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THE IDEAL AND SUBIDEAL STRUCTURE OF
LIE ALGEBRAS

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INTRODUCTION

In this thesis we study infinite-dimensional Lie algebras, drawing inspiration from group theory and ring theory.

Chapter one sets up notation.

Chapter two deals with prime ideals. In the first part of it, we define the concepts of a prime ideal and the radical of an ideal in Lie algebras along the same line as ideals in an associative rings, and investigate some of their properties. In the second we investigate the structure of Lie algebras with certain finiteness conditions on subideals, using the notion of prime ideals and prime algebras. In particular we prove that: If \( X \) is one of \( \text{Max}-\omega^n \) \((n \geq 2)\), \( \text{Max}-\sigma_i \), \( \text{Min}-\omega^n \) \((n \geq 2)\), \( \text{Min}-\sigma_i \), then \( L \in X \) if and only if:

1. \( \sigma(L) \) is a finite-dimensional soluble ideal of \( L \).
2. \( L/\sigma(L) \) is a subdirect sum of a finite number of prime algebras in \( X \).

Chapter three deals with generalizations of the minimal condition on ideals, leading to a new class of "quasi-Artinian" algebras (We say that \( L \) is quasi-Artinian if for every descending chain of ideals \( I_1 \supseteq I_2 \supseteq \ldots \) of \( L \) there exists \( m \in \mathbb{N} \) such that \( [L^{(m)}],I_m \subseteq I_n \) for all \( n \geq m \) which possesses several of the main properties of \( \text{Min}-\sigma \). In particular we prove that the class of quasi-Artinian algebras is \( Q \)-closed and a locally nilpotent quasi-Artinian Lie algebra is soluble.

Chapter four considers the join of subideals. First we prove that:
If $L$ is a Lie algebra over a field of characteristic zero and if $H_{\lambda} \leq L$, $\lambda \in \Lambda$ such that $J = \langle H_{\lambda} \mid \lambda \in \Lambda \rangle$ and $B = \{ B \mid B < J, B \leq L \}$, then $J \leq L$ if and only if $B$ has a maximal element. This result is a counterpart of a group-theoretic one (cf. Wielandt [35]). We also find another condition under which the join of subideals is a subideal by imposing conditions on the circle product $H \circ K = [H,K]_{(H \circ K)^2}$ of subideals. In particular we show that:

If $X$ is an $\{I,N_0\}$-closed and locally coalescent class over any field, and if $H$ and $K$ are $X$-subideals of a Lie algebra $L$ with $J = \langle H,K \rangle$ where $H \circ K | (H \circ K)^2$ is finitely generated, then $J \leq L$ and $J \in X$.

Chapter five considers criteria for subideality and ascendancy generalizing some results of Kawamoto [17], Stitzinger [20]. Our main results are as follows:

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H < L$. Then $H \leq L$ if and only if one of the following conditions holds:

(i) For each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x \rangle^o < h_1^o \cdots < h_n^o \rangle \leq H$ for all $h_1, \ldots, h_n \in H$.

(ii) For each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x \rangle^o_n < h \rangle \leq H$.

(iii) For each $h \in H$ there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x \rangle^o_n < h \rangle \leq H$.

A generalization to infinite-dimensional Lie algebras leads to the following:
Let $L$ be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H \leq L$. Then

(i) If for each $x \in L$ there exists an integer $n = n(x)$ such that $\langle x \rangle \circ \langle h_1 \rangle \circ \langle h_2 \rangle \circ \ldots \circ \langle h_n \rangle \subseteq H$ for any $h_1, \ldots, h_n \in H$, then $H$ asc $L$.

(ii) Suppose that $H$ is finite-dimensional.

(a) If for each $h \in H$, there exists an integer $n = n(h)$ such that for all $x \in L$ we have $\langle x \rangle \circ_n \langle h \rangle \subseteq H$, then $H$ si $L$.

(b) If for each $x \in L$ and $h \in H$, there exists an integer $n = n(x,h)$ such that $\langle x \rangle \circ_n \langle h \rangle \subseteq H$, then $H$ asc $L$.

Finally if $L$ is an ideally finite Lie algebra over a field of characteristic zero and if $H \leq L$, then $H$ asc $L$ if either of the following conditions holds:

(i) For each $x \in L$ and $h \in H$ there exists an integer $n = n(x,h)$ such that $\langle x \rangle \circ_n \langle h \rangle \subseteq H$.

(ii) $H$ asc $\langle H, x \rangle$ for each $x \in L$.

Some of these results are Lie-theoretic analogues of similar results for groups obtained by many authors, especially Wielandt [34] and Peng [22].

Chapter six considers subideals of the join of permutable Lie algebras.

First we consider subideals of the join of permutable finite-dimensional Lie algebras. Then we extend our results
to certain classes of infinite dimension. Our main results are as follows:

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $A, H, K$ be subalgebras of $L$ such that $L = H + K$ and $A \subseteq H, A \subseteq K$. Then $A \subseteq L$ if and only if $A \subseteq H$ and $A \subseteq K$.

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = \langle H_1, H_2, H_3 \rangle$. If $H_i \subseteq \langle H_i, H_j \rangle$ for all $i, j = 1, 2, 3$ and if $\langle H_1, H_2 \rangle$ is permutable with $H_3$, then $H_i \subseteq L$ for all $i$.

Both results are counterparts of group-theoretic ones (see Wielandt [36,37]).

A generalization to infinite dimensions leads to the following:

Let $L$ be a Lie algebra over a field of characteristic zero and let $A, H, K$ be subalgebras of $L$ such that $L = H + K$ and $A \subseteq H, A \subseteq K$. Then

(a) If $L$ is soluble-by-finite and $A \subseteq H, A \subseteq K$, then

$A \subseteq L$.

(b) If $L$ is ideally finite and $A \triangleleft H, A \triangleleft K$, then

$A \triangleleft L$.

Finally, let $L$ be a Lie algebra over any field and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = \langle H_1, H_2, H_3 \rangle$. Suppose that $H_i \subseteq \langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, 2, 3$ and $[H_1, H_2] \subseteq H_1$. 
Then $H_i$ is L for all i.

Two papers [1,2] based on this work have been accepted for publication, and two more [3,4] have been submitted.

All the results in this thesis are original except where explicitly stated otherwise.
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CHAPTER ONE: NOTATION AND TERMINOLOGY

All Lie algebras considered in this thesis will be of finite or infinite dimension, defined over an arbitrary field (unless otherwise stated). Our notation and terminology is based on Amayo and Stewart [5], but for the sake of convenience we will state the more important terms that we use here.

1.1. Preliminaries

Let \( L \) be a Lie algebra. By \( H \subseteq L, H \triangleleft L, H \triangleleft L \), we shall mean that \( H \) is a subset, subalgebra and ideal of \( L \) respectively. Angular brackets \( < > \) denote the subalgebra generated by their contents. If \( X, Y \) are subspaces of \( L \), then \( [X,Y] \) is the subspace spanned by all products \( [x,y] \), \( x \in X \) and \( y \in Y \). Let \( x_1, x_2, \ldots, x_n \in L \). The left-normed products \( [x_1, \ldots, x_n] \) are defined recursively by:

\[
[x_1, \ldots, x_{n+1}] = [[x_1, \ldots, x_n], x_{n+1}].
\]

If \( x_1 = x, x_2 = \ldots = x_{n+1} = y \), we write

\[
[x, y] = [x_1, \ldots, x_n].
\]

Similarly we define, for subsets \( X_1 \subseteq L \),

\[
[x_1, \ldots, x_{n+1}] = [[X_1, \ldots, x_n], x_{n+1}]
\]

and if \( x = x_1, x_2 = x_3 = \ldots = x_{n+1} = y \), we write

\[
[x, y] = [x_1, \ldots, x_n].
\]
If $X, Y \subseteq L$, we say that $X$ is $Y$-invariant if whenever $x \in X$ and $y \in Y$ then $[x, y] \in X$. Alternatively we say that $Y$ idealises $X$. We let $<X^Y>$ denote the smallest subalgebra of $L$ which contains $X$ and is $Y$-invariant and we call it the ideal closure of $X$ under $Y$.

If $X \subseteq L$, then the centralizer of $X$ in $L$ is $C_L(X) = \{y \in L | [X, y] = 0\}$, and the idealiser of $X$ in $L$ is $I_L(X) = \{y \in L | [X, y] \subseteq X\}$.

We write $X \triangleright Y$ to denote the split extension of an ideal $X$ by a subalgebra $Y$ (under a suitably specified $Y$-action on $X$).

1.2 Subideals

Let $H < L$. We say that $H$ is a subideal of $L$ if there is a finite series $H = H_0 < H_1 < \ldots < H_n = L$. We write $H \subseteq L$.

To emphasize the role of the integer $n$ we say that $H$ is an $n$-step subideal of $L$, and write $H \subseteq^n L$. We sometimes refer to $n$ as the subideal index of $H$.

For any subalgebra $H < L$ we define the ideal closure series $(H_i)_{i \in \mathbb{N}}$ recursively by $H_0 = L$, $H_{i+1} = <H_i>$. It follows that $H \subseteq^n L$ if and only if $H_n = H$.

For any ordinal $\lambda$ we say that $H$ is a ($\lambda$-step) ascendant subalgebra of $L$ if there is a series $(H_\alpha)_{\alpha \in \lambda}$ of subalgebras of $L$, such that

(i) $H_0 = H$, $H_\lambda = L$

(ii) $H_\alpha < H_{\alpha+1}$ if $\alpha < \lambda$
(iii) \( H_\beta = \cup_{\alpha<\beta} H_\alpha \) if \( \beta \leq \lambda \) is a limit ordinal. We write, \( H \text{asc} L, H \not\prec^\lambda L \).

1.3 Derivations

If \( x \notin L \) we define the adjoint map \( \text{ad} x : L \to L \) by
\[ y \text{ad} x = [y, x] \ (y \in L). \]

From the Jacobi identity it follows that for any \( y, z \in L \),
\[ [y, z] \text{ad} x = [y \text{ad} x, z] + [y, z \text{ad} x]. \]

Any linear map \( \delta : L \to L \) such that for all \( y, z \in L \)
\[ [y, z] \delta = [y \delta, z] + [y, z \delta] \]
is called a derivation of \( L \). Thus for any \( x \in L \) the adjoint map \( \text{ad} x \) is a derivation, the inner derivation induced by \( x \).

1.4 Central and Derived series

For an ordinal \( \alpha \) we denote by \( L^\alpha, L^{(\alpha)}, \zeta_\alpha(L) \) the \( \alpha \)-th terms of the (transfinite) lower central, derived and upper central series of \( L \) respectively, and define these inductively by \( L^1 = L = L^{(0)}, L^{\alpha+1} = [L^\alpha, L], L^{(\alpha+1)} = [L^{(\alpha)}, L^{(\alpha)}] \) and at limit ordinals \( \lambda \), \( L^\lambda = \cap_{\alpha<\lambda} L^\alpha \) and \( L^{(\lambda)} = \cap_{\alpha<\lambda} L^{(\alpha)} \), we set \( \zeta_0(L) = 0, \zeta_1(L) = C_L(L), \zeta_{\alpha+1}(L)/\zeta_\alpha(L) = \zeta_1(L/\zeta_\alpha(L)) \) and for limit ordinals \( \lambda \), \( \zeta_\lambda = \cup_{\alpha<\lambda} \zeta_\alpha(L) \). We set \( \zeta_\alpha(L) = \cup_{\alpha<\lambda} \zeta_\alpha(L) \) and call it the hypercentre. The Lie algebra \( L \) is hypercentral if \( L = \zeta_\alpha(L) \).
$L^a$, $L^{(a)}$, and $\zeta_a(L)$ are all characteristic ideals of $L$ in the sense that they are invariant under derivations of $L$.
We write $I \text{ ch } L$ to mean that $I$ is a characteristic ideal of $L$.

$L$ is nilpotent (of class $\leq n$) if $L^{n+1} = 0$, is soluble (of derived length $\leq n$) if $L^{(n)} = 0$.

1.5 Classes and closure operations

A class $X$ of Lie algebras over a field $F$ is a collection of Lie algebras over $F$ such that

(i) $X$ contains the 0-dimensional subalgebra.

(ii) If $H = K \in X$, then $H \in X$.

Familiar classes of Lie algebras are:

- $F$: finite-dimensional
- $A$: abelian
- $N$: nilpotent
- $N_c$: nilpotent of class $\leq c$ ($c \in \mathbb{N}$).

Notation for other classes will follow.

If $X$ and $Y$ are classes then $X \subseteq Y$ denotes inclusion.

We denote by $X_y$ the class of all Lie algebras $L$ with an $X$-ideal $H$ such that $L/H \in Y$. We set $X_1 \ldots X_{n+1} = (X_1 \ldots X_n)X_{n+1}$ and write $X^{n+1}$ if all $X_i$'s equal $X$.

A closure operation $A$ assigns to each class $X$ a class $AX$ in such a way that for all classes $X$ and $Y$ the following conditions hold:
(i) \( A(0) = (0) \)
(ii) \( X \subseteq AX \)
(iii) \( A(AX) = AX \)
(iv) \( X \subseteq y \Rightarrow AX \subseteq Ay. \)

We say that \( X \) is \textit{A-closed} if \( X = AX \).

We list some standard closure operations.

\( s : sX \) consists of all subalgebras of \( X \)-algebras.
\( I : IX \) consists of all subideals of \( X \)-algebras.
\( Q : QX \) consists of all quotients of \( X \)-algebras.
\( E : EX \) consists of all algebras \( L \) having a finite series \( 0 = L_0 \prec \ldots \prec L_n = L \) whose factors \( L_{i+1}/L_i \in X \) for \( 0 \leq i \leq n-1 \).
\( L : LX \) consists of those algebras \( L \) such that every finite subset of \( L \) is contained in an \( X \)-subalgebra of \( L \).

\( N_0 : \) A class \( X \) is \( N_0 \)-closed if whenever \( H, K \leq L \) and \( H, K \in X \) then \( H + K \in X \).

\( R : RX \) consists of those algebras \( L \) having a family \( (I_\alpha)_{\alpha \in A} \) of ideals such that \( L/I_\alpha \in X \) for all \( \alpha \in A \) and \( \cap_{\alpha \in A} I_\alpha = 0 \).

The operations \( E, L, R \) are read as "poly", "locally", "residually" respectively. In particular we note the classes

\( EA : \) soluble
\( LN : \) locally nilpotent
\( LF : \) locally finite (-dimensional)
\( RK : \) residually nilpotent
\( RF : \) residually finite (-dimensional).
If $A$ and $B$ are operations we define $AB$ by:

$$ABX = A(BX).$$

In general $AB$ need not be a closure operation. However, let us define an ordering on operations by

$$A \leq B \iff AX \leq BX \text{ for all classes } X.$$

If $BA \leq AB$ then it is easy to see that $AB$ is a closure operation, and is in fact equal to \{A,B\}.

### 1.6 Chain conditions

Let $V$ be a vector space, and $\mathcal{S}$ a collection of subsets of $V$. We say that $V$ has (or satisfies) Max-$\mathcal{S}$ if $\mathcal{S}$ satisfies the maximal condition: every ascending chain $S_0 \subseteq S_1 \subseteq \ldots$ of elements $S_i \in \mathcal{S}$ terminates finitely; so that $S_r = S_{r+1} = \ldots$ for some $r \in \mathbb{N}$.

Similarly $V$ has Min-$\mathcal{S}$ if $\mathcal{S}$ satisfies the minimal condition: every descending chain $S_0 \supseteq S_1 \supseteq \ldots$ terminates.

If $V$ is a Lie algebra $L$ and $\mathcal{S}$ is respectively the set of ideals, subideals, n-step subideals of $L$ we write in place of Max-$\mathcal{S}$

$$\text{Max}-4, \text{Max}-si, \text{Max}-4^n$$

and for Min-$\mathcal{S}$ we write

$$\text{Min}-4, \text{Min}-si, \text{Min}-4^n.$$

We use the same notation for the classes of Lie algebras satisfying the corresponding conditions.
CHAPTER TWO: PRIME IDEALS IN LIE ALGEBRAS

The notion of prime ideals plays an important role in the theory of associative algebras. It is of some interest to know how the corresponding notion behaves in Lie algebras. In the first section of this chapter, which is based on certain results of Kawamoto [18], we define the concepts of prime ideal and radical of an ideal in Lie algebras along the same lines as ideals in an associative ring, and investigate some of their properties. The main result of this section states that: If \( L \in \text{Max-}^d \) and \( I \unlhd L \), then there exist a finite number of minimal prime ideals belonging to \( I \). (Theorem 2.1.13). In Section two of this chapter, which is based on [1], we investigate the structure of Lie algebras with certain finiteness conditions on subideals. The main result states that: If \( X \) is one of \( \text{Max-}^d^n \), \( n \geq 2 \), \( \text{Max-si} \), \( \text{Min-}^d^n \), \( n \geq 2 \), \( \text{Min-si} \), then \( L \in X \) if and only if:

(i) \( \sigma(L) \in F \cap EA \)

(ii) \( L/\sigma(L) \) is a subdirect sum of a finite number of prime algebras in \( X \). (Theorem 2.2.3)

2.1 Prime ideals and radicals

This section is based on certain results of Kawamoto [18], Behrens [7] and McCoy [21]. The proofs are closely related to Behrens [7] and McCoy [21]. We start with the following:
Definition 2.1.1

An ideal $P$ of $L$ is a prime ideal of $L$ if whenever $[a^L, b^L] \subseteq P$ at least one of $a$ and $b$ belongs to $P$.

From this definition we have:

Proposition 2.1.2

Let $P$ be an ideal of $L$. Then $P$ is prime if and only if $[A,B] \subseteq P$ with $A, B$ ideals of $L$ implies $A \subseteq P$ or $B \subseteq P$.

Proof

Let $A, B$ be ideals of $L$ such that $[A,B] \subseteq P$ and $A \not\subseteq P$. Suppose $a \in A$ with $a \not\in P$, and that $b$ is an arbitrary element of $B$. Since $[a^L, b^L] \subseteq [A,B] \subseteq P$ with $a \not\in P$, it follows that $b \in P$. Hence $B \subseteq P$.

The converse is clear. □

To give another characterization of a prime ideal $P$, we shall consider the set-theoretic complement $C(P)$ of $P$ in $L$. This is an $m$-system in the following sense.

Definition 2.1.3 A subset $M$ of $L$ is an $m$-system, if for any $x, y \in M$ there exist in $x^L$ and $y^L$ two elements $x_1$ and $y_1$ respectively such that $[x_1, y_1] \in M$. The empty set $\emptyset$ is to be considered an $m$-system.

This concept plays the same role as the analogous one defined by Behrens [7] and so we can translate some of his
results into ours. First we have from Definition 2.1.1, that an ideal P in L is a prime ideal of L if and only if its complement C(P) is an m-system.

**Definition 2.1.4**

A prime ideal P is called a minimal prime ideal belonging to an ideal I, if P ⊇ I and there is no prime ideal P_1 of L such that I ⊆ P_1 ⊆ P.

The radical rad(I) of an ideal I in L is the intersection of all minimal prime ideals belonging to I. We write rad(L) for rad(0).

Next the following.

**Theorem 2.1.5**

Let I ⊆ L. Then

(i) rad(I) consists of those elements x of L with property that every m-system which contains x contains an element of I.

(ii) rad(I) is the intersection of all prime ideals containing I.

The proof follows after a series of lemmas.

**Lemma 2.1.6**

Let I ⊆ L and let M be an m-system such that I ∩ M = ∅. Then M is contained in an m-system M* which is maximal in the class of m-systems which do not intersect I.
Lemma 2.1.7

Let $M$ be an $m$-system in $L$ and let $I \subseteq L$ be such that $I \cap M = \emptyset$. Then $I$ is contained in an ideal $P$ of $L$ which is maximal in the class of ideals which do not intersect $M$. The ideal $P$ is necessarily a prime ideal of $L$.

Proof

The existence of $P$ follows from Zorn's lemma. We now show that $P$ is a prime ideal of $L$. If $M = \emptyset$, then $P = L$ and $P$ is a prime ideal of $L$. Suppose that $M \neq \emptyset$ and $A, B$ are ideals of $L$ such that $A \notin P, B \notin P$. Then the maximality of $P$ implies that $A + P$ contains an element $x$ of $M$ and $B + P$ contains an element $y$ of $M$. Since $M$ is an $m$-system there exist $x, y \in \langle x \rangle$ and $y \in \langle y \rangle$ such that $[x, y] \in M$. Moreover $[x, y] \in [A + P, B + P]$. Now if $[A, B] \subseteq P$, we would have $[A + P, B + P] \subseteq P$ and it follows that $[x, y] \in P$. But this is impossible since $[x, y] \in M$ and $M \cap P = \emptyset$. Hence $[A, B] \notin P$ and $P$ is therefore a prime ideal of $L$. 

Lemma 2.1.8

A set $P$ of elements of $L$ is a minimal prime ideal belonging to an ideal $I$ in $L$ if and only if its complement $C(P)$ is maximal in the class of $m$-systems which do not intersect $I$. 

Proof

Immediate consequence of Zorn's Lemma. 

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Proof

Let \( P \) be a set of elements of \( L \) with the property that \( M = C(P) \) is maximal among the set of \( m \)-systems which do not intersect \( I \). By Lemma 2.1.7, there is a prime ideal \( P^* \) which contains \( I \) and \( P^* \cap M = \emptyset \). Hence \( C(P^*) \) is an \( m \)-system which does not intersect \( I \) and contains \( M \). From the maximal property of \( M \), it follows that \( C(P^*) = M \), and hence \( P = P^* \). Also this maximal property shows that there is no prime ideal \( P_1 \) such that \( I \subseteq P_1 \subseteq P \), as otherwise \( C(P_1) \) would be an \( m \)-system which does not intersect \( I \) and contains \( M \) as proper subset. Hence \( P \) is a minimal prime ideal belonging to \( I \).

Conversely, if \( P \) is a minimal prime ideal belonging to \( I \), then \( M = C(P) \) is an \( m \)-system which does not intersect \( I \), and Lemma 2.1.6 shows the existence of a maximal \( m \)-system \( M^* \) which contains \( M \) and does not intersect \( I \). By the case just proved, \( C(M^*) = P^* \) is a minimal prime ideal belonging to \( I \). But since \( M^* \supseteq M \), it follows that \( P^* \subseteq P \), and the minimal property of \( P \) then shows that \( P = P^* \). Hence \( M = M^* \), and \( M \) is a maximal \( m \)-system which does not intersect \( I \). \( \Box \)

Proof of Theorem 2.1.5

(i) Suppose that there exists an \( m \)-system \( M \) such that \( x \in M \), but \( M \cap I = \emptyset \). By Lemma 2.1.6, \( M \) is contained in an \( m \)-system \( M^* \) which is maximal in the class of \( m \)-systems which do not intersect \( I \). By Lemma 2.1.8, \( C(M^*) \) is a minimal prime
ideal belonging to I, and clearly \( C(M^*) \) does not contain \( x \). Hence \( x \notin \text{rad}(I) \).

Conversely, if \( P \) is any minimal prime ideal belonging to \( I \), then \( C(P) \) is an \( m \)-system which does not intersect \( I \), and hence \( x \notin C(P) \) by our assumption, that is, \( x \in P \). Thus \( x \notin \text{rad}(I) \).

(ii) It is enough to show that every prime ideal contains \( I \) contains a minimal prime ideal belonging to \( I \). So suppose that \( P \) be a prime ideal of \( L \) containing \( I \). Then \( C(P) \) is an \( m \)-system which does not intersect \( I \). By Lemma 2.1.6, \( C(P) \) is contained in an \( m \)-system \( M^* \) which is maximal in the class of \( m \)-systems which do not intersect \( I \). Lemma 2.1.8 shows that \( C(M^*) \) is a minimal prime ideal belonging to \( I \). Since \( C(P) \subseteq M^* \), it follows that \( I \subseteq C(M^*) \subseteq P \).

Next we shall collect certain results on ideals under the assumption that \( L \) satisfies the maximal condition for ideals, but first we recall the following.

**Definition 2.1.9**

Let \( L \) be a Lie algebra. Then \( \sigma(L) \) is defined to be the sum of the soluble ideals of \( L \) (see Amayo and Stewart [5, p. 180]. We say \( L \) is semi-simple if \( \sigma(L) = 0 \).

**Proposition 2.1.10**

Let \( L \in \text{Max-4} \) and let \( I \triangleleft L \). Then there exists \( n \in \mathbb{N} \) such that \( (\text{rad}(I))^n \subseteq I \).
To prove this we need the following lemmas. The first one is Kawamoto [18, Theorem 7]. We give the proof for completeness.

**Lemma 2.1.11**

\( \sigma(L) \subseteq \text{rad}(L) \). If \( L \in \text{Max-}\sigma \), then \( \sigma(L) = \text{rad}(L) \).

**Proof**

Let \( I \) be a soluble ideal of \( L \). Then there exists \( n \in \mathbb{N} \) such that \( I^{(n)} = 0 \). For any prime ideal \( P \) of \( L \) we have \( I \subseteq P \) since \( I^{(n)} = 0 \subseteq P \). Therefore \( \sigma(L) \subseteq \text{rad}(L) \).

If \( L \in \text{Max-}\sigma \), then \( \sigma(L) \) is the unique maximal soluble ideal of \( L \). Assume that \( \text{rad}(L) = R \) is not soluble. Let \( \mathcal{C} \) be the collection of ideals \( I \) such that \( R^{(n)} \notin I \) for all \( n \geq 0 \). \( \mathcal{C} \neq \emptyset \) because \( 0 \in \mathcal{C} \). Hence \( \mathcal{C} \) has a maximal element \( P \). We claim that \( P \) is prime. If there are ideals \( A, B \) of \( L \) such that \( A \notin P, B \notin P \) and \( [A, B] \subseteq P \), then \( A + P, B + P \notin \mathcal{C} \) by definition of \( P \). Hence \( R^{(n)} \subseteq A + P, R^{(m)} \subseteq B + P \) for some \( m, n \in \mathbb{N} \). Let \( s = \max\{n, m\} \). Then \( R^{(s+1)} \subseteq [A+P, B+P] \subseteq P \). But this contradicts \( P \in \mathcal{C} \). Therefore \( R \) is soluble and \( \text{rad}(L) = \sigma(L) \), which completes the proof. °

**Lemma 2.1.12**

Let \( I \triangleleft L \). Then \( \text{rad}(L/I) = \text{rad}(I)/I \).

**Proof** It follows from Theorem 2.1.5, that \( \text{rad}(L/I) = \bigcap\{P/I : P \) is a prime ideal of \( L \) containing \( I \} \) and hence
rad(L/I) = (∩ \{ P: P is a prime ideal of L containing I\})/I.
Thus rad (L/I) = rad(I)/I. ◻

Proof of Proposition 2.1.10
By Lemma 2.1.12, we have rad(L/I) = rad(I)/I. But by
Lemma 2.1.11, rad(L/I) = o(L/I) which is soluble, hence
rad(I)/I is soluble and so there exists n ∈ N such that
(rad(I))^n ⊆ I. ◻

Finally we prove the following.

Theorem 2.1.13
If L ∈ Max-α and I ⊆ L, then there exists a finite
number of minimal prime ideals belonging to I. Thus rad(I)
is an intersection of a finite number of minimal prime ideals
belonging to I.

Proof
If I is a prime ideal of L, then the assertion is
trivial. We may suppose therefore that I is not a prime
ideal of L. Then we can find ideals A, B of L such that
A ∉ I and B ∉ I but [A,B] ⊆ I. Let us suppose that I has
an infinite number of minimal prime ideals P_i belonging to
it. Then since [A+I, B+I] ⊆ I at least one of A + I and
B + I must be contained in an infinite number of P_i. Without
loss of generality we may assume that the one which is con­
tained in an infinite number of P_i is A + I. It is easily
seen that those P_i which contain A + I are minimal prime
ideas belonging to $A + I$ and moreover $A + I > I$. Continuing
an exactly similar argument, we obtain a strictly increasing
sequence $I < A + I < ...$ of ideals of $L$, which is impossible
from our assumption. □

2.2 On Lie algebras with finiteness conditions

The object of this section is to investigate the structure of Lie algebras with certain finiteness conditions on subideals, using the notion of prime ideals and prime algebras.

We start with the following definition which is the Lie analogue of prime ring.

Definition 2.2.1

We say that a Lie algebra $L$ is prime if whenever $I$ and $J$ are ideals of $L$ and $[I, J] = 0$, then either $I = 0$ or $J = 0$.

It follows that $P$ is a prime ideal of $L$ if and only if $L/P$ is a prime algebra.

Definition 2.2.2

A Lie algebra $L$ is said to be a subdirect sum of a family of Lie algebras $(L_α)_{α ∈ A}$ if there is an injective homomorphism $f: L \rightarrow \bigoplus_{α ∈ A} L_α$ such that for each $β ∈ A$

$ε_β \circ f: L \rightarrow L_β$ is a surjective homomorphism where $ε_β$ is the
projection of $\sum_{\alpha \in A} L_\alpha$ onto $L_\beta$.

The main result in this section is:

**Theorem 2.2.3**

Let $L$ be a Lie algebra and let $X$ be one of Max-$\mathcal{O}^n$ ($n \geq 2$), Max-$\mathcal{O}$, Min-$\mathcal{O}$ ($n \geq 2$), Min-$\mathcal{O}$. Then $L \in X$ if and only if:

(i) $\sigma(L)$ is a finite-dimensional soluble ideal of $L$.

(ii) $L/\sigma(L)$ is a subdirect sum of a finite number of prime algebras in $X$.

The proof follows from a series of lemmas.

**Lemma 2.2.4** Let $I \triangleleft L$ and $H$ be a subideal (resp. n-step subideal) of $L$. Then $H \triangleleft I$.

**Proof**

Let $H = H_n \triangleleft H_{n-1} \triangleleft \cdots \triangleleft H_1 \triangleleft H_0 = L$ be the ideal closure series of $H$ in $L$. Then $H \cap I = H_n \cap I \triangleleft H_{n-1} \cap I \triangleleft \cdots \triangleleft H_1 \cap I \triangleleft L$ and so $H \cap I \triangleleft L$. □

**Lemma 2.2.5**

Let $L$ be a Lie algebra and let $\mathcal{S}$ be respectively the set of ideals, subideals, n-step subideals of $L$. Suppose $I_i \triangleleft L$, $(i = 1, 2, \ldots, m)$ and $\cap_{i=1}^m I_i = 0$. Let $\mathcal{S}_1 = \{ \frac{H+I_i}{I_i} | H \in \mathcal{S} \}$. If $L/I_i \in \text{Max-}\mathcal{S}_1$ (resp. $\text{Min-}\mathcal{S}_1$) for all $i$, then $L \in \text{Max-}\mathcal{S}$ (resp. $\text{Min-}\mathcal{S}$).
Proof

By induction on $m$ we need consider only the case $m = 2$, then $I_1 \cap I_2 = 0$. Let $H_1 \subseteq H_2 \subseteq \ldots$ be an ascending chain of elements $H_i \in \mathcal{S}$. Then $(H_1 + I_1)/I_1 \subseteq (H_2 + I_1)/I_1 \subseteq \ldots$ is an ascending chain of elements of $\mathcal{S}_1$. Therefore there exists $r \in \mathbb{N}$ such that $H_r + I_1 = H_{r+1} + I_1 = \ldots \quad \ldots \quad (1)$

Now $H_1 \cap I_1 \subseteq H_2 \cap I_1 \subseteq \ldots$ is an ascending chain of elements of $\mathcal{S}$ by Lemma 2.2.4. Therefore $\frac{I_1}{I_2} \subseteq \frac{H_2 \cap I_1}{I_2} \subseteq \ldots$

Hence $H_r + I_1 = (H_r \cap I_1) + I_2 = (H_{r+1} \cap I_1) + I_2 = \ldots \quad \ldots \quad (2)$

Now from (1) and (2) we have $H_r = H_{r+1} = \ldots$ and so $L \in \text{Max-}\mathcal{S}$. That $L \in \text{Min-}\mathcal{S}$ can be proved by a similar method. □

Lemma 2.2.6

Let $L$ be a Lie algebra and $(L_a)_{a \in A}$ be a family of Lie algebras. Then $L$ is a subdirect sum of $(L_a)_{a \in A}$ if and only if for each $\beta \in A$, there is a surjective homomorphism $g_\beta : L \longrightarrow L_\beta$ such that $\bigcap_{\beta \in A} \ker g_\beta = 0$.\n
Proof:

This can be proved in the same way as in Gray [12, p.88]. □

An immediate consequence of Lemma 2.2.6 we have the following:

Corollary 2.2.7

Let $L$ be a Lie algebra and let $(I_a)_{a \in A}$ be a family of
ideals of $L$. If $\bigcap A = 0$, then $L$ is a subdirect sum of the family of Lie algebras $\{L/I_\alpha\}_{\alpha \in A}$.

**Lemma 2.2.8**

If $L \in \text{Max}_{\leq 2}$ (resp. $\text{Min}_{\leq 2}$), then $\sigma(L)$ is a finite-dimensional soluble ideal of $L$.

**Proof**

This follows from Amayo and Stewart [5, Corollary 9.1.3(d), p. 183 and Lemma 9.2.1, p. 190].

**Lemma 2.2.9**

Let $L$ be a Lie algebra.

(i) $L$ is semi-simple with $\text{Max}_{\leq n}$, $n \geq 1$ (resp. $\text{Max}_{\text{si}}$) if and only if $L$ is a subdirect sum of a finite number of prime algebras satisfying $\text{Max}_{\leq n}$, $n \geq 1$ (resp. $\text{Max}_{\text{si}}$).

(ii) $L$ is semi-simple with $\text{Min}_{\leq n}$, $n \geq 1$ (resp. $\text{Min}_{\text{si}}$) if and only if $L$ is a subdirect sum of a finite number of prime algebras satisfying $\text{Min}_{\leq n}$ ($n \geq 1$) (resp. $\text{Min}_{\text{si}}$).

**Proof**

(i) Let $L$ be semi-simple with $\text{Max}_{\leq n}$ (resp. $\text{Max}_{\text{si}}$). Then $\sigma(L) = 0$. By Theorem 2.1.13, $\text{rad}(L) = \bigcap_{i=1}^{m} P_i$, where the $P_i$ are prime ideals of $L$. But by Lemma 2.1.11, $\text{rad}(L) = \sigma(L)$, hence $\text{rad}(L) = 0$. Since $P_i$ is a prime ideal of $L$, it follows
that \( \mathbb{L}/P_i \) is a prime algebra and \( \mathbb{L}/P_i \in \text{Max-si} \) (resp. \( \text{Max-si} \)).

Now by Corollary 2.2.7, \( \mathbb{L} \) is a subdirect sum of a finite number of prime algebras satisfying \( \text{Max-si} \), (resp. \( \text{Max-si} \)). To prove the converse suppose that \( \mathbb{L} \) is a subdirect sum of a finite number of prime algebras \( \{ \mathbb{L}_\alpha \}_{\alpha \in A} \), \( A = \{1,2,\ldots,m\} \) satisfying \( \text{Max-si} \) (resp. \( \text{Max-si} \)). Let \( g_\beta : \mathbb{L} \to \mathbb{L}_\beta \) be the surjective homomorphism of Lemma 2.2.6. Then for each \( \beta \), \( \mathbb{L}/\ker g_\beta \cong \mathbb{L}_\beta \) and \( \mathbb{L}_\beta \) is prime. Hence \( \ker g_\beta \) is a prime ideal of \( \mathbb{L} \). Thus \( \text{rad}(\mathbb{L}) \subseteq \ker g_\beta \) for each \( \beta \), and so \( \text{rad}(\mathbb{L}) \subseteq \bigcap_{\beta \in A} \ker g_\beta = 0 \). But by Lemma 2.1.11, \( \sigma(\mathbb{L}) \subseteq \text{rad}(\mathbb{L}) \), hence \( \sigma(\mathbb{L}) = 0 \) and \( \mathbb{L} \) is semi-simple. Now that \( \mathbb{L} \in \text{Max-si} \) (resp. \( \text{Max-si} \)) follows from Lemma 2.2.5.

(ii) Let \( \mathbb{L} \) be semi-simple with \( \text{Min-si} \) (resp. \( \text{Min-si} \)). Then \( \mathbb{L} \) has only a finite number of minimal ideals \( M_1, \ldots, M_r \). Let \( P_1, 1 \leq i \leq r \), be an ideal of \( \mathbb{L} \) which is maximal with respect to not containing \( M_i \). We claim that \( P_i \) is a prime ideal of \( \mathbb{L} \). Suppose not. Then there exists ideals \( I, J \) of \( \mathbb{L} \) such that \( I \notin P_i \), \( J \notin P_i \) and \( [I,J] \subseteq P_i \). Now \( I + P_i \supseteq P_i \) and \( J + P_i \supseteq P_i \), so by the choice of \( P_i \), \( I + P_i \supseteq M_1 \) and \( J + P_i \supseteq M_1 \). Therefore \( M_1^2 \subseteq [I + P_i, J + P_i] \subseteq P_i \). But \( M_1^2 \neq 0 \) for \( \mathbb{L} \) is semi-simple, hence \( M_1^2 = M_1 \subseteq P_i \) which is a contradiction. Therefore \( P_i \) is a prime ideal of \( \mathbb{L} \) and \( \mathbb{L}/P_i \) is a prime algebra. If \( \bigcap_{i=1}^{r} P_i \neq 0 \), then this intersection contains one of the minimal ideals \( M_j \) for some \( j \). But \( M_j \notin P_j \),
so \( M_j \notin \bigcap_{i=1}^{r} P_i \). Hence \( \bigcap_{i=1}^{r} P_i = 0 \) and by Corollary 2.2.7, it follows that \( L \) is a subdirect sum of a finite number of prime algebras satisfying \( \text{Min-}\alpha^n \) (resp. \( \text{Min-}\beta^s \)).

Conversely that \( L \) is semi-simple can be proved as in (i), and that \( L \in \text{Min-}\alpha^n \) (resp. \( \text{Min-}\beta^s \)) follows from Lemma 2.2.5.

**Proof of Theorem 2.3**

The result follows from Lemma 2.2.8 and Lemma 2.2.9.

Theorem 2.2.3 reduces several problems about algebras with chain conditions to the case of prime algebras. We intend to make use of this reduction in future work.
so $M_j \notin \bigcap_{i=1}^{r} P_i$. Hence $\bigcap_{i=1}^{r} P_i = 0$ and by Corollary 2.2.7, it follows that $L$ is a subdirect sum of a finite number of prime algebras satisfying $\text{Min-}^n$ (resp. $\text{Min-}^s$).

Conversely, that $L$ is semi-simple can be proved as in (i), and that $L \in \text{Min-}^n$ (resp. $\text{Min-}^s$) follows from Lemma 2.2.5.

Proof of Theorem 2.3

The result follows from Lemma 2.2.8 and Lemma 2.2.9.

Theorem 2.2.3 reduces several problems about algebras with chain conditions to the case of prime algebras. We intend to make use of this reduction in future work.
CHAPTER THREE : QUASI-ARTINIAN LIE ALGEBRAS

Stewart ([25], pp. 90-92) has shown that the class of Artinian Lie algebras (Lie algebras with Min-v) is \( \{Q, E\} \) closed. Also he shows that a locally nilpotent Artinian Lie algebra is soluble. In this chapter, which is based on [1], we introduce the notion of Quasi-Artinian Lie algebras which generalizes the Artinian Lie algebras in such a way that its main properties are preserved.

**Definition 3.1**

We say that a Lie algebra \( L \) is *quasi-Artinian* if for every descending chain \( I_1 \supseteq I_2 \supseteq \ldots \) of ideals of \( L \) there exist \( r, s \in \mathbb{N} \) such that \( [L^{(r)}, I_s] \subseteq I_n \) for all \( n \), or equivalently there exists \( m \in \mathbb{N} \) such that \( [L^{(m)}, I_m] \subseteq I_n \) for all \( n \).

It is clear that every soluble Lie algebra is quasi-Artinian, but it is easy to construct a soluble Lie algebra which is not Artinian, so quasi-Artinian algebras need not be Artinian. Further we have the following:

**Proposition 3.2**

If \( L \) is a hypercentral Lie algebra and is quasi-Artinian, then \( L \) is soluble.

To prove this we need the following well-known result.
Lemma 3.3
If $L$ is hypercentral then $L^{(\alpha)} = 0$ for some ordinal $\alpha$.

Proof
See Amayo and Stewart [5, Lemma 8.1.1, p. 163]. □

Proof of Proposition 3.2
By Lemma 3.3, $L^{(\alpha)} = 0$ for some ordinal $\alpha$. But $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$ is a descending chain of ideals of $L$ and $L$ is quasi-Artinian, so there exists $m \in \mathbb{N}$ such that $[L^{(m)}, L^{(m)}] \subseteq L^{(n)}$ for all $n$. Hence $L^{(m+1)} = L^{(n)}$ for all $n$. Therefore $L^{(m+1)} = L^{(m+2)} = \ldots$ and $L^{(\alpha)} = L^{(m+1)} = 0$. Thus $L$ is soluble. □

Theorem 3.4
The following are equivalent:

(i) $L$ is quasi-Artinian.

(ii) There exists $m \in \mathbb{N}$ such that for every descending chain $I_1 \supseteq I_2 \supseteq \ldots$ of ideals of $L$, the descending chain of ideals $[L^{(m)}, I_1] \supseteq [L^{(m)}, I_2] \supseteq \ldots$ terminates.

(iii) For every non-empty collection $\mathcal{C}$ of ideals of $L$, there exists an element $I \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $[L^{(m)}, I] \subseteq J$ for every $J \in \mathcal{C}$ with $J \subseteq I$.

Proof
(i) $\Rightarrow$ (ii). Let $L$ be quasi-Artinian. Now $L \supseteq L^{(1)} \supseteq \ldots$
is a descending chain of ideals of $L$, so there exists $m \in \mathbb{N}$ such that $[L^{(m)}, L^{(m)}] \leq L^{(n)}$ for all $n \geq m$. Therefore $L^{(m+1)} = L^{(m+2)} = \ldots$. Also $I_1 \supseteq I_2 \supseteq \ldots$ is a descending chain of ideals of $L$, so there exists $r \in \mathbb{N}$ such that $[L^{(2m)}, I_r] \subseteq I_{r+s}$ for all $s \in \mathbb{N}$. Therefore for all $s \in \mathbb{N}$,

$[L^{(2m)}, I_r] \subseteq [L^{(2m)}, [L^{(2m)}, I_r]] \subseteq [L^{(2m)}, I_{r+s}] \subseteq [L^{(2m)}, I_r]$.

Hence $[L^{(2m)}, I_r] = [L^{(2m)}, I_{r+s}]$. Since the choice of $m$ is independent of the sequence $(I_n)$, the result follows.

$(ii) \implies (iii)$. Suppose that $(iii)$ does not hold for some $C$. Then we can successively find $I_i \in C$, $(i = 1, 2, \ldots)$ such that $I_i \supseteq I_{i+1}, [L^{(i)}, I_i] \notin I_{i+1}$ which implies that $(ii)$ does not hold. Hence $(ii) \implies (iii)$. $(iii) \implies (i)$ is clear. \(

\textbf{Theorem 3.5}

(i) Let $L$ be a quasi-Artinian Lie algebra and let $I \triangleleft L$. Then $L/I$ is quasi-Artinian.

In other words the class of quasi-Artinian algebras is $Q$-closed.

(ii) Let $I \triangleleft L$. Then $L$ is quasi-Artinian if one of the following holds:

(a) $I$ is quasi-Artinian and $L/I$ is soluble.

(b) $L/I$ is quasi-Artinian and if $I \supseteq I_1 \supseteq I_2 \supseteq \ldots$, $I_i \triangleleft L$ then there exists $m \in \mathbb{N}$ such that $[L^{(m)}, I_m] \subseteq I_n$ for all $n \in \mathbb{N}$.

(c) $L/I$ is quasi-Artinian and $I$ is Artinian.
Proof

(i) Let \( \pi : L \rightarrow L/I \) be the natural homomorphism and let \( \bar{I}_1 \supseteq \bar{I}_2 \supseteq \ldots \) be a descending chain of ideals of \( \bar{L} = L/I \). Then \( \pi^{-1}(\bar{I}_1) \supseteq \pi^{-1}(\bar{I}_2) \supseteq \ldots \) is a descending chain of ideals of \( L \). But \( L \) is quasi-Artinian, so there exists \( m \in \mathbb{N} \) such that \( [L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \pi^{-1}(\bar{I}_n) \) for all \( n \geq m \). Therefore

\[
([\pi(L)]^{(m)}(\bar{I}_m) = \pi[L^{(m)}, \pi^{-1}(\bar{I}_m)] \subseteq \bar{I}_n.
\]

Thus \( [L^{(m)}, \bar{I}_m] \subseteq \bar{I}_n \) and \( I \) is quasi-Artinian.

(ii) (a), (b) let \( I_1 \supseteq I_2 \supseteq \ldots \) be a descending chain of ideals of \( L \). Then \( I_1 \cap I \supseteq I_2 \cap I \supseteq \ldots \) is a descending chain of ideals of \( I \) and \( (I_1 + I)/I \supseteq (I_2 + I)/I \supseteq \ldots \) is a descending chain of ideals of \( L/I \). By assumption (a) or (b), there exists \( m \in \mathbb{N} \) such that \( [L^{(m)}, I_m \cap I] \subseteq I_n \cap I \) and

\[
[L^{(m)}, I_m] + I \subseteq [L^{(m)}, I_n] + I \quad \text{for all} \quad n \geq m.
\]

Therefore \( [L^{(m)}, I_m] \subseteq I_n + I \), but \( [L^{(m)}, I_m] \subseteq I_m \). Hence \( [L^{(m)}, I_m] \subseteq (I_n + I) \cap I = I_n + (I_m \cap I) \) and so

\[
[L^{(m)}, (L^{(m)}, I_m)] \subseteq [L^{(m)}, I_n] + [L^{(m)}, (I_m \cap I)].
\]

Therefore \( [L^{(m+1)}, I_m] \subseteq I_n + (I_m \cap I) = I_n \) and \( L \) is quasi-Artinian.

(c) Let \( I_1 \supseteq I_2 \supseteq \ldots \) be a descending chain of ideals of \( L \). Then \( I_1 \cap I \supseteq I_2 \cap I \supseteq \ldots \) is a descending chain of ideals of \( I \) and \( (I_1 + I)/I \supseteq (I_2 + I)/I \supseteq \ldots \) is a descending chain of ideals of \( L/I \). By assumption there exists \( m \in \mathbb{N} \) such that \( I_m \cap I = I_n \cap I \) and \( [L^{(m)}, I_m] \subseteq I_n + I \) for all \( n \geq m \). Hence \( [L^{(m+1)}, I_m] \subseteq I_n \) and \( L \) is quasi-Artinian.

Theorem 3.6

If \( L_1 \) and \( L_2 \) are quasi-Artinian Lie algebras, then
L = L₁ ⊕ L₂ is quasi-Artinian.

Proof

Let I₁ ≥ I₂ ≥ ... be a descending chain of ideals of L. Then [L₁, I₁] ≥ [L₁, I₂] ≥ ... is a descending chain of ideals of L₁ and [L₂, I₁] ≥ [L₂, I₂] ≥ ... is a descending chain of ideals of L₂. But L₁ and L₂ are quasi-Artinian, hence there exists m ∈ N such that [L₁(m), [L₁, I₁]] ⊆ Iₙ and [L₂(m), [L₂, I₁]] ⊆ Iₙ for all n ≥ m. Therefore by the Jacobi identity, [L₁(m+1), I₁] ≤ Iₙ and [L₂(m+1), I₁] ≤ Iₙ. Hence [L(m+1), I₁] ⊆ Iₙ.

Corollary 3.7

A finite direct sum of quasi-Artinian Lie algebras is quasi-Artinian.

Theorem 3.3

Let L be a locally nilpotent quasi-Artinian Lie algebra. Then L is soluble.

Proof

Suppose L is not soluble. Then there is a non-soluble ideal I of L. We claim that I contains a minimal non-soluble ideal of L. Suppose for a contradiction that this is not the case. Let I = I₁. Then 0 ≠ I₁(2) ≤ [L₁, I₁] and [L₁, I₁] is a non-soluble ideal of L, since I₁ is not soluble. So there
is a non-soluble ideal $I_2$ of $L$ such that $I_2 \subseteq [L^{(1)}, I_1] \subseteq I_1$. Now $0 \neq I_2^{(3)} \subseteq [L^{(2)}, I_2]$ and $[L^{(2)}, I_2]$ is a non-soluble ideal of $L$ since $I_2$ is not soluble. So there is a non-soluble ideal $I_3$ of $L$ such that $I_3 \subseteq [L^{(2)}, I_2] \subseteq I_2$. Continuing this process, there is a non-soluble ideal $I_n \subseteq [L^{(n-1)}, I_{n-1}] \subseteq I_{n-1}$. Then $0 \neq I_n^{(n+1)} \subseteq [L^{(n)}, I_n]$ and $[L^{(n)}, I_n]$ is a non-soluble ideal of $L$ since $I_n$ is not soluble. So there is a non-soluble ideal $I_{n+1}$ of $L$ such that $I_{n+1} \subseteq [L^{(n)}, I_n] \subseteq I_n$ and so on. Finally the descending chain $I_1 \supseteq I_2 \supseteq \ldots$ contradicts the hypothesis that $L$ is quasi-Artinian.

Thus there is such a minimal non-soluble ideal, call it $J$. But $J^2 \subseteq J$ and $J^2$ is not soluble, hence $J = J^2$ by the minimality of $J$. Now either $\xi_J(J) = 0$ or $\xi_J(J) \neq 0$.

First, suppose $\xi_J(J) = 0$. Let $\mathcal{C} = \{K \mid L | K \subseteq J$ and $[K, J] \neq 0\}$. Then $\mathcal{C} \neq \emptyset$, since $J \in \mathcal{C}$. We claim that $\mathcal{C}$ has a minimal element. Suppose not. Put $J = J_1$. Then $0 \neq [J_1^{(2)}, J] \subseteq [L^{(1)}, J_1]$, so $[L^{(1)}, J_1] \in \mathcal{C}$. Choose $J_2 \in \mathcal{C}$ such that $J_2 \subseteq [L^{(1)}, J_1] \subseteq J_1$. Then $0 \neq [J, J_2] = [J^2, J_2] \subseteq [J, [J, J_2]] = [J, [J^{(2)}, J_2]] \subseteq [J, [L^{(2)}, J_2]]$. Hence $[L^{(2)}, J_2] \in \mathcal{C}$ and so on. Choose $J_n \in \mathcal{C}$ such that $J_n \subseteq [L^{(n-1)}, J_{n-1}] \subseteq J_{n-1}$. Then $0 \neq [J, J_n] = [J^{(n+1)}, J_n] \subseteq [J, [L^{(n)}, J_n]]$. Therefore $[L^{(n)}, J_n] \in \mathcal{C}$. Repeat this process; then the descending chain of ideals $J_1 \supseteq J_2 \supseteq \ldots$ contradicts the hypothesis that $L$ is quasi-Artinian. Thus $\mathcal{C}$ has a minimal element say $K$. If $K$ is a minimal ideal of $L$, then $K$ is central (see [5], Lemma 7.1.6, p. 137) which is a contradiction.
If $K$ is not a minimal ideal of $L$, then $K \supset H$ and $H \subset L$.

Now either $[H, J] = 0$, or $[H, J] \neq 0$. If $[H, J] = 0$, then $H \subseteq C_L(J)$, but $H \subseteq J$, hence $H \subseteq C_L(J) \cap J = \zeta_1(J) = 0$. If $[H, J] \neq 0$, then $H = K$ by minimality of $K$ and $K$ is a minimal ideal of $L$ and in both cases we get a contradiction.

Hence $\zeta_1(J) \neq 0$. Let $U$ be the hypercentre of $J$. Then $U \subset L$ and $U^{(a)} = 0$ for some infinite $a$. But $L$ is quasi-Artinian, so by Proposition 3.2, $U^{(a)} = U^{(n)} = 0$ for finite $n$ and so $U$ is soluble. Now $J/U$ is a minimal non-soluble ideal of $L/U$ and $J/U = (J/U)^2$ with $\zeta_1(J/U) = 0$, and a similar argument as above again gives a contradiction. Therefore $L$ is soluble.

**Remark**

(1) It appears likely that a theory of prime ideals of quasi-Artinian Lie algebras may exist analogous to that for Min-$\mathcal{A}$. In particular this would be the case if every semi-simple quasi-Artinian Lie algebra were Artinian. We know no example that disproves this, but it remains an open question.

(2) It is possible to define the notion of quasi-Artinian groups in an analogous way and the proofs of Theorems 3.4, 3.5, 3.6 and 3.3 carry over in this case without difficulties.
CHAPTER FOUR: ON THE JOIN OF SUBIDEALS

It is well-known that the join of two subideals of a Lie algebra need not be a subideal (see Amayo and Stewart [5], Lemma 2.1.11, p. 41). This raises the question of finding conditions under which the join is a subideal. The same question arises in group theory. Wielandt [35, Theorem 2.10.5, p. 41] has shown that: If \( \{ H_\lambda \mid \lambda \in \Lambda \} \) is a set of subnormal subgroups of a group \( G \) and \( J \) is their join, then \( J \) is subnormal in \( G \) if and only if the set of subnormal subgroups of \( G \) lying in \( J \) contains a maximal member. Following [1] we shall obtain a similar result for Lie algebras. We also find another condition under which the join of subideals is a subideal by imposing conditions on the circle product of subideals. Our main results are as follows:

Let \( L \) be a Lie algebra over a field of characteristic zero and let \( H_\lambda \in L, \lambda \in \Lambda \). Suppose that \( J = \langle H_\lambda \mid \lambda \in \Lambda \rangle \) and \( B = \{ B \mid B < J, B \in L \} \). Then \( J \) is a subideal if and only if \( B \) has a maximal element (Theorem 4.4).

Suppose that \( X \) is an \( (I,N_0) \)-closed and locally coalescent class over any field. Let \( H \) and \( K \) be \( X \)-subideals of a Lie algebra \( L \) with \( J = \langle H,K \rangle \). If \( H\ast K/(H\ast K)^2 \) is finitely generated then \( J \) is a subideal and \( J \in X \) (Theorem 4.12).

We start with a construction of Lie algebras of power series used in Amayo and Stewart [5, pp. 77, 78, 80].
Let $L$ be a Lie algebra over a field $F$ of characteristic zero and let $F_0 = F\langle t \rangle$ be the field of formal power series in the intermediate $t$. Let $L_+^+$ be the set of all formal power series $x = \sum_{r=n}^{\infty} x_r t^r$, $x_r \in L$ and $n = n(x) \in \mathbb{Z}$. Let $y = \sum_{r=n}^{\infty} y_r t^r \in L_+^+$ and define addition, multiplication and multiplication of elements of $L_+^+$ by scalars from $F_0$ according to the rules:

$$x + y = \sum_{r=n}^{\infty} (x_r + y_r) t^r,$$

$$[x,y] = \sum_{r=n}^{\infty} z_r t^r, \text{ where } z_r = \sum_{i+j=r} [x_i, y_j],$$

$$ax = \sum_{r=n}^{\infty} c_r t^r, \text{ where } c_r = \sum_{i+j=r} a_i x_j.$$

It is easy to verify that this makes $L_+^+$ into a Lie algebra over $F_0$. Let $M < L$ and $M_+^+$ be the set of all elements $x = \sum_{r=n}^{\infty} x_r t^r \in L_+^+$ with $x_r \in M$. Then clearly $M_+^+$ is $F_0$-subalgebra of $L_+^+$ and it is easy to prove the following which is Amayo and Stewart [5, Lemma 4.1.1(b) p.79].

**Lemma 4.1**

Let $L$ be a Lie algebra over a field $F$ of characteristic zero and let $K < H < L$. Then $K_+^+ < H_+^+ < L_+^+$. In particular if $H <^m L$, then $H_+^+ <^m L_+^+$.

Let $L$ be a Lie algebra over $F$ and let $x \neq 0$, $x \in L_+^+$. Then $x$ can be written uniquely in the form $x = \sum_{r=0}^{\infty} x_r t^r$, where $m = m(x) \in \mathbb{Z}$ and $x_0 \neq 0$ is the first non-zero coefficient.
of $x$. $x_0$ is called the first coefficient of $x$. Clearly every non-zero element $\alpha \in F_0$ has a similar expression: 
\[ \alpha = t^s \sum_{r=0}^{\infty} \alpha_r t^r, \] where $\alpha_0 \neq 0$ is called the first coefficient of $\alpha$. Let $y \neq 0, y \in L^+$ so that $y = t^n \sum_{r=0}^{\infty} y_r t^r$ where $y_0 \neq 0$. Clearly $[x, y] = t^{m+n}[x_0, y_0] + t^{m+n} \sum_{r=1}^{\infty} (\sum_{i+j=r} [x_i, y_j]) t^r,$ and for any $\alpha, \beta \in F$, $\alpha x + t^{m-n} \beta y = t^m \sum_{r=0}^{\infty} (\alpha x_r + \beta y_r) t^r.$

Now let $M$ be a subset of $L^+$ and let $M^+ = \{x \in L | x = 0 \text{ or } x \text{ is the first coefficient of some element of } M\}$.

The above equations lead to the following result (see Amayo and Stewart [5, Lemma 4.1.2 (a), (b), (c), (f), p. 80]).

**Lemma 4.2**

Let $L$ be a Lie algebra over a field $F$ of characteristic zero. Then

(a) If $M$ is a subspace (resp. subalgebra) of $L^+$, then $M^+$ is a subspace (resp. subalgebra) of $L$.

(b) If $N < M < L^+$, then $N^+ \prec M^+ \prec L$. In particular if $N \cap L^+$, then $N^+ \prec L$.

(c) Let $M$ and $N$ be subsets of $L^+$. Then $[M^+, N^+] \subseteq [M, N]^+$
\[ (M \cap N)^+ \subseteq M^+ \cap N^+, \quad M^+ + N^+ \subseteq (M + N)^+ \]

(d) If $M$ is a subspace of $L$, then $M^{++} = M$. □

Now we prove the following which is the Lie algebra analogue of Wielandt [35, Lemma 2.10.4, p. 40].
Lemma 4.3

Let $L$ be a Lie algebra over a field of characteristic zero and let $S < L$. Let $B = \{ B | B < S, B \subseteq L \}$ and let $H$ be a maximal element of $B$. Then $H \not< S$ and $H > B$ for every $B \in B$.

Proof

Let $H$ have subideal index $m$ in $S$ and suppose that $m \geq 2$. Denote the $i^{th}$ ideal closure of $H$ in $S$ by $H_i$. It follows that there exists $x \in H_{m-2}$ with $[H,x] \not\subseteq H$. By Lemma 4.1, $H^+ \leq L^+$ and $H^+ < S^+$. Let $\theta = \exp (t \text{ ad } x)$. Then $H^+ \subseteq L^+$ and $H^+ \leq H_{m-1}$. But $H^+ < H_{m-1}$, hence $H^+ \subseteq H_{m-1}$ idealises $H^+$ and

$H^+ + H^{+\theta} \subseteq L^+ + H^{+\theta} < S^+$. By Lemma 4.2, we have

$(H^+ + H^{+\theta})^+ \subseteq L^+$ and $(H^+ + H^{+\theta})^+ < S$. Hence $(H^+ + H^{+\theta})^+ \in B$. By Lemma 4.2, $H^+ = H$ and $H \subseteq (H^+ + H^{+\theta})^+$, but $H$ is maximal, hence $(H^+ + H^{+\theta})^+ = H$. Now take $h \in H$ such that $[h,x] \not\subseteq H$. Then $h^\theta - h = (h,x)t + \ldots$ and therefore $h^\theta - h \in (H^+ + H^{+\theta})^+ = H$ which is a contradiction. Therefore $H \not< S$ and clearly $H > B$ for every $B \in B$. □

Theorem 4.4

Let $L$ be a Lie algebra over a field of characteristic zero and $H_\lambda \subseteq L$, $\lambda \in \Lambda$. Let $J = \langle H_\lambda | \lambda \in \Lambda \rangle$ and $B = \{ B | B < J, B \subseteq L \}$. Then $J \subseteq L$ if and only if $B$ has a maximal element.
Proof

The only if part is clear. To prove the if part, let \( H \) be a maximal element in \( B \). Then by Lemma 4.3, \( H \preceq J \) and each \( H_\lambda \preceq H \) so \( J \preceq H \) and \( J = H \). Therefore \( J \preceq L \).

Corollary 4.5

Let \( L \) be a Lie algebra over a field of characteristic zero and let \( J = \langle H | H \preceq L, \ \lambda \in \Lambda \rangle \). Then \( J \preceq L \) if one of the following holds.

(i) \( L \in \max-si \)

(ii) \( J \in \max-si \)

Proof

Let \( B = \{ B | B \preceq J, B \preceq L \} \). By assumption (i) or (ii) \( B \) has a maximal element. Therefore by Theorem 4.4, \( J \preceq L \).

Next we prove the following, which is the Lie algebra analogue of a well-known result in group theory.

Theorem 4.6

Let \( L \) be a Lie algebra over a field of characteristic zero and let \( H, K \) be subideals of \( L \). Suppose that the set of subideals of \( L \) lying between \( H \) and \( J = \langle H, K \rangle \) contains at least one maximal member. Then \( J \preceq L \).

To prove this we need the following well-known lemma:
Lemma 4.7

Let $L$ be a Lie algebra and suppose that $H \triangleleft_{m} L$, $K \triangleleft_{n} L$ and $J = \langle H, K \rangle$. If $H \triangleleft J$ then $J \triangleleft_{mn} L$.

Proof

See Amayo and Stewart [5, Lemma 2.1.2, p. 33]. □

Proof of Theorem 4.6

Without loss of generality we may assume that $H$ is a maximal member of the set of subideals of $L$ lying in $J$ and containing the original $H$. By Lemma 4.3, $H \triangleleft J$ and by Lemma 4.7, $H \triangleleft_{1} L$. □

As an application of Theorem 4.6, we have the following, which is due to Hartley [13]. It is proved in Amayo and Stewart [5, p. 64] by a different method.

Corollary 4.8

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H, K$ be subideals of $L$. Then $J = \langle H, K \rangle$ is a finite-dimensional subideal of $L$.

Proof

That $J \triangleleft_{1} L$ follows from Theorem 4.6. Further $J$ is finite-dimensional since $L$ is. □
Corollary 4.9

Let $L$ be a Lie algebra over a field of characteristic zero and let $H, K$ be subideals of $L$. If $H$ has finite-codimension in $J = \langle H, K \rangle$, then $J$ is $L$.

Proof

Since $H$ has finite codimension in $J$, it follows that the set of subideals of $L$ lying between $H$ and $J$ has a maximal element and by Theorem 4.6, $J$ is $L$. □

Finally we find another condition under which the join of two subideals of a Lie algebra is a subideal. For this we shall need the following definitions (see Amayo and Stewart [5, pp. 18-20, 30, 67]). A class $X$ is $I$-closed provided every subideal of an $X$-algebra is always an $X$-algebra. A class $X$ is $N_0$-closed if whenever $H, K \leq L$ and $H, K \in X$, then $H + K \in X$. A class $X$ is locally coalescent if and only if whenever $H$ and $K$ are $X$-subideals of a Lie algebra $L$, then to every finitely-generated subalgebra $C$ of $J = \langle H, K \rangle$ there corresponds an $X$-subideal $X$ of $L$ such that $C < X < J$.

Let $H$ and $K$ be a subset of a Lie algebra $L$. The circle product of $H$ and $K$ denoted by $H \circ K$, is defined as $H \circ K = [H, K]^{HUK}$. It is clear that $H \circ K$ is the smallest ideal of $J = \langle H, K \rangle$ containing $[H, K]$.

Now we prove the following, which is the Lie algebra analogue of Robinson [24, Lemma 2.3, p. 149].
Proposition 4.10

Let $L$ be a Lie algebra over any field and let $H, K$ be subideals of $L$ with $J = \langle H, K \rangle$. Then the following are equivalent:

(i) $J \subseteq L$

(ii) $H^K \subseteq L$

(iii) $H*K \subseteq L$.

Proof

(i) $\rightarrow$ (iii). Since $H^K = H + H*K \subseteq J$ and $J \subseteq L$, it follows that $H^K \subseteq L$.

(ii) $\rightarrow$ (iii). From $H^K \subseteq J$, we have $H*K \subseteq H^K$. But $H*K \subseteq L$ hence $H*K \subseteq L$.

(iii) $\rightarrow$ (i). Let $H*K \subseteq L$. Then since $H \subseteq L$ and $H$ idealises $H*K$, it follows from Lemma 4.7, that $H^K \subseteq L$. However $K$ idealises $H^K$ so a further application of Lemma 4.7, shows $H^K + K = J \subseteq L$. □

Corollary 4.11

Let $L \in \mathfrak{NA}$ and $H, K$ are subideals of $L$, then

$J = \langle H, K \rangle \subseteq L$.

Proof

Since $L \in \mathfrak{NA}$, there exists $N \triangleleft L$ with $N \subseteq N$ and $L/N \in \mathfrak{A}$. Hence $H*K \subseteq N$ and so $H*K \subseteq N \subseteq L$. Therefore $H*K \subseteq L$ and by Proposition 4.10, $J \subseteq L$. □
Suppose that \( X \) is an \((I,N_0)\)-closed and locally coalescent class over any field. Let \( H \) and \( K \) be \( X \)-subideals of a Lie algebra \( L \) with \( J = \langle H,K \rangle \). If \( H \cdot K/(H \cdot K)^2 \) is finitely-generated, then \( J \triangleleft L \) and \( J \in X \).

To prove this we need the following well-known results.

**Theorem 4.12**

Suppose that \( X \) is an \((I,N_0)\)-closed and locally coalescent class over any field. Let \( H \) and \( K \) be \( X \)-subideals of a Lie algebra \( L \) with \( J = \langle H,K \rangle \). If \( H \cdot K/(H \cdot K)^2 \) is finitely-generated, then \( J \triangleleft L \) and \( J \in X \).

**Lemma 4.13**

Let \( H \) and \( K \) be subalgebras of a Lie algebra \( L \) such that \( L = H + K^2 \). Then \( L = H + K^{n+1} \) for any \( n \in \mathbb{N} \). If in addition \( H \cap K \triangleleft K \) then for any \( n \in \mathbb{N} \), \( L = H + K^{(n)} \).

**Proof**

See Amayo and Stewart [5, Lemma 2.1.9, p. 40]. \( \spadesuit \)

**Lemma 4.14**

Let \( L \) be a Lie algebra and let \( H_1 \triangleleft h_1 L, H_2 \triangleleft h_2 L \) and \( J \triangleleft \langle H_1,H_2 \rangle \). Then there exists \( \lambda_3 = \lambda_3(h,r) \) such that \( J^{(\lambda_3)} \trianglelefteq H_1^{(r_1)} + H_2^{(r_2)} \) and \( J^{(\lambda_3)} \trianglelefteq \lambda_3 L \) whenever \( h_1 + h_2 \leq h \) and \( r_1 + r_2 \leq r \).

**Proof**

See Amayo and Stewart [5, Theorem 2.2.7, p. 48]. \( \spadesuit \)

**Lemma 4.15**

Let \( X \) be an \((I,N_0)\)-closed class of Lie algebras and
suppose that $H$ and $K$ are $X$-subideals of a Lie algebra $L$ and $J = \langle H, K \rangle$. If $H$ and $K$ are permutable then $J$ is an $X$-subideal of $L$.

Proof

See Amayo and Stewart [5, Theorem 2.2.13, p. 54].

Lemma 4.16

Let $L$ be a Lie algebra and $J = \langle H_1, \ldots, H_n \rangle$ with $H_i \subseteq L$ for $i = 1, \ldots, n$. If each $H_i$ lies in an $(I, N_0)$-closed class $X$ then $J^{(\lambda)} \in X$ and so $J \in \mathcal{X}^\lambda$.

Proof

See Amayo and Stewart [5, Corollary 2.2.17, p. 57].

Proof of Theorem 4.12

Let $M = H \cdot K$. Now there exists a finitely-generated subalgebra $C$ of $M$ such that $M = C + M^2$. By the local coalescence of $X$ there exists an $X$-subideal $X$ of $L$ with $C < X < J$. Thus if $N = X \cap M$, then $N \not\subseteq X \subseteq L$ and so $N \subseteq L$, $N \in \mathcal{IX} = X$ and $M = N + M^2$. From Lemma 4.13, we have $M = N + M^{(r)}$ for all $r$. By Lemma 4.16, we have $J^{(r)} \in X$ for some $r$ and so $M^{(r)} \in \mathcal{IX} = X$. Finally by Lemma 4.15, we have $M = N + M^{(r)} \subseteq X$ and $M \subseteq L$ (for $J^{(r)}$ and so $M^{(r)} \subseteq L$ for some $r$ by Lemma 4.14). We also have by Lemma 4.15, that $H + M$, $K + M \subseteq X$ and $H + M$, $K + M \subseteq L$ and so by the same result $J = H + M + K + M \subseteq X$.
Corollary 4.17

Let $L$ be a Lie algebra over a field of characteristic zero and let $H, K$ be subideals of $L$. If $H \circ K \in \text{Max-si}$ or $H \circ K \in \text{Min-si}$, then $J = \langle H, K \rangle \subseteq L$.

Proof

The hypothesis of Theorem 4.12 are satisfied for these two classes. □
CHAPTER FIVE : SUBIDEALS AND ASCENDANT SUBALGEBRAS
OF LIE ALGEBRAS

Wielandt [34] has shown that a subgroup $H$ of a finite group $G$ is subnormal in $G$ if and only if for each $g \in G$ and $h \in H$ there exists an integer $n$ such that $[g, h] \in H$. This, and related criteria given by Wielandt in the same paper, have been extended to various classes of infinite groups by Peng [22, 23], Hartley and Peng [14], Whitehead [32, 33] and Wehrfritz [31]. The Lie algebra analogue of Wieland's criteria have been investigated in a various class of Lie algebras by Chao and Stitzinger [8], Kawamoto [17], Stewart [27], Stitzinger [28], Tôgô [29], Tôgô, Honda and Sakomoto [30].

Following [4], we give some criteria for subideality and ascendancy in Lie algebras, similar to Wielandt's stated in terms of the circle product. The main results are as follows:

If $L$ is a finite-dimensional Lie algebra over a field of characteristic zero and $H$ is a subalgebra of $L$, then $H \triangleleft L$ if and only if one of the following conditions holds:

(i) For each $x \in L$, there exists an integer $n = n(x)$ such that $\langle x \rangle \circ \langle h_1 \rangle \circ \langle h_2 \rangle \circ \ldots \circ \langle h_n \rangle \subseteq H$ for all $h_1, \ldots, h_n \in H$. 
(ii) for each $x \in L$ and $h \in H$ there exists an integer $n = n(x, h)$ such that $<x> \circ_n <h> \subseteq H$.

(iii) for each $h \in H$ there exists an integer $n = n(h)$ such that for all $x \in L$ we have $<x> \circ_n <h> \subseteq H$.

(Theorem 5.2.1).

A generalization to infinite-dimensional Lie algebras leads to the following.

Let $L$ be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H \triangleleft L$.

(i) If for each $x \in L$ there exists an integer $n = n(x)$ such that $<x> \circ <h_1> \circ <h_2> \circ \ldots \circ <h_n> \subseteq H$ for any $h_1, \ldots, h_n \in H$, then $H \triangleleft L$.

(ii) Suppose that $H$ is finite-dimensional

(a) If for each $h \in H$, there exists an integer $n = n(h)$ such that for all $x \in L$ we have $<x> \circ_n <h> \subseteq H$, then $H \triangleleft L$.

(b) If for each $x \in L$ and $h \in H$, there exists an integer $n = n(x, h)$ such that $<x> \circ_n <h> \subseteq H$, then $H \triangleleft L$. (Theorem 5.3.6).

Finally if $L$ is an ideally finite Lie algebra (see below) over a field of characteristic zero and $H \triangleleft L$, then $H \triangleleft L$ if either of the following conditions holds:
5.1 Definitions and basic results

Let $L$ be a Lie algebra over any field. The **Fitting radical** $v(L)$ is the sum of nilpotent ideals of $L$ (equal to the nil radical in finite dimensions). If $L$ is finite-dimensional over a field of characteristic zero, then $v(L)$ contains every nilpotent subideal of $L$ (see Amayo and Stewart [5], p. 114). A finite-dimensional Lie algebra $L$ is said to be **split** (see Jacobson [16], p. 114), if the characteristic roots of every $\text{ad}_L h$, $h \in H$, where $H$ is a Cartan subalgebra of $L$, are in the base field. The **Hirsch-Plotkin radical** $\mathcal{p}(L)$ of $L$ is the unique maximal locally nilpotent ideal of $L$. If the underlying field has characteristic zero we define the **Gruenberg radical** $\gamma(L)$ to be the subalgebra generated by the nilpotent ascendant subalgebras of $L$ (see Amayo and Stewart [5], pp. 113, 114). An algebra is **locally finite** if every finite set of elements is contained in a finite-dimensional subalgebra. If $H$ is $L$, then $H^\omega = \bigcap_{n=1}^{\infty} H^n \triangleleft L$, where $\omega$ is the first infinite-ordinal (see Amayo and Stewart [5], Lemma 1.3.2, p. 10). Let $A$ and $B$ be subsets of a Lie algebra $L$. The **circle product** $A \circ B$ of $A$ and $B$ is defined (see Amayo and Stewart [5], p. 30), as $A \circ B = [A,B]^{A\cup B}$. We also let
A \circ_{m+1} B = (A \circ_{m} B) \circ B \text{ for all positive integers } m. \text{ We say that } x \in L \text{ is a } \text{left Engel element} \ \text{(see [5], p. 339)} \text{ if for each } y \in L \text{ we can find an integer } n = n(x,y) \geq 0 \text{ such that } [y, x^n] = 0. \text{ If } n \text{ can be chosen independently of } y \text{ we say that } x \text{ is a bounded left Engel element.} \text{ The sets of left Engel, bounded left Engel elements are denoted by } e(L) \text{ and } e(L) \text{ respectively. A local system for a Lie algebra } L \text{ (see Stewart [26], p. 33) is a collection of subalgebras of } L \text{ which generate } L \text{ and have the property that whenever } i,j \in I, \text{ there exists } m \in I \text{ such that } [L_i, L_j] < L_m. \text{ A Lie algebra is said to be ideally finite (see [26], p. 34) if it has a local system of finite-dimensional ideals. A Fitting class is (see Amayo and Stewart [5], p. 259) a subclass } X \text{ of } F \text{ which is } (N_0, I) \text{-closed.} \text{ A subalgebra } H \text{ of a Lie algebra } L \text{ is called serial, written } H \text{ ser } L, \text{ if there is a series from } H \text{ to } L, \text{ (see [5], p. 258). We write } L \in E'A, \text{ see [5], p. 28) if } L \text{ has an ascending abelian series } (L_{\alpha})_{\alpha<\lambda}. \text{ If each } L_{\alpha} \text{ is an ideal of } L, \text{ then } L \in E'(4)A, \text{ L is hyperabelian. Let } L \text{ be a Lie algebra over a field } F. \text{ A universal enveloping algebra of } L \text{ is a pair } (U,i), \text{ where } U \text{ is an associative algebra with } 1 \text{ over } F, \text{ and } i: L \rightarrow U \text{ is a linear map satisfying}

\begin{equation}
i([x,y]) = i(x)i(y) - i(y)i(x) \text{ for } x, y \in L \end{equation}

\text{and the following holds: for any associative } F\text{-algebra } A \text{ with } 1 \text{ and any linear map } j: L \rightarrow A \text{ satisfying (1) there exists a}
unique homomorphism of algebras $\theta: U \to A$ (sending 1 to 1) such that $\theta \circ 1 = j$ (see Humphreys [15], p. 90). We say that $x$ acts nilpotently on $L$ if $[L, x] = 0$ for some $n \in \mathbb{N}$.

Let $H$ be an $L$-module. We say that $H$ locally algebraic if for each $x \in L$ and $a \in H$ there exists a polynomial $P = P_{a,x}$ such that $ap(x) = 0$. Note that Curtis [9] defines the concept of a locally algebraic transformation as follows:

Let $V$ be a vector space over a perfect field $F$. A linear transformation $a$ is locally algebraic if every vector $x \in V$ is contained in a finite-dimensional $a$-invariant subspace of $V$. $a$ is algebraic if $P(a) = 0$, where $P$ is a non-zero polynomial with coefficients in $F$. Now we have the following:

**Lemma 5.1.1**

An $L$-module is locally algebraic in our sense if and only if the transformations induced on it by the $L$-action are locally algebraic in Curtis's sense.

**Proof**

Let $A$ be a locally algebraic $L$-module in our sense and let $a \in A$. Suppose that $\langle a \rangle_\alpha$ is an $a$-invariant subspace of $A$. We claim that $\langle a \rangle_\alpha$ is finite-dimensional. By assumption we have $ap(a) = 0$ for some non-zero polynomial $P$. Therefore

$$ad^m = \sum_{i=0}^{m-1} \lambda_i a^i \quad \text{where } \lambda_i \in F.$$
Therefore by induction \( a^n = \sum_{i=1}^{m-1} \mu_i a^i \) for all \( n > m \).

So \( \langle a \rangle \subseteq \langle a, a^2, \ldots, a^{m-1} \rangle \) and \( \langle a \rangle \) is finite-dimensional.

The converse is clear. °

Finally we define three new relations \( \text{ci}, \text{sci}, \text{nci} \) as follows: Let \( H < L \). Write \( H \text{ ci } L \) if for each \( x \in L \) and \( h \in H \) there exists an integer \( n = n(x, h) \) such that 
\[ \langle x \rangle \circ_n \langle h \rangle \subseteq H. \]
\( H \text{ sci } L \) if for each \( h \in H \) there exists an integer \( n = n(h) \) such that for all \( x \in L \) we have 
\[ \langle x \rangle \circ_n \langle h \rangle \subseteq H. \]
\( H \text{ nci } L \) if for each \( x \in L \) there exists an integer \( n = n(x) \) such that 
\[ \langle x \circ h_1 \circ h_2 \circ \cdots \circ h_n \rangle \subseteq H \]
for any \( h_1, h_2, \ldots, h_n \in H \).

The following implications are clear.

\[
\begin{align*}
H \xleftarrow{\omega} L \quad & \text{ci} \quad \rightarrow \quad H \text{ sci} \quad \rightarrow \quad H \text{ nci} \quad \rightarrow \quad H \text{ ci} \\
\text{ci} \quad & \text{ci} \quad \rightarrow \quad H \text{ sci} \quad \rightarrow \quad H \text{ ci} \quad \rightarrow \quad H \text{ ci}
\end{align*}
\]

The object is to produce partial converses of these implications.

Next we prove the following:

**Lemma 5.1.2**

Let \( L \) be a Lie algebra over any field and let \( H < L \).

Let \( \Delta \) be any of the relations \( \text{ci}, \text{sci}, \text{nci} \). Then
(i) If $K < L$, $H \triangle L$ then $H\cap K \triangle K$.

(ii) If $f$ is a homomorphism of $L$ onto $\tilde{L}$ and $H \triangle L$, then $f(H) \triangle \tilde{L}$. If $\tilde{H} \triangle \tilde{L}$, then $f^{-1}(\tilde{H}) \triangle L$.

(iii) $K \triangle H \triangle L$ implies $K \subset L$.

**Proof**

(i) is clear.

(ii) This follows from the fact that if $H$ and $K$ are subalgebras of $L$ and $f$ is a homeomorphism of $L$ onto $\tilde{L}$, then

$$f(H \cdot K) = f(H) \cdot f(K) \quad \text{and} \quad f^{-1}(\tilde{H} \cdot \tilde{K}) = f^{-1}(\tilde{H}) \cdot f^{-1}(\tilde{K}).$$

(iii) Suppose $K \triangle^m H$. We prove by induction on $m$ that if $K \triangle^m H$ and $H \subset L$, then $K \subset L$. Suppose that $m = 1$. Then since for each $x \in L$ and $y \in K$ it follows that $x \in L$ and $y \in H$, so there exists an integer $n = n(x,y)$ such that

$$\langle x \rangle^m \circ \langle y \rangle \subseteq H.$$ But $K \triangle H$, hence $\langle x \rangle^m \circ \langle y \rangle \circ \langle y \rangle \subseteq K$. Therefore $\langle x \rangle^m \circ \langle y \rangle \circ \langle y \rangle \subseteq K$ and $K \subset L$. Now assume that the result is true for some $i \geq 1$. Then $K \triangle^i H \subset L$ implies $K \subset L$. So if $K \triangle^{i+1} H \subset L$ we have $K \triangle K_i \triangle^i H \subset L$ and by induction $K_i \subset L$. Therefore as in the case $m = 1$, $K \subset L$.

By a similar argument we can show that if $\Delta$ is either of the remaining relations $\triangle$ or $\subset$, and $K \triangle H \triangle L$, then $K \triangle L$. □
5.2 Finite dimensions

Theorem 5.2.1

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H < L$. Then the following are equivalent.

(i) $H$ is $L$
(ii) $H$ is $L$
(iii) $H$ is $L$
(iv) $H$ is $L$

To prove Theorem 5.2.1, we need the following well-known results.

Lemma 5.2.2

If $L$ is a nilpotent Lie algebra of class $c$, and if $H < L$, then $H < C L$.

Proof

See Amayo and Stewart [5, Lemma 1.3.7, p. 12].

Lemma 5.2.3

Let $L$ be a Lie algebra over a field of characteristic zero. If $L \in E'$, then $\rho(L) \subseteq \varepsilon(L) = \gamma(L)$.

Proof

See Amayo and Stewart [5, Theorem 16.4.2(a), p. 341].
Lemma 5.2.4

If $L$ is a finite-dimensional semi-simple Lie algebra over a field of characteristic zero, then every finite-dimensional module for $L$ is completely reducible.

Proof

See Jacobson [16, Theorem 8, p. 79]. □

Lemma 5.2.5

Let $L$ be a Lie algebra of linear transformations in a finite-dimensional vector space $V$ over a field of characteristic zero. Assume that $L$ is completely reducible. Then every non-zero nilpotent element of $L$ can be imbedded in a three-dimensional split simple subalgebra of $L$.

Proof

See Jacobson [16, Theorem 17(1), p. 100]. □

Lemma 5.2.6 (Engel's Theorem)

If $L$ is a finite-dimensional Lie algebra, then $L$ is nilpotent if and only if $\text{ad } x$ is nilpotent for every $x \in L$.

Proof

See Humphreys [15, p. 12]. □
Proof of Theorem 5.2.1

Clearly (i) -> (ii) -> (iii) and (i) -> (iv) => (iii), so we only need to prove that (iii) -> (i). For any \( h \in H \), \( \text{ad}_L h \) induces a linear transformation \( \alpha(h) \) of the space \( L/H \). By assumption each \( \alpha(h) \) is nil. Since the space \( L/H \) is finite-dimensional, \( \alpha(h) \) is nilpotent. Therefore by Lemma 5.2.6 \( \alpha(H) \) is nilpotent. Hence there exists an integer \( m \) such that \( \alpha(h_1) \cdot \alpha(h_2) \cdots \alpha(h_m) = 0 \) for any \( h_1, h_2, \ldots, h_m \in H \).

Therefore \( [L, H] \subseteq H \). So for all \( n \in \mathbb{N} \) \( [L, H^{n+1}] \subseteq H^{n+1} \) from which it follows that \( H^\omega = L \). Now we argue for a contradiction assuming that \( L \) is a counterexample of minimal dimension. We have \( H \subseteq L \) but \( H \) is not a subideal of \( L \).

If \( H^\omega \neq 0 \), then by minimality \( H/H^\omega \subseteq L/H^\omega \), and \( H \subseteq L \), a contradiction. Therefore \( H^\omega = 0 \) and \( H \) is nilpotent. Hence if \( h \in H \), then \( \langle h \rangle \subseteq H \) by Lemma 5.2.2. If the theorem were true for the case \( \dim H = 1 \), it would follow that \( \langle h \rangle \subseteq L \) for all \( h \in H \), hence \( H < \sqrt{L} \). Therefore \( H \) would be a subideal.

It follows that we may assume that \( \dim H = 1 \), so that \( H = \langle h \rangle \) for some \( h \in L \). For all \( x \in L \) we have \( [x, h] = 0 \) for some \( m > 0 \). Since \( L \) has finite dimension \( [L, h] = 0 \) for some \( m > 0 \). Let \( S = \sigma(L) \). If \( S = L \), then every element \( h \) for which \( \text{ad}_L h \) is nilpotent lies in \( \sqrt{L} \) by Lemma 5.2.3. Therefore \( H < \sqrt{L} \), so \( H \subseteq L \), which again is a contradiction. Hence \( S \neq L \). It follows that \( S + H \neq L \), since \( S + H \) is soluble. By minimality we have \( H \subseteq S + H \).
If $S \neq 0$, then $(S + H)/S \cong L/S$ by minimality, which implies that $H \cong L$. Therefore $S = 0$ and $L$ is semi-simple. By Lemma 5.2.4 and Lemma 5.2.5, there is an element $k \in L$ such that $\langle h, k \rangle$ is a three-dimensional split simple Lie algebra. By Lemma 5.1.2, $\langle h \rangle \cong \langle h, k \rangle = T$. Therefore $\langle k \rangle \circ_n \langle h \rangle \subseteq \langle h \rangle$, but $\langle k \rangle \circ_n \langle h \rangle = T$ hence $\langle h \rangle = T$ which is a contradiction, and this completes the proof. $\Box$

As a consequence of Theorem 5.2.1, we have the following, which is Stewart [27, Theorem 1].

**Corollary 5.2.7**

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H \in L$. If $H \cong \langle H, x \rangle$ for all $x \in L$, then $H \cong L$.

**Proof**

Let $K = \langle H, x \rangle$, for some $x \in L$, then by hypothesis, we have $K \circ_n H \subset H$ for some $n$. Hence $H \cong L$ and by Theorem 5.2.1, we have $H \cong L$. $\Box$

**Remark**

The proof of Theorem 5.2.1 fails in characteristic $P > 0$. However we have the following.

**Proposition 5.2.8**

Let $L$ be a finite-dimensional soluble Lie algebra over any field and let $H \leq L$. If $H \cong L$, then $H \cong L$. 
Proof

This follows from Togo [29, Theorem 8 and Corollary (b) of Theorem 2].

5.3 Infinite-dimensions

For infinite dimensional Lie algebras we do not expect a result like Theorem 5.2.1 in general. For example, let $L$ be the Lie algebra of Amayo and Stewart [5, p. 119], that is $L = X + \langle \sigma \rangle$ where $X$ is an abelian Lie algebra with basis $x_0, x_1, x_2, \ldots$ and $\sigma$ is a derivation on $X$ defined by $x_0 \sigma = 0, x_i \sigma = x_{i-1}$ ($i > 0$). Let $H = \langle \sigma \rangle$. Then since $L$ is locally nilpotent, it follows that $\langle x, \sigma \rangle$ is nilpotent for each $x \in L$. Therefore $H \subset L$, but $H$ is not a subideal of $L$. (However, $H$ is ascendant in $L$).

Moreover, let $F$ be any field of characteristic zero, and let $A = F[x_1, x_2, \ldots]$ be the polynomial algebra in a countably infinite set of indeterminates $x_n$ over $F$. Let $I$ be the ideal of $A$ generated by $x_1^2, x_2^2, \ldots, x_1^2, \ldots$. Considered as an abelian Lie algebra, $P = A/I$ has derivations:

$δ_1 : f \longrightarrow f x_1^1, i = 1, 2, \ldots$ for each $f \in P$. Then $δ_1^1 = 0$ and $δ_1 δ_j = δ_j δ_1$ for all $i, j$. Let $H = \langle δ_1 \rangle$ and form the split extension $L = P ∼ H$. Then $L$ is soluble of derived length two and is locally nilpotent; for each $h \in H$, there exists $n = n(h)$ such that for all $x \in L$ we have $\langle x \rangle \sigma_n \langle h \rangle \subseteq H$. Therefore $H \subset L$, but $H = I_L(H)$, so $H$ is neither a subideal nor ascendant. However we have the following:
Let $L$ be a Lie algebra over a field $F$ such that $L = A + H$, where $A$ is an abelian ideal of $L$ and $H \triangleleft L$. If $H \triangleleft L$, then $H \triangleleft L$ if either of the following conditions holds:

(i) $H$ has finite codimension.

(ii) The characteristic of $F$ is zero and $A$ is finite-dimensional.

Proof

(i) For any $h \in H$, $\text{ad}_L h$ induces a linear transformation $\alpha(h)$ of the space $L/H$. By assumption each $\alpha(h)$ is nil. But the space $L/H$ is finite-dimensional, hence $\alpha(h)$ is nilpotent. Therefore by Lemma 5.2.6, $\alpha(H)$ is nilpotent. Hence there exists an integer $m$ such that $\alpha(h_1) \ldots \alpha(h_m) = 0$ for any $h_1, h_2, \ldots, h_m \in H$. Thus $[L, mH] \subseteq H$. Now since $A \cap H \triangleleft L$, we may assume that $A \cap H = 0$. It follows that $[A, mL] = [A, mH] = 0$. Therefore $A \subseteq \mathfrak{z}_m(L)$ and $H \triangleleft mL$.

(ii) This follows from Theorem 5.2.1 applied to $L/C_H(A)$.

In the rest of this section we shall find certain conditions under which $H \vartriangleleft L$ (where $\vartriangleleft$ is one of the relations $\triangleleft, \triangleleft^*, \triangleright$) implies that $H \triangleleft L$ or $H \triangleleft^* L$.

We start with the following.
Theorem 5.3.2

Let $L$ be a hyperabelian Lie algebra over any field, and let $H < L.$

(i) If $H \text{nci } L$, then $H \text{asc } L$

(ii) If $H$ has finite dimension, then $H \text{sci } L$

implies $H \text{si } L$ while $H \text{ci } L$ implies $H \text{asc } L$.

To prove this we need the following results:

Lemma 5.3.3

Let $L$ be a Lie algebra of algebraic linear transformations of a vector space $V$ over a field $F$, and let $U(L)$ be the universal enveloping algebra of $L$. Then $L$ is locally finite if and only if $U(L)$ is locally finite.

Proof

See Curtis [9, Lemma 3.1, p. 456].

Lemma 5.3.4

Let $L$ be a Lie algebra over any field and let $H < L.$ Then if $x \in H$ implies $\langle x \rangle \text{ si } L$, then every finitely generated subalgebra of $H$ is a subideal of $L$.

Proof

See Amayo and Stewart [5, Theorem 7.1.5(c), p. 136].
Let $L$ be a Lie algebra over any field such that $L = A + H$, where $A$ is an abelian ideal of $L$ and $H < L$. Then

(i) If $H \nsubseteq L$, then $H \ast L$.

(ii) If $H$ has finite dimension, then

(a) $H \subseteq L$ implies $H \subseteq L$

(b) $H \subseteq L$ implies $H \ast L$.

Proof

(i) Since $A \cap H \neq L$, we may assume that $A \cap H = 0$. Let $x_1, x_2, \ldots, x_n$ be any elements of $L$. Then $x_i = a_i + h_i$, for some $a_i \in A$ and $h_i \in H$. Now let $a \in A$, then $[a, x_i] = [a, h_i]$ for $A$ is abelian. Hence $[a, x_1, x_2, \ldots, x_n] = [a, h_1, h_2, \ldots, h_n]$. But $[a, h_1, h_2, \ldots, h_n] \in A \cap H = 0$, hence $[a, x_1, x_2, \ldots, x_n] = 0$. Therefore $a \in \zeta_n(L)$. It follows that $A \subseteq \zeta_n(L)$. Hence $H \ast L$.

(ii) (a) Since $A \cap H \neq L$, we may assume that $A \cap H = 0$. It follows that for each $h \in H$ there exists $n = n(h)$ such that for all $a \in A$ we have $[a, n(h)] = 0$. Consider $A$ as an $H$-module, then $A$ is an algebraic $H$-module (because $ad_A h$ is algebraic). Let $E$ be the associative algebra generated by all $\{ad_A h| h \in H\} \cup \{1_A\}$. Then Lemma 5.3.3 implies that $E$ is finite-dimensional. Now if $a \in A$, then $<a>_H = aE = \{ae|e \in E\}$.
where \( <a>_H \) is the \( H \)-module generated by the element \( a \). Hence \( A \) is a locally finite \( H \)-module. Now if \( B \) is any finite-dimensional \( H \)-submodule of \( A \), then for each \( h \in H \) there exists \( n = n(h) \) such that \( B h^n = 0 \). Hence by Lemma 5.2.6, we can find \( m = m(B) \) such that \( B h^m = 0 \). It follows that \( A \leq \zeta_\omega(A + H) \), and \( H^\omega \leq L \). So we may assume that \( H^\omega = 0 \) and \( H \) is nilpotent. By assumption we have \( h \in \bar{\mathfrak{e}}(L) \) for each \( h \in H \). Therefore \( <h> \leq L \). Hence by Lemma 5.3.4, \( H \leq L \).

(b) Since \( A \cap H \leq L \), we may assume that \( A \cap H = 0 \). It follows that for each \( a \in A \) and \( h \in H \), there exists \( n = n(x,h) \) such that \( [a, h]^n = 0 \). Consider \( A \) as an \( H \)-module, then \( A \) is locally algebraic and the argument of Lemma 5.3.3, implies that \( A \) is locally finite. Now if \( B \) is any finite-dimensional \( H \)-submodule of \( A \), then there exists \( n = (B, h) \) such that \( B h^n = 0 \). Hence by Lemma 5.2.6 we can find \( m = m(B) \) such that \( B h^m = 0 \). It follows that \( A \leq \zeta_\omega(A + H) \) and \( H \leq L \).

Proof of Theorem 5.3.2

(i) Let \( (L_{\alpha})_{\alpha<\lambda} \) be an ascending abelian series of ideals of \( L \). Now \( (H + L_{\alpha+1})/L_{\alpha} = (H + L_{\alpha})/L_{\alpha} + L_{\alpha+1}/L_{\alpha} \) and \( L_{\alpha+1}/L_{\alpha} \) is an abelian ideal of \( L/L_{\alpha} \). Therefore

\[
L_{\alpha+1}/L_{\alpha} \leq (H + L_{\alpha+1})/L_{\alpha}.
\]

But by Lemma 5.1.2, \( (H + L_{\alpha})/L_{\alpha} \) nci \( (H + L_{\alpha+1})/L_{\alpha} \), hence by Lemma 5.3.5 (i), \( (H + L_{\alpha})/L_{\alpha} \leq (H + L_{\alpha+1})/L_{\alpha} \).
Therefore $H + L_\alpha \text{ asc } H + L_{\alpha+1}$. This is true for all $\alpha < \lambda$.

So $H = H + L_0 \text{ asc } H + L_\lambda = L$.

(ii) This can be proved exactly the same way as in (i) using Lemma 5.3.5 (ii).

Next we prove the following.

**Theorem 5.3.6**

Let $L$ be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H < L$. Then

(i) If $H \nsubseteq L$, then $H \text{ asc } L$.

(ii) If $H$ has finite dimension, the $H \subseteq L$ implies $H \text{ si } L$, while $H \subseteq L$ implies $H \text{ asc } L$.

**Proof**

(i) Let $S$ be a soluble ideal of $L$ such that $L/S$ is finite-dimensional. We prove that $H \text{ asc } L$ by induction on the derived length $m$ of $S$. If $m = 0$, then $L$ is finite-dimensional and by Theorem 5.2.1, $H \text{ si } L$. We assume therefore that $m \geq 1$. Let $A$ be the last non-zero term of the derived series of $S$. Then $A$ is an abelian ideal of $L$ and $S/A$ has derived length $m-1$. Since $(H + A)/A \nsubseteq L/A$ it follows by induction that $(H + A)/A \text{ asc } L/A$. Therefore $H + A \text{ asc } L$ and by Lemma 5.3.5 (i) $H \text{ asc } H + A$. Hence $H \text{ asc } L$.

(ii) This can be proved exactly the same way as in (i) using Lemma 5.3.5 (ii).
Finally we prove the following which generalizes Stitzinger [28, Theorem 3].

**Theorem 5.3.7**

Let $L$ be an ideally finite Lie algebra over a field of characteristic zero and let $H \vartriangleleft L$. Then $H$ asc $L$ if either of the following conditions holds:

(i) $H \triangleleft L$

(ii) $H$ asc $\langle H, x \rangle$ for each $x \in L$.

The proof follows after the following lemmas.

**Lemma 5.3.8**

Over a field of characteristic zero, let $X$ be an $S$-closed Fitting class and let $L \in \mathcal{LF}$. Then $p_{\mathcal{LX}}(L)$ contains every serial $\mathcal{LX}$-subalgebra of $L$.

**Proof**

See Amayo and Stewart [5, Theorem 13.3.7, p. 261]. □

**Lemma 5.3.9**

A locally nilpotent ideally finite Lie algebra is hypercentral of height $< \omega$.

**Proof**

See Stewart [26, Theorem 3.6, p. 40]. □
Lemma 5.3.10
Let $L$ be an ideally finite Lie algebra over any field and let $H < L$. Then

(i) $H \cap L$ implies $H^\omega < L$

(ii) $H$ asc $<H,x>$ for each $x \in L$ implies $H^\omega < L$.

Proof
(i) By hypothesis we have $L = U \alpha$, where $I_\alpha$ is a finite-dimensional ideal of $L$. For each $n \in \mathbb{N}$, let $H_n = \{x \in L | [x, H] \subseteq H\}$. We claim that $L = UH_n$. Suppose not. Then $I_\alpha \not\subseteq H_n$ for all $n \in \mathbb{N}$. Since $H$ idealises $I_\alpha$ and $H_n$, it follows that $(I_\alpha + H_n)/H_n$ is a non-zero finite-dimensional $H$-module. By assumption $ad_h$ induces a nilpotent transformation on $(I_\alpha + H_n)/H_n$ for each $h \in H$. Hence by Lemma 5.2.6, there exists $y \in I_\alpha \setminus H_n$ such that $[y, H] \subseteq H_n$. Therefore $y \in H_{n+1}$ and $I_\alpha \cap H_n \subseteq I_\alpha \cap H_{n+1}$ for all $n \in \mathbb{N}$. This is a contradiction since $I_\alpha$ is finite-dimensional. Hence $L = UH_n$.

Now let $x \in L$. Then $x \in H_n$ for some $n \in \mathbb{N}$ and so $[x, nH] \subseteq H$. Hence for any $m > 0$, $[x, H^\omega] \subseteq [x, H_n^{m-1}] \subseteq H^m$. It follows that $[x, H^\omega] \subseteq nH^m = H^\omega$ and $H^\omega < L$.

(ii) Since $H$ asc $<H,x>$, then $H <^\omega <H,x>$ for $L$ is ideally finite. Let $H = H_0 < H_1 < \ldots < H_\omega = <H,x>$, where $H_\omega = U H_i$. Let $y \in L$. Then $y \in H_n$ for some $n$ and $H$ idealises $H_n$. Since $H^\omega < H_n$, $[y, H^\omega] \subseteq H^\omega$ and $H^\omega < L$. □
Let $L$ be a locally finite Lie algebra over any field and let $H < L$. Then $H$ ser $L$ if and only if $H \cap K$ si $K$ for every finite-dimensional subalgebra $K$ of $L$.

**Proof**

See Amayo and Stewart [5, Proposition 13.2.4, p. 258].

**Proof of Theorem 5.3.7**

(i) Let $K$ be any finite-dimensional subalgebra of $L$. Then $H \cap K$ ci $K$ by Lemma 5.1.2 (i). By Theorem 5.2.1, we have $H \cap K$ si $K$, hence by Lemma 5.3.11, $H$ ser $L$. By Lemma 5.3.10 (i), $H^\omega < L$, so we may assume that $H^\omega = 0$ and $H$ is locally nilpotent. Therefore $H \leq \rho(L)$ by Lemma 5.3.8. But by Lemma 5.3.9, $\rho(L)$ is hypercentral, hence $H$ asc $\rho(L) < L$. Therefore $H$ asc $L$.

(ii) Let $K$ be any finite-dimensional subalgebra of $L$, and let $x \in K$. Then $H$ asc $\langle H, x \rangle$, so $H \cap K$ asc $\langle H, x \rangle \cap K = \langle H \cap K, x \rangle$. So $H \cap K$ asc $\langle H \cap K, x \rangle$. The latter has finite dimension, so $H \cap K$ si $\langle H \cap K, x \rangle$ for all $x \in K$.

By Corollary 5.2.7, we have, $H \cap K$ si $K$, hence by Lemma 5.3.11, $H$ ser $L$. By Lemma 5.3.10 (ii), $H^\omega < L$, hence we may assume that $H^\omega = 0$ and as in part (i) we deduce that $H$ asc $L$. □
CHAPTER SIX : SUBIDEALS OF THE JOIN OF PERMUTABLE LIE ALGEBRAS

Wielandt [37] has shown that a common subnormal subgroup of two permutable subgroups of a finite group is subnormal in their product. Following [2,3], we shall obtain a similar result for Lie algebras. In particular we prove an analogue of Wielandt's theorem for finite-dimensional Lie algebras over a field of characteristic zero. Chao and Stitzinger [8] proves a similar result for finite-dimensional soluble Lie algebras in arbitrary characteristics without the permutability assumption. For insoluble Lie algebras some such hypothesis is necessary, as we show by examples. We also obtain some analogues of theorems of Wielandt [36].

Finally we extend our results to certain classes of infinite-dimensional Lie algebras. Our main results are as follows:

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $A, H, K$ be subalgebras of $L$ such that $L = H + K$ and $A \subseteq H, A \subseteq K$. Then $A$ si $L$ if and only if $A$ si $H$ and $A$ si $K$ (Theorem 6.2.1).

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = \langle H_1, H_2, H_3 \rangle$. If $H_i$ si $\langle H_i, H_j \rangle$ for all $i, j = 1, 2, 3$ and $\langle H_1, H_2 \rangle$ is permutable with $H_3$, then $H_i$ si $L$ for all $i$ (Theorem 6.2.9). A generalization to infinite-dimensional Lie algebras leads to the following:
Let \( L \) be a Lie algebra over a field of characteristic zero and let \( A, H, K \) be subalgebras of \( L \) such that \( L = H + K \) and \( A \subseteq H, A \subseteq K \). Then

(a) If \( L \) is soluble-by-finite and \( A \subseteq H, A \subseteq K \), then \( A \subseteq L \).

(b) If \( L \) is ideally finite and \( A \subseteq H, A \subseteq K \), then \( A \subseteq L \) (Theorems 6.3.1 and 6.3.8).

Let \( L \) be a Lie algebra over any field and let \( H_1, H_2, H_3 \) be subalgebras of \( L \) such that \( L = \langle H_1, H_2, H_3 \rangle \). If \( [H_1, H_2] \subseteq H_1 \) and if \( H_i \subseteq \langle H_i, H_j \rangle = H_i + H_j \) for all \( i, j = 1, 2, 3 \), then \( H_i \subseteq L \) for all \( i \) (Theorem 6.3.5).

Finally if \( L \) is an ideally finite Lie algebra over a field of characteristic zero and if \( H_1, H_2, H_3 \) are subalgebras of \( L \) such that \( L = \langle H_1, H_2, H_3 \rangle, H_i \subseteq \langle H_i, H_j \rangle \) for all \( i, j = 1, 2, 3 \) and \( \langle H_1, H_2 \rangle \) is permutable with \( H_3 \), then \( H_i \subseteq L \) for all \( i \) (Theorem 6.3.10).

1. Preliminaries

Let \( L \) be a Lie algebra over any field. We recall that the Fitting radical \( \varpi(L) \) is the sum of nilpotent ideals of \( L \) (equal to the nil radical in finite dimensions). If \( L \) has finite dimension and the ground field has characteristic zero, then \( \varpi(L) \) contains every nilpotent subideal of \( L \) (see Amayo and Stewart [5, p. 114]). Let \( H \leq L \), then we write \( H^L \) to
denote the smallest ideal of $L$ which contains $H$ and is called the *ideal closure* of $H$ in $L$. Two subalgebras $H$ and $K$ of a Lie algebra $L$ are said to be *permutable* (see [5, p. 33]) if and only if $[H,K] \subseteq H + K$. If this is so, then $\langle H,K \rangle = H + K$.

Let $H$, $K$ be subalgebras of a Lie algebra $L$. We say that $H$ and $K$ are *cosubideal* if each of them is subideal in their join.

### 6.2 Finite dimensions

**Theorem 6.2.1**

Let $L$ be a finite-dimensional Lie algebra over a field $F$ of characteristic zero and let $A$, $H$, $K$ be subalgebras of $L$ such that $L = H + K$ and $A \subseteq H$, $A \subseteq K$. Then $A$ is $L$ if and only if $A$ is $H$ and $A$ is $K$.

To prove this we need the following well-known results:

**Lemma 6.2.2**

Let $L$ be a finite-dimensional soluble Lie algebra over any field and let $A$, $H$, $K$ be subalgebras of $L$. If $A$ is $H$ and $A$ is $K$, then $A$ is $\langle H,K \rangle$.

**Proof**

See Chao and Stitzinger [8, Theorem 6].
We also need a result due to Dynkin [10], before we state it we recall the following definition:

Let \( L \) be a finite-dimensional semi-simple Lie algebra over a field of characteristic zero. A subalgebra \( H \) of \( L \) is said to be regular (see Dynkin [10, p.142]) if there exists a basis consisting of elements of some Cartan subalgebra \( C \) of the algebra \( L \) and root vectors of the algebra \( L \) relative to \( C \).

Lemma 6.2.3

Let \( L \) be a finite-dimensional semi-simple Lie algebra over algebraically closed field of characteristic zero and let \( \Phi \) be a root system for \( L \) relative to a Cartan subalgebra \( C \). Let \( \pi \) be a system of simple roots, \( \alpha \in \pi \) and \( \pi_1 = \pi \setminus \{\alpha\} \). Let \( \delta \) be the smallest root of \( L \) and let

\[
L(\alpha) = \left< e_\delta, e_{-\delta}, e_\beta, e_{-\beta} \ (\beta \in \pi_1) \right>.
\]

Let

\[
L[\alpha] = \left< c_\alpha, e_\alpha, e_\beta, e_{-\beta} (\beta \in \pi_1) \right>.
\]

Then

(i) Every subalgebra \( L(\alpha) \) except \( L \) is a semi-simple regular maximal subalgebra of \( L \).

(ii) Every subalgebra \( L[\alpha] \) except \( L \) is a non semi-simple regular maximal subalgebra of \( L \).

(iii) Every regular maximal subalgebra is conjugate to one of these subalgebras.
Proof

See Dynkin [10, Theorem 5.5, p. 143]. □

Next the following.

Lemma 6.2.4

Let $\mathcal{L}$ be a finite-dimensional simple Lie algebra over an algebraically closed field of characteristic zero. Then every non-regular maximal subalgebra is semi-simple.

This result is due to Morozov and it is noted in Dynkin [10, p. 213].

Definition

The height of a root $\beta$, which is denoted by $ht\beta$, is defined to be the sum of all the coefficients in the expression of $\beta$ as a linear combination of simple roots.

That is if $\beta = \sum_{\alpha \in \pi} k_{\alpha} \alpha$ then $ht\beta = \sum_{\alpha \in \pi} k_{\alpha}$. See Humphreys [15, p. 47].

Proof of Theorem 6.2.1

If $A \subset \mathcal{L}$, then it is clear that $A \subset H$ and $A \subset K$.

The converse comes in several stages.

(1) We may assume that the field $F$ is algebraically closed. For if not let $\bar{F}$ be the algebraic closure of $F$.

Then $L = H + K$ implies $\bar{L} = \bar{H} + \bar{K}$, where $\bar{L} = L \otimes_F \bar{F}$, $\bar{H} = H \otimes_F \bar{F}$ and $\bar{K} = K \otimes_F \bar{F}$, $\bar{A} = A \otimes_F \bar{F}$. Further, assuming the theorem
in the algebraically closed case, we have $\bar{A} \subseteq \bar{L}$. Hence $A = \bar{A} \cap L \subseteq \bar{L}$.

(ii) Assume that $L$ is a counter-example of minimal dimension and assume that the choice of $A$, $H$, $K$ has been made in such a way that $H$ then has maximal dimension. Since $A^\omega \subseteq H$ and $A^\omega \subseteq K$, it follows that $A^\omega \subseteq L$. If $A^\omega \neq 0$, then by minimality $A/A^\omega \subseteq L/A^\omega$ and $A \subseteq L$, a contradiction. Therefore $A^\omega = 0$ and $A$ is nilpotent. It follows that $a \in A$ implies $\langle a \rangle \subseteq A$, by Lemma 5.2.2. If the theorem were true for the case $\dim A = 1$, it would follow that $\langle a \rangle \subseteq L$ for all $a \in A$, hence $A < \mathcal{V}(L)$. Therefore $A$ would be a subideal. It follows that we may assume that $\dim A = 1$, so that $A = \langle a \rangle$ for some $a \in L$.

(iii) Let $S = \sigma(L)$. If $S = L$, then $A \subseteq L$ by Lemma 6.2.2, which is again a contradiction. Hence $S \neq L$. It follows that $S + A \neq L$. If $S \neq 0$, then $(S + A)/S \subseteq L/S$ by minimality, but $\dim (S + A)/S \leq 1$ and $L/S$ is semi-simple. Therefore $A < S$. Since $\text{ad}(a)$ is nilpotent on $H$ and on $K$, it acts nilpotently on $S$, so $\langle a \rangle \subseteq S \subseteq L$ which again is a contradiction. Therefore $S = 0$ and $L$ is semi-simple.

(iv) We may assume that $L$ is simple. For if not, then $L = L_1 \oplus L_2$ where $L_1$ and $L_2$ are ideals of $L$ and $\dim L_1 > 0$, $\dim L_2 > 0$. Now $(A + L_1)/L_1 \subseteq L/L_1$ by minimality, and since $\dim (A + L_1)/L_1 \leq 1$ we must have $A \subseteq L_1$. Similarly $A \subseteq L_2$, so $A \subseteq L_1 \cap L_2 = 0$, contrary to
hypothesis.

(v) H must be a maximal subalgebra of L. If not, let B be a maximal subalgebra of L with H < B. Then $B = B \cap (H + K) = H + (B \cap K)$. Since $A \leq H$ and $A \leq B \cap K$, it follows that $A \leq B$ by minimality. Now $L = B + K$ and therefore the maximal choice of the dimension of $H$ yields $H = B$.

(vi) We now have an algebraically closed field, and a simple Lie algebra $L = H + K$, where $H$ is a maximal subalgebra of $L$, $A = \langle a \rangle$ and $a \in \nu(H) \cap \nu(K)$. We claim that $a = 0$. We appeal to the classification of maximal subalgebras by Dynkin [10]. Clearly $H$ cannot be semi-simple. Since maximal subalgebras of simple Lie algebras are regular or non-regular and since a non-regular maximal subalgebra of a simple Lie algebra is semi-simple by Lemma 6.2.4, we may assume that $H$ is a non-semi-simple regular maximal subalgebra of $L$. Now by Lemma 6.2.3, $H$ has a basis consisting of the elements $e_a (\alpha \in \Phi^-)$, $c_{\beta} (\beta \in \Pi)$ and $e_\gamma (\gamma \in \Phi^+)$ where $\Phi^-_1$ denotes the negative roots in $\Phi_1 = \Phi \cap \mathbb{R}^\times_1$ and $\Phi^+$ denotes the positive roots in $\Phi$. Also $\nu(H)$ has basis $\{e_\delta | \delta \in \Phi^+_2\}$ where $\Phi^+_2$ denotes the positive roots in $\Phi_2 = \Phi \setminus \Phi_1$. Let $L^+$ be the subalgebra of $H$ spanned by $e_\gamma (\gamma \in \Phi^+)$, and let $J_n$ be the subspace of $L$ spanned by the $e_\alpha$ with $\alpha \in \Phi^-$ such that $ht(\alpha) \leq n$ where $\Phi^-$ denotes the negative roots in $\Phi$. We see that $[J_n, L^+] < J_{n-1}$ and we have, for some $n$,
H = J_0 \leq J_1 \leq \ldots \leq J_n = L$. These are submodules for 
C + L^+ under the adjoint representation. We have 
0 \neq a \in \mathfrak{v}(H) \cap \mathfrak{v}(K)$, so we can write 
a = A_\delta e_\delta + \Sigma \gamma e_\gamma$ with 
\delta \in \Phi_2^+ and $A_\delta \neq 0$, where the sum runs over elements 
\gamma \in \Phi_2^+ such that $ht \gamma \geq ht \delta$. As $L = H + K$ we have 
x = e_{-\delta} + h \in K$ for some $h \in H$, and so \mathfrak{v}(K) contains 
u = [a,x] = A_\delta e_\delta + \Sigma \gamma [e_{-\delta}, e_\gamma] + [a,h].$

Here, $\Sigma \gamma [e_{-\delta}, e_\gamma] + [a,h] \in L^+$. Now $ad u$ acts nil-
potently on the space $L/H = (H + K)/H$. But $e_{-\delta} \notin H$, so 
e_{-\delta} \in J_{i+1} \setminus J_i$ for some $i$. Since $L^+$ annihilates $J_{i+1} \setminus J_i$, we 
have $[e_{-\delta}, u] \equiv [e_{-\delta}, A_\delta c_\delta] \equiv 2A_\delta e_{-\delta} \mod J_i$. So 
e_{-\delta}(ad u)^T \equiv (2A_\delta)^T e_{-\delta} \mod J_i$, a contradiction, as this 
can never belong to $J_i$. Hence $a = 0$ and $A \not\subseteq L$ which is a 
contradiction and the argument is proved.

**Corollary 6.2.5**

Let $L$ be a finite-dimensional Lie algebra over a 
field of characteristic zero and let $A, H, K$ be subalgebras 
of $L$ such that $L = H + K$. Then $A \subseteq L$ if and only if $A \subseteq A^H$ 
and $A \subseteq A^K$.

**Proof**

If $A \subseteq L$, then clearly $A \subseteq A^H$ and $A \subseteq A^K$.

Conversely since $A^H \leq \langle A, H \rangle = H_1$, it follows that 
$A \subseteq H_1$. Similarly $A \subseteq \langle A, K \rangle = K_1$ and by Theorem 6.2.1, 
$A \subseteq H_1 + K_1 = L$.  

\(\Box\)
Corollary 6.2.6

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $A, H_i$ $(i = 1, \ldots, n)$ be subalgebras of $L$ such that $L = H_1 + H_2 + \ldots + H_n$ and $A \subseteq H_i$, $\langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, \ldots, n$. Then $A \subseteq L$ if and only if $A \subseteq H_i$ for all $i$. 

Proof

The proof is easily done by induction on $n$. □

Now we have the following which is the Lie algebra analogue of Wielandt [36, Hilfsatz 2.2].

Corollary 6.2.7

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H_i \subset L$ be such that $H_i \subseteq \langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, 2, \ldots, n$. Then $H_i \subseteq L = \langle H_1, H_2, \ldots, H_n \rangle$ for each $i$. □

Remark

If $L$ is a finite-dimensional Lie algebra over a field of characteristic zero, $H, K$ are subalgebras of $L$ such that $L = \langle H, K \rangle$. If $A \subseteq H$ and $A \subseteq K$, then $A$ need not be a subideal of $L$, as the following example shows.
Let $L$ be the simple Lie algebra of type $A_2$. If $(\alpha, \beta)$ is a system of simple roots (in the terminology of Jacobson [16, pp. 110, 120]), and $H = \langle e_\alpha, e_\beta, e_{\alpha+\beta} \rangle$, $K = \langle e_\alpha, e_{-\beta}, e_{\alpha-\beta} \rangle$, $A = \langle e_\alpha \rangle$ then $A \triangleleft H$ and $A \triangleleft K$, (see Stewart [27]). But $A$ is not a subideal of $L = \langle H, K \rangle$.

Also if $H_1 = \langle e_\alpha \rangle$, $H_2 = \langle e_\beta \rangle$ and $H_3 = \langle e_{-\alpha-\beta} \rangle$, then $A_2 = \langle H_1, H_2, H_3 \rangle$ and it can be checked that $H_1, H_2, H_3$ are pairwise cosubideals. But $A_2$ is simple, so $H_i$ cannot be subideal for any $i$. This example shows that Corollary 6.2.7 is not true without the permutability assumption, however we have the following.

**Proposition 6.2.8**

Let $L$ be a finite-dimensional soluble Lie algebra over any field. Let $A < L$ and let $H_i$ ($i = 1, \ldots, n$) be subalgebras of $L$ with $A < H_i$, and suppose that $L = \langle H_1, H_2, \ldots, H_n \rangle$.

(i) If $A \triangleleft H_i$ for all $i$, then $A \triangleleft L$.

(ii) If $A \triangleleft H_i$ for all $i$, then $A \triangleleft L$.

(iii) If $H_i \triangleleft \langle H_1, H_2 \rangle$ for all $i, j = 1, 2, \ldots, n$, then $H_i \triangleleft L$ for all $i$.

**Proof**

(i) follows from Lemma 6.2.2, and (iii) follows from (i). To prove (ii) we have $A \triangleleft H_i$ for all $i$, but
\[ A^H_1 \subset \langle A, H_1 \rangle \], hence \( A \subset \langle A, H_1 \rangle \). Therefore by (i)
\( A \subset L \). \( \Box \)

**Remark**

Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic \( p > 0 \). Clearly the proof of Theorem 6.2.1 fails in this case. However, for soluble Lie algebras the result remains true by Lemma 6.2.2. The problem remains: is the result true in the insoluble case? It would seem reasonable to seek first a counter-example with \( L \) simple (even though the above reduction argument to this case cannot be justified in characteristic \( P \)). We investigated a variety of simple algebras in characteristic \( P \). Most of the well-known simple Lie algebras do not yield such an example (at least with \( H \) maximal as one might hope possible) as may be established by a lengthy case-by-case analysis. There is, however, a counterexample using a less well-known algebra that exists only in characteristic \( P = 3 \), as follows.

Let \( F \) be a field of characteristic 3. Consider the Jacobson-Witt algebra \( W_3 \) over \( F \). Then \( W_3 \) is spanned (see Frank [11]) by derivations: \( A = (a_1, a_2, a_3) = a_1 \Delta_1 + a_2 \Delta_2 + a_3 \Delta_3 \), where \( a_1 \in F[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3) = P \) say, and \( \Delta_i \) is the derivation of \( P \) defined by \( \Delta_i x_j = \delta_{ij} \). If \( B = (b_1, b_2, b_3) \)
\[ A \mathfrak{h}_1 \leq \langle A, H_1 \rangle, \text{ hence } A \mathfrak{h}_1 \leq \langle A, H_1 \rangle. \] Therefore by (i)
\[ A \mathfrak{h}_1 \leq L. \]

\textbf{Remark}

Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic \( p > 0 \). Clearly the proof of Theorem 6.2.1 fails in this case. However for \textit{soluble} Lie algebras the result remains true by Lemma 6.2.2. The problem remains: is the result true in the insoluble case? It would seem reasonable to seek first a counter-example with \( L \) simple (even though the above reduction argument to this case cannot be justified in characteristic \( P \)). We investigated a variety of simple algebras in characteristic \( P \). Most of the well-known simple Lie algebras do not yield such an example (at least with \( H \) maximal as one might hope possible) as may be established by a lengthy case-by-case analysis. There is, however, a counterexample using a less well-known algebra that exists only in characteristic \( P = 3 \), as follows.

Let \( F \) be a field of characteristic \( 3 \). Consider the Jacobson-Witt algebra \( W_3 \) over \( F \). Then \( W_3 \) is spanned (see Frank [11]) by derivations: \[ A = (a_1, a_2, a_3) = a_1 \Delta_1 + a_2 \Delta_2 + a_3 \Delta_3, \]
where \( a_i \in F[x_1, x_2, x_3]/(x_1^3, x_2^3, x_3^3) = P \) say, and \( \Delta_i \) is the derivation of \( P \) defined by \( \Delta_i x_j = \delta_{ij} \). If \( B = (b_1, b_2, b_3) \)
multiplication in $W_3$ is given by $[A,B] = C = (c_1, c_2, c_3)$, where

$$c_i = \sum_j (\Delta_j a_i)b_j - (\Delta_j b_i)a_j.$$ 

Let $L$ be the subalgebra of $W_3$ generated by the derivations $A_1, A_2, A_3, A_4, B_1, B_2, B_3, \Delta_1, \Delta_2, \Delta_3$, where $A_1 = (x_1, x_2, x_3)$, $A_2 = (0, x_2, -x_3)$, $A_3 = (x_2, x_3, 0)$, $A_4 = (0, x_1, -x_2)$, $B_1 = (x_1x_2, x_1x_3, -x_2x_3)$, $B_2 = (x_1^2, x_1x_2, x_2^2)$, $B_3 = (-x_2^2, x_2x_3, x_3^2)$. Multiplication of $A_1, B_1, \Delta_1$ is given by the following:
<table>
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<tr>
<th></th>
<th>$A_1$</th>
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<td>-A_3</td>
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</table>
Clearly \( \dim L = 10 \). It is shown in Frank [11] that \( L \) is simple. We use this to construct our counter-example.

Let \( H = \langle A_1, A_2, A_3, A_4, B_1, B_2, B_3 \rangle \), \( K = \langle A_3, \Delta_1, \Delta_2, \Delta_3 \rangle \) and \( A = \langle A_3 \rangle \). Clearly \( L = H + K \), and

\[
A < \langle A_3, B_3 \rangle < \langle A_3, B_3, B_1 \rangle < \langle A_1, A_3, B_1, B_2, B_3 \rangle < H,
\]

\[
A < \langle A_3, A_1 \rangle < \langle A_3, A_1, A_2 \rangle < K,
\]

but \( A \) is not a subideal of \( L \) since \( L \) is simple.

Therefore Theorem 6.2.1 does not hold in characteristic 3.

Next we prove the following which are Lie algebra analogues of Wielandt [36].

**Theorem 6.2.9**

Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic zero and let \( H_1, H_2, H_3 \) be subalgebras of \( L \) such that \( L = \langle H_1, H_2, H_3 \rangle \) and suppose that \( H_i \) si \( \langle H_i, H_j \rangle \) for all \( i, j = 1, 2, 3 \). If \( \langle H_1, H_2 \rangle \) is permutable with \( H_3 \), then \( H_i \) si \( L \) for all \( i \).

The proof will be given after the following lemmas.

**Lemma 6.2.10**

Let \( A, B, C \) be subspaces of \( L \) and let \( A \) be permutable with \( B \) and \( C \). Then \( A \) is permutable with \( \langle B, C \rangle \).

**Proof**

It is enough to show that \( [A, B \circ C] \subseteq A + \langle B, C \rangle \).
Let $a \in A$, $x \in B \circ C$. Then $x = [x_1, x_2, x_3, \ldots, x_n]$, where $x_1 \in B$, $x_2 \in C$ and $x_3, \ldots, x_n \in B \cup C$. By the Jacobi identity and by induction on $n$,

$$[[x_1, \ldots, x_n], a] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, a], x_{i+1}, \ldots, x_n].$$

Since $A$ permutes with $B$ and $C$, it follows that for each $i = 1, \ldots, n, [x_i, a] = x_i' + a_i$ according as $x_i \in B$ or $C$. Therefore

$$[[x_1, \ldots, x_n], a] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n] + \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n].$$

Clearly $\sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n] \in <B, C>$. Also by the Jacobi identity and by induction on $n$ and the permutability of $A$ with $B$, $C$, it follows that:

$$\sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n] \subseteq A + <B, C>.$$

Hence $[A, B \circ C] \subseteq A + <B, C>$. □

**Lemma 6.2.11**

Let $L$ be a finite-dimensional Lie algebra over a field of characteristic zero and let $H$, $K$ be subalgebras of $L$ such that $L = H + K$. If $H$, $K$ are nilpotent, then $L$ is soluble.
This follows from Kostrikin [19]. □

Lemma 6.2.12

Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic zero and let \( H_i \triangleleft L \) be such that
\[
L = \langle H_i | i = 1, \ldots, n \rangle.
\]
If \( H_i \triangleleft \langle H_i, H_j \rangle \) for all \( i, j = 1, \ldots, n \), then \( H_i \triangleleft H_i + N \) for all \( i \), where
\[
N = H_1^\omega + H_2^\omega + \ldots + H_n^\omega \triangleleft L.
\]

Proof

Since \( H_i \triangleleft \langle H_i, H_j \rangle \), it follows that \( H_i^\omega \triangleleft \langle H_i, H_j \rangle \). Therefore \( H_i^\omega \triangleleft L \) and
\[
H_1^\omega + \ldots + H_n^\omega = N \triangleleft L.
\]
Now the subalgebras \( H_1^\omega, H_2^\omega, \ldots, H_n^\omega \) satisfies the hypothesis of Corollary 6.2.7. Hence \( H_i \triangleleft \langle H_1, H_2^\omega, \ldots, H_n^\omega \rangle = H_1 + N \). The same is true for \( H_2, \ldots, H_n \). □

Lemma 6.2.13

Over any field of characteristic zero the class \( \mathcal{F} \cap \mathcal{N} \) is coalescent and ascendantly coalescent.

Proof

See Amayo and Stewart [5, Theorem 2.4, p. 62]. □

Proof of Theorem 6.2.9

That \( H_1, H_2 \) are subideals of \( L \) follows from Lemma 6.2.10 and Theorem 6.2.1. To prove that \( H_3 \triangleleft L \) we argue for a
contradiction, assuming \( L \) to be a counter-example of minimal dimension. Consider the subalgebras \( H_3, H_1, H_2 \). It is clear that all pairwise permutable cosubideals and

\[ H_3 = H_3 + N \] by Lemma 6.2.12, where \( N = H_1^{(1)} + H_2^{(1)} + H_3^{(1)} \). If \( N \neq 0 \), then by minimality \( (H_3 + N)/N \) and \( L/N \) which implies that \( H_3 + N \), \( L \). It follows that \( H_3 \), \( L \) which is a contradiction. Hence \( N = 0 \) and all \( H_i \) are nilpotent. By Lemma 6.2.13, \( \langle H_1, H_2 \rangle \) is a nilpotent subideal of \( L \). Therefore by Lemma 6.2.11, \( L = \langle H_1, H_2 \rangle + H_3 \) is soluble and by Lemma 6.2.2, \( H_3 \), \( L \) which is a contradiction and the theorem is proved.

**Remark**

Let \( L \) be a finite-dimensional simple Lie algebra over any field and let \( H_i < L \) such that \( L = \langle H_i \mid i = 1, \ldots, n \rangle \). Suppose that \( H_i \) is \( \langle H_i, H_j \rangle \) for all \( i, j = 1, \ldots, n \). Then either \( H_i = L \) or \( H_i \) nilpotent.

Finally we prove the following which is the algebra analogue of Wielandt [36].

**Theorem 6.2.14**

Let \( L \) be a finite-dimensional Lie algebra over a field of characteristic zero and let \( H_i \) be subalgebras of \( L \) such that \( L = \langle H_i \mid i = 1, \ldots, n \rangle \). Suppose that \( H_i \) is \( \langle H_i, H_j \rangle \) for all \( i, j = 1, \ldots, n \). If \( H_i+1 \) is permutable with
$H_1 + H_2 + \ldots + H_i$ for all $i$, then $H_i \subseteq L$.

**Proof**

That $H_1$ and $H_2$ are subideals of $L$ follows from Lemma 6.2.10 and Theorem 6.2.1. To prove that $H_3 \subseteq L$ we argue for a contradiction assuming $L$ to be a counter-example of minimal dimension, so that $H_3$ is not a subideal of $L$. Clearly $H_3 \subseteq L$. If $H_3 \not= 0$, then by minimality $H_3 / H_3^w \subseteq L / H_3^w$ and $H_3 \subseteq L$ which is a contradiction. Hence $H_3^w = 0$ and $H_3$ is nilpotent. Let $S = \sigma(L)$. If $S = L$, then $H_3 \subseteq L$ by Lemma 6.2.2, which is again a contradiction. Hence $S \not= L$. It follows that $S + H_3 \not= L$. If $S \not= 0$, then $(H_3 + S)/S \subseteq L/S$ by minimality, but $L/S$ is semi-simple and $(H_3 + S)/S$ is soluble, hence $H_3 + S = S$ and $H_3 < S$. Since $H_3$ acts nilpotently on $<H_i,H_3>$ for all $i$, it follows that $H_3$ acts nilpotently on $H_1 + \ldots + H_n$. Therefore $H_3$ acts nilpotently on $S$, so $H_3 \subseteq S \subseteq L$ which again is a contradiction. Therefore $S = 0$ and $L$ is semi-simple. It follows that $H_1, H_2$ are ideals of $L$. But $H_3 \subseteq <H_1,H_3> = H_1 + H_3$ and $H_3 \subseteq H_2 + H_3$. Hence by Theorem 6.2.1, $H_3 \subseteq H_1 + H_2 + H_3$. Now $H_3 \subseteq <H_3,H_4>$ and $<H_3,H_4>$ permutes with $H_1 + H_2 + H_3$, therefore by Theorem 6.2.1 $H_3 \subseteq H_1 + H_2 + H_3 + H_4$. Continuing this process we get $H_3 \subseteq L$, which is a contradiction. Hence $H_3 \subseteq L$. By a
similar argument we can show that $H_4, \ldots, H_n$ are subideals of $L$. \hfill \square

6.3 Infinite dimensions

For infinite-dimensional Lie algebras we do not know whether a result like Theorem 6.2.1 and Theorem 6.2.9 holds in general, and we leave this as an open question. However we shall extend these results for certain classes of infinite-dimensional algebras. We start with the following:

**Theorem 6.3.1**

Let $L$ be a soluble-by-finite Lie algebra over a field of characteristic zero and let $A$, $H$, $K$ be subalgebras of $L$ such that $L = H + K$ and $A \triangleleft H$, $A \triangleleft K$. Then $A \triangleleft L$.

To prove this we need:

**Lemma 6.3.2**

Let $L$ be a soluble Lie algebra over any field and let $A < L$. If $[L, rA] \subseteq A$ for some $r \in \mathbb{N}$, then $A \triangleleft L$.

**Proof**

See Kawamoto [17, Theorem 4]. \hfill \square

**Lemma 6.3.3**

If $L$ is soluble-by-finite and residually nilpotent, then $L$ is soluble.
Proof

Pick $S$ to be a maximal soluble ideal of $L$ and let $R/S$ be the nilpotent residual of $L/S$. Let $I_\alpha < L$ such that $L/I_\alpha$ is nilpotent and $NI_\alpha = 0$. Therefore $L/(I_\alpha + S)$ is nilpotent, and $R < I_\alpha + S$. Hence $R^{(m)} < I_\alpha$ and $R^{(m)} = 0$ for some $m \in \mathbb{N}$. It follows that $R = S$ and $(L/S)^{\omega} = 0$, but $L/S$ is finite-dimensional, hence $(L/S)^{\omega} = (L/S)^n = 0$ for some $n \in \mathbb{N}$. Therefore $L/S$ is nilpotent and $L$ is soluble. □

Proof of Theorem 6.3.1

Since $A \triangleleft H$ and $A \triangleleft K$, it follows that $A^\omega \triangleleft H$ and $A^\omega \triangleleft K$. Therefore $A^\omega \triangleleft L$. Since $A/A^\omega$ is soluble-by-finite and residually nilpotent, it follows from Lemma 6.3.3 that $A/A^\omega$ is soluble. So without loss of generality we may assume that $A$ is soluble. Let $S$ be a soluble ideal of $L$ such that $L/S$ is finite-dimensional. Then $(A + S)/S \triangleleft L/S$ by Theorem 6.2.1. Therefore $A + S \triangleleft L$. But $A + S$ is soluble and $[A + S, m A] \subseteq A$ for some $m \in \mathbb{N}$, hence by Lemma 6.3.2, we have $A \triangleleft A + S$. Therefore $A \triangleleft L$. □

As a consequence of Theorem 6.3.1, we have the following:

Corollary 6.3.4

Let $L$ be a soluble-by-finite Lie algebra over a field of characteristic zero and let $H_1, H_2, H_3$ be subalgebras of
L such that $L = <H_1, H_2, H_3>$. If $H_i \ni <H_i, H_j> = H_i + H_j$ for all $i, j = 1, 2, 3$, then $H_i \ni L$ for all $i$. □

Next we prove:

**Theorem 6.3.5**

Let $L$ be a Lie algebra over any field and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = <H_1, H_2, H_3>$. Suppose that $H_i \ni <H_i, H_j> = H_i + H_j$ for all $i, j = 1, 2, 3$ and suppose that $[H_1, H_2] \subseteq H_1$. Then $H_i \ni L$ for all $i$.

The proof will be given after the following lemmas.

**Lemma 6.3.6**

Let $L$ be a Lie algebra over any field and let $A, H, K$ be subalgebras of $L$ such that $L = H + K$ and $A \ni H, A \ni K$. Then $A \ni L$.

**Proof**

Since $A^L = A^K \ni L$ and $A \ni A^{m-1} A^K$, it follows that $A \ni L$. □

**Lemma 6.3.7**

Let $L$ be a Lie algebra over any field and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = <H_1, H_2, H_3>$ and suppose that $H_i \ni <H_i, H_j> = H_i + H_j$ for all $i, j = 1, 2, 3$. If $h_1 \ni H_1 + H_2$ and $H_3 \ni H_2 + H_3$, then $H_i \ni L$ for all $i$. 
That $H_1$, $H_3$ are subideals of $L$ follows from Lemma 6.3.6.

Suppose that $H_1 <^m H_1 + H_3$. We prove that $H_2 < L$ by induction on $m$. If $m = 1$, then $H_1 < H_1 + H_3$ and

$H_1 = H_1 < H_1 + H_2$. But $H_2 < H_2 + H_3$, hence $H_2 + H_1 < L$ and $H_2 < L$. Now suppose that the result is true for some $m > 1$. Let $I = H_1^m$, then $I = H_1^m < H_1 + H_3$. Let $J = I \cap H_3$ and consider $H_1$, $H_2$ and $J$. We want to show that $H_1$, $H_2$, $J$ are pairwise permutable cosubideals and $H_1 <^{m-1} H_1 + J$. By assumption $H_1 < H_1 + H_2$ and $H_2 < H_1 + H_2$. Since

$I = I \cap (H_1 + H_3) = H_1 + (I \cap H_3) = H_1 + J$ and $I < L$, it follows that $<H_1, J> = H_1 + J$. By assumption $H_1 < H_1 + H_3$, therefore $H_1 < H_1 + J$. Also $J < H_3$ and $H_3 < H_1 + H_3$ implies that $J < H_1 + H_3$, so that $J < H_1 + J$. Since $I < L$ and $H_2$ idealises $H_3$, it follows that $[J, H_2] \subseteq J$. Therefore $<J, H_2> = J + H_2$. But $H_2 < H_2 + H_3$ and $H_2 + J \subseteq H_2 + H_3$, hence $H_2 < H_2 + J$. Also $J < H_3$ and $H_3 < H_2 + H_3$ implies that $J < H_2 + H_3$. Hence $J < H_2 + J$. Therefore $H_1$, $H_2$, $J$ are pairwise permutable cosubideals.

But $H_1 <^{m-1} I = H_1 + J$, hence by induction $H_1$, $H_2$ and $J$ are subideals of $H_1 + H_2 + J = H_2 + I$. Since $H_2 < H_2 + H_3$ and $I < L$, it follows that $H_2 + I < H_2 + H_3 + I = L$. Hence $H_2 < L$ which completes the proof. □
Proof of Theorem 6.3.5

Since $H_3 < H_2 + H_3$ and $H_1^L = H_3 < H_1 + H_3$, it follows that $H_1 + H_3 < H_2 + H_3 = L$. But $H_1$ and $H_3$ are subideals of $H_1 + H_3$, hence $H_1$ and $H_3$ are subideals of $L$.

Suppose that $H_3 <^m H_2 + H_3$. We prove that $H_2 < L$ by induction on $m$. If $m = 1$, then $H_3 < H_2 + H_3$ which implies that $H_2$ idealises $H_3$. But $H_2$ idealises $H_1$ by assumption, hence by Lemma 6.3.7, $H_2 < L$. Now suppose that the result is true for some $m > 1$, and let $I = H_3 = H_2 + H_3$, $J = I \cap H_2$. Consider the subalgebras $H_1$, $H_3$ and $J$. We claim that $H_1$, $H_3$, $J$ are pairwise permutable cosubideals and $H_3 <^m J + H_3$.

Since $H_2$ idealises $H_1$, it follows that $[J, H_1] = [H_2 \cap I, H_1] \subseteq H_1$ and $<H_1, J> = H_1 + J$. Now $J < H_2 < H_1 + H_2$ implies that $J < H_1 + H_2$. But $H_1 + J \subseteq H_1 + H_2$, hence $J < H_1 + J$.

Also $H_1 < H_1 + J$. Since $I = I \cap (H_2 + H_3) = H_3 + (H_2 \cap I) = H_3 + J$ and $I < H_2 + H_3$, it follows that $<H_3, J> = H_3 + J$. Also $H_3 < H_3 + J$ and $J < H_3 + J$. By assumption we have $<H_1, H_3> = H_1 + H_3$ and $H_1 < H_1 + H_3$, $H_3 < H_1 + H_3$.

Therefore $H_1$, $J$, $H_3$ are pairwise permutable cosubideals.

But $H_3 <^m I = H_3 + J$, hence by induction it follows that $J < H_1 + J + H_3 = H_1 + I$. In particular $H_1$, $J$, $H_3$ being subideals of $H_1 + I$ implies $I = J + H_3 < H_1 + I$. Now consider the subalgebras $H_1$, $H_2$ and $I$. We know that $H_2$ idealises $H_1$ and $H_1$, $H_2$ are subideals of $H_1 + H_2$ by assumption.
H₂ idealises I by the definition of I, so \( <H₂, I> = H₂ + I \). Also H₂ si H₂ + I, since H₂ si H₂ + H₃ and H₂ \( \leq \) H₂ + I. Further, I \( \leq \) H₂ + H₃ and I \( \leq \) H₂ + I implies I \( \leq \) H₂ + I. Thus H₁ si H₁ + I and I si H₁ + I. Therefore by Lemma 6.3.7, H₂ si H₁ + H₂ + I = L which completes the proof.

In the rest of this section we shall investigate ascendant subalgebras of the join of permutable subalgebras of ideally finite Lie algebras.

We start with the following which generalizes Stitzinger [28, Lemma 2].

**Theorem 6.3.8**

Let L be an ideally finite Lie algebra over a field of characteristic zero and let A, H, K be subalgebras of L such that L = H + K, and A \( \subseteq \) H, A \( \subseteq \) K. If A asc H and A asc K, then A asc L.

**Proof**

By assumption we have L = UIₐ where Iₐ is a finite-dimensional ideal of L for each α. Let Fₐ = Cₐ(Iₐ). Then since A \( \triangleleft \) A + \( \zeta₁(L) \), we may assume that \( \zeta₁(L) = 0 \) and \( \cap Fₐ = 0 \). Since L/Fₐ is finite-dimensional, it follows that \( (A + Fₐ)/Fₐ \) si L/Fₐ by Theorem 6.2.1. Hence A + Fₐ si L for each α. Now given Iₐ we can find Fₐ such that Iₐ \( \cap \) Fₐ = 0 for if not then Iₐ \( \cap \) Cₐ(Iₐ) \( \neq \) 0, so that Iₐ \( \cap \) \( \zeta₁(L) \) \( \neq \) 0 which is a contradiction. Since A \( \cap \) Iₐ \( \subseteq \) A + Fₐ and
[A \cap I_\alpha, A + F_\beta] \subseteq A \cap I_\alpha$, it follows that $A \cap I_\alpha \leq A + F_\beta$.

But $A + F_\beta \leq L$, hence $A \cap I_\alpha \leq L$ and $A \cap I_\alpha \leq I_\alpha$ for each $\alpha$. Let $X$ be any finite-dimensional subalgebra of $L$.

Then since $L$ is ideally finite, it follows that $X$ is contained in a finite-dimensional ideal of $L$. Therefore $A \cap X \leq X$. Hence by Lemma 5.3.11, $A \leq L$. But $A^\omega < L$, so we may assume that $A^\omega = 0$. Now $A \leq \rho(L)$ by Lemma 5.3.8. By Lemma 5.3.9, $\rho(L)$ is hypercentral. Therefore $A \leq L$.

Indeed $A \leq L$. 

An immediate consequence of Theorem 6.3.6, we have the following:

**Corollary 6.3.9**

Let $L$ be an ideally finite Lie algebra over a field of characteristic zero and let $H_i$ $(i = 1, 2, \ldots, n)$ be subalgebras of $L$ such that $L = \langle H_i | i = 1, \ldots, n \rangle$. Suppose $H_i \leq \langle H_i, H_j \rangle = H_i + H_j$ for all $i, j = 1, \ldots, n$. Then $H_i \leq L$ for all $i$. 

Next we prove the following:

**Theorem 6.3.10**

Let $L$ be an ideally finite Lie algebra over a field of characteristic zero and let $H_1, H_2, H_3$ be subalgebras of $L$ such that $L = \langle H_1, H_2, H_3 \rangle$ and $H_1 \leq \langle H_1, H_j \rangle$ for all $i, j = 1, 2, 3$. If $\langle H_1, H_2 \rangle$ is permutable with $H_3$, then
H_i asc L for all i.

Proof

That H_1, H_2 are ascendant in L follows from Lemma 6.2.10 and Theorem 6.3.8. By assumption we have L = U I_α, where I_α is a finite-dimensional ideal of L for each α.

Let F_α = C_L(I_α). Then since H_3 < H_3 + ζ_1(L) we may assume that ζ_1(L) = 0 and ∩F_α = 0. Since L/F_α is finite-dimensional, it follows that from Theorem 6.2.9 that (H_3 + F_α)/F_α is L/F_α.

Hence H_3 + F_α is L. Now argue as in Theorem 6.3.8, we get H_3 asc L. □

Finally we prove the following:

Theorem 6.3.11

Let L be an ideally finite Lie algebra over a field of characteristic zero and let H_i < L such that

L = <H_i | i = 1,...,n> and H_i asc <H_i,H_j> for all i,j = 1,...,n.

If H_{i+1} is permutable with H_1 + H_2 + ... + H_i for all i, then H_i asc L.

Proof

By Lemma 6.2.10 and Theorem 6.3.8 we have H_1, H_2 ascendant in L. To show that H_i asc L, i ≥ 3 argue as in Theorem 6.3.8 and apply Theorem 6.2.14. □
REFERENCES


