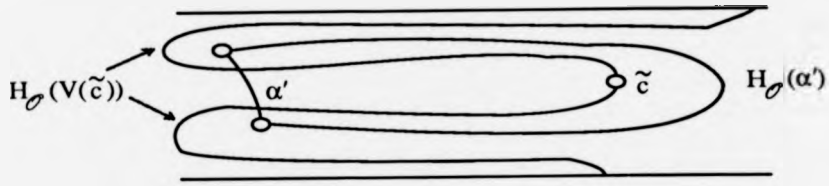

Figure 6.3: The intersection of $H_{\mathcal{O}}(V(\tilde{c}))$ with α'

6.3 The Reidemeister trace and Nielsen fixed point theory

6.3.1 A brief résumé of Nielsen fixed point theory

In this Section, we give a brief account of Nielsen fixed point theory, the Reidemeister trace, and the relation with the free differential calculus of Section 5.2. For a more detailed discussion, the reader is advised to consult [FH, HuJ, J, Ni1].

Let $f : X \to X$ be a map of a compact, connected polyhedron. Given two fixed points of f, we say they are in the same *Nielsen fixed point class* if they can be joined by a path γ such that $f(\gamma) \simeq \gamma$ relative to the endpoints; for example, if f is a pseudo-Anosov homeomorphism, then each fixed point is in a distinct Nielsen class (this follows from property 3 of Section 1.6). Every Nielsen class is an isolated set of fixed points, and so we may define the *Lefschetz index* $Ind(f, \mathsf{F})$ of a class F in a similar way to that in Section 4.2, except that in this case we choose the circle C (about which we calculate the degree of the direction field) to contain precisely those fixed points in F. We say that F is *essential* if $Ind(f, \mathsf{F}) \neq 0$. The number of essential fixed point classes of f is the *Nielsen number* of f, which we denote as $N(f)$, and it is a homotopy invariant of f. Hence any map homotopic to f has at least $N(f)$ fixed points.

Fix a basepoint $x_0 \in X$ and a path η from x_0 to $f(x_0)$. Then f induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$, and η an isomorphism $\eta_* : \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$. Let $f_\pi = \eta_* f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ be their composition. We say that two elements $\alpha, \beta \in \pi_1(X, x_0)$ are *f_π-conjugate* or *Reidemeister equivalent* if there exists $\gamma \in \pi_1(X, x_0)$ such that $\beta = f_\pi(\gamma)\alpha\gamma^{-1}$. This defines an equivalence relation on $\pi_1(X, x_0)$, and we write $[\alpha]$ for the *f_π-conjugacy class* or *Reidemeister class* of α. Let the set of Reidemeister classes be denoted by $R(f_\pi)$, and let $\mathbb{Z}R(f_\pi)$ be the additive Abelian group generated by $R(f_\pi)$.

Fixing a basepoint x_0 and a path η defines a lift $\tilde{f} : \tilde{X} \to \tilde{X}$, and vice versa, where $p : \tilde{X} \to X$ is the universal cover of X, for given $\tilde{x}_0 \in p^{-1}(x_0)$, and the path $\tilde{\eta} = p^{-1}(eta)$ based at $\tilde{x}_0$, we define $\tilde{f}$ so that the image of $\tilde{x}_0$ under $\tilde{f}$ is the other endpoint of $\tilde{\eta}$. Let $\mathsf{D} \cong \pi_1(X)$ be the group of covering translations of $\tilde{X}$. Then for each $x \in$ Fix(f), the set of fixed points of f, there exists a unique $t \in \mathsf{D}$ such that $t\tilde{f}(\tilde{x}) = \tilde{x}$ for any $\tilde{x} \in p^{-1}(x)$ i.e. $\tilde{x} \in$ Fix$(t\tilde{f})$. It can be shown that $x, x' \in$ Fix(f) are in the same Nielsen class if and only if there exists $t \in \mathsf{D}$ such that both $\tilde{x}$ and $\tilde{x}'$ are elements of Fix$(t\tilde{f})$, for $\tilde{x} \in p^{-1}(x)$, $\tilde{x}' \in p^{-1}(x')$ [J]. Then the Reidemeister classes behave as coordinates for the Nielsen classes; two fixed points are in the same Nielsen class if and only if they are in the same Reidemeister class. The choice of basepoint x_0 and path η

define a 'reference frame' for the coordinates. Equivalently, one may specify this frame by the choice of x_0 and a lift $\tilde{f} : \tilde{X} \to \tilde{X}$, where $p : \tilde{X} \to X$ is the universal cover of X.

We define the *Reidemeister trace* of f to be

$$L_R(f) = \sum Ind(f, [\alpha])[\alpha],$$

where $Ind(f, [\alpha])$ is the Lefschetz index of the fixed point class whose coordinate is $[\alpha]$. The sum is over Reidemeister classes, and it can be shown that the number of such classes which are non-empty is finite, and is exactly $N(f)$. It can also be calculated as an alternating sum of traces of the induced maps on cellular chain groups, leading to an equation similar to equation 5.1. Further, the Reidemeister trace is a homotopy invariant, giving fixed point linking information.

We now restrict to the case where $f \in Homeo^+(\mathbf{D}^2, Y)$, and Y is an n-point set as in equation 1.2. Let $\beta \in B_n$ represent the braid type $bt(Y, f)$, and let $\pi_1(\mathbf{D}_n, x_0)$ be identified with the free group $\mathbf{F}_n$ on n generators $x_1, \ldots, x_n$ as in Section 5.2. Fadell and Husseini [FH] showed that

$$L_R(\beta) = L_R(f) = [1] - \sum_{i=1}^{n} \left[\frac{\partial((x_i)\beta)}{\partial x_i} \right] \in \mathbf{Z}R(f_\pi), \tag{6.1}$$

where the derivatives are the free derivatives of Section 5.2.

The one difficulty with this formula is that it may not always be a simple matter to distinguish between elements of distinct Reidemeister classes. However, this whole theory has a homological version, by abelianizing $\pi_1(X)$ into the homology group $H_1(X)$ (factoring out by $[\pi_1(X), \pi_1(X)]$, the commutator subgroup of $\pi_1(X)$), we obtain *homological Nielsen classes*, an *Abelianized Nielsen number*, and an *Abelianized Reidemeister trace*

$$L_H(\beta) = L_H(f) = 1 - \sum_{i=1}^{n} \left[\frac{\partial((x_i)\beta)}{\partial x_i} \right]^{Ab},$$

where Ab denotes the Abelianization operator. Abelianization enables us to distinguish between two elements of $\pi_1(X, x_0)$ in distinct Reidemeister classes.

With $f \in Homeo^+(\mathbf{D}^2, Y)$ and $\beta \in B_n$ as above, Theorem 5.1 implies that

$$L_R(\beta) = [1] - \sum_{i=1}^{n} \left[\frac{\partial}{\partial x_i} (A_i x_{\delta_i} A_i^{-1}) \right],$$

where $(x_i)\beta = A_i x_{\delta_i} A_i^{-1}$ for $1 \leq i \leq n$, $(\delta_1, \ldots, \delta_n) \in \Sigma_n$ is a permutation, and $A_i \in \mathbf{F}_n$.

6.3.2 Pseudo-Anosov homeomorphisms and the fundamental group

After our brief excursion into Nielsen fixed point theory, we now return to the situation in Section 6.2. We shall give the proof of Proposition 6.4, which completes the proof of part 1 of Theorem 6.1.

We will prove Proposition 6.4 in the following way. Assume that no circle $\bar{c} \in \bar{C}$ satisfying the conditions of Proposition 6.3 exists for any iterate of $H_{\mathcal{O}}$. By taking a large finite cover $\mathbf{A}_{m,\mathcal{O}}$ of $\mathbf{A}_{\mathcal{O}}$, we can then translate this assumption into a statement about the action of the map induced by $H_{\mathcal{O}}$ on $\mathbf{A}_{m,\mathcal{O}}$ in terms of $\pi_1(\mathbf{A}_{m,\mathcal{O}})$. The statement says essentially that if we take a particular set $y_1, \ldots, y_k$ of generators of $\pi_1(\mathbf{A}_{m,\mathcal{O}})$, and assume that each point associated with a generator is fixed, then any

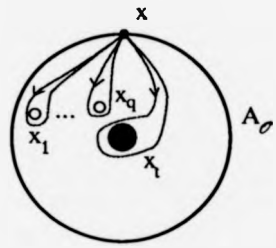

Figure 6.4: The generators of $\pi_1(A_{\mathcal{O}}, x)$

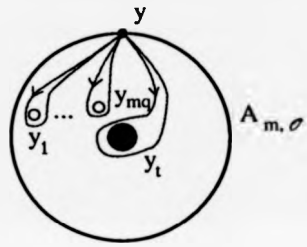

Figure 6.5: The generators of $\pi_1(A_{m,\mathcal{O}}, y)$

iterate of y_i (as a word in $\pi_1(A_{m,\mathcal{O}})$) is made up of just a small subset of $\{y_1, \ldots, y_k\}$. Using the notions of Section 6.3.1, we show that this implies that fixed points of any iterate are uniformly bounded homologically speaking. This contradicts the conclusions of Section 3.2.3 which imply that homology classes of periodic orbits span homology. We now make this more precise.

So assume that there exists no circle $\bar{c} \in \bar{\mathcal{C}}$ satisfying the conditions of Proposition 6.3 for any iterate of $H_{\mathcal{O}}$. Choose a basepoint $x \in \partial^+ A_{\mathcal{O}}$, and identify $\pi_1(A_{\mathcal{O}}, x)$ with the free group F_{q+1}, choosing generators $x_1, \ldots, x_q, x_t$, where x_t corresponds to the loop about $\partial^- A_{\mathcal{O}}$ (see figure 6.4). For any $m \in \mathbb{N}$, let $A_{m,\mathcal{O}} = \tilde{A}_{\mathcal{O}}/m\mathbb{Z}$ be the m-fold cover of $A_{\mathcal{O}}$, with covers $\delta_m : A_{m,\mathcal{O}} \to A_{\mathcal{O}}$ and $\Delta_m : \tilde{A}_{\mathcal{O}} \to A_{m,\mathcal{O}}$. Let $\bar{\mathcal{C}}_m = \delta_m^{-1}(\mathcal{C})$, and let $h_{m,\mathcal{O}} : A_{m,\mathcal{O}} \to A_{m,\mathcal{O}}$ be the induced projection of $H_{\mathcal{O}}$. $h_{m,\mathcal{O}}$ is also pseudo-Anosov; if $(\mathcal{F}^s, \mathcal{F}^u)$ are the invariant foliations for $h_{\mathcal{O}}$, then $\delta_m^{-1}(\mathcal{F}^s, \mathcal{F}^u)$ are the invariant foliations for $h_{m,\mathcal{O}}$. Let $\mathcal{D}_m = \Delta_m(\mathcal{D})$ be a fundamental domain for δ_m.

Let $\bar{\jmath}_1, \ldots, \bar{\jmath}_q$ be the boundary circles in $\bar{\mathcal{C}} \cap \mathcal{D}$, let $y \in \partial^+ A_{m,\mathcal{O}} \cap \mathcal{D}_m$ be the basepoint satisfying $\delta_m(y) = x$. For $i = 1, \ldots, q$, let $y_i \in \mathcal{D}_m$ be such that $\delta_m(y_i) = x_i$ (see figure 6.5). Identify $\pi_1(A_{m,\mathcal{O}}, y)$ with the free group F_{qm+1}, with generators $y_1, \ldots, y_{mq}, y_t$, where y_{jq+i} is the loop based at y about $\Delta_m(T^j(\bar{c}_i))$ for $0 \le j \le m-1$ and $1 \le i \le q$, and y_t is the loop based at y about $\partial^- A_{m,\mathcal{O}}$, as in figure 6.5. Let h_{m*} denote the homomorphism induced by $h_{m,\mathcal{O}}$ on $\pi_1(A_{m,\mathcal{O}}, y)$.

Now take $m \in 2\mathbb{Z}$, $m \gg M$ (in fact, $m \ge 6M$ suffices). For all $k \in \mathbb{N}$, there exists no $i \in \{1, \ldots, mq\}$ satisfying

$$h_{m*}^k(y_i) = B_{i,k} y_i B_{i,k}^{-1},$$

containing both the letters y_1 and $y_{1+mq/2}$, and $B_{i,k}$ is a reduced word, otherwise the conclusions of Proposition 6.3 would be satisfied for some $\bar{c} \in \bar{\mathcal{C}}$ – a contradiction.

Thus

$$\frac{\partial}{\partial x_i}(h^k_{m*}(y_i)) \in \mathbb{Z}\mathbb{F}_n, \quad i = 1, \ldots, mq$$

contains no term containing both the letters y_1 and $y_{1+mq/2}$. From equation 6.1,

$$L_R(h^k_{m,\mathcal{O}}) = [1] - \sum_{i=1}^{mq} \left[\frac{\partial}{\partial y_i}(h^k_{m*}(y_i))\right] - \left[\frac{\partial}{\partial y_t}(h^k_{m*}(y_t))\right].$$

The first two terms on the right-hand side contain no word with both the letters y_1 and $y_{1+mq/2}$ for any $k \in \mathbb{N}$. In fact, the third term satisfies the same condition; for suppose $h^k_{m*}(y_t)$ is a word containing both of these letters. Then we see that the word $h^k_{m*}(y_1 y_t)$ contains both the letters y_1 and $y_{1+mq/2}$. But the loop $y_1 y_t$ retracts to a simple arc from $\Delta_m(\tilde{c}_1)$ to the lower boundary $\partial^- \mathbb{A}_{m,\mathcal{O}}$, and by a similar argument to that above, we obtain a contradiction.

$L_R(h^k_{m,\mathcal{O}})$ is a homotopy invariant giving the linking information of $h^k_{m,\mathcal{O}}$. So there exist no periodic orbits of $h_{m,\mathcal{O}}$ which link with both elements $\tilde{c}, \tilde{c}'$ of $\tilde{C}_m$ corresponding to the loops y_1 and $y_{1+mq/2}$.

However periodic orbits of $h_{m,\mathcal{O}}$ are unremovable under isotopy, and Theorem 3.6 states that homology classes of (a finite set of) periodic orbits span homology. So by concatenating the minimal loops given by Theorem 3.6 appropriately, we can construct a periodic orbit which links with both $\tilde{c}$ and $\tilde{c}'$ in the homological sense, and thus in the homotopic sense. This contradicts the conclusion of the previous paragraph, and so the proof of Proposition 6.4 is complete. □

6.4 Existence of periodic orbits of periodic braid type

In order to prove part 2 of Theorem 6.1, we use an argument of Boyland [Boy3] based on a Theorem of Brunovskii [Bru]. We prove a generalization of a Proposition of Boyland [Boy3].

Proposition 6.5 *Suppose $g : \mathbb{A} \to \mathbb{A}$ is a homeomorphism of the annulus isotopic to the identity, and suppose $o(x)$ is a periodic orbit of period q of g such that $bt(o(x), g)$ is pseudo-Anosov. Then g has a periodic orbit $o(y) \subset$ Int $\mathbb{A}$ such that $bt(o(y), g)$ is periodic, and whose rotation number is equal to that of $o(x)$ (relative to any lift of g).*

Proof

Fix a lift $G : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ of g such that $0 \leq \rho(o(x), G) < 1$, and suppose $o(x)$ is a (p,q)-periodic orbit of G. Suppose first that p and q have a common factor $n > 1$. We show that it reduces to the case where p and q have no common factor. For suppose $q' = q/n$, $p' = p/n$, then

$$G^{q'n}(\tilde{x}) = \tilde{x} + np', \quad \tilde{x} \in \pi^{-1}(x)$$

and so $(T^{-p'}G^{q'})^n(\tilde{x}) = \tilde{x}$, i.e. $\tilde{x}$ is a periodic point of $T^{-p'}G^{q'}$ of period n. It is a consequence of Brouwer's Lemma [Br1, Fa] that $T^{-p'}G^{q'}$ has a fixed point i.e. there exists a periodic orbit of G-rotation type (p', q'). So we shall assume from now on that p and q have no common factor.

Let $o(x)$ be this (p,q)-periodic orbit, and let f be the Thurston representative in the isotopy class of g relative to $o(x)$. If f is periodic relative to $o(x)$, then we are done. By Lemma 4.6, f cannot be reducible relative to $o(x)$. So the only case left to

consider is that where f is pseudo-Anosov relative to $o(x)$. We complete the proof by giving an argument of Boyland [Boy3] based on a Theorem of Brunovskii [Bru]. Recall that a point x is *non-wandering* under f if for all neighbourhoods U of x, there exists $k \in \mathbb{N}$ such that $f^k(U) \cap U \neq \emptyset$ [GuH, Shu]. Every point of $\mathbb{A}$ is non-wandering under f because periodic points are dense in $\mathbb{A}$ (property 6 of Section 1.6); thus given any point w of $\mathbb{A}$ and a neighbourhood U of w, there exists a periodic point lying in Int U, and hence some iterate of U meets U. Since $p/q \in$ Int $\rho(F)$, we may invoke Theorem 4.7 to conclude the existence of at least two (p,q)-periodic orbits of F (of non-zero Lefschetz index under f^q), so neither can be $o(x)$, for each point of $o(x)$ is a 1-pronged singularity, and so $Ind(x, f^q) = 0$. Denote one of these two orbits of non-zero Lefschetz index by $o(x_1)$. Thus given a periodic orbit of pseudo-Anosov braid type, there exists another periodic orbit with the same rotation number.

We now utilize a Theorem of Brunovskii [Bru], which states that given $n_0 > 0$, one may always approximate an isotopy between Kupka-Smale diffeomorphisms by a diffeotopy such that all orbits of period less than n_0 undergo only saddle-node and pitchfork bifurcations, and such bifurcations occur at a finite number of distinct parameter values. Let $\beta = bt(o(x), f)$, let $n_0 > q$, and let f_0, f_1 be Kupka-Smale maps with $\beta \notin bt(f_0)$ and $\beta \in bt(f_1)$, such that both are isotopic to the identity. Then there exists a diffeotopy f_μ between f_0 and f_1 satisfying the conclusions of Brunovskii's Theorem. Let $\mu_0 = \inf\{\mu : \beta \in bt(f_\mu)\}$. There must be a bifurcation at μ_0. If it is a saddle-node bifurcation, then at $\mu = \mu_0$, f_{μ_0} has a single periodic orbit of braid type β. If the bifurcation were a pitchfork, the orbit that persists during the bifurcation existed for $\mu < \mu_0$, so its braid type is not β. Otherwise, the braid type β must come from the doubled orbit. But this would imply that β is reducible – a contradiction. Thus we have shown that there exists a diffeomorphism of the annulus with just one periodic orbit of braid type β.

Let $\gamma = bt(o(x_1), f)$, then $\gamma \neq \beta$, and $\beta \succ \gamma$. Let

$$\mathcal{D} = \{\alpha : \alpha \text{ is a braid type of } F\text{-rotation type } (p,q), \beta \succ \alpha\}.$$

Since f is pseudo-Anosov relative to $o(x)$, it has only finitely many periodic orbits of each period, so $\mathcal{D}$ is finite. Suppose ξ is minimal in $\mathcal{D}$ i.e. $\xi \succ \delta \Rightarrow \xi = \delta$ for all $\delta \in \mathcal{D}$. Thus ξ cannot be pseudo-Anosov by the above argument, and so must be periodic. So there exists a periodic orbit $o(y)$ of f whose braid type is ξ (periodic), and whose F-rotation number is p/q. By unremovability of periodic orbits, g too has such a periodic orbit. This completes the proof of Proposition 6.5, and hence that of part 2 of Theorem 6.1. □

Chapter 7

Dynamics embedded in pseudo-Anosov homeomorphisms

In this Chapter, we show that amongst the set $B_{2,1}$ of pseudo-Anosov braid types of periodic orbits of the annulus of period 2, there exists a 'least' element β_1 (respectively β_0), such that it has the least topological entropy of any braid type in $B_{2,1}$ with rotation type $(1,2)$ (respectively $(0,2)$). Moreover, given any braid type $\beta \in B_{2,1}$ of rotation type $(1,2)$ (respectively $(0,2)$), then $\beta \succ \beta_1$ (respectively $\beta \succ \beta_0$). We demonstrate a similar result for braid types in BT_3, and conjecture analogous statements for BT_n.

7.1 Notation and preliminaries

In this Section, we give some preliminary definitions and notation. Let $B_{2,1}$ be the subset of B_3 such that $\beta \in B_{2,1}$ if and only if

1. β is of pseudo-Anosov braid type, and
2. $\pi_3(\beta)$ is a 2-cycle, where $\pi_3 : B_3 \to \Sigma_3$ is the homomorphism defined in Section 1.4.

Let $\alpha = \sigma_1\sigma_2$, and write $\theta = \theta_3 = \alpha^3$ for the generator of the centre $Z(B_3)$ of B_3. Given $\sigma, \sigma' \in B_3$, define a relation $\sigma \sim \sigma'$ if

$$\sigma' = \theta^j \gamma \sigma \gamma^{-1}$$

for some $j \in \mathbb{Z}$ and $\gamma \in B_3$. It is easy to see that $\sim$ is an equivalence relation. From Section 1.3.2, it follows that σ and σ' represent the same braid type. In particular, the homotopy-invariant linking information given by braid types represented by σ and σ' is the same (the linking numbers mod 3 are equal), and dynamically there is no difference between the two.

Let g_1, g_2 be the horseshoes as shown in figure 7.1. Both g_1 and g_2 have attracting periodic points $p_i \in A_i$ $i = 1,2,3$. Without loss of generality, we shall assume that $p_i = x_i$, $i = 1,2,3$, where $X = \{x_i\}$ is defined as in equation 1.2. To each horseshoe g_1, g_2, we may associate a 2×2 transition matrix M_1, M_2 respectively, where $(M_i)_{jk}$ is the cardinality of the number of connected components of $C_j \cap g_i(C_k)$ for $1 \leq i,j,k \leq 2$.

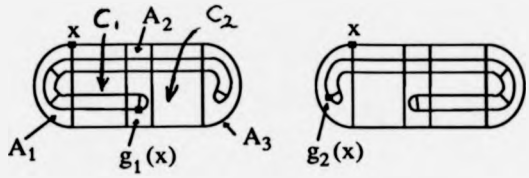

Figure 7.1: The horseshoes g_1 and g_2

So

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For $m \in \mathbf{N}$ and a sequence $J = (j_1, \ldots, j_m) \in \mathbf{Z}^m$, define a matrix

$$M(h_J) = M_1^{j_1} M_2 M_1^{j_2} M_2 \cdots M_1^{j_m} M_2,$$

a 3-braid

$$\sigma(J) = \sigma(j_1, \ldots, j_m) = \sigma_1^{j_1} \sigma_2^{-1} \sigma_1^{j_2} \sigma_2^{-1} \cdots \sigma_1^{j_m} \sigma_2^{-1},$$

conjugacy classes

$$\rho(J) = < \sigma(J) >$$

$$\rho(J; k) = \theta^k \rho(J) \text{ for } k \in \mathbf{Z},$$

and a horseshoe $h_J : (\mathbf{D}^2, X) \to (\mathbf{D}^2, X)$ by

$$h_J = g_1^{j_1} \circ g_2 \circ g_1^{j_2} \circ g_2 \circ \cdots \circ g_1^{j_m} \circ g_2.$$

For $k \in \mathbf{Z}$, let $\rho^k = < \alpha^k >$. Let $\phi_{J,k} : \mathbf{T} \to \mathbf{T}$ be the suspension of h_J such that the conjugacy class of the associated geometric 3-braid (c.f. Section 1.3.1) is $\rho(J; k)$.

7.2 The normal form for B_3

In this Section, we prove the following Theorem, which Matsuoka [Ma2] proved for the case where $\gamma \in BT_3$ (i.e. $\pi_3(\beta)$ is a 3-cycle).

Theorem 7.1 *Let γ be a braid type represented by a braid in B_3. Then we may choose $\beta \in B_3$ representing γ such that exactly one of the following is true:*

1. *γ is reducible.*
2. *$\beta = \rho^k$ for some $k \in \mathbf{Z}$.*
3. *$\beta = \theta^l < \sigma_1 \sigma_2 \sigma_1 >$ for some $l \in \mathbf{Z}$.*
4. *There exists $k \in \mathbf{Z}$ and $m \in \mathbf{N}$ such that $\beta = \rho(J; k)$ where $J \in \mathbf{Z}_+^m \setminus \{0\}$.*

Cases 2 and 3 give braid types of finite order, whilst case 1 gives reducible braid types. So given any pseudo-Anosov braid type γ we may choose an element β of B_3 which represents γ, and has the *normal form* $\sigma(J)$, where J is an m-tuple of non-negative integers. In this way, we may represent γ by a homeomorphism which is in the isotopy class rel. X of the horseshoe h_J.

Proof

Choose a geometric braid $\beta \in B_3$ which represents γ. Since

$$\begin{aligned} \sigma_2 &= \theta^{-1}\alpha\sigma_1\alpha^2 \\ \sigma_1^{-1} &= \theta^{-1}\alpha\sigma_1\alpha \\ \sigma_2^{-1} &= \theta^{-1}\alpha^2\sigma_1 \end{aligned}$$

we may express β as a product of non-negative powers of α and σ_1, and a power (perhaps negative) of θ. We reduce this word in stages as follows. We show that we may write β in the following form:

$$\theta^r(\sigma_1\alpha\sigma_1^{i_1})(\sigma_1\alpha\sigma_1^{i_2})\cdots(\sigma_1\alpha\sigma_1^{i_s}) \tag{7.1}$$

for some $r \in \mathbf{Z}$ and $i_1, \ldots, i_s \in \mathbf{N}$.

1. If β contains a term α^k, where $k > 1$, we utilize the substitution $\alpha^2 = \sigma_1\alpha\sigma_1$ until there are no such terms. So we can reduce β to a product of α, non-negative powers of σ_1 and a power of θ.

2. Suppose that the word obtained in 1 starts or ends (ignoring the power of θ) with α. Then
$$\alpha = \theta^{-1}\alpha^4 = \theta^{-1}\sigma_1\alpha\sigma_1^2\alpha\sigma_1.$$
So we may assume that β starts and ends (ignoring the power of θ) with σ_1.

3. Suppose the word obtained in 2 starts with σ_1^j, $j > 1$, then $\beta = \sigma_1^j\alpha\sigma'$, where σ' is a word in α, positive powers of σ_1 and a power of θ. Then $\sigma_1^j\alpha\sigma' \sim \sigma_1\alpha\sigma'\sigma_1^{j-1}$ so $\beta = \sigma_1\alpha\sigma'\sigma_1^{j-1}$, as γ is a braid type. Thus β starts with $\sigma_1\alpha\sigma_1\ldots$ and ends with σ_1.

4. So we may write β as a word starting with σ_1, then alternating with α and powers of σ_1, and ending with σ_1 (up to a power of θ). The only obstruction to writing β in the form of equation 7.1 is if
$$\beta = \sigma_1\ldots\sigma_1^j\alpha\sigma_1\alpha\sigma_1^k\ldots$$
for $j, k \in \mathbf{N}$. But $\alpha^2 = \sigma_1\alpha\sigma_1$, so $\theta = \alpha^3 = \alpha\sigma_1\alpha\sigma_1 = \sigma_1\alpha\sigma_1\alpha$, thus
$$\beta = \theta\sigma_1\ldots\sigma_1^{j+k-1}\ldots.$$
So wherever a single σ_1 term occurs (except at the beginning and end of β) we can remove it. Hence we may write β in the form of equation 7.1. (This is slightly different from that given in [Ma2]).

Now we reduce equation 7.1 to $\rho(J;k)$ to give the required forms of Theorem 7.1.

Lemma 7.2 ([Ma2]) *1.*

$$\sigma_1 \alpha \sigma_1^j = \theta \sigma_2^{-1} \sigma_1^{j-2} \tag{7.2}$$

2.

$$\sigma(j, -1, k) = \sigma(j-1, k-1) \tag{7.3}$$

3.

$$\sigma(j, -2, k) = \theta^{-1} \sigma(j+k+2) \tag{7.4}$$

for any $j, k \in Z$.

Proof

1.

$$\begin{aligned} \sigma_1 \alpha \sigma_1^j &= \sigma_1 \sigma_1 \sigma_2 \sigma_1^j \\ &= \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1^{j-1} = \theta \sigma_2^{-1} \sigma_1^{j-2}. \end{aligned}$$

2.

$$\begin{aligned} \sigma(j, -1, k) &= \sigma_1^j \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^k \sigma_2^{-1} \\ &= \sigma_1^{j-1} \sigma_2^{-1} \sigma_1^{k-1} \sigma_2^{-1} = \sigma(j-1, k-1). \end{aligned}$$

3.

$$\begin{aligned} \sigma(j, -2, k) &= \sigma_1^j \sigma_2^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^k \sigma_2^{-1} \\ &= \sigma_1^{j+1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_1^{-1} \sigma_1^{k+1} \sigma_2^{-1} \\ &= \theta^{-1} \sigma_1^{j+k+2} \sigma_2^{-1} = \theta^{-1} \rho(j+k+2). \end{aligned}$$

□

So equations 7.1 and 7.2 imply that

$$\begin{aligned} \beta &\sim \theta^{r+s} \rho(i_1 - 2, \ldots, i_s - 2) \\ &= \theta^{r+s} \rho(j_1, \ldots, j_s) \end{aligned} \tag{7.5}$$

where $j_i \geq -1$ for $1 \leq i \leq s$. We now complete the proof of Theorem 7.1, carrying out the following steps.

1. If no $j_i = -1$ in equation 7.5, we are done, for either $J = (j_1, \ldots, j_s) = \mathbf{0}$ and then $\beta = \sigma_2^{-s}$ which implies that γ is a reducible braid type, or $J \in \mathbf{Z}_+^s \setminus \{\mathbf{0}\}$. Otherwise suppose that there exists some $j_i = -1$. Then we use equation 7.3, remembering that we may cyclically permute J, so

$$\beta = \theta^{r+s} \rho(j_1, \ldots, j_{i-1} - 1, j_{i+1} - 1, \ldots, j_s).$$

If one of j_{i-1} or j_{i+1} is equal to -1, then use equation 7.4, e.g.

$$\beta = \theta^{r+s} \rho(j_1, \ldots, j_{i-2} + j_{i+1} + 1, \ldots, j_s) \text{ if } j_{i-1} = -1$$

and each term in this new sequence is at least -1. Continue this process of reduction with equations 7.3 and 7.4 using cyclic permutation if necessary (recall that $\sigma(k_1, \ldots, k_p) \sim \sigma(k_p, k_1, \ldots, k_{p-1})$).

2. With each application of the last step to β, the length of the tuple J is reduced. Eventually we find that one of the following occur.

   (a) $\beta = \theta^k \rho(l_1, \ldots, l_m)$ where $l_i \geq 0$ for $1 \leq i \leq m$, and we are in case 4 of Theorem 7.1.

   (b) $\beta = \theta^k \rho(l_1, \ldots, l_m)$ where $l_i = 0$ for all $1 \leq i \leq m$, β is reducible, and we are in case 1 of Theorem 7.1.

   (c) We run out of terms to carry out the last step. Then either

      i. $\beta = \theta^k \rho(-1, j)$ where $j \geq -1$, or

      ii. $\beta = \theta^k \rho(-2, j)$ where $j \geq -2$, or

      iii. $\beta = \theta^k \rho(j)$ where $j = -1$ or -2.

   We analyze these possibilities. In case 2(c)i,

   - If $j = -1$, then $\rho(-1, -1) = \alpha^{-2} = \theta^{-1}\alpha$ which gives case 2 of Theorem 7.1.
   - If $j = 0$, $\rho(-1, 0) = \theta^{-1}\sigma_1\sigma_2\sigma_1$, which gives case 3 of Theorem 7.1.
   - If $j = 1$, $\rho(-1, 1) = \alpha^{-1} = \theta^{-1}\alpha^2$, which gives case 2 of Theorem 7.1.
   - If $j = 2$, $\rho(-1, 2) = \rho(0)$, which gives case 1 of Theorem 7.1.
   - If $j > 2$, then $\rho(-1, j) = \rho(j-2)$, which gives case 4 of Theorem 7.1.

   In case 2(c)ii, $\rho(-2, j) \sim \theta^{-1}\sigma_1^{j+2}$, which gives case 1 of Theorem 7.1 if $j > -2$, or case 2 of Theorem 7.1 if $j = -2$.
   In case 2(c)iii,

$$\rho(-1, j) = \sigma_1^{-1}\sigma_2^{-1}\sigma_1^{j}\sigma_2^{-1} \sim \rho(j-2)$$

   implies that $\rho(-1) = \rho(-1, 1)$ and $\rho(-2) = \rho(-1, 0)$, which has been dealt with above. □

7.3 Axiom A and pseudo-Anosov homeomorphisms

Given a pseudo-Anosov braid type β represented by a geometric braid in B_3, we see from Theorem 7.1 that we may represent β by the normal form $\rho(J; k)$ for $J \in \mathbb{Z}_+^m \setminus \{0\}$ for some $m \in \mathbb{N}$, $k \in \mathbb{Z}$. The reason for wanting to do this is the following. The normal form is a special product of horseshoes; in particular, the dynamics of the basic set of the Axiom A representative given by h_J coincide with the dynamics of the pseudo-Anosov homeomorphism relative to the three point set X. In this Section, we make this precise by proving the following Theorem.

Theorem 7.3 *Let β be a pseudo-Anosov braid type represented by a geometric braid in B_3. Then*

1. *the following two statements are equivalent*

   (a) $\beta \succ \gamma$.

   (b) There exists a periodic point x of h_J such that $\gamma = bt(x, h_J)$, where β is represented by $\rho(J; k)$, for $J \in \mathbb{Z}_+^m \setminus \{0\}$ and some $m \in \mathbb{N}$ and $k \in \mathbb{Z}$.

2. *let $J = (J_1, J_2)$ where $J_1 \in \mathbb{Z}_+^{m'}, J_2 \in \mathbb{Z}_+^{m-m'}$ and $0 \leq m' \leq m$. Then $M_J = M_{J_1} M_{J_2}$.*

Remark

- Part 1, the main part of this Theorem, is a Theorem of Matsuoka [Ma2]. Part 2, whilst obvious, is important because it implies that we may consider the dynamics of each h_{j_i}, $1 \leq i \leq m$, and by concatenating their orbits in the obvious way, we recover exactly the dynamics of $h(J)$. Then part 1 implies that all such periodic orbits in the basic set for h_J are unremovable – i.e. the dynamics of the Axiom A map h_J on its basic set coincide with the dynamics of the pseudo-Anosov homeomorphism in the isotopy class relative to the 3-point set X. We now prove this result.

Proof

Let β be represented by the conjugacy class of braids $\rho(J;k)$ as given in Theorem 7.1, where $J \in \mathbb{Z}_+^m \setminus \{0\}$, $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. By the definition of the partial ordering, 1a implies 1b. To prove the converse, it suffices to show that if $f \simeq h_J$ relative to X is the pseudo-Anosov homeomorphism in the isotopy class relative to X, and if x is a periodic point of h_J, then it is a periodic point of f. By unremovability of periodic orbits of pseudo-Anosov homeomorphisms,

$$\text{Card Fix}(f^n) \leq \text{Card Fix}(h_J^n)$$

for any $n \in \mathbb{N}$. By the properties of the horseshoe,

$$\text{Card Fix}(h_J^n) = \text{Trace}(M(h_J))^n.$$

Now let $A_J \in SL(2,\mathbb{Z})$ be the matrix determined by the braid $\sigma(J)$ and equation 3.2 in Section 3.5. Since $a_1 = M_1$ and $a_2^{-1} = M_2$, then $A_J = M(h_J)$. But A_J determines the dynamics of f; in particular, equation 3.3 implies that

$$\text{Card Fix}(f^n) = \text{Trace}(A_J)^n.$$

So

$$\text{Card Fix}(f^n) = \text{Card Fix}(h_J^n),$$

thus all periodic orbits of h_J in its basic set are unremovable; but these orbits are precisely those given by the symbolic dynamics of $M(h_J)$. Since part 2 is trivial, the proof is complete. □

7.4 Directed graphs for B_3

7.4.1 Introduction

As we have seen in Sections 7.2 and 7.3, given a pseudo-Anosov braid type which is represented by a braid in B_3, we may write it in the normal form $\rho(J;k)$, and that the dynamics of this braid type is given exactly by the appropriate concatenation of the dynamics of each h_{j_i}, where $1 \leq i \leq m$ and $J = (j_1, \ldots, j_m) \in \mathbb{Z}_+^m \setminus \{0\}$. We are particularly interested in elements of $B_{2,1}$, which correspond to periodic orbits of period 2 of homeomorphisms of the annulus. In this case, it is important to be able to calculate rotation numbers of periodic orbits, and thus it becomes necessary to keep track of the strand of the geometric braid corresponding to the annulus. In this Section, we prove the existence of a directed graph Γ associated with the braid $\sigma(J)$, with the

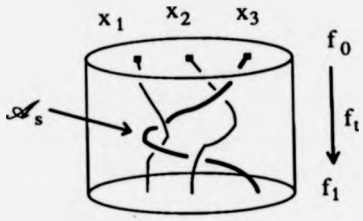

Figure 7.2: $\mathcal{A}_s$ starts at x_3

vertices of the graph made up of subsets of braids $\sigma(j)$ where $j \in \mathbf{Z}_+$. We then show how to reduce this graph to a subgraph, in such a way that if β is represented by a loop in Γ, then there exists a braid β' represented by a loop in the subgraph such that $\beta \succ \beta'$. Using this we prove the assertion made at the beginning of this Chapter, viz.

Theorem 7.4 *In the set $B_{2,1}$ of pseudo-Anosov braid types, there exist two 'minimal' elements β_0,β_1 such that β_1 (respectively β_0) has the least topological entropy of any braid type in $B_{2,1}$ of rotation type $(1,2)$ (respectively $(0,2)$). Further, given any braid type $\beta \in B_{2,1}$ of rotation type $(1,2)$ (respectively $(0,2)$), then $\beta \succ \beta_1$ (respectively $\beta \succ \beta_0$). β_0 (respectively β_1) is represented by the geometric braid $\sigma(4)$ (respectively $\sigma(2)$).*

7.4.2 Construction of the directed graph

We start by defining various subsets of braids in B_3. Let Y be a 2-point invariant set of a homeomorphism f of the annulus isotopic to the identity, then choose a braid $\gamma \in B_3$ so that γ represents the braid type $bt(Y, f)$. We define the *preferred strand* $\mathcal{A}_s$ of γ to be that which does not correspond to either of the points of Y. We do this because we are interested in rotation numbers of periodic orbits. By collapsing $\mathbf{A}$ to the disc $\mathbf{D}^2$, we may consider f to be a homeomorphism of the disc isotopic to the identity. Let $\{f_t\}_{t\in I}$ be the isotopy described in Section 1.3.1 such that $f_0 = Id$, $f_1 = f$ and such that the associated geometric braid is γ. Taking the 3-point invariant set of f to be X (by conjugacy if necessary), we say that the preferred strand $\mathcal{A}_s$ *starts at* x_i if

$$(x_i, 0) \cap \mathcal{A}_s \neq \emptyset \text{ where } 1 \leq i \leq 3$$

in the cylinder $\mathbf{D}^2 \times I$. For example, in figure 7.2, $\mathcal{A}_s$ starts at x_3. Similarly we say that $\mathcal{A}_s$ *finishes at* x_j if

$$(x_j, 1) \cap \mathcal{A}_s \neq \emptyset \text{ where } 1 \leq j \leq 3.$$

Given such a braid type, we can put it into the normal form of Theorem 7.1 i.e. $bt(X, f) = \rho(J; k)$. It suffices to study the dynamics of h_J, which we visualize as the braid $\sigma(J) \in B_3$. Theorem 7.3 implies that *any* of the periodic orbits implied by h_J are unremovable in its isotopy class relative to Y. Consequently, it is sufficient for us to work with the geometric braid, and hence many of the proofs in the remainder of this Section will be pictures, since we know that any periodic orbits which are consistent with the crossings of $\sigma(J)$ really exist for h_J. Further, if $J = (j_1, \ldots, j_m) \in \mathbf{Z}_+^m \setminus \{0\}$, we may concatenate the structure of each $\sigma(j_i)$, $1 \leq i \leq m$, to obtain that of $\sigma(J)$.

Figure 7.3: Possibilities for $\sigma(j_i)$: case (a)

Figure 7.4: Possibilities for $\sigma(j_i)$: case (b)

For this reason, we now consider braids of the form $\sigma(j_i)$. $\sigma(J)$ defines where the preferred strand starts and finishes for each $\sigma(j_i)$; in order to concatenate the $\sigma(j_i)$, we are interested where $\mathcal{A}_s$ starts for each $\sigma(j_i)$, and the parity of j_i (this defines where $\mathcal{A}_s$ finishes for $\sigma(j_i)$, and thus where $\mathcal{A}_s$ starts for $\sigma(j_{i-1})$). So there are the following six possibilities.

(a) $\mathcal{A}_s$ starts at x_2, j_i is even e.g. see figure 7.3.

(b) $\mathcal{A}_s$ starts at x_2, j_i is odd e.g. see figure 7.4.

(c) $\mathcal{A}_s$ starts at x_1, j_i is even e.g. see figure 7.5.

(d) $\mathcal{A}_s$ starts at x_1, j_i is odd e.g. see figure 7.6.

(e) $\mathcal{A}_s$ starts at x_3, j_i is even e.g. see figure 7.7.

(f) $\mathcal{A}_s$ starts at x_3, j_i is odd e.g. see figure 7.8.

For any $\sigma(j_i)$, we say that $\sigma(j_i) \in \Delta_\epsilon$ if it is a braid given by case (ϵ), where $\epsilon \in \Sigma = \{a, b, c, d, e, f\}$. Since $\sigma(j_i)$ finishes where $\sigma(j_{i-1})$ starts, there exists the

Figure 7.5: Possibilities for $\sigma(j_i)$: case (c)

Figure 7.6: Possibilities for $\sigma(j_i)$: case (d)

Figure 7.7: Possibilities for $\sigma(j_i)$: case (e)

Figure 7.8: Possibilities for $\sigma(j_i)$: case (f)

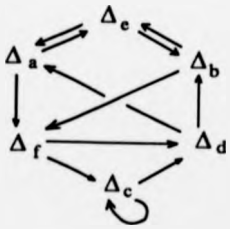

Figure 7.9: The directed graph Γ

Figure 7.10: Linking number $+1$ for a crossing of $\mathcal{A}_{n_i}$ with $\mathcal{A}_s$

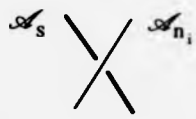

Figure 7.11: Linking number -1 for a crossing $\mathcal{A}_{n_i}$ with $\mathcal{A}_s$

directed graph Γ between these cases (see figure 7.9), where an arrow from Δ_ϵ to $\Delta_{\epsilon'}$ means that any braid in $\Delta_{\epsilon'}$ may be concatenated after any braid from Δ_ϵ ($\epsilon, \epsilon' \in \Sigma$). Further, since $\sigma(j_1)$ finishes where $\sigma(j_m)$ starts, then we may represent $\sigma(J)$ as a loop ξ in Γ, where $\xi = \Delta_{\epsilon_1}\Delta_{\epsilon_2}\ldots\Delta_{\epsilon_m}$ such that there is an arrow from Δ_{ϵ_i} to $\Delta_{\epsilon_{i-1}}$ for all $2 \leq i \leq m$, and an arrow from Δ_{ϵ_1} to Δ_{ϵ_m} in Γ.

For each $\sigma(j_i)$ we define a *rotation number* $n_\rho(j_i)$ of the two non-preferred strands $\mathcal{A}_{n_1}, \mathcal{A}_{n_2}$ to be the sum of the linking numbers of $\mathcal{A}_{n_i}$ with $\mathcal{A}_s$ ($i = 1, 2$) with the usual convention. We count $+1$ for a crossing of the type in figure 7.10, and -1 for a crossing of the type in figure 7.11. Clearly, the rotation number of $\sigma(J)$ is

$$n_\rho(J) = \sum_{i=1}^{m} n_\rho(j_i).$$

Any loop in Γ may be written as the concatenation of the minimal loops of Γ. These minimal loops (up to cyclic permutation) are:

1. Δ_c
2. $\Delta_e\Delta_a$
3. $\Delta_e\Delta_b$
4. $\Delta_d\Delta_f\Delta_a$
5. $\Delta_d\Delta_f\Delta_b$
6. $\Delta_d\Delta_c\Delta_f$
7. $\Delta_d\Delta_c\Delta_f\Delta_b$.

Given two paths $\Delta_1 = \Delta_{\epsilon_1}\ldots\Delta_{\epsilon_q}$, $\Delta_2 = \Delta_{\eta_1}\ldots\Delta_{\eta_{q'}}$ in Γ, we write $\Delta_1 \rhd \Delta_2$ if for any $\gamma = \gamma_{\epsilon_1}\ldots\gamma_{\epsilon_q} \in \Delta_1$ with $\gamma_{\epsilon_i} \in \Delta_{\epsilon_i}$ for $1 \leq i \leq q$, there exists $\xi = \xi_{\eta_1}\ldots\xi_{\eta_{q'}} \in \Delta_2$ such that

1. $\xi_{\eta_i} \in \Delta_{\eta_i}$ for $1 \leq i \leq q'$
2. $\gamma \succ \xi$

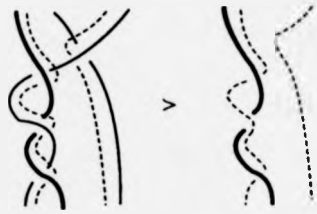

Figure 7.12: The braid $\sigma_1^{2j+1}\sigma_2^{-1} \in \Delta_d$

3. $n_\rho(\gamma) = n_\rho(\xi)$.

Intuitively, this says that given any $\gamma \in \Delta_1$, we can replace it by a braid $\xi \in \Delta_2$ satisfying $\gamma \succ \xi$ without changing the rotation number.

The idea of the proof is the following. Given $\sigma(J)$, we can associate a loop $\gamma(J)$ in Γ with it. Write this loop as a product $M_1 \ldots M_r$ of minimal loops in Γ for some $r \in \mathbb{N}$. By looking at some minimal loop M_i, we can show that $M_i \rhd M$ for some minimal loop M, and replacing M_i by M gives in some way a 'simpler' loop. If $\gamma \in M_i$ and $\gamma \succ \gamma' \in M$, then we may replace γ in $\sigma(J)$ by γ' to give a new braid $\sigma(J')$ such that $\sigma(J) \succ \sigma(J')$ and $n_\rho(\sigma(J)) = n_\rho(\sigma(J'))$. We now elucidate this process.

Proposition 7.5 $\Delta_d\Delta_f \rhd \Delta_e$.

Proof

Let $\sigma_1^{2j+1}\sigma_2^{-1} \in \Delta_d$, $j \in \mathbb{Z}_+$, as in figure 7.12. Then we see that $\sigma_1^{2j+1}\sigma_2^{-1} \succ \sigma_1^{2j+1}$. Suppose $\sigma_1^{2k+1}\sigma_2^{-1} \in \Delta_f$, $k \in \mathbb{Z}_+$, then

$$\begin{aligned} \sigma_1^{2j+1}\sigma_2^{-1}.\sigma_1^{2k+1}\sigma_2^{-1} &\succ \sigma_1^{2j+1}.\sigma_1^{2k+1}\sigma_2^{-1} \\ &= \sigma_1^{2(j+k+1)}\sigma_2^{-1} \in \Delta_e. \end{aligned}$$

Since

$$n_\rho(\sigma_1^{2j+1}\sigma_2^{-1}) + n_\rho(\sigma_1^{2k+1}\sigma_2^{-1}) = n_\rho(\sigma_1^{2(j+k+1)}\sigma_2^{-1})$$

then $\Delta_d\Delta_f \rhd \Delta_e$. □

Proposition 7.6 $\Delta_c\Delta_f \rhd \Delta_f$.

Proof

As in Proposition 7.5, if $\sigma_1^{2j}\sigma_2^{-1} \in \Delta_c$, $j \in \mathbb{Z}_+$, then $\sigma_1^{2j}\sigma_2^{-1} \succ \sigma_1^{2j}$. For $\sigma_1^{2k+1}\sigma_2^{-1} \in \Delta_f$, $k \in \mathbb{Z}_+$, then

$$\begin{aligned} \sigma_1^{2j}\sigma_2^{-1}.\sigma_1^{2k+1}\sigma_2^{-1} &\succ \sigma_1^{2j}.\sigma_1^{2k+1}\sigma_2^{-1} \\ &= \sigma_1^{2(j+k)+1}\sigma_2^{-1} \in \Delta_f \end{aligned}$$

and

$$n_\rho(\sigma_1^{2j}\sigma_2^{-1}) + n_\rho(\sigma_1^{2k+1}\sigma_2^{-1}) = n_\rho(\sigma_1^{2(j+k)+1}\sigma_2^{-1})$$

so $\Delta_c\Delta_f \rhd \Delta_f$. □

Write $\Delta_c^k = \Delta_c \cdots \Delta_c$ (k times).

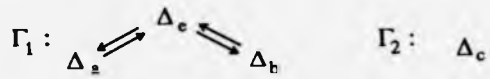

Figure 7.13: The subgraphs Γ_1 or Γ_2 of Γ

Proposition 7.7 $\Delta_c^2 \rhd \Delta_c$.

Proof

In analogy with Proposition 7.5, if $\sigma_1^{2j}\sigma_2^{-1} \in \Delta_c$, $j \in \mathbf{Z}_+$, then $\sigma_1^{2j}\sigma_2^{-1} \succ \sigma_1^{2j}$. Thus if $\sigma_1^{2k}\sigma_2^{-1} \in \Delta_c$, $k \in \mathbf{Z}_+$, then

$$\begin{aligned} \sigma_1^{2j}\sigma_2^{-1}.\sigma_1^{2k}\sigma_2^{-1} &\succ \sigma_1^{2j}.\sigma_1^{2k}\sigma_2^{-1} \\ &= \sigma_1^{2(j+k)}\sigma_2^{-1} \in \Delta_c. \end{aligned}$$

Since

$$n_\rho(\sigma_1^{2j}\sigma_2^{-1}) + n_\rho(\sigma_1^{2k}\sigma_2^{-1}) = n_\rho(\sigma_1^{2(j+k)}\sigma_2^{-1})$$

then $\Delta_c^2 \rhd \Delta_c$. □

A consequence of these Propositions is

Corollary 7.8 *1.* $\Delta_c^k \rhd \Delta_c$ *for* $k \in \mathbf{N}$.

2. $\Delta_c^k\Delta_f \rhd \Delta_f$ *for* $k \in \mathbf{Z}_+$.

3. $\Delta_d\Delta_c^k\Delta_f \rhd \Delta_e$ *for* $k \in \mathbf{Z}_+$. □

Proposition 7.9 *Given $\sigma(J) \in B_3$ a braid in normal form representing a braid type in $B_{2,1}$, then there exists $\sigma(J') \in B_3$ a braid in normal form, such that $\sigma(J) \succ \sigma(J')$, $n_\rho(\sigma(J)) = n_\rho(\sigma(J'))$ and the loop associated with $\sigma(J')$ in Γ is contained within one of the two subgraphs Γ_1 or Γ_2 as in figure 7.13.*

Proof

Consider the loop $\gamma(J)$ in Γ associated with $\sigma(J)$. If $\gamma(J) = \Delta_c^k$, $k \in \mathbf{N}$, then Corollary 7.8 implies that we can choose J' so that $\sigma(J) \succ \sigma(J')$, $\sigma(J') \in \Delta_c$, and $n_\rho(\sigma(J)) = n_\rho(\sigma(J'))$. Otherwise, $\gamma(J)$ contains some vertex of Γ other than Δ_c. If $\gamma(J)$ contains the vertices Δ_d or Δ_f, it must contain the sequence of vertices $\Delta_d\Delta_c^k\Delta_f$ for $k \in \mathbf{Z}_+$. So Corollary 7.8 implies that $\gamma(J) \rhd \gamma(J')$, where $\gamma(J')$ is a loop in Γ_1, $n_\rho(\sigma(J)) = n_\rho(\sigma(J'))$, and $\sigma(J) \succ \sigma(J')$. □

Proposition 7.10 *1.* $\Delta_a \rhd \Delta_{a'}$,

2. $\Delta_b \rhd \Delta_{a'}$,
where $\Delta_{a'}$ consists of the single element $\sigma_2^{-1} = \sigma(0)$ of Δ_a.

Proof

1. The relation $\sigma_1^{2j}\sigma_2^{-1} \in \Delta_a \succ \sigma_2^{-1}$ follows from figure 7.14.
2. The relation $\sigma_1^{2j+1}\sigma_2^{-1} \in \Delta_b \succ \sigma_2^{-1}$ follows from figure 7.14. □

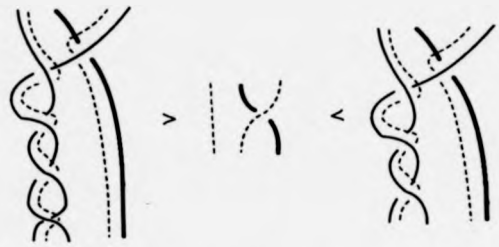

Figure 7.14: $\sigma_1^{2j}\sigma_2^{-1} \in \Delta_a \succ \sigma_2^{-1}$ and $\sigma_1^{2j+1}\sigma_2^{-1} \in \Delta_b \succ \sigma_2^{-1}$

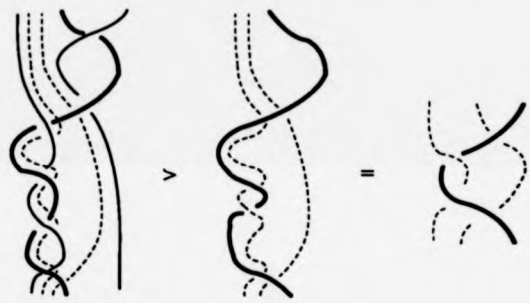

Figure 7.15: $\gamma \succ \gamma'$.

Proposition 7.11 *If $\gamma = \sigma(2j,0) \in \Delta_e\Delta_{a'}$, then $\gamma \succ \sigma_2\sigma_1^{2(j-1)}\sigma_2^{-1}$ and the rotation number is preserved. Hence $(\Delta_e\Delta_{a'})^k \rhd \Delta_e\Delta_{a'}$ for any $k \in \mathbf{N}$.*

Proof

If $\gamma = \sigma(2j,0) = \sigma_1^{2j}\sigma_2^{-2} \in \Delta_e\Delta_{a'}$, then we have the situation as in figure 7.15, and clearly $n_\rho(\gamma) = j - 1 = n_\rho(\gamma')$, where $\gamma' = \sigma_2\sigma_1^{2(j-1)}\sigma_2^{-1}$ as shown. Thus if $\gamma_i = \sigma(2j_i,0) \in \Delta_e\Delta_{a'}$ for $i = 1,\ldots,k$ and $j_i \in \mathbf{N}$, then

$$\begin{aligned}
\gamma_1\ldots\gamma_k &\succ \sigma_1^{2j_1}\sigma_2^{-2}.\sigma_2\sigma_1^{2(j_2-1)}\sigma_2^{-2}.\sigma_2\sigma_1^{2(j_3-1)}\sigma_2^{-2}\ldots\sigma_2\sigma_1^{2(j_k-1)}\sigma_2^{-2} \\
&= \sigma_1^{2j_1}\sigma_2^{-1}\sigma_1^{2(j_2+\cdots+j_k-(k-1))}\sigma_2^{-1} \\
&= \sigma(2j_1,j') \sim \sigma(j',2j_1) \in \Delta_e\Delta_a
\end{aligned}$$

where $j' = 2(j_2 + \cdots j_k - (k-1)) \in \mathbf{Z}_+$. Also,

$$n_\rho(\gamma_1\ldots\gamma_k) = n_\rho(\sigma(j',2j_1)).$$

But Proposition 7.10 implies that $\Delta_e\Delta_a \rhd \Delta_e\Delta_{a'}$, so

$$(\Delta_e\Delta_{a'})^k \rhd \Delta_e\Delta_{a'},$$

for any $k \in \mathbf{N}$. □

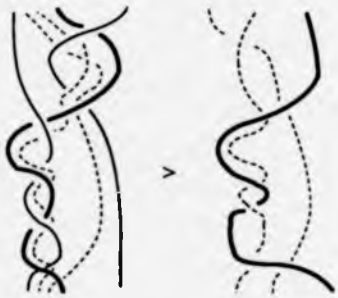

Figure 7.16: $\Delta_e\Delta_{a'} \rhd \Delta_c$.

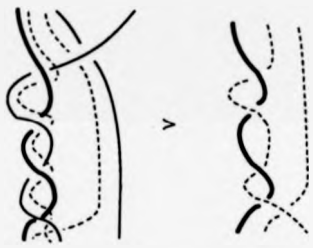

Figure 7.17: $\sigma(2j) \succ \sigma_2\sigma_1^{2j}$.

Proposition 7.12 $\Delta_e\Delta_{a'} \rhd \Delta_c$.

Proof

Suppose $\gamma = \sigma(2j, 0) \in \Delta_e\Delta_{a'}$. We have the situation in figure 7.16. The braid on the right-hand side is

$$\begin{aligned} \sigma_2\sigma_1^{2j-1}.\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} &= \sigma_2\sigma_1^{2j-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} \\ &\sim \gamma' = \sigma_1^{2j-2}\sigma_2^{-1} \in \Delta_c. \end{aligned}$$

Since $n_\rho(\gamma) = n_\rho(\gamma')$, we have that $\Delta_e\Delta_{a'} \rhd \Delta_c$. □

So $(\Delta_e\Delta_{a'})^k \rhd \Delta_c$, and we are reduced to studying $\gamma \in \Delta_c$.

Proposition 7.13 *Suppose* $\gamma = \sigma(2j) \in \Delta_c$, *then* $\gamma \succ< \theta\gamma' >$, *where* $\gamma' = \sigma(2(j-2)) \in \Delta_c$. *Further,* $n_\rho(\gamma) = n_\rho(\theta\gamma')$.

Proof

Let $\gamma = \sigma(2j) \in \Delta_c$, then we have the situation in figure 7.17. So $\gamma \succ \sigma_2\sigma_1^{2j}$, and $n_\rho(\gamma) = n_\rho(\sigma_2\sigma_1^{2j})$. But

$$\begin{aligned} \sigma_2\sigma_1^{2j} &\sim \sigma_1\sigma_2\sigma_1.\sigma_1^{2j-2} \\ &\sim \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2.\sigma_2^{-1}\sigma_1^{2j-4} \\ &\sim \theta\sigma(2(j-2)). \end{aligned}$$

Hence $\gamma \succ\!< \theta\gamma' >$, where $\gamma' = \sigma(2(j-2)) \in \Delta_c$, and the rotation number is preserved. □

We are now able to conclude the proof of Theorem 7.4. Given a braid type $\beta \in B_{2,1}$ and a geometric braid $\sigma(J)$ in normal form which represents β, we have shown that $\beta \succ \gamma$, where $\gamma \in B_{2,1}$ is represented by a geometric braid $\sigma(2j) \in \Delta_c$, and $n_\rho(\sigma(J)) = n_\rho(\sigma(2j))$ for some $j \in \mathbf{N}$. But Proposition 7.13 implies that $\sigma(2j) \succ\!< \theta\sigma(2j-4) >$, and by using this fact repeatedly we may show that either

1. $\sigma(2j) \succ\!< \theta^k\sigma(2) >$ if j is odd, or
2. $\sigma(2j) \succ\!< \theta^k\sigma(4) >$ if j is even,

for some $k \in \mathbf{Z}_+$, and the rotation numbers of the two sides are equal. But $n_\rho(\sigma(2j)) = j$. So if β is of rotation type $(1,2)$ (respectively $(0,2)$), then $\beta \succ \beta_1$ (respectively $\beta \succ \beta_0$), where $\beta_1 \in B_{2,1}$ (respectively $\beta_0 \in B_{2,1}$) is represented by the geometric braid $\sigma(2) \in \Delta_c$ (respectively $\sigma(4) \in \Delta_c$). To show that β_1 (respectively β_0) has the least topological entropy of any braid type in $B_{2,1}$ of rotation type $(1,2)$ (respectively $(0,2)$), we consider the corresponding matrix A_2 (respectively A_4) given by equation 3.2

$$A_2 = a_1^2 a_2^{-1} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$A_4 = a_1^4 a_2^{-1} = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}.$$

Given $A \in SL(2,\mathbf{Z})$, the topological entropy of the associated Anosov homeomorphism of $\mathbf{T}^2$ is $\log \lambda_+$, where λ_+ is the eigenvalue of largest modulus of A. But if we choose A to have strictly positive entries (see Section 3.5),

$$\lambda_+ = \frac{T + \sqrt{T^2 - 4}}{2}$$

where $T =$ Trace(A). Thus the topological entropy is an increasing function of T. Let h_A be the associated pseudo-Anosov homeomorphism of the disc relative to the three ramification points, then its topological entropy is also $\log \lambda_+$. Proposition 3.12 implies that A_2 (respectively A_4) has the least topological entropy for homeomorphisms of the annulus of rotation type $(1,2)$ (respectively $(0,2)$), as all such matrices with trace equal to 4 (respectively 6) are conjugate in $SL(2,\mathbf{Z})$. Hence the proof of the Theorem is completed. □

Given a homeomorphism $f : \mathbf{D}^2 \to \mathbf{D}^2$ of the disc with a 3-point invariant set X such that $bt(X,f) \in B_{2,1}$, it follows from Theorem 7.4 that there exists a 3-point invariant set Y such that $bt(Y,f) \in B_{2,1}$, $bt(X,f) \succ bt(Y,f)$, and $bt(Y,f)$ is one of β_0 or β_1. Suppose that $\bar{f}$ (respectively $\bar{g}$) is the pseudo-Anosov representative in the isotopy class of f rel. X (respectively rel. Y). Then it is clear that the dynamics of $\bar{g}$ are embedded in those of $\bar{f}$, in particular, we have a 'global shadowing'-type result analogous to that of Theorem 1.8:

Corollary 7.14 *Given $\bar{f}, \bar{g}$ as above, there exists a closed subset $S \subset \mathbf{D}^2$ and a continuous surjection $\pi : S \to \mathbf{D}^2$ that is homotopic to inclusion such that $\bar{g} \circ \pi = \pi \circ \bar{f}|_S$.* □

So all of the dynamics of $\bar{g}$ persist for $\bar{f}$, and hence for f.

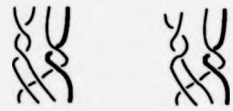

Figure 7.18: Two braids representing the braid types $\beta_\pm \in \mathcal{M}_{p,q}$

7.4.3 Embedded dynamics for higher periods

As we have seen in Chapter 3, we may often exploit the 'self-similarity' inherent in the partial order on braid types using flow-equivalence. We may also utilize it here. Given a braid type β of the annulus, we can define its rotation type $\rho(\beta)$ and its rotation set ρ_β as follows. Choose $f \in Homeo^+(\mathbb{A})$ possessing a periodic orbit $o(x)$ such that $bt(o(x), f) = \beta$ and f is the Thurston representative in its isotopy class relative to $o(x)$. Fix the lift F of f so that $o(x)$ is of F-rotation type (p, q), where $0 \leq p/q < 1$. Then we say β has *rotation type* $\rho(\beta) = (p, q)$, and *rotation set* $\rho_\beta = \rho(F)$.

Now let p,q be coprime integers such that $0 < p/q < 1$. Let $\mathcal{M}_{p,q}$ denote the set of braid types of the annulus such that $\beta \in \mathcal{M}_{p,q}$ if

1. $\rho(\beta) = p/q$ (for some lift of an annulus homeomorphism f satisfying $\beta \in bt(f)$).
2. β is pseudo-Anosov.

Now assume that $q > 2$. We define two special elements $\beta_\pm \in \mathcal{M}_{p,q}$ as follows. Let β_+ be the braid type obtained by making a Dehn twist about two points of the orbit, and then doing a rigid rotation by p/q, and let β_- be the braid type obtained as for β_+, except reversing the direction of the Dehn twist. For example, if $(p, q) = (1, 3)$ we have the two braid types as shown in figure 7.18. We conjecture (as does Boyland [Boy3]) that $\beta_\pm$ are the 'minimal' elements of $\mathcal{M}_{p,q}$ i.e. given any $\alpha \in \mathcal{M}_{p,q}$ then either $\alpha \succ \beta_+$ or $\alpha \succ \beta_-$. Our conjecture is motivated by the following ideas. Suppose we pick some $\alpha \in \mathcal{M}_{p,q}$. If ρ_α is small enough (we define what this means below), we may use flow-equivalence (via Proposition 3.10) to transform α to some $\beta \in \mathcal{M}_{1,2}$. From Theorem 7.4, $\beta \succ \beta_1 \in \mathcal{M}_{1,2}$, and β_1 has the least topological entropy of all braid types in $\mathcal{M}_{1,2}$. Reversing the flow-equivalence transformation, we obtain a braid type $\alpha' \in \mathcal{M}_{p,q}$, such that $\alpha \succ \alpha'$. Further, there does not exist $\gamma \in \mathcal{M}_{p,q}$ such that $\alpha' \succ \gamma$ and $\alpha' \neq \gamma$, for then β_1 would dominate the flow-equivalence transformation of γ, and the latter being pseudo-Anosov would give a contradiction of Theorem 7.4. Since $\rho_{\beta_1} = I(1/2)$, then $\rho_{\alpha'} = I(p/q)$, and we are reduced to showing that $\beta_\pm$ are the only braid types in $\mathcal{M}_{p,q}$ whose rotation sets are exactly $I(p/q)$, and which transform to β_1 under flow-equivalence. We conjecture that this is true.

What exactly do we mean by ρ_α being 'small enough'? Let $A \in SL(2, \mathbb{Z})$ as in Proposition 3.10. Then $A\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and A maps $I(p/q)$ to $I(1/2)$ bijectively. Flow-equivalence fails (by Theorem 3.3) if there exists $m/n \in \rho_\alpha$ such that

$$A\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix} \text{ for some } k \in \mathbb{Z}.$$

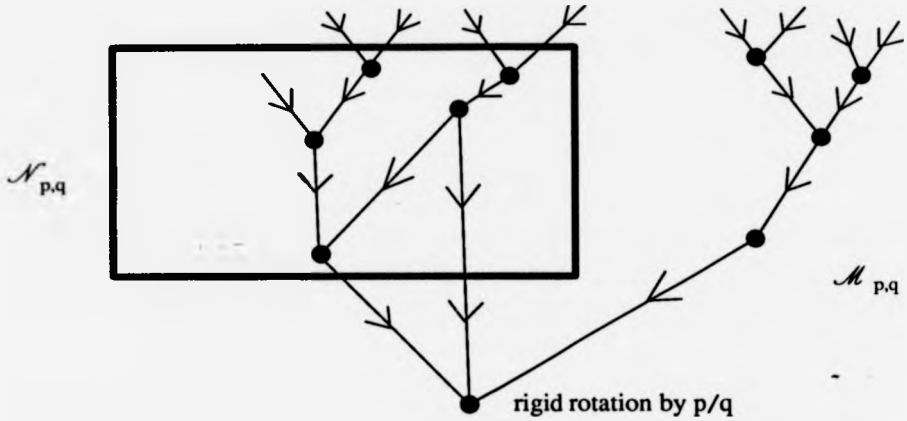

Figure 7.19: The partial ordering in $\mathcal{M}_{p,q}$. Nodes correspond to elements of $\mathcal{M}_{p,q}$, arrows denote domination

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

this occurs if $-d/c \in \rho_\alpha$. ρ_α is a closed interval (by Theorem 3.8), so if $-d/c \notin \rho_\alpha$, we may utilize the argument of the previous paragraph. Let

$$\mathcal{N}_{p,q} = \{\mu : \mu \in \mathcal{M}_{p,q} \text{ and } -d/c \notin \rho_\mu\}.$$

If $\alpha \notin \mathcal{N}_{p,q}$, flow-equivalence fails, but we would expect α to dominate some element ξ of $\mathcal{M}_{p,q}$, since intuitively, we consider the size of the rotation set to be a measure of dynamical complexity. If $\xi \in \mathcal{N}_{p,q}$, then we would pursue the flow-equivalence argument of the previous paragraph. If α dominates no element of $\mathcal{N}_{p,q}$, we are lead to the conclusion that there are two essentially separate strands of the partial ordering in $\mathcal{M}_{p,q}$, consisting of those elements which dominate an element of $\mathcal{N}_{p,q}$, and those which don't (see figure 7.19). We conjecture that this is not the case, and that every element of $\mathcal{M}_{p,q}$ dominates one of $\mathcal{N}_{p,q}$.

7.5 Global shadowing in BT_3

Following on from Theorem 7.4, we may ask if a similar Theorem holds in BT_3 i.e. does there exist some pseudo-Anosov braid type $\beta_m \in BT_3$ such that $\alpha \succ \beta_m$ for all pseudo-Anosov braid types $\alpha \in BT_3$? There exists an obvious candidate for β_m, viz. $\rho(1) =< \sigma_1\sigma_2^{-1} >$, for this induces a 3-cycle on Σ_3, and equation 3.2 gives

$$A_1 = a_1 a_2^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

as the corresponding matrix. Since the trace of any matrix representing a pseudo-Anosov braid type in $\mathbf{D}_3$ is greater than 2 (from Section 3.5), then A_1 has the least

Figure 7.20: $\sigma(2j'+1) \succ \sigma(1)$

possible trace for any such matrix, so $\rho(1)$ is the braid type with the least topological entropy of braid types in $\mathbf{D}_3$. We shall now prove that our choice of β_m is the correct one.

Suppose $\beta \in BT_3$ is a pseudo-Anosov braid type, which without loss of generality we shall assume is represented by an element of B_3 written in the normal form of Theorem 7.1. The strategy is this: firstly, we show that $\beta \succ \gamma \in B_{2,1}$. Theorem 7.4 implies that either $\gamma \succ \beta_0$ or $\gamma \succ \beta_1$, and we complete the proof by showing that $\beta_0 \succ \beta_m$ and $\beta_1 \succ \beta_m$, where β_m is represented by the geometric braid $\sigma(1)$.

So suppose β is represented by a geometric braid $\sigma(J) \in B_3$ for $J \in \mathbf{Z}_+^m \setminus \{0\}$ and $m \in \mathbf{N}$. There are three possible cases.

1. $J = (j)$, $j \in \mathbf{N}$.
2. $J = (j, 0)$, $j \in \mathbf{N}$.
3. $J = (j, J')$, where $j \in \mathbf{N}$ and $J' \neq \mathbf{0}$.

We discuss the cases in turn.

1. By considering the induced permutation, $j = 2j'+1$ for some $j' \in \mathbf{Z}_+$. From figure 7.20,
$$\sigma(J) = \sigma_1^{2j'+1}\sigma_2^{-1} \succ \sigma_1\sigma_2^{-1} = \sigma(1).$$

2. $J = (j, 0) \in \mathbf{Z}_+^m \setminus \{0\}$, where $j \in \mathbf{N}$. By considering the induced permutations, it follows that both m and j are odd. From figure 7.21,
$$\sigma(J) = \sigma_1^j\sigma_2^{-1} \succ \sigma_1\sigma_2^{-1} = \sigma(1).$$

3. There are two subcases to consider.

   (a) j is even. Then $\pi_3(\sigma(j)) = (23)$ so $\pi_3(\sigma(J')) = (12)$ or (13) according as $\pi_3(\sigma(J)) = (132)$ or (123). From figure 7.22,
$$\sigma(j) = \sigma_1^j\sigma_2^{-1} \succ \sigma_1\sigma_2^{-1} = \sigma(1),$$
   so $\sigma(J) \succ \sigma(1, J')$, where
$$\begin{aligned}\pi_3(\sigma(1, J')) &= \pi_3(\sigma(1)).\pi_3(\sigma(J')) \\ &= (13) \text{ or } (23)\end{aligned}$$

Figure 7.21: $\sigma(j) \succ \sigma(1)$

Figure 7.22: $\sigma(j) \succ \sigma(1)$

according as $\pi_3(\sigma(J')) = (12)$ or (13). Thus $\sigma(1, J')$ represents a braid type $\beta' \in B_{2,1}$, so either $\beta' \succ \beta_0$ or $\beta' \succ \beta_1$ by Theorem 7.4.

(b) j is odd. Then $\pi_3(\sigma(j)) = (123)$ so $\pi_3(\sigma(J')) = Id$ or (123) according as $\pi_3(\sigma(J)) = (123)$ or (132). From figure 7.23,

$$\sigma(j) = \sigma_1^j \sigma_2^{-1} \succ \sigma_2^{-1} = \sigma(0).$$

So $\sigma(J) \succ \sigma(0, J')$, where

$$\begin{aligned} \pi_3(\sigma(0, J')) &= \pi_3(\sigma(0)).\pi_3(\sigma(J')) \\ &= (23) \text{ or } (13) \end{aligned}$$

according as $\pi_3(\sigma(J')) = Id$ or (123).Thus $\sigma(0, J')$ represents a braid type $\beta' \in B_{2,1}$, so either $\beta' \succ \beta_0$ or $\beta' \succ \beta_1$ by Theorem 7.4.

Figure 7.23: $\sigma(j) \succ \sigma(0)$

Figure 7.24: $\sigma(2) \succ \sigma(1)$

Figure 7.25: $\sigma(4) \succ \sigma(1)$

So it just remains to show that $\beta_0 \succ \beta_m$ and $\beta_1 \succ \beta_m$, where β_m is represented by the geometric braid $\sigma(1)$. This follows from figure 7.24, where $\sigma(2) \succ \sigma(1)$, and from figure 7.25, where $\sigma(4) \succ \sigma(1)$. Hence we have proved the following Theorem.

Theorem 7.15 *There exists a pseudo-Anosov braid type $\beta_m \in BT_3$ such that $\beta \succ \beta_m$ for any pseudo-Anosov braid type $\beta \in BT_3$. β_m is represented by the geometric braid $\sigma(1)$.* □

We conjecture that a Theorem analogous to Theorem 7.15 holds in BT_n, with $\sigma(1)$ replaced by the braids $\theta_n\sigma_{n-1}^{-2} = \sigma_1\sigma_2\ldots\sigma_{n-2}\sigma_{n-1}^{-1}$ or $\theta_n\sigma_{n-1}^{2} = \sigma_1\sigma_2\ldots\sigma_{n-2}\sigma_{n-1}^{3}$ i.e. any pseudo-Anosov braid type in BT_n dominates (at least) one of these two in the partial order.

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