CENTRES AND INVERTIBLE IDEALS
OF NOETHERIAN RINGS

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CONTENTS

DECLARATION

Most of the original results in this thesis have previously appeared in the author's papers [27] and [28] but have not been included in any material submitted for examination.

ACKNOWLEDGEMENT

I would like to express my gratitude to my supervisor, Dr. C.R. Hajarnavis for his guidance, assistance and persistent encouragement. I also wish to thank all those at the University of Warwick, particularly the members of the Mathematics Institute, for making my stay at Warwick such an enjoyable one.

SUMMARY

This thesis may be divided into two sections. In Chapter 4 we investigate the properties of the Rees ring with respect to an invertible ideal of a non-commutative ring. The technique developed demonstrates the close relationship between invertible ideals and principal ideals. We apply it to derive straightforward proofs of the Invertible Ideal Theorem (Theorem 4.7) and certain results on completions (Chapter 5).

The second half comprises Chapter 6, and concerns Noetherian local rings of finite global dimension which are integral over their centres. Such a ring is shown to be a maximal order, i.e. is "integrally closed".

Earlier chapters deal with the requisite background material which is mostly (but not all) contained in the literature. Greater detail will be found in the introductory chapter (Chapter 1).

NOTATION, TERMINOLOGY AND CONVENTIONS

Throughout, all rings have 1 and all modules are unital. Over-rings are assumed to have the same identity element as the subring concerned.

The following notation is used.

$\mathbb{N}$	The natural numbers $\{1,2,3,\ldots\}$
$\mathbb{Z}$	The integers
$J(R)$	The *Jacobson radical* or R, the intersection of all maximal right (or left) ideals of R.
$N(R)$	The *nilpotent radical* of R (see 2.4)
$Q(R)$	The *right*, *left* or *two-sided quotient ring* of R, when it exists (see 3.2)
$Z(R)$	The *centre* of R.
$l_R(S)$	The *left annihilator* of a subset S of R, i.e. $l_R(S) = \{x \in R \mid xS = 0\}$
$r_R(S)$	The *right annihilator* of a subset S of R, i.e. $r_R(S) = \{x \in R \mid Sx = 0\}$
$C'_R(I)$	The set $\{c \in R \mid cx \in I \Longrightarrow x \in I \text{ for } x \in R\}$ where I is an ideal of R.
$'C_R(I)$	The set $\{c \in R \mid xc \in I \Longrightarrow x \in I \text{ for } x \in R\}$
$C_R(I)$	$'C_R(I) \cap C'_R(I)$

Elements of $C'_R(0)$, $'C_R(0)$ and $C_R(0)$ are called *right regular*, *left regular* and *regular* respectively. A non-regular element of R is termed a *zero-divisor*.

R_S	The localization of R at a right and/or left Ore set *S* (see 3.2, 3.3).
R_P	The localization of R at $C_R(P)$ where P is an ideal of R and $C_R(P)$ is an Ore set (see 3.2, 3.3).
$\hat{R}_{(I)}$	The *I-adic completion* of R with respect to an ideal I (see 3.10).
$\rho(M)$	The *reduced rank* of a module (see 3.5).

1. INTRODUCTION

The substance of this thesis falls into two separate parts, which we discuss individually.

The Rees ring R* of a ring R with respect to an ideal I has recently found many applications in the content of commutative rings (see for example [6, 42, 43]). We apply this concept in Chapter 4 to invertible ideals of non-commutative right Noetherian rings. Under these conditions, R* inherits chain conditions from R (Proposition 4.1), and there is a suitable link between the module structures of R and R* (3.11.2). Certain prime ideals of R are particularly well-behaved, and may be lifted to R* (Lemma 4.4). This link enables us to give a particularly straightforward proof of the Invertible Ideal Theorem of Chatters, Goldie, Hajarnavis and Lenagan [16] which is accomplished via a reduction to the central principal case.

The definition and properties of the ring R* are set out in Section 3.11, and the appropriate results on invertible ideals occupy Section 3.6.

It is also an immediate consequence of Proposition 4.1 that the completion of a right Noetherian ring at an invertible ideal is again right Noetherian. We are thus able to conduct an investigation into such a completion, although stronger conditions are necessary in order to obtain reasonable results. As the Jacobson radical of a hereditary Noetherian prime ring is invertible when it is non-zero, we are able to apply the above to obtain a result of Deshpande [19] which states that such a completion is itself a finite direct sum of complete hereditary Noetherian prime rings (Theorem 5.5). The corresponding results for Dedekind prime rings are virtually immediate (Theorem 5.8).

These results are set out in Chapter 5, at the end of which we investigate completions of invertible ideals under less stringent conditions. If I is a localizable invertible ideal, then the completion at I is semiprime (5.12). These hypotheses are satisfied, for example, when I is an invertible ideal of a maximal order (5.13).

Several preparatory sections are associated with these results, namely those on Invertible Ideals (3.6), hereditary Noetherian prime rings (3.8) and completions (3.10). The last of these deals with completions at ideals satisfying the A.R. property (see Section 3.6), and contains a reasonably comprehensive series of results which, although probably well-known, appear not to have found their way into the literature in this generality.

The second part of this work concerns a generalization of the commutative result that a regular local ring is an integrally closed domain. As a commutative regular local ring is precisely a commutative Noetherian local ring of finite global dimension (Serre, see [39]), we consider non-commutative

Noetherian local rings of finite global dimension which are integral over their centres. By employing results of Brown, Hajarnavis and MacEacharn [8, 9], we are able to show that such a ring is a maximal order in the sense of Asano (Theorem 6.5). This may be regarded as the correct generalization of integral closure. If the above ring also satisfies a polynomial identity, then it is a maximal order in the sense of Fossum [23], (Corollary 6.6).

The relevant results on maximal orders are given in Section 3.9, whilst those on global dimension are contained in Section 3.7.

Other introductory sections appear in Chapter 3. Whilst discussing quotient rings in Section 3.5, we give the following result on centres which does not appear to have been noticed previously: let R be a ring with a semiprimary quotient ring which is ring-indecomposable. Then the centre of R has a primary quotient ring (3.5.3).

I should again like to acknowledge the advice and assistance of my supervisor, Dr. C.R. Hajarnavis. In addition, I would like to thank Dr. A.W. Chatters for many illuminating conversations. Thanks are also due to Terri Moss for her typing, and not least to Amanda Pearce for her tolerance and encouragement.

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2. ELEMENTARY DEFINITIONS

We begin by collating most of the standard and well-known results required. Proofs are not given.

Chain Conditions

Let M be a right R-module and S a non-empty collection of submodules of M ordered under inclusion. We distinguish various properties that this collection might satisfy.

(i) S is said to satisfy the *maximum* (respectively *minimum*) condition if every subset of S has a maximal (minimal) element;

(ii) S is said to satisfy the *ascending chain condition* (ACC) if every chain $M_1 \subset M_2 \subset M_3 \subset \ldots$ with $M_i \in S$ eventually stops; the *descending chain condition* (DCC) is analogously defined.

In the case where S is the set of all submodules of M, modules satisfying the maximum condition are called *Noetherian* whilst those satisfying the minimum condition are termed *Artinian*. A ring is called *right Artinian* if the right module R_R is Artinian; similar definitions may be made on the left and for right and left Noetherian rings. R is Artinian (Noetherian) if it is both right and left Artinian (Noetherian).

The following characterizations are well-known:

Lemma 2.1

The following are equivalent for an R-module M:

(i) M is Noetherian;

(ii) M has ACC on submodules;

(iii) Every submodule of M is finitely generated, i.e. if N is a submodule of M then $N = x_1R + \ldots + x_nR$ for some $x_1,\ldots,x_n \in M$. □

Lemma 2.2

The following are equivalent for an R-module M:

(i) M is Artinian;

(ii) M has DCC on submodules. □

It follows that the Artinian and Noetherian properties are inherited by sub- and factor- modules, and that finitely generated right modules over right Noetherian (Artinian) rings are themselves Noetherian (Artinian). We also have:

Theorem 2.3 (Hopkins)

A right Artinian ring is right Noetherian. □

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Prime and semi-prime ideals

Let I be an ideal of a ring R. I is said to be *semi-prime* if any of the following equivalent conditions hold:

(i) For $a \in R$, $aRa \subset I \Longrightarrow a \in I$

(ii) For an ideal I of R, $A^n \subset I \Longrightarrow A \subset I$.

(iii) For a right ideal I of R, $A^n \subset I \Longrightarrow A \subset I$.

Further, I is said to be *prime* if any of the following equivalent conditions are satisfied:

(i) For $a,b \in R$, $aRb \subset I \Longrightarrow a \in I$ or $b \in I$

(ii) For ideals A and B of R, $AB \subset I \Longrightarrow A \subset I$ or $B \subset I$.

(iii) For right ideals A and B of R, $AB \subset I \Longrightarrow A \subset I$ or $B \subset I$.

R is called semiprime (prime) if 0 is a semiprime (prime) ideal of R. A finite intersection of prime ideals is semiprime. Under Noetherian hypotheses the converse is valid.

Lemma 2.4

Let R be right Noetherian, then:

(i) Every prime ideal of R contains a minimal prime ideal;

(ii) The set of minimal prime ideals is finite, say $T_1,\ldots,T_k$.

(iii) Let $N(R) = T_1 \cap \ldots \cap T_k$. Then $N(R)^t = 0$ for some $t \in \mathbb{N}$, i.e. $N(R)$ is *nilpotent*. $N(R)$ is the unique maximal nilpotent right ideal of R, and is termed the *nilpotent radical* of R.

(iv) $C(N(R)) = C(T_1) \cap \ldots \cap C(T_k)$

(v) A prime ideal P of R is minimal if and only if

$P \cap C(N(R)) = \emptyset$.

Semisimple Artinian Rings

Theorem and Definition 2.5

A ring R is said to be *semisimple Artinian* if it satisfies any of the following equivalent conditions:

(i) R_R is a direct sum of simple modules (i.e. non-zero modules with no proper submodules);

(ii) Every right ideal of R is a direct summand of R;

(iii) R is right Artinian and semi-prime;

(iv) (Wedderburn's Theorem) R is isomorphic to a finite direct sum of matrix rings over division rings;

(v) The left-handed versions of (i), (ii) and (iii). □

If R is in addition prime, then R is called *simple Artinian*. This happens precisely when R has no proper ideals. The above result can be used in order to prove the next theorem.

Theorem 2.6

Let R be a left Noetherian and right Artinian ring. Then R is left Artinian. □

The Jacobson Radical and Idempotent Elements

The definition of the Jacobson radical was given in the "Notation" section. We shall need only the following properties of $J = J(R)$:

(i) J is a semiprime ideal of R;

(ii) (Nakayama's Lemma) If M is a finitely generated right R-module and $MJ = M$, then $M = 0$;

(iii) If S is a simple right R-module, then $SJ = 0$;

(iv) If $x \in R$ and $x + J$ is a unit of R/J, then x is a unit of R;

(v) If R is right Artinian then $J(R) = N(R)$.

A ring R will be called *local* (respectively *semilocal*) if R/J is a simple Artinian (semisimple Artinian) ring.

An element e of R is *idempotent* if $e \neq 0$ and $e^2 = e$. Idempotents e and f of R are *orthogonal* if $ef = 0 = fe$. An idempotent is *primitive* if it cannot be written as a sum of two orthogonal idempotents.

One may prove:

Proposition 2.7

Let I be a nilpotent ideal of a ring R, $\bar{R} = R/I$ and $\bar{1} = e_1 + \dots + e_n$ a decomposition of 1 as a sum of primitive pairwise orthogonal idempotents of $\bar{R}$. Then there are $f_i \in R$

with $1 = f_1 + \ldots + f_n$ and $\bar{f}_i = e_i$ such that the f_i are a pairwise mutually orthogonal set of primitive idempotents of R. □

A ring R is called *primary* (*semiprimary*) if J(R) is nilpotent and R is local (semi-local). Note that in such a ring, idempotent decompositions may be lifted above J(R) = N(R) as above. Any Artinian ring is semiprimary. A Noetherian semiprimary ring is Artinian.

If $1 = e_1 + \ldots + e_n$ is a decomposition of 1 into primitive orthogonal idempotents in a semiprimary ring, it follows from Wedderburn's Theorem that each ring e_iRe_i is *completely primary*, that is a primary ring with $e_iRe_i/J(e_iRe_i)$ a division ring. In a completely primary ring, every element is either a unit or nilpotent.

In closing this subsection we remark that if e is a primitive idempotent, then the module eR cannot be written as a direct sum of non-zero submodules. A module which cannot be written as such a direct sum is called *indecomposable*. A ring will be called *indecomposable* if it is not the direct sum of two-sided ideals. An indecomposable ring may, of course, fail to be indecomposable as a module.

<u>Rank of prime ideals</u>

Consider a fixed prime P of a ring R. We put:

$$\operatorname{rank}(P) = \sup\{k \mid \text{there is a chain } P = P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_k \text{ of primes } P_i \text{ of } R\},$$

assuming that this supremum exists. Note that P is a minimal prime if and only if rank P = 0.

One may define the *Krull dimension* kdim(R) of R by

$$\operatorname{kdim}(R) = \sup\{\operatorname{rank}(P) \mid P \text{ a prime of } R\}.$$

This definition is most appropriate for a commutative ring. It is not in general the most satisfactory definition in the non-commutative setting, and that of [26] may be used in this context. However, in the circumstances where we shall briefly mention Krull dimension, the two definitions coincide [26, 32], so that the above suffices for our purpose.

3. BACKGROUND MATERIAL

3.1 Projective Modules, Flat Modules and Homological Limits

In this section various elementary homological definitions are set out. Throughout, R is an arbitrary ring. Let

$$\cdots \longrightarrow M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} M_{n-2} \longrightarrow \cdots$$

be a sequence of right R-modules and homomorphisms. This sequence is called *exact* if $\ker(f_{n-1}) = \mathrm{Im}(f_n)$ for all n. By a *short exact sequence* we mean an exact sequence of form

$$0 \longrightarrow M'' \xrightarrow{f} M \xrightarrow{g} M' \longrightarrow 0.$$

Note that this sequence is exact if and only if $\ker(g) = \mathrm{Im}(f)$, f is injective and g is surjective.

A right R-module P is *free* if it is a direct sum of copies of R (of arbitrary cardinality) as a right R-module. P is *projective* if for any diagram of R-modules and homomorphisms

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

with the row exact, there exists $h: P \longrightarrow M$ such that $gh = f$. It is elementary to show that any free module is projective.

We have:

<u>Lemma 3.1</u>

If P is a projective right R-module and $g:M \longrightarrow P$ is surjective, then $M \cong P \oplus \ker(g)$.

<u>Proof</u>

The diagram

$$\begin{array}{ccccc} & & P & & \\ & & \big\downarrow 1_P & & \\ M & \xrightarrow[g]{} & P & \longrightarrow & 0 \end{array}$$

yields a map $h:P \longrightarrow M$ with $gh = 1_P$. It is easy to check that $M = \mathrm{Im}(h) \oplus \ker(g)$. As gh is injective, so is h and the result follows. □

By choosing a set of generators for an R-module M and constructing a corresponding free module, one may easily prove:

<u>Lemma 3.1.2</u>

Every R-module M is a homomorphic image of a free module F. If M is finitely generated, then F may be chosen finitely generated too. □

We have:

<u>Lemma 3.1</u>

If P is a projective right R-module and $g: M \longrightarrow P$ is surjective, then $M \cong P \oplus \ker(g)$.

<u>Proof</u>

The diagram

$$\begin{tikzcd} & P \arrow[d, "1_P"] & \\ M \arrow[r, "g"'] & P \arrow[r] & 0 \end{tikzcd}$$

yields a map $h: P \longrightarrow M$ with $gh = 1_P$. It is easy to check that $M = \mathrm{Im}(h) \oplus \ker(g)$. As gh is injective, so is h and the result follows. □

By choosing a set of generators for an R-module M and constructing a corresponding free module, one may easily prove:

<u>Lemma 3.1.2</u>

Every R-module M is a homomorphic image of a free module F. If M is finitely generated, then F may be chosen finitely generated too. □

Combining the last two lemmas, one obtains one implication of the following lemma.

Lemma 3.1.3

An R-module P is projective if and only if it is isomorphic to a direct summand of a free module. □

The above results are sufficient to prove the following result, known as Schanuel's lemma.

Proposition 3.1.4

Let M be a right R-module. Suppose that two short exact sequences

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

and

$$0 \longrightarrow L \longrightarrow Q \longrightarrow M \longrightarrow 0$$

are given, and that P and Q are projective. Then

$$K \oplus Q \cong L \oplus P.$$

Proof

See [5[2], Theorem 3.41]. □

The use of limits will be appropriate at two points in the sequel. We define the notion of *inverse limit*; that of *direct limit* is obtained by the process of "reversing all arrows".

Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of right R-modules, and let Γ be a subset of $\Lambda \times \Lambda$ such that for each $(\lambda,\mu) \in \Gamma$ there is a map $f_{\lambda,\mu}:M_\lambda \longrightarrow M_\mu$. We call the pair $(\{M_\lambda\}_{\lambda\in\Lambda},\Gamma)$ a *directed system* if:

(i) Λ is a partially ordered set under $<$ and for $\lambda, \mu \in \Lambda$ there is some ν with $\lambda,\mu < \nu$;

(ii) For $\lambda < \mu$ we have $(\lambda,\mu) \in \Gamma$;

(iii) For all $\lambda \in \Lambda$, $(\lambda,\lambda) \in \Gamma$ and $f_{\lambda,\lambda} = 1_{M_\lambda}$.

(iv) If (λ,μ) and $(\mu,\nu) \in \Gamma$ then $(\lambda,\nu) \in \Gamma$ and $f_{\lambda,\mu}\ f_{\mu,\nu} = f_{\lambda,\nu}: M_\lambda \longrightarrow M_\nu$.

An *inverse limit* of such a system $(\{M_\lambda\},\Lambda)$ is a right module $\varprojlim M_\lambda$ and set of homomorphisms

$$g_\lambda : \varprojlim M_\lambda \longrightarrow M_\lambda$$

such that for any module X and set of homomorphisms $g'_\lambda : X \longrightarrow M_\lambda$ such that the diagram

$$\begin{tikzcd}
X \arrow[dr, "g'_\lambda"] \arrow[ddr, "g'_\mu"'] & & \varprojlim M_\lambda \arrow[dl, "g_\lambda"'] \arrow[ddl, "g_\mu"] \\
& M_\lambda \arrow[d, "f_{\lambda,\mu}"] & \\
& M_\mu &
\end{tikzcd}$$

commutes, there is a unique map $h:X \longrightarrow \varprojlim M_\lambda$ such that the resulting diagram also commutes.

It can be shown ([52, p. 29]) that direct or inverse limits of such a system may always be constructed.

Finally, we consider the notion of flatness. Let F be a left R-module. F is *flat* if for every short exact sequence of right R-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the induced sequence

$$0 \longrightarrow M' \otimes_R F \longrightarrow M \otimes_R F \longrightarrow M'' \otimes_R F \longrightarrow 0$$

is also exact. Apart from the definition, the only property of flat modules which we shall require is the following.

Lemma 3.1.5

A direct limit of a directed set of flat modules is flat.

Proof

See [52, 3.30]. □

3.2 Quotient Rings

We deal briefly with the generalized technique of commutative localization as applied to non-commutative rings. Throughout, R denotes an arbitrary ring.

As in the commutative case, one hopes to be able to invert sets of regular elements (we shall not here be concerned with inverting sets of elements of R containing zero-divisors).

Definition 3.2.1

Let Q be an over-ring of R. Q is a *right quotient ring* of R with respect to S, a set of regular elements of R, if:

(i) Every element of S has an inverse in Q;

(ii) An arbitrary element q of Q may be written as ac^{-1} for $a \in R$ and $c \in S$.

Q is also called the *right localization* of R with respect to S. It will often be denoted by R_S. The next result determines when such a localization exists.

Theorem 3.2.2 (Ore, Asano)

A ring R has a right quotient ring Q with respect to a set S of regular elements of R if and only if R satisfies the *right Ore condition* with respect to S: that is, for $a \in R$ and $c \in S$ there exist $a_1 \in R$ and $c_1 \in S$ such that

$$ac_1 = ca_1.$$

Proof

This is really a quite elaborate version of the construction of the quotient field of a commutative integral domain. Some details appear in [18]. □

We shall assume Ore sets are non-empty and multiplicatively closed. Two special cases are notable. Firstly, if $S = C(0)$ is a right Ore set, then R_S is simply called the *right quotient ring* of R, denoted Q(R). In these circumstances, R is also called a *right order* in Q. Secondly, if $S = C(I)$ is a right Ore set of regular elements for some two-sided ideal I of R, I is called *right localizable* and the ring $R_I = R_{C(I)}$ is the *right localization* of R at I. In these circumstances, I is usually semiprime.

We recall some elementary properties of localization. In the following, S is a set of regular elements of R.

Lemma 3.2.3

If R satisfies the left and right Ore conditions with respect to S, then the left and right quotient rings at S coincide.

Proof

Let Q be the right quotient ring of R. S satisfies the left Ore condition, and we need only check that each element of Q has the appropriate form. But if $q = ac^{-1}$ then from $c_1a = a_1c$ for suitable a_1 and c_1 we find $q = c_1^{-1}a_1$, as required. □

Lemma 3.2.4

Let I be a right ideal of R, and suppose that S is a right Ore set. Further, let K be a right ideal of $Q = R_S$. Then:

(i) IQ is a right ideal of Q and

$$IQ = \{ac^{-1} | a \in I, c \in S\}.$$

(ii) $K \cap R$ is a right ideal of R and $(K \cap R)Q = K$.

Proof

(i) Simply apply the right Ore condition.

(ii) $K \cap R$ is clearly a right ideal of R. Let $k \in K$; then $k = ac^{-1}$ with $a \in R$ and $c \in S$. Now $kc \in K \cap R$ and so $k = (kc)c^{-1} \in (K \cap R)Q$. □

Ideals are perhaps surprisingly well-behaved under localization.

Lemma 3.2.5

Suppose that R is right Noetherian, S is a right Ore set, and $Q = R_S$.

(i) If I is an ideal of Q, $I \cap R$ is an ideal of R;

(ii) If I is an ideal of R, IQ is an ideal of Q.

(iii) If P is a prime ideal of Q then $P \cap R$ is a prime ideal of R.

(iv) If P is a prime ideal of R, then PQ is a prime ideal of Q. If further $P \cap S = \emptyset$, then $PQ \cap R = P$.

(v) $N(R)Q = N(Q)$ and $N(Q) \cap R = N(R)$.

Proof

See [24] and [17, 1.31]. □

There is one case where we can be sure that a set of elements satisfying the right Ore condition are regular.

Lemma 3.2.6

Suppose that $0 \notin S$ is a left Ore set (possibly containing zero divisors) and that R is prime and right Noetherian. Then the elements of S are regular.

Proof

Let $X = \{r \in R | sr = 0, s \in S\}$. One shows easily using the Ore condition that X is an ideal of R. Write $X = a_1R + \ldots + a_nR$ and choose $s_i \in S$ such that $s_ia_i = 0$. Using the Ore condition one can write $s_i^{-1} = s^{-1}b_i$ for $s \in S$

and $b_i \in R$ (this is the so-called *right common denominator* property). Then $sa_i = b_i s_i a_i = 0$, so that $sX = 0$ and therefore $X = 0$. The elements of S are thus right regular; left regularity follows from [17, 1.13]. □

We end this section by recalling the following result.

<u>Lemma 3.2.7</u>

Let R be a ring, S a right Ore set of regular elements of R. Then the left R-module $Q = R_S$ is flat.

<u>Proof</u>

It is quite easy to see that Q is the direct limit of the set $\{Rc^{-1} | c \in S\}$ of left R-submodules of Q, directed by inclusion. Apply Lemma 3.1.5. □

3.3 Uniform Dimension and Goldie's Theorem

This short section is concerned with the definition of uniform dimension and the statement of Goldie's theorem on orders in semi-simple Artinian rings. Proofs of all of these results may be found in [24] or [17].

Let R be any ring, U a right R-module. U is called *uniform* if any two non-zero submodules of U have non-zero intersection. A submodule M' of a module M is called *essential* in M if M' intersects every non-zero submodule of M non-trivially (hence $U \neq 0$ is uniform if and only if every non-zero submodule of U is essential in U). A module M is said to be of *finite uniform dimension* if M contains no infinite direct sum of non-zero submodules. In fact, if M is of finite uniform dimension then M has a finite direct sum of uniform submodules which is essential in M. The length of such a sum is an invariant for M, called the *uniform dimension* of M and denoted dim (M), [17, 1.9].

Certainly any Artinian or Noetherian R-module is of finite uniform dimension. Further, $\dim (M \oplus M') = \dim (M) + \dim (M')$ for right modules M and M' [17, p. 19].

Let R be a ring, R is called *right Goldie* if R has finite uniform dimension as a right R-module and R has the ascending chain condition on right annihilators. For a semi-prime ring, these conditions are precisely those which are required for the existence of a quotient ring.

<u>Theorem 3.3.1</u> (Goldie, [24])

Let R be any ring. Then R has a right quotient ring Q which is semi-simple Artinian if and only if R is a semi-prime right Goldie ring. Under these conditions, R is prime if and only if Q is simple. □

We state explicitly a result implicit in the proof of the above theorem.

Proposition 3.3.2

Let R be a semiprime right Goldie ring. Then a right ideal E of R is essential in R if and only if it contains a regular element.

Proof [18, p. 450] □

Note that a two-sided non-zero ideal of a prime right Noetherian ring R is such an essential right ideal E, for if T is some right ideal of R with $E \cap T = 0$, then $TE \subset E \cap T = 0$ and hence $T = 0$.

Combining these results with those of the last section on localization, we can now state the well-known result on localization at prime ideals.

Proposition 3.3.3

Let R be a right Noetherian ring and P a semi-prime ideal of R. If P is localizable, then R_P is a semilocal ring with Jacobson radical PR_P. If in addition P is prime, R_P is local.

Proof

Put $Q = R_P$. Then PQ is an ideal of Q by Lemma 3.2.5. Let $x \in PQ$; then $1-x \in C_Q(PQ)$ and $1-x = dc^{-1}$ with $d \in R$ and $c \in C_R(P)$. It follows from Lemma 3.2.5 that $d \in C_R(P)$, so that d is a unit of Q. Thus 1-x is a unit of Q, and therefore $PQ \subset J(Q)$.

Now let $\bar{E}$ be an essential right ideal of $\bar{Q} = Q/PQ$. By Proposition 3.3.2, $\bar{E}$ contains a regular element $\bar{d}$ of $\bar{Q}$, where $\bar{d} = d + PQ$ for some $d \in C_Q(PQ)$. As in the previous paragraph, d is a unit of Q, hence $\bar{E} = \bar{Q}$. As every right ideal of $\bar{Q}$ is a direct summand of an essential right ideal, $\bar{Q}$ is thus a semisimple Artinian ring (Theorem 2.5). We must now have $PQ = J(Q)$, as required. The result now follows from Lemma 3.2.5. □

3.4 Generalizations of Commutativity

The definitions and elementary properties of both rings integral over their centres and PI rings are given here. The latter class will only briefly be mentioned in Chapter 6, and we content ourselves with providing the definition and the statement of Posner's theorem.

Let R be a ring. The *centre* Z(R) of R is of course the subring $\{z \in R \mid zr = rz \text{ for all } r \in R\}$. Let Z be a subring of Z(R). Then R is *integral over* Z if for any element x of R there is some equation of the form

$$x^n + z_{n-1}x^{n-1} + \dots + z_0 = 0$$

with each $z_i \in Z$, $n \in \mathbb{N}$.

In such rings there is a tight relationship between the prime ideals of R and Z.

<u>Lemma 3.4.1</u> [4]

Let R be a right Noetherian ring integral over a central subring Z. Then:

(i) If p is a prime ideal of Z then there is a prime ideal P of R such that $P \cap Z = p$ (lying over). There are only finitely many such P for each p.

(ii) If p is as in (i), and P and Q are prime ideals of R with $P \cap Z = Q \cap Z = p$, then $P \subset Q \Longrightarrow P = Q$ (Incomparability). □

Localization is also well-behaved.

<u>Proposition 3.4.2</u> [12]

Let R be a right Noetherian ring integral over a central subring Z and let p be a prime ideal of Z. Let $P_1,\dots,P_k$ be the set of prime ideals of R lying over p (i.e. $p = P_i \cap Z$). Then if $N = P_1 \cap \ldots \cap P_k$:

(i) $C_R(N)$ is a right and left Ore set of the ring R; and

(ii) $R_N = R_{Z \setminus p}$. □

By a PI-ring we mean a ring satisfying a polynomial identity, that is every element of R satisfies a fixed polynomial of form

$$\sum_{\sigma \in S_d} \alpha_\sigma \, x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(d)}$$

where the x_i are non-commuting indeterminates, S_d is the group of permutations of $\{1,\ldots,d\}$ and α_σ is ± 1. One has

Theorem 3.4.3 (Posner)

Let R be a prime PI ring. Then R has a quotient ring Q obtained by inverting the central regular elements of R. Further, Q is a finite dimensional central simple algebra (that is, a simple Artinian ring finite dimensional as a vector space over its centre).

Proof

See, for example, [18, Theorem 8, p. 465]. □

3.5 Reduced Rank and Artinian Quotient Rings

Although the concept of uniform dimension introduced in the previous section was of sufficient depth to allow the proof of Goldie's theorem to proceed, it is deficient in one respect: namely it fails to be additive over short exact sequences, i.e. there are short exact sequences

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of right R-modules for which $\dim(M) \neq \dim(M') + \dim(M'')$. For example, $\mathbb{Q}$ and $\mathbb{Z}$ are both $\mathbb{Z}$-modules of uniform dimension 1, yet $\mathbb{Q}/\mathbb{Z}$ does not have finite uniform dimension. Worse, $\mathbb{Z}$ and $p_1p_2 \ldots p_k\mathbb{Z}$ ($p_1,\ldots,p_k$ distinct primes) are finitely generated uniform $\mathbb{Z}$-modules, yet $\dim(\mathbb{Z}/p_1 \ldots p_k\mathbb{Z}) = k$. Additivity is repaired in the definition of reduced rank, first introduced in [25].

<u>Proposition and Definition 3.5.1</u>

Let M be a finitely generated right module over a right Noetherian ring R.

(i) Suppose that R is semi-prime. Let T(M) be the submodule

$$\{x \in M \mid xc = 0 \text{ for } c \in C_R(0)\}$$

of M. We define the *reduced rank*

$$\rho_R(M) = \dim (M/T(M)).$$

(ii) In the general case, let N be the nilpotent radical of R and define the *reduced rank*

$$\rho_R(M) = \sum_{i=0}^{k-1} \rho_{R/N} (MN^i/MN^{i+1})$$

where $N^k = 0$.

Then:

(a) If K is a submodule of M we have

$$\rho(M) = \rho(K) + \rho(M/K);$$

(b) $\rho(M) = 0$ if and only for each $m \in M$ there is some $c \in C(N)$ with $mc = 0$;

(c) If $\rho(M) = 0$ then for $m_1, \dots, m_n \in M$ there exists $c \in C(N)$ with $m_i c = 0$ for all $i \leq n$.

<u>Proof</u> [17, Theorem 2.2]. □

The concept of reduced rank has been recently used to provide proofs of many well-known and important results [16]. Among these is the theorem of Small on Artinian quotient rings [54, 55].

<u>Theorem 3.5.2</u>

Let R be a right Noetherian ring with nilpotent radical N. Then R has a right Artinian right quotient ring if and only if $C(0) = C(N(R))$.

<u>Proof</u>

For the reduced rank proof see [16] or [17]. □

At this point we digress to consider the relationship between the existence of an Artinian quotient ring for R and the existence of a similar quotient ring for the centre of R. It is certainly not true that the existence of an Artinian quotient ring of the centre implies that R has an Artinian quotient ring; consider, for example, the Noetherian quotient ring

$$\begin{bmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)} \\ 0 & \mathbb{Z}_{(2)} \end{bmatrix}$$

whose centre is isomorphic to $\mathbb{Z}_{(2)}$. On the other hand, it is easily seen that if R is Noetherian and finitely generated over its centre Z, and if R has an Artinian quotient ring Q, then Z has an Artinian quotient ring, namely the centre of Q. Integrality may replace finite generation here. Moving away from integrality restrictions, if R is an Artinian ring with centre Z, then Z may be considered as the endomorphism ring of the finitely generated $R \otimes R^{op}$ module R, and it may be deduced that Z is semiprimary. We are interested in the following generalization: If R has an Artinian quotient ring, then is Q(Z) semiprimary? Having gone so far, we may as well let R have a semi-primary quotient ring and attempt the same result. We have not been able to obtain the conclusion we seek in all generality: rather, we ask that Q(R) should be indecomposable as a ring.

Proposition 3.5.3

Let R be a ring with a semiprimary quotient ring Q. Suppose further that Q is ring-indecomposable. Then Z, the centre of R, has a primary quotient ring.

Proof

Clearly, after inverting the regular elements of R lying in Z we may immediately reduce to the case where such elements are invertible.

Let $\{e_1,\dots,e_n\}$ be a complete set of orthogonal primitive idempotents of Q. By the discussion following Proposition 2.7, an element of e_iQe_i is either invertible or nilpotent in e_iQe_i. For $z \in Z$ we let z_i denote the element $e_iz = e_ize_i$ of e_iQe_i.

Suppose that z is a non-nilpotent element of Z. Then as $z = 1.z = e_1z + \ldots + e_nz$ and the e_i are orthogonal, not all the z_i can be nilpotent. Re-ordering if necessary, we may assume that z_j is invertible in e_jQe_j for $1 \leq j \leq t$, and $z_{t+1},\ldots,z_n$ are nilpotent. Note $t \geq 1$. There is some $k \geq 1$ such that $z_{t+1}^k = \ldots = z_n^k = 0$.

Choose an inverse $y_j \in e_jQe_j$ for z_j^k in e_jQe_j, $1 \leq j \leq t$. Thus $y_jz_j^k = e_j$. For $q \in Q$ and arbitrary i and j we certainly have

$$e_iqe_jz^k - z^ke_iqe_j = 0.$$

Yet if $i \leq t$ and $j > t$, $e_jz^k = 0$ whilst $z^ke_i = z_i^k$. Therefore

$$z_i^ke_iqe_j = 0 \text{ for } i \leq t \text{ and } j > t.$$

Now $y_iz_i^k = e_i$, so that

$$e_iqe_j = 0.$$

As q was a general element of Q, we have $e_iQe_j = 0$. Put $e = e_1 + \ldots + e_t$ and $f = e_{t+1} + \ldots + e_n$; then e and f are idempotents of Q such that $eQf = 0$ and, by symmetry, $fQe = 0$. Now eQ is an ideal of Q, for

$$\begin{aligned} Q.eQ &= (eQ + fQ).eQ \\ &= eQeQ + fQeQ \\ &= eQeQ \\ &\subset eQ. \end{aligned}$$

Similarly, fQ is an ideal of Q. Yet $Q = eQ \oplus fQ$ and is ring-indecomposable. As $e \neq 0$, we must therefore have $f = 0$. Therefore $t = n$, and z_i is an invertible element of e_iQe_i for all i. It follows easily that z is invertible.

It has been shown that every non-nilpotent element of Z is invertible. Z is thus a primary ring, which proves the result. □

Indecomposable right Noetherian maximal orders in Artinian quotient rings provide non-trivial examples of rings satisfying the conditions of the above Proposition (see [31]).

Of course, we may deduce:

<u>Corollary 3.5.4</u>

The centre of an Artinian ring is semi-primary. □

3.6 <u>Invertible Ideals and the A.R. Property</u>

In this section we set out the definitions and well-known properties of invertible ideals and of ideals satisfying the A.R. property. This material will be of use in Chapters 4 and 5.

Definition 3.6.1

Let R be a ring and I an ideal of R. I is said to satisfy the *right Artin-Rees* (A.R.) *property* if for every right ideal E of R there is some integer $n > 0$ with $E \cap I^n \subset EI$.

Definition 3.6.2

An ideal I of a ring R is said to be *invertible* with respect to an over-ring S if $I^*I = R = II^+$ where

$$I^* = \{s \in S | sI \subset R\}$$

and

$$I^+ = \{s \in S | Is \subset R\}.$$

Under these circumstances it is easy to see that $I^* = I^+$, and this R-bisubmodule of S is called the *inverse* of I, denoted I^{-1}.

We have chosen to treat the A.R. property and invertible ideals simultaneously because of the following result. A proof along different lines from that usually given appears as Corollary 4.3.

<u>Lemma 3.6.3</u> (Ginn, [17, 3.3]).

Let R be a right Noetherian ring. Then any invertible ideal of R satisfies the right A.R. property. □

Other examples of ideals satisfying the right A.R. property are provided by:

(i) An ideal of a right Noetherian ring which has a centralizing set of generators [50];
(ii) The Jacobson radical of a semilocal FBN ring [17,11.3];
(iii) Any ideal of the integral group ring of a finitely generated nilpotent group, or of the universal enveloping algebra of a finite dimensional nilpotent Lie algebra [51, 44].

We give first those results on ideals satisfying the A.R. property which will be of interest to us later. Localizability is perhaps the most important.

<u>Proposition 3.6.4</u> (P.F. Smith)

Let R be a ring with an ideal I which satisfies the right A.R. property. Suppose that for each $n \geqslant 1$ the ring R/I^n satisfies the right Ore condition with respect to $C(I/I^n)$. Then R satisfies the right Ore condition with respect to $C(I)$.

<u>Proof</u>

Let $a \in R$ and $c \in C(I)$. By the right A.R. property, there is some $n \geqslant 1$ such that

$$(aR + cR) \cap I^n \subset (aR + cR)I$$
$$\subset aI + cI.$$

As R/I^n has the stated Ore condition, there exist $a' \in R$ and $c' \in \mathcal{C}(I)$ with $ac' - ca' \in I^n$. But $ac' - ca' = ax + cx'$ for some $x, x' \in I$, and hence $a(c'-x) = c(a' + x')$ with $c' - x \in \mathcal{C}(I)$ as required. □

The second result is clear for ideals contained in the Jacobson radical of R, but is in fact true in greater generality.

<u>Proposition 3.6.5</u>

Let I be a proper ideal of a right Noetherian prime ring R, and suppose that I satisfies the right A.R. property. Then $\bigcap_{n=1}^{\infty} I^n = 0$.

<u>Proof</u> See [56]. □

The A.R. property may be transported to modules: a proof of the following result may be found in [17].

<u>Proposition 3.6.6</u> (Hartley)

Let R be a right Noetherian ring and let I be an ideal of R which has the right A.R. property. If M is a finitely generated right R-module and K is a submodule of M, then there is some $n > 0$ such that $K \cap MI^n \subset KI$. □

The following essentially trivial result is recorded for ease of reference in the sequel.

Proposition 3.6.7

If I is an ideal of a ring R satisfying the right A.R. property, the I^n satisfies the right A.R. property for each $n > 0$.

Proof

Let E be a right ideal of R. Fixing n, we may assume that $I, I^2, \ldots, I^{n-1}$ all satisfy the right A.R. property. Hence there are integers k_i such that:

$$E \cap I^{k_1} \subset EI$$

$$EI \cap I^{k_2} \subset EI^2 \quad (\text{hence } E \cap I^{k_1} \cap I^{k_2} \subset EI^2)$$

$$\vdots$$

$$EI^{n-1} \cap I^{k_n} \subset EI^n \quad (\text{hence } E \cap I^{k_1} \cap \ldots \cap I^{k_n} \subset EI^n).$$

Let $k > \max\{k_i\}$. Then $E \cap I^k \subset EI^n$. As k may be taken to be a multiple of n, the result is proved. □

We move on to invertible ideals. For such ideals there is a precise tie-up between the additive and multiplicative structures of R.

<u>Lemma 3.6.8</u> [17, p. 45]

Let R be a ring, X an invertible ideal of R with respect to some over-ring S, and A and B right R-submodules of S. Then:

(i) $(A \cap B)X = AX \cap BX$; and

(ii) $(A \cap B)X^{-1} = AX^{-1} \cap BX^{-1}$.

<u>Proof</u>

$$\begin{aligned}(A \cap B)X &\subseteq AX \cap BX \\ &\subseteq (AX \cap BX)X^{-1}X \\ &\subseteq (AXX^{-1} \cap BXX^{-1})X \\ &= (A \cap B)X.\end{aligned}$$

Hence $(A \cap B)X = AX \cap BX$. The second equality follows similarly. □

This link has a particularly pleasant manifestation when prime ideals are considered.

<u>Lemma 3.6.9</u> [17].

Let R be a ring, X an invertible ideal of R. Then:

(i) Let P be a prime ideal of R not containing X. Then $PX = P \cap X = XP$.

(ii) Suppose that R is right Noetherian, and let $P_1, \ldots, P_n$ be the primes of R minimal over X. Then conjugation by X permutes the P_i. Moreover, if $N = P_1 \cap \ldots \cap P_n$ then $X^{-1}NX = N$.

<u>Proof</u>

(i) We certainly have $P \cap X \subset P$. Now

$$\begin{aligned} P \cap X \subset P &\Longrightarrow PX^{-1}X \cap X \subset P \\ &\Longrightarrow (PX^{-1} \cap R)X \subset P \\ &\Longrightarrow (PX^{-1} \cap R) \subset P \\ &\Longrightarrow (P \cap X)X^{-1} \subset P \\ &\Longrightarrow P \cap X \subset PX. \end{aligned}$$

The proof is easily completed.

(ii) Notice first that $X^{-1}NX$ is clearly a nilpotent ideal modulo X. It follows that $X^{-1}NX \subset N$, i.e. $NX \subset XN$. By symmetry, $NX = XN$. It is easily seen that $X^{-1}P_iX$ is again a prime of R minimal over X, so that the remaining assertion of the lemma follows. □

There are two further simple properties of invertible ideals which it is necessary to record.

Lemma 3.6.10

An invertible ideal X of a ring R is projective.

Proof

Let X^{-1} be the inverse of X in some over-ring S. Then there are $x_1,\ldots,x_n \in X$ and $y_1,\ldots,y_n \in X^{-1}$ such that $x_1y_1 + \ldots + x_ny_n = 1$. Define R-homomorphisms $f_i : X \to R$ by $f_i(x) = y_ix$ for $x \in X$. Then for $x \in X$,

$$\sum_{i=1}^{n} x_if_i(x) = \sum_{i=1}^{n} x_iy_ix = x.$$

The $\{f_i, x_i\}$ thus form a dual basis for X, and so X is (right) projective by the dual basis lemma, [52, Lemma 4.15]. □

Lemma 3.6.11

Let X be an invertible ideal of a ring R. Then $C'(X) = C'(X^n)$ for all n.

Proof

Suppose that $C'(X) = C'(X^{n-1})$ for some $n \geq 2$. Let $t \in R$, and assume that $ct \in X^n$ with $c \in C'(X)$. As $X^n \subset X^{n-1}$, $t \in X^{n-1}$. Thus

$$ctX^{1-n} \subset X^n.X^{1-n} = X$$

and consequently $tX^{1-n} \subset X$, i.e. $t \in X^n$. Therefore $C'(X) \subset C'(X^n)$. The opposite inclusion may be proved with equal ease, and the result follows by induction. □

3.7 Finite Global Dimension

Only the definition and standard characterizations of global dimension will be required in our work.

Let R be a ring, M a right R-module. By successively applying Lemma 3.1.2, one may form an exact sequence

$$\ldots \longrightarrow P_k \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each P_i is a projective right R-module. Such a sequence is called a *projective resolution* of M. If there is an integer n such that there is a projective resolution of form

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

then M is said to have *finite projective dimension*. The smallest such n is called the *projective dimension* of M, and will be denoted by $pd_R(M)$.

Consider now the entire category of right R-modules. If every right R-module has finite projective dimension, and further all such dimensions are bounded by some integer N, then R is said to have *finite right global dimension*. We define the *right global* *dimension* of R, denoted D(R), to be the least such possible integer, i.e.

$$D(R) = \sup \{pd_R(M) \mid M \text{ a right } R\text{-module}\}.$$

One may, in fact, restrict the number of modules which need to be considered in this supremum. We state without proof the following well-known characterizations.

<u>Theorem 3.7.1</u>

Let R be a ring. Then the following quantities are equal (and if one quantity is undefined, so are the others):

(a) D(R);

(b) $\sup \{pd_R(M) \mid M$ a finitely generated right module$\}$

(c) $\sup \{pd_R(R/I) \mid I$ a right ideal of $R\}$

If $D(R) \neq 0$, then one also has equality with:

(d) $1 + \sup \{pd_R(I) \mid I$ a right ideal of $R\}$. □

In the Noetherian case, a theorem of Auslander allows us to neglect the adjective "left" or "right".

<u>Theorem 3.7.2</u> [52, Theorem 9.20]

Let R be a left and right Noetherian ring. Then the left and right global dimensions coincide (and if one is finite, so is the other). □

Happily, the property of having finite global dimension is preserved under localization.

Proposition 3.7.3

Let R be a ring of finite right global dimension, and S a right Ore set in R. Then R_S has finite right global dimension and $D(R_S) \leq D(R)$.

Proof

One simply forms a projective resolution of a right R_S-module M as an R-module and tensors through by the flat left R-module R_S. As $M \otimes_R R_S \cong M$ and the other modules in the resulting resolution are easily seen to be projective, the result follows. □

The rings of global dimension 0 are precisely the semisimple Artinian rings. Those of (right) global dimension at most 1 are termed (right) hereditary. The latter class of rings are the subject of the next introductory section.

3.8 HNP, Asano and Dedekind Rings

This section is concerned with the generalizations of (commutative) Dedekind domains to the non-commutative case. The first class of rings we mention is the class of hereditary Noetherian prime rings (which we shall refer to as HNP rings).

Recall that in the previous section a right hereditary ring was defined to be a ring of finite right global dimension at most 1. By Theorem 3.7.1, this means precisely that every right ideal of R is a projective right R-module. We shall, in the sequel, exclude semisimple Artinian rings from the class of HNP rings.

The theory of HNP rings is well-developed. Two theorems of Chatters [13, 14] will be required at a later point.

<u>Theorem 3.8.1</u>

Let R be a Noetherian hereditary ring and E an essential right ideal of R. Then R/E is an Artinian R-module (This is the so-called "restricted minimum condition").

<u>Proof</u> [13]. □

<u>Theorem 3.8.2</u> [14]

Let R be a hereditary Noetherian ring. Then R is isomorphic to a finite direct sum of prime and Artinian hereditary rings. □

The main result on HNP rings which it is necessary to give here is taken from [21].

<u>Theorem 3.8.3</u>

Let R be a HNP ring with Jacobson radical J. If $J \neq 0$, then J is an invertible ideal of R. □

The next class of rings of interest at this point comprises those rings for which every non-zero ideal is invertible. A prime, Noetherian ring with this property is called an *Asano order*. A hereditary Asano order is called a *Dedekind prime ring*.

Asano orders are characterized by the following result.

<u>Theorem 3.8.4</u>

Let R be a prime Noetherian ring. Then R is an Asano order if and only if the localization R_P exists for each maximal ideal P of R and is a hereditary ring.

<u>Proof</u> [30]. □

We mention one more class of Noetherian rings. A ring R is called a *principal right ideal ring* (pri-ring) if every right ideal of R is a cyclic R-module. A similar definition of pli-rings may be made on the left. Note that pri-rings are necessarily right Noetherian.

Let I be an ideal of a prime Noetherian ring, and suppose that $aR = I = Rb$. It is easily seen that a and b are regular elements of R, hence that I is an invertible ideal. As the left and right inverses of I coincide (3.6.2), it follows that we may take a = b.

The following result gives a link between HNP rings and pri-pli rings.

Proposition 3.8.5

Let R be a Noetherian local prime ring. Then the following are equivalent.

(i) The Jacobson radical J of R is invertible;

(ii) R is hereditary;

(iii) R is a pli-pri ring;

(iv) R is an Asano order.

Proof

See [30]. □

3.9 Maximal Orders

In Chapter 6 we shall be concerned with proving that the members of a certain class of rings of finite global dimension are maximal orders. This section deals with the remaining definitions and background material required for that chapter.

We begin by considering the classical case. Let D be a Noetherian integrally closed integral domain (typically a Dedekind domain) with field of fractions K, and let Q be a central simple K-algebra. By a *D-order* R we mean a subring R of Q containing D and finitely generated as a module over D such that R has quotient ring Q. A D-order R of Q is called *maximal* if it is not properly contained in any other D-order of Q.

We shall consider generalizations of this definition - in particular that of Asano - which are not so closely tied to a central subring. We could in fact take condition (d) of Proposition 3.9.1 as our defining property, but prefer instead a more orthodox definition which perhaps provides better motivation.

Let S be any ring. We shall call two (right and left) orders R and R' of S *equivalent* if there are units a, b, a' and b' of S with

$$aRb \subset R' \text{ and } a'R'b' \subset R.$$

Then an order R of S is called *maximal* if it is not properly contained in any order of S to which it is equivalent.

An appropriate sub-class of the R-submodules of S is also worthy of mention. If R is an order in S, then a subset I of S is called a *right R-ideal* if:

(i) I is a right R-submodule of S;

(ii) I contains a unit of S;

(iii) There is a unit u of S with $uI \subset R$.

Left R-ideals are appropriately defined using the reader's mirror. A subset I of S is called an *R-ideal* if it is both a right and left R-ideal. Finally, an R-ideal is *integral* if it happens to lie in R. For the case in which we are most interested, R will be a prime ring and S its semisimple Artinian ring of quotients. By Proposition 3.3.2, any non-zero ideal of R is then an integral R-ideal.

Let R be an order in a ring S and let I be a one-sided R-ideal of S. We put

$$O_l(I) = \{x \in S | xI \subset I\}.$$

and

$$O_r(I) = \{x \in S | Ix \subset I\}.$$

The definition of maximal orders may now be rephrased as follows:

<u>Proposition 3.9.1</u> [40, Proposition 3.1, p. 7].

Let R be an order in a ring S. The following conditions are equivalent:

(a) R is a maximal order;

(b) For every right R-ideal I of S, $O_r(I) = R$ and for every left R-ideal I, $O_l(I) = R$.

(c) As in (b), but for integral one-sided R-ideals.

(d) If I is a (two-sided) integral R-ideal, then $O_l(I) = R = O_r(I)$. □

Examples 3.9.2

Examples of maximal orders include universal enveloping algebras of finite dimensional Lie algebras [40, Theorem 3.1, p. 173] and certain generalizations [41], Asano orders [40, p. 47], and prime Noetherian A.R. rings of finite global dimension with enough invertible ideals [8]. In the commutative case a (prime) maximal order is just a completely integrally closed integral domain [40, Proposition 5.1, p. 12]. Thus a commutative Noetherian domain is a maximal order if and only if it is integrally closed. [36, p. 53].

Certain facts on localization in maximal orders will be utilized in Chapter 5. We set out some results which may be found in [11] or [31]. First, a definition. Let R be a maximal order in a simple Artinian ring Q, and let I be an R-ideal of Q. Write

$$I^* = \{q \in Q \mid qI \subseteq R\}.$$

I^* is in fact an R-ideal of Q; further,

$$
\begin{aligned}
q \in I^* &\Longleftrightarrow qI \subset R \\
&\Longleftrightarrow IqI \subset I \\
&\Longleftrightarrow Iq \subset R \qquad (3.9.1)
\end{aligned}
$$

so that the definition of I^* is left-right symmetric. Note that $I^{**} \supset I$. An R-ideal I of R is called *reflexive* if $I = I^{**}$.

The results required may be summarized as:

<u>Proposition 3.9.3</u>

Let R be a prime Noetherian maximal order, T a reflexive ideal of R. Then:

(i) Each prime ideal P of R minimal over T is of rank one and localizable;

(ii) If N/T is the nilpotent radical of R/T, then the localization R_N exists and is hereditary;

(iii) R/T has an Artinian (in fact, a quasi-Frobenius) quotient ring.

<u>Proof</u>

Each such P is rank one by [31, Corollary 3.4]. That P is localizable (and, in fact, reflexive) follows by inspection of the proof of [31, Theorem 3.3], as does part (ii). Part (iii) is the statement of [31, Theorem 3.3]. □

We take this opportunity to observe, somewhat belatedly, that our best example of a reflexive ideal is an invertible ideal. This follows directly from the definitions.

A second and related generalization of classical maximal orders will be of interest in Chapter 6. These are the maximal orders in the sense of Fossum [23].

<u>Definition 3.9.4</u>

Let Z be a Krull domain with quotient field K, and Q a finite dimensional central simple algebra over K. A Z-order is, in the sense of Fossum, a subring R of Q satisfying:

(i) $Z \subset R$;

(ii) $K.R = Q$;

(iii) R is integral over Z.

A Z-order is *maximal* if it is not properly contained in any other Z-order in Q.

The following proposition connects this class with the maximal orders previously discussed.

<u>Proposition 3.9.5</u>

(i) Every maximal Z-order is a (bounded) maximal order;

(ii) Let R be a maximal order in a simple Artinian PI ring, and let Z be the centre of R. Then R is a maximal

Z-order.

Proof

(i) [40, Theorem 7.5 p.17].

(ii) [10, Proposition 1.4] or [40, Theorem 3.2]. □

3.10 Completions

Here most of the useful properties of the completion functor on the category of right R-modules over a general non-commutative ring R are set out. We are primarily interested in the case where the completion is I-adic where I is an ideal of R which satisfies the right Artin-Rees property. Although most of the proofs are well-known, it has proved difficult to locate a systematic source for these results in this generality. Thus many proofs are indicated briefly. To a large extent we follow the treatment of the commutative case of Atiyah and MacDonald [2, Chapter 10]. We shall assume that the reader is familiar with a little basic topology.

Definition 3.10.1

Let M be a right R-module. A *filtration* of M is a sequence

$$M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

of submodules of M.

A filtration $\{M_i\}_{i\in\mathbb{N}}$ of M determines a topology on M. This topology has base $\{x + M_i | x \in M, i \in \mathbb{N}\}$ where $x + M_i = \{x + m | m \in M_i\}$. It is immediate that addition is a continuous operation on M, and that this topology on M is Hausdorff if and only if $\bigcap_{n=1}^{\infty} M_n = 0$.

One may now consider Cauchy sequences in M: a sequence $m_1, m_2, m_3, \ldots$ of elements of M is *Cauchy* if for any k there is some $K \in \mathbb{N}$ such that for $i,j > K$ one has $m_i - m_j \in M_k$. The sequence $\{m_i\}$ is said to *converge* to $m \in M$ if for any k there is some $K \in \mathbb{N}$ such that $m - m_i \in M_k$ for $i > K$. As usual, a module M is *complete* if every Cauchy sequence in M converges to a limit in M.

The object is, of course, to embed a suitably topologised module in a complete module in a canonical and minimal manner.

<u>Definition 3.10.2</u>

Let M be an R-module with filtration $\{M_i\}$. A *completion* $\hat{M}$ of M is an R-module $\hat{M}$ with filtration $\{\hat{M}_i\}$ and an R-homomorphism $\psi : M \to \hat{M}$ which satisfy:

(i) $\hat{M}$ is complete with respect to its filtration;

(ii) ψ is a continuous map;

(iii) Every point of $\hat{M}$ is the limit of a sequence of elements of $\psi(M)$;

(iv) $\text{Ker } \psi = \bigcap_{n=1}^{\infty} M_n$.

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(ii) ψ is a continuous map;

(iii) Every point of $\hat{M}$ is the limit of a sequence of elements of $\psi(M)$;

(iv) $\operatorname{Ker} \psi = \bigcap_{n=1}^{\infty} M_n$.

In practice one usually neglects to mention the homomorphism ψ when referring to completions, particularly when ψ is injective. We note that by [49, p. 395 Theorem 7] the completion of a module is essentially unique. Existence may be established as follows:

Proposition 3.10.3

Let $\{M_i\}$ be a filtration of an R-module M, and consider the directed set of modules and homomorphisms consisting of the canonical projections $M/M_j \to M/M_i$ $(i < j)$. Then the inverse limit $\varprojlim M/M_i$ is a completion for M. Further, any completion of such a module is complete.

Proof

See [2, p. 103] and [2, Proposition 10.5]. □

I-adic topologies and their associated completions are our next topic of interest.

Definition 3.10.4

Let I be an ideal of a ring R, and let M be a right R-module. Put $M_n = MI^n$ for each $n \in \mathbb{N}$. Then $\{M_n\}$ defines a filtration, called the *I-adic filtration*, on M. The completion of M with respect to the topology induced by this filtration is known as the *I-adic completion* of M. We shall denote the I-adic completion of R itself by $\hat{R}$ or $\hat{R}_{(I)}$ when it seems prudent to indicate the ideal in question. Obviously, $\hat{R}_{(I)}$ inherits a canonical ring structure from R.

It is convenient to realize $\hat{R}_{(I)}$ more concretely as a set of sequences.

<u>Proposition 3.10.5</u>

Let R be a ring, and I an ideal of R such that $\bigcap_{n=0}^{\infty} I^n = 0$. Then $\hat{R}_{(I)}$ may be viewed as the set of sequences of form

$$a = (a_1 + I,\ a_2 + I^2,\ a_3 + I^3, \ldots)$$

such that $a_j - a_{j+1} \in I^j$ for all $j \geq 1$. Here multiplication and addition act componentwise. R is embedded in $\hat{R}_{(I)}$ via the map

$$r \longmapsto (r+I,\ r+I^2,\ r+I^3, \ldots).$$

<u>Proof</u>

This is the content of the discussion of [2, p. 103]. □

For I to be sufficiently well-behaved, it is necessary to impose the A.R. property. Some useful properties of the I-adic completion subject to this condition appear in the following propositions.

Proposition 3.10.6

Let R be a right Noetherian ring, and let I be an ideal of R satisfying the right A.R. property. Let M and M' be finitely generated right R-modules with $M' \subset M$. Then the I-adic topology on M' coincides with the subset topology induced from the I-adic topology on M.

Proof

Note that by 3.6.7 I^k has the right A.R. property for all $k \geq 1$. Thus there is an $n \in \mathbb{N}$ such that

$$MI^n \cap M' \subset M'I^k \qquad (3.6.6).$$

Thus the classes of open sets coincide under these topologies. The result follows easily. □

The above proposition may be used to investigate the exactness of the completion functor.

Proposition 3.10.7

Let R be a right Noetherian ring with an ideal I which satisfies the right A.R. property, and such that $\bigcap_{n=1}^{\infty} I^n = 0$. If

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence of finitely generated right R-modules, then the sequence of I-adic completions

$$0 \longrightarrow \hat{M}_1 \longrightarrow \hat{M}_2 \longrightarrow \hat{M}_3 \longrightarrow 0$$

is also exact.

Proof

Exactly as in [2, Proposition 10.12], making use of Proposition 3.10.6. □

The flatness of $\hat{R}_{(I)}$ now follows from the above as in [2, Propositions 10.13 and 10.14].

Proposition 3.10.8

Let R and I be as in 3.10.7. Then:

(i) If M is a finitely generated right R-module, then $M \otimes_R \hat{R} \cong \hat{M}$. This isomorphism is given by the composition

$$M \otimes_R \hat{R} \longrightarrow \hat{M} \otimes_R \hat{R} \longrightarrow \hat{M} \otimes_{\hat{R}} \hat{R} \cong \hat{M}$$

where the first map is $\psi \otimes 1$ and $\psi : M \longrightarrow \hat{M}$ is the natural map.

(ii) $\hat{R}_{(I)}$ is a flat left R-module. □

From the flatness of $\hat{R}_{(I)}$, certain properties of the I-adic completion functor follow immediately:

Proposition 3.10.9

Let R and I be as in 3.10.7, then:

(i) $\hat{I} = I\hat{R}_{(I)} \cong I \otimes_R \hat{R}_{(I)}$;

(ii) $\widehat{I^n} = \hat{I}^n$;

(iii) $\hat{I}^n/\hat{I}^{n+1} \cong I^n/I^{n+1}$;

(iv) $\hat{I}$ is contained in the Jacobson radical of $\hat{R}_{(I)}$.

Proof

The elementary proof of this proposition may be found in [2, 10.15]. □

In particular, the above proposition yields:

Proposition 3.10.10

In the above situation, if $\hat{R}_{(I)}$ is realized as a set of sequences as in 3.10.5, then

(i) $\hat{I}^n = I^n\hat{R}_{(I)} = \{(r_1+I, r_2+I^2,\ldots) \in \hat{R}_{(I)} \mid r_j \in I^j, j=1,\ldots,n\}$;

(ii) $\hat{I}^n \cap R = I^n$.

Proof

Part (i) follows directly from 3.10.9, and part (ii) is then obvious. □

It is convenient at this point to recall the notion of the associated graded ring. Let R be any ring, and I an ideal of R. We define the *graded ring* of R at I, denoted $gr_I(R)$, to be the abelian group

$$R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

with componentwise addition. We may make $gr_I(R)$ into a ring by defining

$$[x + I^m].[y + I^n] = [xy + I^{m+n}]$$

and extending distributively. We recall that:

<u>Proposition 3.10.11</u>

Let R be a right Noetherian ring and I an ideal of R which satisfies the right A.R. property. Suppose in addition that $\bigcap_{n=1}^{\infty} I^n = 0$. Then

$$gr_I(R) \cong gr_{\hat{I}}(\hat{R}).$$

If, further, $gr_{\hat{I}}(\hat{R})$ is a right Noetherian ring, then so is $\hat{R}_{(I)}$.

<u>Proof</u>

That $gr_I(R) \cong gr_{\hat{I}}(\hat{R})$ follows from the Proposition 3. 10. 9. The second part is proved in, for example, [49, Theorem 13, p. 409]. □

Certain completions of rings are semiperfect; definitions necessitate a slight digression.

<u>Definition 3.10.12</u>

A ring R is called *semiperfect* if every idempotent of R/J is the image of an idempotent of R, and R/J is (semisimple) Artinian.

Let M be a right R-module. A *projective cover* P of M is a projective module P and a surjective map $\phi : P \longrightarrow M$ such that Ker ϕ is superfluous in P, i.e. if $P = \ker \phi + N$ for some submodule N of P then $N = P$.

The following results are well-known.

<u>Proposition 3.10.13</u>

Let $\phi : P \longrightarrow M$ be a projective cover of a right R-module M. If $\psi : Q \rightarrow M$ is an epimorphism and Q is projective, there is a homomorphism $\pi : Q \rightarrow P$ which makes the diagram

$$\begin{tikzcd} Q \arrow[dr, "\psi"] \arrow[dd, "\pi"'] & \\ & M \\ P \arrow[ur, "\phi"'] & \end{tikzcd}$$

commute. Hence if M is finitely generated, so is P.

<u>Proof</u>

The existence of π follows directly from the projectivity of Q. Now $P = \text{Ker } \phi + \text{Im } \pi$, so that π must be surjective. The last assertion follows as Q may clearly be chosen finitely generated when M is. $\square$

Proposition 3.10.14

Let R be a semilocal right Noetherian ring and M and P finitely generated right R-modules with P projective. Then $\phi: P \to M$ is a projective cover of M if and only if $\ker \phi \subset P.J$.

Proof

It is not hard to show that P.J is the unique maximal superfluous submodule of P, from which a proof follows easily. □

The following proposition is proved in [3, Theorem 2.1] and explains the relevance of projective covers.

Proposition 3.10.15

A semilocal ring is semiperfect if and only if every finitely generated right (and left) module has a projective cover. □

The link between semiperfect rings and completion is apparent in the next result.

Proposition 3.10.16

Let R be a semilocal ring which is complete in its J(R)-adic topology, and suppose that $\bigcap_{n=1}^{\infty} J^n = 0$. Then R is semiperfect.

Proof

One demonstrates that idempotents may be suitably lifted. See [22, 22.15]. □

The ability to lift idempotents may be refined to prove the following result of Müller, which links localizability of certain primes of R to the ring-decomposability of the completion.

Proposition 3.10.17

Let R be a Noetherian ring, and N a semiprime ideal of R. Denote the primes of R minimal over N by $P_1,\ldots,P_k$. Suppose that N is localizable and that the ideal NR_N satisfies the left and right A.R. properties in R_N. Then there is a one-to-one correspondence between the central idempotents of $\widehat{(R_N)}_{(NR_N)}$ and the subsets of $\{P_1,\ldots,P_k\}$ which have localizable intersection.

Proof

See Theorem 4 of [47]. □

3.11 Rees rings

This short section is devoted to setting out the definitions and elementary properties of Rees rings, which may be regarded to a certain extent as generalizations of polynomial rings in a single commuting indeterminate. This concept will prove useful in Chapter 4, where it is used to give a proof of the Invertible Ideal Theorem, and also yields some information of use in Chapter 5.

Let R be any ring, I an ideal of R and t an indeterminate. Consider the subring of R[t], the (commuting) polynomial extension of R, given by

$$R[It] = \{f \in R[t] \mid f = a_0 + a_1 t + \ldots + a_n t^n,\ a_j \in I^j,\ n > 0\}.$$

This ring will commonly be expressed as

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \ldots$$

for brevity. Notice that R[It] has a natural degree function, denoted deg(·), inherited by restriction from R[t]. We also define the highest coefficient of an element f of R[It] in the obvious fashion, and denote it by hc(f). We let hc(0) = 0 and deg(0) = -1.

The *Rees ring of* R *at* I, which will be denoted by $R[t^{-1}, It]$ is given by

$$R[t^{-1}, It] = R[It][t^{-1}]$$

i.e. the subring of $R[t,t^{-1}]$ generated by R[It] and t^{-1}.

Again, this ring will be more loosely denoted by

$$\ldots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus It \oplus I^2t^2 \oplus \ldots$$

and occasionally by R* when no confusion as to the ideal I involved is likely. One important property of the Rees Ring is immediately apparent:

<u>Lemma 3.11.1</u>

Let R be any ring and I an ideal of R. Then there is an isomorphism

$$gr_I(R) \cong R[t^{-1},It]/t^{-1}R[t^{-1},It],$$

where $gr_I(R)$ is the associated graded ring of R at the ideal I.

<u>Proof</u>

This is clear from the direct sum representation of $R[t^{-1},It]$ shown above. □

We may, of course, extend the definition of R[It] to modules. Specifically, for a right R-module M we may put

$$M[It] = M \oplus MIt \oplus MI^2t^2 \oplus \ldots$$

and make M[It] an R[It]-module in a straightforward manner. Such definitions are of course also possible for the Rees ring $R[t^{-1},It]$. However, for the purposes of Chapter 4 a slightly different approach to lifting right ideals to $R[t^{-1},It]$ will be necessary. The definition which we have in mind provides machinery in the case where the ideal I is invertible.

<u>Definition 3.11.2</u>

Let K be a right ideal of R, X an invertible ideal of R and $R^* = R[t^{-1}, Xt]$. Then define

$$K^* = \ldots \oplus (KX^{-2} \cap R)t^{-2} \oplus (KX^{-1} \cap R)t^{-1} \oplus K \oplus KXt \oplus KX^2t^2 \oplus \ldots$$

K^* becomes an R^*-module in the obvious way. Notice that for right ideals K containing X the definition reduces to that which might be expected, i.e.

$$K^* = \ldots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus KXt \oplus KX^2t^2 \oplus \ldots$$

A more thorough analysis of the correspondence between K and K^* is undertaken in Chapter 4, where the ideal structure of R^* is partially investigated.

4. THE INVERTIBLE IDEAL THEOREM

The much-celebrated principal ideal theorem of Krull occupies a pivotal position in the theory of commutative Noetherian rings. It states that a prime ideal of a commutative Noetherian ring minimal over a non-unit has rank at most one. By localizing and observing that an invertible ideal of a local ring is principal, one easily generalizes this result to primes of a commutative ring minimal over an invertible ideal. In recent years, these results have been extended to non-commutative Noetherian rings (see [33, 34 and 16]). We give here a conceptually simple proof of the so-called Invertible Ideal theorem which is partially analoguous to the commutative proof indicated above, in that the proof is accomplished via a reduction to the case of a prime ideal minimal over a single central element. The technique of localization is, however, not available and its position in the argument is instead occupied by the Rees ring of R (see Section 3.11). The proof is then completed by an adaption of the commutative argument as found, for example, in [36]. The reader is also referred to the papers of Jategaonkar on the subject, [33, 35], where another independent proof of the theorem is given which yields some additional information.

Much of the contents of this chapter have appeared in the author's publication [28].

As our proof effects a reduction by way of passing from R to the Rees ring R* at an invertible ideal, we must clearly begin by lifting certain properties from R to R*. The following proposition - and indeed its proof - are the appropriate variants on the Hilbert Basis Theorem. An independent proof has subsequently appeared in [58].

Proposition 4.1

Let R be a right Noetherian ring and X an invertible ideal of R. Then the rings R[Xt] and $R^* = R[t^{-1}, Xt]$ are also right Noetherian.

Proof

As R* is clearly a homomorphic image of a polynomial extension of R[Xt] in a single commuting indeterminate, it clearly suffices to show that R[Xt] is right Noetherian.

Let A be a right ideal of R[Xt]; we shall show that A is finitely generated and hence that R[Xt] is right Noetherian.

Define:

$$B_n' = \{hc(f) \mid \deg (f) = n \text{ or } f = 0, f \in A\}.$$

and put

$$B_n = B_n' X^{-n} \quad (n \in \mathbb{N}).$$

Then B_n' is a right ideal of R contained in X^n, so that there is a chain

$$B_0 \subset B_1 \subset B_2 \subset \ldots$$

of right ideals of R. Now there is some integer k with $B_k = B_{k+i}$ for all $i > 0$, or equivalently $B_k'X^i = B_{k+i}'$. For $i = 1,\ldots,k$ choose generators $b_{i,1},\ldots,b_{i,m_i}$ for B_i' over R. Further, choose $f_{i,j} \in A$ with $\deg(f_{i,j}) = i$ and $hc(f_{i,j}) = b_{i,j}$. We claim that the $f_{i,j}$ together generate A. Let $g \in A$, and suppose that $hc(g) = b$ and $\deg(g) = v$. Suppose first that $v > k$. Then

$$b \in B_v' = B_k'X^{v-k}$$

and so $b = \sum_j b_{k,j}x_j$ for suitable $x_j \in X^{v-k}$. Thus $x_j t^{v-k} \in R[Xt]$, hence

$$g - \sum_j f_{k,j}x_j t^{v-k}$$

lies in A and has degree strictly less than that of g. We may thus reduce to the case where $v < k$. Repeating a somewhat simpler version of the above argument, we finally obtain

$$g \in \sum_{i,j} f_{i,j}\, R[Xt]$$

as required. □

Then B'_n is a right ideal of R contained in X^n, so that there is a chain

$$B_0 \subset B_1 \subset B_2 \subset \ldots$$

of right ideals of R. Now there is some integer k with $B_k = B_{k+i}$ for all $i > 0$, or equivalently $B'_k X^i = B'_{k+i}$. For $i = 1,\ldots,k$ choose generators $b_{i,1},\ldots,b_{i,m_i}$ for B'_i over R. Further, choose $f_{i,j} \in A$ with $\deg(f_{i,j}) = i$ and $hc(f_{i,j}) = b_{i,j}$. We claim that the $f_{i,j}$ together generate A. Let $g \in A$, and suppose that $hc(g) = b$ and $\deg(g) = v$. Suppose first that $v > k$. Then

$$b \in B'_v = B'_k X^{v-k}$$

and so $b = \sum_j b_{k,j} x_j$ for suitable $x_j \in X^{v-k}$. Thus $x_j t^{v-k} \in R[Xt]$, hence

$$g - \sum_j f_{k,j} x_j t^{v-k}$$

lies in A and has degree strictly less than that of g. We may thus reduce to the case where $v \leq k$. Repeating a somewhat simpler version of the above argument, we finally obtain

$$g \in \sum_{i,j} f_{i,j} R[Xt]$$

as required. □

The Proposition has two worthwhile Corollaries. The first will be of use in Chapter 5.

Corollary 4.2

Let R be a right Noetherian ring, X an invertible ideal of R. Then the graded ring $gr_X(R)$ is right Noetherian.

Proof

This is a direct consequence of the Proposition and Proposition 3.10.11. □

Secondly, one may use the above to derive a previously stated result (3.6.3) via a classical commutative argument, namely:

Corollary 4.3

An invertible ideal X of a right Noetherian ring satisfies the right A.R. property.

Proof

Let E be a right ideal of R. As R[Xt] is right Noetherian, R[Xt] ∩ E[t] is a finitely generated R[Xt]-module. Let t^s be the highest power of t involved in a fixed finite generating set for this module. We have

$$\bigoplus_{n=0}^{\infty} (E \cap X^n)t^n = R[Xt] \cap E[t]$$

$$= (\bigoplus_{j=0}^{s} (E \cap X^j)t^j).R[Xt]$$

Comparing coefficients of t we have for $n > s$

$$E \cap X^n = \bigoplus_{j=0}^{s} (E \cap X^j)X^{n-j}$$

$$= (E \cap X^s)X^{n-s}. \quad \square$$

The slightly stronger condition satisfied by X in the last Corollary is known as the *strong* (right) *A.R. property*.

We return to the properties of the extension $R^* = R[t^{-1},Xt]$ of R where X is an invertible ideal. Recall that a right ideal K of R gives rise to a right ideal K^* of R^* given by

$$\ldots \oplus (KX^{-2} \cap R)t^{-2} \oplus (KX^{-1} \cap R)t^{-1} \oplus K \oplus KXt \oplus KX^2t^2 \oplus \ldots$$

(see Section 3.11). It is clear that were K a two-sided ideal of R, then K^* would be a two-sided ideal of R^*, provided that $KX = XK$. The next lemma concerns the behaviour of prime ideals of R subject to such conditioning, and forms the technical basis for our argument.

Lemma 4.4

Let R be a ring with an invertible ideal X.

(a) If P is a prime ideal of R such that $PX = XP$, then

(i) $P[Xt]$ is a prime ideal of $R[Xt]$;

(ii) P^* is a prime ideal of R^*.

(b) Suppose that R is right Noetherian. Let N/X be the nilpotent radical of R/X and $P_1,\ldots,P_k$ be the prime ideals of R minimal over X. Suppose also that $P_iX = XP_i$ for each i. Then N^* is nilpotent modulo X^* in R^*, and consequently N^*/X^* is the nilpotent radical of R^*/X^*.

Proof

(a) As $PX = XP$, $P[Xt]$ and P^* are easily seen to be ideals of the appropriate rings. We show first that $P[Xt]$ is a prime ideal of $R[Xt]$.

Let $f(t) = a_0 + \ldots + a_nt^n$ and $g(t) = b_0 + \ldots + b_st^s$ be elements of $R[Xt]$ such that

$$g(t)R[Xt]f(t) \subset P[Xt]$$

and $f(t) \notin P[Xt]$. We shall show that $g(t) \in P[Xt]$. We may assume that

$$f(t) = a_mt^m + \ldots + a_nt^n$$

where $a_m \notin PX^m$ and $a_i = 0$ for $i < m$. Now $b_0Ra_m \subset PX^m$, so that $b_0Ra_mX^{-m} \subset P$. As $a_mX^{-m} \not\subset P$, one has $b_0 \in P$. Therefore

$$(g(t) - b_0)R[Xt]f(t) \subset P[Xt]$$

so that

$$\begin{aligned} X^{-1}t^{-1}(g(t)-b_0)R[Xt]f(t) &\subset X^{-1}t^{-1}(P[Xt] \cap (Xt \oplus X^2t^2 \oplus \ldots)) \\ &= X^{-1}t^{-1}(PXt \oplus PX^2t^2 \oplus \ldots) \\ &= P[Xt]. \end{aligned}$$

Each polynomial in the set $X^{-1}t^{-1}(g(t) - b_0)$ has degree strictly less than that of g, so by induction $g \in P[Xt]$. Hence $P[Xt]$ is a prime ideal of $R[Xt]$.

Finally, we show that P^* is a prime ideal of R^*. For if a and b are elements of R^* and $aR^*b \subset P^*$, then one may choose integers r and s so that $X^rt^ra \subset R[Xt]$ and $bt^sX^s \subset R[Xt]$. We obtain

$$\begin{aligned} X^rt^ra\; R[Xt]bt^sX^s &\subset P^* \cap R[Xt] \\ &= P[Xt] \end{aligned}$$

from which it follows easily that P^* is indeed a prime ideal.

(b) By part (a), the P_i^* are prime ideals of R^*. As N^* is their irredundant intersection (Lemma 3.6.8), it is sufficient to check that N^* is indeed nilpotent modulo X^*.

As the direct approach appears notationally complex, we first prove: If A and B are ideals of R with $X \subset B \subset A$ and $XA = AX$, then $A^2 \subset B$ implies $(A^*)^2 \subset B^*$. Recalling that

$$A^* = \ldots \oplus Rt^{-1} \oplus A \oplus AXt \oplus AX^2t^2 \oplus \ldots$$

it can be seen that the coefficient of t^i in $(A^*)^2$ is a sum of three types of product:

(i) $AX^nAX^{i-n}, \quad 0 \leq n \leq i$

(ii) $RAX^{i-n}, \quad n < 0$

(iii) $AX^nR, \quad n > i.$

Each of these clearly lies in BX^i, so that $(A^*)^2 \subset B^*$. We apply the above to the proof at hand.

From $N^2 \subset N^2 + X$, it follows $(N^*)^2 \subset (N^2 + X)^*$, and from $(N^2 + X)^2 \subset N^4 + X$ one obtains $((N^2 + X)^*)^2 \subset (N^4 + X)^*$. Combining those inclusions yields $(N^*)^4 \subset (N^4 + X)^*$. Continuation of this process inevitably produces $(N^*)^k \subset (N^k + X)^* \subset X^*$ for large enough k. □

The next lemma is the pivot of our proof.

<u>Lemma 4.5</u>

Suppose that X is an invertible ideal of a ring R. Then the ideal X^* is generated by a single central element of R^*.

Proof

It is easily seen that $X^* = t^{-1}R^*$. □

A proof of the special case of the theorem is given next; it covers the case where X is generated by a central element of R. The argument is taken almost verbatim from [36], except that reduced rank replaces applications of the length function. The pieces of the general result are then assembled in Theorem 4.7.

Theorem 4.6 (Krull's Principal Ideal Theorem, non-commutative version).

Let R be a right Noetherian ring, $a \in R$ a central element of R and P a prime ideal of R minimal over aR. Then rank (P) $\leq$ 1.

Proof

Suppose that the theorem fails. One may immediately assume that R is prime and that $a \in R$ is non-zero, hence regular. Further, R contains a chain of primes $P \supsetneq P_1 \supsetneq 0$ with $a \notin P_1$. By Goldie's Theorem (3.3.2), P_1 contains a regular element y of R. Replacing a by a power if necessary, one may also assume that

$$ta^2 \in yR \Rightarrow ta \in yR. \quad (*)$$

It is easily seen that there is a module epimorphism

$$\theta : \frac{aR}{a^2R} \longrightarrow \frac{a^2R + yR}{a^2R + ayR}$$

given by $\theta(as + a^2R) = ys + a^2R + ayR$. Further θ is injective; for if $k \in R$ and

$$yk = a^2u + ayv \quad (u,v \in R)$$

we have immediately $ua^2 \in yR$, so that by (*) $ua = yw$ for some $w \in R$. Hence

$$\begin{aligned} yk &= ayw + ayv \\ &= y(aw + av). \end{aligned}$$

Cancelling the regular element y yields $k = a(w+v) \in aR$ as required.

We thus have

$$\frac{a^2R + yR}{a^2R + ayR} \underset{\theta}{\cong} \frac{aR}{a^2R} \cong \frac{R}{aR} \qquad (**)$$

On the other hand, if ρ denotes the reduced rank of right R/aR-modules, then additivity of ρ over short exact sequences (3.5.1(a)) gives

$$\rho\left(\frac{aR + yR}{a^2R + yaR}\right) = \rho\left(\frac{aR}{a^2R + yaR}\right) + \rho\left(\frac{aR + yR}{aR}\right)$$

$$= \rho\left(\frac{R}{aR + yR}\right) + \rho\left(\frac{aR + yR}{aR}\right)$$

$$= \rho(R/aR).$$

Combining with (**) and using additivity again, we obtain

$$\rho\left(\frac{aR + yR}{a^2R + yR}\right) = 0.$$

If N/aR is the nilpotent radical of R/aR, then there is some $c \in \mathcal{C}(N) \subset \mathcal{C}(P)$ with $ac \in a^2y + yR$ (Lemma 2.4 and Proposition 3.5.1(b)). Hence $ac \in a^2R + P_1$. Recalling that $a \notin P_1$, one easily finds that $c \in P_1 + aR \subset P$, a contradiction. □

Finally, we have:

<u>Theorem 4.7</u> (The Invertible Ideal Theorem) [16]

Let R be a right Noetherian ring, X an invertible ideal of R and P a prime ideal of R minimal over X. Then rank $(P) \leq 1$.

<u>Proof</u>

Suppose for a contradiction that rank $(P) > 1$, so that there is a chain of prime ideals $P \supsetneqq Q \supsetneqq Q'$ of R. Let $P_1,\ldots,P_n$ be the prime ideals of R minimal over X with, say,

$P = P_1$. By Lemma 3.6.9 (ii) conjugation by X permutes the P_i. Replacing X by a suitable power (say the lowest common multiple of the lengths of the orbits of the P_i under this conjugation) we may assume that $P_iX = XP_i$ for each i. As X lies in neither Q nor Q', X commutes with these primes by Lemma 3.6.9 (i). Lemma 4.4 now allows us to lift each of these primes to $R^* = R[t^{-1},Xt]$, yielding a chain $P^* \supsetneq Q^* \supsetneq Q'^*$ with P^* minimal over X^*. However, X^* is generated by a central element (Lemma 4.5), and this clearly contradicts Theorem 4.6. The proof is complete. □

5. COMPLETIONS AT INVERTIBLE IDEALS

In this chapter we shall be concerned with exploiting Proposition 4.1 (on the lifting of the right Noetherian property to R[Xt]) in the context of completions at invertible ideals. For the most part R will satisfy certain additional hypotheses. We obtain an easy proof of a result of Deshpande on the structure of the completion of a semilocal hereditary Noetherian prime (HNP) ring. Such a completion is a finite direct sum of complete HNP rings. This result is then specialized to obtain results of Gwynne and Robson and of Kuzmanovitch which deal with the completion of Dedekind prime rings.

The proofs given here differ from those in the literature [19, 37] in that they do not involve techniques related to Morita duality; they are direct ring-theoretic arguments. A further different and independent approach to these problems has recently appeared in [38].

Our first result is well known; it deals with the transfer of finite global dimension from R to its J-adic completion.

Theorem 5.1

Suppose that R is a right Noetherian semilocal ring whose Jacobson radical J satisfies the right A.R. property. Suppose also that the J-adic completion $\hat{R}_{(J)}$ is right Noetherian.

If there is some integer n such that for every simple right R-module S we have $pd_R(S) \leq n$, then $\hat{R}_{(J)}$ has finite global dimension, and indeed this dimension may not exceed n.

<u>Proof</u>

The theorem may be proved by applying Theorem 2.7 of [45] to obtain the right A.R. property for the Jacobson radical of the completion, and then invoking the main result of [5]. Alternatively, Theorem 11 of [20] may be employed to construct a proof. □

Both of the approaches mentioned above require the use of the Ext or Tor functors; for completeness we give a proof of Theorem 5.1 in the case n = 1 which suffices for our limited needs here.

<u>Proof</u> (n = 1 only)

It is well-known (and easy to prove) that $\bigcap_{n=1}^{\infty} J^n = 0$. We recall from Proposition 3.10.9 that $\hat{R}_{(J)}$ is a semilocal ring with Jacobson radical $\hat{J}$, and also that every finitely generated right $\hat{R}_{(J)}$-module has a projective cover (Propositions 3.10.16 and 3.10.15). For brevity we put $S = \hat{R}_{(J)}$. Now $pd_R\ (R/J) \leq 1$, so that by exactness of the completion functor (3.10.7) one may obtain:

$$\mathrm{pd}_S\,(S/\hat{J}) < 1.$$

Thus $\hat{J}$ is a projective right S-module. Let A be an arbitrary right ideal of S, and form an exact sequence

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\beta} A \longrightarrow 0 \qquad (*)$$

where P_S is finitely generated and projective and $K \subset P\hat{J}$ (3.10.15), and where i is an inclusion. As $\hat{J}$ is certainly flat, we derive the exact sequence

$$0 \longrightarrow K \otimes_S \hat{J} \xrightarrow{i\otimes 1} P \otimes_S \hat{J} \xrightarrow{\beta\otimes 1} A \otimes_S \hat{J} \longrightarrow 0.$$

Now $P \otimes_S \hat{J}$ may be identified with $P\hat{J}$ and $A \otimes \hat{J}$ with $A\hat{J}$. Under the former identification, $K \otimes \hat{J}$ becomes identified with $K\hat{J}$. We thus have a commuting diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i"] & P \arrow[r, "\beta"] & A \arrow[r] & 0 \\
0 \arrow[r] & K\hat{J} \arrow[r] & P\hat{J} \arrow[u, "\text{inc.}"'] \arrow[r, "\beta|_{P\hat{J}}"'] & A\hat{J} \arrow[u, "\text{inc.}"'] \arrow[r] & 0.
\end{tikzcd}$$

As $K \subset P\hat{J}$, the diagram immediately yields $K \subset K\hat{J}$. By Nakayama's lemma, $K = 0$. Thus (*) shows that A is projective. Theorem 3.7.1 yields $D(S) < 1$, i.e. S is hereditary, as required. □

Theorem 5.1 requires that the completion of R should be right Noetherian. In practice, this is a difficult condition to fulfill. Our supply of properly conditioned completions flows from Corollary 4.2; we give a result which could have been stated then.

Lemma 5.2

Let R be a right Noetherian ring, and X an invertible ideal of R. Then $\hat{R}_{(X)}$, the X-adic completion of R, is also right Noetherian.

Proof

Combine Lemma 4.2 and Proposition 3.10.11. □

Indeed, more may be said about the extension of X to $\hat{R}_{(X)}$.

Lemma 5.3

Let I and X be ideals of a ring R which are invertible in some over-ring S of R, and suppose that $\bigcap_{n=1}^{\infty} X^n = 0$. Then $I\hat{R}_{(X)}$ is an invertible ideal of $\hat{R}_{(X)}$.

Proof

By Proposition 3.10.8 (ii), $\hat{R}_{(X)}$ is a flat left R-module. There is therefore an injective composition

$$\hat{R}_{(X)} \cong R \otimes_R \hat{R}_{(X)} \hookrightarrow S \otimes_R \hat{R}_{(X)}.$$

to fulfill. Our supply of properly conditioned completions flows from Corollary 4.2; we give a result which could have been stated then.

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Let I and X be ideals of a ring R which are invertible in some over-ring S of R, and suppose that $\bigcap_{n=1}^{\infty} X^n = 0$. Then $I\hat{R}_{(X)}$ is an invertible ideal of $\hat{R}_{(X)}$.

Proof

By Proposition 3.10.8 (ii), $\hat{R}_{(X)}$ is a flat left R-module. There is therefore an injective composition

$$\hat{R}_{(X)} \cong R \otimes_R \hat{R}_{(X)} \hookrightarrow S \otimes_R \hat{R}_{(X)}.$$

Let I^{-1} be the inverse of I in S. Then

$$(I^{-1} \otimes 1)(1 \otimes I\hat{R}_{(X)}) = I^{-1}I \otimes \hat{R}_{(X)}$$

$$= 1 \otimes \hat{R}_{(X)}$$

so that $I\hat{R}_{(X)}$ is invertible in the over-ring $S \otimes_R \hat{R}_{(X)}$ of $\hat{R}_{(X)}$, as required. □

We may already derive the main result of Deshpande [19]. Firstly:

<u>Corollary 5.4</u>

Let R be a hereditary Noetherian prime ring with Jacobson radical J. Then the J-adic completion $\hat{R}_{(J)}$ of R at J is hereditary and Noetherian.

<u>Proof</u>

We dismiss the case $J = 0$, so that by Proposition 3.3.2 J is essential as a right ideal of R and hence R/J is Artinian (Theorem 3.8.1). R is thus semilocal. Theorem 3.8.3 states that J is invertible. As R is right and left Noetherian, two applications of Lemma 5.2 now show that $\hat{R}_{(J)}$ is Noetherian.

Since $\hat{J}$ is invertible (Lemma 5.3 and Proposition 3.10.9), it satisfies the right and left A.R. properties (3.6.3). Theorem 5.1 now yields $D(\hat{R}_{(J)}) \leq 1$. Finally, $D(\hat{R}_{(J)}) = 0$

would imply $J/J^2 \cong \hat{J}/\hat{J}^2 = 0$ (3.10.9), and then $J = 0$ by Nakayama's lemma. We may thus, if we wish, dispense with the possibility $D(\hat{R}_{(J)}) = 0$. □

To complete the main result of [19], we need now only apply the decomposition theorem of Chatters for hereditary Noetherian rings.

Theorem 5.5

Let R be a HNP ring with Jacobson radical J. Then $\hat{R}_{(J)}$ is a finite direct sum of HNP rings.

Proof

It has been shown in 5.4 that $\hat{R}_{(J)}$ is hereditary and Noetherian. Theorem 3.8.2 ensures that R is a finite direct sum of rings of the required type and an Artinian ring. However, as noted during the proof of the preceding result, $J(\hat{R}) = J.\hat{R}$ is an invertible ideal of $\hat{R}$. As any Artinian right ideal of $\hat{R}$ has finite length, some power of $\hat{J}$ annihilates any such right ideal (see Section 2). The invertibility of $\hat{J}$ now shows that R has no non-zero Artinian right ideals, as required. □

The number of summands occurring in this completion is the cardinality of the largest partition of the set of maximal ideals of R into sets with localizable intersection (3.10.17).

It is convenient to record the following obvious corollary.

Corollary 5.6

Let R be an HNP ring and X an ideal of R lying in the Jacobson radical J of R. Then $\hat{R}_{(X)}$ is a finite direct sum of HNP rings; indeed, $\hat{R}_{(X)} \cong \hat{R}_{(J)}$.

Proof

From Proposition 3.3.2 and Theorem 3.8.1, R/X is Artinian; hence $J^n \subset X$ for some $n \geq 1$. Therefore the X-adic and J-adic topologies are equivalent, so that $\hat{R}_{(J)} \cong \hat{R}_{(X)}$. The Corollary now follows directly from the theorem. □

The structure of the complete HNP rings arising from the above theorem has been investigated in [46]. Specifically, A ring R is a complete semi-local HNP ring if and only if R is isomorphic to the ring of n × n matrices

$$\begin{bmatrix} D_{m_1} & M_{m_1,m_2} & M_{m_1,m_3} & \cdots & M_{m_1,m_k} \\ M_{m_2,m_1} & D_{m_2} & M_{m_2,m_3} & \cdots & M_{m_2,m_k} \\ \vdots & & & & \\ M_{m_k,m_1} & M_{m_k,m_2} & \cdots & & D_{m_k} \end{bmatrix}.$$

where D is a complete rank one discrete valuation ring with maximal ideal M, $\sum_j m_j = n$ and D_j and $M_{i,j}$ denote the sets of $j \times j$ matrices over D and $i \times j$ matrices over M respectively.

The integers k, n and m_i are uniquely determined by R.

The results of Gwynne and Robson and of Kuzmanovitch may now be obtained.

Theorem 5.7

Let R be a Dedekind prime ring. If X is a non-zero ideal of R contained in the Jacobson radical of R, then $\hat{R}_{(X)}$ is a finite direct sum of local prime principal right and left ideal rings.

Proof

As noted in Corollary 5.6, we may immediately assume that X = J. As every rank one prime of R is localizable, (Theorem 3.8.4), an application of Proposition 3.10.17 shows that there are precisely n primitive idempotents of the centre of $\hat{R}_{(J)}$, say $e_1,\ldots,e_n$, where n is the (finite) number of rank one primes of R. By Theorem 5.5, $\hat{R}_{(J)}$ is a finite direct sum of HNP rings. As $R/J \cong \hat{R}/\hat{J}$, $\hat{R}$ has precisely n maximal primes. Therefore each ring $e_i\hat{R}_{(J)}$ must be local. But then $e_i\hat{R}_{(J)}$ is a principal right and left ideal ring by Proposition 3.8.5. □

One may pass more or less directly to the full results of [37, 29].

<u>Theorem 5.8</u>

Suppose that X is a non-zero ideal of a Dedekind prime ring R. Then $\hat{R} = \hat{R}_{(X)}$ is a sum of prime local pri-pli rings. The number of summands in this completion coincides with the number of maximal ideals of R containing X.

<u>Proof</u>

Again, $\hat{R}$ is a Noetherian ring. If N is the ideal of R such that $N/X = N(R/X)$ then $\hat{R} = \hat{R}_{(N)}$. From Proposition 3.10.9 it follows that $\hat{N}$ is the Jacobson radical of $\hat{R}$. As $C(N) \subset C(\hat{N})$ and the latter set consists of units of $\hat{R}$, one has $\hat{R} = \widehat{R_{C(N)}}$, this adic completion taken with respect to $NR_{C(N)}$. But $R_{C(N)}$ satisfies the conditions of 5.6. In combination with 3.10.17, the result follows. □

The remainder of this section is concerned with deriving some weaker results (under weaker hypotheses) on the structure of $\hat{R}_{(X)}$ without homological restrictions on R. In order to tackle a more general case of completion at a localizable invertible ideal it is necessary to develop a simple reduction technique. The next three lemmas are preparatory; their thrust is that there is an embedding

$$(\hat{R}_{(X)})_{C(\hat{X})} \hookrightarrow \widehat{(R_{C(X)})}_{XR_{C(X)}}.$$

Lemma 5.9

Let R be a prime right Noetherian ring, X an invertible ideal of R. Realize, as usual, $\hat{R}_{(X)}$ as the set of sequences

$$\{r = (r_1 + X, r_2 + X^2, \ldots) \mid r_i - r_{i+1} \in X^i\}.$$

If $c = (c_1 + X, c_2 + X^2, \ldots) \in \hat{R}_{(X)}$, then the following are equivalent:

(i) $c \in C'(\hat{X})$;

(ii) $c_j \in C'(X)$ for all j;

(iii) $c_1 \in C'(X)$.

Proof

Recall that $C'(X) = C'(X^j)$ for all j, and that $\hat{X}^j = X^j\hat{R}_{(X)} = \{(r_1 + X, r_2 + X^2, \ldots) \in \hat{R}_{(X)} \mid r_i \in X^i, i=1,\ldots,j\}$ (3.6.11).

(iii) $\Rightarrow$ (ii). Let $k \geq 1$ and suppose that $c_k \in C'(X)$. Then $c_k \in C'(X^k)$. Hence $c_{k+1} = c_k + x$ for some $x \in X^k$, so that $c_{k+1} \in C'(X^k) = C'(X)$. The result thus follows by induction.

(ii) $\Rightarrow$ (i). Let $c, r \in \hat{R}_{(X)}$ with representation as above, and assume that $c_j \in C'(X)$ for each j. Then $cr = 0$ forces $c_i r_i \in X^i$, and hence $r_i \in X^i$. Therefore $r = 0$.

(i) $\Rightarrow$ (iii) Let $c \in C'(\hat{X})$ and $r_0 \in R$. Define $r = (r_0 + X, r_0 + X^2, \ldots) \in \hat{R}_X$. Then

$$\begin{aligned} c_1 r_0 \in X &\Rightarrow c_j r_0 \in X \text{ for all } j \\ &\Rightarrow cr \in \hat{X} \\ &\Rightarrow r \in \hat{X} \\ &\Rightarrow r_0 \in X \qquad \square \end{aligned}$$

<u>Lemma 5.10</u>

Let R and X be as in Lemma 5.9. Then:

$$C_{\hat{R}_{(X)}}(\hat{X}) \subset C_{\hat{R}_{(X)}}(0).$$

<u>Proof</u>

If $c \in C_{\hat{R}_{(X)}}(\hat{X})$ then, in the usual notation, $c_i \in C(X^i)$ for all i (Lemma 5.9). Now for $r \in \hat{R}_{(X)}$

$$\begin{aligned} cr = 0 &\Rightarrow c_i r_i \in X^i \text{ for all } i \\ &\Rightarrow r_i \in X^i \text{ for all } i \\ &\Rightarrow r = 0. \end{aligned}$$

Consideration of symmetry proves the result. $\square$

The above lemma is utilized in producing the embedding we seek.

<u>Lemma 5.11</u>

Let R be a prime Noetherian ring and X an invertible ideal of R. Suppose that R/X has an Artinian left and right quotient ring. Then:

(i) X and $\hat{X}$ are localizable ideals of the rings R and $\hat{R}_{(X)}$ respectively.

(ii) Write $\hat{R}$ for $\hat{R}_{(X)}$ and $\widehat{R_{C(X)}}$ for $(\widehat{R_{C(X)}})_{(XR_{C(X)})}$.

Then there is a continuous embedding of $\hat{R}_{C(\hat{X})}$ into $\widehat{R_{C(X)}}$ and an associated diagram of canonical maps

$$\begin{tikzcd} & R_{C(X)} \arrow[dl] \arrow[dr] & \\ \hat{R}_{C(\hat{X})} \arrow[rr] & & \widehat{R_{C(X)}} \end{tikzcd}$$

which is commutative.

<u>Proof</u>

First notice that $C(X) = C(X^i)$ for all i. Small's Theorem (3.5.2), may now be applied to show that R/X^i has an Artinian quotient ring for all $i \geq 1$. As R/X^i and $\hat{R}/\hat{X}^i$ are canonically isomorphic, $C(X^i)$ is in particular an Ore set modulo $\hat{X}^i$. Now $\hat{X}$ is an invertible ideal of $\hat{R}$ (Lemma 5.3), and thus has the right A.R. property (see 5.2 and 4.3). A result of Smith (Proposition 3.6.4) shows that $\hat{X}$ (and indeed X itself) is localizable. The elements of $C(\hat{X})$ are regular by Lemma 5.10.

Let q be an element of $\hat{R}_{C(\hat{X})}$. By Lemma 5.9 we may write $q = ac^{-1}$ with

$$a = (a_1 + X,\ a_2 + X^2, \ldots) \qquad a_i \in R$$

$$c = (c_1 + X,\ c_2 + X^2, \ldots) \qquad c_i \in C_R(X).$$

As $X^i R_{C(X)} \cap R = X^i$, we have an embedding

$$f : \hat{R} \longrightarrow \widehat{R_{C(X)}}$$

under which

$$(r_1 + X,\ r_2 + X^2, \ldots) \longmapsto (r_1 + XR_{C(X)},\ r_2 + X^2 R_{C(X)}, \ldots).$$

As $C(X)$ is a right Ore set and thus satisfies the right common denominator property, it is easily seen that the sequence $\{c_i^{-1}\}$ is Cauchy with respect to the filtration $\{X^i R_{C(X)}\}$, so that f can be extended to a map

$$F : \hat{R}_{C(\hat{X})} \longrightarrow \widehat{R_{C(X)}}$$

which is an embedding by Lemma 5.10. Clearly, yet another embedding

$$G: R_{C(X)} \longrightarrow \hat{R}_{C(\hat{X})}$$

is induced by the inclusion of R into $\hat{R}$, and the composition $F \circ G$ is merely the embedding of $R_{(X)}$ into its completion.

It remains to check that the map F is continuous, and we thus compare the topologies induced on $\hat{R}_{C(\hat{X})}$. Let

$$t \in \widehat{X^n R_{C(X)}} \cap F(\hat{R}_{C(\hat{X})}).$$

Then $t = F((a_1 + X, a_2 + X^2, \ldots)(c_1 + X, c_2 + X^2, \ldots)^{-1})$ with $a_i \in R$ and $c_i \in C(X)$ for all i. Then

$$t = (a_1 c_1^{-1} + XR_{C(X)}, a_2 c_2^{-1} + X^2 R_{C(X)}, \ldots).$$

As $a_i c_i^{-1} \in X^i R_{(X)}$ for $i < n$ (3.10.10), we have $a_i \in X^i$ for such i. Therefore $a \in \hat{X}^n$ and $t \in \hat{X}^n \hat{R}_{C(\hat{X})}$. Thus

$$\widehat{X^n R_{C(X)}} \cap F(\hat{R}_{C(X)}) \subset F(\hat{X}^n \hat{R}_{C(\hat{X})}).$$

The opposite inclusion follows from the definition of F. Thus the natural topology on $\hat{R}_{C(X)}$ coincides with that induced by the inclusion into $\widehat{R_{C(X)}}$. The map F is thus continuous, as required. □

The conditions of Lemma 5.11 are not the weakest possible. By applying a result of McConnell ([45, Theorem 2.7]), one can allow R to be a prime Noetherian ring with an ideal X satisfying:

(i) the left and right A.R. properties;

(ii) $C(X)$ is a left and right Ore set;

(iii) $C(X) = C(X^i)$ for all i.

The proof requires only minor modifications.

For our closing results of this section we consider the existence of nilpotent ideals in the completion under more general hypotheses. Certain common ground is dealt with in the next lemma.

<u>Lemma 5.12</u>

Let R be a prime Noetherian ring and X an invertible ideal of R. If the localization $R_{C(X)}$ of R at X exists and is hereditary, then the completion $\hat{R}_{(X)}$ is a semiprime Noetherian ring.

<u>Proof</u>

As $R_{C(X)}$ is an HNP ring, $\widehat{(R_{C(X)})}_{(XR_{C(X)})}$ is a finite direct sum of HNP rings by Corollary 5.6, and is in particular semiprime. However, Lemma 5.11 ensures that this latter ring is also isomorphic to the completion of $(\hat{R}_{(X)})_{C(\hat{X})}$ in the $\hat{X}(\hat{R}_{(X)})_{C(\hat{X})}$-adic topology. Thus $(\hat{R}_{(X)})_{C(\hat{X})}$ is also semiprime, so that the nilpotent radical of $\hat{R}_{(X)}$ must be torsion with respect to $C(\hat{X})$. Yet $C(\hat{X})$ consists of regular elements by 5.10, so that $\hat{R}_{(X)}$ is semiprime. □

(i) the left and right A.R. properties;

(ii) $C(X)$ is a left and right Ore set;

(iii) $C(X) = C(X^i)$ for all i.

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Proof

As $R_{C(X)}$ is an HNP ring, $(\widehat{R_{C(X)}})_{(XR_{C(X)})}$ is a finite direct sum of HNP rings by Corollary 5.6, and is in particular semiprime. However, Lemma 5.11 ensures that this latter ring is also isomorphic to the completion of $(\hat{R}_{(X)})_{C(\hat{X})}$ in the $\hat{X}(\hat{R}_{(X)})_{C(\hat{X})}$-adic topology. Thus $(\hat{R}_{(X)})_{C(\hat{X})}$ is also semiprime, so that the nilpotent radical of $\hat{R}_{(X)}$ must be torsion with respect to $C(\hat{X})$. Yet $C(\hat{X})$ consists of regular elements by 5.10, so that $\hat{R}_{(X)}$ is semiprime. □

We consider now conditions on X sufficient to satisfy the hypotheses of Lemma 5.12.

Proposition 5.13

Let R be a prime Noetherian ring, X an invertible ideal of R. If either:

(i) X is prime; or

(ii) R is a maximal order

then $\hat{R}_{(X)}$ is a semiprime Noetherian ring.

Proof

(i) Each R/X^n has an Artinian quotient ring (3.5.2) so that as we have previously noted, X is localizable. $R_{C(X)}$ is thus a prime Noetherian local ring with invertible Jacobson radical, so is hereditary by Proposition 3.8.5. Apply Lemma 5.12.

(ii) Follows from Lemma 5.12 and Proposition 3.9.3. □

Note that part (i) of the above Proposition would go through for X a semiprime ideal of R if an appropriate variant of Proposition 3.8.5 could be found. We give such a modification below, based on our results on completions. A still more general result is given in [15, Theorem 1.5] in which projectivity of the Jacobson radical replaces invertibility, but at the expense of an application of a theorem of Strooker.

Recently, Chatters and Hajarnavis (unpublished) have also given a simple proof in this case.

Proposition 5.14

Let R be a Noetherian semilocal ring whose Jacobson radical J is invertible. Then R is hereditary.

Proof

As J satisfies the (right) A.R. property it is easily seen that $\bigcap_{n=1}^{\infty} J^n = 0$, and by Theorem 5.1 $\hat{R}_{(J)}$ is a right Noetherian ring of global dimension at most 1. $\hat{R}_{(J)}$ is thus a finite direct sum of HNP rings and an Artinian ring. By Lemma 5.3, $\hat{J}$ is invertible and it follows as in the proof of Theorem 5.5 that $\hat{R}_{(J)}$ has no Artinian left or right ideals. It now follows that $\hat{R}_{(J)}$, and hence R, is semiprime.

The remainder of the argument is standard: firstly, every maximal right ideal is projective. For if M is a maximal right ideal, then by choosing a complement K/J for M/J in the semisimple module R/J we obtain $J = M \cap K$ and $M + K = R$. From the sequence

$$0 \longrightarrow M \cap K \longrightarrow M \oplus K \longrightarrow M + K \longrightarrow 0$$

it is now immediate that M is projective.

If R fails to be right hereditary, there is a right ideal K maximal amongst non-projective rights ideals. K is an essential right ideal, and hence contains a regular element c (Proposition 3.3.2). We claim that R/K is an Artinian right module. Consider a chain of right ideals

$$R \supsetneqq I_1 \supsetneqq I_2 \supsetneqq \cdots \supsetneqq K$$

and put $I_j^* = \{q \in Q(R) \mid qI_j \subset R\}$. As the left module Rc^{-1} is finitely generated, the chain of submodules

$$I_1^* \subsetneqq I_2^* \subsetneqq I_3^* \subsetneqq \cdots$$

of Rc^{-1} must halt. The dual basis lemma then yields $I_k = I_{k+1}$ for all large k, as required.

Let S be a right ideal of R containing K such that S/K is a simple right R-module. There is a maximal right ideal M of R with $R/M \cong S/K$, and S is projective by maximality of K. By Schanuel's lemma (3.1.4), K is projective. This contradiction proves the result. □

Note that this proposition may be regarded as a converse of Theorem 3.8.3.

6. A CLASS OF MAXIMAL ORDERS INTEGRAL OVER THEIR CENTRES

Let R be a commutative Noetherian local ring. R is called a *regular local ring* if the Krull dimension of R equals the minimal number of generators of the maximal ideal of R. Such rings assume great importance in commutative algebra as they are the natural generalizations of certain rings arising in algebraic geometry, namely those local rings which are associated with non-singular points of irreducible varieties. A famous theorem of Serre characterizes regular local rings as those local Noetherian rings of finite global dimension (equal, in fact, to the Krull dimension of R). Further, a theorem of Auslander and Buchsbaum states that a regular local ring is a unique factorization domain (UFD). In particular, such a ring is an integrally closed domain, a result which may be proved more directly.

Turning now to non-commutative generalizations, it is perhaps not immediately clear what one should take as the correct definition of a "regular local" ring. However, following Serre's characterization in the commutative case, one might consider instead Noetherian local rings R of finite global dimension.

Some additional hypotheses on R would appear to be necessary for the development of a satisfactory theory. Indeed, in a recent paper [9], Brown, Hajarnavis and MacEacharn have considered such rings which are in addition assumed to be integral over their centres. For such a ring they have shown:

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Some additional hypotheses on R would appear to be necessary for the development of a satisfactory theory. Indeed, in a recent paper [9], Brown, Hajarnavis and MacEacharn have considered such rings which are in addition assumed to be integral over their centres. For such a ring they have shown:

(i) R is a prime ring whose Krull and global dimensions coincide;

(ii) $R = \cap R_p$ where p runs through the set of rank one primes of the centre Z(R) of R. Each such R_p is hereditary;

(iii) The centre Z(R) of R is a Krull domain.

We shall use the above results to prove:

(iv) R is a maximal order (Theorem 6.5 (ii)).

This property corresponds to that of integral closure in the commutative case. We deduce that if R is in addition a PI ring with centre Z, then R is a maximal Z-order in the sense of Fossum.

Our result covers the case where R is a Noetherian ring of finite global dimension finitely generated as a module over its centre which has previously been discussed in [59]. The proof given there will not, however, generalize to these circumstances. Moreover, our proof is somewhat more direct than that of [59]. The following example provides a ring satisfying our hypotheses which is not finitely generated as a module over its centre.

<u>Example 6.1</u> [9]

Let D be a division ring which is locally finite dimensional, but not finite dimensional, as a vector space over its centre. Then the localization of the polynomial ring

$D[x_1,\ldots,x_n]$ at the maximal ideal generated by the commuting indeterminates $x_1,\ldots,x_n$ is a local, Noetherian ring of global dimension n which is integral, but not finitely generated, over its centre. The reader will find more detail in [9].

The main result of this chapter has appeared in the author's paper, [27].

To avoid cumbersome notation we shall denote the direct sum of s copies of a right module M by $M^{\oplus s}$. We begin with the following well-known lemma:

<u>Lemma 6.2</u>

Let R be any ring, J its Jacobson radical, and P and Q finitely generated projective right R-modules. If P/PJ is a direct summand of Q/QJ as a right R/J-module, then P is a direct summand of Q.

<u>Proof</u>

We consider the diagram

$$\begin{tikzcd}
 & Q \arrow[d, "\nu_Q"] \arrow[ddl, "\phi"'] & \\
 & Q/QJ \arrow[d, "\pi"] & \\
P \arrow[r, "\nu_P"'] & P/PJ \arrow[r] & 0
\end{tikzcd}$$

where π is the projection onto the direct summand and ν_Q and ν_P are canonical. By projectivity, the map ϕ exists which makes the diagram commute. As $\pi\nu_Q$ is surjective, we obtain $P = \phi(Q) + PJ$. Applying Nakayama's lemma to $P/\phi(Q)$, it follows $P = \phi(Q)$. As P is projective it follows that P is isomorphic to a direct summand of Q. □

Since a local ring has a unique simple right module (up to isomorphism), it follows that such a ring has a unique finitely generated indecomposable projective right module. We shall, however, wish to apply this remark to certain semilocal localizations of a local ring, and thus require:

Proposition 6.3

Let R be a right Noetherian ring of finite right global dimension, and suppose that R has a unique finitely generated indecomposable projective right module P. Let $S = R_S$ be the classical localization of R at a right Ore set S of regular elements. Suppose that S is semilocal. Then S has a unique finitely generated indecomposable projective right module, namely $P \otimes_R S$.

Proof

Let Q be a finitely generated indecomposable projective right S-module. We may write $Q = q_1S + \ldots + q_tS$ with each $q_i \in Q$. Let

$$K = q_1R + \dots + q_tR,$$

and form an R-projective resolution

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow K \longrightarrow 0.$$

Each P_i may be chosen finitely generated, and hence is a finite direct sum of copies of P. Since $K \otimes_R S \cong Q$ and ${}_RS$ is flat, we have an exact sequence of S-modules

$$0 \longrightarrow P_n \otimes_R S \longrightarrow \dots \longrightarrow P_0 \otimes_R S \longrightarrow Q \rightarrow 0.$$

As Q is S-projective an easy induction on the length of this resolution shows that there are integers k and l such that

$$(P \otimes_R S)^{\oplus k} \oplus Q \cong (P \otimes_R S)^{\oplus l}.$$

If J is the Jacobson radical of S we obtain:

$$\frac{(P \otimes_R S)^{\oplus k}}{(P \otimes_R S)^{\oplus k}J} \oplus \frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus l}}{(P \otimes_R S)^{\oplus l}J}$$

Comparing the simple modules occurring on each side we must therefore have:

$$\frac{Q}{QJ} \cong \frac{(P \otimes_R S)^{\oplus(1-k)}}{(P \otimes_R S)^{\oplus(1-k)}J}$$

From Lemma 6.2 and the indecomposability of Q we deduce $Q \cong P \otimes_R S$, as required. □

Proposition 6.3 allows us to deal with certain localizations which happen to be hereditary, via the following result.

<u>Proposition 6.4</u>

Let R be a semilocal HNP ring with a unique finitely generated indecomposable projective right module Q. Then R is a principal right ideal ring.

<u>Proof</u>

Let I be a non-zero right ideal of R. We are to prove that I is principal, and so may assume that I is essential as a right ideal of R. Being projective, $I \cong Q^{\oplus s}$ for some integer s. Also, $R \cong Q^{\oplus t}$ for some t. As the uniform dimensions of I and R are equal, we have $I \cong R$ and I is right principal. □

Assembling these results yields the conclusion we seek.

<u>Theorem 6.5</u>

Let R be a local Noetherian ring of finite global dimension integral over its centre Z, and let P denote the set of all rank one primes of Z. Then:

(i) For each $p \in P$, R_p is a principal right and left ideal ring;

(ii) R is a maximal order.

Proof

Let $p \in P$. By the result quoted in the introduction, R_p is certainly an HNP ring, and is semilocal by Proposition 3.4.2. Further, Lemma 6.3 ensures that R_p has a unique finitely generated indecomposable projective right module, and thus is a principal right ideal ring (Proposition 6.4). Part (i) is now proved by symmetry.

We have $R = \bigcap_{p \in P} R_p$, again by the result in the introduction. If now I is a non-zero ideal of R and q lies in the quotient ring of R, then

$$qI \subset I \Rightarrow qIR_p \subset IR_p \text{ for all } p \in P$$

$$\Rightarrow q \in \bigcap IR_p = R,$$

the second implication following from the invertibility of the ideal IR_p of R_p. R is thus a maximal order by Proposition 3.9.1. □

Theorem 6.5 certainly fails should the requirement that R be local be weakened to one of semilocality. To see this let S be the ring of integers localized at 2, and using the usual notation, put

$$T = \begin{bmatrix} S & 2S \\ S & S \end{bmatrix}$$

Then T is a semilocal HNP ring finitely generated as a module over its centre. However, T is not a maximal order. For if I is the ideal

$$\begin{bmatrix} 2S & 2S \\ S & S \end{bmatrix}$$

of T, and

$$q = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

then $qI \subset I$ and q lies in the quotient ring of T, yet $q \notin T$. T is thus not a maximal order: Notice also that T must fail to satisfy the hypotheses of Proposition 6.4. As noted in [7], this ring is also a homologically homogeneous ring, in the sense of that paper.

Some remarks on the PI case are perhaps in order. It appears to be an open question whether a local, Noetherian prime PI ring of finite global dimension must be a maximal order. However, such a maximal order must in fact be integral over its centre [40, Theorem 3.2]. We indicate without detail a simple proof of this fact based on the trace ring (or characteristic closure of Schelter [53].

Let R be a prime PI ring. R has a simple Artinian quotient ring Q (Theorem 3.4.3). We let T = T(R) be the subring of Q generated by the centre of R and the coefficients of the reduced characteristic polynomials of the elements of R. We abbreviate T(R).R to just T.R; the ring T.R is known as the *trace ring* of R. According to [1, Theorem 2.4], one has:

(i) TR is integral over the central subring T;

(ii) TR is prime;

(iii) If R is Noetherian, then T.R is a finite R-module, hence also Noetherian.

(iv) R and T.R have a common non-zero ideal, denoted V.

Assume that R is a prime PI maximal order. As V is an ideal of T.R, a subring of Q, one has immediately that $T.R.V \subset V$, hence $T.R \subset R$ and therefore $R = T.R$. From condition (i) we see that R is integral over its centre.

There is one related consequence of Theorem 6.5 which is perhaps worth pointing out. It concerns the case where R is known to be both PI and integral over its centre (for example, finitely generated over its centre).

Corollary 6.7

Let R be a local Noetherian PI ring of finite global dimension integral over its centre Z. Then R is a maximal Z-order in the sense of Fossum.

Proof

Immediate from the theorem and Proposition 3.9.5. □

It is known that if p is a rank one prime of the centre Z of a maximal Z-order R, then R has exactly one prime P lying over p ([40, p.147]). We have been unable to prove an analogue of this result for maximal orders integral over their centres, even with additional hypotheses, in particular the case where R is local and of finite global dimension.

We close with some remarks on related results. A second class of local Noetherian rings of finite global dimension known to be maximal orders is provided by the (local) Noetherian A.R. rings of finite global dimension ([8]). To date there seems to be no sensible class of non-commutative local rings of finite global dimension which are known to be suitably generalized UFDs.

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