GENERALIZED DERIVED FUNCTORS

by

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Thesis submitted for a PhD degree

December 1981

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Summary

This thesis is about a construction of derived functors which considerably generalizes the original definition of Cartan and Eilenberg. The new derived functors can be constructed in the following circumstances:- Let $\underline{C}$ be a category with an initial object, I, and let $\underline{M}$ be a subcategory of $\underline{C}$ containing I. If F is a functor from $\underline{M}$ to a category of "based sets with structure" then the derived functors of F can be defined. The derived functors of F have domain $\underline{C}$ and their actions on objects of $\underline{C}$ are expressed as the homotopy groups of a simplicial set. This simplicial set is the nerve of a category so the derived functors can be regarded as the (topological) homotopy groups of the classifying space of a category.

The 0-th derived functor constructed in this manner is the Kan extension of F. The derived functors satisfy an "acyclic model theorem". Taking $\underline{M}$ as the full subcategory of projectives of an abelian category the derived functors agree with those of Cartan and Eilenberg, as is shown using the theory of over categories and Γ-spaces. A previous important generalization of derived functors are the cotriple derived functors. If the cotriple derived functors of F are defined then, for a suitable choice of $\underline{M}$, the new derived functors agree with them. This shows that many well-known sequences of functors arise from the construction, including Eckmann-Hilton homotopy groups, Hochschild's K-relative Tor, the homology of groups and the homology of commutative algebras. The singular homology functors occur as the derived functors of the 0-th reduced singular homology functor and for these new derived functors the derived functors of the 0-th homotopy functor behave like the higher homotopy functors when applied to connected spaces.

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CONTENTS

1. Introduction

1.1 Achievements

The idea behind my research has been to generalize the notion of derived functor. I take a category $\underline{C}$, which has an initial object, I, and a subcategory, $\underline{M}$, whose objects (the models) include I. If F is a functor from $\underline{M}$ to a category, $\underline{A}$, of "based sets with structure", then, for each object C of $\underline{C}$, I construct a simplicial set whose homotopy groups are the derived functors of F acting on C (2.2 - 2.3). If $\underline{M}$ is a full subcategory of $\underline{C}$ then the 0-th derived functor is the Kan extension of F (theorem 3.8). In theorem 6.7 I show that if F is an additive functor between abelian categories and $\underline{M}$ is the full subcategory of projectives of $\underline{C}$ then these functors I have constructed agree with the classical derived functors of Cartan and Eilenberg. Theorem 8.7 demonstrates that, if $\underline{C}$ is a category with a cotriple, $\mathbb{G}$, $\underline{M}$ is the full subcategory of $\mathbb{G}$-projectives or the full subcategory of $\mathbb{G}$-free objects of $\underline{C}$ and $\underline{A}$ is an abelian category then my derived functors coincide with cotriple derived functors and therefore with other generalized derived functor theories.

In 3.10, I show that if F preserves sums then so do its derived functors. Therefore 3.10 and 8.7 together provide a proof of the homology coproduct theorem for cotriple derived functors which Barr and Beck state in [2, chapter 7] they are unable to prove in generality. Another general result my derived functors satisfy is the acyclic model theorem (11.3), which states that in certain circumstances an alternative choice of the models gives the "same" derived functors.

The advantage of my approach is its extensive applicability. I do not need the range of F to be abelian. Thus I am able to construct the derived functors of many functors not covered by previous theories. For example if $\underline{C}$ is the category of based topological spaces and $\underline{M}$ is the full subcategory of spaces homotopy equivalent to finite discrete spaces then I can construct the derived functors of π_0. In theorem 10.5 I show the derived functors of π_0 behave like the homotopy functors on connected spaces.

1.2 History

Derived functors were originally defined for certain functors by Cartan and Eilenberg [3]. Cartan and Eilenberg defined the derived functors of additive functors between categories of modules. Their definition was soon abstracted to make it applicable to any additive functor whose domain was an abelian category with sufficient projectives and whose range was any abelian category. There have been some previous attempts to generalize the notion of derived functors (see section 7 for a discusion of the most significant of these). These generalizations have usually concentrated on generalizing the notion of projective resolution and are therefore only defined for functors whose range is an abelian category. One exception to this method is [10] where Keune does construct derived functors for some functors whose range is a category of based sets but he imposes fairly stringent conditions on the domain of the functor. Verdier [21] used a completely different approach to constructing a general theory incorporating derived functors when he defined the derived category.

The method I have used to define the derived functors is developed from the construction of the torsion products as homotopy groups given by Robinson [16]. The derived functors can be constructed in the following circumstances:- Let $\underline{C}$ be a category with an initial object, I, and let $\underline{M}$ be a subcategory of $\underline{C}$, such that I is an object of $\underline{M}$. If F is a functor from $\underline{M}$ to a category of "based sets with structure" then the derived functors of F can be defined. The derived functors of F have domain $\underline{C}$ and their actions on objects of $\underline{C}$ are expressed as the homotopy groups of a simplicial set.

1.3 Restrictions

In my work I have been concentrating on covariant functors and their left derived functors, but since a contravariant functor can be regarded as a covariant functor with the dual category to its original domain as its new domain this does not impose any serious restriction on my work. For convienience I will refer to derived functors rather than left derived functors throughout this thesis.

I need a functor to have range a category of "based sets with structure" to carry out my construction. Examples of categories of based sets endowed with some structure include the category of based sets and the category of abelian groups. Categories of sets with structure are usually called concrete categories but there does not appear to be any accepted word to describe categories of based sets with structure.

The classical abstract definition of derived functors requires that the functor under consideration should have range an abelian category while my construction needs the range to be a category of based sets with structure; so does my construction exist in all cases when derived functors were previously defined ? There are various imbeding theorems which show that most abelian categories, including all small abelian categories (those whose class of objects is a set) can be imbedded in the category of abelian groups or a category of modules and can therefore be regarded as categories of based sets with structure (the basepoints are provided by the 0 elements). Throughout this work, whenever an abelian category is mentioned, it is assumed that it is a concrete category. Note that there is no problem if $\underline{M}$ is a small category for then the range of F can be replaced by its full subcategory of objects of the form FM which will not affect the construction and the new range of F is a small abelian category. Since every abelian category has a zero object it certainly has an initial object so this condition on $\underline{C}$ is satisfied by every abelian category.

1.4 Summary

My generalized derived functors are constructed in section 2 and some of their properties, including their actions on objects of $\underline{M}$ and a comparison of the 0-th derived functor with the Kan extension, are demonstrated in section 3. Sections 4 and 5 are concerned with proving some technical facts about over and under categories and Γ-spaces. The results of these sections are used in sections 6 and 8 where the equivalence of my derived functors with those of Cartan and Eilenberg and cotriple derived functors is demonstrated. Cotriple derived functors are one of the previous generalizations of derived functors described in section 7. The equivalence of my theory and previous generalized derived functors shows that many interesting sequences of functors are obtained from my construction. Some of these are mentioned in section 9. Section 10 contains a discussion of the derived functors of π_0 , which do not exist for theories that demand that the range of the functor to be derived shall be an abelian category. I show that the derived functors of π_0 acting on connected spaces behave like the homotopy functors. The final section, section 11, contains an acyclic model theorem which demonstrates that in many places in this work alternative choices of the category $\underline{M}$ could have been made without affecting the results.

1.5 Acknowledgements

I would like to thank my supervisor, Alan Robinson, for suggesting that it might be possible to develop a generalization of derived functors from his construction of the torsion products as homotopy groups and for many stimulating discussions during my research. I should also like to thank the Science Research Council for providing me with a grant to enable me to do this research.

2. The Construction

Throughout this work $\underline{C}$ will be a category with an initial object, I, and a subcategory, $\underline{M}$. The objects of $\underline{M}$ will be called models. The category $\underline{M}$ will be the domain of a functor, F, whose range, $\underline{A}$, is a category of based sets with structure.

2.1 The Approach

The classical derived functors of a functor, F, acting on an object, C, are defined by taking a projective resolution of C, applying F to this resolution and taking homology of the resulting complex. To generalize this process one needs a class of objects to use for the projectives which will be the class of objects of the category $\underline{M}$. Note that in the classical case it is only necessary for a functor to be defined on projective objects to construct its derived functors, so it is not surprising that generalized derived functors can be constructed when F is only defined on a subcategory. If the range of F is not an abelian category then the concept of taking homology of a complex does not exist. This process is replaced by taking the homotopy of a simplicial set. The initial object of $\underline{C}$ provides a natural choice of basepoint for these homotopy groups.

2.2 The Category $\tau(C,F)$

Given $\underline{C}$, $\underline{M}$, $\underline{A}$, F and I as described at the start of this section, for each object, C, of $\underline{C}$ construct a new category $\tau(C,F)$. The category $\tau(C,F)$ has as objects pairs, (f,x), where $f:M \longrightarrow C$ is a morphism of $\underline{C}$ with domain, M, an object of $\underline{M}$ and x is an element of the "set" FM. The morphisms of $\tau(C,F)$ from (f_1,x_1) to (f_2,x_2) correspond to $\underline{M}$-morphisms $\xi:M_1 \longrightarrow M_2$ where

$$\begin{tikzcd} \xi:M_1 \arrow[rr] \arrow[dr, "f_1"'] & & M_2 \arrow[dl, "f_2"] \\ & C & \end{tikzcd}$$

commutes and $F(\xi)x_1 = x_2$.

Segal [17] shows how to construct the nerve of a category. The nerve of $\tau(C,F)$, denoted $|\tau(C,F)|$, is the simplicial set with n-simplices the functors from the category $\underset{\sim}{n}$ to $\tau(C,F)$. Here $\underset{\sim}{n}$ is the category with objects $\{0,1,\ldots,n\}$ and has a morphism from i to j if and only if $i<j$ or $i=j$. The face and degeneracy maps of the simplicial category, of ordered sets and order-preserving maps, induce funtors $\underset{\sim}{n} \dashrightarrow \underset{\sim}{n+1}$ and $\underset{\sim}{n} \dashrightarrow \underset{\sim}{n-1}$. The face and degeneracy maps of $|\tau(C,F)|$ are composition with the corresponding functor. Thus 0-simplices of $|\tau(C,F)|$ correspond to objects of $\tau(C,F)$. For $n>0$ the set of n-simplices of $|\tau(C,F)|$ is denoted $|\tau(C,F)|_n$ and corresponds to a chain

$$\begin{tikzcd} M_0 \arrow[r, "\xi_1"] \arrow[drr, "f_0"'] & M_1 \arrow[r] \arrow[dr, "f_1"] & \cdots \arrow[r, "\xi_n"] & M_n \arrow[dl, "f_n=f"] \\ & & C & \end{tikzcd}$$

with specified elements $x_i \in M_i$ for $0 \le i \le n$, satisfying

$$f_{i-1} = f_i \cdot \xi_i \qquad \text{and} \qquad x_i = F(\xi_i)x_{i-1} .$$

So elements of $|\mathcal{T}(C,F)|_n$ are completely determined by

$$(\xi_1,\xi_2,\dots,\xi_n,f,x_0) .$$

2.3 The Derived Functors

Using simplicial homotopy theory the sets $\pi_n|\mathcal{T}(C,F)|$, based at the 0-simplex $0 \longrightarrow (u_C,*)$, (where u_C is the unique map $u_C : I \longrightarrow C$ and $*$ is the basepoint of FI) can be formed. The $\pi_n|(C,F)|$ are groups if n is greater than 0 and abelian groups for n greater than or equal to 2.

If $\psi : C \longrightarrow B$ is a morphism of $\underline{C}$ then ψ induces a functor

$$\Psi : \mathcal{T}(C,F) \longrightarrow \mathcal{T}(B,F)$$

defined by $\quad (f,x) \longmapsto (\psi f,x) \quad$ on objects

and $\quad \xi \longmapsto \xi \quad$ on morphisms.

This Ψ induces, by composition of functors, a morphism of the nerves of the categories in the category of simplicial sets. Therefore there is a morphism

$$\psi_* = \pi_n|\mathcal{T}(\psi,F)| : \pi_n|\mathcal{T}(C,F)| \longrightarrow \pi_n|\mathcal{T}(B,F)|$$

corresponding to ψ and

$$\pi_n|\mathcal{T}(-,F)| : \underline{C} \longrightarrow \underline{\text{Sets}}.$$

is a functor. (Sets$_*$ is the category of based sets.)

2.4 An Alternative Description

The abstract definition of $\pi_n|\mathcal{T}(C,F)|$ given above can be difficult to use in practise. A more concrete definition is available. The geometric realization functor of Milnor [12] can be applied to any simplicial set to give a topological space, whose topological homotopy groups agree with the simplicial homotopy groups of the original simplicial set. The geometric realization of the nerve of a category, $\mathcal{T}$, is called the classifying space of $\mathcal{T}$ and is denoted $B\mathcal{T}$. The topological space $B\mathcal{T}$ is a CW-complex with 0-cells corresponding to objects of the category, $\mathcal{T}$, and n-cells (n>0) corresponding to sequences of n non-identity morphisms of $\mathcal{T}$. Thus the $\pi_n|\mathcal{T}(C,F)|$ are equal to the homotopy groups of the CW-complex, $B\mathcal{T}(C,F)$. In particular

$$\pi_0|\mathcal{T}(C,F)| = \frac{\{\text{ Objects of } \mathcal{T}(C,F)\ \}}{\text{Domain of a morphism} \sim \text{range of the morphism}}.$$

2.5 A Further Generalization

All the information concerning the derived functors of F acting on C is contained in the simplicial object, $|\mathcal{T}(C,F)|$, which is

$$|\mathcal{T}(C,F)|_n \cong \bigcup_{\substack{M_0 \xrightarrow{\xi_1} M_1 \longrightarrow \cdots \xrightarrow{\xi_n} M_n \longrightarrow C \\ M_i \in \mathrm{Ob}\,\underline{M} \\ \xi_i \in \mathrm{Mor}\,\underline{M}}} FM_0 .$$

If $\underline{A}$, the range of F, is a category with sums then it is possible to parallel this construction and form the simplicial $\underline{A}$-object

$$D(C,F)_n \cong \bigoplus_{\substack{M_0 \xrightarrow{\xi_1} M_1 \longrightarrow \cdots \xrightarrow{\xi_n} M_n \longrightarrow C \\ M_i \in \mathrm{Ob}\,\underline{M} \\ \xi_i \in \mathrm{Mor}\,\underline{M}}} FM_0.$$

The problem with considering these simplicial $\underline{A}$-objects instead of the derived functors is that different objects are obtained if, for example, $\underline{A}$ is regarded first as a category of groups and then as a category of based sets with structure. Note however that if $\underline{A}$ is an abelian category the chain complex associated with D(C,F) is the one used to define André homology [1] as described in (7.3).

3. Simple Theorems

3.1 The Models

The models can be thought of as the replacements for the projectives of classical derived functor theory. The models resemble the projectives in having trivial derived functors in the sense of the next theorem.

3.2 Theorem

If $\underline{M}$ is a full subcategory of $\underline{C}$ and N is a model then

$$\pi_n|\tau(N,F)| \cong \begin{cases} FN & n = 0 \\ 0 & n > 0. \end{cases}$$

Proof

Let $\underline{T}(N,F)$ be the full subcategory of $\tau(N,F)$ whose objects are of the form $(1_N,x)$. Then the morphisms of $\underline{T}(N,F)$ from $(1_N,x)$ to $(1_N,y)$ correspond to morphisms $\xi:N \longrightarrow N$ such that

$$\xi.1_N = 1_N \qquad \text{and} \qquad F(\xi)x = y$$

so

$$\xi = 1_N \qquad \text{and} \qquad x = y.$$

This means $\underline{T}(N,F)$ has only identity morphisms and therefore its classifying space has no cells in any dimension except 0. The 0-cells of $B\underline{T}(N,F)$ are in one-one correspondence with objects of the category $\underline{T}(N,F)$ and therefore in one-one correspondence with the elements of FN. (This type of category with only identity morphisms is often called a discrete category.) Thus it is easy to calculate the homotopy groups of

$|\underline{T}(C,F)|$.

$$\pi_n |\underline{T}(N,F)| \cong \begin{cases} FN & n = 0 \\ 0 & n > 0. \end{cases}$$

Define the functors

$$J: \underline{T}(N,F) \dashrightarrow \mathcal{T}(N,F) \qquad \text{is inclusion}$$

and $\quad G: \mathcal{T}(N,F) \dashrightarrow \underline{T}(N,F)$

by $\quad (f,x) \longmapsto (1_N, F(f)x) \qquad$ on objects

and $\quad \xi \longmapsto 1_N \qquad$ on morphisms.

Note that this is where it is necessary for $\underline{M}$ to be a full subcategory of $\underline{C}$, otherwise $F(f)$ may not be defined. Now, since the sets

$$\mathrm{Hom}_{\mathcal{T}(N,F)}((f,x), J(1_N,y)) \quad \text{and} \quad \mathrm{Hom}_{\underline{T}(N,F)}(G(f,x),(1_N,y))$$

are naturally isomorphic, J and G are adjoint functors. Therefore

$$\pi_n |\mathcal{T}(N,F)| \cong \pi_n |\underline{T}(N,F)| \cong \begin{cases} FN & n = 0 \\ 0 & n > 0. \end{cases}$$

by [15, Corollary 1 to Proposition 2].

3.3 Trivial Cases

Theorem (3.2) demonstrates what happens for some extreme choices of $\underline{M}$. In particular if $\underline{M} = \underline{C}$ then all derived functors are trivial. It is also easy to see that if $\underline{M}$ is a discrete category then

$$\pi_n |\mathcal{T}(C,F)| \cong \begin{cases} \coprod_{M \text{ a model}} \{\underline{C}\text{-morphisms } M \dashrightarrow C\} \times FM & n = 0 \\ 0 & n > 0. \end{cases}$$

3.4 Extensions

If $\underline{M}$ is any subcategory of $\underline{C}$ and F is a functor from $\underline{M}$ to $\underline{A}$ then an extension of F to $\underline{C}$ is a functor, $F':\underline{C} \dashrightarrow \underline{A}$, such that the restriction of F′ to $\underline{M}$ is F. If $\underline{A}$ is a category of based sets with structure then F can be regarded as a functor with range the category of based sets, denoted $\underline{\text{Sets}_*}$. Clearly, with this assumption, 3.2 shows that if $\underline{M}$ is a full subcategory of $\underline{C}$ then $\pi_0|\mathcal{T}(-,F)|$ is an extension of F. For any $\underline{M}$ and $\underline{C}$ if F′ is an extension of F then the following lemma holds.

3.5 Lemma

If $\underline{M}$ is a subcategory of $\underline{C}$ and F′ is an extension of $F:\underline{M} \dashrightarrow \underline{\text{Sets}_*}$ then there is a natural transformation between functors from $\underline{C}$ to $\underline{\text{Sets}_*}$.

$$\pi_0|\mathcal{T}(-,F)| \dashrightarrow F'.$$

Proof

Define a map

$$\mu:\{\text{ Objects of } \mathcal{T}(C,F)\ \} \dashrightarrow F'C$$

by $$(f,x) \longmapsto F'(f)x.$$

If two objects of $\mathcal{T}(C,F)$ are connected by a morphism, say $$\xi:(f_1,x_1) \dashrightarrow (f_2,x_2)\ ,$$

so $\xi:M_1 \dashrightarrow M_2$ with $F(\xi)x_1 = x_2$ and $f_1 = f_2\xi$, then

$$
\begin{aligned}
\mu(f_1,x_1) &= F'(f_1)x_1 \\
&= F'(f_2\xi)x_1 \\
&= F'(f_2)x_2 \\
&= \mu(f_2,x_2).
\end{aligned}
$$

Therefore μ induces a well-defined map

$$\mu' : \pi_0|\tau(C,F)| \dashrightarrow F'C.$$

For any morphism $\psi : C \dashrightarrow B$ the following diagram commutes:-

$$
\begin{tikzcd}
\pi_0|\tau(C,F)| \arrow[rrr] \arrow[ddd] & & & \pi_0|\tau(B,F)| \arrow[ddd] \\
& (f,x) \arrow[r, mapsto] \arrow[d, mapsto] & (\psi f,x) \arrow[d, mapsto] & \\
& F'(f)x \arrow[r, mapsto] & F'(\psi)F'(f)x & \\
F'C \arrow[rrr] & & & F'B.
\end{tikzcd}
$$

therefore μ induces the required natural transformation.

3.6 Kan extensions

Let $J : \underline{M} \dashrightarrow \underline{C}$ be any functor (often the inclusion of a subcategory). If $\underline{M}$ and $\underline{C}$ are small categories then, if $\underline{Sets}_*$ is the category of based sets, there exist categories of functors from $\underline{M}$ and $\underline{C}$ to $\underline{Sets}_*$, denoted $[\underline{M},\underline{Sets}_*]$ and $[\underline{C},\underline{Sets}_*]$. The functor J induces a functor between these categories

$$
\begin{aligned}
R : [\underline{C},\underline{Sets}_*] &\dashrightarrow [\underline{M},\underline{Sets}_*] \\
G &\longmapsto G.J
\end{aligned}
$$

The Kan extensions along J are the left and right adjoints of R. Authors differ in which adjoint they call the left Kan extension and which the right Kan extension but since all future references in this paper will be to the left adjoint of R this will simply be referred to as the Kan extension. (Note that if J is not a full functor then the Kan extension need not be an extension.) Clearly for $F:\underline{M}\text{---->}\underline{Sets_*}$ and $G:\underline{C}\text{---->}\underline{Sets_*}$ any functors the Kan extension of F along J,

$$E_J(F):\underline{C}\text{-------->}\ \underline{Sets_*}\ ,$$

satisfies :-

$$(*)\qquad \mathrm{Hom}_{[\underline{M},\underline{Sets_*}]}(F,G.J) \cong \mathrm{Hom}_{[\underline{C},\underline{Sets_*}]}(E_J(F),G)$$

so when $\underline{M}$ or $\underline{C}$ is not small, and so $[\underline{M},\underline{Sets_*}]$ or $[\underline{C},\underline{Sets_*}]$ may not be categories, then the Kan extension of F along J is defined to be the functor $E_J(F)$ which satisfies (*) for every $G:\underline{C}\text{---->}\underline{Sets_*}$, regarding $\mathrm{Hom}_{[\underline{C},\underline{Sets_*}]}$ now as just a class of functors.

3.7 Theorem

If $\underline{M}$ is a subcategory of a category, $\underline{C}$, where $\underline{C}$ has an initial object which is an object of $\underline{M}$ and $J:\underline{M}\text{---->}\underline{C}$ denotes the inclusion functor, then, for any $F:\underline{M}\text{---->}\underline{Sets_*}$ the 0-th derived functor of F is its Kan extension along J, ie

$$E_J(F) = \pi_0|\mathcal{T}(-,F)|\ .$$

Proof

It is necessary to show that

$$\mathrm{Hom}_{[\underline{M},\underline{Sets_*}]}(F,G.J) \cong \mathrm{Hom}_{[\underline{C},\underline{Sets_*}]}(\pi_0|\mathcal{T}(-,F)|,G)$$

for every functor $G:\underline{C}\text{---->}\underline{Sets_*}$. An object of the former class is a natural transformation, θ , which makes

$$\begin{array}{ccc} \theta(M):FM & \longrightarrow & [G.J]M = GM \\ \downarrow F(\zeta) & & \downarrow G(\zeta) \\ \theta(N):FN & \longrightarrow & [G.J]N = GN \end{array}$$ commute,

for every $\underline{M}$-morphism, $\zeta: M \longrightarrow N$. The map

$$\{ \text{ Objects of } \tau(C,F) \} \longrightarrow GC$$

$$(f,x) \longmapsto G(f).\theta(M)x$$

induces a natural transformation between $\pi_0|\tau(-,F)|$ and G. This is because if (f_1,x_1) and (f_2,x_2) are objects of $\tau(C,F)$ connected by the morphism, ξ, then

$$\begin{aligned} G(f_1).\theta(M_1)x_1 &= G(f_2.\xi).\theta(M_1)x_1 \\ &= G(f_2).G(\xi).\theta(M_1)x_1 \\ &= G(f_2).\theta(M_2).F(\xi)x_1 \\ &= G(f_2).\theta(M_2)x_2 \end{aligned}$$

and

$$\begin{array}{ccc} \pi_0|\tau(C,F)| & \longrightarrow & GC \\ (f,x) & \longmapsto & G(f).\theta(M)x \\ \downarrow \psi_* & & \downarrow G(\psi) \\ (\psi f,x) & \longmapsto & G(\psi f).\theta(M)x \\ \pi_0|\tau(B,F)| & \longrightarrow & GB \end{array}$$ commutes.

The map taking θ to this natural transformation has an inverse. Let $\lambda \in \mathrm{Hom}_{[\underline{C},\underline{Sets_*}]}(\pi_0|\tau(-,F)|,G)$, so λ is defined on equivalence classes of objects of $\tau(C,F)$, be the image of θ.

Define $\theta_M(x) = \lambda(M)\{(1_M,x)\}$.

Then
$$\begin{aligned} \theta_M x &= \lambda(M)\{(1_M,x)\} \\ &= G(1_M).\theta(M)x \\ &= \theta(M)x \end{aligned}$$

and
$$\begin{aligned}\lambda_C\{(f,x)\} &= G(f).\theta(M)x \\ &= G(f).\lambda(M)\{(1_M,x)\} \\ &= (\lambda(C).\pi_0|\mathcal{T}(f,F)|)\{(1_M,x)\} \\ &= \lambda(C)\{(f,x)\}.\end{aligned}$$

Thus the classes of natural transformations are isomorphic as required.

3.8 Comments

Note that (3.7) does not require that $\underline{M}$ is a full subcategory of $\underline{C}$ as Ulmer does in [20]. There Ulmer proves that any full and faithful functor (an inclusion is trivially faithful) satisfies $E_J(F).J = F$ so this is an alternative, less direct proof of (3.2). The best existence theorem for Kan extensions is that every full, small, faithful functor has a Kan extension. The above theorem shows that every inclusion functor has a Kan extension provided that the subcategory included by the functor contains an initial object of the larger category.

3.9 Additional Structure

Extra structure on the category $\underline{M}$ or the functor F can give rise to extra properties of the functors $\pi_n|\mathcal{T}(C,F)|$. If $\underline{C}$ has finite sums (+) and $\underline{M}$ is closed under the operation of taking sums then the sum structure induces a functor

$$\mathcal{T}(C,F) \times \mathcal{T}(C,F) \dashrightarrow \mathcal{T}(C,F)$$

defined by $((f_1,x_1),(f_2,x_2)) \longmapsto ((f_1,f_2),(x_1,x_2))$

where (f_1,f_2) is the map $M_1 + M_2 \dashrightarrow C$. This functor induces the group structure in $\pi_n|\mathcal{T}(C,F)|$.

If $\underline{C}$ is an additive category and F is an additive functor then let $\psi_1 : C \dashrightarrow B$ and $\psi_2 : C \dashrightarrow B$ be $\underline{C}$-morphisms.

$((\psi_1)\ ,\ (\psi_2))_*$ is induced by $\mathcal{T}(C,F) \dashrightarrow \mathcal{T}(B,F)$

$$(f,x) \dashrightarrow ((\psi_1 f, \psi_2 f),(x,x))$$

and $(\psi_1 + \psi_2)_*$ is induced by $\mathcal{T}(C,F) \dashrightarrow \mathcal{T}(B,F)$

$$(f,x) \dashrightarrow (\psi_1 f + \psi_2 f, x).$$

Define the natural transformation η by

$$\eta(f,x)((\psi_1 f, \psi_2 f),(x,x)) = (\psi_1 f + \psi_2 f, x).$$

By [17, 2.1] the functors inducing (ψ_1) ,$(\psi_2)_*$ and $(\psi_1 + \psi_2)_*$ induce homotopic maps of classifying spaces and $\pi_n|\mathcal{T}(-,F)|$ is an additive functor.

<u>3.10 Theorem</u>

Assume that $\underline{C}$ is an additive category with finite sums and $\underline{M}$ is a subcategory closed under the operation of taking sums. If F preserves sums then so do the derived functors of F.

<u>Proof</u>

If F preserves sums then the sum structure induces a map

$$\mathcal{T}(C_1,F) \times \mathcal{T}(C_2,F) \dashrightarrow \mathcal{T}(C_1 + C_2,F)$$

$$(f_1,x_1),(f_2,x_2)) \dashrightarrow ((f_1,f_2),x_1 + x_2)$$

where $x_1+x_2 \in FM_1+FM_2 \cong F(M_1+M_2)$. The maps $p_i : C_i \dashrightarrow C_1+C_2$ can be used to construct an adjoint to this functor. Therefore [15, theorem A] the classifying spaces of the categories are homotopy equivalent and

$$B(\mathcal{T}(C_1,F) \times \mathcal{T}(C_2,F)) \simeq B\mathcal{T}(C_1+C_2,F)$$

$$B\mathcal{T}(C_1,F) \times B\mathcal{T}(C_2,F) \simeq B\mathcal{T}(C_1+C_2,F)$$

because the classifying space functor preserves products. But

$$\pi_n|B\mathcal{T}(C_1.F) \times B\mathcal{T}(C_2,F)| \cong \pi_n|\mathcal{T}(C_1,F)| + \pi_n|\mathcal{T}(C_2,F)|$$

Hence

$$\pi_n|\mathcal{T}(C_1+C_2,F)| \cong \pi_n|\mathcal{T}(C_1,F)| + \pi_n|\mathcal{T}(C_2,F)|$$

and $\pi_n|\mathcal{T}(-.F)|$ preserves sums.

4. Over and Under Categories

This section and the next contain some technical results which will be used later, particularly in sections 5 and 7. In those sections the derived functors defined in section 2 will be compared with other definitions of derived functors. It will be shown that the $\pi_n|\Upsilon(C,F)|$ satisfy the characteristic axioms of these other derived functors. These axioms include the existence of a long exact sequence of derived functors. It is to obtain these long exact sequences that most work is needed and the results of this section and the next will be used for this purpose.

4.1 The Over Categories and the Under Categories

When $\Psi:\mathcal{T}\text{---->}\mathcal{T}'$ is any functor and D is an object of the category $\mathcal{T}'$, then the category of $\mathcal{T}$-objects Ψ-over D, denoted Ψ/D , can be constructed. These categories are special cases of the comma categories of Maclane [11, II.6]. The category Ψ/D has as objects pairs, (E,ε), where E is an object of $\mathcal{T}$ and $\varepsilon:\Psi E\text{---->}D$ is a morphism of $\mathcal{T}'$. The morphisms of Ψ/D from (E_1,ε_1) to (E_2,ε_2) correspond to $\mathcal{T}$-morphisms $\xi:E_1\text{---->}E_2$ which satisfy $\varepsilon_1 = \varepsilon_2\Psi(\xi)$. Clearly there is a natural projection from the category of $\mathcal{T}$-objects Ψ-over D to $\mathcal{T}$, given by $(E,\varepsilon)\longmapsto E$.

A $\mathcal{T}'$-morphism $\xi:D_1\text{---->}D_2$ induces a functor, Ψ/ξ, defined by

$$\Psi/\zeta : \Psi/D_1 \longrightarrow \Psi/D_2$$

is $\qquad (E,\varepsilon) \longmapsto (E,\zeta\varepsilon) \qquad$ on objects

and $\qquad \zeta \longmapsto \zeta \qquad$ on morphisms.

Dually, for $\Psi:\mathcal{C}\longrightarrow\mathcal{C}'$ and D an object of $\mathcal{C}'$, there is a category, D/Ψ, of $\mathcal{C}$-objects Ψ-under D, whose objects are pairs, (E,ε), with E an object of $\mathcal{C}$ and $\varepsilon.D\longrightarrow\Psi(E)$ a morphism of $\mathcal{C}'$. Morphisms of D/Ψ, from (E_1,ε_1) to (E_2,ε_2) correspond to $\mathcal{C}$-morphisms $\zeta:E_1\longrightarrow E_2$ such that $\Psi(\zeta)\varepsilon_1 = \varepsilon_2$. A $\mathcal{C}'$-morphism $\zeta:D_1\longrightarrow D_2$ induces a functor

$$\zeta/\Psi : D_2/\Psi \longrightarrow D_1/\Psi.$$

Once again there is a natural projection from the category of $\mathcal{C}$-objects Ψ-under D to $\mathcal{C}$.

The over and under categories which have just been defined are useful because of the existence, in certain circumstances, of cartesian squares involving the over (under) categories and their natural projections. The existence of these cartesian squares is demonstrated in the following theorems, which were proved by Quillen [15, theorem B].

4.2 Theorem

Let $\Psi:\mathcal{C}\longrightarrow\mathcal{C}'$ be any functor. If for every $\mathcal{C}'$-morphism $\zeta:D_1\longrightarrow D_2$, the induced functor,

$$\Psi/\zeta : \Psi/D_1 \longrightarrow \Psi/D_2 ,$$

itself induces a homotopy equivalence of the classifying spaces of these categories, then, for any object, D, of $\mathcal{C}'$ the cartesian square of categories

$$\begin{tikzcd}[row sep=small]
\Psi/D \arrow[rrr] \arrow[dddd] & & & \mathcal{T} \arrow[dddd] \\
& (E,\varepsilon) \arrow[r, mapsto] \arrow[dd, mapsto] & E \arrow[dd, mapsto] & \\
& & & \\
& (\Psi E,\varepsilon) \arrow[r, mapsto] & \Psi E & \\
1_{\mathcal{T}'}/D \arrow[rrr] & & & \mathcal{T}'
\end{tikzcd}$$

induces a homotopy cartesian square of their classifying spaces.

The dual theorem is:-

<u>4.3 Theorem</u>

Let $\Psi:\mathcal{T} \dashrightarrow \mathcal{T}'$ be any functor. If every $\mathcal{T}'$-morphism (say $\xi:D_1 \dashrightarrow D_2$) induces a homotopy equivalence of the relevant classifying spaces

$$\xi/\Psi : D_2/\Psi \dashrightarrow D_1/\Psi$$

then, for any object, D, of $\mathcal{T}'$ the cartesian square of categories,

$$\begin{tikzcd}[row sep=small]
D/\Psi \arrow[rrr] \arrow[dddd] & & & \mathcal{T} \arrow[dddd] \\
& (E,\varepsilon) \arrow[r, mapsto] \arrow[dd, mapsto] & E \arrow[dd, mapsto] & \\
& & & \\
& (\Psi E,\varepsilon) \arrow[r, mapsto] & \Psi E & \\
D/1_{\mathcal{T}'} \arrow[rrr] & & & \mathcal{T}'
\end{tikzcd}$$

induces a homotopy cartesian square of their classifying spaces.

5. Γ-Spaces

5.1 The Category Γ and Γ-Spaces

The concept of Γ-spaces was introduced by Segal in [18]. The category Γ is defined to be the category whose objects are finite sets and whose morphisms $S \dashrightarrow T$ (S and T being finite sets) correspond to maps from S to the set of subsets of T, denoted $P(T)$, such that if s_1 and s_2 are elements of S, $s_1 \neq s_2$, then $\omega : S \dashrightarrow P(T)$ satisfies

$$\omega(s_1) \cap \omega(s_2) = \emptyset .$$

Composition of the morphisms corresponding to

$$\omega_1 : S \dashrightarrow P(T) \qquad \text{and} \qquad \omega_2 : T \dashrightarrow P(U)$$

is the morphism corresponding to

$$s \dashrightarrow \bigcup_{t \in \omega_1(s)} \omega_2(t)$$

A Γ-category is a contravariant functor

$$\underline{X} : \Gamma \dashrightarrow \underline{\text{Categories}}$$

(Categories is the category of small categories) such that if $\underline{n}$ is the set $\{1,2,\ldots,n\}$ then:-

i) $\underline{X}(\underline{0})$ is equivalent to the category with one object and one morphism

and ii) for every integer, n, the functor

$$p_n : X(\underline{n}) \dashrightarrow \underline{X}(\underline{1}) \times \underline{X}(\underline{1}) \times \ldots \times \underline{X}(\underline{1})$$

induced by the maps $i_k : \underline{1} \dashrightarrow \underline{n}$, $i_k(1) = k \in \underline{n}$, is an equivalence of categories.

A Γ-space is a contravariant functor

$$X:\Gamma \dashrightarrow \underline{\text{Topological Spaces}}$$

(Topological Spaces is the category of topological spaces and continuous maps) which satisfies:-

i) $X(\underline{0})$ is contractible

and ii) for every integer, n,

$$p_n : X(\underline{n}) \longrightarrow X(\underline{1}) \times X(\underline{1}) \times \dots \times X(\underline{1})$$

is a homotopy equivalence.

If $\underline{X}$ is a Γ-category then $S \longrightarrow B\underline{X}(S)$ is a Γ-space. A connection between Γ-spaces and the derived functors defined in section 2 is illustrated by the next lemma.

5.2 Lemma

Let $\underline{C}$ be an additive catergory with finite sums and let $\underline{M}$ be a subcategory of $\underline{C}$ which is closed under the operation of taking finite sums. If F is a sum-preserving functor from $\underline{M}$ to the category of abelian monoids then, for every object, C, of $\underline{C}$ there is a Γ-space, X, with

$$X(\underline{1}) = B\Gamma(C,F).$$

Proof

If S is a finite set then define the category $\Gamma_S(C,F)$ to have objects consisting of sets of pairs

$$\{ (f_{S'},x_{S'}) \mid S' \subset S \},$$

where $f_{S'}:M_{S'} \longrightarrow C$ has domain an object of M, $x_{S'}$ is an element of $FM_{S'}$ and the unions of disjoint subsets of S are mapped to the pairs induced by the sum. Morphisms of $\mathcal{T}_S(C,F)$ from $\{(f_{S'},x_{S'})\}$ to $\{(f'_{S'},x'_{S'})\}$ correspond to sets,

$$\{\xi_{S'} \mid S' \subset S\},$$

where $\xi_{S'}$ is a $\underline{M}$-morphism such that

$$\begin{tikzcd} \xi_{S'}:M_{S'} \arrow[rr] \arrow[dr, "f_{S'}"'] & & M'_{S'} \arrow[dl, "f'_{S'}"] \\ & C & \end{tikzcd}$$

commutes and $F(\xi_{S'})x_{S'} = x'_{S'}$.

Clearly if S is a set with one element then $\mathcal{T}_S(C,F)$ is naturally equivalent to $\mathcal{T}(C,F)$.

Now define $\underline{X}:\Gamma \longrightarrow \underline{\text{Categories}}$ as follows:-

$$S \longmapsto \mathcal{T}_S(C,F) \qquad \text{on objects.}$$

If a morphism of Γ corresponds to $\omega:S \longrightarrow P(T)$ then

$$\underline{X}(\omega):\underline{X}(T) \longrightarrow \underline{X}(S)$$

is $\quad \{(f_{T'},x_{T'})\} \longmapsto \{(f_{S'},x_{S'})\}$ on objects

and $\quad \{\xi_{T'}\} \longmapsto \{\xi_{S'}\}$ on morphisms.

where if $T'' = \bigcup_{s\in S'} \omega(s)$ then $f_{S'} = f_{T''}$ and $\xi_{S'} = \xi_{T''}$.

Now $\quad (p_n):\mathcal{T}_{\underline{n}}(C,F) \longrightarrow \mathcal{T}_{\{1\}}(C,F) \times \dots \, \mathcal{T}_{\{n\}}(C,F)$

$$\cong \mathcal{T}(C,F) \times \dots\dots \times \mathcal{T}(C,F)$$

so $\underline{X}$ is a Γ-category and $X:S \longrightarrow B\underline{X}(S)$ is a Γ-space with

$$X(\underline{1}) \simeq B\mathcal{T}(C,F).$$

The Γ-space, X, obtained in (5.2) will be useful because now once it has been proved that $\pi_0|\mathcal{T}(C,F)|$ is an abelian group, then [18] shows that $B\mathcal{T}(C,F)$ is an infinite loop space.

5.3 Theorem

If $\underline{C}$ is a category with an initial object, I, $\underline{M}$ is a subcategory with finite sums and I is an object of M, then $B\tau(C,F)$ is an infinite loop space when C is any object of $\underline{C}$ and F is a functor from $\underline{M}$ to a category of abelian groups.

Proof

By (5.2) there is a Γ-space, X, with $X(\underline{1}) \simeq B\tau(C,F)$. Therefore it is sufficient to prove $\pi_0|\tau(C,F)|$, which is known to be an abelian monoid, is an abelian group. Now

$$\pi_0|\tau(C,F)| \cong \frac{\{\text{ Objects of } \tau(C,F)\ \}}{\text{Domain of } \xi \sim \text{ range of } \xi}$$

where ξ is a $\tau(C,F)$-morphism. Let $[(f,x)]$ denote the class in $\pi_0|\tau(C,F)|$ containing (f,x) . Because the sums in $\underline{M}$ induce the sum in $\tau(C,F)$, as shown in (3.9),

$$[(f,x)] + [(f,-x)] = [(f+f,(x,-x)].$$

Let $p:M + M \longrightarrow M$ be the map such that $p.i_j = 1_M$ for $i_j:M \longrightarrow M + M$, $j = 1,2$. Then consider the diagram :-

$$\begin{tikzcd}
I \arrow[r, "u_M"] \arrow[d, "u_M"'] & M \arrow[d, "i_2"] \arrow[ddrr, "f"] & & \\
M \arrow[r, "i_1"] \arrow[drrr, "f"'] & M + M \arrow[dr, "p"] \arrow[drr, "f+f"'] & & \\
 & & M \arrow[r, "f"] & C .
\end{tikzcd}$$

Since both f.p and f+f make this diagram commute, by the pushout property f+f = f.p. Thus

$$
\begin{aligned}
[(f+f,(x,-x))] &= [(f,F(p)(x,-x))] \\
&= [(f,F(p.i_1)x - F(p.i_2)x)] \\
&= [(f,0)].
\end{aligned}
$$

But

$$
\begin{tikzcd}
I \arrow[rr, dashed, "u_M"] \arrow[dr] & & M \arrow[dl, "f"] \\
& C &
\end{tikzcd}
\qquad \text{commutes and} \quad F(u_M)0 = 0
$$

so $[(f,0)] = [(u_M,0)]$ and $\pi_0|\mathcal{T}(C,F)|$ has inverses given by

$$[(f,x)]^{-1} = [(f,-x)] \ ,$$

which makes it an abelian group. Now, by [18, Theorem 1.4] the Γ-space $B\mathcal{T}(C,F)$ is known to be an infinite loop space.

6. Classical Derived Functors

The situation to be examined in this section is when $\underline{C}$ is an abelian category with sufficient projectives, $\underline{M}$ is the full subcategory of projectives and F is an additive functor from $\underline{M}$ to an abelian category. The derived functors constructed in section 2 will be compared with the classical derived functors of Cartan and Eilenberg [3]. The derived functors defined in section 2 will be denoted by $\pi_n|\tau(-,F)|$ while the classical derived functors will be denoted L_nF. The $\pi_n|\tau(-,F)|$ will be shown to satisfy the characteristic axioms of the classical derived functors [3]. Any sequence of functors and connecting homomorphisms, (G_n,∂_n), which satisfies:-

(i) $G_0 = L_0$

(ii) $G_n(P) = 0$ for P projective and n>0

(iii) if $0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$ is exact then

$$\cdots \longrightarrow G_{n+1}B \xrightarrow{\partial_n} G_nA \xrightarrow{\phi_*} G_nC \xrightarrow{\psi_*} G_nB \xrightarrow{\partial} G_{n-1}A \longrightarrow \cdots$$

is exact and maps of short exact sequences induce natural maps of long exact sequences

has $G_nC = L_nC$.

First consider the 0-th derived functors to prove (i).

6.1 Lemma

If $\underline{C}$ is an abelian category with sufficient projectives, $\underline{M}$ is the full subcategory of projectives and F is an additive functor with domain $\underline{C}$ and range an abelian category then for every object, C, of $\underline{C}$

$$\pi_0|\Upsilon(C,F)| = L_0FC$$

So the Kan extension of the restriction of F to the subcategory of projectives is the 0-th derived functor (3.7). Note that $\pi_0|\Upsilon(C,F)| = FC$ if and only if F is right exact.

<u>Proof</u>

Since $\underline{C}$ has sufficient projectives there is a projective resolution of C, ie an exact sequence:-

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} C \longrightarrow 0$$

where the P_i $(i \in \mathbb{Z}^+)$ are objects of $\underline{M}$. By the definition of L_0FC in [3]

$$L_0FC = FP_0/\mathrm{Im}(Fd_1) \ .$$

Define a map

$$\nu:\{\text{Objects of } \Upsilon(C,F)\} \longrightarrow FP_0$$

as follows:-

If (f,x) is an object of $\Upsilon(C,F)$ then there exists a map f' making

$$\begin{array}{ccccc} f:M & \longrightarrow & C & & \\ \vdots f' & & \| & & \\ P_0 & \xrightarrow{d_0} & C & \longrightarrow & 0 \end{array}$$

commute (because M is projective).

Note that if $f = d_0$ then $f' = 1_{P_0}$. Put

$$\nu(f,x) = F(f')x \ .$$

Now

$$\pi_0|\Upsilon(C,F)| = \frac{\{\text{ Objects of } \Upsilon(C,F)\ \}}{\text{Domain of a morphism} \sim \text{range of the morphism}} .$$

Consider the $\underline{M}$-morphism corresponding to

$$\xi:(f_1,x_1)\longrightarrow(f_2,x_2).$$

This gives a diagram

$$\begin{tikzcd}
 & & M_1 \arrow[dll, "f'_1"'] \arrow[d, "\xi"] \arrow[ddr, "f_1"] \arrow[dll, phantom, "(?)", near end] & \\
P_0 & & M_2 \arrow[ll, "f'_2"'] \arrow[dr, "f_2"] & \\
P_1 \arrow[r, "d_1"] & P_0 \arrow[rr, "d_0"] & & C
\end{tikzcd}$$

which commutes except for the triangle (?).

The map $f'_1 - f'_2\xi:M_1\longrightarrow P_0$ satisfies

$$d_0(f'_1 - f'_2\xi) = f_1 - f_2\xi = 0.$$

Therefore, since M_1 is projective and $P_1\longrightarrow P_0\longrightarrow C$ is exact there exists a map $g:M_1\longrightarrow P_1$ with $d_1g = f'_1 - f'_2\xi$.

$$\begin{tikzcd}
 & M_1 \arrow[dl, "g"'] \arrow[d, dashed, "f'_1-f'_2\xi"] \arrow[dr, "0"] & \\
P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & C
\end{tikzcd}$$

Thus $\nu(f_1,x_1) - \nu(f_2,x_2) = F(d_1).F(g)x_1 \in \mathrm{Im}(Fd_1)$.

Therefore ν induces a well-defined map

$$\bar{\nu}:\pi_0|\mathcal{T}(C,F)|\longrightarrow L_0FC.$$

If $y\in L_0FC$ then there is an $x\in FP_0$ mapped to y. But

$$\bar{\nu}(\text{class of }(d_0,x)) = \text{class of }F(1_{P_0})x = y.$$

so $\bar{\nu}$ is surjective.

Let $$\bar{\nu}(f_1,x_1) = \bar{\nu}(f_2,x_2)\ ,$$

so $$F(f'_1)x_1 = F(f'_2)x_2 + F(d_1)z\ ,$$

for some $z\in FP_1$. In fact f'_1 corresponds to a morphism of

$\mathcal{T}(C,F)$ from (f_1,x_1) to $(d_0,F(f_1')x_1)$. Because M_2 and P_1 are known to be projective and this implies that M_2+P_1 is projective, $(+.(f_2+0),(x_2,z))$ is an object of $\mathcal{T}(C,F)$. The map $(+.(f_2+0))$ is the composition

$$M_2+P_1 \xrightarrow{f_2+0} C+C \xrightarrow{+} C .$$

Since

$$\begin{tikzcd}
M_2+P_1 \arrow[r, "f_2'+d_1"] \arrow[dr, "+.(f_2+0)"'] & P_0+P_0 \arrow[r, "+"] \arrow[d] & P_0 \arrow[dl, "d_0"] \\
& C &
\end{tikzcd}$$

commutes, the map $(+.(f_2'+d_1))$ corresponds to a morphism of $\mathcal{T}(C,F)$ from $(+.(f_2+0),(x_2,z))$ to $(d_0,F(f_2')x_2+F(d_1)z) = (d_0,F(f_1')x_1)$.

Projection $M_2+P_1 \longrightarrow M_2$ corresponds to a $\mathcal{T}(C,F)$-morphism from $(+.(f_2+0),(x_2,z))$ to (f_2,x_2). So there is a string of morphisms:-

$$\begin{tikzcd}
x_2 & (x_2,z) \arrow[l, mapsto] \arrow[r, mapsto] & F(f_1')x_1 & x_1 \arrow[l, mapsto] \\
M_2 \arrow[drr, "f_2"'] & M_2+P_1 \arrow[l] \arrow[r, "f_2'+d_1"] \arrow[dr, "+.f_2+0"] & P_0 \arrow[d, "d_0"] & M_1 \arrow[l, "f_1'"'] \arrow[dl, "f_1"] \\
& & C. &
\end{tikzcd}$$

Therefore (f_1,x_1) and (f_2,x_2) correspond to the same element in $\pi_0|\mathcal{T}(C,F)|$ and $\bar{\nu}$ is injective. So $\bar{\nu}$ is an isomorphism as required.

6.2 The Exact Sequence

The long exact sequence required (iii) will be obtained using the results of sections 4 and 5. If

$$0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$$

is a short exact sequence in the abelian category, $\underline{C}$, then the epimorphism, ψ, induces the functor

$$\bar{\Psi} : \mathcal{T}(C,F) \longrightarrow \mathcal{T}(B,F) \ .$$

Thus there is an over category, $\bar{\Psi}/(h,z)$, for (h,z) an object of $\mathcal{T}(B,F)$. Here h is a morphism, $h:P \longrightarrow B$, and z is an "element" of FP for some projective P. The category $\bar{\Psi}/(h,z)$ has objects $((f,x),v)$ with

$$\begin{array}{ccc} v:M & \longrightarrow & P \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ \psi:C & \longrightarrow & B \end{array} \qquad \text{commuting and} \quad F(v)x = z.$$

To investigate the category $\bar{\Psi}/(h,z)$ it is useful to consider its subcategory, $[\bar{\Psi}/(h,z)]^0$. This $[\bar{\Psi}/(h,z)]^0$ is the full subcategory of $\bar{\Psi}/(h,z)$ whose objects are the $((f,x),v)$ with v an epimorphism . The classifying spaces of $\bar{\Psi}/(h,z)$ and $[\bar{\Psi}/(h,z)]^0$ are actually homotopy equivalent but only the results of the next lemma are needed.

6.3 Lemma

Let $\psi:C \longrightarrow B$ be an epimorphism of $\underline{C}$ inducing

$$\bar{\Psi}:\mathcal{T}(C,F) \longrightarrow \mathcal{T}(B,F) \ .$$

Defining $\bar{\Psi}/(h,z)$ and $[\bar{\Psi}/(h,z)]^0$ as above and letting

$$J:[\bar\Psi/(h,z)]^{o} \dashrightarrow \bar\Psi/(h,z)$$

be the inclusion functor, there is a functor

$$R:\bar\Psi/(h,z) \dashrightarrow [\bar\Psi/(h,z)]^{o}$$

and a natural transformation

$$\eta:JR \dashrightarrow 1_{\bar\Psi/(h,z)}.$$

(The homotopy equivalence of the classifying spaces of $\bar\Psi/(h,z)$ and $[\bar\Psi/(h,z)]^{o}$ can then be demonstrated by exhibiting a natural transformation between RJ and $1_{[\bar\Psi/(h,z)]^{o}}$.)

<u>Proof</u>

Since P is projective there is a map $h':P \dashrightarrow C$ such that

$$\begin{tikzcd} & P \arrow[dl, dashed, "h'"'] \arrow[d, "h"] & \\ C \arrow[r, dashed, "\psi"] & B \arrow[r, dashed] & 0 \end{tikzcd}$$

commutes.

Define the functor

$$R:\ \bar\Psi/(h,z) \dashrightarrow [\bar\Psi/(h,z)]^{o}$$

by $((f,x),v) \longmapsto (((f,h'),(x,0)),(v,1_P))$ on objects

and $\zeta \longmapsto \zeta + 1_P$ on morphisms.

Define $\eta((f,x),v)((f,x),v) = (((f,h'),(x,0)),(v,1_P))$

$= JR((f,x),v).$

If $\zeta:((f_1,x_1),v_1) \dashrightarrow ((f_2,x_2),v_2)$ is a morphism of $\bar\Psi/(h,z)$ then

$$\begin{tikzcd} ((f_1,x_1),v_1) \arrow[r, dashed] \arrow[d, "\eta((f_1,x_1),v_1)"] & ((f_2,x_2),v_2) \arrow[d, "\eta((f_2,x_2),v_2)"] \\ (((f_1,h'),(x_1,0)),(v_1,1_P)) \arrow[r, dashed] & (((f_2,h'),(x_2,0)),(v_2,1_P)) \end{tikzcd}$$

commutes. Therefore $\tilde{\iota}$ is a natural transformation between JR and the identity as required.

To show that the map $B(\underline{\Psi}/\xi)$, for $\xi: Q \dashrightarrow P$ a $\underline{M}$-morphism, is a homotopy equivalence, it is necessary to know something about the pullbacks of models.

<u>6.4 Lemma</u>

In an abelian category, if M, P and Q are projective, $v: M \dashrightarrow P$ is an epimorphism and $\xi: Q \dashrightarrow P$ is any map then

(i) the pullback $M \times_P Q$ is projective

and (ii) for an additive functor, F, whose range is an abelian category,

$$F(M \times_P Q) \cong FM \times_{FP} FQ.$$

<u>Proof</u>

In the pullback diagram:-

$$\begin{tikzcd} u: M \times_P Q \arrow[r, dashed] \arrow[d, "w"] & M \arrow[d, "v"] \\ \xi: Q \arrow[r, dashed] & P \end{tikzcd}$$

v an epimorphism implies that w is an epimorphism [7, 2.54] so, since Q is projective, there exists a $w': Q \dashrightarrow M \times_P Q$ making

$$\begin{tikzcd} & & & Q \arrow[dl, dashed, "w'"'] \arrow[d, equal] & \\ 0 \arrow[r, dashed] & \ker(w) \arrow[r, dashed] & M \times_P Q \arrow[r, dashed] & Q \arrow[r, dashed] & 0 \end{tikzcd}$$

commute.

Since the exact sequence of this diagram splits

$$M \times_P Q \cong Q + \ker(w)$$

(*) $$M \times_P Q \cong Q + \ker(v) \qquad \text{by [7, 2.52].}$$

(**) Similarly $\quad M \cong P + \ker(v)$, say $i:\ker(v) \longrightarrow M$

and $p:M \longrightarrow \ker(v)$.

(i)

Given
$$\begin{tikzcd} & \ker(v) \arrow[d, "\gamma"'] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$
form
$$\begin{tikzcd} & M \arrow[d, shift right, "p"'] \arrow[ddl, "(\gamma p)'"'] & \\ & \ker(v) \arrow[u, shift right, "i"'] \arrow[d, "\gamma"'] & \\ A \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

If $A \longrightarrow B \longrightarrow 0$ is exact then $(\gamma p)':M \longrightarrow A$ exists because M is projective. The existence of

$$(\gamma p)'.i:\ker(v) \longrightarrow A$$

for arbitary exact $A \longrightarrow B \longrightarrow 0$ shows that ker(v) is projective . Therefore $M \times_P Q \cong Q + \ker(v)$, being the direct sum of projective objects, is itself projective.

(ii)
$$\begin{aligned} FM \times_{FP} FQ &\cong F(P + \ker(v)) \times_{FP} FQ && \text{using (*)} \\ &\cong (FP + F(\ker(v))) \times_{FP} FQ && \\ &\cong F(\ker(v)) + FQ && \\ &\cong F(M \times_P Q) && \text{using (**).} \end{aligned}$$

Now it is possible to show that any M-morphism , $\xi:Q \longrightarrow P$, induces a homotopy equivalence $B(\Psi/\xi)$, and this enables the result of (4.2) to be used to obtain a cartesian square of categories.

6.5 Lemma

Let $\underline{C}$ be an abelian category with sufficiently many projectives and let $\underline{M}$ be the full subcategory of projectives. Assume that F is an additive functor. If $\psi: C \dashrightarrow B$ is a morphism of $\underline{C}$ then any morphism of $\mathcal{T}(B,F)$,

say $$\xi:(h_1,z_1) \dashrightarrow (h_2,z_2),$$

induces a homotopy equivalence of the classifying spaces of the over categories

$$\Psi/\xi : \Psi/(h_1,z_1) \dashrightarrow \Psi/(h_2,z_2).$$

Proof

Let $h_1: Q \dashrightarrow B$, $h_2: P \dashrightarrow B$ so that $\xi: Q \dashrightarrow P$. Given an element $((f,x),v)$ of $[\Psi/(h_2,z_2)]^{\circ}$ with v an epimorphism the following pullback can be formed by (6.2) :-

$$\begin{tikzcd} M \times_P Q \arrow[r, dashed, "u"] \arrow[d, "w"] & M \arrow[r, dashed, "f"] \arrow[d, "v"] & C \arrow[d] \\ Q \arrow[r, dashed] & P \arrow[r, dashed, "h_2"] & B \end{tikzcd}$$

Define $(\Psi/\xi)^{*}: [\Psi/(h_2,z_2)]^{\circ} \dashrightarrow \Psi/(h_1,z_1)$

by $$((f,x),v) \dashrightarrow ((fu,(x,z_1)),w) \quad \text{on objects}$$

and $$\xi \dashrightarrow \xi' \quad \text{on morphisms}$$

where ξ' is the unique morphism defined by the pullback property:-

$$\begin{tikzcd}
M_1 \times_P Q \arrow[rr, "u_1"] \arrow[dr, "\xi'"] \arrow[ddr, "w_1"'] & & M_1 \arrow[dl, "\xi"'] \arrow[ddl, "v_1"] \\
& M_2 \times_P Q \arrow[r, "u_2"] \arrow[d, "w_2"] & M_2 \arrow[d, "v_2"] \\
& Q \arrow[r] & P
\end{tikzcd}$$

Consider $\mathrm{Hom}_{\Psi/(h_1,z_1)}(K_1,(\Psi/\xi)^{*}K_2)$

where $K_i = ((f_i,x_i),v_i)$.

Then there is a diagram :-

$$\begin{tikzcd}
M_1 \arrow[d, "\alpha"] \arrow[dr, "\alpha'"] \arrow[drr, "f_1"] \arrow[dd, bend right, "v_1"'] & & \\
M_2 \times_P Q \arrow[r, "u"] \arrow[d, "w"] & M_2 \arrow[r, "f_2"] \arrow[d, "v_2"] & C \arrow[d, "\Psi"] \\
Q \arrow[r, "\xi"] & P \arrow[r, "h_2"] & B
\end{tikzcd}$$

(*)

$F(v_1)x_1 = z_1$

$F(v_2)x_2 = z_2$.

A morphism $((f_1,x_1),v_1) \longrightarrow ((f_2u,(x_2,z_1)),w)$

corresponds to a morphism

$$(f_1,x_1) \longrightarrow (f_2u,(x_2,z_1))$$

and so corresponds to $\alpha: M_1 \longrightarrow M_2 \times_P Q$ with

$$v_1 = w.\alpha \qquad \text{and} \qquad F(\alpha)x_1 = (x_2,z_1).$$

Now define the subcategory $[\Psi/(h_2,z_2)]^{o}$,

the functors $J:[\Psi/(h_2,z_2)]^{o} \longrightarrow \Psi/(h_2,z_2)$,

$R:\Psi/(h_2,z_2) \longrightarrow [\Psi/(h_2,z_2)]^{o}$

and the natural transformation $\eta: JR \longrightarrow 1$ as in (6.1).

The map

$$\mathrm{Hom}_{\Psi/(h_1,z_1)}(K_1,(\Psi/\xi)^{*}K_2) \longrightarrow \mathrm{Hom}_{\Psi/(h_2,z_2)}((\Psi/\xi)K_1,J(K_2))$$

morphism corresponding to $\alpha \longmapsto$ morphism corresponding to α'

has an inverse taking

$$\alpha' : (\Psi/\xi)K_1 \dashrightarrow J(K_2)$$

to the morphism corresponding to the unique map defined by the pullback property in the diagram (*). Therefore

$$\mathrm{Hom}_{\Psi/(h_1,z_1)}(K_1, (\Psi/\xi)^* K_2) \cong \mathrm{Hom}_{\Psi/(h_2,z_2)}((\Psi/\xi)K_1, J(K_2)).$$

Putting $K_1 = (\Psi/\xi)^* K_2$ gives a natural transformation (corresponding to the identity morphism)

$$(\Psi/\xi)(\Psi/\xi)^* \dashrightarrow J .$$

Thus among the functors

$$\begin{tikzcd} \Psi/(h_1,z_1) \arrow[r, "\Psi/\xi"] & \Psi/(h_2,z_2) \arrow[d, shift right, "R"'] \\ & {[\Psi/(h_2,z_2)]^o} \arrow[ul, "(\Psi/\xi)^*"] \arrow[u, shift right, "J"'] \end{tikzcd}$$

there are natural transformations

$$(\Psi/\xi)(\Psi/\xi)^* R \dashrightarrow JR \dashleftarrow 1_{\Psi/(h_2,z_2)} .$$

Therefore $\quad B(\Psi/\xi).B((\Psi/\xi)^*.R) \simeq 1 \qquad$ by [17, 2.1]

Now put $K_2 = R(\Psi/\xi)K_1$ in the above isomorphism of homomorphism groups. The morphism corresponding to the natural transformation $1 \dashrightarrow JR$ gives a natural transformation

$$1 \dashrightarrow (\Psi/\xi)^* R.(\Psi/\xi)$$

and so $\quad 1 \simeq B[(\Psi/\xi)^* R].B(\Psi/\xi) \qquad$ by [17, 2.1]

Thus $B(\Psi/\xi)$ is a homotopy equivalence, having homotopy inverse $B[(\Psi/\xi)^* R]$.

The cartesian square of (4.2) involves certain over categories. To make this square useful it is necessary to find other categories equivalent to these categories.

6.6 Lemma

If $0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$ is a short exact sequence in $\underline{C}$ then there is an object, D, in $\mathcal{T}(B,F)$ such that

$$\Psi/D \cong \mathcal{T}(A,F) .$$

Proof

Since the category $\underline{M}$ is the subcategory of projectives of an abelian category it contains the zero object, I, of $\underline{C}$. Because F is an additive functor $FI=\{0\}$, the trivial based set with one element. For all objects, B, in $\underline{C}$ the set of morphisms $I \longrightarrow B$ consists of $u_B : I \longrightarrow B$ alone. Take $D = (u_B, 0)$. Now Ψ/D has objects of the form $((f,x),v)$, $f:M \longrightarrow C$, $x \in FM$, where

$$\begin{tikzcd} v:M \arrow[rr] \arrow[dr, "\psi f"'] & & I \arrow[dl, "u_B"] \\ & B & \end{tikzcd}$$

commutes and $F(v)x = 0$.

Given (f,x), since I is the zero object, there is a unique map $v:M \longrightarrow I$ and $F(v)x = 0$. Thus there is a one-one correspondence between

$$\begin{aligned} \text{Objects of } \Psi/D \text{ and } \quad & \{ (f,x) \mid \psi f = u_B v \} \\ &= \{ (f,x) \mid \operatorname{Im}(\psi f) = \mathcal{D} \} \\ &= \{ (f,x) \mid \operatorname{Im}(f) \subset A \} \\ &\cong \{ \text{Objects of } \mathcal{T}(A,F) \} \end{aligned}$$

The morphisms of Ψ/D are

$$((f_1,x_1),v_1) \dashrightarrow ((f_2,x_2),v_2)$$

corresponding to

$$w:(f_1,x_1) \longrightarrow (f_2,x_2) \quad \text{with} \quad v_1 = v_2\Psi(w) ,$$

and this later condition is trivially satisfied since v_1 and v_2 are trivial. Therefore morphisms of Ψ/D correspoud to morphisms of $\Upsilon(A,F)$. Thus

$$\Psi/D \simeq \Upsilon(A,F).$$

Now the assertion that the construction of section 2 gives a generalization of the notion of derived functor can be justified.

<u>6.7 Theorem</u>

If F is an additive functor from an abelian category, $\underline{C}$, with sufficient projectives, to an abelian category and $\underline{M}$ is the full subcategory of projectives of $\underline{C}$ then

$$\pi_n|\Upsilon(C,F)| \simeq L_nFC.$$

<u>Proof</u>

The conditions of (5.2) are satisfied so there exists a Γ-space, X, with $X(\underline{1}) = B\Upsilon(C,F)$. Lemma (6.1) shows that $\pi_0|\Upsilon(C,F)| \cong L_0FC$ is an abelian group and therefore $B\Upsilon(C,F)$, being a group-complete Γ-space, is an infinite loop space [18, theorem 1.4], all of whose homotopy groups are abelian groups. The $\pi_n|\Upsilon(-,F)|$ are (3.9) additive functors from the category $\underline{C}$ to the category of abelian groups. They satisfy the following conditions :-

(i) $\pi_0|\tau(C,F)| \cong L_0FC$ by (6.1)

(ii) $\pi_n|\tau(N,F)| \cong 0$ for N projective, $n>0$ by (3.2)

(iii) Let $0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$ be a short exact sequence in $\underline{C}$. The morphism ψ induces

$$\Psi : \tau(C,F) \longrightarrow \tau(B,F) .$$

Define $\Psi/(h,z)$ as in (5.1). The lemmas (6.3) and (6.4) show that (6.4) can be applied so that every morphism

$$\xi : (h_1,x_1) \longrightarrow (h_2,z_2)$$

of $\tau(B,F)$ gives a functor

$$\Psi/\xi : \Psi/(h_1,z_1) \longrightarrow \Psi/(h_2,z_2)$$

which induces a homotopy equivalence of classifying spaces. Therefore (4.2)

$$\begin{tikzcd} B[\Psi/(h,z)] \arrow[r] \arrow[d] & B\tau(C,F) \arrow[d, "B\Psi"'] \\ B[1_{\tau(B,F)}/(h,z)] \arrow[r] & B\tau(B,F) \end{tikzcd}$$ is homotopy cartesian.

Now use (6.6) on the short exact sequences

$$0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$$

and $$0 \longrightarrow 0 \longrightarrow B \xrightarrow{1} B \longrightarrow 0$$

to show that

$$\begin{tikzcd} B\tau(A,F) \arrow[r, "B\Phi"] \arrow[d] & B\tau(C,F) \arrow[d, "B\Psi"'] \\ B\tau(0,F) \arrow[r] & B\tau(B,F) \end{tikzcd}$$ is homotopy cartesian.

Because $B\mathcal{T}(C,F) \longrightarrow B\mathcal{T}(B,F)$ is an infinite loop space map, the homotopy theoretic fibres of this map over any two points are equal. From (3.2) it is clear that $B\mathcal{T}(0,F)$ is contractible and therefore the homotopy theoretic fibres of

$$B\mathcal{T}(\psi,F) : B\mathcal{T}(C,F) \longrightarrow B\mathcal{T}(B,F)$$

are equivalent to

$$B\mathcal{T}(\phi,F) : B\mathcal{T}(A,F) \longrightarrow B\mathcal{T}(C,F) \ .$$

The homotopy boundary of this homotopy fibration of infinite loop spaces gives a natural connecting homomorphism in the long exact sequence

$$\ldots \longrightarrow \pi_n|\mathcal{T}(A,F)| \xrightarrow{\phi_*} \pi_n|\mathcal{T}(C,F)| \xrightarrow{\psi_*} \pi_n|\mathcal{T}(B,F)| \xrightarrow{\partial} \pi_{n-1}|\mathcal{T}(A,F)| \longrightarrow \ldots$$

Maps between short exact sequences

$$0 \longrightarrow A \xrightarrow{\phi} C \xrightarrow{\psi} B \longrightarrow 0$$

induce natural maps of long exact sequences.

The classical derived functors are characterised by the properties (i), (ii) and (iii) [3]. Therefore

$$\pi_n|\mathcal{T}(C,F)| \cong L_n FC.$$

7. Other Generalized Derived Functors

There have been some previous attempts to define derived functors for a larger class of functors than that to which the abstraction of Cartan and Eilenberg's definition applies. This section contains a brief survey of the more significant of these attempts and exhibits various maps between these functors and the derived functors constructed in section 2. When the different structures used to define the derived functors described in this section are suitably related they coincide. It is therefore desirable that, for a suitable choice of models, the $\pi_n|\tau(C,F)|$ defined in section 2 also agree with them. This could be demonstrated by showing that any of the maps exhibited in this section are isomorphisms but in fact it is easier to prove that the derived functors defined in section 2 coincide with the other generalized derived functors by the method of section 8, where the $\pi_n|\tau(C,F)|$ are shown to satisfy the characteristic axioms of cotriple derived functors.

The first important steps in generalizing derived functors were the introduction of the notion of projective class by Eilenberg and Moore [6] and the definition of derived functors for non-additive functors between abelian categories by Dold and Puppe [4].

7.1 Projective Classes

Let $\underline{C}$ be an abelian category. If $\underline{P}$ is a class of objects of $\underline{C}$ then a morphism of $\underline{C}$, ψ:C---->B , is a $\underline{P}$-epimorphism if, for every object Q in $\underline{P}$, the induced map:-

$$\mathrm{Hom}_{\underline{C}}(Q,C)\text{-------->}\ \mathrm{Hom}_{\underline{C}}(Q,B)$$

is surjective. A class, $\underline{P}$, which is closed under coproducts and retractions, is a projective class if, for every object P of $\underline{P}$, there exists a $\underline{P}$-epimorphism e:Q---->P , with Q in $\underline{P}$. A sequence

$$(*)\qquad \ldots\text{---->}X_n\text{---->}X_{n-1}\text{---->}\ldots\text{---->}X_1\text{---->}X_0\text{---->}C\text{---->}0$$

is $\underline{P}$-exact if the sequence of abelian groups

$$\ldots\text{---->}\ \mathrm{Hom}_{\underline{C}}(P,X_n)\text{---->}\ \mathrm{Hom}_{\underline{C}}(P,X_{n-1})\text{--->}\ldots$$

$$\ldots\text{--->}\ \mathrm{Hom}(P,X_0)\text{---->}\ \mathrm{Hom}(P,C)\text{---->}0$$

is exact for every P in $\underline{P}$. If also every X_i is in $\underline{P}$ then the sequence (*) is a $\underline{P}$-projective resolution of C. With these definitions it is possible to define the derived functors of F:$\underline{C}$---->$\underline{A}$ where $\underline{C}$ is a category with a projective class and $\underline{A}$ is an abelian category. To define the derived functors of F acting on an object C of $\underline{C}$ take a P-projective resolution of C (which always exists), apply F and take homology.

7.2 Dold-Puppe Derived Functors

Dold and Puppe [4] defined the n-level derived functors of $F:\underline{C}\longrightarrow\underline{A}$ for $\underline{C}$ and $\underline{A}$ categories of modules. For C an object of $\underline{C}$ form the simplicial abelian group,

$$P = \{ P_j, d^i_j : P_j \longrightarrow P_{j-1}, s^i_j : P_j \longrightarrow P_{j+1} \}$$

with P_j projective and

$$(**) \qquad H_j(P) \cong \begin{cases} C & n = j \\ 0 & n \neq j. \end{cases}$$

For the given F, if P is a simplicial abelian group, then so is $FP = \{FP_j, Fd^i_j, Fs^i_j\}$. Associated with any simplicial abelian group is a chain complex. The chain complex associated with FP, which is also called FP, is $\{FP_j, \Sigma^{i=n}_{i=0}(-1)^i d^i_j\}$. The q-th left derived functor of F of type n is $L^{(q)}_n FC = H_n(FP)$. For F an additive functor $L^{(q)}_n F = L_{n-q} F$.

A theorem of Moore [13] shows that the homotopy groups of a simplicial abelian group equal the homology groups of the associated chain complex. Thus $H_n(FP) \cong \pi_n(FP)$. This means a map from the Dold-Puppe derived functors to the derived functors defined in section 2 can be constructed. Let $\underline{M}$ be the full subcategory of projectives of $\underline{C}$. Then there is a map

$$L^{(0)}_n FC \cong \pi_n(FP) \longrightarrow \pi_n |\tau(C,F)|$$

induced by $FP_j \longrightarrow |\tau(C,F)|_j$

$$x \longrightarrow (\Sigma^{j}_{i=0}(-1)^i d^i_j, \ldots, \Sigma^{i=1}_{i=0}(-1)^i d^i_1, f, x)$$

where $f:P_0\longrightarrow C$ corresponds to the isomorphism (**). The notation used here for the n-simplices of $|\tau(C,F)|$ is that introduced in (2.2).

7.3 André Homology

André [1] considers the situation where a category, $\underline{C}$ has a locally small, full subcategory, $\underline{N}$ and $\underline{A}$ is an abelian category with direct sums such that the direct sum of monomorphisms is a monomorphism. The functor F has domain $\underline{N}$ and range $\underline{A}$. For C an object of the category $\underline{C}$ he constructs the index sets

$$I_n(C,F) = \{(\xi_1,\xi_2,\dots,\xi_n,f)\}$$

where $\xi_i : M_{i-1} \dashrightarrow M_i$ and $f: M_n \dashrightarrow C$ are $\underline{C}$-morphisms and the M_i are objects of $\underline{N}$. Let

$$B_n(C,F) = \bigoplus_{I_n(C,F)} FM_0 .$$

If $f[\xi_1,\dots,\xi_n]$ denotes the inclusion $FM_0 \dashrightarrow B_n(C,F)$ then face and degeneracy maps can be defined by :-

$$d_n^i . f[\xi_1,\dots,\xi_n] = \begin{cases} f\,\xi_n[\xi_1,\dots,\xi_{n-1}] & i=n \\ f[\xi_1,\dots,\xi_{i-1},\xi_{i+1}\xi_i,\dots,\xi_n] & 0<i<n \\ f[\xi_2,\dots,\xi_n]\,F(\xi_1) & i=0. \end{cases}$$

$$s_n^i . f[\xi_1,\dots,\xi_n] = f[\xi_1,\dots,\xi_i,1_{M_i},\xi_{i-1},\dots,\xi_n]$$

to make $B(C,F)$ a simplicial set. André homology $A_n(C,F)$ is defined as the homology of the chain complex whose n-th term is $B_n(C,F)$ and whose maps are $\partial_n = \sum_{i=0}^{n}(-1)^i d_n^i$. Moore's theorem [13] shows $A_n(C,F)$ is the n-th homotopy group of the simplicial set $B(C,F)$, which is the simplicial set mentioned in (2.5). It is possible to define a map

$$\pi_n|\tau(C,F)| \dashrightarrow A_n(C,F).$$

This map is induced by the simplicial set homomorphism

$$|\tau(C,F)|_n \longrightarrow B_n(C,F)$$

$$(\xi_1,\dots,\xi_n,f,x) \dashrightarrow (x \in FM_0) \xrightarrow{[\xi_1 \dots \xi_n]} B_n(C,F).$$

Clearly this map is injective. Theories of derived functors like those defined in section 2 where the functor under consideration need only be defined on a subcategory of models are called model induced theories. André homology is another model induced theory.

7.4 Cotriple Derived Functors

A cotriple, $\mathbb{G} = (G,\varepsilon,\delta)$, in a category, $\underline{C}$, consists of an endofunctor , $G:\underline{C} \dashrightarrow \underline{C}$, and natural transformations $\varepsilon:G \dashrightarrow 1_{\underline{C}}$ and $\delta:G \dashrightarrow G^2$ making the following diagrams commute:-

$$\begin{tikzcd} G^2 \arrow[dr, "G\varepsilon"'] & G \arrow[l, dashed, "\delta"'] \arrow[r, dashed, "\delta"] \arrow[d, equal] & G^2 \arrow[dl, "\varepsilon(G)"] \\ & G & \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, dashed, "\delta"] \arrow[d, "\delta"'] & G^2 \arrow[d, "\delta(G)"] \\ G^2 \arrow[r, dashed, "G\delta"] & G^3 \end{tikzcd} .$$

If $\underline{C}$ is a category containing a cotriple then the cotriple derived functors of a functor, $F:\underline{C} \dashrightarrow \underline{A}$ can be defined for $\underline{A}$ an abelian category. In fact the basic construction does not require the condition that $\underline{A}$ is an abelian category. Given an object, C, in $\underline{C}$ form the simplicial $\underline{C}$-object, $\mathbb{G}C$, with augmentation $\varepsilon(C):\mathbb{G}C \dashrightarrow C$. This $\mathbb{G}C$ is defined by:-

$$(\mathbb{G}C)_n = G^{n+1}C$$
$$d_n^i = G^i\varepsilon(G^{n-i})$$
$$s_n^i = G^i\delta(G^{n-i}).$$

The cotriple derived functors of F are the $\pi_n(F\mathbb{G}C)$. Barr and Beck [2] have studied the cotriple derived functors for the case $\underline{A}$ is an abelian category. Moore's theorem [13] which proves that the homotopy groups of a simplicial abelian group are isomorphic to the homology groups of the associated chain complex shows that the cotriple derived functors of $F:\underline{C}\longrightarrow\underline{A}$, where $\underline{A}$ is an abelian category, are the homology groups of $(F\mathbb{G}C_n, \sum_{i=0}^{n}(-1)^i Fd_n^i)$. These cotriple derived functors will be denoted $H_n(-,F)_{\mathbb{G}}$.

The cotriple derived functors are known to "agree" with André homology. If $\underline{N}$ is the full subcategory of G-free objects of $\underline{C}$, ie the full subcategory with objects

$$\{\ GC \mid C \text{ is any object of } \underline{C}\ \}$$

then [1, 12.1]

$$A_n(C,F) = H_n(C,F)_{\mathbb{G}}.$$

The idea of projective classes is also connected to that of cotriples. If $\mathbb{G} = (G,\delta,\varepsilon)$ is a cotriple in $\underline{C}$ then an object, P, of $\underline{C}$ is G-projective if there is a map $s:P\longrightarrow GP$ satisfying $\varepsilon(P).s = 1_P$. The class of all G-projectives in $\underline{C}$ is a projective class in the sense of (7.1). Actually the cotriple derived functors depend only on the class of G-projectives. More precisely if G and G' are two cotriples such that the classes of G-projectives and of G′-projectives coincide then [2, 5.2]

$$H_n(C,F)_{\mathbb{G}} = H_n(C,F)_{\mathbb{G}'} .$$

Clearly if the class of G-projectives is used in Eilenberg and Moore′s construction then the resulting derived functors are cotriple derived functors. It is also possible to use the full subcategory of projectives for the category $\underline{N}$ in the construction of André homology. Then the relation between these theories becomes [2, 10.6]

$$A_n(C,F) = H_n(C,A_0(-,F))_{\mathbb{G}}.$$

If $\underline{M}$ is any of the full subcategory of G-projectives, the full subcategory of G-free objects or the subcategory of G-free objects and G-free morphisms, then, as in the case of Dold-Puppe derived functors it is possible to define a map

$$H_n(C,F)_{\mathbb{G}} \dashrightarrow \pi_n|\Gamma(C,F)|$$

induced by the simplicial set morphism

$$F(G_C)_n \dashrightarrow (\textstyle\sum(-1)^i d^i_n, \ldots, d_0 = \varepsilon(C), x)$$

7.5 Tierney and Vogel′s Derived Functors

Tierney and Vogel [19] construct derived functors of $F:\underline{C} \longrightarrow \underline{A}$ where $\underline{C}$ is a category with finite limits and a projective class and $\underline{A}$ is an abelian category. These derived functors agree with those of Eilenberg and Moore and they also agree with the 0-th level Dold-Puppe derived functors (7.2). If there is a cotriple in $\underline{C}$ realizing the given projective class then they agree with the cotriple derived functors. If $\underline{C}$ has coproducts then a suitable cotriple realizing the given projective class exists [2, 10.1].

8. Cotriple Derived Functors

8.1 Axioms

Cotriple derived functors are one of the most well-known generalizations of derived functors. When $\underline{C}$ is a category with a cotriple, $\mathbb{G} = (G,\varepsilon,\delta)$, let $\underline{M}$ be any full subcategory containing the full subcategory of $\mathbb{G}$-free objects of $\underline{C}$ and contained in the full subcategory of $\mathbb{G}$-projectives. Hence all models are $\mathbb{G}$-projective and any object in the image of G is a model. In fact the necessary condition on $\underline{M}$-morphisms is slightly weaker than that stated as will be seen later. For a functor $F:\underline{C}\text{---->}\underline{A}$ whose range is an abelian category the $\pi_n|\mathcal{T}(C,F)|$ constructed in section 2 will be compared with the $H_n(C,F)_{\mathbb{G}}$ defined in (7.4). As in the case with classical derived functors, cotriple derived functors are characterised by certain axioms [2, section 3]. The $\pi_n|\mathcal{T}(C,F)|$ will be shown to satisfy these axioms which are:-

(i) $$H_n(-,FG)_{\mathbb{G}} = \begin{cases} FG & n = 0 \\ 0 & n > 0 \end{cases}$$

(ii) If $0 \text{---->} F' \xrightarrow{\lambda} F \xrightarrow{\sigma} F'' \text{---->} 0$ is a short G-exact sequence of functors (ie $0\text{---->}F'GC\text{---->}FGC\text{---->}F''GC\text{---->}0$ is exact for every object C of $\underline{C}$ and this implies that $0\text{---->}F'P\text{---->}FP\text{---->}F''P\text{---->}0$ is exact for every $\mathbb{G}$-projective P) then there is a natural long exact sequence

$$\ldots \dashrightarrow H_n(C,F')_{\mathbb{G}} \dashrightarrow H_n(C,F)_{\mathbb{G}} \dashrightarrow H_n(C,F'')_{\mathbb{G}} \dashrightarrow H_{n-1}(C,F')_{\mathbb{G}} \dashrightarrow \ldots$$

The proof that the $\pi_n|\mathcal{T}(-,FG)|$ are trivial, in the sense that they satisfy (i) is reasonably easy.

8.2 Theorem

Let $\underline{C}$ be a category with a cotriple, $\mathbb{G} = (G,\varepsilon,\delta)$, and let $\underline{M}$ be a subcategory containing the $\mathbb{G}$-free objects and containing all morphisms which can be expressed as either $G\psi$ for ψ a $\underline{C}$-morphism or ε_M for M a model. If $F:\underline{C} \dashrightarrow \underline{A}$ is a functor whose range is an abelian category then the $\pi_n|\mathcal{T}(-,F)|$ satisfy:-

$$\pi_n|\mathcal{T}(-,FG)| \cong \begin{cases} FG & n = 0 \\ 0 & n > 0. \end{cases}$$

Proof

The category $\mathcal{T}(C,FG)$ has objects (f,x) with $f:M \dashrightarrow C$, $x \in FGM$ and morphisms corresponding to $\xi:M_1 \dashrightarrow M_2$ where

$$\begin{tikzcd} M_1 \arrow[rr, dashed, "\xi"] \arrow[dr, "f_1"'] & & M_2 \arrow[dl, "f_2"] \\ & C & \end{tikzcd}$$

commutes and $FG(\xi)x_1 = x_2$.

Let $\underline{T}(C,FG)$ be the category which would be $\mathcal{T}(C,FG)$ if $\underline{M}$ were taken to be $\underline{C}$. Objects of $\underline{T}(C,FG)$ are therefore pairs, (g,y), with $g:B \dashrightarrow C$ any $\underline{C}$-morphism and $y \in FGB$. There are functors defined by:-

$K:\mathcal{T}(C,FG) \dashrightarrow \underline{T}(C,FG)$ is inclusion

$$J:\underline{T}(C,FG) \longrightarrow \mathcal{T}(C,FG)$$

is $\quad (g,y) \longmapsto (g.\epsilon_B, F(\delta_B)y) \quad$ on objects

and $\quad \zeta \longmapsto G\zeta \quad$ on morphisms.

Now $\quad KJ(g,y) = (g.\epsilon_B F(\delta_B)y)$

so if $\quad \eta(g,y) = \epsilon_B$

then η is a natural transformation between $1_{\underline{T}(C,FG)}$ and KJ.

Also $\quad JK(f,x) = (f.\epsilon_M, F(\delta_M)x)$

and letting $\quad \nu(f,x) = \epsilon_M \quad$ ν is a natural transformation

$$\nu: 1_{\mathcal{T}(C,FG)} \longrightarrow JK$$

Thus $\quad \pi_n |\mathcal{T}(C,FG)| \cong \pi_n |\underline{T}(C,FG)|$

But $\underline{C}$ is obviously a full subcategory of itself. Therefore, using (3.2) the homotopy groups of $\underline{T}(C,FG)$ are known and

$$\pi_n |\mathcal{T}(C,FG)| \cong \begin{cases} FGC & n=0 \\ 0 & n>0. \end{cases}$$

8.3 The Under Categories

Let $0 \longrightarrow F' \xrightarrow{\lambda} F \xrightarrow{\sigma} F'' \longrightarrow 0$ be a short $\mathcal{C}$-exact sequence of functors. Then σ induces a functor which, abusing notation, will also be denoted σ.

$\sigma: \mathcal{T}(C,F) \longrightarrow \mathcal{T}(C,F'')$ is defined by

$(f,x) \longmapsto (f, \sigma(M)x)$ on objects

and $\quad \zeta \longmapsto \zeta \quad$ on morphisms.

So when (h,z) is an object of $\mathcal{T}(C,F'')$, with $h:P \longrightarrow C$ and $z \in F''P$, the category of $\mathcal{T}(C,F)$-objects σ-under (h,z) exists as defined in (4.1). The category $(h,z)/\sigma$ has objects $((f,x),\alpha)$

$$J:\underline{T}(C,FG) \longrightarrow \mathcal{T}(C,FG)$$

is $(g,y) \longmapsto (g.\varepsilon_B, F(\delta_B)y)$ on objects

and $\zeta \longmapsto G\zeta$ on morphisms.

Now $KJ(g,y) = (g.\varepsilon_B F(\delta_B)y)$

so if $\eta(g,y) = \varepsilon_B$

then η is a natural transformation between $1_{\underline{T}(C,FG)}$ and KJ.

Also $JK(f,x) = (f.\varepsilon_M, F(\delta_M)x)$

and letting $\nu(f,x) = \varepsilon_M$ ν is a natural transformation

$$\nu: 1_{\mathcal{T}(C,FG)} \longrightarrow JK$$

Thus $\pi_n|\mathcal{T}(C,FG)| \cong \pi_n|\underline{T}(C,FG)|$

But $\underline{C}$ is obviously a full subcategory of itself. Therefore, using (3.2) the homotopy groups of $\underline{T}(C,FG)$ are known and

$$\pi_n|\mathcal{T}(C,FG)| \cong \begin{cases} FGC & n=0 \\ 0 & n>0. \end{cases}$$

8.3 The Under Categories

Let $0 \longrightarrow F' \xrightarrow{\lambda} F \xrightarrow{\sigma} F'' \longrightarrow 0$ be a short $\mathfrak{C}$-exact sequence of functors. Then σ induces a functor which, abusing notation, will also be denoted σ.

$\sigma:\mathcal{T}(C,F) \longrightarrow \mathcal{T}(C,F'')$ is defined by

$(f,x) \longmapsto (f,\sigma(M)x)$ on objects

and $\zeta \longmapsto \zeta$ on morphisms.

So when (h,z) is an object of $\mathcal{T}(C,F'')$, with $h:P \longrightarrow C$ and $z \in F''P$, the category of $\mathcal{T}(C,F)$-objects σ-under (h,z) exists as defined in (4.1). The category $(h,z)/\sigma$ has objects $((f,x),\alpha)$

where $f:M \dashrightarrow C$, $x \in FM$ and $\alpha:(h,z) \dashrightarrow (f,\theta(M)x)$

so
$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, "h"'] & & M \arrow[dl, "f"] \\ & C & \end{tikzcd}$$
commutes and $F''(\alpha)z = \theta(M)x$.

Morphisms of $(h,z)/\theta$ correspond to $\underline{M}$-morphisms $\xi:M_1 \dashrightarrow M_2$ where

$$\begin{tikzcd} & P \arrow[dl, "\alpha_1"'] \arrow[dr, "\alpha_2"] \arrow[dd, "h"] & \\ M_1 \arrow[rr, "\xi"] \arrow[dr, "f_1"'] & & M_2 \arrow[dl, "f_2"] \\ & C & \end{tikzcd}$$
commutes and $F''(\xi)x_1 = x_2$.

<u>8.4 Theorem</u>

Let $\underline{C}$ be a category with a cotriple, $\mathbb{G}$, and an initial object, I. Let $\underline{M}$ be a subcategory of the full subcategory of $\mathbb{G}$-projectives of $\underline{C}$. Assume that $\underline{M}$ has finite sums and that GI=I, so I is an object of $\underline{M}$ (I is then necessarily an initial object of $\underline{M}$). Let F, F' and F" be functors with domain $\underline{M}$ and range an abelian category, $\underline{A}$. Assume that

$$0 \dashrightarrow F' \xrightarrow{\lambda} F \xrightarrow{\theta} F'' \dashrightarrow 0$$

is a short $\mathbb{G}$-exact sequence of functors. If

$$\gamma:(h_1,z_1) \dashrightarrow (h_2,z_2)$$

is a morphism of $\mathcal{T}(C,F'')$, for C an object of $\underline{C}$, then the induced functor of under categories

$$\gamma/\theta:(h_2,z_2)/\theta \longrightarrow (h_1,z_1)/\theta \qquad \text{defined by}$$

$$((f,x),\alpha) \longmapsto ((f,x),\alpha\gamma) \qquad \text{on objects}$$

$$\xi \longmapsto \xi \qquad \text{on morphisms}$$

itself induces a homotopy equivalence of the classifying spaces of these categories.

Proof

It will be useful to fix some notation. Let "0" denote the basepoint of any "set" in $\underline{A}$. Let $u_C:I\longrightarrow C$ be the unique $\underline{C}$-morphism from the initial object to C. Denote the sum in $\underline{M}$ by + and let the natural injections be

$$in_N:N\longrightarrow N+P \qquad \text{and} \qquad in_P:P\longrightarrow N+P.$$

Given $g:N\longrightarrow C$ and $h:P\longrightarrow C$ they induce the map

$$g+h:N+P\longrightarrow C\ .$$

(1) First consider the case when

$$\gamma:(u_C,0)\longrightarrow(h,0)$$

where $h:P\longrightarrow C$ is any $\underline{C}$-morphism. Since γ is a map $I\longrightarrow P$, it is clear that $\gamma = u_C$. Objects of $(u_C,0)/\theta$ are of the form $((g,y),\beta)$ where

$$\begin{tikzcd} I \arrow[rr, "\beta"] \arrow[dr, "u_C"'] & & N \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

commutes and $y\in FN$, $F''(\beta)0 = \theta(N)y$.

So $\beta = u_N$ and $\theta(N)y = 0$.

Define the functor $(\overline{u_C/\theta})$ as follows:-

$$(\overline{u_C/\theta}):(u_C,0)/\theta \text{------------}> (h,0)/\theta$$

is $\qquad ((g,y),u_N) \longmapsto ((g+h,F(in_N)y),in_P)$ on objects

and $\qquad \zeta \longmapsto \zeta + 1_P$ on morphisms.

Given $\omega : N+P \text{----}> M$ which induces a morphism of $(h,0)/\theta$

from $\quad ((\overline{u_C/\theta})((g,y),u_N)$ to $((f,x),\alpha)$

then $\quad \omega' = \omega . in_N \quad$ induces a morphism of $(u_C,0)/\theta$

from $\quad ((g,y),u_N)$ to $(u_C/\theta)((f,x),\alpha)$.

$$\begin{tikzcd}
u_C : I \arrow[r] \arrow[d, "u_P"] & N \arrow[d, "in_N"] \arrow[ddr, "\omega'"] \arrow[dddrr, "g"] & & \\
in_P : P \arrow[r] \arrow[ddrrr, "h"'] & N+P \arrow[dr, "\omega"] & & \\
 & & M \arrow[dr, "f"] & \\
 & & & C
\end{tikzcd}$$

Conversely given ω' let $\omega = \alpha + \omega'$. The maps

$$\omega \longmapsto \omega' \qquad \text{and} \qquad \omega' \longmapsto \omega$$

are inverse. This shows that the functors (u_C/θ) and $(\overline{u_C/\theta})$ are adjoint. Therefore they induce homotopy equivalences of classifying spaces [15].

(2) Next consider $\mathcal{T}(C,F'')$-morphisms of the form

$$\gamma : (h_1,0) \text{------}> (h_2,0) \qquad\qquad h_1 : P_1 \text{----}> C.$$

Then $\quad u_{P_2} = \gamma . u_{P_1}$

So $\quad (u_{P_2}/\theta) = (\gamma u_{P_1}/\theta) = (\gamma/\theta)(u_{P_1}/\theta)$

Since, by the above, both the (u_{P_i}/θ) $(i=1,2)$ induce homotopy equivalences of classifying spaces so does (γ/θ).

(3) Finally let $\gamma:(h_1,z_1)\text{---->}(h_2,z_2)$ be any morphism of $\mathcal{T}(C,F'')$ where

$$h_i:P_i\text{---->}C \qquad z_i\in F''P \qquad (i=1,2).$$

For each $\tilde{z}_i\in FP_i$ with $\theta(P_i)\tilde{z}_i=z_i$ there is a functor, $L_{\tilde{z}_i}$, inducing a homotopy equivalence of classifying spaces. (Since P is a model P is $\mathbb{C}$-projective and $0\text{---->}F'P\text{---->}FP\text{---->}F''P\text{---->}0$ is exact. Therefore such $\tilde{z}_i$ always exist). The functor $L_{\tilde{z}_i}$ is defined by

$$L_{\tilde{z}_i}:(h_i,z_i)/\theta\text{-------->}(h_i,0)/\theta$$

$$((f,x),\alpha)\text{-------->}((f,x-F(\alpha)\tilde{z}_i),\alpha) \qquad \text{on objects}$$

and the identity on morphisms.

It is known that $L_{\tilde{z}_i}$ induces a homotopy equivalence of classifying spaces because it has an inverse given by:-

$$L_{\tilde{z}_i}^*:(h_i,0)/\theta\text{-------->}(h_i,z_i)/\theta$$

$$((g,y),\beta)\text{------>}((g,y+F(\beta)z_i),\beta) \qquad \text{on objects}$$

and the identity on morphisms.

The morphism of $\underline{M}$ which induces the given γ also induces $\gamma^*:(h_1,0)\text{---->}(h_2,0)$ in $\mathcal{T}(C,F'')$.

But
$$\begin{aligned}\theta(P_2)F(\gamma)\tilde{z}_1 &= F''(\gamma)\theta(P_1)\tilde{z}_1\\ &= F''(\gamma)z_1\\ &= z_2\ .\end{aligned}$$

Since the following square commutes :-

$$\begin{tikzcd}
\gamma/\theta : (h_2, z_2)/\theta \arrow[r] \arrow[d, "L_{F(\gamma)\tilde{z}_1}"] & (h_1, z_1)/\theta \arrow[d, "L_{\tilde{z}_1}"] \\
\gamma''/\theta : (h_2, 0)/\theta \arrow[r] & (h_1, 0)/\theta
\end{tikzcd}$$

γ/θ must induce a homotopy equivalence of classifying spaces because the vertical maps and γ''/θ do by the above.

The above theorem (8.4) has been proved because it shows that the conditions necessary to apply (4.3) hold in the given circumstances. To obtain a useful result from applying (4.3) one needs to be able to identify certain under categories with other equivalent categories.

8.5 Lemma

If the assumptions of (8.4) hold then there exists an object, D, of $\mathcal{T}(C,F'')$ with

$$D/\theta \cong \mathcal{T}(C,F') .$$

Proof

The category $\underline{M}$ has an initial object, I. Take

$$D = (u_C, 0) \qquad \text{where} \qquad u_C : I \longrightarrow C .$$

Then objects of D/θ are $((g,y),u_N)$ where

$$\begin{tikzcd}
u_N : I \arrow[rr] \arrow[dr, "u_C"'] & & N \arrow[dl, "g"] \\
& C &
\end{tikzcd}$$

commutes and $\theta(N)y = 0$.

Therefore the map

$$\tau(C,F') \dashrightarrow D/\theta$$

$$(f,x) \longmapsto ((f,\lambda(M)x),u_M)$$

is a bijection on the objects of these categories. Morphisms of D/θ from $((g_1,y_1),u_{N_1})$ to $((g_2,y_2),u_{N_2})$ correspond to $\underline{M}$-morphisms $\zeta : N_1 \dashrightarrow N_2$ where

$$\begin{tikzcd} & I \arrow[dl, "u_{N_1}"'] \arrow[dr, "u_{N_2}"] & \\ N_1 \arrow[rr, "\zeta"] \arrow[dr, "g_1"'] & & N_2 \arrow[dl, "g_2"] \\ & C & \end{tikzcd}$$

commutes and $F(\zeta)y_1 = y_2$.

But if $\xi:(f_1,x_1) \dashrightarrow (f_2,x_2)$ is a $\tau(C,F')$-morphism then

$$F(\xi)\lambda(M_1)x_1 = \lambda(M_2)F'(\xi)x_1$$
$$= \lambda(M_2)x_2$$

and so the functor

$$\tau(C,F') \dashrightarrow D/\theta \qquad \text{defined by}$$
$$(f,x) \longmapsto ((f,\lambda(M)x),u_M) \qquad \text{on objects}$$
$$\xi \longmapsto \xi \qquad \text{on morphisms}$$

is a natural equivalence.

Now it is possible to prove the main theorem of this section.

<u>8.6 Theorem</u>

Let $\underline{C}$ be a category with a cotriple, $\mathfrak{C}$, and an initial object, I. Let $\underline{M}$ be any subcategory of $\underline{C}$ contained in the full subcategory of $\mathbb{G}$-projectives and containing all $\mathbb{G}$-free objects. The morphisms of $\underline{M}$ include all those which can be expressed as either $G\psi$ for ψ some $\underline{C}$-morphism or ε_M for M a model. Assume that GI=I (so I is a model) and that $\underline{M}$ has finite sums. If $F:\underline{C}\text{----}>\underline{A}$ is a functor whose range is a concrete abelian category then

$$H_n(C,F)_{\mathbb{G}} \cong \pi_n|\mathcal{T}(C,F)|.$$

<u>Proof</u>

Cotriple derived functors are characterised by the axioms (i) and (ii) of (8.1). The $\pi_n|\mathcal{T}(C,F)|$ satisfy (i) by (8.2). Given a short exact sequence of functors

$$0 \longrightarrow F' \xrightarrow{\lambda} F \xrightarrow{\theta} F'' \longrightarrow 0$$

there is an induced functor

$$\mathcal{T}(C,\theta):\mathcal{T}(C,F)\longrightarrow\mathcal{T}(C,F'') .$$

Theorem (8.4) shows that (4.3) may be applied to obtain a cartesian square involving this functor. The cartesian square

$$\begin{array}{ccc} D/\theta & \longrightarrow & \mathcal{T}(C,F) \\ \downarrow & & \downarrow \\ D/1_{\mathcal{T}(C,F'')} & \longrightarrow & \mathcal{T}(C,F'') \end{array}$$

where D is any object of $\mathcal{T}(C,F'')$, induces a homotopy cartesian square of classifying spaces. Applying (8.5) to the short G-exact sequences

$$0 \longrightarrow F' \xrightarrow{\lambda} F \xrightarrow{\theta} F'' \longrightarrow 0$$

and $$0 \longrightarrow 0 \longrightarrow F'' \xrightarrow{1} F'' \longrightarrow 0$$

enables some categories in this cartesian square to be replaced by equivalent on ones and so shows that

$$\begin{array}{ccc} B\mathcal{T}(C,F') & \longrightarrow & B\mathcal{T}(C,F) \\ \downarrow & & \downarrow{\scriptstyle B\theta} \\ B\mathcal{T}(C,0) & \longrightarrow & B\mathcal{T}(C,F'') \end{array} \quad \text{is homotopy cartesian.}$$

By (5.3) the spaces and maps involved in this cartesian square are infinite loop spaces. From (8.2) it is clear that $B\mathcal{T}(C,0)$ is contractible and therefore the homotopy theoretic fibre of

$$B\mathcal{T}(C,\theta):B\mathcal{T}(C,F) \longrightarrow B\mathcal{T}(C,F'')$$

over any point is equivalent to

$$B\mathcal{T}(C,\lambda):B\mathcal{T}(C,F') \longrightarrow B\mathcal{T}(C,F) .$$

The homotopy boundary of this homotopy fibration of infinite loop spaces gives natural connecting homorphism in the long exact sequence:-

$$\ldots \longrightarrow \pi_{n+1}|\mathcal{T}(C,F'')| \xrightarrow{\partial} \pi_n|\mathcal{T}(C,F')| \xrightarrow{\lambda_*} \pi_n|\mathcal{T}(C,F)| \xrightarrow{\theta_*} \pi_n|\mathcal{T}(C,F'')| \longrightarrow \ldots$$

Thus the $\pi_n|\mathcal{T}(C,F)|$ also obey axiom (ii) of (8.1) and therefore

$$H_n(C,F)_{\mathfrak{C}} \cong \pi_n|\mathcal{T}(C,F)| .$$

Cotriple derived functors might seem, in some ways easier to construct than the derived functors defined in section 2 but some theorems are much easier to prove using the $\pi_n|\mathcal{T}(-,F)|$. For example (3.10) is the homology coproduct theorem which [2, section 7] is devoted to proving for various specific examples.

The most restrictive condition in (8.6) might appear to be the requirement that GI=I which often does not occur in practice. However this condition is really very weak. The following theorem shows how it can be achieved.

8.7 Theorem

If $\underline{C}$ is a category with a zero object (which is necessarily the initial object, I) and cokernels, then given any cotriple, $\mathbb{G} = (G,\varepsilon,\delta)$, in $\underline{C}$ there exists a cotriple, $\mathbb{G}' = (G',\varepsilon',\delta')$, in $\underline{C}$ such that $G'I=I$ and objects of $\underline{C}$ are $\mathbb{G}'$-projective if and only if they are $\mathbb{G}$-projective. (So $H_n(C,F)_{\mathbb{G}} = H_n(C,F)_{\mathbb{G}'}$ for any object C in $\underline{C}$ and any functor $F:\underline{C}\text{---->}\underline{A}$.)

Proof

Define the endofunctor of the new cotriple, G', on objects by

$$G'C = \operatorname{coker}(GI \xrightarrow{u_C} GC) .$$

Let ω_C be the natural map $\omega_C : GC \text{---->} \operatorname{coker} G(u_C)$. If $\psi : C \text{---->} B$ is a $\underline{C}$-morphism then there is a commutative diagram

$$\begin{array}{ccccc}
GI & \longrightarrow & GC & \xrightarrow{\omega_C} & G'C \\
\downarrow & & \downarrow{\scriptstyle G\psi} & & \downarrow \\
GI & \longrightarrow & GB & \xrightarrow{\omega_B} & G'B
\end{array}$$

and $\quad \omega_B . G\psi . G(u_C) = \omega_B . G(u_B)$

$$= 0.$$

Therefore, since $G'C$ is a cokernel there is an induced map

$G'C \dashrightarrow G'B$ making the above diagram commute. Define $G'\psi$ to be this map.

Now

$$\begin{tikzcd}
G(u_C):GI \arrow[r] \arrow[d, "\varepsilon_I"] & GC \arrow[r] \arrow[d, "\varepsilon_C"] & G'C \\
u_C:\ I \arrow[r] & C &
\end{tikzcd}$$

commutes,

so $\varepsilon_C . G(u_C) = 0$ and by the defining property of cokernels there is a map $\varepsilon'_C : G'C \longrightarrow C$. Clearly these ε'_C induce a natural transformation $\varepsilon' : G' \longrightarrow 1_{\underline{C}}$.

In the following solid arrow diagram

$$\begin{tikzcd}
GI \arrow[r, "Gu_C"] \arrow[d, "\delta_I"] & GC \arrow[r, "\omega_C"] \arrow[d, "\delta_C"] & G'C \arrow[d, dotted] \\
G^2I \arrow[r, "G^2u_C"] \arrow[d, "GO"] & G^2C \arrow[r] \arrow[d, "G\omega_C"] & \mathrm{coker}(G^2u_C) \arrow[d, dotted] \\
GI \arrow[r, "Gu_{G'C}"] & G(G'C) \arrow[r] & (G')^2C = \mathrm{coker}(Gu_{G'C})
\end{tikzcd}$$

the upper square commutes since δ is a natural transformation and the lower square commutes since

$$\begin{aligned}
G\omega_C . G^2u_C &= G(\omega_C Gu_C) \\
&= G(\ 0 : GI \longrightarrow G'C\ ) .
\end{aligned}$$

It is therefore possible to define the dotted maps whose composite is taken to be δ'_C.

If M is $\mathbb{G}$-projective, so there exists a map

$$s : M \longrightarrow GM \qquad \text{with} \qquad \varepsilon_M . s = 1_M .$$

then let $\qquad s' = \omega_M . s : M \longrightarrow G'M.$

So

$$\begin{aligned}
\varepsilon'_M . s' &= \varepsilon'_M . \omega_M . s \\
&= 1_M
\end{aligned}$$

and M is $\mathbb{G}'$-projective. Conversely consider the case when M is $\mathbb{G}'$-projective, and so there is a map $s':M \longrightarrow G'M$. Because I is a zero object the map $u_M:I \longrightarrow M$ has as left inverse the unique map $e_M:M \longrightarrow I$. Therefore the map $Gu_M:GI \longrightarrow GM$ has a left inverse $Ge_M:GM \longrightarrow GI$. Thus the exact sequence

$$I \longrightarrow GI \xrightarrow{Gu_M} GM \xrightarrow{\omega_M} G'M \longrightarrow I$$

splits and ω_M has a right inverse, $\tilde{\omega}_M$.

Let $$s = \tilde{\omega}_M . s'.$$

Then $$\begin{aligned} \varepsilon_M . s &= \varepsilon_M . \tilde{\omega}_M . s' \\ &= \varepsilon'_M . \omega_M . \tilde{\omega}_M . s' \\ &= 1 \end{aligned}$$

and M is $\mathbb{G}$-projective.

In many examples of cotriple derived functors the endofunctor of the cotriple is the composition of two adjoint functors, which are a "forgetful" functor and a "free generation" functor. In this case the above theorem gives a new cotriple whose endofunctor is again the composition of adjoint functors. These functors are "forget structure but remember basepoint" and "freely generate with relation basepoint=0".

9. Examples

The fact that the derived functors defined in section 2 agree with cotriple derived functors gives many examples of sequences of functors which therefore arise from that construction [2,sections 1 and 10].

9.1 Eckmann-Hilton Homotopy Groups

In [5] Eckmann and Hilton define two-space homotopy groups

$$\Pi_n(A,B) = [\Sigma^n A,B] \cong [\Sigma^{n-i}A,\Omega^i B] \qquad 0\leq i\leq n$$

where [,] denotes the set of homotopy classes of maps and Σ and Ω are the usual suspension and loop functors. This process can be applied to the category of R-modules, for any ring R, or any abelian category with sufficient injectives or projectives. If $\underline{C}$ is any abelian category with sufficient projectives a "loop" functor can be defined. For C an object of $\underline{C}$ take an epimorphism $P \longrightarrow C$ with P projective. Then

$$\Omega C = \ker(P \longrightarrow C) .$$

Homotopy groups can now be defined using iterated loops. Similarly if $\underline{C}$ has sufficient injectives homotopy groups are defined using a "suspension" functor. These groups were constructed in an alternative manner by Huber [9]. If G is the composite functor

$$\underline{\text{Unitary R-modules}} \longrightarrow \underline{\text{Sets}}_* \longrightarrow \underline{\text{Unitary R-Modules}}$$

where the second functor is "generate with relation base-point=0" then there is a simplicial set with n-simplices $\mathrm{Hom}(X, G^{n+1}(Y))$, for X and Y any R-modules.

Now $$\pi_n(X,Y) \cong \pi_n(\mathrm{Hom}(X, G^{*}(Y)))$$
$$\cong H_n(\mathrm{Hom}(X, G^{*}(Y))) \qquad \text{by Moore's theorem [13]}$$

so take $\underline{M}$ to be the full subcategory of $\mathbf{G}$-projectives of $\underline{C}$.

Then applying (8.6)

$$\pi_n(X,Y) \cong \pi_n |\tau(Y, \mathrm{Hom}(X,-))|.$$

9.2 Hochschild's K-Relative Tor

In [8] Hochschild takes a map of ring with multiplicative identities which preserves the multiplicative identities, $\alpha: K \longrightarrow R$. (For example K is a subring of R.) This map makes any R-module into a K-module. Now it is possible to define the notions of (R,K)-exact and (R,K)-projective so one can form (R,K)-projective resolutions to be used when constructing the K-relative derived functors. In particular, for any R-module M, there is a (R,K)-projective resolution given by:-

$$\cdots \longrightarrow M \otimes_K R \otimes_K R \longrightarrow M \otimes_K R \longrightarrow M \longrightarrow 0$$

Thus K-relative derived functors are cotriple derived functors for $G_\alpha M$ "=" $M \otimes_K R$, ie G_α is defined by the composition

$$\begin{tikzcd} \underline{\text{R-modules}} \arrow[rr] \arrow[dr, "\text{restrict}"'] & & \underline{\text{R-modules}} \\ & \underline{\text{K-modules}}. \arrow[ur, "\otimes_K R"'] & \end{tikzcd}$$

Taking $\underline{M}$ as the full subcategory of $\mathbf{G}_\alpha$-free modules or as the full subcategory of $\mathbf{G}_\alpha$-projective modules

$$\pi_n|T(C,A\otimes_K -)| \cong \mathrm{Tor}_n^K(A,C).$$

9.3 Homology of Groups

Let Γ be a group and $W \dashrightarrow \Gamma$ be a group over Γ. Then there is an action of W on any left Γ-module, M. A derivation from M to W over Γ is a map $a: W \dashrightarrow M$ such that

$$a(ww') = w.a(w') + a(w).$$

The collection of all such derivations forms a group, $\mathrm{Der}(W,M)_\Gamma$ and there is a functor

$$\mathrm{Der}(\ ,M)_\Gamma : (\text{Groups over } \Gamma)^o \dashrightarrow \underline{\text{Abelian Groups}}.$$

The augmentation ideal of W is $IW = \ker(\mathbb{Z}W \dashrightarrow \mathbb{Z})$. Now

$$\mathrm{Der}(W,M)_\Gamma \cong \mathrm{Hom}_W(IW,M)$$
$$\cong \mathrm{Hom}_\Gamma(\mathbb{Z}\Gamma \otimes_W IW, M).$$

But the Γ-module of representations of Der(W,M) are the differentials of W over Γ, $\mathrm{Diff}_\Gamma(W) = \mathbb{Z}\Gamma \otimes_W IW$. Alternatively Diff_Γ can be regarded as the "free abelian group over Γ" functor which is left adjoint to the forgetful functor

$$\underline{\Gamma\text{-modules}} = \underline{\text{Abelian groups over } \Gamma} \dashrightarrow \underline{\text{Groups over } \Gamma}.$$

The homology of $W \dashrightarrow \Gamma$ with coefficients in the right Γ-module M is the homology of

$$\ldots \dashrightarrow M \otimes_{\mathbb{Z}\Gamma} \mathrm{Diff}_\Gamma(G^{n+1}W) \dashrightarrow \ldots \dashrightarrow M \otimes_{\mathbb{Z}\Gamma} \mathrm{Diff}_\Gamma(GW) \dashrightarrow 0$$

where GW is the free group on the elements of W. Altering this functor to preserve the basepoint as described in (8.7) before choosing $\underline{M}$

$$\pi_n |\mathcal{T}(W, M \otimes_{\mathbb{Z}\Gamma} \mathrm{Diff}_\Gamma(-)| \cong \begin{cases} M \otimes_{\mathbb{Z}\Gamma} IW & n=0 \\ H^{E-M}_{n+1}(W,M) & n>0 \end{cases}$$

where H^{E-M} is Eilenberg-Maclane Homology and is known to be

$$\mathrm{Tor}_n^{\mathbb{Z}W}(\mathbb{Z},M) = \mathrm{Tor}_n^{\mathbb{Z}W}(IW,M).$$

The ordinary homology of a group Γ with coefficients in a Γ-module M is obtained by taking $W=\Gamma$ and regarding Γ as a group over itself using the identity map.

9.4 Homology of Commutative Algebras

All rings will be commutative rings with a 1 and ring homomorphisms will preserve the multiplicative identities. Let A be such a ring and let B be a commutative A-algebra, ie B is a ring and there is a given map $A \longrightarrow B$. Let W be a B-module. Then Der(A,B,W) is the abelian group of A-derivations of B into W so its elements are A-homomorphisms $q: B \longrightarrow W$ satisfying

$$q(bb') = bq(b') + b'q(b).$$

Let Diff(A,B) be the B-module of A-differentials of B. It is completely determined by the isomorphism

$$\mathrm{Der}(A,B,W) \cong \mathrm{Hom}_B(\mathrm{Diff}(A,B),W).$$

Also $\qquad \mathrm{Diff}(A,B,W) \cong \mathrm{Diff}(A,B) \otimes_B W.$

If $I = \ker(B \otimes_A B \longrightarrow B)$ then $\mathrm{Diff}(A,B) = I/I^2$. Letting $\mathbb{G}$ be the cotriple induced by the adjoint functors

$$\underline{\text{Commutative A-Algebras}} \longrightarrow \underline{\text{Sets}}_* \longrightarrow \underline{\text{Commutative A-Algebras}}$$

then the $\mathbb{G}$-free objects are the polynomial A-algebras with elements of any given object as variables. Take $\underline{M}$ to be the full subcategory on these objects. Then since the $\mathbb{G}$-derived functors are the homology of the commutative A-algebra B over W so are the $\pi_n|\mathcal{T}(B,\mathrm{Diff}(A,-,W))|$.

9.5 Singular Homology

Let $\underline{C}$ be the category of based topological spaces and continuous based maps. Let Δ_n denote the standard affine n-simplex and if a space X is given a disjoint basepoint the new space formed will be X^+. Singular Homology is realized as a cotriple derived functor for the cotriple $\mathbb{G}=(G,\varepsilon,\delta)$ where, for X a based topological space,

$$GX = \bigcup_{\substack{\Delta_p \to X \\ p \geq 0}} \Delta_p$$

and ε and δ are induced by the obvious maps [2, 10.2(d)]. By (8.7) this cotriple can be replaced by G′ where

$$G'X = \left(\bigcup_{\substack{\Delta_p^+ \to X, \text{ non-trivial} \\ p \geq 0}} \Delta_p \right)^+$$

then the $\mathbb{G}'$-free objects are spaces of disjoint simplices.

Since $$H_n(X, H_0^{\mathrm{Sing}}(-,\Gamma))_G \cong H_n^{\mathrm{Sing}}(X,\Gamma)$$

then, taking $\underline{M}$ as the subcategory of spaces of disjoint simplices and face inclusions of simplices,

$$\pi_n|\mathcal{T}(X, H_0^{\mathrm{Sing}}(-,\Gamma))| \cong H_n^{\mathrm{Sing}}(X,\Gamma).$$

10. The Derived Functors of π_0

In this section $\underline{C}$ will be the category of based topological spaces and continuous maps. The models will be spaces homotopy equivalent to finite discrete spaces and $\underline{M}$ will be the full subcategory of $\underline{C}$ with models as objects. The derived functors of π_0 will be compared with the homotopy functors, π_n. A space, X, with a disjoint base point will be denoted X^+.

First consider the 0-th derived functor (which is its Kan extension by (3.7)).

10.1 Theorem

For any topological space, with the above choice of $\underline{M}$,

$$\pi_0 C \cong \pi_0 |\tau(C,\pi_0)|.$$

Proof

In (3.5) a map $\mu : \pi_0 |\tau(C,\pi_0)| \longrightarrow \pi_0 C$ is defined. There is an isomorphism

$$\pi_0 |\tau(C,\pi_0)| \cong \frac{\{\text{ Objects of } (C,\pi_0)\ \}}{\text{Domain of a morphism} \sim \text{range of the morphism}}.$$

so that the elements of $\pi_0 C$, which are components of C, correspond to classes of objects of $\tau(C,\pi_0)$. Let M′ be a component of M, then

μ(Class of (f, M′)) = Component containing f(M′).

The map μ is surjective for given $Y \in \pi_0 C$ define

$$f : (\text{point})^+ \longrightarrow C$$

by
$$(\text{point}) \longrightarrow y \in Y .$$

Then

$$(f,\text{point}) = \text{Component containing } f(\text{point})$$
$$= Y.$$

Assume $\mu(f_1,M'_1) = \mu(f_2,M'_2)$ ie $f_1(M'_1)$ is contained in the same component as $f_2(M'_2)$. Let $m_i \in M'_i$ $(i=1,2)$. Consider the two-point space $(\text{point})^+$.

Define $\xi_i : (\text{point})^+ \dashrightarrow M_i$, $f'_i : (\text{point})^+ \dashrightarrow C$

by $(\text{point}) \dashrightarrow m_i$ $(\text{point}) \dashrightarrow f_i(m_i)$.

Then
$$\begin{tikzcd}
\xi_i : (\text{point})^+ \arrow[rr] \arrow[dr, "f'_i"'] & & M_i \arrow[dl, "f_i"] \\
& C &
\end{tikzcd}$$
commutes ,

so ξ_i corresponds to a morphism of $\tau(C,\pi_0)$ from (f'_i,point) to (f_i,M'_i). Now $f_1(M'_1)$ and $f_2(M'_2)$ are contained in the same path component of C so there exists a path $\rho : [0,1] \dashrightarrow C$ ($[0,1]$ is the unit interval) with

$$\rho(0) = f_1(m_1) \qquad \text{and} \qquad \rho(1) = f_2(m_2) .$$

Define $\xi'_i : (\text{point}) \dashrightarrow [0,1]$ $(i=1,2)$

by $(\text{point}) \dashrightarrow i - 1$

Then ξ'_i corresponds to a $\tau(C,\pi_0)$-morphism from (f'_i,point) to $(\rho^+,[0,1])$. Thus there is a chain of morphisms between (f_1,M'_1) and (f_2,M'_2) which proves that μ is injective . Therefore, since μ is both injective and surjective it is the required isomorphism.

10.2 The Higher Derived Functors

Since $\underline{C}$ is the category of based topological spaces and when defining the derived functors in section 2 a preferred basepoint of the simplicial set was chosen it seems sensible to restrict consideration to connected spaces when comparing the higher derived functors with the higher homotopy functors. Trying to compare $\pi_n|\tau(C,\pi_0)|$ and $\pi_n C$ when the former is a (simplicial) homotopy group of a simplicial set and the latter is a (topologial) homotopy group of a topological space seems unnatural. The obvious answer might appear to be to apply the geometric realization functor to the simplicial set so that the groups to be compared are both topological homotopy groups but in fact it is more productive to consider the singular simplicial functor, which is right adjoint to the geometric realization functor. This means that the objects whose homotopy groups are to be compared both become simplicial sets.

10.3 The Singular Simplicial Functor

For n>0 let Δ_n be the affine n-simplex, ie

$$\{(x_0,x_1,\dots,x_n)\in\mathbb{R}^{n+1} \mid x_i\geqslant 0\ ,\ \Sigma_0^n x_i=1\}.$$

The vertices of Δ_n have a natural ordering given by

$$e_i^n = (0,0,\dots,0,1,0,\dots,0,0)$$

where the "1" appears in the i-th position. This enables affine maps $\delta_n^i:\Delta_{n-1}\dashrightarrow\Delta_n$ and $\sigma_n^i:\Delta_{n+1}\dashrightarrow\Delta_n$ to be defined by

$$\delta_n^j(e_i^{n-1}) = \begin{cases} e_i^n & i < j \\ e_{i+1}^n & i \geqslant j \end{cases}$$

$$\sigma_n^j(e_i^{n+1}) = \begin{cases} e_i^n & i \leqslant j \\ e_{i-1}^n & i > j \end{cases}$$

Given a topological space, C, the singular simplicial set, Sing(C), has as n-simplices the set of all continuous maps, $g:\Delta_n \dashrightarrow C$, and its face and degeneracy maps are

$$d_n^j(g) = g.\delta_n^j$$
$$s_n^j(g) = g.\sigma_n^j \ .$$

A composite of any number of maps of the form $d_n^j(g)$ will be referred to as a face inclusion. The identity of an n-simplex is a face inclusion since it is the composition of 0 factors. Any continuous map of spaces induces, by composition, a simplicial map of the associated singular simplicial sets, so Sing is a functor. Milnor [12] shows that Sing is adjoint to the geometric realization and that $\pi_n C \cong \pi_n \mathrm{Sing}(C)$.

10.4 The First Derived

Associated with any simplicial set, X, is another simplicial set, X′, called the first derived of X, whose 0-simplices are all the simplices of X (of any dimension) and whose n-simplices are chains of n face inclusions (simplices whose chain contains an identity map are degenerate). The i-th face map of X′ is given by inserting an identity map after the i-th map in the chain and the i-th degeneracy is given by composing the i-th and (i+1)-th maps of the chain. Considering the geometric realizations of the simplicial sets it can be seen that this process is the simplicial set analog of taking a barricentric subdivision of a complex and that $\pi_n X \to \pi_n X'$.† The first derived construction will be used when C is a connected based space and X = Sing(C). Note that the n-simplices of (Sing(C))′ correspond to chains

$$\Delta_{i_0} \dashrightarrow \Delta_{i_1} \dashrightarrow \cdots \dashrightarrow \Delta_{i_n} \dashrightarrow C$$

with $i_j < i_{j+1}$. Thus there is an obvious inclusion map

$$(\mathrm{Sing}(C))' \dashrightarrow |\Gamma(C,\pi_0)| \ .$$

Let $\underline{S}$ be the category whose objects are maps of the form

$$(\Delta_r)^+ \dashrightarrow C \qquad \text{for} \qquad r \in \{-1,0,1,2,\ldots\} \ .$$

Morphisms of S from $[(\Delta_{r_1})^+ \dashrightarrow C]$ to $[(\Delta_{r_2})^+ \dashrightarrow C]$ can only exist if $r_1 < r_2$ or $r_1 = r_2$ and correspond to face inclusions of simplices $\Delta_{r_1} \dashrightarrow \Delta_{r_2}$ satisfying

$$\begin{tikzcd}
(\Delta_{r_1})^+ \arrow[rr] \arrow[dr] & & (\Delta_{r_2})^+ \arrow[dl] \\
& C &
\end{tikzcd}$$

commutes.

The category $\underline{S}$ is interesting because its nerve "is" (Sing(C))′.

† For if $\|X\|$ is the modified geometric realization without degeneracies collapsed [18, Appendix A] clearly $|X'| \simeq \|X\|$. But $\|X\| \simeq |X|$ [18, A.1(iv)].

<u>10.5 Theorem</u>

Let $\underline{C}$ be the category of based topological spaces and continuous maps. Let $\underline{M}$ be the full subcategory of spaces homotopy equivalent to finite discrete spaces. If C is a based connected space then

$$\pi_n |\tau(C,\pi_0)| \cong \pi_n C.$$

<u>Proof</u>

Define the category $\underline{S}$, with nerve $(\mathrm{Sing}(C))'$, as in (10.4) above. Construct a new category, $\underline{Q}$, with obvious maps

$$\tau(C,\pi_0) \longrightarrow \underline{Q} \rightleftarrows \underline{S}\ .$$

The category $\underline{Q}$ has as objects pairs of continuous maps (which are morphisms of $\underline{C}$), (g,h), where $g:(\Delta_r)^+ \longrightarrow M$, $h:M \longrightarrow C$ for M an object of $\underline{M}$. The component of M containing the image of Δ_r under g will correspond to an element y of $\pi_0 M$. Morphisms of $\underline{Q}$ from (g,h) to (g',h') correspond to pairs, (α,β) where α is a face inclusion of the simplices $\alpha:\Delta_r \longrightarrow \Delta_{r'}$ and β is a morphism of $\tau(C,\pi_0)$ from (h,y) to (h',y') such that

$$\begin{tikzcd}
\alpha:(\Delta_r)^+ \arrow[rr] \arrow[d] & & (\Delta_{r'})^+ \arrow[d] \\
\beta:\ M \arrow[rr] \arrow[dr, "h"'] & & M' \arrow[dl, "h'"] \\
& C &
\end{tikzcd}$$

commutes.

There is a functor, $J:\underline{Q} \longrightarrow \tau(C,\pi_0)$, which is

$$(g,h) \longrightarrow (h,y).$$

For any object (f,x) of $\mathcal{T}(C,\pi_0)$ the category of $\underline{Q}$-objects J-over (f,x) can be formed. This category, (J/(f,x) in the notation of section 4) has objects corresponding to triples, (g,h,ξ) where

$$\begin{tikzcd} (\Delta_r)^+ \arrow[d, "g"] & \\ M' \arrow[r, dashed, "\xi"] \arrow[dr, "h"'] & M \arrow[d, "f"] \\ & C \end{tikzcd}$$

commutes and $\xi(y) = x$.

Since this means $h = f.\xi$ it is not necessary to specify h and so objects of J/(f,x) correspond to pairs (g,ξ). Morphisms of J/(f,x) from (g,ξ) to (g',ξ') correspond to $\underline{Q}$-morphisms from (g,h) to (g',h') with

$$\begin{tikzcd} (\Delta_r)^+ \arrow[rr, dashed, "\alpha"] \arrow[d, "g"] & & (\Delta_{r'})^+ \arrow[d, "g'"] \\ N \arrow[rr, dashed, "\beta"] \arrow[dr, "\xi"] \arrow[ddr, "h"'] & & N' \arrow[dl, "\xi'"'] \arrow[ddl, "h'"] \\ & M \arrow[d, "f"] & \\ & C & \end{tikzcd}$$

commuting.

Let $[J/(f,x)]^o$ be the full subcategory of J/(f,x) consisting of the (g,ξ) with $\xi = 1_M$. Then the functors

inclusion: $[J/(f,x)]^o \longrightarrow J/(f,x)$

and $\quad J/(f,x) \dashrightarrow [J/(f,x)]^{o}$ $\quad$ defined by

$(g,\xi) \dashrightarrow (\xi g, 1_{M})$ $\quad$ on objects

$(\alpha,\beta) \dashrightarrow (\alpha, 1_{M})$ $\quad$ on morphisms

are adjoint. Therefore [15, Corollary 1 to Proposition 2] the classifying spaces of these categories are homotopy equivalent. But $[J/(f,x)]^{o}$ has objects corresponding to maps $g:\Delta_r \dashrightarrow M$ and morphisms corresponding to inclusions $\alpha:\Delta_r \dashrightarrow \Delta_{r'}$ such that

$$\begin{tikzcd} \alpha:\Delta_r \arrow[rr, dashed] \arrow[dr, "g"'] & & \Delta_{r'} \arrow[dl, "g'"] \\ & M & \end{tikzcd}$$

commutes

ie $|[J/(f,x)]^{o}|$ is $(\mathrm{Sing}(M_x))$ where M_x is the component of M which corresponds to $x\in\pi_0 M$. But $(\mathrm{Sing}(M_x))'$ is homotopic to $(\mathrm{Sing}(M_x))$ which is weakly homotopic to M_x. By the choice of $\underline{M}$, M_x is contractible. Therefore $|[J/(f,x)]^{o}|$ and thus $|J/(f,x)|$ are weakly contractible and [15, Theorem A] J is a homotopy equivalence. There are functors

$\underline{Q} \dashrightarrow \underline{S}$ $\quad$ defined by

$(g,h) \dashrightarrow h.g$ $\quad$ on objects

and $\quad (\alpha,\beta) \dashrightarrow \alpha$ $\quad$ on morphisms ;

$\underline{S} \dashrightarrow \underline{Q}$ $\quad$ defined by

$\underline{f} \dashrightarrow (1_{\Delta_r^+}, f)$ $\quad$ on objects

and $\quad \alpha \dashrightarrow (\alpha,\alpha)$ $\quad$ on morphisms.

These functors are adjoint, therefore they induce homotopic maps of classifying spaces [15]. So combining the above the functors

$$\mathcal{T}(C,\pi_0) \longleftarrow \underline{Q} \rightleftarrows \underline{S}$$

all induce homotopic maps of classifying spaces and

$$\begin{aligned}\pi_n |\mathcal{T}(C,\pi_0)| &\cong \pi_n |\underline{S}| \\ &\cong \pi_n(\mathrm{Sing}(C))' \\ &\cong \pi_n(\mathrm{Sing}(C)) \\ &\cong \pi_n C.\end{aligned}$$

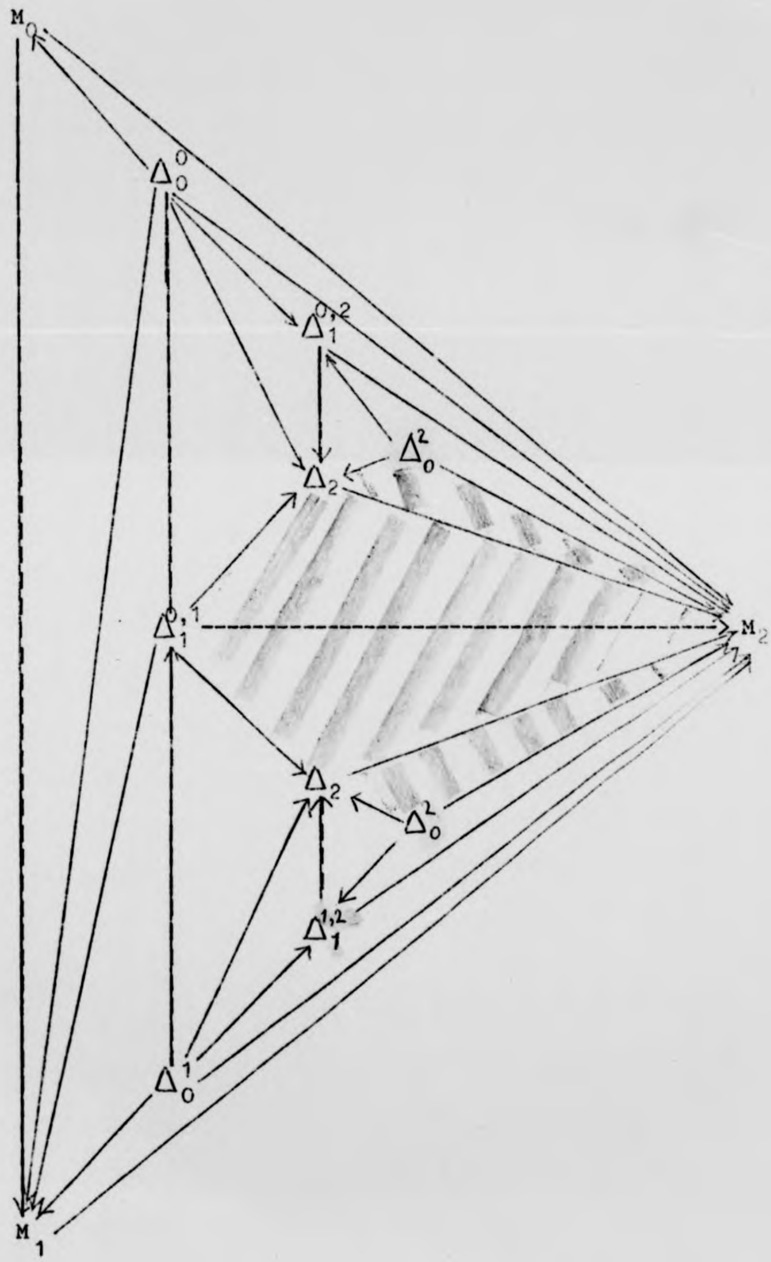
10.6 Direct Methods

The category $\underline{Q}$ was introduced in the proof of (10.5) to simplify the notation. It is possible to describe a simplicial map $|\mathcal{T}(C,\pi_0)| \longrightarrow (\mathrm{Sing}(C))'$ which is the homotopy inverse to the inclusion mentioned in (10.4) but to prove that the maps are homotopy inverses is very messy. The map is defined by choosing, for each object, (f,x), of $\mathcal{T}(C,\pi_0)$, a point of M in the component corresponding to x. Then for each map, $\xi : M_0 \longrightarrow M_1$, choose a path from $\xi(m_0)$ to m_1, to each triangle of maps

$$\begin{tikzcd} \xi_1 : M_0 \arrow[rr] \arrow[dr] & & M_1 \arrow[dl, "\xi_2"] \\ & M_2 & \end{tikzcd}$$

choose a map $\Delta_2 \longrightarrow M_2$ such that the vertices of the 2-simplex are mapped to $\xi_2 \cdot \xi_1(m_0)$, $\xi(m_1)$ and m_2 and its edges are mapped to the relevant paths. Similarly for each n-gon of maps choose a suitable $\Delta_n \longrightarrow M_n$ with the correct boundary.

Then using these simplices and their face inclusions an n-simplex of $|\mathcal{T}(C,\pi_0)|$ can be homotoped into $(\mathrm{Sing}(C))'$ as shown overleaf for the case n=2. If this homotopy is taken relative to the boundary then the simplices obtained which do not lie in $(\mathrm{Sing}(C))'$ cancel out when a group of simplices which are the image of a sphere are considered.


M_0
Δ^0_0
$\Delta^{0,2}_1$
Δ^2_0
Δ_2
$\Delta^{0,1}_1$
M_2
Δ_2
Δ^2_0
$\Delta^{1,2}_1$
Δ^1_0
M_1

11. Acyclic Models

In section 8, when comparing the derived functors defined in section 2 with $\mathbb{G}$-derived functors (for a cotriple $\mathbb{G}$) the category $\underline{M}$ could be a full subcategory containing the full subcategory of $\mathbb{G}$-free objects of $\underline{C}$ and contained in the full subcategory of $\mathbb{G}$-projectives. Within these limitations the choice of $\underline{M}$ did not affect the derived functors. In the previous section while investigating the derived functors of π_0 the models were taken to be spaces homotopic to finite discrete spaces, but later attention was restricted to the affine n-simplices and face inclusions between them. This effective alteration of the subcategory $\underline{M}$ was easier to deal with for π_0 than for the general functor but it remains true in the general case that under certain conditions the category $\underline{M}$ can be altered without affecting the resulting derived functors. Consequently there is a choice of constructions of the derived functors and the one most easily worked with can be used in any particular set of circumstances.

11.1 A Choice of Models

Consider the situation $\underline{M} \overset{K}{\subset} \underline{L} \subset \underline{C}$ where $F:\underline{L} \dashrightarrow \underline{A}$. If $FK:\underline{M} \dashrightarrow \underline{A}$ is the restriction of F to $\underline{M}$ then there are three different derived functors defined, the derived functors of F, the derived functors of FK having domain $\underline{L}$ and those of FK having domain $\underline{C}$. If the objects of $\underline{L}$ are acyclic then it will be shown that the first and last of these derived functors coincide. The objects of $\underline{L}$ are acyclic if, for every object, P, in $\underline{L}$, the classifying space of $\tau(P,FK)$ is homotopy equivalent to the discrete space FP, the homotopy equivalence being induced by the map $(f,x)\dashrightarrow F(f)x$. In particular there are isomorphisms

$$\pi_n|\tau(P,FK)| \cong \begin{cases} FP & n = 0 \\ 0 & n > 0. \end{cases}$$

11.2 A Category Q

The category $\underline{Q}$ is defined so that its objects are triples, (l,f,x) where $l:M\dashrightarrow L$ is an $\underline{L}$-morphism, $f:L\dashrightarrow C$ is an $\underline{C}$ morphism and $x \in FM$. Morphisms of $\underline{Q}$ are pairs, (α,β) where $\alpha:M_1\dashrightarrow M_2$ is an $\underline{M}$-morphism and $\beta:L_1\dashrightarrow L_2$ is an $\underline{L}$-morphism such that

$$\begin{tikzcd} M_1 \arrow[r, "\alpha"] \arrow[d, "l_1"] & M_2 \arrow[d, "l_2"] \\ L_1 \arrow[r, "\beta"] \arrow[dr, "f_1"'] & L_2 \arrow[d, "f_2"] \\ & C \end{tikzcd}$$

commutes and $F(\alpha)x_1 = x_2$.

Clearly there are natural transformations as indicated below;

$$\begin{array}{ccccc} \tau(C,FK) & \rightleftarrows & \underline{Q} & \longrightarrow & \tau(C,F) \\ (fl,x) & \longleftarrow & (l,f,x) & & \\ (g,y) & \longrightarrow & (1,g,y) & & \\ & & (l,f,x) & \longrightarrow & (f,F(l)x). \end{array}$$

11.3 Theorem

If $\underline{M}$ is a subcategory of $\underline{L}$ and the objects of $\underline{L}$ are acyclic for a functor $F:\underline{L}\dashrightarrow\underline{A}$, then the derived functors of F are equivalent to the derived functors of the restriction of F to $\underline{M}$.

Proof

Define the category $\underline{Q}$ as above. Since the natural transformations $\tau(C,FK) \leftrightarrows \underline{Q}$ are adjoint they induce homotopic maps of classifying spaces. Let $J:\underline{Q}\dashrightarrow\tau(C,F)$ be the map given in (11.2). Construct the over category $J/(h,z)$ (4.1) where (h,z) is an object of $\tau(C,F)$, so $h:P\dashrightarrow C$ and $z\in FP$. Objects of $J/(h,z)$ are $((l,f,x),k)$ where

$$\begin{tikzcd} M \arrow[r, dashed, "l"] & L \arrow[r, dashed, "f"] \arrow[dr, "k"'] & C \\ & & P \arrow[u, "h"'] \end{tikzcd}$$

commutes $x\in FM$ and $F(kl)x = z$.

Morphisms of $J/(h,z)$ correspond to $\underline{Q}$-morphisms (α,β) where

$$\begin{tikzcd} \alpha:M_1 \arrow[r, dashed] \arrow[d, "l_1"] & M_2 \arrow[d, "l_2"] \\ \beta:L_1 \arrow[r, dashed] \arrow[dr, "k_1"] \arrow[ddr, "f_1"'] & L_2 \arrow[d, "k_2"'] \arrow[dd, bend left, "f_2"] \\ & P \arrow[d, "h"] \\ & C \end{tikzcd}$$

commutes

Let $[J/(h,z)]^{o}$ be the full subcategory of $J/(h,z)$ with $k = 1_P$ (and therefore $f = h$). The inclusion

$$[J/(h,z)]^{o} \longrightarrow J/(h,z)$$

has an adjoint given by :-

$$J/(h,z) \longrightarrow [J/(h,z)]^{o}$$

$$((l,f,x),k) \longmapsto ((kl,h,x),1_P)$$

$$(\alpha,\beta) \longmapsto (\alpha,1_P)$$

and so the classifying spaces of these categories are homotopy equivalent. Now clearly objects of $[J/(h,z)]^{o}$ correspond to pairs (l,x) with $l:M \longrightarrow P$, $x \in FM = FKM$ and $F(l)x = z$. Morphisms of $[J/(h,z)]^{o}$ correspond to $\alpha:M_1 \longrightarrow M_2$ where

$$\begin{tikzcd} \alpha: M_1 \arrow[rr] \arrow[dr, "k_1"'] & & M_2 \arrow[dl, "k_2"] \\ & P & \end{tikzcd}$$

commutes.

Thus, if all objects of $\underline{L}$ are acyclic with respect to $\underline{M}$, then $[J/(h,z)]^{o}$ is the component of $\mathcal{T}(P,FK)$ corresponding to z under the acyclic isomorphism . Therefore $[J/(h,z)]^{o}$ is contractible, as is $J/(h,z)$ and [15, Theorem A] J induces a homotopy equivalence of classifying spaces. Therefore

$$\begin{aligned} \pi_n|\mathcal{T}(C,FK)| &\cong \pi_n|\underline{Q}| \\ &\cong \pi_n|\mathcal{T}(C,F)|. \end{aligned}$$

11.4 Comments

In section 8 it was shown that cotriple derived functors, for a cotriple $\mathbb{G}=(G,\varepsilon,\delta)$, agree with the derived functors defined in section 2 if $\underline{M}$ is taken to be any subcategory of the full subcategory of $\mathbb{G}$-projectives containing the full subcategory of $\mathbb{G}$-free objects. The full subcategory of $\mathbb{G}$-projectives is acyclic with respect to the full subcategory of $\mathbb{G}$-free objects (since the $\mathbb{G}$-projectives have trivial cotriple derived functors). This is why choosing any intermediate category as the models does not affect the resulting derived functors. It is pleasing to know that this model theory agrees with that of André (7.3) for the same choice of models.

In (9.5) and in section 10 derived functors defined on the category of based topological spaces and continuous based maps were considered. In the former, for the derived functors of H_0, $\underline{M}$ was the category of affine n-simplices with disjoint basepoints and face inclusions but to calculate the derived functors of π_0 $\underline{M}$ was the full subcategory of spaces homotopy equivalent to finite discrete spaces. However the full subcategory of based spaces homotopy equivalent to finite discrete spaces is acyclic with respect to the category of affine simplices with disjoint basepoints and face inclusions for both functors H_0 and π_0 . Therefore either category of models could be used for both functors.

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