ISSN 1560-3547, Regular and Chaotic Dynamics, 0000, Vol. 00, No. 0, pp. 1–6. © RCD Editorial Office, 2016.

Finding the complement of the invariant manifolds transverse to a given foliation for a 3D flow

Robert S. MacKay[1*]

[1] *Mathematics Institute, University of Warwick*
Gibbet Hill Road, Coventry CV4 7AL, U.K.
Received September 8, 2018; accepted Month XX, 20XX

Abstract—A method is presented to establish regions of phase space for 3D vector fields through which pass no co-oriented invariant 2D submanifolds transverse to a given oriented 1D foliation. Refinements are given for the cases of volume-preserving or Cartan-Arnol'd Hamiltonian flows and for boundaryless submanifolds.

MSC2010 numbers: 37J30, 70H07

DOI: 10.0000/S1560354700000012

Keywords: Converse KAM theory, conefields, volume-preserving, Hamiltonian

1. INTRODUCTION

Jürgen Moser was a great influence on me, through his publications, his kind interest in my work and his helpful suggestions. In particular, I owe to him the genesis of my paper [9] on a method[1)] to establish regions through which pass no rotational invariant tori of a positive-definite[2)] Hamiltonian system.

In the present paper I present an extension of [9] in the 3D case to general 3D systems with no twist assumption. My main motivation is to establish regions of a magnetic field through which pass no flux surfaces (invariant 2-tori) transverse to a given 1D foliation, without assuming shear, and then to do the same for guiding-centre motion. The method does not require Hamiltonian structure, and indeed [3] already made an application of the idea of [9] to a dissipative system, but the principal intended applications are to systems that can be viewed as 3D Hamiltonian vector fields (in the odd-dimensional approach of Cartan-Arnol'd).

KAM theory (named after Kolmogorov, Arnol'd and Moser) constructs a set (often of positive volume) of invariant tori. Although some current versions [4] are able to obtain nearly all the numerically believed invariant tori of a given class, they still require high sophistication in both theory and application.

A converse KAM theory for the related case of area-preserving twist maps of the annulus or cylinder, first put forward by Mather [12] and extended by Percival and me [11], establishes regions through which pass no rotational[3)] invariant circles (RICs). It uses the result that every orbit segment on an RIC for an exact area-preserving twist map minimises the action sum subject to fixed ends. The paper [11] also showed that the method can be interpreted geometrically in terms of conefields: for any orbit on an RIC there are neighbourhoods of the upward and downward vertical vector fields such that it is impossible to iterate from one to the other by the derivative of the map. Thus if one finds a tangent orbit which does cross from one to the other then the underlying orbit

---

[*] E-mail: R.S.MacKay@warwick.ac.uk

[1)] I called the method a "criterion" but later learnt that in mathematics it is usual to reserve the word "criterion" for a necessary and sufficient condition, whereas all I meant was a sufficient condition.

[2)] meaning the second derivative with respect to momenta is positive-definite

[3)] "rotational" means homotopic to a graph of momentum as a function of the angle variable does not lie on an RIC. Stark [14] proved that if augmented in a simple way then the method will, with enough work, find any point of the complement of the set of RICs.

When I presented the method of [11] at a conference [8], Moser suggested I could extend the approach to continuous-time positive-definite Hamiltonian systems of arbitrary degrees of freedom (DoF) by using a theorem of Weierstrass, that every orbit on an invariant Lagrangian graph for such a system is minimising[4)]. Indeed this suggestion worked nicely [9], though I tested it on only 1.5 DoF systems (time-periodic Hamiltonians in 1 DoF). The idea is related to that of Green bundles in Riemannian geometry [5]. Green proved that if a geodesic is minimising (equivalently, has no conjugate points) then it has a stable bundle and an unstable bundle to which respectively the backward and forward tangent orbits of the vertical converge.

I extended the method in discrete-time to higher dimensional symplectic twist maps with Meiss and Stark [10]. I also made a geometric interpretation of the continuous-time method in arbitrary DoF in terms of the induced flow on Lagrange planes and the Maslov cycle [1], which I presented at a Los Alamos workshop in 1991 but never published.

Back in 1983, Percival and I wanted to extend the method of [11] for area-preserving maps to more general surfaces than the cylinder, where the concept of twist may not apply, and in particular to treat invariant circles in islands. We started writing a paper on this but it was never finished. Here I return to this problem and solve it in the context of 3D vector fields, in particular for the Hamiltonian case.

2. THE METHOD

Although the principal intended application is to Hamiltonian systems (time-periodic 1DoF systems or the restriction to a regular[5)] energy level of an autonomous 2DoF Hamiltonian system or the 3D case of the Cartan-Arnol'd notion of odd-dimensional Hamiltonian system to be recalled shortly), the method can be formulated for any 3D C^1 flow ϕ generated by a C^1 vector field v:

$$\partial_t \phi_t(x) = v(\phi_t(x)), \tag{2.1}$$

where ∂_t denotes the derivative with respect to time t.

We suppose a 1D C^1 oriented foliation $\mathcal{F}$ of the 3D space and want to find a large part of the set of points through which there is no co-oriented invariant 2D submanifold transverse to $\mathcal{F}$. The submanifolds need not be C^1; topological transversality is enough. Although our primary interest is in eliminating KAM tori, the method makes no such restriction. Thus it can also eliminate homoclinic separatrices to periodic orbits, for example. Boundaryless surfaces with genus not equal to 1 have to include equilibrium points of v, however, so they may be excluded in many contexts (e.g. 1.5DoF Hamiltonian systems).

In the case of a 1.5DoF Hamiltonian system $H(q, p, t)$ positive definite in the momentum p, the standard choice of foliation is by the verticals, $q, t =$ constant, as used in [9]. In other contexts, however, other choices are relevant. For example, in a toroidal magnetic confinement device for plasma the relevant foliations are called "radial", being by curves of constant poloidal and toroidal angles relative to a magnetic axis (a closed field line that is a coordinate singularity for the poloidal angle). There is not in general a natural choice of such angle coordinates nor foliation, but the goal is to have many magnetic flux surfaces transverse to such a foliation. The method presented here determines points which do not lie on such flux surfaces, which we'd like to form a small set. Even for 1.5DoF positive-definite Hamiltonian systems, if one wishes to study vibrational tori around a periodic orbit then one needs to choose a "radial" foliation centred on the periodic orbit and when its multipliers are not near 1 there may be no foliation with twist [13]. Similarly, to treat the "banana" tori of guiding centre motion requires a careful choice of foliation.

The method works by contradiction. Suppose there is a co-oriented invariant 2D submanifold M transverse to $\mathcal{F}$. Take a point $x \in M$ and a tangent vector ξ at x to $\mathcal{F}$ in the positive direction.

4) which incidentally forms the content of Hilbert's 23rd problem, the unique one in his list that was not really a problem but rather an excuse to talk about his new proof of Weierstrass' result.

5) i.e. containing no critical points of the Hamiltonian

Let it flow to its pushforward $\phi_{t*}\xi$ $(= D\phi_t(x)\xi)$ under the tangent flow after time $t \in \mathbb{R}$. This can be computed by integrating

$$\partial_t \ \phi_{t*}\xi = Dv_{\phi_t(x)} \ \phi_{t*}\xi \tag{2.2}$$

in any coordinate system, where $Dv_{\phi_t(x)}$ is the derivative of v at $\phi_t(x)$. Then $\phi_{t*}\xi$ must remain on the same side of M as ξ. In particular, it can never become $\eta + cv$ for any η tangent to $\mathcal{F}$ at $\phi_t(x)$ in the negative direction and $c \in \mathbb{R}$. Choose an oriented 2D foliation $\mathcal{P}$ containing $\mathcal{F}$ and transverse to v, let π be the projection of a tangent vector along v into the tangent space to the leaf of $\mathcal{P}$ at $\phi_t(x)$ and choose an inner product on the leaves of $\mathcal{P}$. If there is an interval I of t for which the oriented angle θ of $\pi\phi_{t*}\xi$ with a negative tangent η to $\mathcal{F}$ at $\phi_t(x)$ changes sign, then by continuity there is a $t \in I$ for which $\phi_{t*}\xi = k\eta + cv$ for some $k > 0$, $c \in \mathbb{R}$. Hence x does not lie on such an M.

We summarise this as:

Theorem 1. *Given $x \in M$, positively oriented tangent ξ to $\mathcal{F}$ at x and the projection π, if there is an interval I of t in which the oriented angle θ of $\pi\phi_{t*}(x)$ to a negative tangent η to $\mathcal{F}$ at $\phi_t(x)$ changes sign, then x does not lie on any co-oriented invariant 2D submanifold of M transverse to $\mathcal{F}$.*

There are several elements of this method that require construction besides the foliation $\mathcal{F}$, namely the 2D foliation $\mathcal{P}$ and the inner product on its leaves.

In the Hamiltonian case, we can avoid some of this arbitrariness. Specifically, let us consider Cartan-Arnol'd Hamiltonian systems [2]. They are defined by a 1-form α on an odd-dimensional manifold ($3D$ in the present case) such that the kernel of the exterior derivative $d\alpha$ is 1D everywhere.[6] The dynamics is given by the integral curves of the kernel (up to time-parametrisation).

The most familiar class of examples is time-periodic Hamiltonian systems $H(q, p, t)$ in canonical coordinates, for which

$$\alpha = p.dq - H(q, p, t) \ dt, \tag{2.3}$$

defined on the product of phase space and a circle for time t. Volume-preserving flows such as along magnetic fields, however, give another class of examples: for nowhere-zero[7] divergence-free field B in Euclidean space with a vector potential A (so that $B = \text{curl } A$), take $\alpha = A.dx$.[8] Thirdly, for an autonomous (even-dimensional) Hamiltonian system in canonical coordinates (q, p) on a regular component of an energy level $H(q, p) = E$, take $\alpha = p.dq$ restricted to the energy level. Fourthly, guiding centre motion (to first order) for a particle with charge e, mass m, energy E and magnetic moment μ corresponds to

$$\alpha = (eA + mv_\parallel b).dx \tag{2.4}$$

on the 3D (regular components of) $H^{-1}(E)$, where

$$H(x, v_\parallel) = \frac{1}{2}mv_\parallel^2 + \mu|B(x)| \tag{2.5}$$

for $x \in \mathbb{R}^3$ and $v_\parallel \in \mathbb{R}$, and $b = B/|B|$ [7].

Note that the definition of a Cartan-Arnol'd system does not determine a time-parametrisation of the trajectories. One can usually make a natural choice of time-parametrisation, however, and hence of vector field v. For a time-periodic Hamiltonian then t is the natural choice of time-parametrisation. For a 1-form corresponding to a vector field B preserving a volume-form Ω, it is usual to take B as the vector field, though for some purposes it is better to take $b = B/|B|$, which then preserves the volume-form $|B|\Omega$. For an autonomous Hamiltonian system on a regular energy level, v can be determined by $i_v\omega = dH$, where ω is the symplectic form $-d\alpha$ and H the

[6] As the manifold is odd-dimensional, $d\alpha$ is degenerate, so the kernel being 1D just says $d\alpha$ is minimally degenerate.

[7] Zeroes of B correspond to places where $d\alpha = 0$ so the kernel is 3D there.

[8] This can be extended to 3D manifolds with arbitrary Riemannian metric: $i_B\Omega = dA^\flat$, where Ω is a volume-form and $\alpha = A^\flat$ is the 1-form defined by $A^\flat(u) = \langle A, u\rangle$ for all vectors u. Note that volume-preservation ($L_B\Omega = 0$ and $\int_S i_B\Omega = 0$ for all oriented closed surfaces S) implies existence of a vector potential A.

Hamiltonian function, so it uses gradient information of H on the energy level. For guiding centre motion, we similarly consider the motion for all E, given by letting $\omega = -d\alpha$ on the 4D space of $(x, v_\parallel)$ and H as above; this gives the natural choice that $v_\parallel = b.\dot{x}$.

For a 3D Cartan-Arnol'd system and a choice of associated vector field v, we can detect the above sign change of θ as a sign change of

$$K(t) = \Omega(v, \phi_{t*}\xi, \eta), \tag{2.6}$$

where Ω is a volume-form. There is a natural volume form, namely

$$\Omega = \frac{v^\flat}{|v|^2} \wedge d\alpha. \tag{2.7}$$

Note that it is preserved by v:

$$L_v\Omega = di_v\Omega = d(d\alpha - v^\flat \wedge i_v d\alpha) = 0,$$

because $d^2 = 0$ and $i_v d\alpha = 0$. For the choice (2.7) then

$$K(t) = d\alpha(\phi_{t*}\xi, \eta). \tag{2.8}$$

In the case of a divergence-free vector field B one can choose Ω to be the standard volume-form (such that $di_B\Omega = 0$). Actually, this agrees with the choice (2.7), because for $\alpha = A.dx$ with curl $A = B$, then $d\alpha = i_B\Omega$ with Ω the standard volume-form.

The only problem with this way of detecting sign changes of θ is that it equally well detects θ crossing π. So it has to be supplemented by restricting attention to intervals of trajectory for which $\phi_{t*}\xi$ is closer to η than $-\eta$. One way to do this is to choose a rotation R around v by roughly $\pi/2$, and restrict attention to intervals of t for which $d\alpha(\phi_{t*}\xi, R\eta)$ has the same sign as $d\alpha(\eta, R\eta)$. This saves computing the projection π because $i_v d\alpha = 0$.

With this supplement, the method is ready to apply to 3D Cartan-Arnol'd systems without needing to choose the foliation $\mathcal{P}$, nor an inner product on the leaves of $\mathcal{P}$, nor to compute the projection π, just choosing the rotation R.

3. EXHAUSTIVENESS

One question is whether with enough work the condition of section 2 would obtain all but an arbitrarily small volume of the complement of the union of the invariant submanifolds transverse to the foliation. There are two immediate obstructions.

The first is that in the positive-definite case with vertical foliation, there can be other minimising orbits besides those on invariant tori, for example, minimising periodic orbits and orbits on cantori. For these the image of the vertical tangent never points in the opposite direction[9)], so they are not eliminated by the condition, though if they are hyperbolic then they occupy zero volume (except if the dynamics is Anosov, as in [6], but that can happen only on special manifolds).

The second is that for a locally symplectic but not Hamiltonian system (i.e. vector field v such that $i_v\omega$ is closed but not exact) there are no invariant graphs, but the condition of section 2 might not eliminate everything; consider, for example, an integrable system composed with a commuting vertical drift.

We can take care of these obstructions by supplementing the condition of section 2, provided we restrict attention to boundaryless submanifolds.

Firstly, I need to reformulate the condition so as to obtain bounds on the slope of possible invariant submanifolds transverse to $\mathcal{F}$ through a given point. Given a point x calculate $Q_t(x) = D\phi_t(x)^{-1}$ for an interval T of t around 0, using the tangent equations. Concretely, integrate the matrix equation

$$\partial_t Q_t(x) = -Q_t(x) Dv_{\phi_t(x)}, \tag{3.1}$$

from $Q_0(x) = I$. Apply $Q_t(x)$ to the tangent ξ to the foliation $\mathcal{F}$ at $\phi_t(x)$, obtaining the pullback $\phi_t^*\xi$ of ξ to x. Project it along the vector field $v(x)$ to a transverse plane $\mathcal{P}$ at x and call the result

9)this would make a pair of conjugate points and then any longer orbit segment would not be minimising

ξ_t. Then the slope in $\mathcal{P}$ of any invariant submanifold through x transverse to $\mathcal{F}$ (in the sense of all possible limit points of sequences of finite differences, in case it is not differentiable) is contained between the set S of directions of $\{\xi_t : t \in T\}$ and its negative. If the complement to $S \cup (-S)$ is empty then there is no such invariant submanifold, which reformulates the condition of section 2.

We summarise this as:

Theorem 2. *Given vector field v, 1D foliation $\mathcal{F}$, $x \in M$, interval $T \subset \mathbb{R}$ containing 0 and plane $\mathcal{P}$ transverse to v in the tangent space at x, let S be the set of directions of ξ_t for $t \in T$ obtained above. Then the slope of any co-oriented invariant 2D submanifold of M transverse to $\mathcal{F}$ is in the complement of $S \cup (-S)$ in $\mathcal{P}$. In particular, if the complement is empty then there is no such submanifold.*

In the case with twist the motion of ξ_t is monotone in rotational order on $\mathcal{P}$, so it suffices to follow it at the endpoints of T, but the present generalisation is more powerful when there is not twist.

The advantage of this reformulation is that it constrains boundaryless invariant submanifolds through points that have not been eliminated by the condition of section 2, as follows. The requirement that the slope in $\mathcal{P}$ of an invariant submanifold lie in the complement of $S \cup (-S)$ defines a differential inclusion. Specifically, the tangent plane to the submanifold contains v and its intersection with $\mathcal{P}$ is in the complement of $S \cup (-S)$. Suppose that integrating the resulting differential inclusion from the boundary of regions that have already been eliminated rules out any such invariant submanifold passing through x, then x can be added to the eliminated set. This is a version of the procedure called “killends” in [11], but not requiring twist.

In the case of a Hamiltonian system or a volume-preserving vector field, the combination of these two steps eventually eliminates any point not in the union of the boundaryless invariant submanifolds transverse to $\mathcal{F}$, by extending the result of [14].

In more general cases, we have to add another test. The method so far looks only at tangent information, so for example does not pay attention if there is a net drift along $\mathcal{F}$. For Hamiltonian or volume-preserving systems the average drift along $\mathcal{F}$ is zero so we don't have to consider net drift, but in general there may be a drift that prevents invariant submanifolds and this might not be detected by looking at tangent orbits.

The additional test is to see if any point $\phi_t(x)$ of the orbit of x lands in the region above or below x given by integrating the field of directions S or its negative, plus arbitrary amount of v or its negative, from x. Then x can be added to the set of points through which pass no invariant submanifolds transverse to $\mathcal{F}$. With the addition of this test, I believe that any point not on such an invariant submanifold will eventually be eliminated.

4. CONCLUSION

I have presented a method, not requiring twist, to determine regions in a 3D flow through which pass no co-oriented invariant 2D submanifolds transverse to a given oriented 1D foliation. Specialisations are given for volume-preserving flows, Hamiltonian flows and boundaryless invariant submanifolds.

Subsequent papers will present tests of the method for magnetic fields and guiding centre motion, and develop an extension to higher dimensional Hamiltonian systems.

ACKNOWLEDGEMENTS

This work was supported partly by a grant from the Simons Foundation/SFARI (601970, RSM) and partly by the National Science Foundation under Grant No. DMS-1440140 while the author was in residence at the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2018 semester.

REFERENCES

1. Arnol'd VI, On a characteristic class entering into conditions of quantisation, Functional Analysis and Its Applications 1 (1967) 1–14.
2. Arnol'd VI, Mathematical methods of classical mechanics (Springer, 1978).
3. Celletti A, MacKay RS, Regions of nonexistence of invariant tori for spin-orbit models, Chaos 17 (2007) 043119
4. Figueras J-L, Haro A, Luque A, Rigorous computer-assisted application of KAM theory: a modern approach, Found Comput Math 17 (2017) 1123–93.
5. Green LW, A theorem of E.Hopf, Michigan Math J 5 (1958) 31–4.
6. Hunt TJ, MacKay RS, Anosov parameter values for the triple linkage and a physical system with a uniformly chaotic attractor, Nonlinearity 16 (2003) 1499–1510.
7. Littlejohn RG, Variational principles of guiding centre motion, J Plasma Phys 29 (1983) 111–125.
8. MacKay RS, Transition to Chaos for area-preserving maps, in: Nonlinear Dynamics Aspects of Particle Accelerators, JM Jowett, M Month, S Turner (eds), Springer Lecture Notes in Physics 247 (1986) 390–454.
9. MacKay RS, A criterion for non-existence of invariant tori for Hamiltonian systems, Physica D 36 (1989) 64-82.
10. MacKay RS, Meiss JD, Stark J, Converse KAM theory for symplectic twist maps, Nonlinearity 2 (1989) 555–70.
11. MacKay RS, Percival IC, Converse KAM: theory and practice, Commun Math Phys 98 (1985) 469–512.
12. Mather JN, Non-existence of invariant circles, Ergod Th Dyn Sys 4 (1984) 301–9.
13. Moeckel R, Generic bifurcations of the twist coefficient, Ergod Th Dyn Sys 10 (1990) 185–195.
14. Stark J, An exhaustive criterion for the non-existence of invariant circles for area-preserving twist maps, Commun Math Phys 117 (1988) 177–89.