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Representation of Homothetic Forward Performance Processes in Stochastic Factor Models via Ergodic and Infinite Horizon BSDE*

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Abstract. In an incomplete market, with incompleteness stemming from stochastic factors imperfectly correlated with the underlying stocks, we derive representations of homothetic (power, exponential, and logarithmic) forward performance processes in factor-form using ergodic BSDE. We also develop a connection between the forward processes and infinite horizon BSDE, and, moreover, with risk-sensitive optimization. In addition, we develop a connection, for large time horizons, with a family of classical homothetic value function processes with random endowments.

Key words. homothetic forward performance processes, ergodic BSDE, infinite horizon BSDE

AMS subject classifications. 91G40, 91G80, 60H30

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1. Introduction.

This paper contributes to the study of homothetic forward performance processes, namely, of power, exponential and logarithmic type, in a stochastic factor market model. Stochastic factors are frequently used to model the predictability of stock returns, stochastic volatility, and stochastic interest rates. (For an overview of the literature, we refer the reader to the review paper [38].) Forward performance processes were introduced and developed in [27], [28], and [30] (see, also, [29], [31], and [32]). They complement the classical expected utility paradigm in which the utility is a deterministic function chosen at a single point in time (terminal horizon). The value function process is, in turn, constructed backwards in time, as the dynamic programming principle yields. As a result, there is limited flexibility to incorporate updating of risk preferences, rolling horizons, learning, and other realistic "forward in nature" features if one requires that time-consistency is being preserved at all times. Forward investment performance criteria alleviate some of these shortcomings and offer the construction of a genuinely dynamic mechanism for evaluating the performance of investment strategies as the market evolves across (arbitrary) trading horizons.

In [33] a stochastic PDE (cf. (10) herein) was proposed for the characterization of forward performance processes in a market with Itô-diffusion price processes. It may be viewed as the forward analogue of the finite-dimensional classical Hamilton–Jacobi–Bellman (HJB) equation that arises in Markovian models of optimal portfolio choice. Like the HJB equation, the forward SPDE is fully nonlinear and possibly degenerate. In addition, however, it is ill-posed

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and its volatility coefficient is an input that the investor chooses while, in the classical case, the corresponding volatility is uniquely determined from the Itô decomposition of the value function process. These features result in significant technical difficulties and, as a result, the use of the forward SPDE for general Itô-diffusion market dynamics has been limited. Results for time-monotone processes (zero forward volatility) can be found in [32], and a connection between the forward performance process and optimal portfolios has been explored in [12] (see, also [11]). In semi-martingale markets, an axiomatic construction for exponential preferences can be found in [40].

When the market coefficients depend explicitly on stochastic factors, as herein, there is more structure that can be explored by seeking performance criteria represented as *deterministic functions* of these factors. As was first noted in [33], the SPDE reduces to a finite-dimensional HJB equation (see equation (51) therein) that these functions are expected to satisfy. Still, however, this HJB equation remains ill-posed and how to solve it is an open problem.

For a single stochastic factor, two cases have been so far analyzed, specifically, for the power and exponential cases. The power case was treated in [35], where the homotheticity reduces the forward HJB to a semilinear PDE which is, in turn, linearized using a distortion transformation. One then obtains a one-dimensional ill-posed linear equation with state dependent coefficients, which is solved using an extension of Widder's theorem. The exponential case was studied in [29] (see, also, [28] and [23]) in the context of forward exponential indifference prices.

Multifactor modeling of forward performance processes is considered in [34], where the complete market setting is analyzed in detail. Because of market completeness, the Legendre–Fenchel transformation linearizes the forward SPDE, and a multidimensional ill-posed linear equation with space/time dependent coefficients arises. Its solutions are, in turn, characterized via an extension of Widder's theorem developed by the authors. More recently, multifactors of different (slow and fast) scales in incomplete markets were studied in [37], and asymptotic expansions were derived for the limiting regimes. Therein, the leading order terms are expressed as time-monotone forward performances with appropriate stochastic time-rescaling, resulting from averaging phenomena. The first order terms reflect compiled changes in the investor's preferences based on market changes and her past performance.

Herein, we initiate a study to generalize the existing results on forward processes in factor-form allowing for market incompleteness, multistocks, and multistochastic factors. We first focus on homothetic processes (power, exponential, and logarithmic), for these are also the popular choices of risk preferences in the classical setting.

For such cases, the homotheticity reduces the SPDE to an ill-posed multidimensional semilinear PDE (cf. (13), (40)), which, however, *cannot* be linearized. To our knowledge, no results exist to date for such ill-posed equations. The main contribution herein is that we bypass the difficulties generated by the ill-posedness by constructing factor-form forward processes *directly* from *Markovian* solutions of a family of *ergodic BSDE*. While the form of their driver is suggested by the operator appearing in the ill-posed PDE, we use exclusively results from ergodic equations to construct the forward solutions and not from (forward) stochastic optimization. As a byproduct, we use these findings to construct a smooth solution to the ill-posed multidimensional semilinear PDE. To our knowledge, this approach is new. It is quite direct and requires mild assumptions on the dynamics of the factors, essentially the ergodicity condition (4).

The second contribution is that we provide a connection with risk-sensitive optimization and the constant appearing in the solution of the ergodic BSDE. Thus, we provide a new interpretation, in the context of forward optimization, of the classical results of [5], [14], and [15] on the optimal growth rate of long-term utility maximization problems.

In a different direction, we develop a connection of the homothetic forward processes with *infinite horizon BSDE*. Our contribution is threefold. First, we establish that the solutions of the latter are themselves homothetic forward processes, albeit not Markovian. Second, we show that as the parameter ρ, which appears naturally in these BSDE, converges to zero, the relevant solutions will converge to their Markovian ergodic counterparts. Third, we use these infinite horizon BSDE to establish a connection among the homothetic forward processes we construct and classical analogues, specifically, finite-horizon value function processes with an appropriately chosen terminal endowment. We show that these value functions converge to the homothetic processes as the trading horizon tends to infinity.

In the finite horizon setting, (quadratic) BSDE were first studied in [22] and have been subsequently analyzed by a number of authors. They constitute one of the most active areas of research in financial mathematics, for they offer direct applications to risk measures [2], indifference prices [1], [18], [24], and value functions for homothetic utilities [19]. Several extensions to the latter line of applications include, among others, [25] and [3], where the results of [19] were, respectively, generalized to a continuous martingale setting and to jump-diffusions. We note that in the traditional framework, prices, portfolios, risk measures, and value functions are intrinsically constructed "backward" in time and, thus, BSDE offer the ideal tool for their analysis.

Despite the popularity of (quadratic) BSDE in the finite horizon setting, neither their ergodic or infinite horizon counterparts have received much attention to date. In an infinite-dimensional setting, an ergodic Lipschitz BSDE was introduced in [16] for the solution of an ergodic stochastic control problem; see also [8], [10], [36] and more recently [9] and [20] for various extensions. The infinite horizon quadratic BSDE was first solved in [6] by combining the techniques used in [7] and [22].

To our knowledge, both types of ergodic and infinite horizon equations have been so far motivated mainly from theoretical interest. Our results show, however, that both types of equations are natural candidates for the characterization of forward performance processes and their associated optimal portfolios and wealths. It is worth mentioning that both the ergodic and infinite horizon BSDE we consider actually turn out to be Lipschitz, since one can show that the parts corresponding to the relevant processes Z are bounded. In other words, the quadratic growth, which is the standard assumption in the current setting, does not play a crucial role. Indeed, as we show in the appendix, the existing results from the ergodic Lipschitz BSDE [16] and the infinite horizon Lipschitz BSDE [7] can be readily adapted to solve the forward equations at hand.

We conclude by mentioning that while we focus on forward processes in factor-form, most of the results also apply for non-Markovian forward processes (e.g., the results in section 3.1.3). Furthermore, we stress that a measure transformation (see the examples in 3.1.3) might indicate that one can construct new homothetic forward processes directly from the ones with zero volatility, thus making the results herein redundant. However, this is *not* the case. On the one hand, changing measure corresponds to changing the risk premia, which essentially amounts to changing the original market model. Therefore, one does not produce any genuinely new forward processes within the original market. More importantly, zero volatility forward processes are decreasing in time and path-dependent with regards to the stochastic factors. It is *not* possible to produce from them their Markovian counterparts using a measure change transformation.

The paper is organized as follows. In section 2, we introduce the market model and review the notion of forward performance process and the forward SPDE. In sections 3, 4, and 5, we construct the corresponding forward performance processes in factor-form and the associated optimal portfolios and wealth processes. In each section, we also present the connection with an ill-posed semilinear PDE as well as with the solutions of the related infinite horizon BSDE and with finite horizon counterparts. For the reader's convenience, we present the technical background results on the ergodic and infinite horizon BSDE in the appendix.

2. The stochastic factor model and its forward performance process. The market consists of a riskless bond and n stocks. The bond is taken to be the numeraire and the individual (discounted by the bond) stock prices S_t^i, $t \geq 0$, solve, for $i = 1, \ldots, n$,

$$\frac{dS_t^i}{S_t^i} = b^i(V_t)dt + \sum_{j=1}^{d} \sigma^{ij}(V_t)dW_t^j \tag{1}$$

with $S_0^i > 0$. The process $W = (W^1, \cdots, W^d)^T$ is a standard d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. The superscript T denotes the matrix transpose.

The d-dimensional process $V = (V^1, \cdots, V^d)$ models the stochastic factors affecting the dynamics of stock prices, and its components are assumed to solve, for $i = 1, \ldots, d$,

$$dV_t^i = \eta^i(V_t)dt + \sum_{j=1}^{d} \kappa^{ij} dW_t^j \tag{2}$$

with $V_0^i \in \mathbb{R}$.

We introduce the following model assumptions.

Assumption 1.

(i) The market coefficients $b(v) = (b^i(v))$ and $\sigma(v) = (\sigma^{ij}(v))$, $1 \leq i \leq n, 1 \leq j \leq d$, $v \in \mathbb{R}^d$, are uniformly bounded and the volatility matrix $\sigma(v)$ has full row rank n.

(ii) The market price of risk vector $\theta(v)$, $v \in \mathbb{R}^d$, defined as the solution to the equation $\sigma(v)\theta(v) = b(v)$ and given by $\theta(v) = \sigma(v)^T[\sigma(v)\sigma(v)^T]^{-1}b(v)$, is uniformly bounded and Lipschitz continuous.

Assumption 2. The drift coefficients of the stochastic factors satisfy the dissipative condition

$$(\eta(v) - \eta(\bar{v}))^T(v - \bar{v}) \leq -C_\eta |v - \bar{v}|^2 \tag{3}$$

for any $v, \bar{v} \in \mathbb{R}^d$ and a constant C_η large enough. The volatility matrix $\kappa = (\kappa^{ij})$, $1 \le i, j \le d$, is a constant matrix with $\kappa\kappa^T$ positive definite and normalized to $|\kappa| = 1$.

The "large enough" property of the above constant C_η will be refined later on when we introduce another auxiliary constant C_v (cf. (57) and the example in section 3.1.3) related to the drivers of the upcoming BSDE.

The dissipative condition (3) implies that the stochastic factor process V admits a unique invariant measure, and it is, thus, ergodic. Indeed, a direct application of Gronwall's inequality yields that V satisfy, for any $v, \bar{v} \in \mathbb{R}^d$, the *exponential ergodicity* condition

$$|V_t^v - V_t^{\bar{v}}|^2 \le e^{-2C_\eta t}|v - \bar{v}|^2, \tag{4}$$

where the superscript v denotes the dependence on the initial condition.

Inequality (4) states that any two distinct paths of the process V will converge to each other exponentially fast. We note that (4) is the only condition needed to be satisfied by the stochastic factors. Any diffusion process satisfying inequality (4) may serve as a stochastic factor vector.

Next, we consider an investor who starts at time $t = 0$ with initial endowment x and trades among the $(n+1)$ assets. We denote by $\tilde{\pi} = (\tilde{\pi}^1, \cdots, \tilde{\pi}^n)^T$ the proportions of her total (discounted by the bond) wealth in the individual stock accounts. Assuming that the standard self-financing condition holds and using (1), we deduce that her (discounted by the bond) wealth process solves

$$dX_t^\pi = \sum_{i=1}^n \tilde{\pi}_t^i X_t^\pi \frac{dS_t^i}{S_t^i} = X_t^\pi \tilde{\pi}_t^T \left(b(V_t)dt + \sigma(V_t)dW_t\right)$$

with $X_0 = x \in \mathbb{D}$, where the set $\mathbb{D} \subseteq \mathbb{R}$ denotes the wealth admissibility domain.

For mere convenience, we will be working throughout with the trading strategies *rescaled by the volatility*, namely,

$$\pi_t^T = \tilde{\pi}_t^T \sigma(V_t). \tag{5}$$

Then, the wealth process solves

$$dX_t^\pi = X_t^\pi \pi_t^T(\theta(V_t)dt + dW_t). \tag{6}$$

For any $t \ge 0$, we denote by $\mathcal{A}_{[0,t]}$ the set of admissible strategies in the trading interval $[0, t]$, given by

$$\mathcal{A}_{[0,t]} = \{(\pi_u)_{u\in[0,t]} : \pi \in \mathcal{L}^2_{BMO}[0, t],\ \ \pi_u \in \Pi \text{ and } X_u^\pi \in \mathbb{D},\ \ u \in [0, t]\}. \tag{7}$$

The set $\Pi \subseteq \mathbb{R}^d$ is closed and convex, and the space $\mathcal{L}^2_{BMO}[0, t]$ defined as

$$\mathcal{L}^2_{BMO}[0,t] = \Bigg\{(\pi_u)_{u\in[0,t]} : \pi \text{ is } \mathbb{F}\text{-progressively measurable and}$$
$$\operatorname*{ess\,sup}_\tau E_{\mathbb{P}}\left(\int_\tau^t |\pi_u|^2 du \,\middle|\, \mathcal{F}_\tau\right) < \infty \text{ for any } \mathbb{F}\text{-stopping time } \tau \in [0,t]\Bigg\}.$$

The above integrability condition is also called the *BMO-condition*, since for any $\pi \in \mathcal{L}^2_{BMO}[0,t]$,

$$\operatorname*{ess\,sup}_{\tau\in[0,t]} E_{\mathbb{P}}\left(\int_\tau^t \pi_u^T dW_u \middle| \mathcal{F}_\tau\right)^2 = \operatorname*{ess\,sup}_{\tau\in[0,t]} E\left(\int_\tau^t |\pi_u|^2 du \middle| \mathcal{F}_\tau\right) < \infty,$$

and, hence, the stochastic integral $\int_0^s \pi_u^T dW_u$, $s \in [0,t]$, is a BMO-martingale.

In turn, we define the set of *admissible strategies* for all $t \geq 0$ as $\mathcal{A} := \cup_{t\geq 0}\mathcal{A}_{[0,t]}$.

Next, we review the notion of forward performance process, introduced and developed in [28, 29, 30, 31, 32, 33]. Variations and relaxations of the original definition can be also found in [4], [12], [17], and [34].

Definition 2.1. *A process $U(x,t)$, $(x,t) \in \mathbb{D}\times[0,\infty)$, is a forward performance process if*

(i) *for each $x \in \mathbb{D}$, $U(x,t)$ is $\mathbb{F}$-progressively measurable;*

(ii) *for each $t \geq 0$, the mapping $x \mapsto U(x,t)$ is strictly increasing and strictly concave;*

(iii) *for any $\pi \in \mathcal{A}$ and $0 \leq t \leq s$,*

$$E_{\mathbb{P}}\left(U(X_s^\pi, s)|\mathcal{F}_t\right) \leq U\left(X_t^\pi, t\right), \tag{8}$$

and there exists an optimal portfolio $\pi^* \in \mathcal{A}$ such that, for $0 \leq t \leq s$,

$$E_{\mathbb{P}}\left(U_s(X_s^{\pi^*}, s)|\mathcal{F}_t\right) = U\left(X_t^{\pi^*}, t\right). \tag{9}$$

As mentioned earlier, it was shown in [33] that $U(x,t)$ is associated with an ill-posed fully nonlinear SPDE, which plays the role of the HJB equation in the classical finite-dimensional setting. Formally, this forward SPDE is derived by first assuming that $U(x,t)$ admits the Itô decomposition

$$dU(x,t) = b(x,t)dt + a(x,t)^T dW_t$$

for some $\mathbb{F}$-progressively measurable processes $a(x,t)$ and $b(x,t)$, and that all involved quantities have enough regularity so that the Itô–Ventzell formula can be applied to $U(X_s^\pi, s)$ for all admissible π. The requirements (8) and (9) then yield that, for a chosen volatility process $a(x,t)$, the drift $b(x,t)$ must have a specific form.

In the setting herein, the forward performance SPDE takes the form

$$\begin{aligned} dU(x,t) = \Bigg(&-\frac{1}{2}x^2U_{xx}(x,t)dist^2\left(\Pi, -\frac{\theta(V_t)U_x(x,t)+a_x(x,t)}{xU_{xx}(x)}\right) \\ &+\frac{1}{2}\frac{|\theta(V_t)U_x(x,t)+a_x(x,t)|^2}{U_{xx}(x,t)}\Bigg)dt + a(x,t)^T dW_t, \end{aligned} \tag{10}$$

where $dist(\Pi, x)$ represents the distance function from $x \in \mathbb{R}^d$ to Π. Furthermore, if a strong solution to (6) exists, say, $X_t^{\pi^*}$, when the feedback policy

$$\pi_t^* = Proj_\Pi\left(-\frac{\theta(V_t)U_x(X_t^{\pi^*},t)}{X_t^{\pi^*}U_{xx}(X_t^{\pi^*},t)} - \frac{a_x(X_t^{\pi^*},t)}{X_t^{\pi^*}U_{xx}(X_t^{\pi^*},t)}\right) \tag{11}$$

is used, then the control process π_t^* is optimal. We note that these arguments are formal and a general verification theorem is still lacking.

Herein, we bypass these difficulties and construct homothetic forward performance processes in factor-form using directly the Markovian solutions of associated ergodic BSDE. The SPDE is merely used to guess the appropriate form of the latter.

3. Power case.

We start with the construction of forward performance processes that are homogeneous of degree $\delta \in (0,1)$ and have the factor-form

$$U(x,t) = \frac{x^\delta}{\delta} e^{f(V_t,t)}, \tag{12}$$

where $f : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ is a (deterministic) function to be specified. For this range of δ, the admissible wealth domain is taken to be $D = \mathbb{R}_+$.

Using the form (12) and the SPDE (10), we deduce that f must satisfy, for $(v,t) \in \mathbb{R}^d \times [0,\infty)$, the semilinear PDE

$$f_t + \frac{1}{2} Trace\left(\kappa\kappa^T \nabla^2 f\right) + \eta(v)^T \nabla f + F(v, \kappa^T \nabla f) = 0 \tag{13}$$

with

$$F(v,z) := -\frac{1}{2}\delta(1-\delta) dist^2\left(\Pi, \frac{z+\theta(v)}{1-\delta}\right) + \frac{1}{2}\frac{\delta}{1-\delta}|z+\theta(v)|^2 + \frac{1}{2}|z|^2. \tag{14}$$

The above equation, however, is ill-posed with no known solutions to date. On the other hand, as we demonstrate below, the process $f(V_t,t)$ can be actually constructed directly from the Markovian solution of an ergodic BSDE whose driver is of the above form (cf. (16)).

3.1. Construction via ergodic BSDE.

We first introduce the underlying ergodic BSDE and provide the main existence and uniqueness result for Markovian solutions. For the reader's convenience, we present the proof in the appendix.

Proposition 3.1. *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1(ii) and let the set Π be as in (7). Then, the ergodic BSDE*

$$dY_t = (-F(V_t,Z_t) + \lambda)dt + Z_t^T dW_t \tag{15}$$

with the driver $F(\cdot,\cdot)$ defined as

$$F(V_t,Z_t) := -\frac{1}{2}\delta(1-\delta) dist^2\left(\Pi, \frac{Z_t+\theta(V_t)}{1-\delta}\right) + \frac{1}{2}\frac{\delta}{1-\delta}|Z_t+\theta(V_t)|^2 + \frac{1}{2}|Z_t|^2 \tag{16}$$

admits a unique Markovian solution (Y_t, Z_t, λ), $t \geq 0$.

Specifically, there exist a unique $\lambda \in \mathbb{R}$ and functions $y : \mathbb{R}^d \to \mathbb{R}$ and $z : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t, Z_t) = (y(V_t), z(V_t))$. The function $y(\cdot)$ is unique up to a constant and has at most linear growth, and $z(\cdot)$ is bounded with $|z(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η and C_v are as in (3) and (57), respectively.

We next present one of the main results.

Theorem 3.2. *Let $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda), t \geq 0$, be the unique Markovian solution of (15). Then,*

(i) *the process $U(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by*

$$U(x,t) = \frac{x^\delta}{\delta} e^{y(V_t)-\lambda t} \ , \tag{17}$$

is a power forward performance process with volatility

$$a\,(x,t) = \frac{x^\delta}{\delta} e^{y(V_t)-\lambda t} z(V_t). \tag{18}$$

(ii) *The optimal portfolio weights π_t^* and the associated wealth process X_t^* (cf. (5) and (6)) are given, respectively, by*

$$\pi_t^* = Proj_\Pi \left(\frac{z(V_t) + \theta(V_t)}{1-\delta} \right) \ \text{and} \ X_t^* = X_0 \mathcal{E} \left(\int_0^\cdot (\pi_s^*)^T (\theta(V_s)ds + dW_s) \right)_t . \tag{19}$$

Proof. It is immediate that the process $U(x,t)$ is $\mathbb{F}$-progressively measurable, strictly increasing, and strictly concave in x and homogeneous of degree δ. To show that it also satisfies requirements (ii) and (iii) of Definition 1, we will establish that, for $0 \leq t \leq s$, if $\pi \in \mathcal{A}$,

$$E_\mathbb{P} \left(\frac{(X_s^\pi)^\delta}{\delta} e^{Y_s - \lambda s} | \mathcal{F}_t \right) \leq \frac{(X_t^\pi)^\delta}{\delta} e^{Y_t - \lambda t},$$

while for π^* given by (19),

$$E_\mathbb{P} \left(\frac{(X_s^{\pi^*})^\delta}{\delta} e^{Y_s - \lambda s} | \mathcal{F}_t \right) = \frac{(X_t^{\pi^*})^\delta}{\delta} e^{Y_t - \lambda t}.$$

To this end, the wealth equation (6) and Itô's formula yield

$$(X_s^\pi)^\delta = (X_t^\pi)^\delta \exp \left(\int_t^s \delta \left(\pi_u^T \theta(V_u) - \frac{1}{2} |\pi_u|^2 \right) du + \int_t^s \delta \pi_u^T dW_u \right).$$

On the other hand, from the ergodic BSDE (15), we have

$$Y_s - \lambda s = Y_t - \lambda t - \int_t^s F(V_u, z(V_u))du + \int_t^s z(V_u)^T dW_u. \tag{20}$$

Combining the above yields

$$\begin{aligned}(X_s^\pi)^\delta e^{Y_s - \lambda s} = (X_t^\pi)^\delta e^{Y_t - \lambda t} \exp &\left(\int_t^s \left(\delta \left(\pi_u^T \theta(V_u) - \frac{1}{2} |\pi_u|^2 \right) - F(V_u, z(V_u)) \right) du \right. \\ &\left. + \int_t^s \left(\delta \pi_u^T + z(V_u)^T \right) dW_u \right).\end{aligned}$$

Therefore,

$$
\begin{aligned}
&E_{\mathbb{P}}\left((X_s^{\pi})^{\delta}e^{Y_s-\lambda s}|\mathcal{F}_t\right)\\
&=(X_t^{\pi})^{\delta}e^{Y_t-\lambda t}E_{\mathbb{P}}\left(\exp\left(\int_t^s\left(\delta\left(\pi_u^T\theta(V_u)-\frac{1}{2}|\pi_u|^2\right)-F(V_u,z(V_u))\right)du\right.\right.\\
&\quad\left.\left.+\int_t^s\left(\delta\pi_u^T+z(V_u)^T\right)dW_u\right|\mathcal{F}_t\right).
\end{aligned}
$$

Next, for $s\geq 0$ and $\pi\in\mathcal{A}$, we define a probability measure, say, $\mathbb{Q}^{\pi}$, by introducing the Radon–Nikodym density process $\mathcal{Z}_u, u\in[0,s]$,

$$
\mathcal{Z}_u=\left.\frac{d\mathbb{Q}^{\pi}}{d\mathbb{P}}\right|_{\mathcal{F}_u}=\mathcal{E}(N)_u \quad \text{with} \quad N_u=\int_0^u\left(\delta\pi_t^T+Z_t^T\right)dW_t. \tag{21}
$$

We recall that both processes π_u and $z(V_u)$, $u\in[0,s]$, satisfy the BMO-condition (up to time s). Therefore, the process N_u, $u\in[0,s]$, is a BMO-martingale and, in turn, $\mathcal{E}(N)$ is in Doob's class $\mathcal{D}$ and, thus, uniformly integrable. In turn,

$$
\begin{aligned}
&E_{\mathbb{P}}\left(\exp\left(\int_t^s\left(F^{\pi}(V_u,z(V_u))-F(V_u,z(V_u))\right)du\right)\frac{\mathcal{Z}_s}{\mathcal{Z}_t}\right|\mathcal{F}_t\Bigg)\\
&=E_{\mathbb{Q}^{\pi}}\left(\exp\left(\int_t^s\left(F^{\pi}(V_u,z(V_u))-F(V_u,z(V_u))\right)du\right)\right|\mathcal{F}_t\Bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
F^{\pi}(V_t,z(V_t)):&=-\frac{1}{2}\delta(1-\delta)|\pi_t|^2+\delta\pi_t^T(z(V_t)+\theta(V_t))+\frac{1}{2}|z(V_t)|^2\\
&=-\frac{1}{2}\delta(1-\delta)\left|\pi_t-\frac{z(V_t)+\theta(V_t)}{1-\delta}\right|^2+\frac{1}{2}\frac{\delta}{1-\delta}|z(V_t)+\theta(V_t)|^2+\frac{1}{2}|z(V_t)|^2.
\end{aligned}
$$

Using that $F^{\pi}(V_t,z(V_t))\leq F(V_t,z(V_t))$, we easily deduce that

$$
E_{\mathbb{P}}\left((X_s^{\pi})^{\delta}e^{Y_s-\lambda s}|\mathcal{F}_t\right)\leq(X_t^{\pi})^{\delta}e^{Y_t-\lambda t}.
$$

Moreover, for $\pi=\pi^*$ as in (19), $F^{\pi^*}(V_t,z(V_t))=F(V_t,z(V_t))$ and, thus,

$$
E_{\mathbb{P}}\left((X_s^{\pi^*})^{\delta}e^{Y_s-\lambda s}|\mathcal{F}_t\right)=(X_t^{\pi^*})^{\delta}e^{Y_t-\lambda t}.
$$

To show (18), we recall the SPDE (10) and observe that representation (17) yields

$$
dU(x,t)=U(x,t)(-F(V_t,z(V_t))+\frac{1}{2}|z(V_t)|^2)dt+U(x,t)z(V_t)^TdW_t.
$$

The rest of the proof follows easily. ■

3.1.1. Connection with risk-sensitive optimization. We provide an interpretation of the constant λ, appearing in the representation of the forward performance process (17), as the solution of the risk-sensitive control problem (23). It turns out that the constant λ is also the optimal growth rate of the long-term utility maximization problem as considered in [5], [14], and [15] (see (24) below).

Proposition 3.3. *Let $T > 0$ and $\pi \in \mathcal{A}$ and define the probability measure $\mathbb{P}^{\pi}$ using the Radon–Nikodym density process $\mathcal{Z}_u$, $u \in [0,T]$,*

$$\mathcal{Z}_u = \left.\frac{d\mathbb{P}^{\pi}}{d\mathbb{P}}\right|_{\mathcal{F}_u} = \mathcal{E}\left(\int_0^{\cdot} \delta\pi_u^T dW_u\right)_u \tag{22}$$

and the stochastic functional

$$L(V_s,\pi_s) := -\frac{1}{2}\delta(1-\delta)|\pi_s|^2 + \delta\theta(V_s)^T\pi_s$$

for $s \in [0,T]$.

Let $(y(V_t), z(V_t), \lambda)$, $t \ge 0$, be the unique Markovian solution of the ergodic BSDE (15) and X^{π} solving the wealth equation (6). Then, λ is the long-term growth rate of the risk-sensitive control problem

$$\lambda = \sup_{\pi\in\mathcal{A}} \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}^{\pi}}\left(e^{\int_0^T L(V_s,\pi_s)ds}\right), \tag{23}$$

or, alternatively,

$$\lambda = \sup_{\pi\in\mathcal{A}} \limsup_{T\uparrow\infty} \frac{1}{T} \ln E_{\mathbb{P}}\left(\frac{(X_T^{\pi})^{\delta}}{\delta}\right). \tag{24}$$

For both problems (23) and (24), the associated optimal control process π_t^*, $t \ge 0$, is as in (19).

Proof. We first observe that the driver $F(\cdot,\cdot)$ in (16) can be written as

$$F(V_t,Z_t) = \sup_{\pi_t\in\Pi}\left(L(V_t,\pi_t) + Z_t^T\delta\pi_t\right) + \frac{1}{2}|Z_t|^2.$$

Therefore, for arbitrary $\tilde{\pi} \in \mathcal{A}$, we rewrite the ergodic BSDE (15) under the probability measure $\mathbb{P}^{\tilde{\pi}}$ as

$$dY_t = \left(-\sup_{\pi_t\in\Pi}\left(L(V_t,\pi_t) + Z_t^T\delta\pi_t\right) + Z_t^T\delta\tilde{\pi}_t + \lambda - \frac{1}{2}|Z_t|^2\right)dt + Z_t^T dW_t^{\mathbb{P}^{\tilde{\pi}}},$$

where the process $W_t^{\mathbb{P}^{\tilde{\pi}}} := W_t - \int_0^t \delta\tilde{\pi}_u du$, $t \ge 0$, is a Brownian motion under $\mathbb{P}^{\tilde{\pi}}$. In turn,

$$\begin{aligned} &e^{\lambda T + Y_0}e^{-Y_T}\mathcal{E}\left(\int_0^{\cdot} Z_u^T dW_u^{\mathbb{P}^{\tilde{\pi}}}\right)_T \\ &= \exp\left(\int_0^T \left(\sup_{\pi_t\in\Pi}\left(L(V_t,\pi_t) + Z_t^T\delta\pi_t\right) - \left(L(V_t,\tilde{\pi}_t) + Z_t^T\delta\tilde{\pi}_t\right)\right)dt\right) e^{\int_0^T L(V_t,\tilde{\pi}_t)dt}. \end{aligned}$$

Next, we observe that for any $\tilde{\pi} \in \mathcal{A}$, the first exponential term on the right-hand side is bounded below by 1. Taking expectation under $\mathbb{P}^{\tilde{\pi}}$ then yields

$$e^{\lambda T + Y_0} E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{-Y_T} \mathcal{E} \left(\int_0^{\cdot} Z_s^T dW_s^{\mathbb{P}^{\tilde{\pi}}} \right)_T \right) \geq E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_s, \tilde{\pi}_s) ds} \right).$$

Using the measure $\mathbb{Q}^{\tilde{\pi}}$, defined in (21), we deduce that

$$\lambda + \frac{Y_0}{T} + \frac{1}{T} \ln E_{\mathbb{Q}^{\tilde{\pi}}} \left(e^{-Y_T} \right) \geq \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_s, \tilde{\pi}_s) ds} \right).$$

Note, however, that there exists a constant, say, C, independent of T, such that

$$\frac{1}{C} \leq E_{\mathbb{Q}^{\pi}} \left(e^{-Y_T} \right) \leq C.$$

This follows from the linear growth property of the function $y(\cdot)$ and the ergodicity condition (4) (see, for example, [13]).

Sending $T \uparrow \infty$ then yields that, for any $\tilde{\pi} \in \mathcal{A}$,

$$\lambda \geq \limsup_{T \uparrow \infty} \frac{1}{T} \ln E_{\mathbb{P}^{\tilde{\pi}}} \left(e^{\int_0^T L(V_s, \tilde{\pi}_s) ds} \right)$$

with equality choosing $\tilde{\pi}_s = \pi_s^*$, with π_s^* as in (19).

To show that λ also solves (24), we observe that for $\pi \in \mathcal{A}$, we have

$$\begin{aligned} E_{\mathbb{P}} \left(\frac{(X_T^{\pi})^{\delta}}{\delta} \right) &= \frac{X_0^{\delta}}{\delta} E_{\mathbb{P}} \left(e^{\int_0^T L(Vs, \pi_s) ds} \mathcal{E} \left(\int_0^{\cdot} \delta \pi_s^T dW_s \right)_T \right) \\ &= \frac{X_0^{\delta}}{\delta} E_{\mathbb{P}^{\pi}} \left(e^{\int_0^T L(V_s, \pi_s) ds} \right), \end{aligned}$$

and the rest of the arguments follow. ■

3.1.2. Connection with an ill-posed multi-dimensional semilinear PDE. A byproduct of the previous result is the construction of a smooth solution to the ill-posed semilinear PDE given in (25) below. Recall that the latter was derived from (10) as a necessary requirement when we seek forward processes of the form (12). We establish below that for an appropriate initial datum, this ill-posed PDE has a solution, which is separable in time and space.

We note that the well-posed analogue of this semilinear equation, as well as of the one appearing in the exponential case (cf. (40)), have been extensively analyzed and used for the representation of indifference prices, risk measures, power and exponential value functions, etc. To our knowledge, however, their ill-posed versions have not been studied, with the exception of the one-dimensional case studied in [35]. This case, on the other hand, can be linearized and the solution is constructed using an extension of Widder's theorem. We refer in detail to this case in 3.1.3. However, the multidimensional case cannot be linearized and, to our knowledge, no results for this case exist to date.

Proposition 3.4. *Consider the ill-posed semilinear PDE*

$$f_t + \mathcal{L} f + F(v, \kappa^T \nabla f) = 0, \tag{25}$$

$(v,t) \in \mathbb{R}^d \times [0, \infty)$ *with* $F(\cdot, \cdot)$ *as in* (14) *(or* (16)*) and* $\mathcal{L}$ *being the infinitesimal generator of* *the factor process V,*

$$\mathcal{L} = \frac{1}{2} Trace\left(\kappa\kappa^T\nabla^2\right) + \eta(v)^T\nabla. \tag{26}$$

For initial condition $f(v,0) = y(v)$, where $y(\cdot)$ is the function appearing in the Markovian solution $(y(V_t), z(V_t), \lambda)$ of the ergodic BSDE (15), (25) admits a smooth solution given by

$$f(v,t) = y(v) - \lambda t.$$

Proof. First, assume that the function $y(\cdot)$ appearing in Proposition 2 is in $C^2(\mathbb{R}^d)$. Itô's formula then gives

$$dy(V_t) = \mathcal{L}y(V_t)dt + \left(\kappa^T\nabla y(V_t)\right)^T dW_t,$$

which combined with (15) yields that $Z_t = z(V_t) = \kappa^T\nabla y(V_t)$ and

$$-\lambda + \mathcal{L}y(V_t) + F(V_t, \kappa^T\nabla y(V_t)) = 0.$$

It therefore remains to show that $y(\cdot) \in C^2(\mathbb{R}^d)$. Indeed, for any $\rho > 0$, consider the semilinear elliptic PDE

$$\rho y^\rho = \mathcal{L}y^\rho + F\left(v, \kappa^T\nabla y^\rho\right). \tag{27}$$

Classical PDE results yield that the above equation admits a unique bounded solution $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$. Using arguments similar to the ones in the appendix, we deduce that $|y^\rho(v)| \leq \frac{K}{\rho}$ and $|\nabla y^\rho(v)| \leq \frac{C_v}{C_\eta - C_v}$.

Therefore, for any reference point, say, $v^0 \in \mathbb{R}^d$, we have that $\rho y^\rho(v_0)$ is uniformly bounded and, moreover, that the difference $y^\rho(v) - y^\rho(v_0)$ is equicontinuous. Using a diagonal argument (cf. (74) in the appendix), we deduce that there exists a subsequence $\rho_n \downarrow 0$ such that $\rho_n y^{\rho_n}(v_0) \to \lambda$ and $y^{\rho_n}(v) - y^{\rho_n}(v_0) \to y(v)$, uniformly on compact sets of $\mathbb{R}^d$. Since, however, both $\rho_n y^{\rho_n}(v)$ and $\nabla y^{\rho_n}(v)$ are bounded uniformly in ρ_n, $\nabla^2 y^{\rho_n}(v)$ is also bounded on compact sets, as it follows from (27) above. In turn, this yields a Hölder estimate for $\nabla y^{\rho_n}(v)$, uniformly on compact sets. Standard arguments for elliptic equations then give that the limit $y(\cdot) \in C^2(\mathbb{R}^d)$ (see, for example, [13, Theorem 3.3]). ■

3.1.3. Example: Single stock and single stochastic factor. For the state equations (1) and (2), let $n = 1$ and $d = 2$. Then, the stock and the stochastic factor processes follow, respectively,

$$dS_t = b(V_t)S_t dt + \sigma(V_t)S_t dW_t^1,$$
$$dV_t^1 = \eta(V_t)dt + \kappa^1 dW_t^1 + \kappa^2 dW_t^2 \quad \text{and} \quad dV_t^2 = 0$$

with $\min(\kappa^1, \kappa^2) > 0$, $|\kappa^1|^2 + |\kappa^2|^2 = 1$ and $\sigma(\cdot)$ bounded by a positive constant.

Let $\Pi = \mathbb{R} \times \{0\}$ so that $\pi_t^2 \equiv 0$. Then, the wealth equation (6) reduces to $dX_t^\pi = X_t^\pi \pi_t^1 \left(\theta(V_t)dt + dW_t^1\right)$ with $\theta(V_t) = b(V_t)/\sigma(V_t)$. In turn, the driver of (15) takes the form

$$F(V_t, Z_t^1, Z_t^2) = \frac{1}{2}\frac{\delta}{1-\delta}|Z_t^1 + \theta(V_t)|^2 + \frac{1}{2}|Z_t^1|^2 + \frac{1}{2}|Z_t^2|^2.$$

By Theorem 3.2, the optimal portfolio weights are $\left(\pi_t^{*,1}, \pi_t^{*,2}\right) = \left(\frac{Z_t^1 + \theta^1(V_t)}{1-\delta}, 0\right)$.

Next, note that if C_θ and K_θ are, respectively, the Lipschitiz constant and the bound for the market price of risk $\theta(v)$ (cf. Assumption 1(ii)), then

$$\begin{aligned}|F(v,z^1,z^2)-F(\bar{v},z^1,z^2)| &\le \frac{\delta}{1-\delta}|z^1+\theta(v)||\theta(v)-\theta(\bar{v})|\\ &\le \frac{\delta}{1-\delta}\max\{1,K_\theta\}C_\theta(1+|z|)|v-\bar{v}|.\end{aligned}$$

Hence, we may take in inequality (57) the constant C_v to be defined as $C_v=\delta\max\{1,K_\theta\}C_\theta/(1-\delta)$.

To find the processes Z_t^1 and Z_t^2, we set $Z_t^i=\kappa^i Z_t$, $i=1,2$, for some process Z_t to be determined. Then, (15) further reduces to

$$\begin{aligned}dY_t=&\left(-\frac{\hat{\delta}}{2}|Z_t|^2-\frac{\delta\kappa^1}{1-\delta}\theta(V_t)Z_t-\frac{\delta}{2(1-\delta)}|\theta(V_t)|^2+\lambda\right)dt\\ &+Z_t\left(\kappa^1 dW_t^1+\kappa^2 dW_t^2\right)\end{aligned}$$

with $\hat{\delta}=\frac{1-\delta+\delta|\kappa^1|^2}{1-\delta}$.

Next, let $\tilde{Y}_t:=e^{\hat{\delta}(Y_t-\lambda t)}$ and $\tilde{Z}_t:=\hat{\delta}\tilde{Y}_tZ_t$. Then,

$$d\tilde{Y}_t=-\hat{\delta}\frac{\delta}{2(1-\delta)}|\theta(V_t)|^2\tilde{Y}_tdt+\tilde{Z}_td\tilde{W}_t,$$

where $\tilde{W}_t:=\kappa^1W_t^1+\kappa^2W_t^2-\int_0^t\frac{\delta\kappa^1}{1-\delta}\theta(V_u)du$, $t\ge 0$, is a Brownian motion under some probability measure equivalent to $\mathbb{P}$.

Let $\beta_t:=\exp\left(\int_0^t\hat{\delta}\frac{\delta}{2(1-\delta)}|\theta(V_u)|^2du\right)$. Applying Itô's formula to $\tilde{Y}_t\beta_t$ yields

$$\tilde{Y}_t=\frac{\beta_0}{\beta_t}\tilde{Y}_0+\int_0^t\frac{\beta_u}{\beta_t}\tilde{Z}_ud\tilde{W}_u.$$

The power forward performance process can be then written as

$$U(x,t)=\frac{x^\delta}{\delta}(\tilde{Y}_t)^{1/\hat{\delta}}=\frac{x^\delta}{\delta}\left(\frac{\beta_0}{\beta_t}\tilde{Y}_0+\int_0^t\frac{\beta_u}{\beta_t}\tilde{Z}_ud\tilde{W}_u\right)^{1/\hat{\delta}}.$$

The above result yields an alternative representation to the solution derived in [35], where the same market model is considered, bypassing various lengthy steps for the reduced linearized forward SPDE. Indeed, one can easily deduce that writing $\tilde{Y}_t=\tilde{y}\,(V_t,t)$ and using the dynamics of the stochastic factor (2) yields that $\tilde{y}\,(v,t)$ must satisfy

$$\tilde{y}_t\,(v,t)+\frac{1}{2}\tilde{y}_{vv}\,(v,t)+\left(\eta\,(v)+\frac{\delta\kappa^1}{1-\delta}\theta\,(v)\right)\tilde{y}_v\,(v,t)+\frac{\hat{\delta}\delta}{2\,(1-\delta)}\theta^2\,(v)\,\tilde{y}\,(v,t)=0,$$

recovering directly the result of [35].

3.2. Connection with infinite horizon BSDE. In this section, we build a connection between power forward processes and the solutions of a family of infinite horizon BSDE. The contribution is threefold.

First, these solutions are themselves power forward processes, albeit not in a factor-form. Second, we consider their limit as the parameter ρ, appearing naturally in the infinite horizon BSDE, converges to zero. We establish that appropriately discounted, they provide an approximation to the process $U(x,t)$ as $\rho \downarrow 0$. Third, we build a connection with a family of classical value function processes in finite horizon, say, $[0,T]$, when the horizon is long $(T \uparrow \infty)$.

We start with some background results on infinite horizon BSDE. Among others, we recall that [7] is one of the first papers in which Girsanov's transformation is used to solve infinite horizon BSDE with Lipschitz driver, while the quadratic driver case was solved in [6]. We refer the reader to [6] for further references.

Proposition 3.5. *Let $\rho > 0$, and consider the infinite horizon BSDE*

$$dY_t^\rho = (-F(V_t, Z_t^\rho) + \rho Y_t^\rho)\, dt + (Z_t^\rho)^T\, dW_t, \tag{28}$$

where the driver $F(\cdot,\cdot)$ is given in (15) with $\theta(\cdot)$, Π, and V satisfying the assumptionthes in section 1. Then, (28) admits a unique Markovian solution (Y_t^ρ, Z_t^ρ), $t \geq 0$.

Specifically, for each $\rho > 0$, there exist unique functions $y^\rho : \mathbb{R}^d \to \mathbb{R}$ and $z^\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t^\rho, Z_t^\rho) = (y^\rho(V_t), z^\rho(V_t))$ with $|y^\rho(\cdot)| \leq \frac{K}{\rho}$ and $|z^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η as in (3), and C_v, K given in (57) and (59), respectively.

The solvability of (28) is an intermediate step to solve (15) and is included in the proof of Proposition 3.1 in the appendix.

Theorem 3.6. *Let $(y^\rho(V_t), z^\rho(V_t))$, $t \geq 0$, be the unique Markovian solution to the infinite horizon BSDE (28). Then,*

(i) *the process $U^\rho(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, given by*

$$U^\rho(x,t) = \frac{x^\delta}{\delta} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds} \tag{29}$$

is a power forward performance process with volatility

$$a^\rho(x,t) = \frac{x^\delta}{\delta} e^{y^\rho(V_t) - \int_0^t \rho y^\rho(V_s) ds} z^\rho(V_t).$$

(ii) *The optimal portfolio weights $\pi_t^{*,\rho}$ and the associated wealth process $X_t^{*,\rho}$ (cf. (5), (6)), $t \geq 0$, are given, respectively, by*

$$\pi_t^{*,\rho} = Proj_\Pi \left(\frac{z^\rho(V_t) + \theta(V_t)}{1-\delta} \right) \quad \text{and} \quad X_t^{*,\rho} = X_0 \mathcal{E} \left(\int_0^\cdot (\pi_s^{*,\rho})^T (\theta(V_s) ds + dW_s) \right)_t.$$

The proof is similar to the one of Theorem 3.2, and it is thus omitted.

The next result relates the factor-from forward process $U(x,t)$ (cf. Theorem 3.2) and the path-dependent one $U^\rho(x,t)$ (cf. Theorem 3.6) and their corresponding optimal portfolio strategies.

We use the superscript v to denote dependence on the initial condition.

Proposition 3.7. *For $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, let $U(x,t)$ and $U^\rho(x,t)$ be the power forward processes given in (17) and (29), and $y(V_t)$ the component of the Markovian solution to the ergodic BSDE (15). Then, for an arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,*

$$\lim_{\rho_n \downarrow 0} \frac{U^{\rho_n}(x,t)e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1. \tag{30}$$

Moreover, for each $t \geq 0$, the associated optimal portfolio weights π^{*,ρ_n} and π^* satisfy

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t |\pi_s^{*,\rho_n} - \pi_s^*|^2 \, ds = 0. \tag{31}$$

Proof. For an arbitrary reference point $v_0 \in \mathbb{R}^d$, from the representations (17) and (29), we have that

$$\frac{U^\rho(x,t)e^{-y^\rho(v_0)}}{U(x,t)} = \exp\left((y^\rho(V_t^v) - \int_0^t \rho y^\rho(V_u^v)\,du) - (y(V_t^v) - \lambda t) - y^\rho(v_0) \right)$$

$$= \exp\left((y^\rho(V_t^v) - y^\rho(v_0) - y(V_t^v)) - \int_0^t \rho\,(y^\rho(V_u^v) - y^\rho(v_0))\,du - (\rho y^\rho(v_0) - \lambda)t \right).$$

On the other hand, the limits (74) and (75), established in the appendix, yield that there exists a subsequence $\rho_n \downarrow 0$ such that

$$\lim_{\rho_n \downarrow 0} (y^{\rho_n}(V_t^v) - y^{\rho_n}(v_0) - y(V_t^v)) = 0,$$

$$\lim_{\rho_n \downarrow 0} \rho_n (y^{\rho_n}(V_t^v) - y^{\rho_n}(v_0)) = 0 \quad \text{and} \quad \lim_{\rho_n \downarrow 0} (\rho_n y^{\rho_n}(v_0) - \lambda) = 0,$$

and we conclude.

To show assertion (31), we use the Lipschitz continuity of the projection operator on the convex set Π and the convergence

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t |z^{\rho_n}(V_s^v) - z(V_s^v)|^2 \, ds = 0 \tag{32}$$

for $t \geq 0$. The latter is established in the appendix. ■

3.3. Connection with the classical power expected utility for long horizons. We examine whether the forward processes $U(x,t)$ and $U^\rho(x,t)$ can be interpreted as long-term limits of the classical value function process. We show that this is indeed the case for a family of expected utility models with appropriately chosen terminal random (multiplicative) payoffs.

To this end, let $[0,T]$ be an arbitrary trading horizon and introduce, for $\rho > 0$, the value function process

$$u^\rho(x,t;T) = \operatorname*{ess\,sup}_{\pi \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left(\frac{(X_T^\pi e^{\xi_T})^\delta}{\delta} | \mathcal{F}_t, X_t^\pi = x \right) \tag{33}$$

for $(x,t) \in \mathbb{R}_+ \times [0,T]$ and the wealth process X_s^π, $s \in [t,T]$, solving (6).

The payoff ξ_T is defined as

$$\xi_T := -\frac{1}{\delta}\int_0^T \rho Y_t^{\rho,T} dt, \tag{34}$$

where $Y_t^{\rho,T}$ is the solution of the finite-horizon quadratic BSDE

$$Y_t^{\rho,T} = \int_t^T \left(F(V_s, Z_s^{\rho,T}) - \rho Y_s^{\rho,T}\right) ds - \int_t^T \left(Z_s^{\rho,T}\right)^T dW_s \tag{35}$$

with the driver $F(\cdot,\cdot)$ given in (16). The associated optimal portfolio weights are denoted by $\pi_s^{*,\rho,T}$, $s \in [t,T]$.

We recall that the classical optimal investment problem with power utility has been solved using quadratic BSDE methods in [19] for a Brownian motion setting and in [25] for a general semi-martingale framework.

Proposition 3.8.

(i) *Let $u^\rho(x,t;T)$ and $U^\rho(x,t)$ be given in* (33) *and* (29)*, respectively. Then, for each $\rho > 0$ and $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,*

$$\lim_{T\uparrow\infty} \frac{u^\rho(x,t;T)}{U^\rho(x,t)} = 1,$$

and the optimal portfolio weights satisfy, for $s \in [t,T)$,

$$\lim_{T\uparrow\infty} E_{\mathbb{P}} \int_t^s \left|\pi_u^{*,\rho,T} - \pi_u^{*,\rho}\right|^2 du = 0.$$

(ii) *Let $U(x,t)$ be the power forward process as in* (17)*. Then, for each arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,*

$$\lim_{\rho_n\downarrow 0}\lim_{T\uparrow\infty} \frac{u^{\rho_n}(x,t;T)e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1,$$

and the optimal portfolio weights satisfy, for $s \in [t,T)$,

$$\lim_{\rho_n\downarrow 0}\lim_{T\uparrow\infty} E_{\mathbb{P}} \int_t^s \left|\pi_u^{*,\rho_n,T} - \pi_u^{*}\right|^2 du = 0.$$

Proof. We only show part (i). From Theorem 3.3 in [6] we have that $|Y_t^{\rho,T}| \leq \frac{K}{\rho}$, and therefore, the quantity $\delta\xi_T = -\int_0^T \rho Y_u^{\rho,T} du$ is bounded. On the other hand, the driver $F(\cdot,\cdot)$ satisfies properties (58) and (59). Therefore, using similar arguments to the ones used in [19, section 3], it follows that the value function process is given by $u^\rho(x,t;T) = \frac{x^\delta}{\delta} e^{Y_t}$ with Y_t being the *unique* solution of the quadratic BSDE

$$Y_t = \delta\xi_T + \int_t^T F(V_s, Z_s)ds - \int_t^T (Z_s)^T dW_s \tag{36}$$

for $t \in [0,T]$. In addition, the optimal portfolio weights are given by $\pi_t^{*,\rho,T} = Proj_\Pi(\frac{Z_t+\theta(V_t)}{1-\delta})$.

Note, however, that the pair of processes $(Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds, Z_t^{\rho,T})$, $t \in [0,T]$, with $(Y_t^{\rho,T}, Z_t^{\rho,T})$ solving (35) also satisfies the above quadratic BSDE (36). Therefore, we must have $Y_t = Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds$, $t \in [0,T]$ and, as a consequence,

$$u^\rho(x,t;T) = \frac{x^\delta}{\delta} \exp\left(Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds \right).$$

In turn,

$$\frac{u^\rho(x,t;T)}{U^\rho(x,t)} = \exp\left((Y_t^{\rho,T} - \int_0^t \rho Y_s^{\rho,T} ds) - (Y_t^\rho - \int_0^t \rho Y_s^\rho ds) \right).$$

Using (65) we deduce that $\lim_{T\uparrow\infty} Y_t^{\rho,T} = Y_t^\rho$, and we easily conclude.

The convergence of the optimal portfolio weights follows from the Lipschitz continuity of the projection operator on the convex set Π and the convergence of $Z^{\rho,T}$ to Z^ρ in $\mathcal{L}_\rho^2[t,\infty)$. The space $\mathcal{L}_\rho^2[t,\infty)$ is defined in (66) and the latter limit is shown in (67) in the appendix. ∎

3.4. General (non-Markovian) power forward performance processes and ergodic BSDE. Departing from factor-form power forward performance processes, we may still use the ergodic BSDE approach we developed earlier to construct such processes of the general form

$$U(x,t) = \frac{x^\delta}{\delta} e^{K_t}$$

for some $\mathbb{F}$-progressively measurable process $K_t, t \geq 0$, independent of x.

Indeed, consider an arbitrary process $Z \in \mathcal{L}_{BMO}^2$, and, in turn, choose (Y_t, λ), $t \geq 0$, $\lambda \in \mathbb{R}$ and Y being $\mathbb{F}$-progressively measurable, such that the triplet (Y_t, Z_t, λ) solves the ergodic BSDE (15). Using similar arguments as the ones in the proof of Theorem 3.2, we may deduce that the process

$$U(x,t) = \frac{x^\delta}{\delta} e^{Y_t - \lambda t}, \tag{37}$$

$(x,t) \in \mathbb{R}_+ \times [0,\infty)$, satisfies Definition 2.1. Then, the SPDE (10) will yield that the forward volatility is given by the process $a(x,t) = U(x,t) Z_t, t \geq 0$. One can also develop similar connections with infinite horizon BSDE and the value function processes with terminal (multiplicative) payoff, as in sections 3.2 and 3.3.

The analysis of general power forward processes is beyond the scope of this paper and will be carried out separately. Herein, we only comment on three examples, cast in the absence of portfolio constraints, $\Pi = \mathbb{R}^d$.

(i) *Time-monotone case.* Let $Z_t \equiv 0$, $t \geq 0$, and choose (Y_t, λ) as

$$Y_t - \lambda t := Y_0 - \int_0^t \frac{1}{2} \frac{\delta}{1-\delta} |\theta(V_s)|^2 ds$$

for any constant $Y_0 \in \mathbb{R}$. Then, $(Y_t, 0, \lambda)$ satisfies (15). In turn, we deduce, using (37) and the above, that the process

$$U(x,t) := e^{Y_0} \frac{x^\delta}{\delta} e^{-\frac{1}{2}\frac{\delta}{1-\delta} A_t}$$

with $A_t = \int_0^t |\theta(V_s)|^2 ds$ is a power performance process. This process has zero volatility ($a(x,t) \equiv 0$), and it is decreasing in time and path-dependent (see [32] for a general study).

Variations of this solution with nonzero forward volatility can be constructed, as is shown below. We stress, however, that these forward processes essentially correspond to a fictitious market with *different risk premia*, and thus they do *not* constitute genuine new solutions for the original market.

(ii) *Market view case.* Let $Z_t = \phi_t$ with $\phi \in \mathcal{L}^2_{BMO}$, and choose (Y_t, λ), $t \geq 0$, as

$$Y_t - \lambda t := Y_0 - \frac{1}{2}\frac{\delta}{1-\delta}\int_0^t |\phi_s + \theta(V_s)|^2 ds - \frac{1}{2}\int_0^t |\phi_s|^2 ds + \int_0^t \phi_s^T dW_s.$$

We can then verify that (Y_t, ϕ_t, λ) satisfies (15). Using (37) and rearranging terms, we deduce the representation

$$\begin{aligned} U(x,t) &= \frac{x^\delta}{\delta} e^{Y_0 - \frac{1}{2}\frac{\delta}{1-\delta}\int_0^t |\phi_s + \theta(V_s)|^2 ds} \mathcal{E}\left(\int_0^\cdot \phi_s^T dW_s\right)_t \\ &= e^{Y_0}\frac{x^\delta}{\delta} e^{-\frac{1}{2}\frac{\delta}{1-\delta}A_t^\phi} M_t \end{aligned}$$

with $A_t^\phi = \int_0^t |\phi_s + \theta(V_s)|^2 ds$ and $M_t = \mathcal{E}(\int_0^\cdot \phi_s^T dW_s)_t$.

(iii) *Benchmark case.* A different parametrization yields an alternative representation and interpretation of the solution. Let $Z_t = \delta\phi_t$ with $\phi \in \mathcal{L}^2_{BMO}$, and choose (Y_t, λ), $t \geq 0$, as

$$Y_t = Y_0 + \lambda t - \int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\delta\phi_s + \theta(V_s)|^2 ds - \frac{1}{2}\int_0^t |\delta\phi_s|^2 ds + \int_0^t \delta\phi_s^T dW_s.$$

Then, $(Y_t, \delta\phi_t, \lambda)$ solves (15), and, in turn, (37) yields the power forward process

$$\begin{aligned} U(x,t) &= \frac{x^\delta}{\delta} e^{Y_0 - \int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\delta\phi_s + \theta(V_s)|^2 ds} \mathcal{E}\left(\int_0^\cdot \delta\phi_s^T dW_s\right)_t \\ &= e^{Y_0}\frac{x^\delta}{\delta} e^{-\int_0^t \frac{1}{2}\frac{\delta}{1-\delta}|\phi_s + \theta(V_s)|^2} \left(\mathcal{E}\left(\int_0^\cdot -\phi_s^T (dW_s + \theta(V_s) ds)\right)_t\right)^{-\delta} \\ &= e^{Y_0}\frac{1}{\delta}\left(\frac{x}{M_t}\right)^\delta e^{-\frac{1}{2}\frac{\delta}{1-\delta}A_t^\phi} \end{aligned}$$

with $A_t^\phi = \int_0^t |\phi_s + \theta(V_s)|^2 ds$ and $M_t = \mathcal{E}\left(\int_0^\cdot -\phi_s^T(\theta(V_s)ds + dW_s)\right)_t$. We may then view this process as measuring the performance of investment strategies in relation to a "benchmark," represented by the process M_t.

For more details about the above processes and further interpretations, as well as the specification of the associated myopic and nonmyopic portfolio components, and the corresponding wealth processes, we refer the reader to [33].

4. Exponential case. We examine forward performance processes in the exponential factor-form

$$U(x,t) = -e^{-\gamma x + f(V_t, t)}, \tag{38}$$

where f is a (deterministic) function to be specified.

For exponential forward performance processes, it is more convenient for the control policy to represent the discounted amount (and not the proportions of the discounted wealth) invested in the individual stock accounts. Hence, we set $\tilde{\alpha}_t = \tilde{\pi}_t X_t^{\pi}$. In turn, we rescale $\tilde{\alpha}_t$ by the stocks' volatility and deduce that the wealth process solves, for $t \geq 0$,

$$dX_t^{\alpha} = \alpha_t^T(\theta(V_t)dt + dW_t) \tag{39}$$

with $\alpha_t^T = \tilde{\alpha}_t^T \sigma(V_t)$. The set of admissible policies is $\mathcal{A}$, and we take the admissible wealth domain to be $\mathbb{D} = \mathbb{R}$.

As in the power case, (38) and (10) yield that f must satisfy, for $(v,t) \in \mathbb{R}_+ \times [0,\infty)$, a semilinear PDE, given by

$$f_t + \frac{1}{2} Trace\left(\kappa\kappa^T \nabla^2 f\right) + \eta(v)^T \nabla f + G(v, \kappa^T \nabla f) = 0 \tag{40}$$

with

$$G(v,z) = \frac{1}{2}\gamma^2 dist^2\left(\Pi, \frac{z+\theta(v)}{\gamma}\right) - \frac{1}{2}|z+\theta(v)|^2 + \frac{1}{2}|z|^2, \tag{41}$$

which is ill-posed with no known solutions to date. On the other hand, as in the former case, we will construct the process $f(V_t,t)$ itself directly from the Markovian solution of an ergodic BSDE whose driver is of the above form (cf. (43)).

4.1. Construction via ergodic BSDE. The results are similar to the ones derived in the previous section and are, thus, stated without proofs.

Proposition 4.1. *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1(ii) and let the set Π be as in (7). Then, the ergodic BSDE*

$$dY_t = (-G(V_t,Z_t) + \lambda)dt + Z_t^T dW_t \tag{42}$$

with the driver $G(\cdot,\cdot)$ is given by

$$G(V_t,Z_t) = \frac{1}{2}\gamma^2 dist^2\left(\Pi, \frac{Z_t+\theta(V_t)}{\gamma}\right) - \frac{1}{2}|Z_t+\theta(V_t)|^2 + \frac{1}{2}|Z_t|^2 \tag{43}$$

and admits a unique Markovian solution $(Y_t,Z_t,\lambda), t \geq 0$.

Specifically, there exist a unique $\lambda \in \mathbb{R}$ and functions $y : \mathbb{R}^d \to \mathbb{R}$ and $z : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t,Z_t) = (y(V_t), z(V_t))$. The function $y(\cdot)$ is unique up to a constant and has at most linear growth, and $z(\cdot)$ is bounded with $|z(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η and C_v are as in (3) and (57), respectively.

Theorem 4.2. *Let $(Y_t,Z_t,\lambda) = (y(V_t), z(V_t), \lambda)$, $t \geq 0$, be the unique Markovian solution of the ergodic BSDE (42). Then,*

(i) *the process $U(x,t)$, given, for $(x,t) \in \mathbb{R} \times [0,\infty)$, by*

$$U(x,t) = -e^{-\gamma x + y(V_t) - \lambda t} \tag{44}$$

is an exponential forward performance process with volatility

$$a\,(x,t) = -e^{-\gamma x + y(V_t) - \lambda t} z(V_t).$$

(ii) *The optimal portfolios α_t^* and the optimal wealth process X_t^* are given, respectively, by*

$$\alpha_t^* = Proj_{\Pi}\left(\frac{z(V_t)+\theta(V_t)}{\gamma}\right) \text{ and } X_t^* = X_0 + \int_0^t (\alpha_t^*)^T(\theta(V_t)dt + dW_t). \tag{45}$$

An axiomatic construction of exponential performance processes was developed in [40] for semi-martingale markets. These processes have been also used for the construction of forward indifference prices (see, among others, [26], [28], [30], and [17]) as well as for the axiomatic construction and characterization of the so-called maturity-independent entropy risk measures in [39].

As in the power case, we may prove the following result.

Proposition 4.3. *Consider the ill-posed semilinear PDE*

$$f_t + \mathcal{L}f + G(v, \kappa^T \nabla f) = 0, \tag{46}$$

$(v,t) \in \mathbb{R}^d \times [0,\infty)$, with $G(\cdot,\cdot)$ as in (41) (or (43)) and $\mathcal{L}$ as in (26). For initial condition $f(v,0) = y(v)$, where $y(\cdot)$ is the function appearing in the Markovian solution $(y(V_t), z(V_t), \lambda)$ of the ergodic BSDE (42), (46) admits a smooth solution given by

$$f(v,t) = y(v) - \lambda t.$$

4.2. Representation via infinite horizon BSDE. In analogy to the results of section 3.2, we derive an alternative representation of the exponential forward performance process using an infinite horizon BSDE. The proof follows along similar arguments and is, thus, omitted.

Proposition 4.4. *Assume that the market price of risk vector $\theta(v)$ satisfies Assumption 1*(ii) *and let the set Π be as in (7). Let $\rho > 0$. Then, the infinite horizon BSDE*

$$dY_t^\rho = \left(-G(V_t, Z_t^\rho) + \rho Y_t^\rho\right) dt + \left(Z_t^\rho\right)^T dW_t, \tag{47}$$

$t \geq 0$, with the driver $G(\cdot,\cdot)$ as in (42), admits a unique Markovian solution. Specifically, for each $\rho > 0$, there exist unique functions $y^\rho : \mathbb{R}^d \to \mathbb{R}$ and $z^\rho : \mathbb{R}^d \to \mathbb{R}^d$ such that $(Y_t^\rho, Z_t^\rho) = (y^\rho(V_t), z^\rho(V_t))$, with $|y^\rho(\cdot)| \leq \frac{K}{\rho}$ and $|z^\rho(\cdot)| \leq \frac{C_v}{C_\eta - C_v}$, where C_η as in (3), and C_v, K given in (57) and (59), respectively.

Theorem 4.5. *Let $(y^\rho(V_t), z^\rho(V_t))$, $t \geq 0$, be the unique Markovian solution to the infinite horizon BSDE (47). Then,*

(i) *the process $U^\rho(x,t)$, $(x,t) \in \mathbb{R} \times [0,\infty)$, given by*

$$U^\rho(x,t) = -e^{-\gamma x + y^\rho(V_t) - \int_0^t \rho y^\rho(V_u)du} \tag{48}$$

is an exponential forward performance process with volatility

$$a^\rho(x,t) = -e^{-\gamma x + y^\rho(V_t) - \int_0^t \rho y^\rho(V_u)du} z^\rho(V_t).$$

(ii) *The optimal portfolios* $\alpha_t^{*,\rho}$ *and optimal wealth process* $X_t^{*,\rho}$ *(cf.* (39)*),* $t \geq 0$*, are given, respectively, as in* (45) *with* $z(V_t)$ *replaced by* $z^{\rho}(V_t)$*.*

In line with Proposition 3.7, we have the following connection between the ergodic and infinite horizon representations for exponential forward performance processes.

Proposition 4.6. *Let* $U^{\rho}(x,t)$ *and* $U(x,t)$ *be the exponential forward performance processes* (44) *and* (48)*, respectively. Then, for any reference point* $v_0 \in \mathbb{R}^d$*, there exists a subsequence* $\rho_n \downarrow 0$ *(depending on* v_0*) such that, for* $(x,t) \in \mathbb{R} \times [0,\infty)$*,*

$$\lim_{\rho_n \downarrow 0} \frac{U^{\rho_n}(x,t) e^{-y^{\rho_n}(v_0)}}{U(x,t)} = 1.$$

Moreover, for $t \geq 0$*, the associated optimal portfolios satisfy*

$$\lim_{\rho_n \downarrow 0} E_{\mathbb{P}} \int_0^t |\alpha_u^{*,\rho_n} - \alpha_u^*|^2 \, du = 0.$$

4.3. Connection with the classical exponential expected utility for long horizons. As in section 3.3, we discuss the relationship between the exponential forward performance process $U(x,t)$ and its traditional finite horizon expected utility analogue with the latter incorporating a terminal random endowment.

To this end, let $\rho > 0$ and $[0,T]$ be an arbitrary trading horizon. Consider a family of maximal expected utility problems

$$u^{\rho}(x,t;T) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{[t,T]}} E_{\mathbb{P}} \left(-e^{-\gamma(X_T^{\alpha} + \xi_T)} | \mathcal{F}_t, X_t^{\alpha} = x \right) \tag{49}$$

for $(x,t) \in \mathbb{R} \times [0,T]$ and the wealth process X_s^{α}, $s \in [t,T]$, solving (39). The payoff ξ_T is defined as $\xi_T = \frac{1}{\gamma} \int_0^T \rho Y_t^{\rho,T} dt$, where $Y_t^{\rho,T}$ is the solution of the finite horizon quadratic BSDE

$$Y_t^{\rho,T} = \int_t^T \left(G(V_s, Z_s^{\rho,T}) - \rho Y_s^{\rho,T} \right) ds - \int_t^T \left(Z_s^{\rho,T} \right)^T dW_s$$

with the driver $G(\cdot,\cdot)$ given in (43). The optimal portfolios are denoted by $\alpha_s^{*,\rho,T}$ for $s \in [t,T]$. We have the following convergence result.

Proposition 4.7.

(i) *Let* $u^{\rho}(x,t;T)$ *and* $U^{\rho}(x,t)$ *be given in* (49) *and* (48)*, respectively. Then, for each* $\rho > 0$*, and* $(x,t) \in \mathbb{R} \times [0,\infty)$*,*

$$\lim_{T \uparrow \infty} \frac{u^{\rho}(x,t;T)}{U^{\rho}(x,t)} = 1,$$

and the optimal portfolios satisfy, for $s \in [t,T)$*,*

$$\lim_{T \uparrow \infty} E_{\mathbb{P}} \int_t^s \left| \alpha_u^{*,\rho,T} - \alpha_u^{*,\rho} \right|^2 du = 0.$$

(ii) *Let* $U(x,t)$ *be the exponential forward process as in* (44). *Then, for any reference point* $v_0 \in \mathbb{R}^d$, *there exists a subsequence* $\rho_n \downarrow 0$ *(depending on* v_0*) such that, for* $(x,t) \in \mathbb{R} \times [0,\infty)$,

$$\lim_{\rho_n \downarrow 0} \lim_{T \uparrow \infty} \frac{u^{\rho_n}(x,t;T) e^{-y^{\rho_n}}(v_0)}{U(x,t)} = 1,$$

and the optimal portfolios satisfy, for $s \in [t,T)$,

$$\lim_{\rho_n \downarrow 0} \lim_{T \uparrow \infty} E_{\mathbb{P}} \int_t^s \left| \alpha_u^{*,\rho_n,T} - \alpha_u^* \right|^2 du = 0.$$

5. Logarithmic case.

We conclude with logarithmic forward performance processes in factor-form, namely, of the form

$$U(x,t) = \ln x + f(V_t, t) \tag{50}$$

for a function f to be determined. The "additive" format is more appropriate for the logarithmic class, given the "myopic" character of the latter in the classical setting. Then, (50) and (10) yield that $f : (v,t) \in \mathbb{R}^d \times [0,\infty)$ must satisfy the ill-posed linear equation

$$f_t + \frac{1}{2} Trace\left(\kappa\kappa^T \nabla^2 f\right) + \eta(v)^T \nabla f + \tilde{F}(v) = 0 \tag{51}$$

with

$$\tilde{F}(v) = -\frac{1}{2} dist^2 \{\Pi, \theta(v)\} + \frac{1}{2} |\theta(v)|^2.$$

The results that follow are similar to the ones in section 3 and, because of this, they are stated in an abbreviated manner. To this end, the associated ergodic BSDE is given by

$$dY_t = (-\tilde{F}(V_t) + \lambda) dt + Z_t^T dW_t \tag{52}$$

with the driver

$$\tilde{F}(V_t) = -\frac{1}{2} dist^2 \{\Pi, \theta(V_t)\} + \frac{1}{2} |\theta(V_t)|^2,$$

as it is easily guessed by the form of the operator appearing in (51) above. Working as in the proof of Proposition 3.1 we deduce that (52) has a unique Markovian solution, say, $(Y_t, Z_t, \lambda) = (y(V_t), z(V_t), \lambda)$, for some functions $y(\cdot)$ and $z(\cdot)$ with similar properties to the ones therein.

We verify that the process

$$U(x,t) := \ln x + y(V_t) - \lambda t \tag{53}$$

is a logarithmic forward performance process in factor-form. The SPDE (10) then yields volatility $a(x,t) = z(V_t)$. Moreover, the optimal policy and the wealth it generates are given, respectively, by $\pi_t^* = Proj_\Pi \theta(V_t)$, and

$$X_t^* = X_0 \mathcal{E}\left(\int_0^\cdot (Proj_\Pi \theta(V_s))^T (\theta(V_s) ds + dW_s) \right)_t .$$

The constant λ has the interpretation

$$\lambda = \sup_{\pi\in\mathcal{A}} \limsup_{T\uparrow\infty} \frac{1}{T} E_{\mathbb{P}}\left(\ln X_T^{\pi}\right).$$

A byproduct of this result is that the ill-posed linear PDE (51) has a smooth solution for initial data $f(v,0) = y(v)$, given by $f(v,t) = y(v) - \lambda t$.

There is also a connection with infinite horizon BSDE. Indeed, we easily deduce that the infinite horizon BSDE

$$dY_t^{\rho} = \left(-\tilde{F}(V_t) + \rho Y_t^{\rho}\right) dt + \left(Z_t^{\rho}\right)^T dW_t \tag{54}$$

has a unique Markovian solution $(y^{\rho}(V_t), z^{\rho}(V))$, and, in turn, the process $U^{\rho}(x,t)$, $(x,t) \in \mathbb{R}_+ \times [0,\infty)$, defined as

$$U^{\rho}(x,t) := \ln x + y^{\rho}(V_t) - \int_0^t \rho y^{\rho}(V_s)\, ds, \tag{55}$$

is a path-dependent logarithmic forward performance process.

The process $U(x,t)$ and $U^{\rho}(x,t)$ in (53) and (55) are connected in a similar way as their power analogues in Proposition 3.7. Namely, for an arbitrary reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n\downarrow 0} \left(U^{\rho_n}(x,t) - y^{\rho_n}(v_0) - U(x,t)\right) = 0.$$

Finally, in order to connect $U(x,t)$ and $U^{\rho}(x,t)$ with their classical counterparts, we introduce the logarithmic expected utility problem

$$u^{\rho}(x,t;T) = \operatorname*{ess\,sup}_{\pi\in\mathcal{A}_{[t,T]}} E_{\mathbb{P}}\left(\ln X_T^{\pi} + \xi_T | \mathcal{F}_t, X_t^{\pi} = x\right), \tag{56}$$

where $\xi_T = -\int_0^T \rho Y_u^{\rho,T} du$ and $Y_t^{\rho,T}$, $t \in [0,T]$, is the unique solution of the BSDE on $[0,T]$,

$$Y_t^{\rho,T} = \int_t^T \left(\tilde{F}(V_u) - \rho Y_u^{\rho,T}\right) du - \int_t^T \left(Z_u^{\rho,T}\right)^T dW_u.$$

Using similar arguments to the ones in Proposition 3.8, we deduce that for any reference point $v_0 \in \mathbb{R}^d$, there exists a subsequence $\rho_n \downarrow 0$ (depending on v_0) such that, for $(x,t) \in \mathbb{R}_+ \times [0,\infty)$,

$$\lim_{\rho_n\downarrow 0} \lim_{T\uparrow\infty} \left(u^{\rho_n}(x,t;T) - y^{\rho_n}(v_0) - U(x,t)\right) = 0.$$

Appendix A. Solving ergodic and infinite horizon BSDE.

We present background results for Markovian solutions of the ergodic BSDE (15) and (42). We also obtain the existence and uniqueness of bounded Markovian solutions to the infinite horizon BSDE (28) and (47) as intermediate steps in the proofs of Propositions 3.1 and 4.1.

Equations (52) and (54) appearing in the logarithmic case are degenerate versions on (15) and (28), so they are not discussed.

We start with the key observation that, using Assumption 1(ii) on the market price of risk process as well as the definition of the admissible set $\mathcal{A}$ and the Lipschitz continuity of the distance function $dist\,(\Pi, \cdot)$, we deduce that the drivers $H = F$, G appearing in (16) and (43) satisfy

$$|H(v,z) - H(\bar{v},z)| \leq C_v(1+|z|)|v-\bar{v}|, \tag{57}$$

$$|H(v,z) - H(v,\bar{z})| \leq C_z(1+|z|+|\bar{z}|)|z-\bar{z}|, \tag{58}$$

and

$$|H(v,0)| \leq K \tag{59}$$

for any $v, \bar{v}, z, \bar{z} \in \mathbb{R}^d$, and C_v, C_z, $K > 0$ being positive constants.

The main ideas for establishing existence and uniqueness of solutions come from Theorem 3.3 in [6], Theorem 3.3 in [7], Theorem 4.4 in [16], and Theorem 2.3 in [22]. To this end, we first define the truncation function $q : \mathbb{R}^d \to \mathbb{R}^d$,

$$q(z) := \frac{\min\left(|z|, C_v/(C_\eta - C_v)\right)}{|z|} z \mathbf{1}_{\{z \neq 0\}}, \tag{60}$$

and consider the truncated ergodic BSDE,

$$dY_t = \left(-H(V_t, q(Z_t)) + \lambda\right) dt + Z_t^T dW_t, \tag{61}$$

$t \geq 0$, where q is as in (60), and the driver $H(\cdot,\cdot)$ satisfies conditions (57)-(59). We easily obtain the Lipschitz continuity conditions

$$|H(v,q(z)) - H(\bar{v},q(z))| \leq \frac{C_\eta C_v}{C_\eta - C_v}|v-\bar{v}| \tag{62}$$

and

$$|H(v,q(z)) - H(v,q(\bar{z}))| \leq C_z \frac{C_\eta + C_v}{C_\eta - C_v}|z-\bar{z}|. \tag{63}$$

If, therefore, we can show that the BSDE (61) admits a Markovian solution denoted, say, by (Y_t, Z_t, λ) with $|Z_t| \leq \frac{C_v}{C_\eta - C_v}$, $t \geq 0$, then $q(Z_t) = Z_t$, $t \geq 0$. In turn, this process (Y_t, Z_t, λ) would also solve the ergodic BSDE (15) in Proposition 3.1 and (42) in Proposition 4.1, respectively.

We first establish existence of Markovian solutions of (61). For this, we adapt the perturbation technique and the Girsanov's transformation used in [16, section 4] in an infinite-dimensional setting. To this end, let $n > 0$, and consider the discounted BSDE with a small discount factor, say, $\rho > 0$, on the finite horizon $[0, n]$,

$$Y_t^{\rho,v,n} = \int_t^n \left(H(V_s^v, q(Z_s^{\rho,v,n})) - \rho Y_s^{\rho,v,n}\right) ds - \int_t^n (Z_s^{\rho,v,n})^T\, dW_s, \tag{64}$$

where we use the superscript v to emphasize the initial dependence of the stochastic factor process on its initial data $V_0^v = v$.

From [7, section 3.1], we deduce that BSDE (64) admits a unique solution $(Y_t^{\rho,v,n}, Z_t^{\rho,v,n}) \in \mathcal{L}^2_{[0,n]}$ with $|Y_t^{\rho,v,n}| \le \frac{K}{\rho}$, $0 \le t \le n$, where

$$\mathcal{L}^2\,[0,n] = \left\{ (Y_t)_{t\in[0,n]} : Y \text{ is } \mathbb{F}\text{-progressively measurable and } E_{\mathbb{P}}\left(\int_0^n |Y_t|^2 dt\right) < \infty \right\}.$$

On the other hand, parameterizing (64) by the auxiliary horizon n, we obtain (cf. [7, section 3.1]) that there exists a process $Y_t^{\rho,v}$, $t \ge 0$, such that

$$\lim_{n\uparrow\infty} Y_t^{\rho,v,n} = Y_t^{\rho,v} \tag{65}$$

for a.e. $(t,\omega) \in [0,\infty) \times \Omega$, and moreover that for each $\rho > 0$, both $\{Y_t^{\rho,v,n}\}$ and $\{Z_t^{\rho,v,n}\}$ are Cauchy sequences in $\mathcal{L}^2_\rho\,[0,\infty]$, where

$$\begin{aligned} \mathcal{L}^2_\rho[0,\infty) = \big\{ (Y_t)_{t\in[0,\infty)} &: Y \text{ is } \mathbb{F}\text{-progressively measurable} \\ &\text{and } \ E_{\mathbb{P}}\left(\int_0^\infty e^{-2\rho t}|Y_t|^2 dt\right) < \infty \right\}. \end{aligned} \tag{66}$$

Therefore, there exist limiting processes $(Y_t^{\rho,v}, Z_t^{\rho,v})$, $t \ge 0$, belonging to $\mathcal{L}^2_\rho[0,\infty)$, such that

$$\lim_{n\uparrow\infty} (Y_t^{\rho,v,n}, Z_t^{\rho,v,n}) = (Y_t^{\rho,v}, Z_t^{\rho,v}) \tag{67}$$

in $\mathcal{L}^2_\rho[0,\infty)$ with $|Y_t^{\rho,v}| \le \frac{K}{\rho}$. It is, then, easy to show that the process $(Y_t^{\rho,v}, Z_t^{\rho,v})$, $t \ge 0$, is a solution to the infinite horizon BSDE

$$dY_t^{\rho,v} = (-H(V_t^v, q(Z_t^{\rho,v})) + \rho Y_t^{\rho,v})\, dt + (Z_t^{\rho,v})^T\, dW_t. \tag{68}$$

Moreover, we recall that the solution is Markovian in the sense that there exist functions, say, $y^\rho(\cdot)$ and $z^\rho(\cdot)$, such that

$$(Y_t^{\rho,v}, Z_t^{\rho,v}) = (y^\rho(V_t^v), z^\rho(V_t^v))\,.$$

Next, using the Girsanov's transformation and adapting the argument in [16, Lemma 4.3], we claim that the Lipschitz continuity property

$$|y^\rho(V_t^v) - y^\rho(V_t^{\bar v})| \le \frac{C_v}{C_\eta - C_v}|V_t^v - V_t^{\bar v}| \tag{69}$$

holds, for any $v, \bar v \in \mathbb{R}^d$, with the constants C_v and C_η as in (57) and (3), respectively.

Indeed, define, for $t \ge 0$,

$$\Delta Y_t := Y_t^{\rho,v} - Y_t^{\rho,\bar v} \quad \text{ and } \quad \Delta Z_t := Z_t^{\rho,v} - Z_t^{\rho,\bar v}.$$

Then,

$$\begin{aligned} d\left(\Delta Y_t\right) &= -\left(H(V_t^v, q(Z_t^{\rho,v})) - H(V_t^{\bar v}, q(Z_t^{\rho,\bar v}))\right)dt + \rho\Delta Y_t dt + \left(\Delta Z_t\right)^T dW_t \\ &= -\Delta H_t dt + \rho \Delta Y_t dt + \left(\Delta Z_t\right)^T\left(dW_t - m_t dt\right), \end{aligned}$$

where $\Delta H_t := H(V_t^v, q(Z_t^{\rho,v})) - H(V_t^{\bar v}, q(Z_t^{\rho,v}))$ and

$$m_t := \frac{H(V_t^{\bar v}, q(Z_t^{\rho,v})) - H(V_t^{\bar v}, q(Z_t^{\rho,\bar v}))}{|\Delta Z_t|^2}\Delta Z_t \mathbf{1}_{\{\Delta Z_t \neq 0\}}.$$

The process m_t is bounded, as it follows from (63). Therefore, we can define the process $\bar W_t := W_t - \int_0^t m_u du$, $t \geq 0$, which is a Brownian motion under some measure $\mathbb{Q}^m$ equivalent to $\mathbb{P}$. Hence, for $0 \leq t \leq s < \infty$, taking conditional expectation on $\mathcal{F}_t$ under $\mathbb{Q}^m$ yields

$$\Delta Y_t = \frac{\beta_s}{\beta_t} E_{\mathbb{Q}^m}\left(\Delta Y_s | \mathcal{F}_t\right) + E_{\mathbb{Q}^m}\left(\int_t^s \frac{\beta_u}{\beta_t}\left(\Delta H_u\right) du \middle| \mathcal{F}_t\right),$$

where $\beta_t = e^{-\rho t}$. Note, however, that the first expectation above is bounded by $2K/\rho$, and thus, it converges to zero as $s \uparrow \infty$. Moreover, by (62), the second expectation is bounded by

$$\begin{aligned} E_{\mathbb{Q}^m}\left(\int_t^s \frac{\beta_u}{\beta_t}\left(\Delta H_u\right) du \middle| \mathcal{F}_t\right) &\leq \frac{C_\eta C_v}{C_\eta - C_v} E_{\mathbb{Q}^m}\left(\int_t^s e^{-\rho(u-t)}|V_u^v - V_u^{\bar v}| du \middle| \mathcal{F}_t\right) \\ &\leq \frac{C_\eta C_v}{C_\eta - C_v}\frac{e^{\rho t}\left(e^{-(\rho + C_\eta)t} - e^{-(\rho+C_\eta)s}\right)}{\rho + C_\eta}|v - \bar v|, \end{aligned}$$

where we used the exponential ergodicity condition (4). Then, inequality (69) follows by letting $s \uparrow \infty$.

Next, assume that $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$. Applying Itô's formula to $y^\rho(V_t^v)$ yields

$$dy^\rho(V_t^v) = \mathcal{L}y^\rho(V_t^v)dt + \left(\kappa^T \nabla y^\rho(V_t^v)\right)^T dW_t, \tag{70}$$

where $\mathcal{L}$ is as in (26). In turn, from (68) and (70) we deduce that

$$\kappa^T \nabla y^\rho(V_t^v) = Z_t^{\rho,v} \tag{71}$$

and (with a slight abuse of notation) that

$$\rho y^\rho(v) = \mathcal{L}y^\rho(v) + H\left(v, q\left(\kappa^T \nabla y^\rho(v)\right)\right) \tag{72}$$

for $v \in \mathbb{R}^d$. Equation (72) is a standard semilinear elliptic PDE, and classical PDE results yield that it admits a unique bounded solution $y^\rho(\cdot) \in C^2(\mathbb{R}^d)$ with $|y^\rho(v)| \leq \frac{K}{\rho}$. In addition, recall that (69) yields $|\nabla y^\rho(v)| \leq \frac{C_v}{C_\eta - C_v}$, and thus, using (71) and Assumption 2 on the matrix κ, we obtain that, for $t \geq 0$,

$$|Z_t^{\rho,v}| \leq \frac{C_v}{C_\eta - C_v}. \tag{73}$$

Next, we fix a reference point, say, $v_0 \in \mathbb{R}^d$. Define the process $\bar{Y}_t^{\rho,v} := Y_t^{\rho,v} - Y_0^{\rho,v_0}$, and consider the perturbed version of the infinite horizon BSDE (68), namely,

$$\bar{Y}_t^{\rho,v} = \bar{Y}_s^{\rho,v} + \int_t^s \left(H(V_u^v, q(Z_u^{\rho,v})) - \rho\bar{Y}_u^{\rho,v} - \rho Y_0^{\rho,v_0}\right) du - \int_t^s (Z_u^{\rho,v})^T \, dW_u$$

for $0 \le t \le s < \infty$. Then $\bar{Y}_t^{\rho,v} = \bar{y}^\rho(V_t^v)$ with $\bar{y}^\rho(\cdot) = y^\rho(\cdot) - y^\rho(v_0)$.

Since, on the other hand, $y^\rho(\cdot)$ is Lipschitz continuous, uniformly in ρ, we deduce that $|\bar{y}^\rho(v)| \le \frac{C_v}{C_\eta - C_v} |v - v_0|$. Moreover, $|\rho y^\rho(v)| \le K$. Hence, there exists a sequence $\rho_{0n} \downarrow 0$ such that

$$\lim_{\rho_{0n}\downarrow 0} \rho_{0n} y^{\rho_{0n}}(v_0) = \lambda$$

for some constant λ.

Next, we take a dense subset, say, $S = \{v_1, \cdots, v_n, \cdots\} \in \mathbb{R}^d$. Since $\bar{y}^{\rho_{0n}}(v_1)$ is bounded, there exists a subsequence of $\{\rho_{0n}\}$, denoted as $\{\rho_{1n}\}$, such that

$$\lim_{\rho_{1n}\downarrow 0} \bar{y}^{\rho_{1n}}(v_1) = y(v_1)$$

for some $y(v_1)$. Proceeding this way, we obtain a sequence $\{\rho_{0n}\} \supset \{\rho_{1n}\} \supset \cdots$. Taking its diagonal sequence $\{\rho_{nn}\}$, denoted as $\{\rho_n\}$, we deduce that, for $v \in S$,

$$\lim_{\rho_n\downarrow 0} \rho_n y^{\rho_n}(v_0) = \lambda \quad \text{and} \quad \lim_{\rho_n\downarrow 0} \bar{y}^{\rho_n}(v) = y(v)\,. \tag{74}$$

Moreover, since the function $\bar{y}^\rho(\cdot)$ is Lipschitz continuous uniformly in ρ, the limit $y(\cdot)$ can be extended to a Lipschitz continuous function defined for all $v \in \mathbb{R}^d$,

$$\lim_{\rho_n\downarrow 0} \bar{y}^{\rho_n}(v) = y(v)\,. \tag{75}$$

Thus, we have $\lim_{\rho_n\downarrow 0} \bar{Y}_t^{\rho_n,v} = y(V_t^v)$ and $\lim_{\rho_n\downarrow 0}\left(\rho_n \bar{Y}_t^{\rho_n}\right) = 0$.

Next, define the process $Y_t^v = y(V_t^v)$, $t \ge 0$. It is then standard to show that there exists $Z_u^v = z(V_u^v)$, $u \in [t,s]$, in $\mathcal{L}^2[t,s]$ such that $\lim_{\rho_n\downarrow 0} Z^{\rho_n,v} = Z^v$ in $\mathcal{L}^2[t,s]$, and moreover, that the triplet $(Y_t^v, Z_t^v, \lambda) = (y(V_t^v), z(V_t^v), \lambda)$ is a solution to the truncated ergodic BSDE (61).

Finally, using the latter limit and the fact that $|Z_t^{\rho,v}| \le C_v/(C_\eta - C_v)$, as it follows from (73), we obtain that $|Z_t^v| \le C_v/(C_\eta - C_v)$. Therefore, $q(Z_t^v) = Z_t^v$, $t \ge 0$, and in turn, the triplet (Y^v, Z^v, λ) is also a solution to the ergodic BSDEs (15) and (42) in Propositions 3.1 and 4.1, respectively.

From the above arguments, it follows, as a byproduct, the existence of Markovian solutions to the infinite horizon BSDEs (28) and (47), respectively.

It remains to show the uniqueness of Markovian solutions to the ergodic BSDE (15) and (42). Indeed, since Z_t, $t \ge 0$, is bounded by $C_v/(C_\eta - C_v)$ for both (15) and (42), the uniqueness can be proved along similar arguments used in [16, Theorem 4.6] and [10, Theorem 3.11].

The uniqueness of the Markovian solutions to the infinite horizon BSDE (28) and (47) follows easily from [7, section 3.1] and [6, Theorem 3.3].

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