A Thesis Submitted for the Degree of PhD at the University of Warwick

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Surfaces with $pg = 3$ and $K^2 = 4$
and
extension-deformation theory

by

Duncan Dicks.

University of Warwick 1983

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Duncan Dicks.

Thesis submitted for the degree of
Doctor of Philosophy
at the University of Warwick

Sept. 1988
Acknowledgements

Many thanks to my supervisor Dr. Miles Reid for all his time and ideas. Also to my family and friends and everyone at the Mathematics Institute of the University of Warwick, especially Catharine. This work was funded by Science and Engineering Research Council grant (studentship no. 83301467).

D.D.

-Щи да каша, пища наша.-
-Old Russian proverb.
Summary

In Chapter 2 we give a description of an algorithm suggested by Reid for studying extensions of a variety $C \subset \mathbb{P}^n$ as a hyperplane section of a variety in $\mathbb{P}^{n+1}$.

In Chapter 3 we use this method to study surfaces with numerical invariants $p_g = 3$ and $K^2 = 4$. We find that there are 5 families of such surfaces and produce explicitly the canonical ring for a generic member of each family. In [Ho 1] there is a geometric study of surfaces with these invariants.

Proposition (15.2) is an example of an obstruction to the extension deformation algorithm which appears in degree 4.

In Chapter 4 we write down some one parameter deformations between the families. We conjecture that there are no degenerations, $\Pi \rightarrow \Pi_a$ or $\Pi \rightarrow \Pi_b$. We draw some geometric conclusions, from the algebraic descriptions, about the branch locus of surfaces of type $\Pi$, $\Pi_a$ and $\Pi_b$ as double covers of $\mathbb{P}^2$ [Ho 1]. It is also shown that a surface of type $\Pi$ is the resolution of a numerical quintic with an elliptic Gorenstein singularity of type $k = 1$. 

Abstract

The aims of this thesis are 2-fold.

We consider the extension-deformation method of Reid in some detail. In section 6 we explain its equivalence, as a practical tool of calculation, to the cotangent complex as discussed in [LS]. We reproduce the calculation of Pinkham [P] and the result of Griffin [G] as expository examples of this method.

Secondly, we use this method to analyse surfaces with $p_g = 3$ and $K^2 = 4$. We produce explicitly their canonical rings and use this information to draw some conclusions about the geometry of the surfaces and their moduli space.

Extension-deformation Theory

Since the cotangent complex was first introduced many people have made use of it to explore the deformation theory of singularities (e.g. [P], [Sc] and others). In [LS] Lichtenbaum and Schlessinger explain the definitions of the spaces $T^1$ and their role in producing infinitesimal extensions and deformations of rings. More recently Reid suggested a method to study extensions of a variety considered as a hyperplane section of another variety. This extension-deformation method is equivalent to the calculation of $T^1$. Griffin [G] used this method to solve the problem of giving an explicit family of numerical quintics, $\{X_t\}$, such that

1) $X_t$ is a smooth quintic in $\mathbb{P}^3$, for $t \neq 0$,

2) $X_0$ is a smooth numerical quintic whose canonical system has a single base point.

In Chapters 3 and 4 we use the method to classify surfaces with $p_g = 3$ and $K^2 = 4$. In this case, as with the quintics it is possible to find a format of the relations in each ring which survives the extension-deformation process. This should lead to a good understanding of the space of extensions and the moduli space of deformations in each case.
Introduction

In Proposition (15.2) we give an example of an obstruction occurring very late in the process. There are two formats for the ring in this case which seem only to differ when there are nilpotents in the extension. This means that one of the formats may not be present in the intermediate stages of the calculation but conditions are imposed which force it to return for the final result.

Surfaces of General Type

Much work has been done on the problem of describing surfaces with small \( K^2 \) (see for example [R3], [C1], [C2], [Ci], [Ho1], [Ho3]). In particular in [Ho1] there is a geometric description of 3 classes of surfaces with \( p_g = 3 \) and \( K^2 = 4 \). Horikawa’s approach here is predominantly geometric in nature. There are two main differences between the results here and those in [Ho1]. Due to a mistake on Horikawa’s part, surfaces of our type II were omitted from his classification. This mistake was corrected only in a footnote to [Ho1]. Consequently the notation differs and our surfaces of type III correspond to the Horikawa surfaces of type II. Secondly we consider separately surfaces of type IIIa and IIIb which are indistinguishable using Horikawa’s description of their branch loci [Ho1, surfaces of type II.1].

Catanese ([C1], [C2]) has developed an algebraic approach to the study of surfaces with small numerical invariants. The geometric interpretation of this method is related to the study of generic projections to \( \mathbb{P}^3 \). Using this method Catanese and Debarre [CD] have studied regular surfaces with \( p_g = 1, K^2 = 2 \). The extension-deformation theory of Reid has now been applied in several different cases, and gives some hope of proving the conjecture that the moduli space is irreducible in the torsion zero case of [CD].
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2. Notation

A curve is a projective variety of dimension 1 over \( \mathbb{C} \).

A surface is a projective variety of dimension 2 over \( \mathbb{C} \).

Given a nonsingular surface \( S \), divisors \( D, D_1, D_2 \) on \( S \) and the corresponding invertible sheaf \( \mathcal{O}_S(D) \), we write:

- \( D_1D_2 = \) intersection number on \( S \).
- \( h^i(D) = h^i(\mathcal{O}_S(D)) = \dim \mathbb{C} H^i(S, \mathcal{O}_S(D)) \) and
- \( \chi(D) = \chi(\mathcal{O}_S(D)) = h^0(D) - h^1(D) + h^2(D) \).
- \( |D| = \) the linear system in which \( D \) moves,
- \( r = \dim |D| = h^0(D) - 1 \) and

for \( h^0(D) \geq 1 \), we have \( \varphi_{|D|} : S \to \mathbb{P}^r \), the corresponding rational map.

A linear system of degree \( d \) and dimension \( r \) is denoted by the symbol \( g^r_d \).

- \( K_S = \) canonical divisor of \( S \).
- \( \mathcal{O}_S(K_S) = \wedge^2 \Omega^1_S \).
- \( p_g = p_g(S) = h^0(K_S) = \) geometric genus of \( S \).
- \( q(S) = h^1(K_S) = \) irregularity of \( S \).

- \( R(S,D) = R(S,\mathcal{O}_S(D)) = \bigoplus_{n=0}^{\infty} H^0(S, \mathcal{O}_S(nD)) \).
- \( R(S,K_S) = \) canonical ring of \( S \).
Suppose we have a set of elements \( \{x_i \in H^0(rK_x) \mid i = 1, \ldots, k \} \). Then we use the notation \( S^n(x_i) \) to mean the set of elements \( \{x_i^{r_1} \cdots x_i^{r_j} \in H^0(nrK_x) \mid \sum_{i=1}^j r_i = n, i_s \in \{1, \ldots, k\} \} \).

We are interested mostly in surfaces of general type i.e. surfaces \( S \) such that \( \dim \text{Proj } R(S,K_S) = 2 \).

For the formal definitions of the theory of the cotangent complex see [LS]. Similarly for the theory of weighted projective spaces see [D].

3. Facts and formulas

Let \( S \) be a nonsingular surface. We recall the following well known results, for future reference.

Riemann–Roch Formulas (3.1) (see [B])

a) \( \chi(D) = 1 - g + \deg(D) \)

for a divisor \( D \) on a nonsingular curve of genus \( g \).

b) \( \chi(D) = \chi(O_S) + \frac{1}{2}(D^2 - DK_S) \)

for any divisor \( D \) on \( S \).

Adjunction Formula (3.2) (see [B])

\[ C^2 + CK_S = 2g - 2 \]

for an irreducible curve of genus \( g \) on \( S \). Notice that

\[ C^2 = CK_S \pmod{2} \]

as a useful corollary.

Base Point Free Pencil Trick (3.3) (see [ACGH, p.126])

Let \( C \) be a smooth curve, \( L \) an invertible sheaf on \( C \), and \( F \) a torsion free \( \mathcal{O}_C \)-module. Suppose that \( s_1 \) and \( s_2 \) are linearly independent sections of
L and denote by $V$ the subspace of $H^0(C, L)$ they generate. Then the kernel of the cup-product map

$$V \otimes H^0(C, L) \to H^0(C, F \otimes L)$$

is isomorphic to $H^0(C, F \otimes L^\ast(B))$, where $B$ is the base locus of the pencil spanned by $s_1$ and $s_2$.

M. Noether's Theorem (3.4) (see [ACGH, p.117])

If $C$ is a non-hyperelliptic curve, then the homomorphisms

$$\text{Sym}^d H^0(C, K) \to H^0(C, dK)$$

are surjective for $d \geq 1$.

The Index Theorem (3.5) (see e.g. [BPVdeV (2.15)])

Let $D$ and $E$ be divisors with rational coefficients on the algebraic surface $X$. If $D^2 > 0$ and $DE = 0$ then $E^2 \leq 0$ and $E^2 = 0$ if and only if $E$ is rationally equivalent to 0.

Bertini's Theorem (3.6) (see e.g. [Ha] III.10.9.2)

Let $X$ be a nonsingular variety over an algebraically closed field of characteristic 0. Let $\delta$ be a linear system on $X$ with no fixed part. Then the general member of $\delta$ can only have singularities at the base points.

The Snake Lemma (3.7) (see e.g. [Co2] p.129)

Let $R$ be a ring and consider the following commutative diagram of exact sequences of $R$-modules:

$$
\begin{array}{cccccc}
A & \rightarrow & A' & \rightarrow & A'' & \rightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \rightarrow & B & \rightarrow & B' & \rightarrow & B'' \\
\end{array}
$$

Then there is a map, $\Delta : \ker \gamma \rightarrow \coker \alpha$, such that

$$
\ker \alpha \rightarrow \ker \beta \rightarrow \ker \gamma \rightarrow \coker \alpha \rightarrow \coker \beta \rightarrow \coker \gamma
$$

is exact.
4. Structure theorems for Gorenstein rings

Definition (4.1)

Let \( R \) be a Noetherian ring and \( M \neq 0 \) a finite \( R \)-module. Then we define

\[
\text{grade } M = \inf \{ i | \text{Ext}^i_R(M, A) \neq 0 \}.
\]

This definition extends the geometric concept of codimension for an ideal.

Definitions (4.2) (see [Ma,p.142])

An \( n \)-dimensional Noetherian local ring \( R \) with residue field \( k \) is Gorenstein if \( \text{Ext}^i_R(k, R) = 0 \) for \( i \neq n \) and \( = k \) for \( i = n \).

A Noetherian ring is Gorenstein if its localisation at every maximal ideal is Gorenstein.

For a Gorenstein ring, \( R \) and perfect ideal \( I \subset R \) we have the equality

\[
\text{grade } I = \text{proj dim } (R/I).
\]

Let \( R/I \) be the canonical ring of a regular surface of general type. Then \( R/I \) is Gorenstein [GW] and the codimension of the canonical model of the surface is equal to the grade of \( I \) [BE]. In this section we state the structure theorems for ideals of Gorenstein rings of grade 2 and 3.

Proposition (4.1) (see [S])

Let \( I \) be a grade 2 ideal in \( R \) such that \( R/I \) is Gorenstein. Then \( R/I \) is a complete intersection.

Before stating the theorem for grade 3 ideals we shall say a few words about Pfaffians. It is well-known (e.g. [Co1] p.220) that for every \((2n) \times (2n)\) antisymmetric matrix, \( M \), there can be found \( \text{Pf}(M) \), a polynomial in the entries of \( M \), such that \( \text{Pf}(M)^2 = \det M \). We shall usually be interested in the case \( n = 2 \) and in this case we have by convention:
Preliminaries

5

\[
\begin{bmatrix}
0 & A & B & C \\
-A & 0 & D & E \\
-B & -D & 0 & F \\
-C & -E & -F & 0 \\
\end{bmatrix}
= -AF + BE - CD.
\]

Proposition (4.2) [BE Theorem 4.2]

Let \( I \) be a grade 3 ideal in \( R \), such that \( R/I \) is Gorenstein. Then \( I \) is generated by the diagonal Pfaffians of an antisymmetric \((2n+1) \times (2n+1)\) matrix, \( M \). That is the Pfaffians of the antisymmetric \(2n \times 2n\) matrices formed by removing the \( i \)th row and \( i \)th column from \( M \), \( i = 1, \ldots, 2n+1 \).

\[
\square
\]

5. Canonical Linear Systems

We collect together the results of several theorems concerning surfaces of general type with given values of \( p_g \) and \( K_S^2 \) into two lemmas relevant to the particular surfaces which we study.

Lemma (5.1)

Let \( S \) be a surface of general type with \( p_g = 3 \) and \( K_S^2 = 4 \). Let \( C \in |K_S| \) be a general element. Then either

1) \( |K_S| \) has no fixed part, \( g(C) = 5 \) and deg. base locus \( \leq 2 \)

or

2) \( |K_S| = \Gamma + E \) where \( \Gamma \) has no fixed part and \( E \) is a fixed \((-2)\)-cycle.

Proof

We know that \( |K_S| \) is not composed of a pencil [Ho 1, XG]. Then let \( K_S = \Gamma + E \), where \( \Gamma \) is free and \( E \) is fixed. By [Ho 2, Lemma 2]

\[ \Gamma^2 \geq 2 \]

and by 2-connectedness of \( K_S \) either \( E = 0 \) or
\[ \Gamma E \geq 2. \]

Now \( 4 = K_S^2 \geq \Gamma^2 + \Gamma E \geq 2. \) Therefore either \( \Gamma^2 = \Gamma E = 2 \) and hence \( E^2 = -2 \) or, alternatively, \( \Gamma = K_S. \) We consider these 2 cases separately:

1) \( K_S = \Gamma. \) By the Adjunction Formula (3.2) \( g(C) = 5. \) Now \( \deg K_S|_{\Gamma} = 4 \) and \( \dim K_S|_{\Gamma}^1 = 1. \) If the degree of the base locus of \( K_S \geq 3 \) there would be a complete linear system of degree \( \leq 1 \) and dimension \( = 1. \) This contradicts genus \( C = 5, \) and so deg base locus \( \leq 2 \) as required.

2) \( K_S = \Gamma + E. \) We must show that \( |\Pi| \) is base point free. Considering \( K_S|_{\Pi} \) we see as before that the degree of the base locus \( \leq 2 \) and since \( \deg E|_{\Pi} = 2, \) \( |\Pi|^1, \) and hence \( |\Pi|, \) is base point free.

\[ \square \]

**Lemma (5.2)**

Let \( S \) be as above. Then \( S \) is in one of the following mutually exclusive classes:

I. \( |K_S| \) is free.

II. \( |K_S| \) has one base point.

III. \( |K_S| \) has two distinct base points.

III'. \( |K_S| = |\Pi| + E \) where \( |\Pi| \) is free and \( E \) is a fixed \((-2)\)-cycle.

\( E|_{\Pi} = P_1 + P_2 \) for distinct points \( P_1 \) and \( P_2. \)

III'\( _b \). \( K_S = \Gamma + E \) where \( \Gamma \) is free and \( E \) is a fixed \((-2)\)-cycle.

\( E|_{\Gamma} = 2P. \)

**Proof**

From (5.1) we have 2 cases:

1) \( |K_S| \) has no fixed part and \( \deg base locus \leq 2. \)
   
   I. \( \deg base locus = 0 \) and \( |K_S| \) is free.
   
   II. \( \deg base locus = 1 \) and \( |K_S| \) has one base point.
III. deg. base locus = 2. Suppose $|K_S|$ does not have distinct base points. Then $|K_{SI_C}| = D + 2P$ where $D$ is a $g_2^1$ and $2P$ is fixed. By the Adjunction Formula

$$2K_{SI_C} = K_C$$

and so (e.g. [ACGH], Appendix B) $P$ is a Weierstrass point and $|2P| = g_2^1$. This contradicts $2P$ being fixed.

2) $K_S = \Gamma + E$ where $E^2 = -2$. As pointed out in (5.1) $|K_{SI}|^{-1}$ has base locus of degree 2. This gives cases III$_a$ and III$_b$. 

\[\square\]


CHAPTER 2:
Extension-Deformation Theory

6. Extension - Deformation Theory

Let $X$ be a canonically embedded surface of general type, and $C \in |K_X|$ a hyperplane section given by the vanishing of $x_0 \in H^0(X, K_X)$. If we can write down the ring

$$R(C, K_{XC}) = \bigoplus_{n=0}^{\infty} H^0(C, nK_{XC})$$

explicitly then we can calculate all possibilities for the ring

$$R(X, K_X) = \bigoplus_{n=0}^{\infty} H^0(X, nK_X).$$

We split this section into two parts, one explaining the general strategy behind the above calculation, and one explaining the practical algorithm by which this strategy is carried out in terms of the cotangent complex of $[LS]$. A more satisfactory theory to accompany this essentially practical approach is nearing completion and is expected to appear in $[R2]$.

General Strategy (6.1)

Suppose $C$ and $X$ are as above, and we are given the ring $R(C, K_{XC})$, then our aim is to produce the ring $R(X, K_X)$. Let $D = K_{XC}$, then the strategy (due to Reid) is to produce a sequence of families of rings,

$$\{R(C, D)\}, \{R(2C, D^{(2)})\}, \ldots, \{R(nC, D^{(n)})\}, \{R(X, K_X)\}$$

where,

$$R(C, D) = \mathcal{O}[x_1, \ldots, x_k]/I,$$

$$R(nC, D^{(n)}) = \mathcal{O}[x_0, x_1, \ldots, x_k]/(I^{(n)}, x_0^\ell),$$

$$R(X, K_X) = \mathcal{O}[x_0, x_1, \ldots, x_k]/(I^{(n)}).$$

Each $R(nC, D^{(n)})$ is to depend in a linear way on the one before and
we have

$$(R(lC, D^{(l)})/(x^l_0 - 1)) \subset (R((l - 1)C, D^{(l-1)})).$$

For example, if $C$ is smooth, in computing $R(2C, D^{(2)})$ the theory is essentially that of Kodaira and Spencer [KS].

In particular, the ring $R(X, K_X)$ will be required to satisfy,

$$R(C, D) \subset R(X, K_X)/(x_0).$$

The following proposition goes some way towards describing the generators and relations of such a ring.

Proposition (6.1) (compare [R2])

Suppose $R = \mathbb{C}[x_1, ..., x_k]/I$ and $R'$ is such that $R'/x^q_0 = R$ for some $x^q_0 \in R'$ with $\deg x_0 > 0$. Then

$$R' = \mathbb{C}[x_0, ..., x_k]/I'$$

and:

i) If $x_0$ is a non-zero divisor and $I = (f_1, ..., f_r)$, where the $f_i$ are homogeneous, then there are homogeneous $F_1, ..., F_r \in I'$ such that $I' = (F_1, ..., F_r)$ where $F_i$ reduces mod $x_0$ to $f_i$.

ii) With the notation of i); $x_0$ is a non-zero divisor of $R'$ if and only if for every syzygy $\sigma_i$: $\Sigma f_i = 0 \in \mathbb{C}[x_1, ..., x_k]$, there are $L_1 \in \mathbb{C}[x_0, ..., x_k]$ such that $\Sigma: L_1 F_i = 0 \in \mathbb{C}[x_0, ..., x_k]$ where $L_1$ reduces mod $x_0$ to $L_i$.

Proof

We proceed by induction on the degree of $g \in R'$. Firstly $x_0, ..., x_k$ generate because given $g \in R'$ we have $g \mod x_0 \in \mathbb{C}[x_1, ..., x_k]$ so we can write $g = g_0(x_1, ..., x_k) + x_0g'$, where $g' \in R'$, $\deg g' < \deg g$.

i) Consider the commutative diagram:
We see that $I' \rightarrow I$ is surjective by the Snake Lemma (3.7). Thus for all $f \in I$, homogeneous of degree $d$, there exists $F \in I'$ and $F' \in \mathbb{C}[x_0, \ldots, x_k]$ such that $f = F - x_0 F'$ with $F$ homogeneous of degree $d$.

Choose $F_1, \ldots, F_r$ in this way and let $G \in I'$ be any element. Then $G \rightarrow \Sigma_i f_i$ with $l_i \in \mathbb{C}[x_1, \ldots, x_k]$, and thus $G - \Sigma_i F_i \in (x_0) \cap I'$.

Since $x_0$ is a non-zerodivisor mod $I'$, $(x_0) \cap I' = x_0 I'$ and $G - \Sigma_i F_i = x_0 G'$ for some $G'$ of lower degree than $G$ in $I'$. Continuing by induction on degree proves i).

ii) Write $F_i = f_i + x_0 g_i$ and suppose $\sigma_i : \Sigma_i f_i = 0$. Then

$$\Sigma_i F_i = x_0 \Sigma_i g_i$$

Since $\Sigma_i F_i \in I'$ and $x_0 \Sigma_i g_i \in (x_0)$, $x_0$ a non-zerodivisor implies that $\Sigma_i g_i \in I'$, and we can write $\Sigma_i g_i = \Sigma m_i F_i$. This gives us $\Sigma L_i F_i = 0$ where $L_i = l_i - x_0 m_i$.

Conversely, if $g \in \mathbb{C}[x_0, \ldots, x_k]$ and $x_0 g = \Sigma l_i F_i$ then $\Sigma l_i f_i = 0$. Thus there exists $L_i$ with $\Sigma L_i F_i = 0$, and $x_0 g = \Sigma (l_i - L_i) F_i = x_0 \Sigma m_i F_i \in \mathbb{C}[x_0, \ldots, x_k]$, where $L_i = l_i - x_0 m_i$, therefore $g \in I$.

□

This theorem tells us that the generators, relations and syzygies of
R' = R(X, K_X) occur in the same degrees as those of R = R(C, D). To calculate the relations and syzygies explicitly, we observe that given f_i \in I we are looking for

\[ F_i = f_i + x_0f_i' + x_0^2f_i'' + \ldots + x_0^{d_i(4)}f_i^{(4)} + \ldots, \]

such that f_i^{(4)} \in \mathbb{C}[x_1, \ldots, x_k]_d, and the F_i satisfy syzygies as in Proposition (6.1) ii). We get the sequence of rings

\{R(C, D)\}, \{R(2C, D^{(2)})\}, \ldots, \{R(nC, D^{(n)})\}, \{R(X, K_X)\}

by calculating the F_i in stages allowing successively higher powers of x_0.

Practical Details (6.2)

Proposition (6.2) is to be found in a more general form in [LS, Theorem (4.2.5)]. First we make some definitions to explain the terminology found in the diagram (S_d) in the statement of Proposition (6.2).

Definitions

Suppose we are given

\[ R^{(1)} = \mathbb{C}[x_1, \ldots, x_k]/I \]

and

\[ R^{(4)} = \mathbb{C}[x_0, x_1, \ldots, x_k]/(I^{(4)}, x_0^d) \]

such that R^{(4)} \mod x_0 is R^{(1)}.

1. Define S_R to be the graded \mathbb{C}[x_1, \ldots, x_k]-module with grading \[ S_R \] = R^{(1)}_{d-1} and with multiplication defined by

\[ fg = 0 \text{ for all } f, g \in S_R. \]

2. Define T_{R^{(4)}} to be the graded \mathbb{C}[x_0, \ldots, x_k]-module with grading \[ T_{R^{(4)}} \] = [R^{(4)}]_d and with multiplication defined by

\[ \left( \sum_{i=0}^{d-1} x_0^i p_i \right) \left( \sum_{i=0}^{d-1} x_0^i q_i \right) = x_0 \left( \sum_{i=0}^{d-2} x_0^i \sum_{j=0}^{d-1} p_j q_k \right) \]
LS, (4.2.5)] gives the following result in a highly theoretical setting. The diagram (Eₜ) in Proposition (6.2) has been taken from [LS, (4.2.5)] and rewritten in a more pragmatic notation.

**Proposition (6.2)**

Suppose that we are given $R^{(1)}$ and $R^{(4)}$ as above. Then $R^{(4+1)}$ exists if and only if there exists a degree preserving module homomorphism $\beta_4$ in the diagram (Eₜ), such that $i_{g'} \circ \beta_4 = \alpha_4$.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & x_{i} & \rightarrow & \mathbb{C}[x_1, \ldots, x_k] & \rightarrow & R^{(1)} & \rightarrow & 0 \\
& & \alpha_4 & \downarrow \beta_4 & & \alpha_4 & & & \\
0 & \rightarrow & S_R & \rightarrow & T_{R^{(4)}} & \rightarrow & R^{(4)} & \rightarrow & R^{(1)} & \rightarrow & 0
\end{array}
\]

The rows of (Eₜ) are exact. $(f_1, \ldots, f_r)$ is the free $\mathbb{C}[x_1, \ldots, x_k]$-module generated by $f_1, \ldots, f_r$. $\Sigma$ is the $\mathbb{C}[x_1, \ldots, x_k]$-module generated by $\sigma_1, \ldots, \sigma_4$ such that the top row of (Eₜ) is exact. The maps $i_4: S \rightarrow T$ and $j_4: T \rightarrow R^{(4)}$ are given by

\[
i_4(p) = x_0^{2-1}p \text{ and } j_4(\sum_{i=0}^{4-1}x_0^i p_i) = x_0 \sum_{i=0}^{4-2}x_0^i p_i.
\]

The maps $\alpha_4$ and $\alpha_4'$ are defined inductively, indeed they depend on the map $\beta_{4-1}$ in the diagram (Eₜ⁻¹).
Step 1: $(R^{(1)} \rightarrow R^{(2)})$.

We have the following diagram:

\[
\begin{array}{cccccccc}
0 & \rightarrow & \Sigma & \rightarrow & (f_1, \ldots, f_r) & \rightarrow & \mathbb{C}[x_1, \ldots, x_k] & \rightarrow & R^{(1)} & \rightarrow & 0 \\
| & & | & & \alpha_1 & \searrow & | & & | & & (g_1) \\
0 & \rightarrow & R^{(1)} & \rightarrow & R^{(1)} & \rightarrow & R^{(1)} & \rightarrow & R^{(1)} & \rightarrow & 0 \\
\end{array}
\]

Here the maps $\alpha_1$, $\alpha_1'$, and $j$ are all zero and $i$ is the identity map. Thus we must find $\beta_1 : (f_1, \ldots, f_r) \rightarrow S_R$ such that $\deg \beta_1(f_i) = \deg f_i - 1$ and $\Sigma_i \beta_1(f_i) = 0$ whenever $i'(\sigma_j) = \Sigma_i f_i$ for some $\sigma_j$. The zero map is a solution here and corresponds to the fact that first order extensions are always unobstructed. The construction of $\beta_1$ allows us to define,

\[
R^{(2)}(\beta_1) = \mathbb{C}[x_0, \ldots, x_k] / (1^{(2)}, x_0^2),
\]

where $1^{(2)}$ is generated by $F_1, \ldots, F_r$ given by $F_i = f_i + x_0 \beta_1(f_i)$.

Let $\beta : (f_1, \ldots, f_r) \rightarrow S_R$ be a 'general' such map. That is $\{\beta(f_1), \ldots, \beta(f_r)\}$ are generic polynomials satisfying the required conditions.

Step 2: $(R^{(2)} \rightarrow R^{(3)})$.

Suppose we have a ring $R^{(2)}(\beta)$ from step 1. That is, we have a map $\beta : (f_1, \ldots, f_r) \rightarrow S_R$ such that $\Sigma_i \beta(f_i) = 0$ whenever $i'(\sigma_j) = \Sigma_i f_i$ for some $\sigma_j$. $\beta : (f_1, \ldots, f_r) \rightarrow S_R$ can then be lifted to $\beta^- : (f_1, \ldots, f_r) \rightarrow \mathbb{C}[x_1, \ldots, x_k]$ where $\Sigma_i \beta^-(f_i) \in 1$ whenever $i'(\sigma_j) = \Sigma_i f_i$ for some $\sigma_j$.

Then $\alpha_2 : (f_1, \ldots, f_r) \rightarrow T_R \mathcal{O}$ is given by $\alpha_2(f_i) = \beta(f_i)$. Consider $\sigma_j \in (\sigma_1, \ldots, \sigma_i)$, where $i'(\sigma_j) = \Sigma_i f_i$ then step 1 has given the expression,

\[
\Sigma_i \beta^-(f_i) = \Sigma \sigma_k f_k \in 1.
\]
We define $\alpha_2^{-1}(\sigma_j) = \Sigma p_k \beta(f_k)$, and then $\beta_2$ in the diagram (8.2) must satisfy:

$$\Sigma_2 \beta_2(f_i) - \Sigma p_k \beta(f_k) = 0.$$ 

As in step 1 the construction of $\beta_2$ allows us to put

$$R^{(3)}(\beta_2) = \mathfrak{C}(x_0,\ldots,x_k)/\langle I^{(3)} \rangle,$$

where $I^{(3)}$ is generated $F_1,\ldots,F_r$ given by $F_i = f_i + x_0 \beta(f_i) + x_0^2 \beta_2(f_i)$.

If there are no maps $\beta_2$ making the diagram (8.2) commute, the extension is obstructed. In this case, in trying to construct $\beta_2$ we are forced to put conditions on $\beta$, that is we replace $\beta$ by the most general $\beta_1$ such that (8.2) can be made to commute.

**Step 2:** $(R^{(4)} \rightarrow R^{(4+1)}).$

To go from $R^{(4)} \rightarrow R^{(4+1)}$ we repeat the above procedure. The example in the next section shows how this algorithm works in a fairly simple case and how obstructions occur in the calculations.

It is precisely because of the occurrence of obstructions that

$$\{R^{(4)}/(x_0^{d-1})\}$$

is a subset of, not necessarily equal to $\{R^{(4-1)}\}$.

7. Pinkham's Example.

In [P. Section 8] Pinkham computes directly the deformations of the cone over the rational curve, $C_4$ of degree 4 in $\mathbb{P}^4$, using the theory outlined in section 6. In this section we reproduce his calculation, by way of example, pointing out the similarities with the later calculations in Chapter 4.

**Example (7.1)**

Let $C_4 \subset \mathbb{P}^4$ be the rational, normal curve defined by:
where $x_1,\ldots,x_5$ are the coordinates in $\mathbb{P}^4$.

This yields the 6 equations,

\begin{align*}
  r_1: & \quad x_1x_2 = \frac{x_2^2}{x_1}, \\
  r_2: & \quad x_1x_4 = \frac{x_2x_3}{x_1}, \\
  r_3: & \quad x_1x_5 = \frac{x_2x_4}{x_1}, \\
  r_4: & \quad x_2x_4 = \frac{x_3^2}{x_2}, \\
  r_5: & \quad x_2x_5 = \frac{x_3x_4}{x_2}, \\
  r_6: & \quad x_3x_5 = \frac{x_4^2}{x_3}.
\end{align*}

There are 8 syzygies between these 6 relations, namely:

\begin{align*}
  \sigma_1: & \quad x_1r_4 - x_2r_2 + x_3r_1 = 0, \\
  \sigma_2: & \quad x_1r_5 - x_2r_3 + x_4r_1 = 0, \\
  \sigma_3: & \quad x_1r_6 - x_3r_3 + x_4r_2 = 0, \\
  \sigma_4: & \quad x_2r_6 - x_3r_5 + x_4r_1 = 0, \\
  \sigma_5: & \quad x_2r_4 - x_3r_2 + x_4r_1 = 0, \\
  \sigma_6: & \quad x_2r_5 - x_3r_5 + x_4r_1 = 0, \\
  \sigma_7: & \quad x_2r_6 - x_4r_3 + x_4r_2 = 0, \\
  \sigma_8: & \quad x_3r_6 - x_4r_5 + x_3r_4 = 0.
\end{align*}

Let $R_C = \mathbb{C}(x_1,\ldots,x_5)/I$ where $I = (r_1,\ldots,r_6)$. To write down the first order extensions we must compute $\beta(r_i)$ for $i=1,\ldots,6$ such that the syzygies $\sigma_1,\ldots,\sigma_8$ are lifted in the way prescribed in (6.2). Let us suppose that
Then the ideal \( \mathfrak{I}^{(2)} \) (notation as in (6.2)) will be generated by

\[
\mathfrak{I}^{(2)} : r_i + x_0 \sum_{j=1}^{5} a_{ij} x_j, \quad i = 1, \ldots, 6,
\]

where the \( a_{ij} \) are subject to restrictions imposed by the syzygies. We shall now calculate the effects of these restrictions one syzygy at a time.

First we make some simplifications using changes of coordinates in \( x_1, \ldots, x_5 \). For example, if we set

\[ x_1' = x_1 + a_{13} x_0, \]

then we can rewrite \( \mathfrak{I}^{(2)} \) with the coefficient of \( x_0 x_3 \) equal to 0. In this way before we begin our calculation we can make the assumptions that \( a_{13} = a_{42} = a_{43} = a_{44} = a_{63} = 0. \)

Consider \( \sigma_1 : x_1 x_4 - x_2 x_2 + x_3 x_1 = 0. \) We must have

\[ x_1 \beta(r_4) - x_2 \beta(r_2) + x_3 \beta(r_1) \in (r_1, \ldots, r_6), \]

if \( \beta \) is going to make the diagram (81) commute. That is there exist \( p_1, \ldots, p_6 \in \mathbb{C} \) such that:

\[
x_1 (a_{41} x_4 + a_{45} x_5) - x_2 \sum_{j=1}^{5} a_{2j} x_j
\]

\[ + x_3 (a_{11} x_1 + a_{12} x_2 + a_{14} x_4 + a_{15} x_5), \]

\[ = p_1 (x_1 x_3 - x_2^2) + p_2 (x_1 x_4 - x_2 x_3) + p_3 (x_1 x_5 - x_2 x_4) + p_4 (x_2 x_4 - x_3^2) + p_5 (x_2 x_5 - x_3 x_4) + p_6 (x_3 x_5 - x_2^2). \]

Since this is an identity in \( \mathbb{C}[x_1, \ldots, x_5] \), we can equate coefficients. We find the following relations hold between the \( a_{ij} \).
Similarly, consider the syzygy
\[ \sigma_2: x_1 r_5 - x_2 r_3 + x_4 r_1. \]
As before there must be \( p_1, \ldots, p_6 \in \mathbb{C} \) (not the same \( p_1, \ldots, p_6 \) as before, of course) such that:

\[
\sum_{j=1}^{5} a_{3j} x_j + x_4 (a_{11} x_1 + a_{12} x_2 + a_{14} x_4)
\]
\[= p_1(x_1 x_3 - x_2^2) + p_2(x_1 x_4 - x_2 x_3) + p_3(x_1 x_5 - x_2 x_4)
\]
\[+ p_4(x_2 x_4 - x_3^2) + p_5(x_2 x_5 - x_3 x_4) + p_6(x_3 x_5 - x_4^2).\]

Equating coefficients we can extend the list of relations between the \( a_{ij} \) with those below:

\[ a_{14} = 0, \]
\[ a_{51} = 0, \]
\[ a_{32} = a_{31}, \]
\[ a_{53} = a_{32}, \]
\[ a_{54} + a_{11} = a_{33}, \]
\[ a_{55} + a_{12} = a_{34}, \]
\[ a_{35} = 0. \]

If we treat the syzygy \( \sigma_3: x_1 r_6 - x_3 r_3 + x_4 r_2 \) in the same way we get the following polynomials for \( \beta(r_i) \) which make the diagram \((E_1)\) commute. This is easy, but tedious, to check. Put \( a_{11} = a, a_{12} = b, a_{33} = c, a_{34} = d, \) then,
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\[ \beta(r_1) = ax_1 + bx_2, \]
\[ \beta(r_2) = ax_2 + bx_3, \]
\[ \beta(r_3) = cx_2 + (a+d)x_3 + bx_4, \]
\[ \beta(r_4) = 0, \]
\[ \beta(r_5) = cx_3 + dx_4, \]
\[ \beta(r_6) = cx_4 + dx_5. \]

Thus we have computed the rings which correspond to first order deformations of the rational, normal quartic in \( \mathbb{P}^4 \). They depend on 4 free parameters, \( a, b, c \) and \( d \).

To find the space of second order deformations we again consider the syzygies one at a time and find \( \beta_2: (r_1, \ldots, r_6) \rightarrow \mathbb{R}^c \) which makes the diagram (B2) commute. For example, reconsider the syzygy \( \sigma_1 \) and suppose that \( \beta_2(r_i) = a_i, i = 1, \ldots, 6 \). Then we must have:
\[ x_1a_4 - x_2a_2 + x_3a_1 - (a_3 \beta(r_1)) = 0. \]
Substituting for \( \beta(r_1) \) and equating coefficients we find that:
\[ a_1 = 0, \]
\[ a_2 = ab, \]
\[ a_4 = -a^2. \]

Treating \( \sigma_2 \) in the same way we must have:
\[ x_1a_5 - x_2a_3 + x_4a_1 - (a_3 \beta(r_1) - (a+d)\beta(r_2)) = 0. \]
Equating coefficients again gives:
\[ (a+d)b = 0, \]
\[ a_5 = -ac, \]
\[ a_3 = bc + a(a+d). \]

After considering \( \sigma_3 \) we have the whole story:
\[ a_1 = 0, \]
\[ a_2 = ab, \]
Extension-deformation theory

\[ a_3 = bc + a(a+d), \]
\[ a_4 = -a^2, \]
\[ a_5 = -ac, \]
\[ a_6 = 0. \]

Moreover, we have \((a+d)b = (a+d)c = (a+d)(a-d) = 0\) as a requirement for first order deformations to extend to second order. It can be seen (e.g. from determinantal presentations given below or by continuing the calculation) that the algorithm terminates here, in the sense that these new relations satisfy \(\sigma_1, \ldots, \sigma_5\) after removing the condition on the order of \(x_0\). It is important to notice that in general the algorithm will terminate in \(\leq N\) steps where \(N\) is the maximum degree of the generators of the module of syzygies \((\sigma_1, \ldots, \sigma_5)\) (see section 11). As Pinkham remarks [P], this gives two components of the space of deformations of the cone. If \(a+d = 0\), we get the rings given by,

\[ R = \mathbb{C}[x_0, \ldots, x_5]/I', \]

where \(I'\) is generated by,

\[
\begin{align*}
\text{rank} & \begin{bmatrix}
x_1 & x_2 & x_3 + ax_0 & x_4 + cx_0 \\
x_2 + bx_0 & x_3 - ax_0 & x_4 & x_5
\end{bmatrix} \leq 1.
\end{align*}
\]

If \(b = c = a-d = 0\), we get the rings given by,

\[ R = \mathbb{C}[x_0, \ldots, x_5]/I', \]

where \(I'\) is generated by,

\[
\begin{align*}
\text{rank} & \begin{bmatrix}
x_1 & x_2 & x_3 + ax_0 \\
x_2 & x_3 - ax_0 & x_4 \\
x_3 + ax_0 & x_4 & x_5
\end{bmatrix} \leq 1.
\end{align*}
\]

Notice that these two rings coincide when

\[ a = b = c = d = 0. \]

That is exactly at the ring of the cone over \(C_4\).
8. Griffin's Example and Deformation Theory.

In section 7 we saw how the relations in the ring $R_C$ could be formatted in two different ways. That is, the relations $r_1, \ldots, r_6$ can be generated by either,

$$\text{rank} \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & x_3 & x_4 & x_5 \end{bmatrix} \leq 1,$$

or,

$$\text{rank} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{bmatrix} \leq 1.$$

In each case the syzygies in the ring depend only on the shape of the matrix, in the sense that if we replace each $x_i$ by a (reasonably general) linear term $l_i(x_0, x_1, \ldots, x_5)$ the resulting ideal will have an isomorphic resolution as a $\mathbb{C}[x_0, \ldots, x_5]$-module. More precisely the terms in the resolution will have the same number of generators in each degree and the maps will be given by matrices of the same shape. In such cases we say that the given presentation of the relations is formative (the ring structure depends only on the format of the equations not on the individual coefficients). The idea is that a formative presentation determines the homological algebra of the ring.

For example, any generic determinantal ring has the formative presentation given by a matrix with a rank condition.

It is sometimes the case that a formative presentation of the relations gives rise to all the extensions of a ring. In the notation of Proposition (6.1), if

$$R'/(x_0) = R$$
for some non-zerodivisor \( x_0 \in R' \), \( \deg x_0 = d > 0 \), then it is possible that the relations in \( R' \) can be placed in the same format as those in \( R \). If this is true for all extensions \( R' \) we say that the presentation is \textit{informative} in degree \(-d\) (the information about the extensions is completely contained in the format of the equations of \( R \)).

In [G], Griffin treated the case of numerically quintic surfaces, \( X \), using the method outlined in section 6. In the case where \( \mathcal{K}_X \) has a single transverse base point he takes a nonsingular curve \( C \in \mathcal{K}_X \) given by \( (x_0 = 0) \) and calculates the ring \( R(C, \mathcal{K}_X) \). Proposition 8.1 is a summary of Griffins results in the above notation. Indeed in [R2] there is an extremely beautiful presentation of the numerical quintics in terms of formative presentations which has a lot in common with the case of Pinkham and the examples given in chapter 4 of this thesis.

**Proposition (8.1)**

i). Let \( X \) and \( C \) be as above and let

\[
A = \begin{bmatrix}
    x_1 & x_2 & x_3^2 & z_1 \\
    x_2 & x_3 & y & z_2
\end{bmatrix}
\]

Then,

\[
R(C, \mathcal{K}_X) = \mathbb{C}[x_1, x_2, x_3, y, z_1, z_2]/I,
\]

where \( I \) is generated by,

\[
\text{rank } A \leq 1,
\]

giving \( \tau_1, \ldots, \tau_6 \), and

\[
AM(A)^t = 0,
\]

where
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\[
M = \begin{bmatrix}
Q & a_{11}y^2 & 0 & 0 \\
0 & a_{11}y^2 & a_{12}y^2 & 0 \\
0 & 0 & a_{13}y & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

giving \( r_7, r_8 \) and \( r_9 \). Here \( a_{ij} \in \mathbb{C} \) and \( Q \) is a weighted, homogeneous quartic \( Q(x_1,y) \).

ii). The presentation of the ring given in i) is formative.

iii). The presentation of the ring given in i) is informative.

Proof

i), ii) and iii) are effectively the results of the calculations in [G].

\( \square \).
CHAPTER 3:
The canonical linear system.


Let $X$ be a surface of general type with $p_g = 3$ and $K_X^2 = 4$. In Lemma (5.2) we saw that $X$ belongs to one of five different classes depending on the behaviour of its canonical linear system $|K_X|$. $|K_X|$ can have 0, 1 or 2 base points or a fixed $(-2)$-cycle. In this section we shall analyse the nature of a general element $C \in |K_X|$. We collect together general facts about the canonical curve into a lemma.

Lemma (9.1)

Let $X$ be a surface as above and $C \in |K_X|$ be a general element.

I. If $X$ is of type I then $C$ is a nonsingular, non-hyperelliptic, genus 5 curve.

II. If $X$ is of type II then $C$ is a nonsingular, non-hyperelliptic, genus 5 curve.

III. If $X$ is of type III then $C$ is a nonsingular, hyperelliptic genus 5 curve.

IIIa and IIIb. If $X$ is of type IIIa or IIIb then we can write $C = \Gamma + E$ where $E$ is a $(-2)$-cycle and $\Gamma$ is a nonsingular, genus 4 hyperelliptic curve with $\Gamma.E = 2$.

Proof

I. By the Adjunction Formula (3.2) $g(C) = 5$ and by Bertini's Theorem (3.6) $C$ is nonsingular. Suppose $C$ is hyperelliptic and let $D = K_X|C|$. Then $2D = K_C$ and so $D = \frac{1}{2} + P_1 + P_2$, where $P_1$ and $P_2$ are
Weierstrass points [ACGH, p.288, Ex.31,32]. This is true for surfaces of all types, but contradicts the number of base points in Cases I and II.

II. As before \( g(C) = 5 \) and \( C \) is non-hyperelliptic. Bertini's Theorem (3.6) says that \( C \) is nonsingular away from the base point. But from Lemma (5.2) \( \deg \text{base locus} = 1 \) in this case and so \( C \) is nonsingular at the base point.

III. \( g(C) = 5 \) and \( |K_X|_C| = g_2^1 + P_1 + P_2 \), where \( P_1 \) and \( P_2 \) are the distinct base points. Bertini's Theorem (3.6) says that \( C \) is nonsingular away from the base points and since \( \deg \text{base locus} = 2 \) \( C \) is nonsingular at the base points.

III_a and III_b. By the Adjunction Formula (3.2) \( g(\Gamma) = 4 \). The rest follows from Lemmas (5.1) and (5.2)

\[ \square \]

The eventual aim is to describe the ring

\[ R(X, K_X) = \bigoplus_{n=0}^\infty H^0(X, nK_X) \]

in terms of generators and relations. We do this in Chapter 4 but first we calculate a ring associated to the canonical curve of \( X \) (cf. [Mu, lecture 1]).

In cases I, II and III we consider a nonsingular curve \( C \in |K_X| \) and the divisor \( D = K_X|_C \) on \( C \), and calculate the graded ring \( R(C, D) = \bigoplus_{n=0}^\infty H^0(C, nD) \). In cases III_a and III_b, we consider a nonsingular curve \( \Gamma \in |K_X - E| \), where \( E \) is the fixed \((-2)\)-cycle, and an appropriate divisor on \( \Gamma \). We make heavy use of the Base Point Free Pencil Trick (BPFPT (3.3)).
Case I. $|K_X|$ has no base points.

Theorem (9.2)

Consider $C \in |K_X|$, a nonsingular curve and $D = K_X|_C$ a divisor on $C$. Then

$$R(C, D) = \mathbb{C}[x_1, x_2, y_1, y_2] / <Q_1, Q_2>$$

where the generators are $\deg (x_i, y_i) = (1, 2)$ and the two polynomials $Q_1$, $Q_2$ are weighted homogeneous of degree 4 in $x_1, x_2, y_1, y_2$.

Proof.

Since $\deg D = 4$, we can calculate, using the Riemann–Roch formula (3.1a), that

$$\dim H^0(D) = 2, \quad \dim H^0(2D) = 5$$

and

$$\dim H^0(nD) = 4(n-1) \text{ for } n > 2.$$ 

We aim to verify the information in Table 1 below. The notation is explained in section 2.

Table 1.

<table>
<thead>
<tr>
<th>degree</th>
<th>generators</th>
<th>new relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_1, x_2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$S^2(x_i), y_1, y_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$S^3(x_i), x_i y_j$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$S^4(x_i), S^2(x_i) y_j, S^2(y_i)$</td>
<td>$Q_1, Q_2$</td>
</tr>
<tr>
<td>$\geq 5$</td>
<td>no new generators</td>
<td>no new relations</td>
</tr>
</tbody>
</table>
The canonical linear system
Let $x_1, x_2$ be generators of $H^0(C, D)$. Then

$$x_1^2, x_1 x_2, x_2^2 \in H^0(C, 2D)$$

are linearly independent (a relation between them would have to factorise, contradicting the independence of $x_1$ and $x_2$), so choose $y_1, y_2$ to complete a basis for $H^0(C, 2D)$.

To show that $S^3(x_i) \cup \{x_i y_j\}_{j=1,2}^{i=1,3}$ form a basis of $H^0(C, 3D)$ we use the BPFPT (3.3). Consider the natural map

$\varphi: H^0(C, D) \otimes H^0(C, 2D) \to H^0(C, 3D)$.

Since $|D|$ is a base point free pencil, by the BPFPT (3.3), $\text{Ker}(\varphi) = H^0(C, D)$ and hence $\dim \text{Ker} \varphi = 2$. As $\dim H^0(C, 3D) = 8$, $\varphi$ must be surjective and $\{x_1^3, x_1^2 x_2, x_1 x_2^2, x_2^3, x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2\}$ is a basis for $H^0(C, 3D)$.

As $C$ is not hyperelliptic (Lemma 9.1), we can appeal to M. Noether's Theorem (3.4) to see the degree 4 generators. This tells us that the 14 monomials in Table 1 generate this 12 dimensional space and hence we can write down two relations $Q_1$ and $Q_2$ between them.

Let

$$\varphi_n: H^0(C, D) \otimes H^0(C, nD) \to H^0(C, (n+1)D).$$

Then by BPFPT

$$\dim \text{Im} \varphi_n = (2(4n-4)) - [4n-8] = 4n = h^0(C, (n+1)D).$$

This shows that the $x_i, y_j$ generate $R(C, D)$ as an algebra, and the only relations are generated by $Q_1$ and $Q_2$, proving the theorem.
Case II. $|K_X|$ has a single transversal base point.

Theorem (9.3)

Consider $C \in |K_X|$ and $D = K_X|C$, then

$$R(C,D) = \mathbb{C}[x_1, x_2, y_1, y_2, z]/I$$

generated by $x_i, y_i$ and $z$ with $\deg (x_i, y_i, z) = (1, 2, 3)$ and related by $I$

which is generated by

$$r_1 : x_1 y_2 - x_2 y_1,$$

$$r_2 : x_1 z - y_1^2,$$

$$r_3 : x_2 z - y_1 y_2,$$

$$r_4 : y_1 z - x_2 A - x_1 B,$$

$$r_5 : z^2 - y_2 A - y_1 B,$$

where

$$A = a_1 y_2^2 + a_2 x_2^2 y_2 + a_3 x_2^4, \ a_i \in \mathbb{C},$$

$$B = B(x_1, x_2, y_1, y_2)$$

is a degree 4 weighted homogeneous polynomial in $x_1, x_2, y_1, y_2$.

Moreover $r_1, ..., r_5$ are the diagonal 4x4 Pfaffians of the following matrix [see Section 4].

$$M = \begin{pmatrix}
0 & 0 & x_1 & x_2 & y_1 \\
0 & 0 & y_1 & y_2 & z \\
-x_1 & -y_1 & 0 & -z & -A \\
-x_2 & -y_2 & z & 0 & B \\
-y_1 & -z & A & -B & 0
\end{pmatrix}$$
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Proof

Our strategy is to choose sections of $\mathcal{O}_C(D)$ according to their geometrical properties, (in fact, with respect to the order of their vanishing at the base point $P$). Notice that $C$ is trigonal because we can write $D = D_1 + P$ where $P$ is the base point and $|D_1| = g_3^1$ on $C$.

We calculate the dimensions in Table 2 by Riemann–Roch. Let $u$ be a generator for $H^0(D-2P)$ and $t : \mathcal{O}_C \to \mathcal{O}_C(P)$. Then $ut \in H^0(D-P)$ and we let $v$ be the second element of a basis. Since $ut^2$ and $vt$ vanish to different order at $P$ they are linearly independent in $H^0(D)$ and so form a basis.

Table 2.

<table>
<thead>
<tr>
<th>Space</th>
<th>dimension</th>
<th>generators</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^0(D-2P)$</td>
<td>1</td>
<td>$u$</td>
</tr>
<tr>
<td>$H^0(D-P)$</td>
<td>2</td>
<td>$ut, v$</td>
</tr>
<tr>
<td>$H^0(D)$</td>
<td>2</td>
<td>$ut^2, vt$</td>
</tr>
<tr>
<td>$H^0(D+P)$</td>
<td>3</td>
<td>$ut^3, vt^2, w$</td>
</tr>
</tbody>
</table>

Similarly we need one extra basis element $w \in H^0(D+P)$ which does not vanish at $P$.

We are aiming to establish Table 3 where $x_1 = ut^2$, $x_2 = vt$, $y_1 = utw$, $y_2 = vw$ and $z = uw^2$. 
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Table 3.

<table>
<thead>
<tr>
<th>n</th>
<th>generators of $H^0(nD)$</th>
<th>new relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_1, x_2$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$S^2(x_1), y_1, y_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$S^3(x_1), x_i y_j, z$</td>
<td>$r_1$</td>
</tr>
<tr>
<td>4</td>
<td>generated by monomials in $x_i y_j z$</td>
<td>$r_2, r_3$</td>
</tr>
<tr>
<td>5</td>
<td>generated by monomials in $x_i y_j z$</td>
<td>$r_4$</td>
</tr>
<tr>
<td>6</td>
<td>generated by monomials in $x_i y_j z$</td>
<td>$r_5$</td>
</tr>
</tbody>
</table>

Notice that $u, v \in H^0(D-P)$ form a free pencil so we can use the BPFPT (3.3) on the maps

$\varphi_{n,m} : H^0(nD+mP) \otimes H^0(D-P) \to H^0((n+1)D+(m-1)P)$

This tells us that $\text{Ker } \varphi_{n,m} = H^0((n-1)D+(m+1)P)$ so that we can calculate whether $\varphi_{n,m}$ is surjective.

Table 2 tells us that $x_1, x_2$ generate $H^0(D)$. Consider

$\varphi_{1,1} : H^0(D+P) \otimes H^0(D-P) \to H^0(2D)$,

then $\text{Ker } \varphi_{1,1} = H^0(2P)$ which is generated by $t^2$. This means that $\dim \text{Im } \varphi_{1,1} = 3x^2 - 1 = 5$, and so $\varphi_{1,1}$ is surjective and $H^0(2D)$ is generated by $u^2 t^4, u t^3 v, t^2 v^2, w u t^3, w u t v, w t v^2$. However $u w^2 \in$
The canonical linear system

$H^0(3D)$ and is independent of those elements given since it does not vanish at $P$. Thus we have the generators of $H^0(3D)$ given in table 3. The relation $r_1$ is given because $x_1y_2 = ut^2vw$ and $x_2y_1 = ut^2vw$ and so

$$r_1 : x_1y_2 = x_2y_1$$

For higher degrees we note first that $\varphi_{n,1}$ is surjective for $n \geq 3$ (by Riemann–Roch for curves (3.1a))

$$h^0(nD+P) = 4n-3$$

and

$$\dim \ker \varphi_{n,1} = h^0((n-1)D+2P) = 4n-6,$$

so

$$\dim \im \varphi_{n,1} = 2(4n-3) - (4n-6) = 4n$$

which is $h^0((n+1)D))$. The generators in degree 4 come immediately from this, and the relations $r_2$ and $r_3$ again follow from the substitution of the definitions of $x_1, x_2, y_1, y_2, z$, giving

$$r_2 : x_1z = y_1^2,$$
$$r_3 : x_2z = y_1y_2.$$  

It remains to derive the relations $r_4$ and $r_5$. These are inherited from a single relation in $H^0(5D-P)$ which we construct now. The map

$$\varphi_{4,0} : H^0(4D) \otimes H^0(D-P) \rightarrow H^0(5D-P)$$

has kernel $H^0(3D+P)$ and so $\dim \im \varphi_{4,0} = 15 = h^0(5D-P)$. However we can write down explicitly the monomials in $H^0(5D-P)$ as follows:

There are the 15 linearly independent monomials

$$u^9t^9, u^8t^8v, u^2t^7v^2, u^2t^6v^3, ut^5v^4,$$
$$u^4t^5, u^4t^6w, u^3t^5vw, u^2t^4v^2w, ut^3v^3w,$$
$$u^2t^3w, u^3t^3w^2, u^2t^2vw^2, utv^2w^2, v^3w^2 \in \im \varphi_{4,0}$$

and one more $u^2w^3$. Therefore we must have a relation
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\[ R: u^2w^3 = g, \]

where \( g \) is a linear combination of the other 15 monomials by surjectivity of \( \varphi_{4,0} \).

It follows that in \( H^0(5D) \) we must have the relation

\[ tR: tu^2w^3 = tg. \]

Rewriting this in terms of \( x_1, x_2, y_1, y_2 \) and \( z \) gives

\[ r_4: y_1z - x_2A - x_1B = 0, \]

where

\[ A = a_1y_2^2 + a_2x_2^2y_2 + a_3x_2^4, \quad a_i \in \mathbb{C}, \]

\[ B = B(x_1, x_2, y_1, y_2) \text{ is a polynomial in } x_1, x_2, y_1, y_2. \]

In the same way, in \( H^0(6D) \) we have the relations

\[ ut^3R = x_1r_4, \]
\[ vt^2R = x_2r_4 \]

and

\[ wR = r_5: z^2 - y_2A - y_1B = 0. \]

That the above relations do indeed suffice to give the ring \( R(C,D) \) is a check left to the reader. It can be seen (and follows from the theory of Pfaffian rings) that the relations given satisfy the syzygies \( s_1, \ldots, s_5 \) below.

\[ s_1: \quad x_1r_3 - x_2r_2 + y_1r_1 = 0, \]
\[ s_2: \quad -y_1r_3 + y_2r_2 = zr_1 = 0, \]
\[ s_3: \quad -x_1r_5 + y_4r_4 + zr_2 = Ar_1 = 0, \]
\[ s_4: \quad x_2r_5 - y_2r_4 - zr_3 = Br_1 = 0, \]
\[ s_5: \quad -y_1r_5 + zr_4 + Ar_3 + Br_2 = 0. \]
Case III. Two distinct base points.

Theorem 9.4

Let $X$ be a surface of type III and $C \in |K_X|$ a nonsingular curve; let $D=K_XC$. Then

$$R(C,D) = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I,$$

where $\deg (x_1, y_1, z_k) = (1, 2, 3)$ and $I$ is the ideal generated by the relations $r_1, \ldots, r_9$ below:

Degree 3

\begin{align*}
r_1 & : x_1^3 - x_2y_1, \\
r_2 & : x_1y_2 - x_2^3.
\end{align*}

Degree 4

\begin{align*}
r_3 & : y_1y_2 - x_1^2x_2, \\
r_4 & : x_1z_2 - x_2z_1.
\end{align*}

Degree 5

\begin{align*}
r_5 & : y_1z_2 - x_1^2z_1, \\
r_6 & : x_2^2z_2 - y_2z_1.
\end{align*}

Degree 6

\begin{align*}
r_7 & : -z_1^2 + \lambda y_1^3 + \mu x_2^3y_2 + x_1^2h, \\
r_8 & : -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h, \\
r_9 & : -z_2^2 + \lambda x_1^2y_1 + \mu y_2^3 + x_1^2h,
\end{align*}

where $\mu, \lambda \in \mathbb{C}$ and $h = h_C$ is some polynomial of degree 4.

Alternatively, we can write $r_1, \ldots, r_9$ in a quasi-determinantal form as follows:
The canonical linear system

\[ f_1, \ldots, f_6 \text{ are given by} \]
\[ \text{rank } A \leq 1. \]

where

\[
A = \begin{bmatrix}
  x_1 & y_1 & x_2^2 & z_1 \\
  x_2 & x_1^2 & y_2 & z_2
\end{bmatrix}
\]

\[ r_7, r_8, r_9 \text{ are given by} \]

\[
A = \begin{bmatrix}
  h & 0 & 0 & 0 \\
  0 & \lambda y_1 & 0 & 0 \\
  0 & 0 & \mu y_2 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix}
\]

\[ A^T = \begin{bmatrix}
  r_7 & r_8 \\
  r_8 & r_9
\end{bmatrix}
\]

Proof

The canonical curve \( C \) is hyperelliptic in this case, as

\[ |D| = g_2^1 + P_1 + P_2, \]
where \( P_1 \) and \( P_2 \) are the base points of \( |K_X| \) (\( P_1 \) and \( P_2 \) are Weierstrass points, as remarked in Lemma (9.1)). There is a standard method of calculating \( R(D') \) for a divisor \( D' \) made up of Weierstrass points on a hyperelliptic curve (see [R2], [Gl]) and so we shall present the computation with little proof.

Let \( u: \mathcal{O}_C \rightarrow \mathcal{O}_C(P_1+P_2), \ v: \mathcal{O}_C \rightarrow \mathcal{O}_C(P_3+\ldots+P_{12}) \) where \( P_1, \ldots, P_{12} \) are the Weierstrass points on \( C \). Then we choose \( t_1, t_2 \in H^0(g_2^1) \) forming a basis, such that \( u^2 = t_1 t_2 \) and \( v^2 = p(t_1, t_2) \), where \( p \) is the degree 10 polynomial in \( t_1, t_2 \) defining the 10 branch points.
Table 4.

<table>
<thead>
<tr>
<th>Space</th>
<th>Generators</th>
<th>New names</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^0(D)$</td>
<td>$ut_1, ut_2$</td>
<td>$x_1, x_2$</td>
</tr>
<tr>
<td>$H^0(2D)$</td>
<td>$t_1^2, t_1t_2, t_2^2$</td>
<td>$y_1 = t_1^2, y_2 = t_2^2$</td>
</tr>
<tr>
<td>$H^0(3D)$</td>
<td>$ut_1^3, ut_2^3, vt_1, vt_2$</td>
<td>$z_1 = vt_1, z_2 = vt_2$</td>
</tr>
</tbody>
</table>

Given Table 4, the relations $r_1, ..., r_6$ are simply identities following from the definitions of $x_1, x_2, y_1, y_2, z_1$ and $z_2$, and the equality $u^2 = t_1t_2$.

For the last 3 relations it is necessary to study the relation $v^2 = p(t_1, t_2)$ in more detail. Since $p(t_1, t_2)$ is degree 10 in $t_1$ and $t_2$, using the definitions of $y_1$ and $y_2$ we can write

$$p(t_1, t_2) = \lambda y_1^2 + \mu y_2^2 + t_1t_2h_8(t_1, t_2)$$

where $\lambda, \mu \in \mathbb{C}$ are non-zero and $h_8$ is of degree 8. Let

$$h_8(t_1, t_2) = a_0 t_1^8 + a_1 t_1^7 t_2 + ... + a_8 t_2^8, a_i \in \mathbb{C},$$

then we can write $h_8(t_1, t_2) = h(x_1, x_2, y_1, y_2)$, a uniquely determined quartic in $R(C, D)$. Since $z_1^2 = v^2 t_1^2$, $z_1 z_2 = v^2 t_1 t_2$ and $z_2^2 = v^2 t_2^2$ we get the 3 degree 6 relations in the statement:

$$r_7: -z_1^2 + \lambda y_1^3 + \mu x_2^2 y_2 + x_1^2 h,$$
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\[ r_8: -z_1 z_2 + \lambda x_1^2 y_1^2 + \mu x_2^2 y_2^2 + \alpha x_1 x_2 h. \]

\[ r_9: -z_2^2 + \lambda x_1^4 y_1 + \mu y_2^3 + x_2 h. \]

It is easy to check that these relations satisfy the matrix equation given. That is we can put

\[
\begin{bmatrix}
  x_1 & y_1 & x_2^2 & z_1 \\
  x_2 & x_1^2 & y_2 & z_2
\end{bmatrix} \begin{bmatrix}
  h & 0 & 0 & 0 \\
  0 & \lambda y_1 & 0 & 0 \\
  0 & 0 & \mu y_2 & 0 \\
  0 & 0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  r_7 \\
  r_8 \\
  r_8 \\
  r_9
\end{bmatrix}
\]

The syzygies holding between these 9 relations can be split into two groups. The determinantal ones following purely from the determinantal form of the first 6 relations are as follows:

\[ \Sigma_1 : x_1 r_3 - y_1 r_2 + x_2^2 r_1 = 0, \]

\[ \Sigma_2 : x_2 r_3 - x_1^2 r_2 + y_2 r_1 = 0, \]

\[ \Sigma_3 : x_1 r_6 - x_2^2 r_4 + z_1 r_2 = 0, \]

\[ \Sigma_4 : x_1 r_5 - y_1 r_4 + z_1 r_1 = 0, \]

\[ \Sigma_5 : x_2 r_5 - x_1^2 r_4 + z_2 r_1 = 0, \]

\[ \Sigma_6 : x_2 r_6 - y_2 r_4 + z_2 r_2 = 0, \]

\[ \Sigma_7 : y_1 r_6 - x_2^2 r_5 + z_1 r_3 = 0. \]
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\[ \Sigma_8 : x_1^2 r_6 - y_2 r_5 + z_2 r_3 = 0. \]

The second group depend only on the matrix form of the 9 relations in the following sense. These syzygies can be constructed using the following observation (due to Reid). Consider the following matrix \( A^* \)

\[ A^* = \begin{bmatrix} -x_2 & x_1 \\ -x_1^2 & y_1 \\ -y_2 & x_2^2 \\ -z_2 & z_1 \end{bmatrix} \]

Then observe that

\[ (A^*)A = \begin{bmatrix} 0 & r_1 & r_2 & r_4 \\ -r_1 & 0 & r_3 & r_5 \\ -r_2 & -r_3 & 0 & r_6 \\ -r_4 & -r_5 & -r_6 & 0 \end{bmatrix} \]

The equality

\[ ((A^*)A)M A^T = A^*(A M A^T) \]

can be written as an equality of relations i.e. syzygies.

\[ \Sigma_9 : -x_2 r_7 + x_1 r_8 = \lambda y_1^2 r_1 + \mu x_2^2 y_2 r_2 - z_1 r_4. \]

\[ \Sigma_{10} : -x_2 r_8 + x_1 r_9 = \lambda x_1^2 y_1 r_1 + \mu y_2^2 r_2 - z_2 r_4. \]

\[ \Sigma_{11} : -x_1^2 r_7 + y_1 r_8 = -x_1 h r_1 + \mu x_2^2 y_2 r_3 - z_1 r_5. \]
The canonical linear system

\[ \Sigma_{12} : -x_1^2 r_8 + y_1 r_9 = -x_2 r_1 + \mu y_2 r_3 - z_2 r_5. \]

\[ \Sigma_{13} : -y_2 r_7 + x_2^2 r_8 = -x_1 r_2 - \lambda y_1^2 r_3 - z_1 r_6. \]

\[ \Sigma_{14} : -y_2 r_7 + x_2^2 r_9 = -x_2 r_2 - \lambda x_1^2 y_1 r_3 - z_2 r_6. \]

\[ \Sigma_{15} : -z_2 r_7 + z_1^2 r_8 = -x_1 r_4 - \lambda y_1 r_5 - \mu x_2^2 y_2 r_6. \]

\[ \Sigma_{16} : -z_2 r_8 + z_1 r_9 = -x_2 r_4 - \lambda x_1^2 y_1 r_5 - \mu y_2^2 r_6. \]

\( \Sigma_1, \ldots, \Sigma_{16} \) generate the module of syzygies, though the proof of this (and similarly in (9.4a)) is a calculation best carried out by computer (e.g. [BSJ]).

\( \square \)

**Cases III\( a \) and III\( b \) \( |K_X| \) has a fixed \((-2)\)-cycle.**

**Theorem (9.4a)**

Let \( X \) be a surface with \( p_g = 3, \ k_X^2 = 4 \) and such that the canonical linear system contains a fixed \((-2)\)-cycle, \( E \). Suppose \( C = \Gamma + E \in |K_X| \) as in Lemma (9.1) and put \( D = K_X(C) \), then there are two possibilities:

\[ R(C, D) = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I \]

where \( \deg (x_1, y_1, z_i) = (1, 2, 3) \) and \( I \) is generated by \( r_1, \ldots, r_6, r_7, r_8, r_9 \) as follows:

**Degree 3**

\[ r_1 : x_1^2 - x_2 y_2. \]

\[ r_2 : x_1 y_2 - x_2 y_1. \]

**Degree 4**

\[ r_3 : y_2^2 - x_1^2 y_1. \]
The canonical linear system

Degree 5

\[ r_4: x_1 z_2 - x_2 z_1, \]

\[ r_5: y_2 z_2 - x_1 z_1, \]

\[ r_6: y_1 z_2 - y_2 z_1. \]

Degree 6

Case a)

\[ r_7: -z_1^2 + \lambda y_1^3 + \mu y_1 y_2^2 + x_1^2 h, \]

\[ r_8: -z_1 z_2 + \lambda y_1 y_2^2 + \mu y_1 y_2^2 + x_1 x_2 h, \]

\[ r_9: -z_2^2 + \lambda y_1 y_2^2 + \mu y_2^2 + x_2^2 h. \]

Case b)

\[ r_7: -z_1^2 + \mu y_1^3 y_2 + x_1 h, \]

\[ r_8: -z_1 z_2 + \mu y_1 y_2^2 + x_1 x_2 h, \]

\[ r_9: -z_2^2 + \mu y_2^2 + x_2^2 h. \]

\[ \lambda, \mu \in \mathbb{C}, \]

\[ h = \lambda_1 x_1^4 + \lambda_2 x_1^3 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1 x_2^3 + \lambda_5 y_1^2 + \lambda_6 x_1 y_2 + \lambda_7 y_2^2 + \lambda_8 x_1 y_2, \lambda_4 \in \mathbb{C}. \]

Alternatively, we can write \( r_1, \ldots, r_9 \) in the following form:

\( r_1, \ldots, r_6 \) are given by

\[
\begin{bmatrix}
x_1 & y_2 & y_1 & z_1 \\
x_2 & x_1^2 & y_2 & z_2
\end{bmatrix}
\]

\[ \text{rank} \leq 1 \]
The canonical linear system

and \( r_7, r_8, r_9 \) are given by

\[
\begin{bmatrix}
    x_1 & y_2 & y_1 & z_1 \\
    x_2 & x_1^2 & y_2 & z_2 \\
    & & &
\end{bmatrix}
\begin{bmatrix}
    h & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & -1 
\end{bmatrix}
\begin{bmatrix}
    x_1 & x_2 \\
    y_2 & x_1^2 \\
    y_1 & y_2 \\
    z_1 & z_2 
\end{bmatrix}
= \begin{bmatrix}
    r_7 \\
    r_8 \\
    r_9 
\end{bmatrix}
\]

In Case b we have \( \lambda = 0 \).

Proof

Let \( E \) be the fixed \((-2)\) - cycle and \( |K_X| = |\Gamma| + E \). Then, by Lemma (9.1), we can choose \( \Gamma \) to be a nonsingular genus 4 curve and \( K_{X|C} = g_2^1 + P_1 + P_2 \) where \( E|_{\Gamma} = P_1 + P_2 \). The two cases in the statement of the theorem account for the possibilities:

Case a \( P_1, P_2 \) distinct points

Case b \( P_1 = P_2 = P \).

Notice that \( |P_1 + P_2| = g_2^1 \) (since \( 3g_2^1 = |K_{\Gamma}| = |2K_{X|C} - P_1 - P_2| = 2g_2^1 + |P_1 + P_2| \)), so that in Case b \( P \) is a Weierstrass point.

We shall calculate \( R(C, D) \) in 3 main steps.

Step 1: we write down the ring \( R(\Gamma, Q) \) where \( Q \) is a Weierstrass point.

Step 2: we produce from Step 1, \( R(\Gamma, 4Q) = R(\Gamma, 2g_2^1) \), because \( D = g_2^1 + P_1 + P_2 \), and so we have \( R(C, D) \subset R(\Gamma, 2g_2^1) \).
Step 3: we write down the condition which characterises \( R(C, D) \) as a subring of \( R(\Gamma, 2g^1_2) \) and subsequently \( R(C, D) \).

Step 1

Let \( Q \) be a Weierstrass point on \( \Gamma \). As in Theorem (9.4),

\[
R(\Gamma, Q) = C[s, t, u]/F_{18},
\]

where degree \((s, t, u) = (1, 2, 9)\) and \( F_{18} \) is the degree 18 relation given by

\[
F_{18} : u^2 = \varphi_{18}(s, t).
\]

Step 2

We can pick the generator \( t \in g^1_2 \) of the form \( t = s_1s_2 \) where \( s_i \) vanishes at \( Q_i \) for \( |Q_1 + Q_2| = g^1_2 \). We choose \( Q_1 = P_j \) in Case a and \( Q_1 = Q_2 = P \) in Case b. Then we can write down bases for \( H^0(4nQ) = H^0(2ng^1_2) \) for small \( n \) as in Table 5.

Table 5.

<table>
<thead>
<tr>
<th>( n )</th>
<th>generators of ( H^0(2ng^1_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( s^4, s^2t, t^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( s^{2n}4-n, n=0,\ldots,4 )</td>
</tr>
<tr>
<td>3</td>
<td>( s^{2n}6-n, n=0,\ldots,6, s^3u, stu )</td>
</tr>
<tr>
<td>4</td>
<td>( s^{2n}8-n, n=0,\ldots,8, s^{2k+1}t^3-ku, k=0,\ldots,3 )</td>
</tr>
<tr>
<td>5</td>
<td>( s^{2n}10-n, n=0,\ldots,10, s^{2k+1}t^5-ku, k=0,\ldots,5 )</td>
</tr>
<tr>
<td>6</td>
<td>( s^{2n}12-n, n=0,\ldots,12, s^{2k+1}t^7-ku, k=0,\ldots,7 )</td>
</tr>
</tbody>
</table>
The canonical linear system

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Remembering the relation \( F_{18} \) the table is self-explanatory and can be computed independently or with reference to \([R2]\).

Step 3

Recall that \( D = g_2^1 + P_1 + P_2 \) where \( |P_1 + P_2| = g_2^1 \) and that \( 2D = 2g_2^1 \).

Then \( R(C,D) \subset R(\Gamma, 2g_2^1) \) and moreover if \( f \in H^0(C, D) \) then \( f^2 \in R(\Gamma, 4g_1^1) \). It follows that

\[
\frac{f^2(P_1)}{s^8} = \frac{f^2(P_2)}{s^8}
\]

and so

\[
\frac{f(P_1)}{s^4} = \mp \frac{f(P_2)}{s^4}.
\]

By consideration of the dimensions of \( H^0(C, nD) \) we must have

\[
\frac{f(P_1)}{s^4} = -\frac{f(P_2)}{s^4}.
\]

It can then be seen that \( f \in R(\Gamma, 2g_2^1) \) is an element of \( R(C,D) \) if and only if

\[
\frac{f(P_1)}{s^{4d}} = -\frac{f(P_2)}{s^{4d}},
\]

where \( d \) = degree of \( f \)

Bearing in mind our choice of \( t \) we get the following table of generators.

<table>
<thead>
<tr>
<th>( n )</th>
<th>generators of ( H^0(nD) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_1 = s^2t, x_2 = t^2 )</td>
</tr>
<tr>
<td>2</td>
<td>( S^2(x_1), y_1 = s^8, y_2 = s^6t )</td>
</tr>
<tr>
<td>3</td>
<td>( S^3(x_1), x_1 y_1, z_1 = s^3u, z_2 = stu )</td>
</tr>
</tbody>
</table>

Using the definitions in Table 6 the following relations are seen to hold:

\[
\begin{align*}
& r_1 : x_1^2 - x_2 y_2, \\
& r_2 : x_1 y_2 - x_2 y_1.
\end{align*}
\]
As usual the degree 6 generators for $R(C,D)$ are made up from lower degrees. However at this stage the relation $F_{18}$ from the ring $R(\Gamma, Q)$ makes its presence felt in $R(C,D)$. Thus we get the 3 degree 6 relations (with a slight abuse of notation),

$$r_7: z_1^2 = s^6 \psi_{18},$$
$$r_8: z_1 z_2 = s^4 \psi_{18},$$
$$r_9: z_2^2 = s^2 \psi_{18}.$$

Case a

We can write out $\psi_{18}(s, t)$ as follows:

$$\psi_{18} = \lambda s^{18} + \mu s^{16} t + \lambda_1 s^{14} t^2 + \ldots + \lambda_5 t^8.$$

Then $r_7, r_8, r_9$ can be rewritten,

$$r_7: -z_1^2 + \lambda y_1^3 + \mu y_1^2 y_2 + x_1^2 h,$$
$$r_8: -z_1 z_2 + \lambda y_1^2 y_2 + \mu y_1 y_2^2 + x_1 x_2 h,$$
$$r_9: -z_2^2 + \lambda y_1 y_2^2 + \mu y_2^3 + x_2^2 h,$$
The canonical linear system

where

\[ h = \lambda_1 x_1^4 + \lambda_2 x_1^3 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1 x_2^3 + \lambda_5 y_1^2 + \lambda_6 y_1 y_2 + \lambda_7 y_2^2 + \lambda_8 x_1 y_2, \]

after renumbering the \( \lambda_i \in \mathbb{C} \).

Case b

Previously we chose \( t = \tau^2 \) for \( \tau \) vanishing at \( P \). In this case \( u = \tau v \) (\( v \) vanishes at the Weierstrass points missing \( P \) and \( Q \)) and so \( \varphi_{18} = t \varphi_{16} \).

This means that we can write

\[ \varphi_{18} = \mu s^{16} + \lambda_4 s^{14} t^2 + \ldots + \lambda_8 t^8. \]

The form of \( r_7, r_8, r_9 \) in Case b now follows from Case a.

Thus we have written

\[ R(C,D) = \mathbb{C}(x_1, x_2, y_1, y_2, z_1, z_2)/I, \]

where \( I \) is generated as in the statement of the theorem. It is a simple check to see that the determinant given does indeed produce the relations \( r_1, \ldots, r_6 \).

These relations satisfy the following syzygies \( \Sigma_1, \ldots, \Sigma_{16} \):

\[ \Sigma_1: x_1 r_3 - y_2 r_2 + y_1 r_1 = 0, \]
\[ \Sigma_2: x_2 r_3 - x_1^2 r_2 + y_2 r_1 = 0, \]
\[ \Sigma_3: x_1 r_6 - y_1 r_4 + z_1 r_2 = 0, \]
\[ \Sigma_4: x_1 r_5 - y_2 r_4 + z_1 r_1 = 0, \]
\[ \Sigma_5: x_2 r_5 - x_1^2 r_4 + z_2 r_1 = 0, \]
\[ \Sigma_6: x_2 r_6 - y_2 r_4 + z_2 r_2 = 0, \]
\[ \Sigma_7: y_2 r_6 - y_1 r_5 + z_1 r_3 = 0. \]
The canonical linear system

\[ \Sigma_8: x_1^2 r_6 - y_2 r_5 + z_2 r_3 = 0, \]
\[ \Sigma_9: -x_2 r_7 + x_1 r_8 = (\lambda y_1 + \mu y_2)y_1 r_2 - z_1 r_6, \]
\[ \Sigma_{10}: -x_2 r_8 + x_1 r_9 = (\lambda y_1 + \mu y_2)y_2 r_2 - z_2 r_4, \]
\[ \Sigma_{11}: -x_1^2 r_7 + y_2 r_8 = -x_1 h r_1 + (\lambda y_1 + \mu y_2)y_1 r_3 - z_1 r_5, \]
\[ \Sigma_{12}: -x_1^2 r_8 + y_2 r_9 = -x_2 h r_1 + (\lambda y_1 + \mu y_2)y_2 r_3 - z_2 r_5, \]
\[ \Sigma_{13}: -y_2 r_7 + y_1 r_8 = -x_1 h r_2 - z_1 r_6, \]
\[ \Sigma_{14}: -y_2 r_8 + y_1 r_9 = -x_2 h r_2 - z_2 r_6, \]
\[ \Sigma_{15}: -z_2 r_7 + z_1 r_8 = -x_1 h r_4 - (\lambda y_1 + \mu y_2)y_1 r_6, \]
\[ \Sigma_{16}: -z_2 r_8 + z_1 r_9 = -x_2 h r_4 - (\lambda y_1 + \mu y_2)y_2 r_6. \]

Remark (9.5)

The rings in Theorems (9.4) and (9.4a) have been presented in a format depending on certain matrices. That is in each case the relations are generated by

\[ \text{rank } A \leq 1 \]

and

\[ \text{AMA}^T = 0 \]

for a given 2x4 matrix A and a symmetric matrix M.

Given a polynomial ring P we make the observation that there is a variety defined by

\[ \text{rank } \begin{bmatrix} l_{11} & \ldots & l_{14} \\ l_{21} & \ldots & l_{24} \end{bmatrix} \leq 1 \]

and
The canonical linear system

\[
\begin{bmatrix}
R_7 & R_8 \\
R_8' & R_9
\end{bmatrix}
= \begin{bmatrix}
l_{11} & \cdots & l_{14} \\
l_{21} & \cdots & l_{24}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{14} & A_{24}
\end{bmatrix}
= 0
\]

for \(l_{ij}, A_{ij} \in \mathbb{P}\).

If \(I \subset \mathbb{P}\) is generated by the matrix conditions above we can write out the generators explicitly as follows:

\[R_{ij} : l_{1j}l_{2j} - l_{1i}l_{2i}, \quad 1 \leq i < j \leq 4, \text{ gives the 6 determinantal relations}\]

\[R_7 : \sum_{i=1}^{4} l_{1i}A_{1i},\]

\[R_8 : \sum_{i=1}^{4} l_{1i}A_{2i},\]

\[R_8' : \sum_{i=1}^{4} l_{2i}A_{1i},\]

\[R_9 : \sum_{i=1}^{4} l_{2i}A_{2i},\]

with the condition that \(R_8 = R_8' \text{ in } \mathbb{P}\). We make the obvious convention that \(R_{ij} = -R_{ji} \text{ for } i > j\).

The point is that these 9 relations will then satisfy the syzygies below:

\[S_{ijk} : l_{1j}R_{jk} - l_{1j}R_{ik} + l_{1k}R_{ij}, \quad 1 \leq i < j < k \leq 4,\]

\[T_{ijk} : l_{2j}R_{jk} - l_{2j}R_{ik} + l_{2k}R_{ij}, \quad 1 \leq i < j < k \leq 4,\]

give the 8 determinantal syzygies, and
The canonical linear system

\[ U_j : -l_{2j}R_8 + l_{1j}R_6 = \sum_{i,j} R_{ij}A_{1j}, \ 1 \leq j \leq 4, \]

\[ U_j' : -l_{2j}R_8 + l_{1j}R_9 = \sum_{i,j} R_{ij}A_{2j}, \ 1 \leq j \leq 4, \]

give the remaining 8. As remarked in the proof of Theorem (9.4) these last 8 syzygies are constructed from the identity

\[ ((A^\ast)A)MAT = A^\ast(AMA^T) \]

where

\[
A^\ast = \begin{bmatrix}
-l_{21} & l_{11} \\
& \\
& \\
-l_{24} & l_{14}
\end{bmatrix}
\]

The rings of Theorem (9.4) and (9.4a) are now of this form if we put the product

\[
MAT = \begin{bmatrix}
A_{11} & A_{12} \\
& \\
& \\
A_{14} & A_{24}
\end{bmatrix}
\]
10. Deformations of canonical curves.

In section 16 we calculate the number of moduli for surfaces of each of the 5 types and in some cases write down concretely a degeneration between surfaces of two different types. In this section we do the same thing for the canonical curves of the surfaces. These calculations serve as something of a model for the larger (surface) ones.

Theorem (10.1)

In the following diagram

\[ \text{I} \rightarrow \text{II} \rightarrow \text{III} \rightarrow \text{III}_a \rightarrow \text{III}_b, \]

for any given canonical curve of a surface of type B, C, there can be found a family \( \{C_t\}_t \) of curves, such that \( C_t \) is the canonical curve of a surface of type A for \( t \neq 0 \), and \( C_0 = C \) is the canonical curve of a surface of type B, whenever there is an arrow from A to B.

Proof.

In each case we simply write down a family of rings, \( R_t \), which are rings of the type calculated for curves of type A when \( t \neq 0 \) and a general ring of type B when \( t = 0 \).

In order for the corresponding deformation to be flat, the syzygies in \( R_t \) reduced modulo \( t \), must be equal to the syzygies in \( R_0 \).

Type I \( \rightarrow \) Type II

Consider the ring,

\[ R_t = \mathbb{C}[x_1, x_2, y_1, y_2, z]/I_t, \]

where \( I_t \) is generated by \( r_1^1, ..., r_5^2 \) given by the 4X4 Pfaffians of the following matrix (see Section 4):
The canonical linear system

\[
M_t = \begin{pmatrix}
0 & t & x_1 & x_2 & y_1 \\
t & 0 & y_1 & y_2 & z \\
-x_1 & -y_1 & 0 & -z & -A \\
-x_2 & -y_2 & z & 0 & B \\
-y_1 & -z & A & -B & 0
\end{pmatrix}
\]

Here \(A\) and \(B\) are as in Theorem (9.3).

First notice that this guarantees that \(R_t \mod t\) is the ring of a curve of type II. Let us write down the relations \(r_1, \ldots, r_5\).

\[
\begin{align*}
r_1^t : & \quad x_1y_2 - x_2y_1 + tz, \\
r_2^t : & \quad x_1z - y_1^2 + tA, \\
r_3^t : & \quad x_2z - y_1y_2 - tB, \\
r_4^t : & \quad y_1z - x_2A - x_1B, \\
r_5^t : & \quad z^2 - y_2A - y_1B.
\end{align*}
\]

Now, for \(t \neq 0\), \(r_1^t\) allows us to eliminate \(z\) from \(R_t\). If we do this systematically in the rest of the equations we get two degree 4 relations,

\[
\begin{align*}
r_2^t : & \quad -(x_1y_2 - x_2y_1)/t - y_1^2 + tA, \\
r_3^t : & \quad -(x_1y_2 - x_2y_1)/t - y_1y_2 - tB.
\end{align*}
\]
and find that these span $I_t$. In particular

\[ r_4 = \frac{x_1 r_3}{t} - x_2 r_2 / t, \]

\[ r_5 = \frac{y_1 r_3}{t} - y_2 r_2 / t. \]

So we see that for $t \neq 0$, $R_t = \mathbb{C}[x_1, x_2, y_1, y_2]/\langle Q_1, Q_2 \rangle$, as required.

**Type II → Type III**

In this case there are several *ad hoc* ways of writing down families of rings which satisfy our requirements. We give one more or less at random.

Let $R_t = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I_t$, where $I_t$ is generated by

\[ r_1^t : x_1^3 - x_2 y_1 + tz_1, \]
\[ r_2^t : x_2^3 - x_1 y_2 - t^2 \alpha x_2 y_1, \]
\[ r_3^t : y_1 y_2 - x_1^2 x_2^2 - tx_2 z_2, \]
\[ r_4^t : x_1 z_2 - x_2 z_1 + \alpha y_1^2, \]
\[ r_5^t : x_1^2 z_1 - y_1 z_2 + t \alpha x_1 x_2 y_1 z_1, \]
\[ r_6^t : x_2 z_2 - y_2 z_1 + t \alpha x_1^2 x_2 y_1, \]
\[ r_7^t : z_1^2 - \lambda y_1^3 - \mu x_1 x_2 y_1^2 - x_1^2 h, \]
\[ r_8^t : -z_1 z_2 + \lambda x_1^2 y_1^2 + \mu x_2^2 y_1^2 + x_1 x_2 h, \]
The canonical linear system

\[ r^1_0 : z^2 \lambda x^4 y - \mu y^3 - x^3 h + t^2 \lambda h y. \]

Again the notation used here corresponds to that in section 9. Using \( r^1 \) to eliminate \( z_1 \) when \( t \neq 0 \) we find that \( I_1 \) can be generated by the 4x4 Pfaffians of the following matrix.

\[
M_t = \begin{bmatrix}
0 & 0 & x_1 & x_2 & y_1 \\
0 & 0 & x^2_2 - t^2 \lambda y_1 & x_2 & x_1 x_2 + tz \\
-x_1 & -x_2^2 + t^2 \lambda y_1 & 0 & -x_1^2 x_2 + tz & t^2 \mu y^2_2 \\
-x_2 & -y_2 & x_1^2 x_2 - tz & 0 & -t^2 h + x^4_1 \\
-y_1 & -x_1^2 x_2 - tz & -t^2 \mu y^2_2 & t^2 h - x^4_1 & 0
\end{bmatrix}
\]

where we are writing \( z \) for \( Z_j \). \( M_t \) gives us the relations \( r^1_2, r^1_3, r^1_4, r^1_5, r^1_6 \) and we can write

\[
r^1_6 = -x_2 r^1_3 / t - x^2_1 r^1_2 / t.
\]

\[
r^1_7 = -x_1 r^1_4 / t - y_1 r^1_3 / t.
\]

\[
r^1_8 = x^2_1 r^1_4 / t + x_2 r^1_5 / t.
\]

Thus \( R_1 \) is a ring of type II as required.

Type III \( \rightarrow \) Type III_a

Consider the ring,

\[
R_1 = \mathbb{C}[x_1, x_2, y_1, y_2, x_1, x_2] / I_1
\]
The canonical linear system

where \( I_t \) is generated by

\[
\text{rank } M = \text{rank } \begin{bmatrix} x_1 & y_1 & y_2 & z_1 \\ x_2 & x_1^2 & y_1 + ty_2 & z_2 \end{bmatrix} \leq 1,
\]

and \( MPM^T = 0 \) for some symmetric \( P \).

For \( t = 0 \) this is clearly a ring of type IIIa. For \( t \neq 0 \) we can perform a sequence of basis changes and row and column operations on the matrix \( M \) until we are left with the presentation given in Theorem (9.4) for a ring of type III.

**Type IIIa \( \to \) Type IIIb**

The presentation of the rings given in Theorem (9.4a) makes the deformation clear in this case. Consider the rings

\[
R_\lambda = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I_\lambda,
\]

where \( I_\lambda \) is generated by \( r_{1,\lambda}, \ldots, r_{9,\lambda} \):

\[
r_{1,\lambda}, \ldots, r_{6,\lambda} \text{ given by }
\]

\[
\text{rank } \begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1,
\]

\[
r_{7,\lambda}, r_{8,\lambda}, r_{9,\lambda} \text{ given by }
\]

\[
r_7: z_1^2 - \lambda y_1^3 - \mu y_1^2 y_2 - x_1^2 h,
\]

\[
r_8: z_2 - \lambda y_1^2 y_2 - \mu y_1^2 - x_1 x_2 h,
\]

\[
r_9: z_2^2 - \lambda y_1^2 y_2 - \mu y_2^3 - x_2^2 h.
\]

Then \( R_\lambda \) is a ring of type IIIa and \( R_0 \) is a ring of type IIIb as required.

\[\Box\]
The degeneration II $\rightarrow$ III given is not a structured one. However we can write down a degeneration I $\rightarrow$ III in a structured way (10.2). We hope to find a 2-parameter degeneration I $\rightarrow$ III factoring through II at a later date but as yet this remains undiscovered.

**Proposition (10.2)**

The degeneration of type I curves to type III curves can be expressed by the ring

$$\mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I_t$$

where degree $(x_j, y_j, z_j) = (1, 2, 3)$ and $I_t$ is generated by 15 relations $\{r_{ij}\}_{1 \leq i < j \leq 1}$. $r_{ij}$ is derived from an antisymmetric $6 \times 6$ matrix, $M_t$, by deleting the $i$th and $j$th rows and columns and taking the Pfaffian of the $4 \times 4$ antisymmetric matrix remaining (see section 4). The matrix $M_t$ is shown below:

$$
\begin{bmatrix}
0 & t & y_1 & z_1 & x_1 & z_2 \\
. & 0 & y_2 & x_1^2 & x_2 & z_2 \\
. & . & 0 & -\lambda x_1 y_1^2 + \lambda x_1^2 x_2 y_1 - x_2 y_1 - \mu x_1 x_2^2 y_2 - x_1 y_2 + \lambda y_1^2 + \mu y_2^2 & -x_2 z_2 + x_1 z_1 + \theta & z_2 + t\lambda x_2 y_1 \\
. & . & . & 0 & -x_2 z_2 + x_1 z_1 + \theta & z_1 - t\mu x_1 y_2 \\
. & . & . & . & 0 & 0
\end{bmatrix}
$$

**Proof**

Let
be as in section (9.4).

It suffices to show that the ideal generated by $r_{12}, \ldots, r_{56}$ is equal to $I$ when $t = 0$ but when $t \neq 0$ we can eliminate $z_1$ and $z_2$ and be left with two quartics. Below is a list of the 15 relations generated from $M_4$:

$$r_{56} : \lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 + y_2 z_1 - x_2^2 z_2,$$

$$r_{46} : \mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 + x_1^2 z_1 - y_1 z_2,$$

$$r_{36} : t(x_2 z_2 - x_1 z_1 - th) + x_1^2 z_2 - y_1 z_2,$$

$$r_{26} : z_1(x_2 z_2 - x_1 z_1 - th) - x_2^2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 + x_1^2 z_1 - y_1 z_2),$$

$$\quad + y_1 (\lambda x_1 y_1^2 - \lambda x_1 x_2 y_1 + x_2 + y_2 z_1 - x_2^2 z_2),$$

$$r_{16} : z_2(x_2 z_2 - x_1 z_1 - th) - y_2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 + x_1^2 z_1 - y_1 z_2),$$

$$\quad + x_1^2(\lambda x_1 y_1^2 - \lambda x_1 x_2 y_1 + x_2 + y_2 z_1 - x_2^2 z_2),$$

$$r_{45} : -t(\lambda y_1^2 + \mu y_2^2) + x_2 z_1 - x_1 z_2,$$

$$r_{35} : -t(z_2 + t\lambda x_2 y_1) + x_2^3 - x_1 y_2,$$

$$r_{25} : -z_1(z_2 + t\lambda x_2 y_1) + x_2^2(\lambda y_1^2 + \mu y_2^2) + x_1(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 + y_2 z_1 - x_2^2 z_2),$$

$$r_{15} : -z_2(z_2 + t\lambda x_2 y_1) + y_2(\lambda y_1^2 + \mu y_2^2) + x_2(\lambda x_1 y_1^2 - \lambda x_1^2 x_2 y_1 + x_2 + y_2 z_1 - x_2^2 z_2),$$

$$r_{34} : -t(z_1 - t\lambda x_1 y_2) + x_2 y_1 - x_1^3.$$
The canonical linear system

\[ r_{24} : -z_1(z_1 - \mu x_1 y_2) + y_1(\lambda y_1^2 + \mu y_2^2) + x_1(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h), \]
\[ r_{14} : -z_2(z_1 - \mu x_1 y_2) + x_2(\lambda y_1^2 + \mu y_2^2) + x_2(\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h), \]
\[ r_{23} : -x_2(z_1 - \mu x_1 y_2) + y_1(z_2 + \lambda x_2 y_1) + x_1(x_2 z_2 - x_1 z_1 - th), \]
\[ r_{13} : -y_2(z_1 - \mu x_1 y_2) + x_1(z_2 + \lambda x_2 y_1) + x_2(x_2 z_2 - x_1 z_1 - th), \]
\[ r_{12} : (\lambda x_1 y_1^2 - \lambda x_2^2 x_2 y_1 + x_2 h)(z_1 - \mu x_1 y_2) - (\mu x_2 y_2^2 - \mu x_1 x_2^2 y_2 + x_1 h)(z_2 + \lambda x_2 y_1) + (\lambda y_1^2 + \mu y_2^2)(x_2 z_2 - x_1 z_1 - th). \]

We make the following definitions:

\[ r_1^1 : -r_{34}, \]
\[ r_2^1 : r_{35}, \]
\[ r_3^1 : -r_{45}, \]
\[ r_4^1 : -r_{36}, \]
\[ r_5^1 : -r_{46}, \]
\[ r_6^1 : -r_{56}, \]
\[ r_7^1 : -r_{24} - \mu x_2 y_2^2 r_{35} - \mu y_2^2 r_{36}, \]
\[ r_8^1 : -r_{25} + \lambda x_2 y_1 r_{34}. \]
The canonical linear system

\[ r_5^5 = -r_{15} + \lambda x_1 y_1 r_{34} - r_{36}. \]

It is now easy to check that \( r_i^0 = r_i \) and that those \( r_{ij} \) which we have not used in the above definitions are generated by the others. The Proposition follows easily.

\[ \square \]
11. Form of extension calculations

As can be seen in section 7 extension-deformation calculations, when worked through fully, are long even in simple cases (such as Pinkham's example). In this chapter we work through the calculations extending the relatively complicated rings given in section 9. This makes the proofs of Theorems (13.1), (14.1) and (15.1) extremely long. For this reason we have kept to a strict scheme when making the statements and proofs of these theorems and to aid the reader we present this scheme below.

Scheme for Theorems in Chapter 4.

Statement of theorem is of the form

Let \( X \) be a surface of type I (II, III, IIIa, IIIb). Then the presentation for \( R(C, D) \) in Section 9 is formative and informative in degree \(-1\) (see section 8). In fact

\[
R(X, K_X) = \mathfrak{C}x_0, \ldots, 1/\mathfrak{I}
\]

where \( \mathfrak{I} \) is generated by \( \mathfrak{r}_1, \ldots, \mathfrak{r}_n \) as follows:

Description of \( \mathfrak{r}_1, \ldots, \mathfrak{r}_n \)

Proof of theorem is of the form

Reproduction of \( R(C, D) \)

The first item in the proof is a reproduction of the format of the ring \( R = R(C, D) \) for \( C \in K_X \), as given in Section 9. We then observe that this format is formative. To prove the theorem we need to extend this ring. That is, if \( R \) is related by \( r_1, \ldots, r_n \) with syzygies \( \Sigma_1, \ldots, \Sigma_m \), we want to write
such that the \( \tilde{f}_i \) satisfy syzygies \( \Sigma_j \) which reduce mod \( x_0 \) to \( \Sigma_j \).

**Step 0 : Simplifications**

It is possible to make simplifying coordinate changes to reduce the possible choices of the \( a_i \). This has the effect of shortening the calculations considerably.

**Step 1 : \( R \rightarrow R^{(2)} \)**

We make the extension \( R \rightarrow R^{(2)} \) where \( R^{(2)}/(x_0) = R \) as outlined in Section 6. This involves taking the syzygies one at a time and finding the \( r^{(2)}_i \) (\( i = 1,\ldots,n \)) which will lift them. It is usual that after lifting just 5 or 6 syzygies the expressions for \( r^{(2)}_i \) (\( i = 1,\ldots,n \)) can be written in the same formative presentation as the \( t_i \) (\( i = 1,\ldots,n \)). This means that the other syzygies will automatically be lifted by these expressions.

**Step 2 : \( R^{(2)} \rightarrow R^{(3)} \)**

We make the extension \( R^{(1)} \rightarrow R^{(2)} \) where \( R^{(2)}/(x_0) = R \). Again we put the relations in the ring \( R^{(2)} \) into the format of the relations in \( R \), if this is possible.

**Step 3 : \( R^{(d)} \rightarrow R^{(d+1)} \)**

Having made the extension up to \( R^{(d)} \) we extend to \( R^{(d+1)} \). The algorithm will certainly have terminated in \( n \) steps, where \( n \) is the maximum degree of the generators of the module of syzygies. In fact keeping a formative presentation will guarantee that it terminates in many fewer steps than this.
12. Surfaces of type I.

Theorem (12.1)

Let $X$ be a surface of general type with $p_g=3$, $K_X^2=4$ and such that the canonical linear system $|K_X|$ has no base points. Then

$$R(X, K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2]/\langle R_1^{(4)}, R_2^{(4)} \rangle$$

where degree $(x_i, y_j) = (1, 2)$ and $R_1^{(4)}, R_2^{(4)}$ are weighted homogeneous polynomials of degree 4. Hence $X$ is the complete intersection $X_{4,4} \subset \mathbb{P}(1^3, 2^2)$.

Remark (12.2)

The generalised hyperplane section principle (Proposition (6.1)) tells us that in each degree there are the same number of generators and relations in $R(X, K_X)$ as there are in $R(C, D)$. This means that $R(X, K_X)$ is a Gorenstein grade 2 ring and the theorem follows immediately from this (Proposition (4.1)). However since the result is such an easy one the proof given below has been included to to illustrate the extension–deformation method of Section 6 in a transparent way.

Proof of (12.1)

Let $C \in |K_X|$ be a nonsingular curve, given by the vanishing of $x_0 \in H^0(X, K_X)$. In Theorem (9.2) we found that:

$$R(C, D) = \mathbb{C}[x_1, x_2, y_1, y_2]/\langle Q_1, Q_2 \rangle,$$

where $Q_1(x_1, x_2, y_1, y_2), Q_2(x_1, x_2, y_1, y_2)$ are weighted homogeneous degree 4 polynomials.

Step 0 : Simplifications

As the calculation is very simple we do not need to simplify by making coordinate changes.
Step 1: $R \rightarrow R^{(2)}$

We make an extension of the ring $R(C,D)$ to one $R(2C,D^{(2)})$. That is we must find the most general $\beta: \langle Q_1, Q_2 \rangle \rightarrow R(C, D)$ such that $\beta(Q_1)$ and $\beta(Q_2)$ are degree 3 polynomials and

$$Q_1 \beta(Q_2) - Q_2 \beta(Q_1) \in \langle Q_1, Q_2 \rangle.$$

Of course this equation holds for arbitrary degree 3 polynomials so we can write,

$$R(2C, D^{(2)}) = C[\xi_0, \xi_1, \xi_2, \xi_3, \xi_4] / (\Pi^{(2)}, x_0^2),$$

where $\Pi^{(2)}$ is generated by

$$r_1^{(2)}: Q_1 - \xi_0 \beta(Q_1),$$

$$r_2^{(2)}: Q_2 - \xi_0 \beta(Q_2).$$

Step 2: $R^{(2)} \rightarrow R^{(3)}$

Similarly for the second order extension we find $\beta_2: \langle Q_1, Q_2 \rangle \rightarrow R(C, D)$ such that $\beta_2(Q_1)$ and $\beta_2(Q_2)$ are degree 2 polynomials and

$$Q_1 \beta_2(Q_2) - Q_2 \beta_2(Q_1) \in \langle Q_1, Q_2 \rangle.$$ 

Again this is true for any choice of degree 2 polynomials.

Step 3: $R^{(4)} \rightarrow R^{(4+1)}$

Repeating the process twice more gives us $R^{(5)}$ of the form,

$$R^{(5)} = C[\xi_0, \xi_1, \xi_2, \xi_3, \xi_4] / (R_1^{(4)}, R_2^{(4)}, x_0^5),$$

where $R_1^{(4)}$ and $R_2^{(4)}$ are weighted degree 4 polynomials.

This yields the theorem immediately.
Theorem (13.1)

Let $X$ be a surface of general type with $p_g=3$, $K_X^2=4$ and such that the canonical linear system has one base point. Then the Pfaffian presentation of the ring $R(C, D)$ calculated in Theorem (9.3) is formative and informative in degree $-1$. The canonical ring of a surface of type II is of the form:

$$R(X, K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/\mathcal{I}$$

where $\mathcal{I}$ is generated by $t_1, \ldots, t_5$ corresponding to the $4 \times 4$ Pfaffians of the matrix $M$ [cf. curve case (9.3)],

$$M = \begin{bmatrix}
0 & 0 & x_1 & x_2 & y_1 + \lambda_3 x_0^2 \\
0 & 0 & y_1 & y_2 & z \\
-x_1 & -y_1 & 0 & -z + x_0 f_1 & -A + x_0 f_2 \\
-x_2 & -y_2 & z - x_0 f_1 & 0 & B + x_0 f_3 \\
-y_1 - \lambda_3 x_0^2 & -z & A - x_0 f_2 & -B - x_0 f_3 & 0
\end{bmatrix}$$

Here $A$ and $B$ are as in Theorem (9.3), $\lambda_3 \in \mathbb{C}$ and $f_1, f_2, f_3$ are all polynomials with coefficients in $\mathbb{C}$ written out in full below.

$$f_1 = t_3 x_1^2 + u_2 x_0^2.$$
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\[ f_2 = -t_4x_1^3 - t_5x_1^2x_2 - t_6x_1x_2^2 - t_7x_2^3 - t_8x_1y_2 - t_9x_2y_2 - t_{10}z - t_{11}x_0^3 - u_3x_0^2x_1 - q_6x_0x_2^2 - q_7x_0x_1x_2 - q_8x_0x_1^2 - q_{10}x_0y_2. \]

\[ f_3 = t_1x_1y_2 + t_2x_2y_2 + t_{11}x_1^3 + v_3x_0 + u_1x_0^2x_1 + q_11x_0x_1^2 + q_2x_0y_2 + q_4x_0y_1. \]

The relations thus given are precisely

\[ \tilde{r}_1 : x_1y_2 - x_2y_1. \]
\[ \tilde{r}_2 : x_1z - y_1^2 - \lambda_5x_0^2y_1. \]
\[ \tilde{r}_3 : x_2z - y_1y_2 - \lambda_5x_0^3y_2. \]
\[ \tilde{r}_4 : (y_1 + \lambda_5x_0^2)(z - x_0f_1) - x_2(A - x_0f_2) - x_1(B + x_0f_3). \]
\[ \tilde{r}_5 : z(z - x_0f_1) - y_2(A - x_0f_2) - y_1(B + x_0f_3). \]

These relations satisfy the syzygies, \( \tilde{r}_1, \ldots, \tilde{r}_5 \) below given by:

\[ \tilde{s}_1 : (y_1 + \lambda_5x_0^2)r_1 - x_2r_2 + x_1r_3 = 0 \]
\[ \tilde{s}_2 : zr_1 - y_2r_2 + y_1r_3 = 0 \]
\[ \tilde{s}_3 : (A - x_0f_2)r_1 - (z - x_0f_1)r_2 + y_1r_4 - x_1r_5 = 0 \]
\[ \tilde{s}_4 : (B + x_0f_3)r_1 + (z - x_0f_1)r_3 - y_2r_4 + x_2r_5 = 0 \]
\[ \tilde{s}_5 : -(B + x_0f_3)r_2 - (A - x_0f_2)r_3 + zr_4 - (y_1 + \lambda_5x_0^2)r_5 = 0 \]
Remark (13.2)

By the general hyperplane section principle (Proposition (6.1)) we know the number of generators and relations in each degree for the ring $R(X, K_X)$. Observing that the ring must be Gorenstein and codimension 3 the structure theorems in section 4 tell us that the ideal of relations is generated by Pfaffians. We prove this explicitly by using the extension-deformation algorithm.

Proof of (13.1)

Let $C \in |K_X|$ be the general nonsingular curve given by the vanishing of $x_0 \in H^0(X, K_X)$, and $D = K_XC$. In Theorem (9.3) we showed that,

$$R(C,D) = \bigoplus_{n=0}^{\infty} H^0(C, nD) = \mathbb{C}(x_1, x_2, y_1, y_2, z)/I,$$

where $\deg (x_1, y_1, z) = (1, 2, 3)$ and $I$ is generated by

$\begin{align*}
   r_1 &: x_1 y^2 - x_2 y_1 \\
   r_2 &: x_1 z - y_1^2 \\
   r_3 &: x_2 z - y_1 y_2 \\
   r_4 &: y_1 z - x_2 A - x_1 B, \\
   r_5 &: z^2 - y_2 A - y_1 B,
\end{align*}$

where

$\begin{align*}
   A &= a_1 y_2^2 + a_2 x_2^2 y_2 + a_3 x_2^4, \quad a_i \in \mathbb{C}
\end{align*}$

and

$\begin{align*}
   B &= B(x_1, x_2, y_1, y_2) \text{ is a general degree 4 polynomial in the weighted coordinates } x_1, x_2, y_1, y_2.
\end{align*}$

We have already written these down in Theorem (9.3) as the diagonal 4x4 Pfaffians of the following matrix,
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\[
\begin{pmatrix}
0 & 0 & x_1 & x_2 & y_1 \\
0 & 0 & y_1 & y_2 & z \\
-x_1 & -y_1 & 0 & -z & -A \\
-x_2 & -y_2 & z & 0 & B \\
-y_1 & -z & A & -B & 0
\end{pmatrix}
\]

The syzygies between the relations are,

\begin{align*}
s_1 &: x_1 r_3 - x_2 r_2 + y_1 r_1 = 0, \\
s_2 &: y_1 r_3 - y_2 r_2 + z r_1 = 0, \\
s_3 &: x_1 r_5 - y_1 r_4 - z r_2 + A r_1 = 0, \\
s_4 &: x_2 r_5 - y_2 r_4 - z r_3 - B r_1 = 0, \\
s_5 &: y_1 r_5 - z r_4 - A r_3 - B r_2 = 0.
\end{align*}

It is easy to notice here that the Pfaffian format is formative (section 8).

Having constructed \( R(C,D) \) we use the algorithm in Proposition (6.2) to calculate the ring

\[ R(X,K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/I, \]

where \( I \) is generated by,

\begin{align*}
\tilde{r}_1 &= x_1 y_2 - x_2 y_1 - \alpha x_0, \\
\tilde{r}_2 &= x_1 z - y_1^2 - \beta_1 x_0, \\
\tilde{r}_3 &= x_2 z - y_1 y_2 - \beta_2 x_0, \\
\tilde{r}_4 &= y_1 z - x_2 A - x_1 B - y x_0, \\
\tilde{r}_5 &= z^2 - y_2 A - y_1 B - \delta x_0.
\end{align*}
Here deg (α, β, γ, δ) = (2, 3, 4, 5), and the $\tilde{s}_i$ satisfy syzygies $\tilde{s}_i$ which reduce mod $x_0$ to $s_i$.

Step 0: Simplifications.

We make some coordinate changes. For example consider

$$\lambda_1 : x_1 y_2 - x_2 y_1 - \alpha x_0.$$  

By judicious changes of coordinates in $x_1, x_2, y_1$ and $y_2$ we can ensure that

$$\lambda_1 : x_1 y_2 - x_2 y_1 - \alpha x_0^3$$

where $\alpha \in \mathbb{C}$.

Similarly, looking at

$$\lambda_2 : x_1 x_2 - y_1^2 - \beta_1 x_0$$

if $\beta_1 = \beta x_2 y_1 + b_1'$ then using $\lambda_1$ we can write

$$\beta_1 = \beta x_1 y_2 - \alpha \beta x_0^3 + b_1'$$

so we may assume that $\beta_1$ has no terms in $x_2 y_1$. By change of coordinates in $z$ we may also assume that $\beta_1$ has no monomials in which $x_1$ appears.

So, we have arranged that

$$\lambda_2 : x_1 x_2 - y_1^2 - \lambda_1 x_0 x_2^3 - \lambda_2 x_0 x_2 y_1 - \lambda_3 x_0 z - \lambda_4 x_2^2 - \lambda_5 x_1 y_1 - \lambda_6 x_0 y_2 - \lambda_7 x_0 x_2 - \lambda_8 x_0^4, \lambda_1 \in \mathbb{C}.$$
By using \( \tau_1 \) we can assume that \( \beta_2 \) has no monomial in \( x_2y_1 \) and using all the relations we can assume that \( x_2y_1, y_1^2, y_1y_2 \) and \( y_1z \) do not appear in \( \gamma \) or \( \delta \).

**Step 1 :** \( R \to R^{(2)} \)

We begin our construction of

\[
R(2C,D^{(2)}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/(x_0^2, I^{(2)})
\]

where \( I^{(2)} \) is generated by,

\[
\begin{align*}
\tau_1^{(2)} : x_1y_2 - x_2y_1 - \alpha'x_0, \\
\tau_2^{(2)} : x_1z - y_1^2 - \beta'x_0, \\
\tau_3^{(2)} : x_2z - y_1y_2 - \beta_2x_0, \\
\tau_4^{(2)} : y_1z - x_2A - x_1B - \gamma'x_0, \\
\tau_5^{(2)} : z^2 - y_2A - y_1B - \delta'x_0,
\end{align*}
\]

where \( \deg(\alpha'(x_1,x_2,y_1,y_2)), \beta'(x_1,x_2,y_1,y_2,z), \gamma'(x_1,x_2,y_1,y_2,z), \delta'(x_1,x_2,y_1,y_2,z) \) = (2, 3, 4, 5). These satisfy the syzygies \( s_1^{(2)} .... s_5^{(2)} \) such that \( s_1^{(2)} \) reduced mod \( x_0 \) is \( s_1 \). By Step 0 we notice that:

\[
\begin{align*}
\tau_1^{(2)} : x_1y_2 - x_2y_1, \\
\tau_2^{(2)} : x_1z - y_1^2 - (\lambda_1x_0x_2^3 + \lambda_2x_0x_2y_2 + \lambda_3x_0z).
\end{align*}
\]
Let us look at the syzygy $s_1$ which must lift to a syzygy, which we call $s^{(2)}_1$. This means that

$$x_1 r_3^{(2)} - x_2 r_2^{(2)} + y_1 r_1^{(2)} = x_0 S$$

where $S \in I^{(2)}$, that is,

$$x_1 \beta_2' - \lambda_1 x_2^4 - \lambda_2 x_2^2 y_2 - \lambda_3 x_2 z \in I$$

is of degree 4. Equating the left-hand side of this with a general degree 4 element of $I$ we have

$$x_1 \beta_2' - \lambda_1 x_2^4 - \lambda_2 x_2^2 y_2 - \lambda_3 x_2 z = p_1 r_3 + p_2 r_2 + x_1 p_3 r_1 + x_2 p_4 r_1$$

$$= p_1 x_2 z - p_1 y_1 y_2 + p_2 x_1 z - p_2 y_1^2$$

$$+ p_3 x_1^2 y_2 - p_3 x_1 x_2 y_1 + p_4 x_2^2 y_1$$

Comparing coefficients it is easy to see that $\lambda_1 = \lambda_2 = \lambda_3 = \beta_2' = 0$.

We have shown that in $R(2C,D^{(2)})$ we are forced to put $\alpha' = \beta_1 = \beta_2' = 0$ in order for the syzygy $s_1$ to lift to a syzygy in $R(2C,D^{(2)})$. In a similar way consider the syzygy $s_3$ which lifts to a syzygy $s^{(2)}_3$. This tells us that

$$x_1 \delta - y_1 y \in I$$

is of degree 6. That is
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\[ x_1 \delta - y_1 \gamma = t_1 r_5 + t_2 x_1 r_4 + t_3 r_4 + t_4 x_1^2 r_3 + t_5 x_1 x_2 r_3 + \]
\[ t_6 x_2^2 r_3 + t_7 y_1 r_3 + t_8 y_2 r_3 + t_9 x_1^2 r_2 + t_{10} x_1 x_2 r_2 + \]
\[ + t_{11} x_2^2 r_2 + t_{12} y_1 r_2 + t_{13} y_2 r_2 + t_{14} x_1^3 r_1 + \]
\[ t_{15} x_1^2 x_2 r_1 + t_{16} x_1 x_2^2 r_1 + t_{17} x_2^3 r_1 + t_{18} x_1 y_1 r_1 + \]
\[ t_{19} x_1 y_2 r_1 + t_{20} x_2 y_1 r_1 + t_{21} x_2 y_2 r_1 + t_{22} y r_1. \]

We can assume that
\[ \delta = \delta' + \delta'', \]
\[ \gamma = \gamma' + \gamma'', \]
where
\[ x_1 \delta' = y_1 \gamma'. \]

So there is some \( f \in C[x_1, x_2, y_1, y_2, z] \) of degree 3, such that \( \delta' = y_1 f \) and \( \gamma' = x_1 f \). Given our restrictions on \( \delta \) we can assume that \( f = \phi x_1^3, \phi \in C \).

So we equate coefficients in,
\[ x_1 \delta'' - y_1 \gamma'' = t_1 r_5 + t_2 x_1 r_4 + t_3 r_4 + t_4 x_1^2 r_3 + t_5 x_1 x_2 r_3 + \]
\[ t_6 x_2^2 r_3 + t_7 y_1 r_3 + t_8 y_2 r_3 + t_9 x_1^2 r_2 + t_{10} x_1 x_2 r_2 + \]
\[ + t_{11} x_2^2 r_2 + t_{12} y_1 r_2 + t_{13} y_2 r_2 + t_{14} x_1^3 r_1 + \]
\[ t_{15} x_1^2 x_2 r_1 + t_{16} x_1 x_2^2 r_1 + t_{17} x_2^3 r_1 + t_{18} x_1 y_1 r_1 + \]
\[ t_{19} x_1 y_2 r_1 + t_{20} x_2 y_1 r_1 + t_{21} x_2 y_2 r_1 + t_{22} y_1 r_1. \]
and discover that $t_1 = 0, t_2 = 0, t_3 = 0, t_6 = 0, t_7 = 0, t_8 = 0, t_{10} = 0, t_{11} = 0, t_{12} = 0, t_{13} = 0, t_{18} = 0$ and $t_{20} = 0$.

Also

$$
\gamma = \varphi x_1^4 + t_4 x_1^2 y_2 + t_5 x_1 x_2 y_2 + t_9 x_1^2 y_1 + t_{14} x_1^3 x_2 + t_{15} x_1^2 x_2^2 + t_{16} x_1 x_2^3 + t_{17} x_2^4 + t_{19} x_1 x_2 y_2 + t_{21} x_2^2 y_2 + t_{22} x_2 y_2 z.
$$

$$
\delta = \varphi x_1^3 y_1 + t_4 x_1 x_2 z + t_5 x_2^2 z + t_9 x_1^2 z + t_{14} x_1^3 y_2 + t_{15} x_1^2 x_2 y_2 + t_{16} x_1 x_2^3 y_2 + t_{17} x_2^3 y_2 + t_{19} x_1 y_2 z + t_{21} x_2 y_2^2 + t_{22} y_2 z.
$$

It is now a relatively simple matter to check that this $\gamma$ and $\delta$ do in fact give a lift of $s_4$ and $s_5$ to $s_4^{(2)}$ and $s_5^{(2)}$ respectively:

$$
s_4^{(2)} : x_2 \delta - y_2 \gamma \in 1,
$$

$$
s_5^{(2)} : y_1 \delta - z \gamma \in 1.
$$

Thus we have computed

$$
R(2C,D^{(2)}) = C[x_0, x_1, x_2, y_1, y_2, z]/(x_0, 1^{(2)})
$$

where, after relabelling the coefficients in $\gamma$ and $\delta$, $1^{(2)}$ is generated by

$$
r_1^{(2)} : x_1 y_2 - x_2 y_1,
$$

$$
r_2^{(2)} : x_1 z - y_1^2,
$$

$$
r_3^{(2)} : x_2 z - y_1 y_2.
$$
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\[ r^{(2)}_4 : y_1 z - x_2 A - x_1 B - t_1 x_0 x_2^2 y_2 - t_2 x_0 x_1 x_2 y_2 - t_3 x_0 x_1^2 y_1 - \]
\[ t_4 x_0 x_1^3 x_2 - t_5 x_0 x_1^2 x_2^2 - t_6 x_0 x_1 x_2^3 - t_7 x_0 x_2^4 - t_8 x_0 x_1 x_2 y_2 - t_9 x_0 x_2^2 y_2 - t_{10} x_0 x_2 z - t_{11} x_0 x_4, \]

\[ r^{(2)}_5 : z^2 - y_2 A - y_1 B - t_1 x_0 x_1 x_2 z - t_2 x_0 x_2^2 z - t_3 x_0 x_1^2 z - t_4 x_0 x_1^3 y_2 - t_5 x_0 x_1^2 x_2 y_2 - t_6 x_0 x_1 x_2^2 y_2 - t_7 x_0 x_2^3 y_2 - t_8 x_0 x_1 y_2^2 - t_9 x_0 x_2 y_2^2 - t_{10} x_0 y_2 z - t_{11} x_0 x_1^3 y_1. \]

These relations satisfy the following syzygies

\[ s^{(2)}_1 : x_1 r^{(2)}_3 - x_2 r^{(2)}_2 + y_1 r^{(2)}_1 = 0, \]

\[ s^{(2)}_2 : y_1 r^{(2)}_3 - y_2 r^{(2)}_2 + z r^{(2)}_1 = 0, \]

\[ s^{(2)}_3 : x_1 r^{(2)}_3 - y_1 r^{(2)}_4 - z r^{(2)}_2 + A r^{(2)}_1 - t_1 x_0 x_1^2 r^{(2)}_3 - t_2 x_0 x_1 x_2 r^{(2)}_3 - t_3 x_0 x_1^2 r^{(2)}_2 - t_4 x_0 x_1 x_2 r^{(2)}_2 - t_5 x_0 x_1^2 r^{(2)}_1 - t_6 x_0 x_1 x_2 r^{(2)}_1 - t_7 x_0 x_1 y_2 r^{(2)}_1 - t_8 x_0 x_2 y_2 r^{(2)}_1 - t_{10} x_0 z r^{(2)}_1 = 0. \]
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\[ s_4^{(2)} : x_2 r_3^{(2)} - y_2 r_4^{(2)} - z r_3^{(2)} + B r_1^{(2)} + t_1 x_0 x_1 y_2 r_1^{(2)} - 
\]
\[ t_1 x_0 x_1 x_2 r_3^{(2)} + t_2 x_0 x_2 y_2 r_1^{(2)} - t_2 x_0 x_2 r_3^{(2)} - t_3 x_0 x_1 r_1^{(2)} + 
\]
\[ t_1 x_0 x_1 r_1^{(2)} = 0, 
\]
\[ s_3^{(2)} : y_1 r_3^{(2)} - z r_4^{(2)} - A r_3^{(2)} - B r_2^{(2)} + t_1 x_0 x_1 z r_1^{(2)} + t_2 x_0 x_2 z r_1^{(2)} 
\]
\[ + t_4 x_0 x_1^2 r_3^{(2)} + t_5 x_0 x_1^2 x_2 r_3^{(2)} + t_6 x_0 x_1 x_2^2 r_3^{(2)} + t_7 x_0 x_1^2 r_3^{(2)} + 
\]
\[ t_8 x_0 x_1 y_2 r_3^{(2)} + t_9 x_0 x_2 y_2 r_3^{(2)} + t_{10} x_0 z r_3^{(2)} + t_{11} x_0 x_1^3 r_2^{(2)} 
\]
\[ = 0. 
\]

Notice that these relations can be written in the Pfaffian form as previously. The following matrix yields \( r_1^{(2)}, \ldots, r_3^{(2)} \) as its diagonal 4x4 Pfaffians:

\[
M^{(2)} = \begin{vmatrix}
0 & 0 & x_1 & x_2 & y_1 \\
0 & 0 & y_1 & y_2 & z \\
-x_1 & -y_1 & 0 & -z + x_0 p_1 & -A + x_0 p_2 \\
-x_2 & -y_2 & z - x_0 p_1 & 0 & B + x_0 p_3 \\
-y_1 & -z & A - x_0 p_2 & -B - x_0 p_3 & 0 \\
\end{vmatrix}
\]

where

\[ p_1 = t_3 x_1^2. \]
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\[ p_2 = -t_4x_1^3 - t_5x_1^2x_2 - t_6x_1x_2^2 - t_7x_2^3 - t_8x_1y_2 - t_9x_2y_2 - t_{10}z. \]

\[ p_3 = t_1x_1y_2 + t_{11}x_1^3. \]

**Step 2 :** \( R(2) \to R(3) \)

Using this ring we extend \( R(2C,D^{(2)}) \) to \( R(3C,D^{(3)}) = C[x_0, x_1, x_2, y_1, y_2, z]/(x_0^3, I^{(3)}) \), where \( I^{(3)} \) is generated by,

\[ r_1^{(3)} : x_1y_2 - x_2y_1, \]
\[ r_2^{(3)} : x_1z - y_1^2 - \lambda_4x_0^2x_2 - \lambda_5x_0^2y_1 - \lambda_6x_0^2y_2, \]
\[ r_3^{(3)} : x_2z - y_1y_2 - \beta^2x_0^2, \]
\[ r_4^{(3)} : y_1z - x_2A - x_1B - t_1x_0x_1y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1z - t_4x_0x_1^2z - t_5x_0^2x_2 - t_6x_0^2x_2z - t_7x_0^2y_2 - t_8x_0^2x_1z - t_9x_0^2x_1z - t_{10}x_0x_1^2y_2 - t_{11}x_0x_2 - t_{11}x_0x_1^2. \]
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satisfying $s^{(3)}_1$ which reduce mod $x_0^2$ to $s^{(2)}_1$, $i = 1, \ldots, 5$.

If we consider $s^{(3)}_1$ we find that $\beta''_1 \left( = \lambda_4 x_0^2 x_2^2 + \lambda_5 x_0^2 y_1 + \lambda_6 x_0^2 y_2 \right)$
and $\beta''_2$ must satisfy

$$x_1 \beta''_2 - x_2 \beta''_1 \in I$$

which is of degree 3. Hence it can easily be seen that $\lambda_4 = \lambda_6 = 0$ and $\beta''_2$
$= \lambda_5 y_2$. Similarly $s_2$ gives

$$x_1 \beta''_2 - x_2 \beta''_1 \in I$$

but

$$x_1 \beta''_2 - x_2 \beta''_1 = \lambda_5 y_1 y_2 - \lambda_5 y_1 y_2 = 0,$$

as required.

We must also check that $s_3$, $s_4$ and $s_5$ can be lifted using these values
and calculate $\gamma'$ and $\delta''$. $s_3$ renders

$$x_1 \delta'' - y_1 \gamma'' + \lambda_5 z y_1 \in I$$

of degree 5. That is

$$x_1 \delta'' - y_1 \gamma'' + \lambda_5 z y_1 = q_1 r_4 + q_2 x_1 r_3 + q_3 x_2 r_3 + q_4 x_1 r_2$$

$$+ q_5 x_2 r_2 + q_6 x_1^2 r_1 + q_7 x_1 x_2 r_1 +$$

$$q_8 x_2 r_1 + q_9 y_1 r_1 + q_{10} y_2 r_1.$$ 

Equating coefficients we have

$$q_1 = q_3 = q_5 = q_9 = 0,$$
\[ \gamma'' = -\lambda_5 z + q_2 x_1 y_2 + q_4 x_1 y_1 + q_6 x_1^2 x_2 + q_7 x_1^2 x_2 + q_8 x_2^2 + q_{10} x_2 y_2 + q_{11} x_1^2. \]

\[ \delta'' = q_2 x_2 z + q_4 x_1 z + q_6 x_1^2 y_2 + q_7 x_1 x_2 y_2 + q_8 x_2^2 y_2 + q_{10} y_2^2 + q_{11} x_1^2 y_1. \]

Substituting these values into similar equations derived from \( s_4 \) and \( s_5 \) shows that we have indeed constructed

\[ R(3C,D^{(3)}) = \mathcal{C}(x_0, x_1, x_2, y_1, y_2, z) / \langle s_0, l^{(3)} \rangle, \]

where \( l^{(3)} \) is generated by,

\( r^{(3)}_1: x_1 y_2 - x_2 y_1. \)

\( r^{(3)}_2: x_1 z - y_1^2 - \lambda_3 x_0^2 y_1. \)

\( r^{(3)}_3: x_2 z - y_1 y_2 - \lambda_5 x_2^2 y_2. \)

\( r^{(3)}_4: (y_1 - \lambda_3 x_2)(z - t_3 x_0 x_1^2) - x_2(A - x_0(-t_4 x_1^3 - t_5 x_1^2 x_2 - t_6 x_1 x_2^2 - t_7 x_2^3 - t_8 x_1 y_2 + t_9 x_2 y_2 - t_{10} y_2^2 - q_6 x_0 x_1^2 - q_7 x_0 x_1 x_2 - q_8 x_0 x_2^2 - q_{10} x_0 y_2)) - x_1(B + x_0(t_1 x_1 y_2 + t_2 x_2 y_2 + t_3 x_1^3 + q_{11} x_0 x_2^2 + q_2 x_0 y_2 + q_4 x_0 y_1)), \)
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\[ r_{2}^{(3)}: z(z - t_{3}x_{0}x_{1}) - y_{2}(A - x_{0}(-t_{4}x_{1}^{3} - t_{5}x_{1}^{2} - t_{6}x_{1}x_{2} - t_{7}x_{2}^{2} - t_{8}x_{1}y_{2} - t_{9}x_{2}y_{2} - t_{10}z - q_{6}x_{0}x_{1}^{2} - q_{7}x_{0}x_{1}x_{2} - q_{8}x_{0}x_{2}^{2} - q_{9}x_{0}y_{2})) - y_{1}(B + x_{0}(t_{1}x_{1}y_{2} + t_{2}x_{2}y_{2} + t_{3}x_{1}^{3} + t_{4}x_{1}^{2} + q_{4}x_{0}y_{2} + q_{5}x_{0}y_{1})). \]

It can be seen quite clearly that these relations can be given in Pfaffian form as before. The matrix, \( M^{(3)} \), can be formed by picking the entries \( (M^{(3)})_{ij} \) equal to the entries \( M_{ij} \) reduced mod \( x_{0}^{3} \). The reader will by now, no doubt, have understood both the mechanics of the calculation and be able to guess the solution at the end of each step. We show the rest of the calculation only for the sake of completeness.

**Step 3: \( R^{(3)} \to R^{(4)} \)**

We now extend to

\[ R^{(4)} = C[x_{0}, x_{1}, x_{2}, y_{1}, y_{2}, z]/(x_{0}^{4}, I^{(4)}). \]

By consideration of \( s_{1} \) we see easily that

\[ r_{1}^{(4)}: x_{1}y_{2} - x_{2}y_{1}, \]

\[ r_{2}^{(4)}: x_{1}z - y_{2}^{2} - \lambda_{2}x_{0}^{2}y_{1}, \]

\[ r_{3}^{(4)}: x_{2}z - y_{1}y_{2} - \lambda_{3}x_{0}^{2}y_{2}. \]
To compute \( \gamma'''' \) and \( \delta'''' \) consider the lift of \( s_3 \),

\[
s_3 : x_1 \delta'''' - y_1 \gamma'''' - t_1 \lambda_5 x_1^2 y_2 - t_2 \lambda_5 x_1 x_2 y_2 - t_3 \lambda_5 x_1^2 y_1 \in \mathcal{I}.
\]

This is a degree 4 element of \( \mathcal{I} \).

Notice that we can write

\[
\gamma'''' = u_1 x_1^2 + \bar{\gamma}''''
\]

\[
\delta'''' = u_1 x_1 y_1 + \bar{\delta}''''
\]

Then a similar calculation to the previous ones shows us that

\[
\bar{\gamma}'''' = u_2 y_1 + u_3 x_1 x_2 - t_3 \lambda_5 x_1^2
\]

\[
\bar{\delta}'''' = u_2 z + (u_3 + t_1 \lambda_5) x_1 y_2 - t_2 \lambda_5 x_2 y_2
\]

It is again a simple task to check that these new relations do allow us to lift all the syzygies as required. In fact we get

\[
R(4C.D^{(4)}) = \mathcal{C}(x_0, x_1, x_2, y_1, y_2, z) / (x_0^4, \mathcal{I}^{(4)}),
\]

where \( \mathcal{I}^{(4)} \) is generated by

\[
\begin{align*}
\gamma^{(4)}_1 & : x_1 y_2 - x_2 y_1, \\
\gamma^{(4)}_2 & : x_1 z - y_1^2 - \lambda_5 x_0^2 y_1, \\
\gamma^{(4)}_3 & : x_2 z - y_1 y_2 - \lambda_5 x_0^2 y_2.
\end{align*}
\]
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\[ r^{(4)}: y_1z - x_2A - x_1B - t_1x_0x_2y_2 - t_2x_0x_1x_2y_2 - t_3x_0x_1x_2 - t_5x_0x_1x_2 - t_6x_0x_1x_2^3 - t_7x_0x_2 - t_8x_0x_1x_2y_2 - t_9x_0x_2y_2 - t_{10}x_0x_2z - t_{11}x_0x_1^4 - \lambda_5x_0x_1y_2 - q_2x_0x_1y_2 - q_4x_0x_1y_1 - q_6x_0x^2y_2 - q_7x_0x_1x_2 - q_8x_0x_2^3 - q_{10}x_0x_2y_2 - q_{11}x_0x_1^3 - u_1x_0x_1 - u_2x_0y_1 - u_3x_0x_1x_2 + t_3\lambda_5x_0x_1^2.\]

\[ r^{(5)}: z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_2z - t_3x_0x_1x_2 - t_5x_0x_1x_2 - t_6x_0x_1x_2y_2 - t_7x_0x_1y_2 - t_8x_0x_1y_2^2 - t_9x_0x_2y_2^2 - t_{10}x_0y_2z - t_{11}x_0y_1^3 - q_2x_0y_2z - q_4x_0y_2z - q_6x_0y_2z^2 - q_7x_0y_1x_2y_2 - q_8x_0y_2^3 - q_{10}x_0y_2^2 - q_{11}x_0y_1^3 - u_1x_0y_1 - u_2x_0z^3 - (u_3 + t_1\lambda_5)x_0y_1^3 - t_2\lambda_5x_0y_1x_2.\]

These relations satisfy

\[ s_1^{(4)}: x_1r_3^{(4)} - x_2r_2^{(4)} + y_1r_1^{(4)} - \lambda_5x_0r_1^{(4)} = 0,\]

\[ s_2^{(4)}: y_1r_3^{(4)} - y_2r_2^{(4)} + x_1r_1^{(4)} = 0,\]

\[ s_3^{(4)}: t_1r_3^{(4)} - t_2r_2^{(4)} + y_1r_1^{(4)} - \lambda_5x_0r_1^{(4)} = 0,\]

\[ s_4^{(4)}: t_1r_3^{(4)} - t_2r_2^{(4)} + x_1r_1^{(4)} = 0.\]
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\[ s_3^{(3)}: x_1 r_5^{(4)} - y_1 r_5^{(4)} - z r_2^{(4)} + A r_1^{(4)} - t_1 x_0 x_1^2 r_3^{(4)} - t_2 x_0 x_1^2 r_3^{(4)} - \\
- t_3 x_0 x_1^2 r_2^{(4)} - t_4 x_0 x_1^2 r_2^{(4)} - t_5 x_0 x_1^2 r_2^{(4)} - t_6 x_0 x_1^2 r_2^{(4)} - t_7 x_0 x_1^2 r_1^{(4)} - \\
- t_8 x_0 x_1^2 r_1^{(4)} - t_9 x_0 x_2 r_1^{(4)} - t_{10} x_0 z r_1^{(4)} - q_2 x_0 x_1 r_3^{(4)} - q_4 x_0 x_1 r_2^{(4)} - \\
- q_6 x_0 x_1 r_1^{(4)} - q_7 x_0 x_1 r_1^{(4)} - q_8 x_0 x_1 r_1^{(4)} - q_9 x_0 x_1 r_1^{(4)} - u_2 x_0 r_2^{(4)} - \\
u_3 x_0 x_1 r_1^{(4)} = 0. \\
\]

\[ s_4^{(4)}: x_2 r_5^{(4)} - y_2 r_4^{(4)} - z r_3^{(4)} + B r_1^{(4)} + t_1 x_0 x_1 y_2 r_1^{(4)} - t_1 x_0 x_1 y_2 r_1^{(4)} + \\
t_2 x_0 x_2 y_2 r_1^{(4)} - t_2 x_0 x_2 y_2 r_1^{(4)} - t_3 x_0 x_2 y_2 r_1^{(4)} + t_1 x_0 x_1^2 r_3^{(4)} - q_2 x_0 x_2 r_3^{(4)} + \\
q_3 x_0 x_2 y_2 r_1^{(4)} - q_4 x_0 x_2 y_2 r_1^{(4)} + q_{11} x_0 x_1 r_1^{(4)} + u_1 x_0 x_1 r_1^{(4)} - u_2 x_0 r_3^{(4)} - \\
u_3 x_0 x_2 r_1^{(4)} = 0. \\
\]

\[ s_3^{(3)}: y_1 r_3^{(4)} - z r_4^{(4)} - A r_3^{(4)} - B r_2^{(4)} + t_1 x_0 x_1 z r_1^{(4)} + t_2 x_0 x_2 z r_1^{(4)} + \\
t_4 x_0 x_1^2 r_3^{(4)} + t_5 x_0 x_1^2 r_3^{(4)} - t_6 x_0 x_1^2 r_3^{(4)} + t_7 x_0 x_1^2 r_3^{(4)} + t_8 x_0 x_1^2 r_3^{(4)} + \\
t_9 x_0 x_2 r_3^{(4)} + t_{10} x_0 z r_3^{(4)} + t_{11} x_0 x_1 r_2^{(4)} - q_2 x_0 r_3^{(4)} + q_2 x_0 x_2 r_1^{(4)} + \\
q_6 x_0 x_1 r_3^{(4)} + q_7 x_0 x_1 r_3^{(4)} + q_8 x_0 x_1 r_3^{(4)} + q_{10} x_0 x_2 r_3^{(4)} + q_{11} x_0 x_2 r_3^{(4)} + \\
u_1 x_0 x_1 r_2^{(4)} + u_3 x_0 x_1 r_3^{(4)} + t_1 x_1 x_0 x_1 r_3^{(4)} + y_2 x_0 x_2 r_3^{(4)} = 0. \\
\]
Step 4: $\mathbb{R}^{(4)} \rightarrow \mathbb{R}^{(4+1)}$

As we extend from $\mathcal{S} \mathcal{C}$ to $(4+1)\mathcal{C}$ and work modulo higher powers of $x_0$ there are fewer monomials of lower degrees. The calculations get somewhat easier and so the final ring required shall be written down here.

We conclude that,

$$\mathbb{R}(7\mathcal{C},\mathcal{D}) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/(x_0^7, \overline{I}),$$

where $\overline{I}$ is generated by

$$\overline{r}_1 : x_1y_2 - x_2y_1,$$
$$\overline{r}_2 : x_1z - y_1^2 - \lambda x_0^2y_1,$$
$$\overline{r}_3 : x_2z - y_1y_2 - \lambda x_0^2y_2,$$
$$\overline{r}_4 : y_1z - x_2A - x_1B - t_1x_0x_1^2y_2 - t_2x_0x_1x_2y_2 - t_3x_0^2y_1 -
\quad t_4x_0x_1^3x_2 - t_5x_0x_1^2x_2^2 - t_6x_0x_1x_2^3 - t_7x_0^2x_2^4 - t_8x_0x_1x_2y_2
\quad - t_9x_0^2y_2 - t_{10}x_0x_2z - t_{11}x_0x_1^4 - \lambda x_0^2z - q_2x_0^2x_1y_2 -
\quad q_4x_0^2x_1y_1 - q_6x_0^2x_1^2x_2 - q_7x_0^2x_1x_2^2 - q_8x_0^3x_2^3 - q_9x_0^2x_2y_2 -
\quad q_{11}x_0^3x_2^3 - u_1x_0^3x_1^2 - u_2x_0^3y_1 - u_3x_0^3x_1x_2 + t_3\lambda x_0^3x_2^2 -
\quad v x_0^4x_1 - v_1x_0^4x_2 + u_2\lambda x_0^5.
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\[ \tilde{r}_5 : \quad z^2 - y_2A - y_1B - t_1x_0x_1x_2z - t_2x_0x_1^2z - t_3x_0x_1^2z - \]
\[ 4x_0x_1^3y_2 - t_5x_0x_1^2x_2y_2 - t_6x_0x_1x_2y_2 - t_7x_0x_1^3y_2 - \]
\[ 8x_0x_1y_2 - t_9x_0x_2y_2 - t_{10}x_0y_2z - t_{11}x_0x_1^3y_1 - q_{2}x_0^2x_2z - \]
\[-q_{4}x_0^2x_1z - q_6x_0^2x_1y_2 - q_{7}x_0^2x_1x_2y_2 - q_{8}x_0^2x_2y_2 - \]
\[ q_{10}x_0y_2^2 - q_{11}x_0^2x_1y_1 - u_{1}x_0x_1y_1 - u_{2}x_0z - \]
\[ (u_3 - t_1\lambda_5)x_0^3x_1y_2 - t_2\lambda_3x_0^3x_2y_2 - (v_1 - \lambda_5q_1)x_0^4y_2 - \]
\[ (v - \lambda_5q_2)x_0^4y_1 \]

These relations satisfy the syzygies below

\[ \tilde{s}_1 : \quad x_1\tilde{r}_3 - x_2\tilde{r}_2 + y_1\tilde{r}_1 + \lambda_3x_0\tilde{r}_1 = 0, \]
\[ \tilde{s}_2 : \quad y_1\tilde{r}_3 - y_2\tilde{r}_2 + z\tilde{r}_1 = 0, \]
\[ \tilde{s}_3 : \quad x_1\tilde{r}_5 - y_1\tilde{r}_4 - z\tilde{r}_2 + Ar_{1} + t_1x_0x_1\tilde{r}_3 + t_2x_0x_1x_2\tilde{r}_3 + t_3x_0x_1^2\tilde{r}_2 + \]
\[ 4x_0x_1^3\tilde{r}_1 + t_5x_0x_1^2x_2\tilde{r}_1 + t_6x_0x_1x_2\tilde{r}_1 + t_7x_0x_1^3\tilde{r}_1 + t_8x_0x_1y_2\tilde{r}_1 + \]
\[ t_{9}x_0x_2y_2\tilde{r}_1 + t_{10}x_0^{2}r_{1} + q_{2}x_0^{2}x_1r_{3} + q_{4}x_0^{2}x_1r_{2} + q_{6}x_0^{2}x_1r_{1} + \]
\[ q_{7}x_0^{3}x_1x_2r_{1} + q_{8}x_0^{3}x_2r_{1} + q_{10}x_0y_2r_{1} + u_{2}x_0r_{2} + u_{3}x_0x_1r_{1} = 0 \]
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\[s_4: x_2 \tilde{r}_5 - y_2 \tilde{r}_4 - z \tilde{r}_3 + B r_1 - t_1 x_0 x_1 y_2 \tilde{r}_1 + t_1 x_0 x_1 x_2 \tilde{r}_3 - t_2 x_0 x_2 y_2 \tilde{r}_1 + t_2 x_0 x_2 \tilde{r}_3 + t_3 x_0 x_3 \tilde{r}_3 - t_1 x_0 x_1 \tilde{r}_1 + q_2 x_0 x_2 \tilde{r}_3 - q_2 x_0 y_2 \tilde{r}_1 + q_4 x_0 x_1 \tilde{r}_3 - q_1 x_0 x_1 \tilde{r}_1 - u_1 x_0 x_1 \tilde{r}_1 + u_2 x_0 \tilde{r}_3 + u_3 x_0 x_2 \tilde{r}_1 = 0,
\]

\[s_5: y_1 \tilde{r}_5 - z \tilde{r}_4 - A \tilde{r}_3 - B \tilde{r}_2 - t_1 x_0 x_1 z \tilde{r}_1 - t_2 x_0 x_2 \tilde{r}_1 - t_4 x_0 x_1 \tilde{r}_3 - t_5 x_0 x_1 x_2 \tilde{r}_3 - t_6 x_0 x_1 x_2 \tilde{r}_3 - t_7 x_0 x_1 \tilde{r}_3 - t_8 x_0 x_1 y_2 \tilde{r}_3 - t_9 x_0 x_2 y_2 \tilde{r}_3 - t_{10} x_0 \tilde{r}_3 - t_{11} x_0 x_1 \tilde{r}_2 + q_2 x_0 \tilde{r}_5 - q_2 x_0 \tilde{r}_1 - q_3 x_0 x_1 \tilde{r}_3 - q_4 x_0 x_1 x_2 \tilde{r}_3 - q_8 x_0 \tilde{r}_3 - q_{10} x_0 \tilde{r}_3 - q_{11} x_0 \tilde{r}_2 - u_1 x_0 x_1 \tilde{r}_2 - u_3 x_0 x_1 \tilde{r}_3 - t_{11} x_0 x_1 \tilde{r}_3 - y_2 \tilde{r}_3 = 0.
\]

At this stage the algorithm terminates and we can write:

\[R(X, K_X) = C_{x_0, x_1, x_2, y_1, y_2, z} / \tilde{t}.
\]

As with the curve we can fit the structure of this ring into a Pfaffian one. At the moment the relations are not quite written in such a way that a skew symmetric matrix with appropriate Pfaffians can be found. By adding multiples of \(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3\) to \(\tilde{r}_4\) and \(\tilde{r}_5\) this may be remedied and the matrix \(M\), in the statement of the theorem will deliver a generating set for \(\tilde{t}\).
Surfaces of type III

14). Surfaces of type III.

Theorem (14.1)

Let $X$ be a surface with $p_g = 3$ and $K_X^2 = 4$ such that the canonical system has two base points. Then the presentation of the ring $R(C, D)$ given in Theorem (9.4) is formative and informative in degree -1. In fact

$$R(X, K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/I.$$ 

The degrees of the generators are,

$$\text{deg} (x_i, y_j, z_k) = (1, 2, 3) \text{ respectively.}$$

$I$ is generated by $r_1, \ldots, r_9$ as follows:

Let $A = \begin{bmatrix} x_1 & y_1 & x_2 - \alpha_2 & \alpha_3 x_0^2 & z_1 \\ x_2 & x_1 & -\alpha_1 x_0 - \alpha_3 x_0 & y_2 & z_2 \end{bmatrix}$.

Then $r_1, \ldots, r_6$ are given by,

$$\text{rank} A \leq 1$$

$r_7, r_8, r_9$ are given by,

$$\begin{bmatrix} r_7 \\ r_8 \\ r_9 \end{bmatrix} = APA^t,$$

where $P = (P_{ij})$ is a symmetric matrix with entries
\begin{align*}
P_{11} &= -h - \delta_1 x_0 x_2 y_1 - \delta_2 \alpha_{21} x_0^2 \delta x_1^2 - \delta_3 x_0 x_2 y_2 - \delta_{11} x_0 x_1 y_2 + \\
&\delta_{11} x_1^2 \delta y_2 - \delta_{13} x_0^2 y_1 - \delta_{18} x_0^2 y_2 - \delta_{13} x_0 x_2 - \delta_{16} x \delta x_1 + \\
&\delta_{24} x_0 x_1 + \delta_{25} x_0 x_2 - \delta_{23} x_0^2 - \delta_{26} x_0^3.
\end{align*}

\begin{align*}
P_{12} &= \frac{1}{2} (-\delta_3 x_0 y_1 - \delta_6 x_0 y_2 + \delta_{27} x_0^2), \\
P_{13} &= \frac{1}{2} (-\delta_2 x_0 y_1 - \delta_4 x_0 y_2 + \delta_{26} x_0^3), \\
P_{14} &= \frac{1}{2} (-\delta_{13} \alpha_{13} x_0^2 + \delta_{23} x_0^2), \\
P_{22} &= -\lambda y_1 - \delta_{10} x_0 x_2 - \delta_{21} x_0^2, \\
P_{23} &= -\frac{1}{2} \delta_{17} x_0^2, \\
P_{24} &= -\frac{1}{2} \delta_{12} x_0, \\
P_{33} &= -\mu y_2 - \delta_4 \alpha_{21} x_0^2 - \delta_{19} x_0^2, \\
P_{34} &= -\frac{1}{2} \delta_9 x_0, \\
P_{44} &= 1.
\end{align*}

\textbf{Remark (14.2)}

As was remarked in the proof of Theorem (9.4) relations generated by rank \( M \leq 1 \) and \( \text{MPMT} \) satisfy syzygies given by \((M^*M)\text{PMT} = M^*(\text{MPMT})\). This shows that this presentation is formative. For reference the relations \( \tilde{r}_1, \ldots, \tilde{r}_6 \) in the statement of the theorem are reproduced below:
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\[
\begin{align*}
\tilde{r}_1 &: x_1^3 - x_2 y_1 - \alpha_{11} x_0 x_1^2 - \alpha_{13} x_0^2 x_1, \\
\tilde{r}_2 &: x_2^3 - x_1 y_2 - \alpha_{21} x_0 x_2^2 - \alpha_{23} x_0^2 x_2, \\
\tilde{r}_3 &: y_1 y_2 - \alpha_{11} x_0 x_1^2 x_2 + \alpha_{12} x_0 x_1^2 x_1 + \alpha_{13} x_0^2 x_2 + \\
&+ \alpha_{23} x_0^2 x_1^2 - \alpha_{11} x_2 x_1^2 - \alpha_{21} x_1^3 x_2 - \alpha_{23} x_1^3 x_1 - \alpha_{23} x_1^3 x_0^2, \\
\tilde{r}_4 &: x_1 z_2 - x_2 z_1, \\
\tilde{r}_5 &: x_1 z_1 - y_1 z_2 - \alpha_{11} x_0 x_1^2 z_1 - \alpha_{13} x_0^2 z_1, \\
\tilde{r}_6 &: x_2 z_2 - y_2 z_1 - \alpha_{21} x_0 x_2^2 z_2 - \alpha_{23} x_0^2 z_2.
\end{align*}
\]

Proof of (14.1)

Let \( C \in \mathbb{P}K_X \) be given by the vanishing of \( x_0 \in H^0(X, K_X) \) and let
\( D = K_X C \). In Theorem (9.4) we showed that
\[
R(C, D) = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/I,
\]
where degree \( x_1, y_1, z_1 = 1, 2, 3, \) and \( I \) is generated by \( r_1, \ldots, r_9 \) as follows.

The relations below are \( \pm \) those given in (9.4).

Degree 3
\[
\begin{align*}
\tilde{r}_1 &: x_1^3 - x_2 y_1, \\
\tilde{r}_2 &: x_2^3 - x_1 y_2.
\end{align*}
\]

Degree 4
\[
\begin{align*}
\tilde{r}_3 &: y_1 y_2 - x_1^3 x_2, \\
\tilde{r}_4 &: x_1 z_2 - x_2 z_1.
\end{align*}
\]
Surfaces of type III

Degree 5

\[ r_5 : x_1^2 z - y_1 z_2, \]
\[ r_6 : x_2^2 z - y_2 z_1. \]

Degree 6

\[ r_7 : z_1^2 - \lambda y_1^3 - \mu x_2^4 y_2 - x_1^2 h, \]
\[ r_8 : -z_1 z_2 + \lambda x_1 y_1^2 + \mu x_2 y_2^2 + x_1 x_2 h, \]
\[ r_9 : z_2^2 - \lambda x_1 y_1 - \mu y_2^3 - x_2^2 h. \]

The first 6 relations were written determinantly as follows,

\[
\text{rank } M = \text{rank } \begin{bmatrix} x_1 & y_1 & x_2^2 & z_1 \\ x_2 & x_1^2 & y_2 & z_2 \end{bmatrix} \leq 1
\]

These give rise to determinantal syzygies as follows,

\[ \Sigma_1 : x_1 r_3 + y_1 r_2 + x_2^2 r_1 = 0, \]
\[ \Sigma_2 : -x_2 r_3 - x_1^2 r_2 - y_2 r_1 = 0, \]
\[ \Sigma_3 : x_1 r_6 - x_2^2 r_4 - z_1 r_2 = 0, \]
\[ \Sigma_4 : x_1 r_5 + y_1 r_4 - z_1 r_1 = 0, \]
\[ \Sigma_5 : -x_2 r_5 - x_1^2 r_4 + z_2 r_1 = 0, \]
\[ \Sigma_6 : -x_2 r_6 + y_2 r_4 + z_2 r_2 = 0, \]
\[ \Sigma_7 : -y_1 r_6 - x_2^2 r_5 - z_1 r_3 = 0. \]
and it was found that the syzygies involving \( r_7, r_8 \) and \( r_9 \) are the following,

\[
\Sigma_8 : x_1^2 r_6 + y_2 r_5 + z_2 r_3 = 0,
\]

\[
\Sigma_9 : x_2 r_7 + x_1 r_8 + z_1 r_4 - \lambda y_1^2 r_1 + \mu x_2^2 y_2 r_2 = 0,
\]

\[
\Sigma_{10} : -x_2 r_8 - x_1 r_9 + z_2 r_4 - \lambda x y_2 r_1 + \mu y_2^2 r_2 = 0,
\]

\[
\Sigma_{11} : x_1^2 r_7 + y_1 r_8 - z_1 r_5 - \mu x_2^2 y_2 r_3 + x_1 h r_1 = 0,
\]

\[
\Sigma_{12} : -x_1^2 r_8 - y_1 r_9 - z_2 r_5 + \mu y_2 r_3 + x_2 h r_1 = 0,
\]

\[
\Sigma_{13} : y_2 r_7 + x_2^2 r_8 + z_1 r_6 + \lambda x y_1 r_3 - x_2 h r_2 = 0,
\]

\[
\Sigma_{14} : -y_2 r_8 - x_2^2 r_9 + z_2 r_6 + \lambda x y_2 r_3 = x_2 h r_1 = 0,
\]

\[
\Sigma_{15} : z_2 r_7 + z_1 r_8 - \lambda y_1^2 r_5 + \mu x_2^2 y_2 r_6 + x_1 h r_4 = 0,
\]

\[
\Sigma_{16} : -z_2 r_8 - z_1 r_9 - \lambda x_1^2 y_1 r_5 + \mu y_2^2 r_6 + x_2 h r_4 = 0.
\]

Remember that we wrote down these relations in a format dependent only on the matrix \( M \) and a symmetric matrix. We shall use this format to shorten the extension-deformation calculations.

We now use the extension-deformation theory to extend this ring to \( R(2C, D^{(2)}) \) then to \( R(3C, D^{(3)}) \), \( R(4C, D^{(4)}) \) etc. If at each stage we can show that the relations lift to ones in the same format for some \( l_{ij}', P_{ij}' \) where \( l_{ij}' \) reduces mod \( x_0 \) to \( l_{ij} \), \( P_{ij}' \) reduces mod \( x_0 \) to \( P_{ij} \) then the lifted relations will automatically satisfy syzygies \( \Sigma_1', ..., \Sigma_{16}' \) such that \( \Sigma_i' \) reduces mod \( x_0 \) to \( \Sigma_i \) as required. This will shorten some of the checking procedures along...
Step 0: Simplifications
First we do some basis changes to simplify the calculations. Suppose

\[ R(X, K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/\mathfrak{I}, \]

where \( \mathfrak{I} \) is generated by,

\[
\begin{align*}
\bar{r}_1 & : x_1^3 - x_2y_1 - a_1x_0, \\
\bar{r}_2 & : x_2^3 - x_1y_2 - a_2x_0, \\
\bar{r}_3 & : y_1y_2 - x_1^2x_2 - b_1x_0, \\
\bar{r}_4 & : x_1z_2 - x_2z_1 - b_2x_0, \\
\bar{r}_5 & : x_1^2z_1 - y_1z_2 - c_1x_0, \\
\bar{r}_6 & : x_2^2z_2 - y_2z_1 - c_2x_0, \\
\bar{r}_7 & : x_1^2 - \lambda y_1^2 - \mu x_2y_2 - x_1^2h - d_1x_0, \\
\bar{r}_8 & : -z_1z_2 + \lambda x_1^2y_1^2 + \mu x_2^2y_2^2 + x_1x_2h - d_2x_0, \\
\bar{r}_9 & : x_2^2 - \lambda y_1^2y_1 - \mu y_2^3 - x_2^2h - d_3x_0.
\end{align*}
\]

Here the \( a_i, b_i, c_i \) and \( d_i \) are polynomials in \( x_0, x_1, y_1, y_2, z_1 \) and \( z_2 \) so that the \( \bar{r}_i \) satisfy syzygies \( \mathcal{L}_i \) such that \( \mathcal{L}_i \) reduces mod \( x_0 \) to \( \Sigma_i \). By changing basis we can assume that

\[ a_1 = \alpha_{11}x_1^2 + \alpha_{12}y_2 + \alpha_{13}x_0x_1 + \alpha_{14}x_0^2. \]
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\[ a_2 = \alpha_{21}x_2^2 + \alpha_{22}y_1 + \alpha_{23}x_0x_2 + \alpha_{24}x_0^2, \]

\[ b_2 = \beta_{21}z_1 + \beta_{22}z_2 + \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0^3. \]

The \( c_i \) are degree 4 and so (with a chosen basis for \( H^0(X,4K_X) \)) we can write

\[ c_1 = \gamma_{1,1}x_1^2y_1 + \gamma_{1,2}x_1x_2y_1 + \gamma_{1,3}x_2^2y_1 + \gamma_{1,4}x_1y_2 + \gamma_{1,5}x_1x_2y_2 + \gamma_{1,6}x_2y_2 + \gamma_{1,7}y_1^2 + \gamma_{1,8}y_1y_2 + \gamma_{1,9}y_2^2 + \gamma_{1,10}x_1z_1 + \gamma_{1,11}x_2z_1 + \gamma_{1,12}x_2z_2 + \gamma_{1,13}x_0x_1^2 + \gamma_{1,14}x_0x_2x_1 + \gamma_{1,15}x_0x_1y_1 + \gamma_{1,16}x_0x_1y_2 + \gamma_{1,17}x_0x_2y_1 + \gamma_{1,18}x_0x_2y_2 + \gamma_{1,19}x_0x_2z_1 + \gamma_{1,20}x_0x_2z_2 + \gamma_{1,21}x_0x_1x_2 + \gamma_{1,22}x_0x_2x_1 + \gamma_{1,23}x_0^2 + \gamma_{1,24}x_0y_1 + \gamma_{1,25}x_0^2y_2 + \gamma_{1,26}x_0^3z_1 + \gamma_{1,27}x_0^3z_2 + \gamma_{1,28}x_0^4. \]

and similarly for \( c_2 \) with \( \gamma_{1,i} \) replaced by \( \gamma_{2,i} \). Since \( h^0(X,5K_X) = 44 \) we could also write down expressions for the \( d_i \) depending on 44 parameters before we begin the calculations (but we won't).

**Step 1 : \( R \rightarrow R^{(2)} \)**

We begin the calculation of \( R(2C,D^{(2)}) \). Consider

\[ \Sigma^{(2)}_1: x_1r^{(2)}_3 + y_1r^{(2)}_2 + x_2r^{(2)}_1 \in x_0l^{(2)}. \]

Then

\[ x_1b' + y_1a' + x_2a' \in I. \]

That is,
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\[ x_1 b'_1 + y_1 a'_2 + z_2 a'_1 = p_1 r_3 + p_2 r_4 + p_3 x_1 r_1 + p_4 x_2 r_1 + p_5 x_1 r_2 + p_6 x_2 r_2, \]

where \( p_i \in \mathbb{C}. \)

Comparing coefficients we see at once that:

\[ \alpha_{12} = \alpha_{22} = p_2 = p_3 = p_5 = p_6 = 0 \text{ and } \alpha_{11} = -p_1, \alpha_{21} = -p_4. \]

\[ b'_1 = -\alpha_{11} x_1^2 + \alpha_{21} x_2^2. \]

These coefficients also satisfy

\[ \Sigma_2^{(2)}: x_2 r_3^{(2)} + x_1 r_2^{(2)} + y_2 r_1^{(2)} \in x_0 l^{(2)}. \]

In the same way \( \Sigma_3 \) must lift to a syzygy in \( R(2C,D^{(2)}): \)

\[ \Sigma_3: x_1 c_2' - x_2 b_2' - z_1 a'_2 \in l. \]

Substituting what we already know from Step 0 about \( a'_2 \) and \( b'_2 \) we get

\[ x_1 c_2' - \beta_{21} x_2^2 z_1 - \beta_{22} x_2 z_2 - \alpha_{21} x_2^2 z_1 \in l. \]

This leads us to the equations between the coefficients

\[ \beta_{22} = 0, \ c_2' = (\beta_{21} + \alpha_{21}) x_2 z_2. \]

However, looking at

\[ \Sigma_4: -x_1 c_1' - y_1 b_2' + z_1 a'_2 \in l, \]

we find that

\[ \beta_{21} = 0, \ c_1' = \alpha_{11} x_1 z_1. \]

So far we have deduced that,

\[ a'_1 = \alpha_{11} x_1^2. \]
Surfaces of type III

\[ a'_2 = \alpha_{21}x_2^2. \]

\[ b'_1 = -\alpha_{11}x_1x_2^2 - \alpha_{21}x_1^2x_2, \]

\[ b'_2 = 0. \]

\[ c'_1 = \alpha_{11}x_1z_1, \]

\[ c'_2 = \alpha_{21}x_2z_2. \]

Incidentally, the relations \((2)\) are still determinantal in form. In fact they are given by,

\[ \text{rank } \begin{bmatrix} x_1 & y_1 & x_2^2 - \alpha_{21}x_0x_2 & z_1 \\ x_2 & x_1^2 - \alpha_{11}x_0x_1 & y_2 & z_2 \end{bmatrix} \leq 1. \]

From this we can see that all the determinantal syzygies are lifted by these choices of polynomial.

The calculations get much longer when we consider the nondeterminantal syzygies, \(\Sigma_9, \ldots, \Sigma_{16}\). In order to lift \(\Sigma_9\) we need,

\[ x_2d'_1 + x_1d'_2 + z_1b'_2 - \lambda y_2^2a'_1 + \mu x_2^2y_2a'_2 \in I. \]

We can write

\[ d'_1 = \delta_1x_1y_2^2 + d'_1, \]

\[ d'_2 = \lambda \alpha_{11}x_1y_2^2 - (\mu \alpha_{21} + \delta_1)x_2y_2^2 + d'_2, \]

where

\[ x_2d'_1 + x_1d'_2 \in I. \]
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Since \( d_1 \) and \( d_2 \) are degree 5, this equation is degree 6. Taking a basis of \( H^0(5K_X|C) \) we can write

\[
d'_i = \delta_{i,1}x_1^2x_2y_1 + \delta_{i,2}x_1x_2^2y_1 + \delta_{i,3}x_1^2x_2y_2 + \delta_{i,4}x_1x_2^2y_2 + \delta_{i,5}x_1y_1^2 + \\
\delta_{i,6}x_1y_1y_2 + \delta_{i,7}x_1y_2^2 + \delta_{i,8}x_1x_2z_1 + \delta_{i,9}x_1^2z_1 + \delta_{i,10}x_2y_1^2 + \delta_{i,11}x_2y_1y_2 + \\
\delta_{i,12}x_2y_2^2 + \delta_{i,13}y_1z_1 + \delta_{i,14}y_1z_2 + \delta_{i,15}y_2z_1 + \delta_{i,16}y_2z_2.
\]

Similarly taking a basis for the degree 6 part of \( I \) we find that we have to solve,

\[
x_2d'_i + x_id'_2 = t_1y_1r_3 + t_2y_2r_3 + t_3x_1^2r_4 + t_4x_1x_2r_4 + t_5x_2^2r_4 + t_6y_1r_4 + \\
t_7y_2r_4 + t_8x_1^3r_1 + t_9x_1x_2^2r_1 + t_{10}x_1^2x_2r_1 + t_{11}x_2^3r_1 + t_{12}x_1y_1r_1 + \\
t_{13}x_1y_2r_1 + t_{14}x_2y_1r_1 + t_{15}x_2y_2r_1 + t_{16}z_1r_1 + t_{17}z_2r_1 + \\
t_{18}x_1^2r_2 + t_{19}x_1x_2^2r_2 + t_{20}x_2^3r_2 + t_{21}x_1y_1r_2 + t_{22}x_1y_2r_2 + \\
t_{23}x_2y_1r_2 + t_{24}x_2y_2r_2 + t_{25}z_1r_2 + t_{26}z_2r_2
\]

It is found that

\[ t_i = 0 \text{ for } i = 1, 2, 4, 5, 8, 9, 10, 11, 12, 13, 16, 18, 19, 20, 23, 26. \]

After renumbering we see that

\[
d'_i = \delta_{1}x_1^2x_2y_1 + \delta_{2}x_1x_2^2y_1 + \delta_{3}x_1^2x_2y_2 + \delta_{4}x_1x_2^2y_2 + \delta_{5}x_1y_1^2 + \\
\delta_{6}x_1y_1y_2 + \delta_{7}x_1y_2^2 + \delta_{8}x_1x_2z_1 + \delta_{9}x_1^2z_1 + \delta_{10}x_2y_1^2 + \\
\delta_{11}x_2y_1y_2 + \delta_{12}y_1z_1 + \delta_{13}y_1z_2 + \delta_{14}y_2z_1.
\]
Surfaces of type III

\[ d'_2 = -\delta_1 x_1 x_2^2 y_1 - \delta_2 x_1 y_1 y_2 - \delta_3 x_1 x_2^3 y_2 - \delta_4 x_1 y_2^2 - \delta_5 x_2 y_1^2 - \]

\[ \delta_6 x_2 y_1 y_2 - (\delta_7 + \mu_\alpha_{21}) x_2 y_2^2 - \delta_8 x_2^2 z_1 - \delta_9 y_2 z_1 - \delta_{10} x_1^2 x_2 y_1 \]

\[ - \delta_{11} x_1^2 x_2 y_2 - \delta_{12} y_1^2 z_2 - \delta_{13} x_1 x_2 z_1 - \delta_{14} y_2 z_2 + \lambda_{\alpha_{11}} x_1 y_1^2. \]

Now some restrictions are placed on \( d'_3 \) using our calculated values for \( a'_1, a'_2, b'_2, d'_2 \) and the syzygy \( \Sigma_{10} \).

\[ \Sigma_{10} : -x_2 d'_2 - x_1 d'_3 - \lambda_{\alpha_{11}} x_1^4 y_1 + \mu_{\alpha_{21}} x_2 y_2^2 \in I \]

That is,

\[ \delta_1 x_1 x_2^3 y_1 + \delta_2 x_1 x_2 y_1 y_2 + \delta_3 x_1 x_2^2 y_2 + \delta_4 x_1 x_2 y_2^2 + \delta_5 x_2^2 y_1^2 + \delta_6 x_2^2 y_1 y_2 + (\delta_7 + \mu_{\alpha_{21}}) x_2 y_2^2 + \delta_8 x_2^3 z_1 + \delta_9 x_2 y_2 z_1 + \delta_{10} x_1^2 x_2^2 y_1 + \delta_{11} x_1^2 x_2 y_2 + \delta_{12} x_2 y_1 z_2 + \delta_{13} x_1 x_2 z_1 + \delta_{14} x_2 y_2 z_2 - \lambda_{\alpha_{11}} x_1 x_2 y_1^2 - x_1 d'_3 - \lambda_{\alpha_{11}} x_1^4 y_1 + \mu_{\alpha_{21}} x_2 y_2^2 \in I \]

Writing out the right-hand side of this equation in full and comparing coefficients gives,

\[ d'_2 = -\delta_1 x_1 x_2^2 y_1 - \delta_2 x_1 y_1 y_2 - \delta_3 x_1 x_2^3 y_2 - \delta_4 x_1 y_2^2 - \delta_5 x_2 y_1^2 - \]

\[ \delta_6 x_2 y_1 y_2 - \delta_9 y_2 z_1 - \delta_{10} x_1^2 x_2 y_1 - \delta_{11} x_1^2 x_2 y_2 - \delta_{12} y_1^2 z_2 - \]

\[ \delta_{13} x_1 x_2 z_1 + \lambda_{\alpha_{11}} x_1 y_1^2 + \mu_{\alpha_{21}} x_2 y_2^2. \]
Surfaces of type III

\[ d' = -2\lambda x_1 x_2 y_1^2 + \delta_1 x_1 y_1 y_2 + \delta_2 x_2 y_1 y_2 + \delta_3 x_1 y_2^2 + \delta_4 x_2 y_2^2 + \]
\[ \delta_5 x_1^2 x_2 y_1 + \delta_6 x_1^2 x_2 y_2 + \delta_9 y_2 z_2 + \delta_{10} x_1 x_2^2 y_1 + \delta_{11} x_1 x_2^2 y_2 + \]
\[ \delta_{12} x_1 x_2 z_1 + \delta_{13} x_2 z_1. \]

Consequently,
\[ \delta_8 = \delta_{14} = 0, \delta_7 = -2\mu \alpha_2. \]

We have shown that the following expressions for \( r_1^{(2)}, \ldots, r_6^{(2)} \) are necessary to extend the syzygies \( \Sigma_1, \ldots, \Sigma_{10} \). They have been written in the formative presentation of the statement of the theorem. This will prove that they also extend the syzygies \( \Sigma_1, \ldots, \Sigma_{16} \).

\( r_1^{(2)}, \ldots, r_6^{(2)} \) are given by
\[
\text{rank } M^{(2)} = \text{rank } \begin{bmatrix} x_1 & -y_1 & x_2^2 - \alpha_{21} x_0 x_2 & -z_1 \\ -x_2 & x_1^2 - \alpha_{11} x_0 x_1 & -y_2 & z_2 \end{bmatrix} \leq 1.
\]

\( r_7^{(2)}, r_8^{(2)}, r_9^{(2)} \) are given by
\[
M^{(2)} P^{(2)} M^{(2)T} = \begin{bmatrix} r_7^{(2)} & r_8^{(2)} \\ r_8^{(2)} & r_9^{(2)} \end{bmatrix},
\]
where \( P^{(2)} \) is the symmetric matrix with coefficients,
\[ P_{11} = -h - \delta_1 x_0 x_2 y_1 - \delta_3 x_0 x_2 y_2 - \delta_{13} x_0 z_1 - \delta_{11} x_0 x_1 y_2. \]
\[ P_{12} = \frac{1}{2} (\delta_3 x_0 y_1 + \delta_6 x_0 y_2). \]
\[ P_{13} = \frac{1}{2} (\delta_3 x_0 y_1 - \delta_4 x_0 y_2). \]
Surfaces of type III

\[ P_{14} = 0, \]
\[ P_{22} = -\lambda y_1 - \delta_{10} x_0 x_2, \]
\[ P_{23} = 0, \]
\[ P_{24} = -\frac{1}{2} \delta_{12} x_0, \]
\[ P_{33} = -\mu y_2, \]
\[ P_{34} = \frac{1}{2} \delta_9 x_0, \]
\[ P_{44} = 1. \]

These relations satisfy the syzygies \( \Sigma^{(2)}_1,...,\Sigma^{(2)}_{16} \) following from the format (see Remark (9.5)). Below are listed \( \Sigma^{(2)}_1, \Sigma^{(2)}_2, \Sigma^{(2)}_3, \Sigma^{(2)}_9, \) and \( \Sigma^{(2)}_{10} \) which we use in step 2.

\[ \Sigma^{(2)}_1 : x_1 r^{(2)}_3 + y_1 r^{(2)}_2 + x_2 r^{(2)}_1 - \alpha_{21} x_0 x_2 r^{(2)}_1 = 0, \]
\[ \Sigma^{(2)}_2 : -x_2 r^{(2)}_3 - x_1 r^{(2)}_2 - y_2 r^{(2)}_1 + \alpha_{21} x_0 x_1 r^{(2)}_1 = 0, \]
\[ \Sigma^{(2)}_3 : x_1 r^{(2)}_6 - x_2 r^{(2)}_4 - z_1 r^{(2)}_2 + \alpha_{21} x_0 x_2 r^{(2)}_4 = 0, \]
\[ \Sigma^{(2)}_5 : -x_2 r^{(2)}_3 - x_1 r^{(2)}_4 + z_2 r^{(2)}_1 + \alpha_{11} x_0 x_1 r^{(2)}_4 = 0, \]
\[ \Sigma^{(2)}_9 : x_2 r^{(2)}_7 + x_1 r^{(2)}_8 + z_1 r^{(2)}_6 - \lambda y_1 r^{(2)}_1 + \mu x_2 y_2 r^{(2)}_2 + \delta_2 x_0 x_1 y_1 r^{(2)}_2 + \delta_4 x_0 x_1 y_2 r^{(2)}_2 + \delta_9 x_0 x_2 r^{(2)}_2 - \delta_{10} x_0 x_2 y_1 r^{(2)}_1 - \delta_{11} x_0 x_2 y_2 r^{(2)}_1 - \delta_{12} x_0 y_1 r^{(2)}_4 - \delta_{13} x_0 x_2 r^{(2)}_5 + \mu \alpha_{21} x_0 x_2 y_2 r^{(2)}_2 = 0. \]
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\[ \Sigma_{10}^{(2)} : -x_2 r_8^{(2)} - x_1 r_9^{(2)} + z_2 r_4^{(2)} - \lambda x_1 y_1 r_1^{(2)} + \mu y_2 r_2^{(2)} + \delta_1 x_0 x_1 y_1 r_1^{(2)} + \delta_2 x_0 x_1 y_2 r_2^{(2)} - \delta_3 x_0 x_2 y_1 r_1^{(2)} - \delta_4 x_0 x_2 y_2 r_2^{(2)} - \delta_5 x_0 y_2 r_4^{(2)} - \delta_6 x_0 x_2 r_5^{(2)} \]

\[ - \lambda \alpha_{11} x_0 x_1 y_1 r_1^{(2)} = 0. \]

Hence we have extended \( R(C,D) \) to

\[ R(2C,D^{(2)}) = C[x_0, x_1, x_2, y_1, y_2, z_1, z_2, l/(x_0, l^{(2)})] \]

where \( l^{(2)} \) is generated by \( r_1^{(2)}, \ldots, r_9^{(2)} \).

Step 2 : \( R^{(2)} \rightarrow R^{(3)} \)

We extend \( R(2C,D^{(2)}) \) to \( R(3C,D^{(3)}) \) in the same way. First we extend the determinantal syzygies.

Begin by extending \( \Sigma_1 \) to a syzygy \( \Sigma_1^{(3)} \) in \( R(3C,D^{(3)}) \)

\[ \Sigma_1^{(3)} : x_1 r_3^{(3)} + y_1 r_2^{(3)} + x_2 r_1^{(3)} - \alpha_{23} x_0 x_2 r_1^{(3)} \in x_0 l^{(3)} \]

That is,

\[ x_1 b''_1 + y_1 \alpha_{23} x_2 + x_2 \alpha_{13} x_1 - \alpha_{11} \alpha_{21} x_1^2 x_2 \in l \]

is of degree 3. We deduce that

\[ b''_1 = -\alpha_{13} x_2^2 + \alpha_{11} \alpha_{21} x_1 x_2 - \alpha_{23} x_1^2. \]

This expression for \( b''_1 \) also lifts \( \Sigma_2 \) to \( \Sigma_2^{(3)} \). To lift \( \Sigma_3 \) we need

\[ x_1 c''_2 - x_2 b''_2 - z_1 x_2 c''_2 = l. \]
Bearing in mind our simplifications at step 0 we set about finding solutions for this expression. To begin with let
\[ c''_2 = c'_2 + x^2f, \quad b''_2 = b'_2 + x_1f \]
where \( f \) is degree 1. Since the possibilities that \( c''_2 \) contains a term in \( x^2 \) and that \( b''_2 \) contains a term in \( x\) have been excluded by our previous simplifications, \( f = 0 \).

Thus we need
\[ x_1c''_2 - x^2b''_2 - z_2\alpha_{23} = I. \]
Comparing with a general degree 4 element of \( I \) gives,
\[ c''_2 = \alpha_{23}z_2 + \gamma_2x_1^2, \]
\[ b''_2 = \gamma_2y_1. \]

To lift to \( \Sigma^{(3)} \) we need
\[ x_2c''_1 + x^2b''_2 - z_2\alpha_{13} = I, \]
that is,
\[ x_2c''_1 + \gamma_2x^2y_1 - z_2\alpha_{13} = I. \]
It is easy to deduce that,
\[ \gamma_2 = 0 \]
\[ c''_1 = \alpha_{13}z_1. \]

A simple check shows that these expressions for \( a''_1, a''_2, b''_1, b''_2, c''_1 \) and \( c''_2 \) also lift the remaining determinantal syzygies.
Now we see if they will lift the nondeterminantal syzygies and calculate expressions for \(d''_1, d''_2\) and \(d''_3\). First consider

\[
\Sigma^2_9: \quad x_2d''_1 + x_1d''_2 + z_1b''_2 - \lambda y_1^2a''_1 + \mu x_2^2y_2a''_2 + \delta_2\alpha_21x_1z_2y_1
\]

\[
+ \delta_4\alpha_21x_1^2y_2 + \delta_9\alpha_21x_2^2z_1 - \delta_10\alpha_11x_1^2y_2 - \delta_11\alpha_11x_1^2z_2y_1 - \delta_13\alpha_11x_1x_2z_1 + \mu\alpha_21x_2^2y_2 \in I.
\]

Comparing this with a general degree 5 element of \(I\) gives

\[
x_2d''_1 + x_1d''_2 - \lambda\alpha_13x_1y_1^2 + \mu\alpha_23x_2^2y_2 + \delta_2\alpha_21x_1^2y_1^2 + \\
\delta_4\alpha_21x_1^2y_2 + \delta_9\alpha_21x_2^2z_1 - \delta_10\alpha_11x_1^2y_2 - \delta_11\alpha_11x_1^2z_2y_1 - \delta_13\alpha_11x_1x_2z_1 + \mu\alpha_21x_2^2y_2
\]

Equating coefficients we get

\[
t_1r_6 + t_2r_5 + t_3x_1r_4 + t_4x_2r_4 + t_5x_1^2r_1 + t_6x_1x_2r_1 + \\
t_7x_2^2r_1 + t_8y_1r_1 + t_9y_2r_1 + t_{10}x_1^2r_2 + t_{11}x_1x_2r_2 + \\
t_{12}x_2^2r_2 + t_{13}y_1r_2 + t_{14}y_2r_2
\]

Equating coefficients we get

\[
t_1 = t_2 = t_3 = t_5 = t_6 = t_7 = t_{10} = t_{11} = t_{12} = 0.
\]

\[
d''_1 = \delta_{15}x_1^2y_1 + \delta_{16}x_1x_2y_1 + \delta_{17}x_2^2y_1 + \delta_{18}x_1^2y_2 + \delta_{19}x_1x_2y_2 + \\
\delta_{20}x_2^2y_2 + \delta_{21}y_1^2 + \delta_{22}y_1x_2y_2 + \delta_{23}x_1x_2 + \delta_{24}x_2^2x_1
\]
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\[ d''_2 = (\delta_{10}\alpha_{11} - \delta_{15})x_1x_2y_1 - (\delta_{2}\alpha_{21} + \delta_{16})x_2^2y_1 - \delta_{17}y_1y_2 + \]

\[ (\delta_{11}\alpha_{11} - \delta_{18})x_1x_2y_2 - (\delta_{19} + \delta_{4}\alpha_{21})x_2^2y_2 - \]

\[ (\mu\alpha_{21}^2 + \mu\alpha_{23} + \delta_{20})y_2 - \delta_{21}x_2^2y_1 - \delta_{22}x_2^2y_2 + \]

\[ (\delta_{13}\alpha_{11} - \delta_{23})x_2z_1 - (\delta_{9}\alpha_{21} + \delta_{24})x_2z_2 + \lambda\alpha_{13}y_1^2 \]

Now by considering

\[ \Sigma^{(3)}: x_2d''_2 + x_1d''_3 - z_2b''_2 + \lambda x_1^2y_1a''_1 - \mu y_2^2a''_2 - \delta_{31}\alpha_{21}x_1x_2^2y_1 \]

\[ - \delta_{32}\alpha_{21}x_1x_2^2y_2 + \delta_{5}\alpha_{11}x_1^2x_2y_1 + \delta_{6}\alpha_{11}x_1^2x_2y_2 + \]

\[ \delta_{12}\alpha_{11}x_1x_2z_1 + \lambda\alpha_{11}^2x_1^2y_1^2 \in I \]

and, in the same way as before, comparing coefficients we find that

\[ \delta_{20} = -2\mu\alpha_{23} - \mu\alpha_{21}^2, \delta_{24} = -\delta_{9}\alpha_{21}, \]

and are left with

\[ d''_1 = \delta_{15}x_1^2y_1 + \delta_{16}x_1x_2y_1 + \delta_{17}x_2^2y_1 + \delta_{18}x_2^3y_1 + \delta_{19}x_1x_2y_2 - \]

\[ (2\mu\alpha_{23} + \mu\alpha_{21}^2)x_2^3y_2 + \delta_{21}y_1^2 + \delta_{22}y_1y_2 + \delta_{23}x_1z_1 - \]

\[ \delta_{9}\alpha_{21}x_2z_1. \]
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\[ d''_2 = (\delta_{10} a_{11} - \delta_{13})x_1x_2y_1 - (\delta_{22} a_{21} + \delta_{16})x_2^2y_1 - \delta_{17} y_1y_2 + \]
\[ (\delta_{11} a_{11} - \delta_{18})x_1x_2y_2 - (\delta_{19} + \delta_4 a_{21})x_2^2y_2 + \mu a_{23} y_2^2 - \]
\[ \delta_{21} x_1^2y_1 - \delta_{22} x_1^2y_2 + (\delta_{13} a_{11} - \delta_{23})x_2z_1 + \lambda a_{13} y_1^2. \]

\[ d''_3 = (\delta_1 a_{21} - \delta_{10} a_{11} + \delta_{15})x_2^2y_1 + (\delta_2 a_{21} + \delta_{16})y_1y_2 + \]
\[ \delta_{17} x_1 y_2 + (\delta_3 a_{21} - \delta_{11} a_{11} + \delta_{18})x_1^2y_2 + (\delta_4 a_{21} + \delta_{19})y_2^2 + \]
\[ (-\delta_5 a_{11} + \delta_{21})x_1x_2 y_1 + (-\delta_6 a_{11} + \delta_{22})x_1^2x_2 y_2 + \]
\[ (\delta_{23} - \delta_{13} a_{11})x_2z_2 - \delta_{12} a_{11}x_1^2z_1 - \delta(2 a_{13} + a_{11}^2)x_1^2 y_1. \]

So we have proved that \( r^{(3)}_1, \ldots, r^{(9)}_9 \) must be of the following form in order for the syzygies \( \Sigma_1, \ldots, \Sigma_{10} \) to lift:

\[ \text{rank } M^{(3)} = \]
\[
\begin{bmatrix}
  x_1 & -y_1 & x_2^2 - \alpha_{21} x_0 x_2 - \alpha_{23} x_0^2 & -z_1 \\
  -x_2 & x_1 - \alpha_{11} x_0 x_1 - \alpha_{13} x_0^2 & -y_2 & z_2
\end{bmatrix}
\]
gequivalent to

\[ r^{(9)}_1 : x_1^3 - x_2 y_1 - \alpha_{11} x_0 x_1^2 - \alpha_{13} x_0^2 x_1. \]

\[ r^{(9)}_2 : x_2^3 - x_1 y_2 - \alpha_{21} x_0 x_2^2 - \alpha_{23} x_0^2 x_2. \]
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\[ r^{(3)}_3 : y_1y_2 - x_1^2x_2 + \alpha_{11}x_0x_1x_2 + \alpha_{21}x_0x_1^2 \]
\[ - \alpha_{11}\alpha_{21}x_0^2x_1x_2. \]

\[ r^{(3)}_4 : x_1z_2 - x_2z_1. \]

\[ r^{(3)}_5 : x_1^2z_1 - y_1z_2 - \alpha_{11}x_0x_1z_1 - \alpha_{13}x_0^2z_1. \]

\[ r^{(3)}_6 : x_2^2z_2 - y_2z_1 - \alpha_{21}x_0x_2z_2 - \alpha_{23}x_0^2z_2. \]

and we may write, \[ r^{(3)}_7, r^{(3)}_8, r^{(3)}_9, \] as

\[ M^{(3)}P^{(3)}M^{(3)T} = \begin{bmatrix}
  r^{(3)}_7 & r^{(3)}_8 \\
  r^{(3)}_8 & r^{(3)}_9
\end{bmatrix}. \]

Here \( P^{(3)} \) is the symmetric matrix given by reducing the coefficients in the matrix \( P \) mod \( x_0^3 \) (\( P \) as in the statement of the theorem).

Since these relations have been presented in the form given in the statement of the theorem they satisfy syzygies \( \Sigma^{(3)}_1, ..., \Sigma^{(3)}_{16} \) such that \( \Sigma^{(3)}_1 \) reduces mod \( x_0 \) to \( \Sigma_1 \).

So we have demonstrated that

\[ R(3C,D^{(3)}) = \mathcal{C}(x_0,x_1,x_2,y_1,y_2,z_1,z_2)/(x_0^3, I^{(3)}). \]
where $\mathfrak{T}^{(3)}$ is generated by $r_1^{(3)}, \ldots, r_9^{(3)}$.

Step 4 : $R^{(4)} \rightarrow R^{(4+1)}$

We continue algorithmically extending $R(3C,D^{(3)})$ to $R(4C,D^{(4)})$, $R(4C,D^{(4)})$ to $R(5C,D^{(5)})$, etc until we have reached $R(7C,D^{(7)})$.

At this point the algorithm ends and we will, in fact have the ideal $\mathfrak{T}$ generated as in the statement of the theorem. The remainder of the calculations are similar to the above and the result has been stated in Theorem (14.1).

\[ \square \]

15). Surfaces of type IIIa and IIIb.

Theorem (15.1)

Let $X$ be a surface with $p_g = 3$, $K^2 = 4$ and such that the canonical linear system contains a fixed $(-2)$-cycle. Then Lemma (9.1) shows that $X$ is a surface of type IIIa or IIIb. In each case the presentation of the ring $R(C, D)$ given in Theorem (9.4a) is formative and informative in degree $-1$. In fact for $X$ a surface of type IIIa we have

$$R(X, K_X) = \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2] / \mathfrak{T}.$$  

The degrees of the generators are,

$$\text{deg } (x_1, y_1, z_1) = (1, 2, 3)$$ respectively.

$\mathfrak{T}$ is generated by $r_1, \ldots, r_9$ as follows:

$$\mathbf{A} = \begin{bmatrix} x_1 & y_2 & y_1 & z_1 \\ x_2 & x_1^2 - \alpha_{11} x_0 x_1 - \alpha_{13} x_0^2 & y_2 - \alpha_{23} x_0^2 & z_2 \end{bmatrix}.$$
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Then \( \tilde{r}_1, \ldots, \tilde{r}_6 \) are given by,

\[
\text{rank } A \leq 1
\]

\( \tilde{r}_7, \tilde{r}_8, \tilde{r}_9 \) are given by

\[
\begin{bmatrix}
\tilde{r}_7 \\
\tilde{r}_8 \\
\tilde{r}_9
\end{bmatrix} = APA^t,
\]

where \( P \) is a symmetric matrix, (cf. 14.1), and the \( P_{ij} \) are,

\[
P_{11} = h + \delta_6 x_0 x_2 (x_1 - \alpha_{11} x_0) + \delta_1 x_0 x_2 + \delta_{11} x_0 x_2 +
\]
\[
\delta_1 x_0 x_2 (x_1 - \alpha_{11} x_0) + \delta_1 x_0 x_2 + \delta_4 y_1 + \delta_6 x_0 x_2 (x_1 - \alpha_{11} x_0)
\]
\[
+ \delta_1 x_0 x_2 + \delta_1 x_0 x_2,
\]

\[
P_{12} = \frac{1}{2} (\delta_5 x_0 x_2 + \delta_6 x_0 x_2 + \delta_5 x_0),
\]

\[
P_{13} = \frac{1}{2} (\delta_4 x_0 x_2 - f x_0 (x_1 - \alpha_{11} x_0) + \delta_5 x_0 x_2 + \delta_4 x_0),
\]

\[
P_{14} = \frac{1}{2} (\delta_1 x_0 (x_1 - \alpha_{11} x_0) + \delta_9 x_0 x_2 + \delta_{10} x_0),
\]

\[
P_{22} = f x_0,
\]

\[
P_{23} = \frac{1}{2} (\delta_9 x_0),
\]

\[
P_{24} = \frac{1}{2} (\delta_{13} x_0),
\]

\[
P_{33} = \lambda y_1 + \mu y_2 + \delta_8 x_0,
\]

\[
P_{34} = \frac{1}{2} (\delta_{14} x_0),
\]

\[
P_{44} = -1.
\]

In the above, \( h, \lambda \) and \( \mu \) are as in Theorem (9.4_2) and \( \delta_1, \delta_1', \ldots, f \in \mathbb{C} \).
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The case where $X$ is a surface of type III_b is analogous.

Remark (15.2)

We only present the calculation for surfaces of type III_a here as this calculation illustrates some points of interest. In the previous cases the presentations we have considered have been formative and in fact all extensions of $R(C, D)$ in degree $-1$ have had a presentation with the same format. That is all the extensions $R(nC, D^{(n)})$ are given by an ideal $I^{(n)}$ which is generated by $x_0^8$ and relations given by the same presentation as those in $R(C, D)$. In Theorem (15.1) however, as in the example of Pinkham, this is not true. The ring $R(X, K_X)$ has the same presentation as $R(C, D)$ but not all the intermediate rings $R(nC, D^{(n)})$ have. There is an obstruction at step 7 forcing exactly the conditions upon the ideal which are required to give $R(X, K_X)$ the formative presentation. The format given in Remark (9.5) does survive at every step however.

Proof of (15.1)

Let $X$ be a surface of type III_a, $E$ be the fixed $(-2)$-cycle and $K_X = \Gamma + E$. In Theorem (9.4_a) we calculated the ring $R(C,D)$ for $D=K_X\mid C$ and found that

$$R(C,D) = C[x_1, x_2, y_1, y_2, z_1, z_2]/I,$$

where degree $(x_1, y_1, z_1) = (1, 2, 3)$ and $I$ is generated by

$$\begin{bmatrix}
x_1 & y_1 & z_1 \\
x_2 & y_1 & z_2 \\
x_1^2 & y_1 & z_1
\end{bmatrix} \leq 1$$

giving

degree 3

$$r_1: x_1^3 - x_2y_2,$$

$$r_2: x_1y_2 - x_2y_1.$$
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degree 4

\[ r_3: y_2^2 - x_1^2 y_1, \]
\[ r_4: x_1 x_2 - x_2 x_1, \]

degree 5

\[ r_5: y_2 z_2 - x_1^2 z_1, \]
\[ r_6: y_1 z_2 - y_2 z_1, \]

and

degree 6

\[ r_7: -z_1^2 + \lambda y_1^3 + \mu y_1^2 y_2 + x_1^2 h, \]
\[ r_8: -z_1 z_2 + \lambda y_1^2 y_2 + \mu y_1 y_2^2 + x_1 x_2 h, \]
\[ r_9: -z_2^2 + \lambda y_1 y_2^2 + \mu y_2^3 + x_2^2 h, \]

where, \( \lambda, \mu \in \mathbb{C} \)

and

\[ h = \lambda_1 x_1^4 + \lambda_2 x_1^3 x_2 + \lambda_3 x_1^2 x_2^2 + \lambda_4 x_1 x_2^3 + \lambda_5 y_1^2 + \lambda_6 y_1 y_2 + \lambda_7 y_2^2 + \lambda_8 x_1^2 y_2, \lambda_1 \in \mathbb{C}. \]

These relations satisfy the syzygies \( \Sigma_1, ..., \Sigma_{16} \) following

\[ \Sigma_1: x_1 r_3 - y_2 r_2 + y_1 r_1 = 0, \]
\[ \Sigma_2: x_2 r_3 - x_1 r_2 + y_2 r_1 = 0, \]
\[ \Sigma_3: x_1 r_6 - y_1 r_4 + z_1 r_2 = 0, \]
\[ \Sigma_4: x_1 r_5 - y_2 r_4 + z_1 r_1 = 0, \]
\[ \Sigma_5: x_2 r_5 - x_1 r_4 + z_2 r_1 = 0, \]
\[ \Sigma_6: x_2 r_6 - y_2 r_4 + z_2 r_2 = 0, \]
\[ \Sigma_7: y_2 r_6 - y_1 r_5 + z_1 r_3 = 0, \]
\[ \Sigma_8: x_1 r_6 - y_2 r_5 + z_2 r_3 = 0. \]
Surfaces of type III\textsubscript{a} and III\textsubscript{b}

\[ \Sigma_9: -x_2 r_7 + x_1 r_8 = (\lambda y_1 + \mu y_2) y_1 r_2 - z_1 r_4, \]
\[ \Sigma_{10}: -x_2 r_8 + x_1 r_9 = (\lambda y_1 + \mu y_2) y_2 r_2 - z_2 r_4. \]
\[ \Sigma_{11}: -x_1 r_7 + y_2 r_8 = -x_1 h r_1 + (\lambda y_1 + \mu y_2) y_1 r_3 - z_1 r_5, \]
\[ \Sigma_{12}: -x_1 r_8 + y_2 r_9 = -x_2 h r_1 + (\lambda y_1 + \mu y_2) y_2 r_3 - z_2 r_5. \]
\[ \Sigma_{13}: -y_2 r_7 + y_1 r_8 = -x_1 h r_2 - z_1 r_6. \]
\[ \Sigma_{14}: -y_2 r_8 + y_1 r_9 = -x_2 h r_2 - z_2 r_6. \]
\[ \Sigma_{15}: -z_2 r_7 + z_1 r_8 = -x_1 h r_4 - (\lambda y_1 + \mu y_2) y_1 r_6. \]
\[ \Sigma_{16}: -z_2 r_8 + z_1 r_9 = -x_2 h r_4 - (\lambda y_1 + \mu y_2) y_2 r_6. \]

**Step 0: Simplifications**

We begin our extension at this point after first making some simplifying basis changes. We wish to produce a ring

\[ R(\mathbb{X},K_{\mathbb{X}}) = \mathbb{C}[x_0,x_1,x_2,y_1,y_2,z_1,z_2]/\mathfrak{T} \]

with \( \mathfrak{T} \) generated by (with a few sign changes in comparison with the curve ring):

\[ \mathfrak{T}_1: x_1^3 - x_2 y_2 - x_0 a_1, \]
\[ \mathfrak{T}_2: x_1 y_2 - x_2 y_1 - x_0 a_2, \]
\[ \mathfrak{T}_3: x_1 z_2 - x_2 z_1 - x_0 b_1, \]
\[ \mathfrak{T}_4: x_1^2 y_1 - y_2^2 - x_0 b_2, \]
\[ \mathfrak{T}_5: x_1^2 z_2 - y_2 z_2 - x_0 c_1, \]
\[ \mathfrak{T}_6: y_1 z_2 - y_2 z_1 - x_0 c_2, \]
\[ \mathfrak{T}_7: x_1^2 - \lambda y_1^3 - \mu y_1^2 y_2 - x_1 h - x_0 d_1. \]
Surfaces of type III\textsubscript{a} and III\textsubscript{b}

\[ r_8: z_1z_2 - \lambda y_1^2y_2 - \mu y_1y_2^2 - x_1x_2h - x_0d_2, \]
\[ r_9: z_2^2 - \lambda y_1y_2^2 - \mu y_2^3 - x_2^2h - x_0d_3, \]

where \( \text{deg} (a_i, b_i, c_i, d_i) = (2, 3, 4, 5) \) such that the \( r_i \) satisfy syzygies \( \Sigma \) and \( \Sigma \) reduces mod \( x_0 \) to \( \Sigma \).

By changing coordinates we can assume that

\[ a_1 = \alpha_{11}x_1^2 + \alpha_{12}y_1x_1 + \alpha_{13}x_0x_1 + \alpha_{14}x_0, \]
\[ a_2 = \alpha_{21}x_1^2 + \alpha_{22}y_1x_1 + \alpha_{23}x_0x_1 + \alpha_{24}x_0, \]
\[ b_2 = \beta_{21}x_1 + \beta_{22}z_2 + \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0. \]

The calculations are in practice somewhat more manageable if we begin by calculating the determinantal relations so that they lift the syzygies \( \Sigma_1, ..., \Sigma_8 \) to any order. In other words we begin by using the algorithm of Section 6 on the ring

\[ R = \mathbb{C}[x_1, x_2, y_1, y_2, z_1, z_2]/\bar{I}, \]

where \( \bar{I} = (r_1, ..., r_6) \). Having lifted \( r_1, ..., r_6 \) in this ring the calculation to lift them from \( R \) is simplified.

**Step 1 : \( R \to R(2) \)**

Taking \( \Sigma_1 \), we produce expressions for \( a_1, a_2 \) and \( b_1 \) which will lift this single syzygy.

\[ \Sigma_1 : x_1b_1 - y_2a_2 + y_1a_1 \in \bar{I} \]

that is mod \( x_0 \).
Surfaces of type $\mathrm{III}_a$ and $\mathrm{III}_b$

\[ \beta_{11}x_1^3x_2 + \beta_{12}x_1^2x_2^2 + \beta_{13}x_1x_2^3 + \beta_{14}x_1^2y_1 + \beta_{15}x_1x_2y_1 + \]
\[ \beta_{16}x_1^2y_2 + \beta_{17}x_1z_1 + \beta_{18}x_2z_1 - \alpha_{21}x_1^2y_2 - \alpha_{22}y_1y_2 + \]
\[ \alpha_{11}x_1^2y_1 + \alpha_{12}y_1^2 + \]
\[ t_1y_2^2 - t_1x_1^2y_1 + t_2x_1z_2 - t_2x_2z_1 + t_3x_1^2y_2 - t_3x_1x_2y_1 + \]
\[ t_4x_1x_2y_2 - t_4x_2^2y_1 + t_5x_4 - t_5x_1x_2y_2 + t_6x_1^2x_2 - t_6x_2^2y_2 \]

where $t_1 \in \mathbb{C}$. Equating coefficients we get:

\[ a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1 + \alpha_{14}x_0^2, \]
\[ a_2 = \alpha_{21}x_1^2 + \alpha_{23}x_0x_1 + \alpha_{24}x_0^2, \]
\[ b_1 = -\alpha_{11}x_1y_1 + \alpha_{21}x_2y_1 + \beta_{11}'x_0x_1^2 + \beta_{12}'x_0x_1x_2 + \]
\[ \beta_{13}'x_0x_2^2 + \beta_{14}'x_0y_1 + \beta_{15}'x_0y_2 + \beta_{11}''x_0^2 + \]
\[ \beta_{12}''x_0x_2 + \beta_{1}'x_0^3. \]

Substituting this back into

\[ \Sigma_1 : x_1b_1 - y_2a_2 + y_1a_1 \in \tilde{I} \]

and continuing the calculation eventually leaves

\[ a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1, \]
\[ a_2 = \alpha_{21}x_1^2 + \alpha_{23}x_0x_1. \]
Surfaces of type III<sub>a</sub> and III<sub>b</sub>

\[ b_1 = -\alpha_{11}x_1y_1 + \alpha_{21}x_2y_1 + \alpha_{23}x_0x_1^2 - \alpha_{13}x_0y_1 + \alpha_{23}x_0y_2 + \alpha_{21}\alpha_{23}x_0y_1. \]

We must check that these formulas satisfy

\[ \Sigma_2 : x_2b_1 - x_1^2a_2 + y_2a_1 \in \mathcal{I} \]

It is easy to check that this is the case only if \( \alpha_{21} = 0 \).

Now using these formulas and the remaining syzygies \( \Sigma_3, \ldots, \Sigma_8 \) we calculate \( b_2, c_1 \) and \( c_2 \).

\[ \Sigma_3 : x_1c_2 - y_1b_2 + z_1a_2 \in \mathcal{I} \]

gives

\[ \gamma_{2,1}x_1x_2 + \gamma_{2,2}x_1^2x_2^2 + \gamma_{2,3}x_1x_2^3 + \gamma_{2,4}x_1^3y_1 + \gamma_{2,5}x_1^2x_2y_1 + \gamma_{2,6}x_1x_2^2y_1 + \gamma_{2,7}x_1^2x_2y_2 + \gamma_{2,8}x_1y_1^2 + \gamma_{2,9}x_1y_1y_2 + \gamma_{2,10}x_1^2z_1 + \gamma_{2,11}x_1x_2z_1 + \gamma_{2,12}x_1x_2z_2 - \beta_{21}y_1z_1 - \beta_{22}y_1z_2 + \alpha_{11}x_1^2z_1 \in \mathcal{I} \]

Setting the above expression equal to a general degree 5 element of \( \mathcal{I} \) and equating coefficients gives

\[ c_2 = x_0c_2^{'}, \]

\[ b_2 = \beta_{23}x_0y_1 + \beta_{24}x_0y_2 + \beta_{25}x_0^3 \]

Continuing in this way with \( \Sigma_3 \) and \( \Sigma_4 \) gives us

\[ a_1 = \alpha_{11}x_1^2 + \alpha_{13}x_0x_1, \]

\[ a_2 = \alpha_{23}x_0x_1, \]

\[ b_1 = -\alpha_{11}x_1y_1 - \alpha_{13}x_0y_1 + \alpha_{23}x_0y_2. \]
Surfaces of type IIIa and IIIb

\[ b_2 = 0, \]
\[ c_1 = -\alpha_{11}x_1z_1 - \alpha_{13}x_0z_1, \]
\[ c_2 = -\alpha_{23}x_0z_1. \]

The relations corresponding to these formulas are given by the following matrix and rank condition which means we do not need to check the remaining determinantal syzygies as they follow automatically from this form of the equations.

\[
\begin{bmatrix}
x_1 & y_2 & y_1 & z_1 \\
x_2 & x_1^2 - \alpha_{11}x_0x_1 - \alpha_{13}x_0^2 & y_2 - \alpha_{23}x_0^2 & z_2
\end{bmatrix}
\]

\[ \text{rank} \leq 1, \]

gives the 6 relations we have thus far calculated though we do not of course know yet whether the full generality of these equations will lift the remaining (non-determinantal) syzygies for any values of \( d_1, d_2 \) and \( d_3 \).

In fact the remainder of this calculation goes through as in the previous cases except for the existence of an obstruction in the final stage. To make this clear we produce a ring in Proposition (15.2) which extends \( R \) modulo \( x_0^6 \) but cannot be extended further. This ring exhibits all the obstructions occurring in the calculation.

\[ \square \]

**Proposition (15.2)**

Let \( R = R(C, D) \) be as in Theorem (9.4a). In the notation of the extension algorithm, we denote by \( R^{(n)} \) any ring which is formed from \( R \) at step \( (n-1) \). The rings given below (dependent on \( \alpha, \beta, \gamma, \delta, \varepsilon \)) are extensions of \( R \)

\[ R^{(6)} = \mathbb{C}(x_0, x_1, x_2, y_1, y_2, z_1, z_2)/(I^{(6)}, x_0^6) \]

where \( I^{(6)} \) is generated by:
Surfaces of type III\(_a\) and III\(_b\)

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\(r_1^{(6)}, \ldots, r_6^{(6)}\) given by

\[
\begin{bmatrix}
x_1 & y_2 & y_1 & z_1 \\
x_2 & x_1^2 - \alpha x_0 x_1 - \beta x_0^2 & y_2 - \gamma x_0^2 & z_2
\end{bmatrix}
\]

\text{rank} \leq 1.

and \(r_7^{(6)}, r_8^{(6)}, r_9^{(6)}\) given by

\(r_7^{(6)} : z_1 \rightarrow (\lambda y_1^2 + \mu y_1 y_2) y_1 - x_2 h - \delta x_1 \delta y_1 - \delta x_1 y_2\)

\(r_8^{(6)} : z_1 z_2 \rightarrow (\lambda y_1^2 + \mu y_1 y_2)(y_2 - \gamma x_0^2) - x_1 x_2 h - \delta x_1 \delta y_2\)

\(r_9^{(6)} : z_1^2 \rightarrow (\lambda y_1 + \mu y_2)(y_2 - \gamma x_0^2)^2 - x_1^2 h - \delta x_1^2(x_1^2 - \alpha x_0 x_1)\)

The above rings \(R^{(6)}\) will extend to some \(R^{(7)}\) if and only if \(\delta y = -\epsilon \beta\).

**Proof**

\(R^{(6)}\) extends \(R\) because it is given in the formative presentation of Remark (9.5).

To extend \(R^{(6)}\) to a ring \(R^{(7)}\) the following syzygies must lift to syzygies in \(R^{(7)}\):

\[\Sigma_9 : -x_2 r_7^{(6)} + x_1 r_8^{(6)} - z_1 r_4^{(6)} + (\lambda y_1^2 + \mu y_1 y_2) r_2^{(6)} + \delta x_1^2 r_2^{(6)}\]

\[+ \delta x_0^2 r_1^{(6)},\]

\[\Sigma_{10} : -x_2 r_8^{(6)} + x_1 r_9^{(6)} - z_2 r_4^{(6)} + (\lambda y_1 + \mu y_2)(y_2 - \gamma x_0^2) r_2^{(6)}\]

\[+ \delta x_1^2 r_1^{(6)}\].
Surfaces of type III_a and III_b

Let

\[ r_7' = r\Phi + x^i d_1, \]
\[ r_8' = r\Phi + x^i d_2, \]
\[ r_9' = r\Phi + x^i d_3. \]

It is easy to see that to lift \( \Sigma_9 \) we must have

\[ \Sigma_9 : -x^i f x_2 d_1 + x^i f x_1 d_2 - \delta \gamma x^i f x_1 - \varepsilon \beta x^i f x_1 \in x^i g. \]

This forces

\[ d_1 = 0 \]

and

\[ d_2 = \delta \gamma + \varepsilon \beta. \]

To lift \( \Sigma_{10} \) we need

\[ \Sigma_{10} : -x^i f x_2 d_2 + x^i f x_1 d_3 - \delta \beta x^i f x_1 \in x^i g. \]

In turn this forces

\[ d_2 = 0 \]

and

\[ d_3 = \delta \beta. \]

These conditions imply that the ring \( R^{(6)} \) can only be extended to some \( R^{(7)} \) if

\[ \delta \gamma = -\varepsilon \beta. \]

\[ \square \]
16. Deformations of surfaces and their moduli space.

In this section we draw some geometrical conclusions about surfaces with \( p_g = 3, K^2 = 4 \) and their moduli space. We determine in some cases whether surfaces of one type deform to surfaces of another. We can produce explicit families of surfaces \( \{X_t\}_{t \in \mathbb{C}} \) parametrised by \( t \in \mathbb{C} \) such that \( \forall t \neq 0, X_t \) is of type A but for \( t = 0, X_t \) is of type B whenever A and B are connected by an arrow in the following diagram:

These are produced using the explicit form of the canonical rings given in Chapter 4.

**Theorem (16.1)**

The above hierarchy holds between surfaces with \( p_g = 3, K^2 = 4 \).

**Proof.**

\( I \rightarrow II \).

We exhibit an explicit family of surfaces \( \{X_t\}_{t \in \mathbb{T}} \), with \( X_t \) of type I when \( t \neq 0 \), and of type II when \( t = 0 \).
Let

$$X_t = \text{Proj} \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/I_t$$

where $I_t$ is generated by $r_1, \ldots, r_5$ as follows,

$$r_1: x_1 y_2 - x_2 y_1 + t(z - x_0 f_1),$$

$$r_2: x_1 z - y_1 (y_1 + \lambda_5 x_0^2) + t(A - x_0 f_2),$$

$$r_3: x_2 z - y_2 (y_1 + \lambda_5 x_0^2) - t(B + x_0 f_3),$$

$$r_4: (y_1 + \lambda_5 x_0^2) (z - x_0 f_1) - x_2 (A - x_0 f_2) - x_1 (B + x_0 f_3),$$

$$r_5: (z - x_0 f_1) z - y_2 (A - x_0 f_2) - y_1 (B + x_0 f_3).$$

Here $A, B, f_1, f_2, f_3, \lambda_5$ are as in Theorem (13.1). This is in fact the ideal given by the $4 \times 4$ Pfaffians of the matrix (see section 4):

$$M_t = \begin{bmatrix}
0 & t & x_1 & x_2 & y_1 + \lambda_5 x_0^2 \\
-x_1 & -y_1 & 0 & -z & -x_0 f_1 \\
-x_2 & -y_2 & z - x_0 f_1 & 0 & B + x_0 f_3 \\
-y_1 - \lambda_5 x_0^2 & z & A - x_0 f_2 & -B & -x_0 f_3 & 0
\end{bmatrix}.$$

When $t \neq 0$, we can eliminate $z$ from these relations using $r_1$, this gives

$$z = x_0 f_1 - (x_1 y_2 - x_2 y_1)/t.$$

Substituting into all the other relations we get that

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\[ r_4^t = (x_1 r_3^t - x_2 r_2^t)/t, \]

\[ r_5^t = (y_1 r_3^t - y_2 r_2^t)/t \]

and \( r_2^t \) and \( r_3^t \) are general degree 4 relations in \( x_0, x_1, x_2, y_1, y_2 \). Thus for \( t \neq 0 \) we have a surface of type I but for \( t = 0 \) we have the standard ring for a surface of type II.

In this case the Pfaffian form of the relations for a surface of type II lends itself to a quick and easy way of writing down a deformation, however for surfaces of type III writing down an explicit flat deformation is rather a large calculation.

**I \rightarrow III**

In Proposition (10.2) we have exhibited a structured deformation for the canonical curves in this case. However for the surfaces such a deformation has not yet been found. We therefore refer to [H01] Theorem (2.3) to show that a deformation does exist. We hope to find an explicit example at a later date.

**III \rightarrow III_a**

Let

\[ X_t = \mathcal{C}(x_0, x_1, x_2, y_1, y_2, z_1, z_2)/I_t, \]

where \( I_t \) is generated by \( r_1^t, ..., r_6^t \) as follows:

\[ r_1^t, ..., r_6^t \] are given by
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\[ \text{rank } M_t = \text{rank } \begin{bmatrix} x_1 & y_1 & y_2 & z_1 \\ x_2 & g_1 & y_1 + \alpha x_0^2 + ty_2 & z_2 \end{bmatrix} \leq 1. \]

Here \( g_1 = x_1^2 + \alpha_{11} x_0 x_1 + \alpha_{13} x_0^2, \alpha_{11}, \alpha_{13} \in \mathbb{C}. \)

\( r_1, r_8, r_9 \) are given by

\[ M_t P(M_t)^T = \begin{bmatrix} r_1 & r_8 \\ r_8 & r_9 \end{bmatrix}, \]

for some symmetric matrix \( P. \)

Notice that \( X_0 \) is precisely a surface of type \( \text{III}_a. \) For \( t \neq 0 \) it is easy to see that a series of row and column operations with changes of basis will put the matrix \( M_t \) into the form which gives a surface of type \( \text{III} \) (see (10.1)).

\( \text{III}_a \rightarrow \text{III}_b \)

A deformation can be found analogous to the one for curves in the canonical linear system (see (10.1)). That is, considering the form of the rings given in Theorem (15.1), the deformation is given by the parameter \( \lambda. \)

\[ \square \]

Number of moduli.

There are several possible ways of calculating the dimension of the moduli spaces. In [Ho1] Horikawa computed the dimension of \( H^1(X, \Theta_X) \). It is also possible to count the number of parameters needed for the construction of the curves in \( P^2 \) forming the branch locus of the double cover in cases \( \text{III}_a, \text{III}_b \) and \( \text{III}_b \) (see section 17). Since we have calculated the
form of the canonical ring for all these surfaces, we can count the number of parameters on which this construction depends. We aim to show the table below. In general the calculations do not give a rigorous proof of the dimensions of the moduli spaces, but a lower bound. [Ho1] gives 32 for the dimension of the moduli space for surfaces of type I which then serves as an upper bound for the other types.

<table>
<thead>
<tr>
<th>Type of surface</th>
<th>Number of moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>32</td>
</tr>
<tr>
<td>II</td>
<td>31</td>
</tr>
<tr>
<td>III</td>
<td>31</td>
</tr>
<tr>
<td>IIIa</td>
<td>30</td>
</tr>
<tr>
<td>IIIb</td>
<td>29</td>
</tr>
</tbody>
</table>

Type I

\( X = \text{Proj} \mathbb{C}(x_0, x_1, x_2, y_1, y_2)/\left<Q_1, Q_2\right> \)

where \( Q_1 \) and \( Q_2 \) are generic degree 4 polynomials. We must count the number of parameters on which \( Q_1 \) and \( Q_2 \) depend and take into account that we are only interested in the subvariety of the weighted projective space \( \mathbb{P}(1^3, 2^2) \) which they define. We do not wish to count surfaces which only differ from each other by an automorphism of \( \mathbb{P}(1^3, 2^2) \).

The total number of degree 4 monomials in \( x_0, x_1, x_2, y_1, y_2 = 30 \)

Number of parameters for \( Q_1 = 30 \)

We can divide \( Q_1 \) by any nonzero coefficient and reduce it modulo \( Q_2 \), therefore we need only count 28 parameters. In the same way we count 28 for \( Q_2 \).

Let \( A = \text{group of automorphisms of} \mathbb{P}(1^3, 2^2) \). Then we can check that \( A \) acts faithfully on \( \{(Q_1, Q_2)\} \) and rank \( A = 24 \).
Therefore the number of moduli $= 28 + 28 - 24 = 32$.

Type II

$X = \text{Proj } \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z]/I$, where $I$ is generated by the $4 \times 4$ Pfaffians of the following matrix:

$$
\begin{vmatrix}
0 & 0 & l_1 & l_2 & q_1 \\
0 & 0 & q_2 & q_3 & c_1 \\
-l_1 & -q_2 & 0 & c_2 & f_1 \\
-l_2 & -q_3 & -c_2 & 0 & f_2 \\
-q_1 & -c_1 & -f_1 & -f_2 & 0
\end{vmatrix}
$$

where the $l_i$ are generic degree 1, $q_i$ are generic degree 2, $c_i$ are generic degree 3 and the $f_i$ are generic degree 4. By exactly analogy to the previous calculation we compute the number of variables on which the matrix depends and take into account that we are only interested in the subvariety of $\mathbb{P}(1^3, 2^2, 3)$ which they define. We then consider the number of automorphisms of the ambient weighted projective space and whether it acts faithfully on the set of ideals.

Number of parameters which entries in matrix depend upon $= 130$

Taking each relation in turn we are allowed to divide through by a nonzero coefficient and reduce modulo the linearly independent relations of the same degree. A short calculation shows that this allows us to take off 58.

Rank of group of automorphisms of $\mathbb{P}(1^3, 2^2, 3) = 41$

Assuming that the group acts faithfully on our set of ideals, the number of moduli $= 130 - 58 - 41 = 31$. 

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Type III

By Remark (9.5) we have that
\[ X = \text{Proj} \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2]/I, \]
where I is generated by
\[
\begin{bmatrix}
  l_1 & q_1 & q_2 & c_1 \\
  l_2 & q_3 & q_4 & c_2
\end{bmatrix}
\]
and
\[
\begin{align*}
  r_7: & l_1 A_1 + q_1 A_2 + q_2 A_3 + c_1 A_4 \\
  r_8: & l_2 A_1 + q_3 A_2 + q_4 A_3 + c_2 A_4
\end{align*}
\]
which is identically equal (in \( \mathbb{C}[x_0, x_1, x_2, y_1, y_2, z_1, z_2] \)) to
\[
\begin{align*}
  r_8': & l_1 B_1 + q_1 B_2 + q_2 B_3 + c_1 B_4 \\
  r_9: & l_2 B_1 + q_3 B_2 + q_4 B_3 + c_2 B_4
\end{align*}
\]
where degree \( l_i = 1 \), degree \( q_i = 2 \), degree \( c_i \), \( A_4 \), \( B_4 = 3 \), degree \( A_2, B_2, A_3, B_3 = 4 \), and degree \( A_1, B_1 = 5 \). All the polynomials can be general under the restriction imposed by \( r_8 = r_8' \).

Now we count the number of parameters required to construct the matrix and \( A_1, ..., A_4 \) and then the freedom of choice in forming \( B_1, ..., B_4 \).

- Number of parameters required for \( l_1, q_1, q_2, c_1 = 37 \),
- Number of parameters required for \( l_2, q_3, q_4, c_2 = 37 \),
- Number of parameters required for \( A_1, ..., A_4 = 156 \).

The expression \( l_1 B_1 + q_1 B_2 + q_2 B_3 + c_1 B_4 \) has 193 separate monomials. Thus the freedom of choice for \( B_1, ..., B_4 = 193 - \) number of degree 6 monomials
\[ = 193 - 115 \]
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= 78.

Therefore we have, so far, $156 + 78 + 37 + 37 = 308$ parameters.

As before, taking each relation in turn we can divide through by a nonzero coefficient and then reduce modulo the linearly independent relations of the same degree. This means that we can take off

$$2 + 2 + 8 + 8 + 22 + 22 + 51 + 51 + 51 = 217.$$ 

Rank of group of automorphisms of $\mathbb{P}(1^3,2^2,3^2) = 60$.

Assuming that the group acts faithfully, the total number of parameters on which our construction depends is $308 - 217 - 60 = 31$ as required.

Types $\text{III}_a$ and $\text{III}_b$

These follow in the same way as case $\text{III}$.

17. The $1$–canonical map.

In [Ho1] Horikawa shows that surfaces of type $\text{III}$ and $\text{III}_a$ are birationally equivalent to double covers of $\mathbb{P}^2$, and he describes the branch locus in each case. We show the same result by an explicit calculation of the branch locus and do the case $\text{III}_b$ in addition (see Introduction).

Theorem (17.1).

Let $X$ be a surface of type $\text{III}$, $\text{III}_a$ or $\text{III}_b$. Then $X$ is birationally equivalent to a double cover of $\mathbb{P}^2$ with branch curve as follows:

Type $\text{III}$. A degree 12 curve, made up of a degree 10 curve $f_{10}$ and two lines $l_1 : (x_1 = 0)$, $l_2 : (x_2 = 0)$. $f_{10}$ has two triple points (possibly infinitely near) on each $l_i$ and a quadruple point at $l_1 \cap l_2$. 
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Type IIIa. A degree 10 curve, having two triple tacnodes (possibly infinitely near) and a quadruple point all lying on a line.

Type IIIb. A degree 10 curve, made up of a degree 9 curve \( f_9 \) and a line \( l : (x_2 = 0) \). \( f_9 \) has 3 triple points lying on \( l \) as in the diagram below.

Proof

In each case the 1-cannotical map gives a double cover of \( \mathbb{P}^2 \) and we can recover the branch locus from the canonical ring by completing the square. We rewrite the relation \( \pi_7 \) as

\[
\pi_7 : z^2 - h_6
\]

and eliminate the variables \( y_1, y_2, z_2 \). By making these eliminations we find the generators and relations of the integral closure of \( \mathbb{C}[x_0, x_1, x_2] \) in \( R(X, K_X) \).
We consider each case in turn.

**Type III**

Refering to Theorem (14.1) we see that we can write

\[ r_7 : (z_1 - \frac{1}{2} \delta_{12} x_0 x_1 y_1 - \frac{1}{2} \delta_{08} x_0 - \frac{1}{2} \lambda_{15} x_1^3 - \frac{1}{2} \lambda_{16} x_1^2 x_2 - \frac{1}{2} \lambda_{18} x_1 x_2^2 - \]
\[ \frac{1}{2} \delta_{13} x_0 x_1^2 + \frac{1}{2} \delta_{13} \alpha_{11} x_0 x_1 - \frac{1}{2} \delta_{23} x_0 x_1^2 \]
\[ = \frac{1}{4} \delta_{12} x_0 x_1^2 + \frac{1}{4} \delta_{08} x_0 x_1 + \frac{1}{4} \lambda_{15} x_1^3 + \frac{1}{4} \lambda_{16} x_1^2 x_2 + \frac{1}{4} \lambda_{18} x_1 x_2^2 + \frac{1}{4} \delta_{13} x_0 x_1^2 + \frac{1}{4} \delta_{23} x_0 x_1^2 + \]
\[ \frac{1}{4} \delta_{13} \alpha_{11} x_0 x_1^2 + \frac{1}{4} \delta_{23} x_0 x_1^2 + x_1 \Lambda_1' + y_1 A_2' + g_2 \Lambda_3', \]

where

\[ g_1 = x_1^2 - \alpha_{11} x_0 x_1 - \alpha_{13} x_0^2, \]
\[ g_2 = x_2^2 - \alpha_{21} x_0 x_2 - \alpha_{23} x_0^2 \]

and

\[ \Lambda_1' = x_1 (x_1 P_{11} + y_1 P_{12} + g_2 P_{13} + z_1 P_{14}), \]
\[ A_2' = y_1 (x_1 P_{12} + y_1 P_{22} + g_2 P_{23} + z_1 P_{24}), \]
\[ A_3' = g_2 (x_1 P_{13} + y_1 P_{23} + g_2 P_{33} + z_1 P_{34}) \]

after the removal of all monomials involving \( z_1 \) and \( z_2 \). Now we use \( r_1 \) and \( r_2 \) to write \( y_1 = x_1 g_1 / x_2 \) and \( y_2 = x_2 g_2 / x_1 \) respectively. Substituting this into the above and clearing the denominator gives

\[ r_7' : x_1 x_2 (x_2 z_1 - g)^2 = f_{10}, \]

for some degree 10 polynomial \( f_{10}(x_0, x_1, x_2) \) and some \( g \) derived from the expression for \( r_7 \) above.
Geometrical consequences

By multiplying both sides by $x_1 x_2$ we produce the required equation,

$$17'' : (z')^2 = f_{12}$$

where $f_{12} = x_1 x_2 f_{10}$.

Consider the intersection of the curve $f_{10}$ with the two lines $l_1 : (x_1 = 0)$ and $l_2 : (x_2 = 0)$. We find that

$$l_1 \cap f_{10} = \{ [x_0, x_1, x_2] \in \mathbb{P}^2 | x_1 = 0, -\mu x_2^3 = 0 \},$$

and

$$l_2 \cap f_{10} = \{ [x_0, x_1, x_2] \in \mathbb{P}^2 | x_2 = 0, -\lambda x_1^4 = 0 \}.$$  

Thus $f_{10}$ intersects $l_1$ and $l_2$ as stated in the theorem.

Type IIIa

Working as in the previous case we find that we can write

$$r_9' : x_2 (z_2 - g)^2 = f_{10},$$

which we can write down as

$$r_9'' : (z')^2 = f_{10}$$

where

$$f_{10} = (x_1 g_1 - \alpha_{23} x_0^2 x_2)^2 (\lambda x_1^2 g_1 - \lambda \alpha_{23} x_0^2 x_1 x_2 + \mu x_1 g_1 x_2 + \delta g_2 x_0 x_2^2) + x_2 \text{ (other terms)}$$

Considering the line $l : (x_2 = 0)$ we find that it meets $f_{10} = 0$ where $-\lambda x_1^4 g_1^3 = 0$. This shows that there is a quadruple point at $[1:0:0]$ and triple points at $[1: P: 0]$ and $[1: Q: 0]$ where $P$ and $Q$ are the roots of $x_1^2 - \alpha_{11} x_1 - \alpha_{13} = 0$.

The tangent lines at $[1: P: 0]$ are given by

$$P \lambda (x_1 (P - Q) P - \alpha_{23} x_2^3 = 0.$$
Similarly at \([1:Q:0]\) the tangent lines are given by

\[ Q\lambda(x_1(Q - P)Q - \alpha_{23}x_2)^3 = 0. \]

This proves the theorem in this case.

**Type III_b**

Working as in case III_a we now have \(\lambda = 0\). Thus \(r_9\) becomes

\[ r_9'' : (z')^2 = x_2 f_9 \]

where

\[ f_9 = (x_1y_1 - \alpha_{23}x_0^2x_2)^2(\mu x_1y_1 + \delta x_0^2x_2) + x_2 (\text{other terms}). \]

Considering the line \((x_2 = 0)\) we find that it meets \(f_9 = 0\) where \(\mu x_1^3y_1^3 = 0\). This shows that there are triple points at \([1:0:0]\), \([1:P:0]\) and \([1:Q:0]\) where \(P\) and \(Q\) are the roots of \(x_1^2 - \alpha_{11}x_1 - \alpha_{13} = 0\). The tangent lines at \([1:P:0]\) are given by

\[ (Px_1(P - Q) - \alpha_{23}x_2)^2(\mu Px_1(P - Q) + \delta x_0^2x_2) = 0, \]

and at \([1:Q:0]\) by

\[ (Qx_1(Q - P) - \alpha_{23}x_2)^2(\mu Qx_1(Q - P) + \delta x_0^2x_2) = 0. \]

This proves the theorem in this case.

\[ \square \]

Another consequence of our calculations is that it is possible to find specific curves on the surfaces. Surfaces of type II, for example, can be shown to contain an elliptic curve which can be contracted to an elliptic Gorenstein singularity of type \(k=1\), [R1]. We show this in the next theorem.
Theorem (17.2)

Let $X$ be a surface of type II. Then there is an elliptic curve $E \subset X$ and a numerical quintic $\tilde{X}$ with a single elliptic Gorenstein singularity $P$ of type $k = 1$ and a contraction map $\pi : X \to \tilde{X}$ such that $\pi(E) = P$.

Proof

Refer to Theorem (13.1) for our description of the canonical ring of $X$. Consider the subvariety of $X$ given by

$$x_1 = x_2 = y_1 + \lambda_5 x_0^2 = 0.$$ 

Then $r_1, \ldots, r_4$ are all trivially satisfied by these conditions and $r_5$ gives the following equation,

$$r_5 : \lambda_5 x_0^2 (B + v x_0^4 + q_2 x_0^2 y_2 - \lambda_5 q_4 x_0^4) +$$

$$y_2 (-A - 10 x_0 z - v_1 x_0^4 - q_3 10 x_0^3 y_2) - z (-z + u_2 x_0^3).$$

This is the equation of a degree 6 curve, $E$ in $\mathbb{P}(1,2,3)$. It is easy to check that it is nonsingular and hence elliptic. The base point of the canonical system is given by the equation $z^2 = \lambda y_2^3$. Of course, this point lies on $E$ and $|K_X|$ goes through $E$ only at this single point and is trivial off it. This is enough to prove the theorem (see [R1]).
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Surfaces with $pg = 3$ and $K^2 = 4$
and
extension-deformation theory

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Duncan Dicks.

University of Warwick 1988

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