

RESEARCH ARTICLE

Rendezvous search with markers that can be dropped at chosen times

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Abstract

The Rendezvous search problem asks how two noncommunicating players, who move at unit-speed after a random placement in the search region (in this article, the line), can minimize their expected meeting time. Baston and Gal (2001) solved a version where each player leaves a marker at his starting point, for example parachutes left after jumping into the search region. Here we consider that one or both players have markers which they can drop at chosen times. When the players are placed facing random directions at a distance D on the line the optimal expected meeting times R are known to be as follows: With no markers, $R = 13D/8$; with one each dropped at the start, $R = 3D/2$. Here we show that when only one player has a marker, but it can be dropped at any time, we still have $R = 3D/2$, obtained by dropping the marker at time $D/4$. Having both players drop markers at chosen times does not further reduce R . We adopt a new algorithmic approach which first fixes the dropping times and then reduces the resulting problem to a finite one. We also consider evanescent markers, which are detectable for a specified time T after being dropped, modeling pheromone scent markers used by some species in mate search or stain dropped by sailors. From a less theoretical and more practical point of view, we can see our problem as that faced by two hikers who get separated when they are walking not too far from a coast. A reasonable Rendezvous strategy is for each of them to head in the general direction of the coastline. On reaching the sea, they can calculate (based on their speed and time taken to reach to coast) the maximum distance along the coastline where the other one arrives. Then they can use our solution, taking that distance as our parameter D . Of course if it turns out that they reached the sea at points closer than D , their Rendezvous time will only be shorter. So our solution can be interpreted as a worst case expected meeting time. The marker might be a note scribbled on the sand, or a pile of rocks. Perhaps before getting separated they would have decided which roles (I or II) each would take up, being careful and cautious hikers. So in particular the hiker taking the role of the marker placer would put down a marker of some sort when he had walked along the coast one quarter the maximum distance to his partner. If the other finds it, he would know for sure the direction of the hiker who dropped it. Another thing to take away from this article is our result that says “two markers are not better than one.” This observation has some significance in

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the mate search context, where males and females use pheromone deposits to help find each other. In most species, it is only the females who can make such deposits, and the males can detect it. Giving both species the depositing facility would be more costly in terms of energy and, given our negative result, might not significantly reduce Rendezvous times. So in a speculative sense, our result has some explanatory value in this context.

KEYWORDS

algorithmic solution, markers, optimal solution, Rendezvous problem, Rendezvous search on the line

1 | INTRODUCTION

The asymmetric Rendezvous problem on the line, introduced by Alpern and Gal (1995) seeks the least expected time R (called the Rendezvous value) required for two unit-speed players to meet, after initially placed a distance D apart on the line and faced in random directions. Alpern and Gal found a strategy pair which minimizes the meeting time with $R = 13D/8$. It is a modification of the “wait for Mommy” strategy where one player searches and the other stays still. Subsequently Baston and Gal (2001) considered a variation they called “markstart Rendezvous,” where both players leave markers at their starting points which can be detected by the other player on arriving at that location. They found that the associated markstart Rendezvous value is $R_{ms} = 3D/2$. Here we consider a generalization of the Baston-Gal idea by giving one or *both* players a marker which can be dropped at any time. Thus a player’s strategy consists of a path (relative to his starting point and direction) and a dropping time. We find that when only one player has a marker which can be dropped at *any* time, the Rendezvous value is given by $R_1 = 3D/2$ and the optimal dropping time is $D/4$. If that marker must be dropped off at the start, it is worthless in that the Rendezvous value remains at $13D/8$, the Rendezvous time with no markers. On the other hand, if we give both players a marker which they can each drop off at a time of their choice, the Rendezvous value is no better than the $3D/2$ which we would have if only one of them had a marker, or if they both had to drop at the start.

We obtain these results in part by a new algorithmic approach in which we first fix the dropping time(s) and then reduce this problem to a finite set of admissible strategies which can be solved by computational enumeration. This approach might enable similar problems in higher dimensions to be solved. We also consider a variation in which the markers have a limited shelf life T , after which they cannot be detected. One interpretation is that the markers are pheromones which evaporate over time. Another is stain which can be dropped by sailors but won’t last forever, or flares that can be seen for a short period of time.

2 | LITERATURE REVIEW

The Rendezvous search problem was first proposed by Alpern in a seminar given at the Institute for Advanced Study, Vienna (Alpern, 1976). Many years passed before the problems presented there were properly modeled. The first model, where the players could only meet at a discrete set of locations, was analyzed in Anderson and Weber (1990). This difficult problem was later solved for three locations in Weber (2012). Rendezvous-evasion on discrete locations was studied (Lim, 1997) and solved for two locations (boxes) in Gal and Howard (2005).

The Rendezvous search problem for continuous space and time, including the infinite line, was introduced in Alpern (1995). The player-asymmetric form of the problem (used in this article), where players can adopt distinct strategies, was introduced in Alpern and Gal (1995). In Baston and Gal (2001), the authors allowed the players to leave markers at their starting points. The last two articles form the starting point of the present article. The corresponding player-symmetric problem on the line was developed in Anderson and Essegiaier (1995) and successively in Baston (1999), Gal (1999), Han, Du, Vera, and Zuluaga (2008). These articles assumed that the initial distance between the players on the line was known. The version where the initial distance between the players is unknown was studied in Baston and Gal (1998), Alpern and Beck (1999), Alpern and Beck (2000) and Ozsoyeller, Beveridge, and Isler (2013).

The continuous Rendezvous problem has also been extensively studied on finite networks: the unit interval and circle (Howard, 1999); arbitrary networks (Alpern, 2002b); planar grids (Anderson & Fekete, 2001; Chester & Tütüncü, 2004); and the star graph (Kikuta & Ruckle, 2007).

An application of Rendezvous to robotic exploration is given in Roy and Dudek (2001). An application of Rendezvous to the communications problem of finding a common channel is given in Chang, Liao, and Lien (2015). A survey of the Rendezvous search problem is given in Alpern (2002a).

3 | FORMALIZATION OF THE PROBLEM(S)

We begin by presenting the formalization of the problem when there are no markers, as given in Alpern and Gal (1995).

Two players, *I* and *II*, are placed a distance D apart on the real line, and faced in random directions which they call “forward.” Their common aim is to minimize the expected amount of time required to meet. They each know the distance but not the direction to the other player. They are restricted to moving at unit-speed, so their position, relative to their starting point and measured in their “forward” direction, is given by a function $f(t) \in \mathcal{F}$ where,

$$\mathcal{F} = \{f : [0T] \rightarrow \mathbb{R}, f(0) = 0, |f(t) - f(t')| \leq |t - t'|\}, \quad (1)$$

for some T sufficiently large so that Rendezvous will have taken place. In fact optimal paths turn out to be much simpler. We will see that optimal paths are piecewise linear with slopes ± 1 and so they can be specified by their turning points. Suppose player *I* chooses path $f \in \mathcal{F}$ and Player *II* chooses path $g \in \mathcal{F}$. The meeting time depends on which way they are initially facing. If they are facing each other, the meeting time is given by,

$$t^1 = t^{\rightarrow\leftarrow} = \min\{t : f(t) + g(t) = D\}.$$

If they are facing away from each other, the meeting time is given by,

$$t^2 = t^{\leftarrow\rightarrow} = \min\{t : -f(t) - g(t) = D\}.$$

If they are facing the same way, say both left, and *I* is on the left, the meeting time is given by,

$$t^3 = t^{\leftarrow\leftarrow} = \min\{t : -f(t) + g(t) = D\}.$$

If *I* is on the left and they are both facing right, the meeting time is given by,

$$t^4 = t^{\rightarrow\rightarrow} = \min\{t : +f(t) - g(t) = D\}.$$

To summarize, the four meeting times when strategies (paths) f and g are chosen are given by the four values,

$$\min\{t : \pm f(t) \pm g(t) = D\}.$$

The Rendezvous time for given strategies is their expected meeting time

$$R(f, g) = \frac{1}{4}(t^1 + t^2 + t^3 + t^4). \quad (2)$$

The Rendezvous value \bar{R} is the optimum expected meeting time,

$$\bar{R} = \min_{f, g \in \mathcal{F}} R(f, g) = R(\bar{f}, \bar{g}). \quad (3)$$

We now illustrate these ideas by exhibiting two specific strategy pairs. We first consider the so called “Wait for Mommy” strategy pair $[f_w, g_w]$ in which Baby (Player *I*) stays still ($f_w(t) \equiv 0$) while Mommy looks for him by first going to $+D$ and then to $-D$ ($g_w(t) = t$ for $t \leq D$ and $g_w(t) = D - (t - D) = 2D - t$ for $t \geq D$). In this case $t^1 = t^4 = D$ while $t^2 = t^3 = 3D$, with a mean Rendezvous time of $2D$. This strategy pair was improved in Alpern and Gal (1995) to the strategy pair $[f^*, g^*]$, called the Modified Wait for Mommy, where Mommy still plays $g^* = g_w$ (goes to $+D$ and then to $-D$) but Baby goes out at time 0 to meet a possibly approaching Mommy at time $D/2$ and then returns to his start to meet

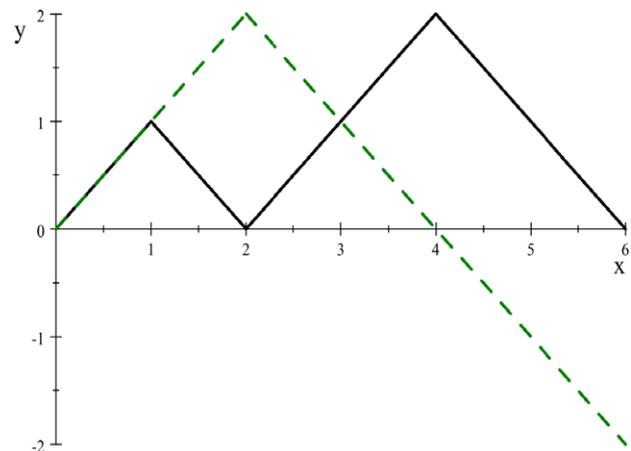


FIGURE 1 Plots of f^* (green dashed) and g^* (black), $D = 2$ [Color figure can be viewed at wileyonlinelibrary.com]

Mommy if she is coming from the other direction. If they haven’t met by time D , Baby knows that Mommy went in the wrong direction and is now approaching him, but again does not know from which direction. More precisely, we have,

$$f^*(t) = \begin{cases} t & t \leq D/2, \\ D/2 - (t - D/2) & D/2 \leq t \leq D, \\ t - D & D \leq t \leq 2D, \\ D - (t - 2D) & 2D \leq t \leq 3D = \bar{T}. \end{cases} \quad (4)$$

$$g^*(t) = \begin{cases} t & t \leq D, \\ D - (t - D) & t \geq D. \end{cases}$$

The pair $[f^*, g^*]$ keeps the Wait for Mommy meeting times $t^2 = 3D$ and $t^3 = D$ but advances the other two to $t^1 = D/2$ and $t^4 = 2D$. The strategies f^* and g^* are drawn together in Figure 1 for initial distance $D = 2$.

A useful way to see the meeting times for a strategy pair, say $[f, g]$, is to plot the Player *I* strategy starting at location (height) 0 and to plot four versions of the Player *II* strategy: starting at $+D$ and facing up or down; starting at $-D$ and facing up or down. We call these four motions respectively the four “agents” of Player *II*. In this way, the problem is derandomized and player *I* must Rendezvous with each deterministic agents instead of with random player *II*. Specifically, the agents 1, 2, 3, 4 of *II* respectively follow paths $D - g(t)$, $-D - g(t)$, $-D + g(t)$ and $+D + g(t)$. If we solve for $f(t^i)$ in the definition of t^i we see that t^i is the first time that $f(t)$ (Player *I*) meets agent i of Player *II*. For the pair $[f^*, g^*]$ the meeting times t^i are given by,

$$t^1 = D/2, t^2 = 3D, t^3 = D \text{ and } t^4 = 2D,$$

with Rendezvous time

$$R(f^*, g^*) = (D/2 + 3D + D + 2D)/4 = 13D/8.$$

We have plotted f^* and the four agents $\pm D \pm g^*(t)$ of Player *II* in Figure 2 (the “Wait for Mommy” strategy pair), circling the meeting points of Player *I* with the four agents of *II*.

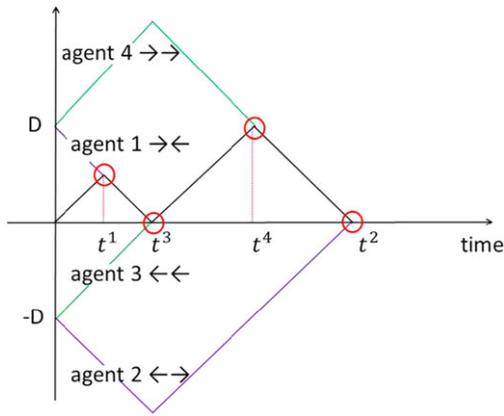


FIGURE 2 Plot of f^* and four agents of g^* [Color figure can be viewed at wileyonlinelibrary.com]

It is sometimes useful to label the four meeting times $\{t^1, t^2, t^3, t^4\}$ of a strategy pair in increasing order. We do this by using subscripts rather than superscripts. In general we have $t_1 \leq t_2 \leq t_3 \leq t_4$, where $\{t_1, t_2, t_3, t_4\} = \{t^1, t^2, t^3, t^4\}$. For example, the pair $[f^*, g^*]$ has increasing meeting times of $t_1 = t^1 = D/2$, $t_2 = t^3 = D$, $t_3 = t^4 = 2D$, $t_4 = t^2 = 3D$. Thus when meeting times are given in subscripted form, they are being listed in the order of meeting.

Another notational convention, for strategies $[f, g]$ that are piecewise linear with slopes ± 1 , is to assume that they start in the forward direction and simply to list their turning points in a square bracket. Thus $f^* = [D/2, D, 2D]$ and $g^* = [D]$. It has been shown in the original “no marker” case that optimal paths are of the form,

$$f = [f_1, f_2, \dots, f_k], \tag{5}$$

where the numbers f_j are the turning times of the path $f(t)$. When a player has a marker to drop, the strategy will be given in the form,

$$f = [x; f_1, f_2, \dots, f_k], \tag{6}$$

where the marker is dropped at time x .

If a player comes upon a marker dropped by the other player, he learns the direction of the other player. So we can assume the following.

Remark 3.1 If a player comes across a marker left by the other, he moves at unit-speed in the direction of the other player until they meet.

We summarize the two known results in the following theorem.

Theorem 1 (Alpern & Gal, 1995; Baston & Gal, 2001)

1. [I0] If neither player has a marker, an optimal strategy pair is given by the “Modified Wait for Mommy” pair as defined in (4), and the Rendezvous value is $13D/8$.
2. [I7] If both players have a marker which must be dropped at time $x = 0$, it is optimal for both

players to adopt the strategy $[0; D]$. In this case the Rendezvous value is $3D/2$. (See Theorem 9 and Figure 7 for more details.)

We prove in Proposition 3 that optimal strategies are of the form of (6). The computational implication of this result is that there are only a finite number of strategies that may be optimal, provided that x is fixed. Hence, by checking all of them we are able to find out the optimal strategies when x is known. By computing the optimal solutions for x belonging to a mesh we identify upper bound for our problems, see Section 4. In Section 5, we prove the pivotal Proposition 3 and describe how implementation of the program can be done. Finally, in Section 6 we show how to derive lower bound of the values of the games given the optimal solutions computed for fixed values of x . In Section 7 we present the results obtained when both players have a marker to drop off, showing that two markers are no better than one. In Section 9 we present results for evanescence markers which can be detected within a fixed time of being dropped.

4 | SOLUTION TO THE ONE MARKER GAME

In this section we present our solution, Theorem 2, to the single marker game, where only Player II has the single marker. We present a strategy pair (\hat{f}, \hat{g}) where the marker is dropped by II at time $D/4$, which is easily shown to have Rendezvous time $3D/2$. The second part of the proof of Theorem 2, that no other strategy does better, is given in Section 8 with the additional assumption that Player II always turns when dropping the marker. This supplementary assumption is very reasonable and likely to be true. However, in this article we prefer to keep advantage of the results of numerical computations and do not delve into a too much complicated proof. An numerical proof that no strategy does much better will be given in Section 6.

Theorem 2 (Single Marker Game) A solution for the Rendezvous problem on the line with a single marker (possessed by II) is as follows:

$$\hat{f} = [3D/4], \hat{g} = [D/4; D/4, 3D/4, 7D/4].$$

That is, the Player II with the marker drops it at time $D/4$. The Rendezvous value for this problem is given by

$$\bar{R} = R(\hat{f}, \hat{g}) \leq 3D/2.$$

For turn-on-drop strategies (see Section 8), we have

$$\bar{R} = R(\hat{f}, \hat{g}) = 3D/2.$$

Proof It is easily calculated, as in Figure 4, that the meeting times are given by

$$t_1 = t^4 = 3D/4, t_2 = t^1 = D, t_3 = t^3 = 7D/4, t_4 = t^2 = 5D/2.$$

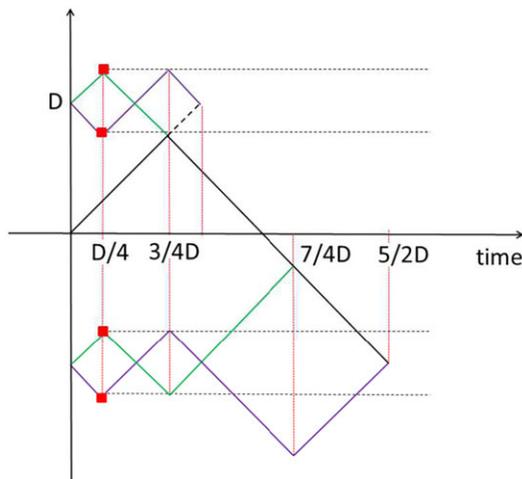


FIGURE 3 Solution of the one marker game. Horizontal dotted lines indicate stationary location of dropped marker by various agents. Dashed line indicates I's motion upon finding marker [Color figure can be viewed at wileyonlinelibrary.com]

Consequently we have

$$R(\hat{f}, \hat{g}) = \frac{3D/4 + D + 7D/4 + 5D/2}{4} = 3D/2.$$

This proves that $\bar{R} \leq 3D/2$. ■

The second part of Theorem 2, for strategies which turn when dropping a marker (turn-on-drop strategies), is proved in Section 8.

In Section 6 we show that the Rendezvous value for this problem is very close (or equal) to $3D/2$.

Notice that only the player playing \hat{g} has a marker at disposal. In Figure 3, Player II plays \hat{g} and the marker is dropped off at time $D/4$. We observe that at time $3D/4$ the Player I trajectory forks given that he finds the marker of Player II or not.

5 | PROPERTIES OF SOLUTION TO ONE OR TWO MARKER GAMES

In this section we present a general result on the turning points of optimal solutions to the one or two marker (one marker for each player) games. We show moreover that if a strategy pair is a Nash equilibrium in such a game then a player following either of these strategies can only turn at a limited number of times: when he drops or finds a marker, or (calling him Player I) when he meets an agent of the other player. Since there are four meeting times and only one marker there are at most six turning times and hence at most 4096 strategies to check for optimality.¹ The following result, which restricts turning times, is similar in spirit to previous results such a Lemma 5.1 of Alpern and Gal (1995) or Theorem 16.10 of Alpern and Gal (2003). The optimality condition of

Proposition 3 holds not only for optimal strategies (those minimizing the Rendezvous time) but for strategy pairs where if either unilaterally changes the Rendezvous time cannot decrease (Nash equilibrium of a common interest game).

Proposition 3 *Let G be any Rendezvous game on the line where each player has at most one marker. Then in any Nash equilibrium (NE) for G (and in particular at any optimal strategy pair) each player moves at unit-speed in a fixed direction (no turns) on each of the time intervals J determined by times 0 , and the following times c :*

- 1 *The meeting times $c = t^i$ when he meets the agent i of the other player*
- 2 *The times c when he finds the marker dropped by agent i .*
- 3 *The time $c = x$ that he drops off a marker*

Proof Assume on the contrary that for some NE strategy pair (f, g) , Player I (say) fails the condition on some time interval $J = [b, c]$ of the asserted type. Suppose his path is given by $f(t)$. There are three cases, depending on the what happens at time c .

1 At time $c = t^i$ Player I first meets agent i .

Since the stated condition fails on J , Player I can modify his strategy inside the interval J so that he arrives at the meeting location $f(c)$ at an earlier time $c - e$. At time $c - e$, agent i of Player II is either at location $f(c)$ or lies in some direction (call this i 's direction) from $f(c)$. In the former case the meeting with i is moved forward to time $c - e$. So Player I can stay there in until time c and then resume his original strategy, so all other meeting times are unchanged. Otherwise, Player I goes in i 's direction at unit-speed on interval $[c - e, c - e/2]$ and then back to $f(c)$ at time c , when he resumes his original strategy. This brings the meeting time with i no later than $c - e/2$, without changing any other meeting times, lowering the expected meeting time. In either case the expected Rendezvous time is lowered, contradicting the assumption that f was an optimal response to g .

2 At time c , Player I finds the marker dropped by agent i .

If the marker has just been dropped off at time c , then I also meets agent i at time c , so the previous case applies. Otherwise there is some earliest time $c - e$ that I can arrive at location $f(c)$. Suppose Player I modifies his strategy (path) on J so that he arrives at $f(c)$ at time $c - e$. If the

¹Indeed, at each turning point there are at most 4 possible motions resulting from Player I and Player II turning or keeping the same direction. Hence, the upper bound results $4^6 = 4096$.

marker is not yet present, then Player I continues until meeting agent i and goes back at $f(c)$ before time c and then resumes with the original strategy. The meeting must occur because agent i drops off the marker at $f(c)$ before time c . If the marker is present, after time $c - e$ Player I continues towards II (he now knows the direction) and will reach him earlier than in his original strategy.

- 3 **At time $c = x$, Player I drops off a marker found at later time.** Suppose Player I modifies f to get earlier to the dropoff location $f(c)$ at time $c - e$, drops off the marker, and then stays still until time c , and resumes with the original strategy f . After time $c - e$, case 1. or 2. occurs and, because I is not moving at full speed on the time interval starting at time $c - e$ the strategy can be further refined. This contradicts the assumption that f was a best response. ■

6 | SINGLE MARKER STRATEGY WHEN DROPPING TIME IS KNOWN

In this section we consider a simpler game $G(x)$ where the player (II) possessing the marker must drop it at a fixed time x , and we show how to calculate the Rendezvous value $\bar{R}(x)$ for this restricted game. A corollary of Proposition 3, Corollary 4 below, shows that for any fixed dropping time x , there are only a finite number of potentially optimal strategy pairs for the game $G(x)$, and we develop an algorithm that quickly optimizes over these possibilities. By letting x vary within a fine mesh, we can approximate the solution to the unrestricted single marker game. Bounds on the values $\bar{R}(x)$ when x does not belong to the mesh of calculated values can be attained by our variational result Proposition 5.

We begin by applying Proposition 3 to show that the search for a solution to the game $G(x)$ can be restricted to a finite number of strategy pairs.

Corollary 4 *In the restricted game $G(x)$ there are only a finite number of candidates for optimality and these admit representations as in (5) and (6).*

Proof First assume the marker must be dropped at time x . The first potential turning time is given by $c = \min(x, D/2)$, according to Proposition 3. Assume $x < D/2$. Then we branch into two cases, depending on whether Player II turns when he drops the marker. If he doesn't turn, the next potential turning time is $D/2$, which is a meeting time. By similarly branching and looking at all cases, we generate a finite number of potentially optimal strategy

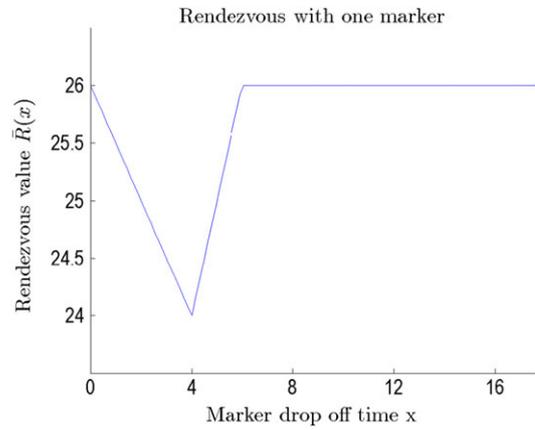


FIGURE 4 Value of the marker game computed dropping times in a regular mesh with initial distance $D = 16$ [Color figure can be viewed at wileyonlinelibrary.com]

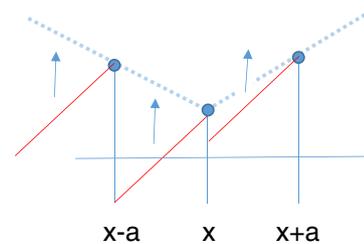


FIGURE 5 Illustration of Proposition 5. The dots represent the values of $\bar{R}(x)$ computed on the mesh $[x - a, x, x + a]$, the dotted line is obtained by linear interpolation between the points and the lines with slopes α represent lower bound of the function $\bar{R}(x)$ for values outside the mesh [Color figure can be viewed at wileyonlinelibrary.com]

pairs, and determine a restricted optimal solution satisfying (5) and (6). ■

We note that this result also applies to the two marker game, where we shall use it in Section 7.

We know from Corollary 4 that the solution can be computed if the dropping time is fixed. To bound the optimal Rendezvous values \bar{R} for the marker game we compute $\bar{R}(x)$ for x at a discrete mesh of values for the dropping time x , as plotted in Figure 4,² and use these values to bound \bar{R} . Indeed, Proposition 5 makes possible the derivation of bounds for $\bar{R}(x)$ for dropping times x that do not belong to the mesh of computed values. This proposition is illustrated in Figure 5.

Proposition 5 *The Rendezvous value $\bar{R}(x)$ of the game if the marker is dropped off at time x satisfies for $\alpha \geq 0$*

$$\bar{R}(x + \alpha) \leq \bar{R}(x) + \alpha. \tag{7}$$

That is, dropping off the marker a little later does not increase the Rendezvous times very much.

²The program may be found at <http://cui.unige.ch/~leonep/AlpernLeoneJournal/prgRDVOneMarkPlotArticle.java>.

Proof Let $(f(t), g(t))$ be an optimal strategy with dropoff time x , with Rendezvous time $\bar{R}(x)$. Consider a new strategy where both players wait for a time α and then follow the original strategy, that is, they adopt $(f(t - \alpha), g(t - \alpha))$. The new strategy has dropoff time $x + \alpha$ and Rendezvous time $\bar{R}(x) + \alpha$, which implies inequality 7. ■

Corollary 6 *The function $\bar{R}(x)$ is left continuous. It follows that for any interval $(x, y]$ the function $\bar{R}(z)$ has a lower bound depending on its values at the endpoint given by*

$$\bar{R}(z) \geq m(x, y) = \bar{R}(y) + (x - y), \quad (8)$$

with $z \in (x, y]$.

To estimate the minimum of $\bar{R}(x)$, we apply Corollary 6 on each interval of the form $(0.00016n, 0.00016(n + 1)]$ forming a partition of $(0, D]$ and take the minimum. This is the interval $(3.99984, 4]$, with,

$$m(3.99984, 4) = 24 - .00016 = 23.99984$$

Using this bound we obtain a global bound for the Rendezvous value \bar{R} of the single marker game. This result is stated in Proposition 7.

Figure 4 shows the plot of the Rendezvous values for various dropping times. The minimal value computed is 24 for dropping time 4.0. We show in Table 1 the values computed on the regular mesh around the better dropping times observed.

Proposition 7 *For initial distance $D = 16$ the Rendezvous value satisfies $23.99984 \leq \bar{R} \leq 24$ and the optimal time to drop the marker lies in the interval $[3.99984, 4]$. This implies that for a general initial distance D , we have that the Rendezvous value satisfies $\left(\frac{3}{2} - 10^{-5}\right)D \leq \bar{R} \leq \frac{3D}{2}$ and the optimal dropping time lies in the interval $\left[\left(\frac{1}{4} - 10^{-5}\right)D, \frac{D}{4}\right]$.*

An anonymous referee has suggested that our numerical work could be improved by using the Divide Best Sector (DiBS) algorithm (Craparo, Karatas, & Kuhn, 2017) to refine the computations in sectors where the lower bound obtained with Corollary 6 does not prevent the existence of an optimal solution.

7 | TWO MARKERS ARE NO BETTER THAN ONE

In this section we show that giving *both* agents a marker does not result in a lower Rendezvous time than just giving one a marker (the case already analyzed). Taking our usual normalization of initial distance as $D = 16$, either model (one or two

markers) gives a Rendezvous value of $\bar{R} = 24$. More accurately, since we will calculate Rendezvous values only for a grid of dropoff time pairs, analogous to what we did in Section 6, we show that with two markers the Rendezvous value is not much less than 24. Of course we know that the Rendezvous value for two markers cannot be greater than for one marker, so cannot be greater than 24. This fact is also shown by Baston and Gal (2001), who showed that this is the Rendezvous time when both drop their markers at time 0. Our method is to apply Proposition 3 to calculate the minimum Rendezvous time $\bar{R}(l_1, l_2)$ when the players drop off their respective markers at times l_1 and l_2 , so that the Rendezvous value \bar{R}_2 in the two marker case is given by $\bar{R}_2 = \min_{l_1, l_2} \bar{R}(l_1, l_2)$. Baston and Gal find that $\bar{R}(0, 0) = 24$ and we have shown earlier that $\bar{R}(\infty, 4) = 24$, whereby $l_1 = \infty$ we mean that Player I does not drop off his marker until all meetings have taken place (or not at all). We find that,

$$\min_{l_1, l_2} \bar{R}(l_1, l_2) = 24, \text{ where } l_1, l_2 \text{ are taken in a fine grid.}$$

Of course lower values of $\bar{R}(l_1, l_2)$ are possible for l_1, l_2 not in the grid, but the same type of interpolation estimates as in Section 6, show that the minimum over all l_1 and l_2 cannot be much less than 24. We determine all regions of l_1, l_2 space where the minimum is close to 24.³

Proposition 8 *When the agents have initial distance D and each one has a marker which can be dropped at any time the least expected meeting time \bar{R} satisfies*

$$\begin{aligned} \bar{R}_2 &\geq 24 - 0.00032, \text{ when } D = 16, \text{ and more generally,} \\ \bar{R}_2 &\geq \left(\frac{3}{2} - 0.00002\right)D, \text{ for arbitrary } D. \end{aligned} \quad (9)$$

Proof Proposition 3 is still valid when both players have a marker and hence, optimal strategies are of the form (6) as well. Proceeding similarly to Sections 5 and 6 we can reduce the complexity of this problem and obtain approximate solutions with computer simulations. In Figure 6 we plot the level lines of the Rendezvous value of the problem when both players have a marker to drop off. We observe minimal values around the marker dropping times $(0, 0)$, $(8, 8)$, $(4, x)$ and $(x, 4)$ (the first one being the solution of Baston and Gal (2001)) corresponding to a Rendezvous value of 24. With the step size of the mesh of 0.00016 we get that the optimal solution must belong in the interval $[23.99968, 24]$. This requires a two-dimensional generalization of Proposition 5. ■

³The program may be found at <http://cui.unige.ch/~leonep/AlpernLeoneJournal/prgRDVOneMarkPlotArticle.java>.

TABLE 1 The Rendezvous values of the marker game computed at a regular mesh of 0.00016. The minimal value computed is 24 for a marker dropping time of 4

Dropping time (x)	3.99952	3.99968	3.99984	4	4.00016	4.00032
$\bar{R}(x)$	24.00024	24.00016	24.00008	24	24.00016	24.00032

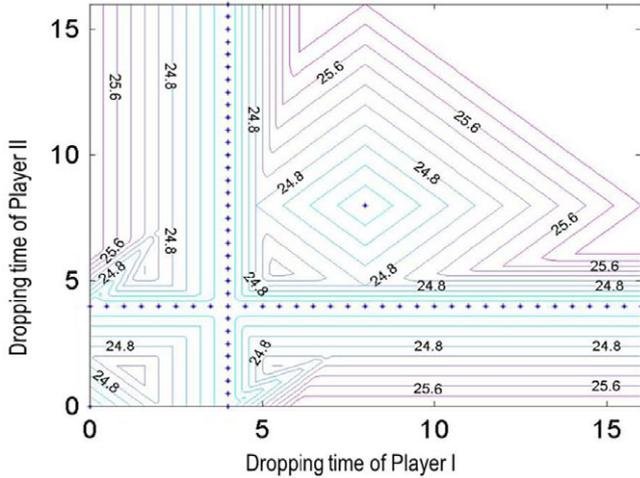


FIGURE 6 Contour plot of the values of the Rendezvous value for the game with two markers. The minimal value computed is 24 for dropping times $(0, 0)$, $(8, 8)$, $(4, x)$ and $(x, 4)$ [Color figure can be viewed at wileyonlinelibrary.com]

To end this section, we summarize the strategies found in the next theorem.

Theorem 9 (Two Marker Games) *Solutions for the Rendezvous problem on the line with two markers (one for each player), with initial distance D , are given by, see Figure 7,*

$$\begin{aligned} \bar{f}_1 &= [0; D], \bar{g}_1 = [0; D], \\ \bar{f}_2 &= [D/4; D/4, 5D/4], \bar{g}_2 = [D/4; 3D/4], \\ \bar{f}_3 &= [D/2; D/2, 3D/2], \bar{g}_3 = [D/2; D/2, 3D/2]. \end{aligned}$$

The corresponding times are

$$t_1 = t^1 = D/2, t_2 = t^3 = 3/2D,$$

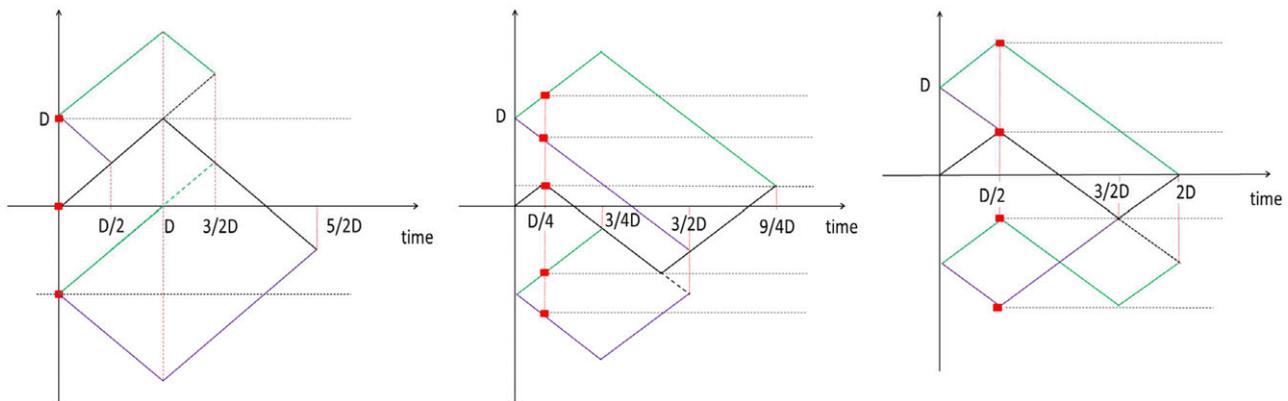


FIGURE 7 Solutions of the two markers game. From left to right the dropping times are $(0, 0)$, $(D/4, D/4)$, $(D/2, D/2)$. The up-left solution is already known, see (Baston & Gal, 2001) [Color figure can be viewed at wileyonlinelibrary.com]

$$\begin{aligned} t_3 = t^4 = 3/2D, t_4 = t^2 = 5/2D, \\ t_1 = t^3 = 3/4D, t_2 = t^1 = 3/2D, \\ t_3 = t^2 = 3/2D, t_4 = t^4 = 9/4D, \\ t_1 = t^1 = D/2, t_2 = t^2 = 3/2D, \\ t_3 = t^3 = 2D, t_4 = t^4 = 2D, \end{aligned} \tag{10}$$

The Rendezvous value for all these solutions is,

$$\bar{R} = R(\bar{f}, \bar{g}) = 24D/16.$$

Notice that the solution provided in Theorem 2 is a solution for the two markers game as well, the Rendezvous Value being the same.

8 | PROOF OF THEOREM FOR TURN-ON-DROP

In Theorem 2 we gave an explicit strategy pair (\hat{f}, \hat{g}) , where Player II drops the marker at time $D/4$, with expected meeting time $3D/2$. This established that the Rendezvous value \bar{R} satisfies $\bar{R} \leq 3D/2$. Our numerical work in Sections 5 and 6, where we calculate the restricted Rendezvous value $\bar{R}(x)$ where the dropoff time is fixed at x , for a grid of x values, shows that $\bar{R}(x)$ is minimized at or very near the time $x = D/4$, and its value is very close to $3D/2$, see Figure 4.

We recall that in Proposition 3 we showed that we could restrict strategies to those with turning points at restricted times, including case 3 when a player drops off a marker. In this section we further restrict strategies to those for which a player turns when he drops off a marker. This seems reasonable because one would drop a marker at the furthest point reached in a certain direction. Moreover, all the strategies we have numerically computed satisfy this assumption. However

proving that optimal strategies have this property seems difficult. With this restriction, we are able to identify the equations of the two straight lines in Figure 4 and thus establish that they meet at the minimum value of $x=4$. In the text we use the term “turn-on-drop strategies” to refer to strategies that satisfy the assumption that a player always turns at the time he drops the marker. Accordingly we speak of “turn-on-drop Rendezvous value” to refer to the optimal Rendezvous value for such strategies.

In general this numerical method, based on our categorization of optimal strategy pairs in Proposition 3 and bound for Rendezvous value that are not explicitly computed in Section 6, is how we obtain approximations to the Rendezvous value. We use this method in Section 7 where we analyze the problem of two markers. However for the single marker game, since we know from Figure 4 that the optimal dropping time is certainly in the interval $[0, 3D/8]$, we can algebraically evaluate $\bar{R}(x)$ separately for the intervals $[0, D/4]$ and $[D/4, 3D/8]$ to show that the minimum is uniquely attained at $D/4$ and that the Rendezvous value is $\bar{R} = \bar{R}(D/4) = 3D/2$. Some readers may be more than willing to accept this result from a perusal of Figure 4, and if so they may skip this section.

Theorem 10 *The turn-on-drop Rendezvous value $\bar{R}(x)$, for a single marker dropped at time x , $0 \leq x \leq 3D/8$, is given by,*

$$\bar{R}(x) = \begin{cases} \frac{13D-4x}{8} & \text{if } 0 \leq x \leq D/4, \text{ and} \\ \frac{5D+4x}{4} & \text{if } D/4 \leq x \leq 3D/8. \end{cases} \quad (11)$$

It follows that the strategy pair (\hat{f}, \hat{g}) given in Theorem 2, with marker dropoff time $D/4$, is optimal. The Rendezvous value is therefore given by $\bar{R} = \bar{R}(D/4) = 3D/2$.

The proof of this result is obtained by combining the following two lemmas which give the two parts of the formula (11) for $\bar{R}(x)$.

Lemma 11 *For $0 \leq x \leq D/4$ the optimal restricted strategy is given by the single turn path $[D-x]$ for Player I and the three turn strategy $[x; x, D/2+x, 2D-x]$ for Player II. Player I turns (at time $D-x$) when he fails to find the marker dropped by agent 1. The Rendezvous times are given by $t_1 = x + D/2$, $t_2 = D$, $t_3 = 2D - x$ and $t_4 = 3D - 2x$. Hence $\bar{R}(x) = 13D/8 - x/2$, see Figure 8(A).*

Proof First note that this strategy has expected meeting time $13D/8 - x/2$, as calculated in Figure 8(A). We now use Proposition 3 to show that this is the minimum. Since the dropoff time x occurs before any possible meeting time, x is the earliest that Player II can

turn. The general assumption of this section is that Player II turns when dropping his marker at time x . So the first meeting time will be $t_1 = t^4 = D/2 + x$. The next meeting is either $t_2 = t^1$ or $t_2 = t^3$.

In the latter case we have $t_2 = t^3 = D + 2x$. Then it makes no difference which of the two remaining agents is met next and the continuation is as shown in Figure 8(B).

One could calculate the remaining meeting times and show the average is above $13D/8 - x/2$ but an easier method is to simply remark that the strategy in Figure 8(B) is available in the no-marker game of Alpern and Gal (1995), so cannot be below the no-marker Rendezvous time of $13D/8$.

So we can assume that the next meeting is at time $t_2 = t^1 = D$, and hence for I to continue and for II to turn. However if Player I fails to find the marker at time $D-x$, he knows the other player is below him, so he turns for the first and last time at time $D-x$. The next two Rendezvous times are $t_3 = t^3 = 2D - x$ and $t_4 = t^2 = 3D - 2x$. Thus

$$\begin{aligned} \bar{R}(x) &= \frac{1}{4}(t_1 + t_2 + t_3 + t_4) \\ &= \frac{(D/2 + x) + (D) + (2D - x) + (3D - 2x)}{4} \\ &= \frac{13}{8}D - \frac{1}{2}x, \end{aligned}$$

as claimed and shown in Figure 8(A). ■

We now consider the later dropoff times $D/4 \leq x \leq 3D/8$.

Lemma 12 *Suppose that $D/4 \leq x \leq 3D/8$. Then the optimal restricted strategies are $[x + D/2]$ for Player I, who makes his only turn when he meets agent 4; and*

$$[x; x, x + D/2, D, x + 3D/2]$$

for II, with a final meeting time of $t_4 = 2x + 2D$, see Figure 9. Hence the Rendezvous time is given by

$$\bar{R}(x) = \frac{5D + 4x}{4}. \quad (12)$$

Proof We first prove the inequality

$$\begin{aligned} \bar{R}(x) &\geq \frac{5D + 4x}{4}, \quad \text{or equivalently} \\ \bar{R}(x) &\geq 20 + x, \quad \text{for } D = 16. \end{aligned} \quad (13)$$

We begin by using our program for evaluating $\bar{R}(x)$ to calculate

$$\bar{R}(6) = 26,$$

the same Rendezvous value as with no marker found in Alpern and Gal (1995). For $a \geq 0$,

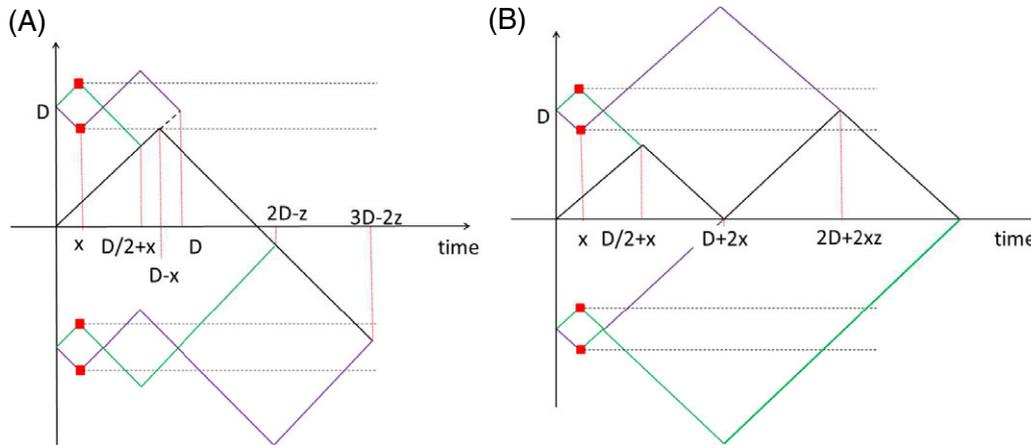


FIGURE 8 The marker game, strategies for $x \leq D/4$ [Color figure can be viewed at wileyonlinelibrary.com]

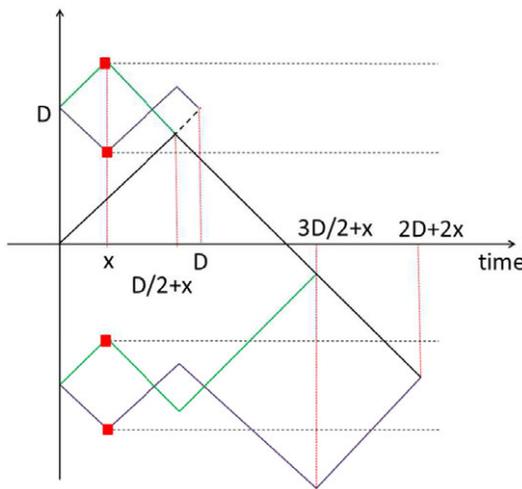


FIGURE 9 The marker game, a strategy for $x \geq D/4$ [Color figure can be viewed at wileyonlinelibrary.com]

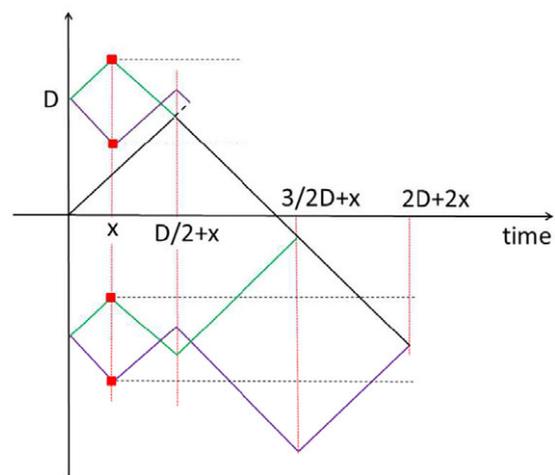


FIGURE 10 The solution of the Rendezvous problem with one marker and marker duration time $D/4 \leq T \leq D/2$. The optimal dropping time $x = (D - T)/2$ ($x \in [D/4, 3D/8]$). The optimal Rendezvous time is given by $7D - 2T$ or equivalently by $5D + 4x$ [Color figure can be viewed at wileyonlinelibrary.com]

Proposition 6 gives,

$$\begin{aligned} \bar{R}(6 - a) &\geq \bar{R}(6) - a = 26 - a, \text{ so} \\ \bar{R}(x) &= \bar{R}(6 - (6 - x)) \geq 26 - (6 - x) \geq 20 + x, \text{ giving (13).} \end{aligned}$$

To obtain the reverse inequality it suffices to exhibit a Rendezvous strategy with this expected time. The strategy pair stated in the Lemma and drawn in Figure 9 has expected meeting time

$$\frac{(x + D/2) + (D) + (x + 3D/2) + (2x + 2D)}{4} = \frac{5D + 4x}{4}.$$

Since there might be better strategies, this shows that,

$$\bar{R}(x) \leq \frac{5D + 4x}{4}, \text{ establishing (12).} \quad \blacksquare$$

We now complete the proof of the weakened version of Theorem 2 where we consider only restricted strategies.

Proof of Theorem 2 for turn-on-drop strategies ($\bar{R} \geq 3D/2$) As the Rendezvous value is linear in the initial distance D , we take $D = 16$ and show that $\bar{R} \geq 24$. Given our

calculations shown in Figure 4, it is enough to show that $\bar{R}(x) \geq 24$ for all x in $[0, 6]$. The formula (12) shows that $\bar{R}(x)$ is decreasing on $[0, 4]$ and hence less than $\bar{R}(x) \leq \bar{R}(4) = 24$; similarly $\bar{R}(x)$ is increasing on $[4, 6]$ and so $\bar{R}(x) \leq \bar{R}(4) = 24$. Hence dropping the marker at time $4 = D/4$ is optimal and $\bar{R} = 24 = 3D/2$.

9 | EVANESCENT MARKERS

We now consider that there is a single marker which, once dropped, lasts for an additional time T . This is realistic in a biological scenario in which the marker is a scent mark which is an evanescent odor. If we look back at the solution (\hat{f}, \hat{g}) to the permanent single marker problem, as drawn in Figure 3, we see that the marker is dropped off at time $D/4$ and found at time $3D/4$. So for $T \geq 3D/4 - D/4 = D/2$, an evanescent marker is as good as a permanent marker, and hence we restrict our attention to $T \leq D/2$. The best solution we find numerically for $T < D/4$ has Rendezvous value $13D/8$. This

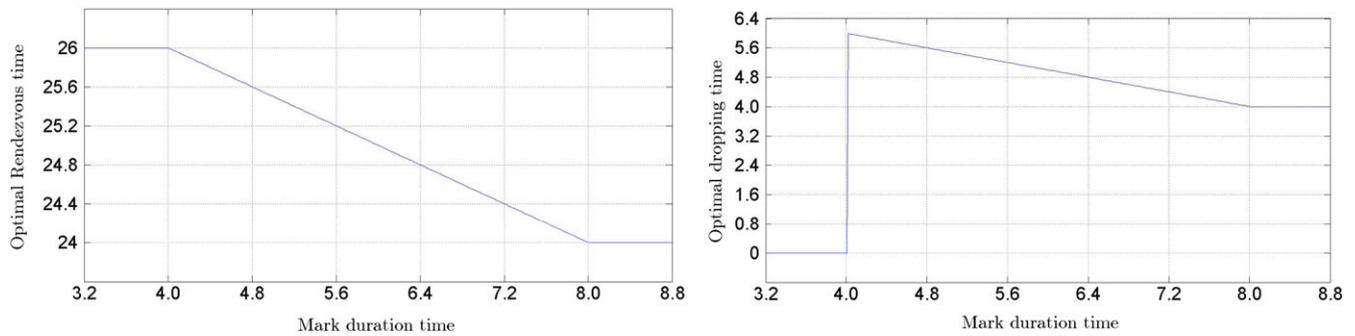


FIGURE 11 *Left:* Plot of the optimal Rendezvous time as a function of the mark duration time T when $D = 16$. If $T \leq D/4 = 4$ or $T \geq D/2 = 8$ the solutions are the ones stated in Theorems 1 and 2 respectively. *Right:* Plot of the optimal dropping time as a function of the mark duration time [Color figure can be viewed at wileyonlinelibrary.com]

is the Rendezvous value of the game without marker stated in Theorem 1, hence we conclude that a marker duration of $T < D/4$ is useless.

As the marker duration time varies in the interval $[D/4, D/2]$ the Rendezvous value changes linearly, see Figures 10 and 11 and Theorem 13.

Theorem 13 *Consider the one marker Rendezvous problem on the line, with initial distance D and marker duration time T . Then,*

- If $T \leq D/4$ then the marker has no use. The solution is the same as for the no-marker game given in Theorem 1, that is,

$$f^* = [D/2, D/2, D], g^* = [D], \text{ see Figure 2.}$$

- If $T \geq D/2$ the solution is the same as for the marker game in Theorem 2, that is,

$$f^* = [3D/4], g^* = [D/4; D/4, 3D/4, 7D/4], \text{ see Figure 3.}$$

- If $D/4 \leq T \leq D/2$ then the best dropping off time is given by $x = (D - T)/2$ and the better strategy given by

$$f^* = [D/2 + x], g^* = [x; x, D/2, D], \text{ see Figure 10.}$$

The corresponding meeting times are

$$\begin{aligned} t_1 = t^1 &= D/2 + x, t_2 = t^2 = D, \\ t_3 = t^3 &= 3D/2 + x, t_4 = t^4 = 2D + 2x, \end{aligned}$$

and the Rendezvous time is given by

$$\bar{R}_T = (20D + 16x)/16 = (22D - 2T)/16.$$

For $D = 16$, we have

$$\bar{R}_T = \begin{cases} 26 & T \leq 4, \\ 22 - 2T & 4 \leq T \leq 8, \\ 24 & 8 \leq T. \end{cases}$$

In Figures 11 we plot⁴ the upper bound for the total Rendezvous time if $D = 16$ and the better dropping time as functions of the marker duration time. Interestingly, this result emphasizes that there is a relationship between the marker duration time and the distance D between the two players. In a biological setting, this could bound the domain of the players.

10 | DISCUSSION

It is well known that when attempting to Rendezvous, agents (people or animals) often leave markers in various forms: notches cut on trees, stains put into water, pheromone deposits (scent markers). The first attempt at mathematical modeling of markers in a Rendezvous context was the classic article of Baston and Gal (2001). Their notion of “markstart Rendezvous” restricted the dropping times to the start of the problem and showed that when both players could drop a marker at the start, this reduced the Rendezvous time. Our advance on that model is that we allow the players to drop their marker or markers at times of their choice. When only one player has a marker to drop, we find that forcing him to drop it at the start gives the marker no value, but allowing him to drop it at an arbitrary start gives a Rendezvous time the same as when both players have markers but must drop them at the start. However we find that also giving the other player a marker to drop at a time of his choice has no further value in reducing the Rendezvous time. Perhaps more important than all these particular results is a new computational technique which gives a numerical solution to more general Rendezvous problems (could be adapted to two dimensional grid networks): we fix the vector of dropping times and then, similarly to some earlier articles, reduce the resulting problem to a finite one. Then we find a fast algorithm to solve the finite problem. We believe this technique will allow us to attack problems hitherto considered too difficult.

An important area of application of marker Rendezvous problems is that of mate search in animal populations of low density, where it may be reasonable for males and females to

⁴The program may be found at <http://cui.unige.ch/~leonep/AlpernLeoneJournal/prgRDVTwoMarksEvapArticle.java>.

assume there is a single individual of the opposite sex in the local habitat. For this application we introduced the notion of an evanescent marker in Section 9. Our model adopts a simple linear search region, which is often used in the Rendezvous literature. This could be seen as an idealization of a river habitat, for example woodlands that are associated with the seasonal watercourses characteristic of the right bank of the Rio Sao Francisco for Rendezvous of two birds of the Spix's Macaw (*Cyanopsitta spixii*) species. A captive bird from a breeding program, of the opposite sex to the last observed wild bird (sexed from a feather found by researchers) was released in the vicinity of the feather. They were later seen flying together, having successfully Rendezvoused. See Griffiths and Tlwarl (1995) and Juniper and Yamashita (1991).

It is well documented (Mizumoto, Abe, & Dobata, 2017) that pheromone markers are used in many species for this type of spatial mate search, sometimes by both sexes and sometimes by only one (usually the females). An important issue raised recently by Mizumoto et al. (2017) is the use of sexually dimorphic movement in mate search. Our model of what is called the player-asymmetric version of Rendezvous allows the two agents to jointly optimize their movements to minimize expected meeting time. So the optimal solution might be symmetric (same strategies employed) or asymmetric (distinct strategies). When only one agent can drop a marker, our unique solution is asymmetric: one agent turns once while the other turns three times (see Theorem 2). This observation goes in the same direction as in Mizumoto et al. (2017). On the other hand, we also find symmetric solutions, for example the (\bar{f}_3, \bar{g}_3) solution found in Theorem 9 for the two marker problem. The evanescent marker problem has asymmetric solutions, as shown in Theorem 13. The expected time to find a mate, based on such models, can then feed into mate choice models involving choosiness, as in the models of Alpern and Reyniers (2005) and Etienne, Rousset, Godelle, and Courtiol (2014), or the non-zero sum common interest games of Alpern, Fokkink, Lidbetter, and Clayton (2012).

A final comment is that our Rendezvousers use deterministic strategies, compared to earlier models ranging from the Brownian walker to the straight line walker in Mizumoto et al. (2017b).

In a different field of research the use of marker is proposed in Kündig, Leone, and Rolim (2016), Leone and Muñoz (2013), Muñoz and Leone (2014) in order to build routing paths in communication networks for the publish/subscribe communication pattern. In the setting, markers are built of information on published data and “pointer” towards the publisher (or the subscriber), providing a way of retrieving the emitter of the marker efficiently.

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How to cite this article: Leone P, Alpern S. Rendezvous search with markers that can be dropped at chosen times. *Naval Research Logistics* 2018;65:449–461. <https://doi.org/10.1002/nav.21818>