# Renewal theorems and mixing for non Markov flows with infinite measure

Ian Melbourne [*] Dalia Terhesiu [†]

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**Abstract**

We obtain results on mixing for a large class of (not necessarily Markov) infinite measure semiflows and flows. Erickson proved, amongst other things, a strong renewal theorem in the corresponding i.i.d. setting. Using operator renewal theory, we extend Erickson's methods to the deterministic (i.e. non-i.i.d.) continuous time setting and obtain results on mixing as a consequence.

Our results apply to intermittent semiflows and flows of Pomeau-Manneville type (both Markov and nonMarkov), and to semiflows and flows over Collet-Eckmann maps with nonintegrable roof function.

# 1 Introduction

Recently, there has been increasing interest in the investigation of mixing properties for infinite measure-preserving dynamical systems [2, 13, 24, 26, 27, 30, 31, 32, 33, 34, 36, 39, 40, 42]. Most of these results are for discrete time noninvertible systems.

For results on semiflows preserving an infinite measure, we refer to [36] (the Markov case) and [13] (which does not assume a Markov structure). The setting is that $F : Y \to Y$ is a mixing uniformly expanding map defined on a probability space $(Y, \mu)$ and $\tau : Y \to \mathbb{R}^+$ is a nonintegrable roof function with regularly varying tails:

$$\mu(y \in Y : \tau(y) > t) = \ell(t)t^{-\beta} \quad \text{for various ranges of } \beta \in [0, 1]. \tag{1.1}$$

Here, $\ell : [0, \infty) \to [0, \infty)$ is a measurable slowly varying function (so $\lim_{t\to\infty} \ell(\lambda t)/\ell(t) = 1$ for all $\lambda > 0$). Consider the suspension $(Y^\tau, \mu^\tau)$ and suspension semiflow $F_t : Y^\tau \to Y^\tau$ (the standard definitions are recalled in Section 3).

[*]Mathematics Institute, University of Warwick, Coventry, CV4 7AL, UK

[†]Department of Mathematics, University of Exeter, Exeter EX4 4QF, UK

The aim is to prove a mixing result of the form

$$\lim_{t\to\infty} a_t \int_{Y^\tau} v\, w\circ F_t\, d\mu^\tau = \int_{Y^\tau} v\, d\mu^\tau \int_{Y^\tau} w\, d\mu^\tau,$$

for a suitable normalisation $a_t \to \infty$ and suitable classes of observables $v, w : Y^\tau \to \mathbb{R}$.

Under certain hypotheses, [13, 36] obtained results on mixing and rates of mixing for such semiflows. The hypotheses were of two types: (i) assumptions on "renewal operators" associated to the transfer operator of $F$ and the roof function $\tau$, and (ii) Dolgopyat-type assumptions of the type used to obtain mixing rates for finite measure (semi)flows [17].

As pointed out to us by Dima Dolgopyat, Péter Nándori and Doma Szász, mixing for indicator functions can be regarded as a local limit theorem and hence hypotheses of type (ii) should not be necessary.

In this paper, we show that operator renewal-theoretic assumptions (i) are indeed sufficient for obtaining the mixing results in [13, 36]. The abstract framework in [13] turns out again to be flexible enough to cover nonMarkov situations. Moreover, our main results extend to flows and we are able to treat large classes of observables $v, w$. (Conditions of type (i) alone are not sufficient for obtaining rates of mixing; the best results remain those in [13].)

The analogous probabilistic results go back to Erickson [20] who obtained *strong renewal theorems* in an i.i.d. continuous time framework under the assumption $\beta \in (\frac{1}{2}, 1]$. (In the discrete time setting, see [22] for the i.i.d. case and [34] for the deterministic case.) Our results on mixing when $\beta \in (\frac{1}{2}, 1]$ for semiflows (Corollary 3.1 and the extensions in Section 8) and for flows (Theorem 9.5), are proved by adapting Erickson's methods to the deterministic setting.

For $\beta \le \frac{1}{2}$, additional hypotheses are needed on the tail of $\tau$ to obtain a strong renewal theorem (and hence mixing) even for discrete time; see [15, 19, 22] for i.i.d. results and [24] for deterministic results (see also [40] for higher order theory in both the i.i.d. and deterministic settings). For the continuous time case, Dolgopyat & Nándori [18] obtain strong renewal theorems for a class of Markov semiflows including the range $\beta \le \frac{1}{2}$ (again under extra hypotheses on the tail $\mu(\tau > t)$), though our main examples seem beyond their framework. In the absence of additional tail hypotheses, [20] showed how to obtain a partial result in the probabilistic setting with limit replaced by lim inf. In Corollary 3.5, we obtain such a lim inf result for semiflows with $\beta \in (0, \frac{1}{2}]$.

The remainder of this paper is organised as follows. In Section 2, we describe the operator renewal-theoretic hypotheses required in this paper and we state a strong renewal theorem for $\beta \in (\frac{1}{2}, 1]$ as well as related results for $\beta \le \frac{1}{2}$. In Section 3, we show how these results lead to mixing properties for semiflows. Sections 4 and 6 are devoted to the proof of the strong renewal theorem, and Section 7 contains the proofs of the remaining results in Section 2. Section 5 contains prerequisites from operator renewal theory

Corollary 3.1 (mixing for semiflows) is stated for observables that are certain indicator functions. This restriction is relaxed considerably in Section 8. The corresponding result for flows is stated and proved in Section 9.

The methods in this paper are illustrated by the examples in Section 10. We treat nonMarkovian intermittent flows and suspensions over unimodal maps with nonintegrable roof functions, significantly relaxing the conditions in [13] (Dolgopyat-type assumption; regularity of observables).

**Notation** We use "big O" and $\ll$ notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ if there is a constant $C > 0$ such that $a_n \le C b_n$ for all $n \ge 1$. Also, we write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

# 2 Strong renewal theorem for continuous time deterministic systems

Let $(Y, \mu)$ be a probability space and let $F : Y \to Y$ be an ergodic and mixing measure-preserving transformation. Let $\tau : Y \to \mathbb{R}^+$ be a measurable nonintegrable function bounded away from zero. For convenience, we suppose that $\operatorname{ess\,inf} \tau > 1$. Throughout we assume the regularly varying tail condition (1.1).

Let $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$. Given measurable sets $A, B \subset Y$, define the renewal measure

$$U_{A,B}(I) = \sum_{n=0}^{\infty} \mu(y \in A \cap F^{-n}B : \tau_n(y) \in I), \tag{2.1}$$

for intervals $I \subset \mathbb{R}$. We write $U_{A,B}(x) = U_{A,B}([0,x])$ for $x > 0$. Our aim is to generalise [20, Theorems 1 and 2] to this set up. That is, we want to obtain the asymptotics of $U_{A,B}(t+h) - U_{A,B}(t)$ for any $h > 0$.

With the same notation as in [13], let $\overline{\mathbb{H}} = \{\operatorname{Re} s \ge 0\}$. Given $\delta > 0$ and $L > 0$, let $\mathbb{H}_{\delta,L} = (\overline{\mathbb{H}} \cap B_\delta(0)) \cup \{ib : |b| \le L\}$. Define the family of operators for $s \in \overline{\mathbb{H}}$,

$$\hat{R}(s) : L^1(Y) \to L^1(Y), \qquad \hat{R}(s)v = R(e^{-s\tau}v).$$

Here $R : L^1(Y) \to L^1(Y)$ is the transfer operator for $F$ (so $\int_Y Rv\, w\, d\mu = \int_Y v\, w \circ F\, d\mu$ for all $v \in L^1(Y)$, $w \in L^\infty(Y)$).

We assume that there exists $p_0 \ge 1$, and for each $p \in (p_0, \infty)$, $\gamma \in (0, \beta)$ and $L > 0$ there exists a Banach space $\mathcal{B} = \mathcal{B}(Y)$ containing constant functions, with norm $\|\ \|_{\mathcal{B}}$, and constants $\delta \in (0, L)$, $\alpha_0 \in (0,1)$ and $C > 0$ such that

**(H)** (i) $\mathcal{B}$ is compactly embedded in $L^p$.

(ii) $\|\hat{R}(s)^n v\|_{\mathcal{B}} \le C(|v|_p + \alpha_0^n \|v\|_{\mathcal{B}})$ for all $s \in \mathbb{H}_{\delta,L}$, $v \in \mathcal{B}$, $n \ge 1$.

(iii) $|R(\tau^\gamma v)|_p \le C\|v\|_{\mathcal{B}}$ for all $v \in \mathcal{B}$.

(iv) The spectrum of $\hat{R}(ib) : \mathcal{B} \to \mathcal{B}$ does not contain 1 for all $b \in [-L, L]$, $b \neq 0$.

Hypotheses (H)(i)–(iii) are similar to [13, hypothesis (H1)]. Hypotheses (H)(iv) is a significant weakening of [13, hypothesis (H4)] and the diophantine ratio assumption used in [36] (Dolgopyat-type condition). The remaining hypotheses in [13], namely (H2) and (H3) (re-inducing), are not required.

**Remark 2.1** (a) For ease of exposition, hypothesis (H) is stated on the half-plane $\overline{\mathbb{H}}$, though we only use $s$ real and $s$ imaginary in this paper. For Theorems 2.3 and 2.4, we can take $s = ib$, $b \in [-L, L]$ in (H)(ii). For Theorem 2.6, (H)(iv) is not required and we can take $s = a$, $a \in [0, \delta)$ in (H)(ii).

(b) In practice, it is not necessary for $\gamma \in (0, \beta)$ to be arbitrary. For our main results Theorem 2.3 and Corollary 3.1, it suffices that $\gamma > 1 - \beta$ (this is possible since $\beta > \frac{1}{2}$ in those results). For our other results which include $\beta \leq \frac{1}{2}$, it suffices that $\gamma > 0$.

In addition, as in [13], there exists $p_0 \geq 1$ depending only on $\beta$ and $\gamma$ such that (H) is required to hold only for one value of $p > p_0$.

**Remark 2.2** In the simplest setting, studied in [36], where the map $F : Y \to Y$ is Gibbs-Markov [1, 3], hypothesis (H) is satisfied (with $\mathcal{B}$ a symbolic Hölder space) provided, for instance, there exist two periodic orbits for $F_t$ with periods $q_1, q_2$ such that $q_1/q_2$ is irrational. In both cases, we can take $p = \infty$. See [13, Remark 2.4] and [36, Proposition 3.5] for further details. This includes the case of Markovian intermittent semiflows.

As explained in Section 10.1, this situation generalizes to the case when $F$ is a mixing AFU map (with $\mathcal{B}$ consisting of bounded variation functions) enabling us to include nonMarkovian intermittent semiflows.

Define

$$d_\beta = \begin{cases} \frac{1}{\pi}\sin\beta\pi & \beta < 1 \\ 1 & \beta = 1 \end{cases}, \qquad m(t) = \begin{cases} \ell(t)t^{1-\beta} & \beta < 1 \\ \int_1^t \ell(s)s^{-1}\,ds & \beta = 1 \end{cases}.$$

Throughout we suppose that $A, B \subset Y$ are measurable and that $1_A \in \mathcal{B}$.

Our main result generalizes [20, Theorem 1] to the present non i.i.d. set up:

**Theorem 2.3 (Strong renewal theorem)** *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$. Suppose that (H) holds. Then for any $h > 0$,*

$$\lim_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = d_\beta \mu(A)\mu(B)h.$$

As discussed in the introduction, additional hypotheses are needed to obtain a strong renewal theorem when $\beta \leq \frac{1}{2}$. However, generalizing [20, Theorem 2] to the present non i.i.d. set up, we still obtain a lim inf result.

**Theorem 2.4** *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in (0,1)$. Suppose that (H) holds. Then for any $h > 0$,*

$$\liminf_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = d_\beta \mu(A)\mu(B)h.$$

**Remark 2.5** In the i.i.d. setting, results of this type are first due to [22] for discrete time and $\beta < 1$. The results of [20] extended [22] to continuous time and incorporated the case $\beta = 1$.

For the proof of Theorem 2.4, we will need the following result which gives the asymptotics of $U_{A,B}$ for the entire range $\beta \in [0,1]$. This implies a property for the semiflow $F_t$ known as *weak rational ergodicity* [1, 4] (see Corollary 3.3 below) and thus is of interest in its own right.

**Theorem 2.6** *Assume $\mu(\tau > t) = \ell(t)t^{-\beta}$ where $\beta \in [0,1]$. Suppose that (H) holds. Then*

$$\lim_{t\to\infty} t^{-1} m(t) U_{A,B}(t) = D_\beta\, \mu(A)\mu(B),$$

*where $D_\beta = \{\Gamma(1-\beta)\Gamma(1+\beta)\}^{-1}$ if $\beta \in (0,1)$ and $D_0 = D_1 = 1$.*

**Alternative hypotheses** In certain examples, such as those where $F : Y \to Y$ is modelled by a Young tower with exponential tails [43], hypothesis (H)(iii) is problematic. In such cases, it is necessary as in [13] to consider alternative hypotheses.

We assume that for every (sufficiently large) $p \in (1,\infty)$, there exists a Banach space $\mathcal{B}$ containing constant functions, with norm $\| \|_{\mathcal{B}}$, and constants $\delta > 0$, $\alpha_0 \in (0,1)$ and $C > 0$ such that

**(A)** (i) $\mathcal{B}$ is compactly embedded in $L^p$.
(ii) $\|\hat{R}(s)^n v\|_{\mathcal{B}} \le C(|v|_{L^1} + \alpha_0^n \|v\|_{\mathcal{B}})$ for all $s \in \overline{\mathbb{H}}_{\delta,L}$, $v \in \mathcal{B}$, $n \ge 1$.

It follows from these assumptions (see Lemma 5.1(c) below), that (after possibly shrinking $\delta$) there is a continuous family of simple eigenvalues $\lambda(s)$ for $\hat{R}(s) : \mathcal{B} \to \mathcal{B}$, $s \in \overline{\mathbb{H}} \cap B_\delta(0)$, with $\lambda(0) = 1$. Let $\zeta(s) \in \mathcal{B}$ be the corresponding family of eigenfunctions normalized so that $\int_Y \zeta(s)\, d\mu = 1$. We assume further that there exists $\beta_+ \in (\beta, 1)$ such that

**(A)** (iii) $\big|\int_Y (e^{-s\tau} - 1)(\zeta(s) - 1)\, d\mu\big| \le C|s|^{\beta_+}$ for all $s \in \overline{\mathbb{H}} \cap B_\delta(0)$.

Finally, as before we assume

**(A)** (iv) The spectrum of $\hat{R}(ib) : \mathcal{B} \to \mathcal{B}$ does not contain 1 for all $b \in [-L, L]$, $b \ne 0$.

**Theorem 2.7** *Suppose that hypothesis (H) is replaced by hypothesis (A). Then Theorems 2.4 and 2.6 remain valid. If in addition $\mu(\tau > t) = ct^{-\beta} + O(t^{-q})$ where $c > 0$, $\beta \in (\frac{1}{2}, 1)$, $q > 1$, then Theorem 2.3 remains valid.*

## 3 Mixing for infinite measure semiflows

In this section, we obtain various mixing results for semiflows as consequences of the results in Section 2.

Let $F : Y \to Y$ and $\tau : Y \to \mathbb{R}^+$ be as in Section 2. Define the suspension $Y^\tau = \{(y,u) \in Y \times \mathbb{R} : 0 \le u \le \tau(y)\}/\sim$ where $(y,\tau(y)) \sim (Fy,0)$. The suspension semiflow $F_t : Y^\tau \to Y^\tau$ is given by $F_t(y,u) = (y, u+t)$, computed modulo identifications. The measure $\mu^\tau = \mu \times \text{Lebesgue}$ is ergodic, $F_t$-invariant and $\sigma$-finite. Since $\tau$ is nonintegrable, $\mu^\tau$ is an infinite measure.

Throughout this section, we suppose that $A_1 = A \times [a_1, a_2]$, $B_1 = B \times [b_1, b_2]$ are measurable subsets of $\{(y,u) \in Y \times \mathbb{R} : 0 \le u \le \tau(u)\}$ (so $0 \le a_1 < a_2 \le \operatorname{ess\,inf}_A \tau$, $0 \le b_1 < b_2 \le \operatorname{ess\,inf}_B \tau$), and that $1_A \in \mathcal{B}$. Also, we continue to suppose that $\mu(\tau > t) = \ell(t)t^{-\beta}$ for various ranges of $\beta \in [0,1]$.

**Corollary 3.1** *Assume the setting of Theorem 2.3 (alternatively Theorem 2.7), so in particular $\beta \in (\frac{1}{2}, 1]$. Then $\lim_{t\to\infty} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta \mu^\tau(A_1)\mu^\tau(B_1)$.*

**Proof** Recall that $\operatorname{ess\,inf} \tau > 1$. Let $h \in (0,1)$ and note using (2.1) that

$$\begin{aligned} U_{A,B}(t+h) - U_{A,B}(t) &= \mu(y \in A : F^n y \in B \text{ and } \tau_n(y) \in [t, t+h] \text{ for some } n \ge 0) \\ &= \mu(y \in A : F_{t+h}(y,0) \in B \times [0,h]). \end{aligned}$$

After dividing rectangles into smaller subrectangles, we can suppose without loss that $b_2 - b_1 < 1$. Set $h = b_2 - b_1$. Then

$$\begin{aligned} \mu^\tau(A_1 \cap F_t^{-1}B_1) &= \mu^\tau\{(y,u) \in A \times [a_1,a_2] : F_t(y,u) \in B \times [b_1,b_2]\} \\ &= \mu^\tau\{(y,u) \in A \times [a_1,a_2] : F_{t+u-b_1}(y,0) \in B \times [0,h]\} \\ &= \int_{a_1}^{a_2} \mu\{y \in A : F_{t+u-b_1}(y,0) \in B \times [0,h]\}\,du \\ &= \int_{a_1}^{a_2} (U_{A,B}(t+u-b_1) - U_{A,B}(t+u-b_1-h))\,du. \end{aligned} \tag{3.1}$$

Hence

$$\begin{aligned} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) &= \int_{a_1}^{a_2} m(t)(U_{A,B}(t+u-b_1) - U_{A,B}(t+u-b_1-h))\,du \\ &= \int_{a_1}^{a_2} \frac{m(t)}{m(t+u-b_1-h)}\,\chi(t+u-b_1-h)\,du, \end{aligned}$$

where $\chi(t) = m(t)(U_{A,B}(t+h) - U_{A,B}(t))$ is bounded by Theorem 2.3. Also $m(t)/m(t+u-b_1-h)$ is bounded by Potter's bounds (see for instance [11]). Since $m(t)$ is regularly varying, we have $\lim_{t\to\infty} m(t)/m(t+u-b_1-h) = 1$ for each $u \in [0,1]$. By Theorem 2.3, $\lim_{t\to\infty} \chi(t+u-b_1-h) = d_\beta\mu(A)\mu(B)h = d_\beta\mu(A)\mu^\tau(B_1)$ for each $u \in [0,1]$. Hence the result follows from the bounded convergence theorem. ∎

**Remark 3.2** The result also holds for all sets of the form $F_r^{-1}A_1$ and $F_s^{-1}B_1$ for fixed $r, s > 0$. Indeed, by Corollary 3.1, using that $m(t) \sim m(t+s-r)$,

$$\begin{aligned} m(t)\mu^\tau(F_r^{-1}A_1 \cap F_{t+s}^{-1}B_1) &= m(t)\mu^\tau(A_1 \cap F_{t+s-r}^{-1}B_1) \\ &\to \mu^\tau(A_1)\mu^\tau(A_2) = \mu^\tau(F_r^{-1}A_1)\mu^\tau(F_s^{-1}A_2). \end{aligned}$$

**Corollary 3.3 (Weak rational ergodicity)** *Assume the setting of Theorem 2.6 (alternatively Theorem 2.7), with $\beta \in [0,1]$. Then*

$$\lim_{t\to\infty} t^{-1}m(t)\int_0^t \mu^\tau(A_1 \cap F_x^{-1}B_1)\,dx = D_\beta\mu^\tau(A_1)\mu^\tau(B_1).$$

**Proof** Continuing from (3.1) (with $h = b_2 - b_1$),

$$\begin{aligned} \int_0^t \mu^\tau(A_1 \cap F_x^{-1}B_1)\,dx &= \int_{a_1}^{a_2}\int_0^t (U_{A,B}(x+u-b_1) - U_{A,B}(x+u-b_1-h))\,dx\,du \\ &= \int_{a_1}^{a_2}\int_0^t U_{A,B}(x+u-b_1)\,dx\,du - \int_{a_1}^{a_2}\int_{-h}^{t-h} U_{A,B}(x+u-b_1)\,dx\,du \\ &= \int_{a_1}^{a_2}\int_{t-h}^t U_{A,B}(x+u-b_1)\,dx\,du - \int_{a_1}^{a_2}\int_{-h}^{0} U_{A,B}(x+u-b_1)\,dx\,du = I_1 + I_2. \end{aligned}$$

Now

$$t^{-1}m(t)I_1 = t^{-1}m(t)U_{A,B}(t)\int_{a_1}^{a_2}\int_{-h}^0 \frac{U_{A,B}(x+t+u-b_1)}{U_{A,B}(t)}\,dx\,du.$$

By Theorem 2.6, $U_{A,B}(t)$ is regularly varying so the integrand $U_{A,B}(x+t+u-b_1)/U_{A,B}(t)$ is bounded for $x, u$ bounded and converges pointwise to 1 as $t \to \infty$. Hence

$$\lim_{t\to\infty}\int_{a_1}^{a_2}\int_{-h}^0 \frac{U_{A,B}(x+t+u-b_1)}{U_{A,B}(t)}\,dx\,du = (a_2-a_1)h = (a_2-a_1)(b_2-b_1).$$

By Theorem 2.6, $t^{-1}m(t)U_{A,B}(t) = D_\beta\mu(A)\mu(B)(1+o(1))$. Hence, $\lim_{t\to\infty} t^{-1}m(t)I_1 = D_\beta\mu(A)\mu(B)(a_2-a_1)(b_2-b_1) = \mu^\tau(A_1)\mu^\tau(B_1)$. A simpler argument shows that $t^{-1}m(t)I_2 = o(1)$. ∎

**Proposition 3.4** *Let $f : [0,\infty) \to \mathbb{R}$ be bounded and integrable on compact sets, and let $K \in \mathbb{R}$. Suppose that $\beta \in (0,1)$, that $\ell(t)$ is slowly varying, and that*

*(a)* $\liminf_{t\to\infty} \ell(t)t^{1-\beta}f(t) \geq K$,

*(b)* $\lim_{t\to\infty} \ell(t)t^{-\beta}\int_0^t f(x)\,dx = \beta^{-1}K$.

*Then there exists a set $E \subset [0,\infty)$ of density zero such that* $\lim_{t\to\infty,\, t\notin E} \ell(t)t^{1-\beta}f(t) = K$.

*In particular,* $\liminf_{t\to\infty} \ell(t)t^{1-\beta}f(t) = K$.

**Proof** This is the continuous time analogue of [34, Proposition 8.2] (which is itself a version of [37, p. 65, Lemma 6.2]). We list the main steps which are proved exactly as in [34].

**Step 1**. Without loss of generality, $K = 0$ and $\ell(t)t^{1-\beta}$ is increasing.

**Step 2**. Define the nested sequence of sets $E_q = \{t > 0 : \ell(t)t^{1-\beta}f(t) > 1/q\}$, $q = 1, 2, \ldots$ Then $E_q$ has density zero for each $q$, i.e. $\lim_{t\to\infty} \frac{1}{t}\int_0^t 1_{E_q}(x)\,dx = 0$.

**Step 3**. By Step 2, we can choose $0 = i_0 < i_1 < i_2 < \cdots$ such that $\frac{1}{t}\int_0^t 1_{E_q}(x)\,dx < 1/q$ for $t \ge i_{q-1}$, $q \ge 2$. Define $E = \bigcup_{q=1}^{\infty} E_q \cap (i_{q-1}, i_q)$. Then $E$ has density zero and $\lim_{t\to\infty,\, t\notin E} \ell(t)t^{1-\beta}f(t) = 0$. ∎

**Corollary 3.5** *Assume the setting of Theorem 2.4 (alternatively Theorem 2.7), with* $\beta \in (0, 1)$. *Then*

*(i)* $\liminf_{t\to\infty} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta\mu^\tau(A_1)\mu^\tau(B_1)$, *and*

*(ii) There exists a set* $E \subset [0, \infty)$ *of density zero such that* $\lim_{t\to\infty,\, t\notin E} m(t)\mu^\tau(A_1 \cap F_t^{-1}B_1) = d_\beta\mu^\tau(A_1)\mu^\tau(B_1)$.

**Proof** We start from the conclusion of Theorem 2.4. Arguing as in the proof of Corollary 3.1, but with lim replaced by lim inf and using Fatou's lemma instead of the bounded convergence theorem, we obtain

$$\liminf_{t\to\infty} \ell(t)t^{1-\beta}\mu^\tau(A_1 \cap F_t^{-1}B_1) \ge d_\beta\mu^\tau(A_1)\mu^\tau(B_1).$$

This is condition (a) in Proposition 3.4, and Corollary 3.3 is condition (b). Hence the result follows from Proposition 3.4. ∎

# 4 Main results used in the proof of Theorem 2.3

The first result needed in the proof of the strong renewal theorem, Theorem 2.3, is an inversion formula for the symmetric measure

$$V_{A,B}(I) = \tfrac{1}{2}(U_{A,B}(I) + U_{A,B}(-I)).$$

Here, $U(-I) = U(\{x : -x \in I\})$ (with $U(-I) = 0$ if $I \subset [0, \infty]$). We find it convenient to adapt the formulation in [20, Section 4], but such an inversion formula goes back to [21] (see also [12, Chapter 10]).[1]

By (H), $\hat{T}(s) = (I - \hat{R}(s))^{-1}$ is a bounded operator on $\mathcal{B}$ for all $s \in \overline{\mathbb{H}} \setminus \{0\}$. Let $A$, $B \subset Y$ be measurable with $1_A \in \mathcal{B}$.

[1] The result does not require any regular variation assumptions on $\mu(\tau > t)$, but we use the extra structure for simplicity.

**Proposition 4.1 ( Analogue of [20, Inversion formula, Section 4]. )** *Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous compactly supported function with Fourier transform $\hat{g}(x) = \int_{-\infty}^{\infty} e^{ixb} g(b)\, db$ satisfying $\hat{g}(x) = O(x^{-2})$ as $x \to \infty$. Then for all $\lambda, t \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} e^{-i\lambda(x-t)} \hat{g}(x-t)\, dV_{A,B}(x) = \int_{-\infty}^{\infty} e^{-itb} g(b+\lambda) \operatorname{Re} \int_B \hat{T}(ib) 1_A \, d\mu \, db.$$

The second result required in the proof of Theorem 2.3 comes directly from [20] and does not require any modification in our set up. To state this result, for each $a > 0$ we let $\hat{g}_a(0) = 1$ and for $x \neq 0$, define

$$\hat{g}_a(x) = \frac{2(1-\cos ax)}{a^2 x^2}.$$

**Proposition 4.2 ( [20, Lemma 8] )** *Let $\{\mu_t,\, t > 0\}$ be a family of measures such that $\mu_t(I) < \infty$ for every compact set $I$ and all $t$. Suppose that for some constant $C$,*

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{g}_a(x)\, d\mu_t(x) = C \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{g}_a(x)\, dx,$$

*for all $a > 0$, $\lambda \in \mathbb{R}$. Then $\mu_t(I) \to C|I|$ for every bounded interval $I$, where $|I|$ denotes the length of $I$.* ∎

Next, note that $\hat{g}_a$ is the Fourier transform of

$$g_a(b) = \begin{cases} a^{-1}(1-|b|/a), & |b| \le a \\ 0, & |b| > a \end{cases}.$$

The final result required in the proof of Theorem 2.3 is as follows.

**Proposition 4.3** *For all $a > 0$ and $\lambda \in \mathbb{R}$,*

$$\lim_{t\to\infty} m(t) \int_{-\infty}^{\infty} e^{-itb} g_a(b+\lambda) \operatorname{Re} \int_B \hat{T}(ib) 1_A \, d\mu \, db = \pi d_\beta g_a(\lambda) \mu(A) \mu(B).$$

**Proof of Theorem 2.3** With the convention $I + t = \{x : x - t \in I\}$, let

$$\mu_t(I) = 2m(t) V_{A,B}(I+t) = m(t)(U_{A,B}(I+t) + U_{A,B}(-I-t))$$

and note that for $I = [0, h]$ with $h > 0$,

$$m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = \mu_t(I).$$

Now,

$$\begin{aligned} m(t) \int_{-\infty}^{\infty} e^{-i\lambda(x-t)} \hat{g}_a(x-t)\, dV_{A,B}(x) &= m(t) \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{g}_a(x)\, dV_{A,B}(x+t) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{g}_a(x)\, d\mu_t(x). \end{aligned}$$

Since $\hat{g}_a$ satisfies the assumptions of Proposition 4.1,

$$\int_{-\infty}^{\infty} e^{-i\lambda x}\hat{g}_a(x)\,d\mu_t(x) = 2m(t)\int_{-\infty}^{\infty} e^{-itb} g_a(b+\lambda)\,\mathrm{Re}\int_B \hat{T}(ib)1_A\,d\mu\,db.$$

By Proposition 4.3 together with the Fourier inversion formula $\int_{-\infty}^{\infty} e^{-i\lambda x}\hat{g}_a(x)\,dx = 2\pi g_a(\lambda)$,

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} e^{-i\lambda x}\hat{g}_a(x)\,d\mu_t(x) = 2\pi d_\beta g_a(\lambda)\mu(A)\mu(B) = d_\beta\int_{-\infty}^{\infty} e^{-i\lambda x}\hat{g}_a(x)\,dx\,\mu(A)\mu(B).$$

Hence, we have shown that the hypothesis of Proposition 4.2 holds with $C = d_\beta\mu(A)\mu(B)$. It now follows from Proposition 4.2 with $I = [0,h]$ that

$$m(t)(U_{A,B}(t+h) - U_{A,B}(t)) = \mu_t([0,h]) \to d_\beta\mu(A)\mu(B)h,$$

as $t\to\infty$. ∎

The proof of Propositions 4.1 and 4.3 are given in Section 6.

# 5 Prerequisites from operator renewal theory

In this section, we establish some estimates for $\hat{T} = (I-\hat{R})^{-1}$. The arguments closely follow [13, Section 4] (which was restricted to the case $\ell(t) = c+o(1)$ for some constant $c>0$ and did not include the case $\beta = 1$).

The estimates are carried out under hypothesis (H) in Subsection 5.1. The analogous results required under hypothesis (A) are obtained in Subsection 5.2.

## 5.1 Estimates under hypothesis (H)

Throughout this subsection, $\beta\in(0,1]$ and $L>0$ are fixed. We begin with $\gamma\in(0,\beta)$ and $p$ large as in (H). During the subsection, the values of $\gamma$ and $p$ change finitely many times; the changes in $\gamma$ are arbitrarily small. Also $C>0$ is a constant whose value changes finitely many times.

Define $\tilde{\ell}(t) = \begin{cases} \ell(t) & \beta<1 \\ \int_1^t \ell(s)s^{-1}\,ds & \beta = 1 \end{cases}$ and $c_\beta = \begin{cases} i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\,d\sigma & \beta<1 \\ 1 & \beta=1 \end{cases}$.

**Lemma 5.1** *(a)* $\|\hat{R}(s_1)-\hat{R}(s_2)\|_{\mathcal{B}\to L^p} \le C\,|s_1-s_2|^\gamma$ *for all* $s_1,s_2\in\overline{\mathbb{H}}$.
*(b) For any* $L>\delta>0$*, there exists* $C>0$ *such that for all* $\delta\le b<b+h<L$,

$$\|\hat{T}(ib)\|_{\mathcal{B}\to L^p}\le C \qquad \textit{and} \qquad \|\hat{T}(i(b+h))-\hat{T}(ib)\|_{\mathcal{B}\to L^p}\le Ch^\gamma.$$

*(c) There exists* $\delta\in(0,1)$ *and a continuous family* $\lambda(s)$*,* $s\in\overline{\mathbb{H}}\cap B_\delta(0)$*, of simple eigenvalues for* $\hat{R}(s):\mathcal{B}\to\mathcal{B}$ *with* $\lambda(0)=1$*. In addition, the corresponding family of*

*spectral projections $P(s)$ are bounded linear operators on $\mathcal{B}$ for all $s \in \overline{\mathbb{H}} \cap B_\delta(0)$ and* $\sup_{s\in\overline{\mathbb{H}}\cap B_\delta(0)} \|P(s)\|_{\mathcal{B}} < \infty$. *Moreover,*

$$\|P(s_1) - P(s_2)\|_{\mathcal{B}\to L^p} \le C|s_1 - s_2|^\gamma \quad \textit{for all } s_1, s_2 \in \overline{\mathbb{H}} \cap B_\delta(0).$$

**Proof** (a) Recall that $\hat{R}(s)v = R(e^{-s\tau}v)$. Since $R$ is a positive operator,

$$|(\hat{R}(s_1) - \hat{R}(s_2))v| \le R(|e^{-s_1\tau} - e^{-s_2\tau}||v|) \le 2|s_1 - s_2|^\gamma R(\tau^\gamma|v|).$$

By (H)(iii), $|(\hat{R}(s_1) - \hat{R}(s_2))v|_p \le 2|s_1 - s_2|^\gamma |R(\tau^\gamma|v|)|_p \ll |s_1 - s_2|^\gamma \|v\|_{\mathcal{B}}$.

(b) Fix $b > 0$. It is immediate from hypothesis (H)(iv) that $\|\hat{T}(ib)\|_{\mathcal{B}} < \infty$. Using also part (a), it follows from (H)(i,ii) and [28, Theorem 1] that there exists $h_0 > 0$ and $C > 0$ such that $\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B}\to L^p} \le C|h|^\gamma$ for all $|h| < h_0$. The result follows from compactness of $[\delta, L]$.

(c) This follows from (H)(i,ii) by [28, Corollary 1] exactly as in [13, Lemma 4.4] (with $\beta - \epsilon$ replaced by $\gamma$). ∎

Let $\zeta(s)$ denote the corresponding family of eigenfunctions normalized so that $\int_Y \zeta(s)\, d\mu = 1$. In particular, $\zeta(0) \equiv 1$ and $P(0)v = \int_Y v\, d\mu$ for all $v \in \mathcal{B}$. Also, define the complementary projections $Q(s) = I - P(s)$. It is immediate that $\zeta(s)$ and $Q(s)$ inherit the estimates obtained for $P(s)$. In particular $|\zeta(s) - \zeta(0)|_p \ll |s|^\gamma$.

Following [23] (see [13, Equation (4.2)]),

$$\lambda(s) = \int_Y e^{-s\tau}\, d\mu + \chi(s) \quad \text{where} \quad \chi(s) = \int_Y (e^{-s\tau} - 1)(\zeta(s) - \zeta(0))\, d\mu. \tag{5.1}$$

From now on, we fix $\delta \in (0, 1)$ as in Lemma 5.1(c).

**Proposition 5.2** *Write $s = a + ib \in \overline{\mathbb{H}}$.*

*(a)* $1 - \int_Y e^{-s\tau}\, d\mu \sim c_\beta \tilde{\ell}(1/|s|)s^\beta$ *as $s \to 0$.*

*(b) When $\beta = 1$,* $\mathrm{Re}\left(1 - \int_Y e^{-ib\tau}\, d\mu\right) \sim \frac{\pi}{2}\ell(1/|b|)|b|$ *as $b \to 0$.*

*(c)* $|\int_Y (e^{-i(b+h)\tau} - e^{-ib\tau})d\mu| \le C\tilde{\ell}(1/h)h^\beta$ *for $0 < h < b < \delta$.*

**Proof** Part (a) is proved in [35, Lemma 2.4] for $\beta < 1$. Suppose that $\beta = 1$ and let $G(x) = \mu(\tau > x)$. Then $1 - \int_Y e^{-s\tau}\, d\mu = s \int_0^\infty e^{-sx}(1 - G(x))\, dx = sI_C(s) - isI_S(s)$, where

$$I_C(s) = \int_0^\infty e^{-ax} \cos bx\, (1 - G(x))\, dx, \quad I_S(s) = \int_0^\infty e^{-ax} \sin bx\, (1 - G(x))\, dx.$$

By [34, Proposition 6.2], we have for $a \ge |b|$ that

$$I_C(s) = \tilde{\ell}(1/a)(1+o(1)) + O(|b|a^{-1}\ell(1/a)) = \tilde{\ell}(1/|s|)(1+o(1)) + O(\ell(1/|s|)) \sim \tilde{\ell}(1/|s|).$$

Similarly, for $a \le |b|$, we have $I_C(s) = \tilde{\ell}(1/|b|)(1+o(1)) + O(a|b|^{-1}\ell(1/|b|)) \sim \tilde{\ell}(1/|s|)$. Hence $I_C(s) \sim \tilde{\ell}(1/|s|)$ as $s \to 0$. In the same way, it follows from [34, Proposition 6.2] that $|I_S(s)| \ll \ell(1/|s|)$. Part (a) for $\beta = 1$ follows immediately from these estimates. Moreover, $I_S(ib) \sim \frac{\pi}{2}\ell(1/|b|)\,\mathrm{sgn}\, b$ as $b \to 0$ by the proof of [34, Lemma 6.8]. Since $\mathrm{Re}\left(1 - \int_Y e^{-ib\tau}\, d\mu\right) = bI_S(ib)$, part (b) follows.

Finally, part (c) follows by the argument used in the proof of [22, Lemma 3.3.2]. ∎

**Proposition 5.3** *(a)* $|\chi(s)| \le C|s|^{\beta+\gamma}$ *for* $s \in \overline{\mathbb{H}} \cap B_\delta(0)$,

*(b) When* $\beta > \frac{1}{2}$, $|\chi(i(b+h)) - \chi(ib)| \le Cb^\beta h^\gamma$ *for* $0 < h < b < \delta$.

**Proof** Choose $\epsilon > 0$ arbitrarily small and $r > 1$ such that $(\beta - \epsilon)r < \beta$ with conjugate exponent $r'$. Then $\tau^{(\beta-\epsilon)r} \in L^1$ and it follows from Hölder's inequality that

$$|\chi(s)| \le 2|s|^{\beta-\epsilon}|\tau^{\beta-\epsilon}(\zeta(s)-1)|_1 \le 2|s|^{\beta-\epsilon}|\tau^{\beta-\epsilon}|_r|\zeta(s)-1|_{r'} \ll |s|^{\beta-\epsilon+\gamma},$$

yielding part (a). Here we used that $|\zeta(s) - 1|_p = O(|s|^\gamma)$ for $p$ as large as desired. Similarly,

$$\begin{aligned} |\chi(i(b+h)) - \chi(ib)| &\le |(e^{i(b+h)\tau} - 1)(\zeta(i(b+h)) - \zeta(ib))|_1 + |(e^{ih\tau} - 1)(\zeta(ib) - 1)|_1 \\ &\ll (b+h)^{\beta-\epsilon}h^\gamma + h^{\beta-\epsilon}b^\gamma \ll b^\beta h^{\gamma-\epsilon}, \end{aligned}$$

(Note that $h^{\beta-\epsilon}b^\gamma = h^{\gamma-\epsilon}h^{\beta-\gamma}b^\gamma \le h^{\gamma-\epsilon}b^\beta$ since $\gamma < \beta$ and $h < b$.) This proves part (b). ∎

**Corollary 5.4** *Write* $s = a + ib \in \overline{\mathbb{H}}$.

*(a)* $1 - \lambda(s) \sim c_\beta \tilde{\ell}(1/|s|)s^\beta$ *as* $s \to 0$.

*(b) When* $\beta = 1$, $\mathrm{Re}(1 - \lambda(ib)) \sim \frac{\pi}{2}\ell(1/|b|)|b|$ *as* $b \to 0$.

*(c) When* $\beta > \frac{1}{2}$, $|\lambda(i(b+h)) - \lambda(ib)| \le C(\tilde{\ell}(1/h)h^\beta + b^\beta h^\gamma)$, *for* $0 < h < b < \delta$.

**Proof** Parts (a) and (b) are immediate from (5.1) and Propositions 5.2(a,b) and 5.3(a). Part (c) follows from (5.1) and Propositions 5.2(c) and 5.3(b). ∎

**Lemma 5.5** $\hat{T}(s) = c_\beta^{-1}\tilde{\ell}(1/|s|)^{-1}s^{-\beta}(P(0) + E(s))$ *for* $s \in \overline{\mathbb{H}} \cap B_\delta(0)$, *where* $E(s)$ *is a family of operators satisfying* $\lim_{s\to 0} \|E(s)\|_{\mathcal{B}\to L^1} = 0$.

**Proof** By Corollary 5.4(a), $(1 - \lambda(s))^{-1} \sim c_\beta^{-1}\tilde{\ell}(1/|s|)^{-1}s^{-\beta}$ as $s \to 0$. Also,

$$\hat{T}(s) = (1 - \lambda(s))^{-1}P(s) + (I - \hat{R}(s))^{-1}Q(s) = (1 - \lambda(s))^{-1}(P(0) + E(s)),$$

where

$$E(s) = P(s) - P(0) + (1 - \lambda(s))(I - \hat{R}(s))^{-1}Q(s). \tag{5.2}$$

By (H), $\|(I - \hat{R}(s))^{-1}Q(s)\|_{\mathcal{B}} = O(1)$. By Lemma 5.1, $\|P(s) - P(0)\|_{\mathcal{B}\to L^1} = O(|s|^\gamma)$. Hence $\|E(s)\|_{\mathcal{B}\to L^1} \ll |s|^\gamma + |s|^{\beta-\epsilon}$. ∎

**Lemma 5.6** *Let $\beta = 1$. Then $\operatorname{Re}\hat{T}(ib) = \frac{\pi}{2}\ell(1/|b|)\tilde{\ell}(1/|b|)^{-2}|b|^{-1}(P(0) + E(b))$ for $b \in \mathbb{R}$, $0 < |b| < \delta$, where $\lim_{b\to 0}\|E(b)\|_{\mathcal{B}\to L^1} = 0$.*

**Proof** By Corollary 5.4(a,b),

$$\operatorname{Re}((1 - \lambda(ib))^{-1}) = \operatorname{Re}(1 - \lambda(ib))|1 - \lambda(ib)|^{-2} \sim \frac{\pi}{2}\ell(1/|b|)\tilde{\ell}(1/|b|)^{-2}|b|^{-1}.$$

As in the proof of Lemma 5.5, $\operatorname{Re}\hat{T}(ib) = \{\operatorname{Re}((1 - \lambda(ib))^{-1})\}(P(0) + E(b))$ where

$$E(b) = \operatorname{Re}(1 - \lambda(ib))\operatorname{Re}\{(1 - \lambda(ib))^{-1}(P(ib) - P(0)) + (I - R(ib))^{-1}Q(ib)\},$$

and

$$\|E(b)\|_{\mathcal{B}\to L^1} \ll \|P(ib) - P(0)\|_{\mathcal{B}\to L^1} + |1 - \lambda(ib)|\|(I - R(ib))^{-1}Q(ib)\|_{\mathcal{B}\to L^1}) \ll |b|^{1-\epsilon},$$

completing the proof. ∎

**Remark 5.7** (a) It follows from Lemmas 5.5 and 5.6 that for each $\beta \le 1$ there is a constant $C > 0$ such that $\|\operatorname{Re}\hat{T}(ib)\|_{\mathcal{B}\to L^1} \le C\psi_\beta(|b|)$ for $0 < |b| \le L$, where

$$\psi_\beta(x) = \begin{cases} \ell(1/x)^{-1}x^{-\beta} & \beta < 1 \\ \ell(1/x)\tilde{\ell}(1/x)^{-2}x^{-1} & \beta = 1 \end{cases}.$$

Note that $\psi_\beta$ is integrable on $[0, L]$ for all $\beta \le 1$. This is clear for $\beta < 1$ while $\tilde{\ell}(1/x)^{-1}$ is an antiderivative for $\psi_1$.
(b) By Karamata's theorem on integration of regularly varying sequences [11], $\tilde{\ell}$ is slowly varying and $\ell(x) = o(\tilde{\ell}(x))$ as $x \to \infty$ when $\beta = 1$. In particular, $\psi_\beta(b) \ll \tilde{\ell}(1/|b|)^{-1}|b|^{-\beta}$ for all $\beta \le 1$.

**Lemma 5.8** *Let $\beta \in (\frac{1}{2}, 1]$. For $0 < h < b < \delta$,*

$$\|\hat{T}(i(b+h)) - \hat{T}(ib)\|_{\mathcal{B}\to L^1} \le C\big\{\tilde{\ell}(1/b)^{-2}b^{-2\beta}\tilde{\ell}(1/h)h^\beta + b^{-\beta}h^\gamma\big\}.$$

**Proof** Recall as in Lemma 5.5 that $\hat{T}(ib) = A_1(b) + A_2(b)$, where

$$A_1(b) = (1 - \lambda(ib))^{-1}P(ib), \qquad A_2(b) = (I - \hat{R}(ib))^{-1}Q(ib).$$

Using Lemma 5.1(c) and Corollary 5.4(a,c),

$$\begin{aligned}\|A_1(b+h)-A_1(b)\|_{\mathcal{B}\to L^1} &\ll |1-\lambda(i(b+h))|^{-1}\|P(i(b+h))-P(ib)\|_{\mathcal{B}\to L^1} \\ &\quad + |1-\lambda(ib)|^{-1}|1-\lambda(i(b+h))|^{-1}|\lambda(i(b+h))-\lambda(ib)|\|P(ib)\|_{\mathcal{B}\to L^1} \\ &\ll \tilde\ell(1/b)b^{-\beta}h^\gamma+\tilde\ell(1/b)^{-2}b^{-2\beta}(\tilde\ell(1/h)h^\beta+b^\beta h^\gamma) \\ &\ll \tilde\ell(1/b)^{-2}b^{-2\beta}\tilde\ell(1/h)h^\beta+b^{-\beta}h^{\gamma-\epsilon}.\end{aligned}$$

An argument from [32, Proposition 3.8] shows that $\|A_2(b+h)-A_2(b)\|_{\mathcal{B}\to L^1}\ll h^{\gamma-\epsilon}$, completing the proof. ∎

## 5.2 Estimates under hypothesis (A)

Let $\epsilon\in(0,\beta)$. Since $R:L^1\to L^1$ is a contraction,

$$|(\hat R(s_1)-\hat R(s_2))v|_1\le|(e^{-s_1\tau}-e^{-s_2\tau})v|_1\le 2|s_1-s_2|^{\beta-\epsilon}|\tau^{\beta-\epsilon}v|_1.$$

Choose $r>1$ such that $(\beta-\epsilon)r<\beta$ with conjugate exponent $r'$. By Hölder's inequality and (A)(i), $|\tau^{\beta-\epsilon}v|_1\le|\tau^{\beta-\epsilon}|_r|v|_{r'}\ll\|v\|_{\mathcal{B}}$ Hence $\|\hat R(s_1)-\hat R(s_2)\|_{\mathcal{B}\to L^1}\ll |s_1-s_2|^{\beta-\epsilon}$ for all $s_1,s_2\in\overline{\mathbb{H}}$.

Using [28] as before, we deduce that the conclusions of Lemma 5.1 hold with $L^p$ replaced by $L^1$ and $\gamma$ replaced by $\beta-\epsilon$.

**Proposition 5.9** *The conclusions of Lemmas 5.5 and 5.6 as well as Remark 5.7 are unchanged under hypothesis (A).*

**Proof** It is immediate from hypothesis (A)(iii) that $|\chi(s)|\ll|s|^{\beta_+}$ where $\beta_+>\beta$, and hence the proofs of Lemmas 5.5 and 5.6 are unchanged. ∎

Lemma 5.8 becomes:

**Lemma 5.10** $\|\hat T(i(b+h))-\hat T(ib)\|_{\mathcal{B}\to L^1}\le Cb^{-2\beta}h^{\beta-\epsilon}$ *for all* $0<h<b<\delta$.

**Proof** Since $\|\zeta(s)\|_{\mathcal{B}}$ is bounded, it follows again from Hölder's inequality that

$$\begin{aligned}|\chi(i(b+h))-\chi(ib)| &\le |(e^{i(b+h)\tau}-1)(\zeta(i(b+h))-\zeta(ib))|_1+|(e^{ih\tau}-1)(\zeta(ib)-1)|_1 \\ &\le 2|\zeta(i(b+h))-\zeta(ib)|_1+2h^{\beta-\epsilon}|\tau^{\beta-\epsilon}|_r|(\zeta(ib)-1)|_{r'}\ll h^{\beta-\epsilon}.\end{aligned}$$

Hence by (5.1) and Proposition 5.2(c), $|\lambda(i(b+h))-\lambda(ib)|\ll h^{\beta-\epsilon}$. Now proceed as in the proof of Lemma 5.8. ∎

The presence of the $\epsilon$ in Lemma 5.10 necessitates some alterations to the strategy in [20]. As in [13], we make use of the following refinement of Lemma 5.5.

**Lemma 5.11** *Assume that* $\mu(\tau>t)=ct^{-\beta}+O(t^{-q})$ *where* $c>0$, $\beta\in(\frac12,1)$, $q>1$. *Then* $c\hat T(ib)=c_\beta^{-1}b^{-\beta}P(0)+\tilde E(b)$ *for* $b\in[0,\delta)$, *where* $\|\tilde E(b)\|_{\mathcal{B}\to L^1}\le Cb^{-(2\beta-\beta_+)}$.

**Proof** A calculation using only the expression for $\mu(\tau > t)$ shows that $1 - \int_Y e^{-s\tau}\,d\mu = cc_\beta b^\beta + O(b)$ (see [13, Eq. (4.4)]). By (5.1) and the estimate $|\chi(s)| \ll |s|^{\beta_+}$ where $\beta_+ \in (\beta,1)$, we obtain $1-\lambda(ib) = cc_\beta b^\beta(1+O(b^{\beta_+-\beta}))$. Hence

$$c(1-\lambda(ib))^{-1} = c_\beta^{-1} b^{-\beta}(1+O(b^{-(2\beta-\beta_+)})).$$

By (5.2), $c\hat{T}(ib) = c(1-\lambda(ib))^{-1}P(0) + c(1-\lambda(ib))^{-1}E(ib) = c_\beta^{-1}b^{-\beta}P(0) + c\tilde{E}(b)$ where $\tilde{E}(b) = (1-\lambda(ib))^{-1}E(ib) + O(b^{-(2\beta-\beta_+)})$ and

$$E(ib) = P(ib) - P(0) + (1-\lambda(ib))(I-\hat{R}(ib))^{-1}Q(ib) = O(b^{\beta-\epsilon}).$$

Hence $\tilde{E}(b) \ll b^{-\epsilon} + b^{-(2\beta-\beta_+)}$. Recall that $2\beta - \beta_+ > 2\beta - 1 > 0$, so we can choose $\epsilon \in (0, 2\beta - \beta_+)$ completing the proof. ∎

# 6 Completion of the proof of Theorem 2.3

In this section, we give the proof of Propositions 4.1 and 4.3, thereby completing the proof of Theorem 2.3. In Subsections 6.1 and 6.2, we assume hypothesis (H). In Subsection 6.3, we show that the results remain true under hypothesis (A).

## 6.1 Proof of Proposition 4.1

Assume hypothesis (H) with $\beta \le 1$. For $n \ge 0$, the Fourier transform of the distribution $G_n(x) = \mu(\tau_n(y) \le x,\, y \in A \cap F^{-n}B)$ is given by $\int_Y 1_A\, 1_B \circ F^n\, e^{ib\tau_n}\,d\mu = \int_B \hat{R}(-ib)^n 1_A\,d\mu$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} e^{ibx}\,dV_{A,B}(x) &= \operatorname{Re}\int_0^{\infty} e^{ibx}\,dU_{A,B}(x)\\ &= \sum_{n=0}^{\infty}\operatorname{Re}\int_B \hat{R}(ib)^n 1_A\,d\mu = \operatorname{Re}\int_B \hat{T}(ib)1_A\,d\mu.\end{aligned}$$

Let $\hat{g}$ and $g$ be as in the statement of Proposition 4.1. Choose $L > 0$ such that $\operatorname{supp} g \in [-L, L]$. By Remark 5.7(a) $|\operatorname{Re}\hat{T}(ib)1_A|_1 \ll \psi_\beta(b)\|1_A\|_{\mathcal{B}}$ for $|b| \le L$, where $\psi_\beta$ is integrable. It follows from Fubini's theorem that

$$\int_{-\infty}^{\infty}\hat{g}(x)\,dV_{A,B}(x) = \int_{-\infty}^{\infty} g(b)\operatorname{Re}\int_B \hat{T}(ib)1_A\,d\mu\,db.$$

Let $g_1(b) = e^{-ibt}g(b+\lambda)$ and set $\hat{g}_1(x) = \int_{-\infty}^{\infty} e^{ibx}\,g_1(b)\,db = e^{-i\lambda(x-t)}\hat{g}(x-t)$. Replacing $\hat{g}, g$ with $\hat{g}_1, g_1$, we obtain the conclusion of Proposition 4.1.

## 6.2 Proof of Proposition 4.3

Assume hypothesis (H) with $\beta \in (\frac{1}{2}, 1]$. We follow the proof of [20, Theorem 1] (an adaptation of the argument in [22]). Let $W(b) = \operatorname{Re} \int_B \hat{T}(ib)1_A \, d\mu$.

Fix $\omega > 1$ and write $\int_{-\infty}^{\infty} e^{-itb} g_a(b+\lambda) \operatorname{Re} \int_B \hat{T}(ib)1_A \, d\mu \, db = I_1(t,\omega) + I_2(t,\omega)$ where

$$I_1(t,\omega) = \int_{-\omega/t}^{\omega/t} e^{-itb} g_a(b+\lambda) W(b)\, db, \quad I_2(t,\omega) = \int_{|b|>\omega/t} e^{-itb} g_a(b+\lambda) W(b)\, db.$$

Proposition 4.3 follows immediately from the estimates for $I_1(t,\omega)$ and $I_2(t,\omega)$ below.

**Lemma 6.1** $\lim_{\omega\to\infty} \lim_{t\to\infty} m(t) I_1(t,\omega) = \pi d_\beta g_a(\lambda)\mu(A)\mu(B)$.

**Proof** It follows from the definition of $g_a$ that $|g_a(b_1) - g_a(b_2)| \le a^{-2}|b_1 - b_2|$. Hence

$$\begin{aligned} \Big| I_1(t,\omega) - g_a(\lambda) \int_{-\omega/t}^{\omega/t} e^{-itb} W(b)\, db \Big| &\le \int_{-\omega/t}^{\omega/t} |g_a(b+\lambda) - g_a(\lambda)|\, |W(b)|\, db \\ &\le 2a^{-2}\omega t^{-1} \int_0^{\omega/t} |W(b)|\, db. \end{aligned}$$

By Remark 5.7(a), $\int_0^{\omega/t} |W(b)|\, db \ll \|1_A\|$ for $t > \omega/\delta$. Hence

$$\lim_{t\to\infty} m(t) I_1(t,\omega) = 2 g_a(\lambda) \lim_{t\to\infty} m(t) \int_0^{\omega/t} W(b) \cos tb\, db.$$

For $\beta < 1$, define $\xi(b) = \mu(A)\mu(B) + \int_B \hat{E}(ib) 1_A \, d\mu$ where $\hat{E}$ is as in Lemma 5.5. In particular, $|\xi(b)| \le |1_A|_1 + |\hat{E}(ib)1_A|_1 \ll \|1_A\|$ and $|\xi(b) - \mu(A)\mu(B)| \le \|\hat{E}(ib)\|_{\mathcal{B}\to L^1} \|1_A\| \to 0$ as $b \to 0$. Hence

$$\begin{aligned} m(t) \int_0^{\omega/t} W(b) \cos tb\, db &= \ell(t) t^{1-\beta} \operatorname{Re} \Big\{ c_\beta^{-1} \int_0^{\omega/t} \ell(1/b)^{-1} b^{-\beta} \xi(b) \cos tb\, db \Big\} \\ &= \ell(t) t^{1-\beta} \operatorname{Re} \Big\{ c_\beta^{-1} \int_0^{\omega/t} \ell(1/b)^{-1} b^{-\beta} \xi(b) \cos tb\, db \Big\} \\ &= \operatorname{Re} \Big\{ c_\beta^{-1} \int_0^{\omega} [\ell(t)/\ell(t/b)] b^{-\beta} \xi(b/t) \cos b\, db \Big\}. \end{aligned}$$

By the dominated convergence theorem,

$$\lim_{t\to\infty} m(t) \int_0^{\omega/t} W(b) \cos tb\, db = (\operatorname{Re} c_\beta^{-1}) \int_0^{\omega} b^{-\beta} \cos b\, db\, \mu(A)\mu(B),$$

and the result for $\beta < 1$ follows.

Now suppose that $\beta = 1$ and recall that $\psi_1(b) = \ell(1/b)\tilde{\ell}(1/b)^{-2}b^{-1}$. By Lemma 5.6,

$$m(t)\int_0^{\omega/t} W(b)\cos tb\,db = \tilde{\ell}(t)\frac{\pi}{2}\int_0^{\omega/t} g(b)\xi(b)\cos tb\,db.$$

where $\xi(b)$ has the same properties as before. Now

$$\begin{aligned}\tilde{\ell}(t)\int_0^{\omega/t} g(b)\xi(b)\,db &= \tilde{\ell}(t)\int_0^{\omega/t} g(b)\,db\,(\mu(A)\mu(B)+o(1))\,db \\ &= \tilde{\ell}(t)\tilde{\ell}(t/\omega)^{-1}(\mu(A)\mu(B)+o(1)) \to \mu(A)\mu(B).\end{aligned}$$

Next,

$$\tilde{\ell}(t)\int_0^{\omega/t} g(b)\xi(b)(\cos tb - 1)\,db = \int_0^{\omega} \frac{\tilde{\ell}(t)}{\tilde{\ell}(t/\sigma)}\frac{\ell(t/\sigma)}{\tilde{\ell}(t/\sigma)}\xi(\sigma/t)\frac{\cos\sigma - 1}{\sigma}\,d\sigma.$$

By Remark 5.7(b), $\tilde{\ell}$ is slowly varying and $\ell(x) = o(\tilde{\ell}(x))$ as $x \to \infty$. By Potter's bounds, the integrand is dominated by $\sigma^{1-\epsilon}$ for any $\epsilon > 0$, so the integrand converges to zero pointwise and $\tilde{\ell}(t)\int_0^{\omega/t} g(b)\xi(b)(\cos tb - 1)\,db \to 0$ as $t \to \infty$. Hence $\lim_{t\to\infty} m(t)\int_0^{\omega/t} W(b)\cos tb\,db = \frac{\pi}{2}\mu(A)\mu(B)$ yielding the result for $\beta = 1$. ∎

**Lemma 6.2** *Let* $\beta' \in (\frac{1}{2}, \beta)$. *Then* $\limsup_{t\to\infty} m(t)I_2(t,\omega) = O(\omega^{-(2\beta'-1)})$.

**Proof** It follows from evenness of $g_a$ and $W(b)$, together with the fact that $\operatorname{supp} g_a \in [-a,a]$, that

$$I_2(t,\omega) = \int_{b>\omega/t} [e^{-itb}g_a(b+\lambda) + e^{itb}g_a(b-\lambda)]W(b)\,db = \int_{\omega/t}^{a+|\lambda|} h(b)W(b)\,db,$$

where $h(b) = e^{-itb}g_a(b+\lambda) + e^{itb}g_a(b-\lambda)$. Continuing as on [20, p. 278] down as far as [20, Equation (5.14)], we obtain $m(t)|I_2(t,\omega)| \le a^{-1}J_1(t,\omega) + \pi a^{-2}J_2(t,\omega) + a^{-1}J_3(t,\omega)$, where

$$J_1(t,\omega) = m(t)\int_{(\omega-\pi)/t}^{\omega/t} |W(b+\pi/t)|\,db, \quad J_2(t,\omega) = m(t)t^{-1}\int_{\omega/t}^{a+|\lambda|} |W(b)|\,db,$$
$$J_3(t,\omega) = m(t)\int_{\omega/t}^{a+|\lambda|} |W(b+\pi/t) - W(b)|\,db.$$

By Remark 5.7(a), $W$ is integrable on $[0, a+|\lambda|]$ so $J_2(t,\omega) \ll \tilde{\ell}(t)t^{-\beta} \to 0$ as $t \to \infty$. By Lemma 5.5, for $\beta < 1$,

$$J_1(t,\omega) \ll \ell(t)t^{1-\beta}\int_{\omega/t}^{(\omega+\pi)/t} \ell(1/b)^{-1}b^{-\beta}\,db = \int_{\omega}^{\omega+\pi} (\ell(t)/\ell(t/\sigma))\sigma^{-\beta}\,d\sigma \ll \omega^{-(\beta-\epsilon)},$$

for any $\epsilon > 0$ by Potter's bounds. By Lemma 5.6 and Remark 5.7(b), for $\beta = 1$,

$$J_1(t, \omega) \ll \tilde{\ell}(t) \int_{\omega/t}^{(\omega+\pi)/t} \psi_1(b)\, db = \tilde{\ell}(t)\{\tilde{\ell}(t/(\omega+\pi))^{-1} - \tilde{\ell}(t/\omega)\} \to 0 \quad \text{as } t \to \infty.$$

By Lemma 5.8 with $h = \pi/t$,

$$J_3(t, \omega) \ll \tilde{\ell}(t)^2 t^{1-2\beta} \int_{\omega/t}^{\infty} \tilde{\ell}(1/b)^{-2} b^{-2\beta}\, db + t^{1-\beta+\epsilon-\gamma} \int_0^{a+|\lambda|} b^{-\beta}\, db = J_{3,1} + J_{3,2}.$$

By Potter's bounds,

$$J_{3,1} = \int_{\omega}^{\infty} [\tilde{\ell}(t)/\tilde{\ell}(t/\sigma)]^2 \sigma^{-2\beta}\, d\sigma \ll \int_{\omega}^{\infty} \sigma^{-2\beta'}\, d\sigma \ll \omega^{-(2\beta'-1)}.$$

Finally, since we are in the case $\beta > \frac{1}{2}$, we can choose $\gamma \in (1-\beta, \beta)$ in hypothesis (H). Hence $J_{3,2} \ll t^{1-\beta+\epsilon-\gamma} = o(1)$ as $t \to \infty$ for $\epsilon > 0$ sufficiently small. ∎

## 6.3 Modified argument under hypothesis (A)

Assume hypothesis (A) and that $\mu(\tau > t) = ct^{-\beta} + O(t^{-q})$ where $c > 0$, $\beta \in (\frac{1}{2}, 1)$, $q > 1$. Recall that $\beta_+ > \beta$.

First, we note by Proposition 5.9 that Remark 5.7 is unchanged under hypothesis (A). Hence the proof of Proposition 4.1 is unchanged.

For Proposition 4.3, we adopt a different strategy from before. Instead of considering $\lim_{\omega\to\infty} \limsup_{t\to\infty} I_r(t, \omega)$ for $r = 1, 2$, we consider $\lim_{t\to\infty} I_r(t, t^\kappa)$ for a suitable choice of $\kappa > 0$.

**Lemma 6.3** $\lim_{t\to\infty} m(t) I_1(t, t^\kappa) = \pi d_\beta g_a(\lambda)\mu(A)\mu(B)$ *for all* $\kappa > 0$.

**Proof** Following the proof of Lemma 6.1 and using Lemma 5.5 and Proposition 5.9,

$$\Big| m(t) I_1(t, \omega) - 2m(t) g_a(\lambda) \int_0^{\omega/t} W(b) \cos tb\, db \Big| \ll \omega t^{-\beta} \int_0^{\omega/t} b^{-\beta}\, db \ll \omega^{2-\beta} t^{-1}.$$

By Lemma 5.11,

$$\begin{aligned} m(t) \int_0^{\omega/t} W(b) \cos tb\, db &= t^{1-\beta} \int_0^{\omega/t} (\operatorname{Re} c_\beta^{-1} b^{-\beta} \mu(A)\mu(B) + O(b^{-(2\beta-\beta_+)}) \cos tb\, db \\ &= \operatorname{Re} c_\beta^{-1} \int_0^{\omega} b^{-\beta} \cos b\, db\, \mu(A)\mu(B) + O(t^{-(\beta_+-\beta)} \omega^{1-2\beta+\beta_+}). \end{aligned}$$

Finally, a calculation (see for example [34, Proposition 9.5]) shows that $\int_0^\omega b^{-\beta} \cos b\, db = \Gamma(1-\beta)\sin(\beta\pi/2) + O(\omega^{-\beta})$. Hence the result follows with $\omega = t^\kappa$ for any $\kappa > 0$. ∎

**Lemma 6.4** $\lim_{t\to\infty} m(t)I_2(t,t^\kappa) = 0$ *for all* $\kappa > 0$ *sufficiently large.*

**Proof** We use the same decomposition $m(t)|I_2(t,\omega)| \le a^{-1}J_1(t,\omega) + \pi a^{-2}J_2(t,\omega) + a^{-1}J_3(t,\omega)$ as in the proof of Lemma 6.2. By Proposition 5.9, we still have $J_1(t,\omega) \ll \omega^{-(\beta-\epsilon)}$ and $J_2(t,\omega) \ll t^{-\beta}$. By Lemma 5.10 with $h = \pi/t$,

$$J_3(t,\omega) \ll t^{1-\beta}t^{-(\beta-\epsilon)}\int_{\omega/t}^{\infty} b^{-2\beta}\,db \ll t^\epsilon\omega^{-(2\beta-1)},$$

for any choice of $\epsilon > 0$. Now take $\omega = t^\kappa$ with $\epsilon < \kappa(2\beta - 1)$. ∎

# 7 Proof of Theorems 2.4 and 2.6

In this section, we prove Theorem 2.4 by establishing separately an upper bound (Corollary 7.3) and a lower bound (Corollary 7.5). In the process of obtaining the upper bound, we prove Theorem 2.6.

For ease of exposition, we assume hypothesis (H) throughout. However, Lemma 5.8 is not required in this section, so we can just as well use hypothesis (A) by Proposition 5.9.

## 7.1 Upper bound for $\liminf$

In this subsection, the only parts of (H) that are required are (i)–(iii) with $s \in \mathbb{R}^+$ in part (ii). A simplified version of the argument used in the proof of Lemma 5.5 can be used to obtain

**Proposition 7.1** *Assume the setting of Theorem 2.6 with* $\beta \in [0,1]$. *For* $\sigma > 0$,

$$\hat{T}(\sigma) = D_\beta{}'\,\tilde{\ell}(1/\sigma)^{-1}\sigma^{-\beta}(P(0) + E(\sigma)),$$

*where* $D_\beta{}' = \Gamma(1-\beta)^{-1}$ *for* $\beta \in (0,1)$ *and* $D_0{}' = D_1{}' = 1$, *and* $E(\sigma)$ *is a family of operators satisfying* $\lim_{\sigma\to 0}\|E(\sigma)\|_{\mathcal{B}\to L^1} = 0$. ∎

We can now complete

**Proof of Theorem 2.6** For $n \ge 0$, the real Laplace transform of the distribution $G_n(x) = \mu(\tau_n(y) \le x,\ y \in A \cap F^{-n}B)$ is given by $\int_Y 1_A\, 1_B \circ F^n\, e^{-\sigma\tau_n}\,d\mu = \int_B \hat{R}(e^{-\sigma})^n 1_A\,d\mu$. Hence,

$$\int_{-\infty}^{\infty} e^{-\sigma t}\,dU_{A,B}(t) = \sum_{n=0}^{\infty}\int_B \hat{R}(e^{-\sigma})^n 1_A\,d\mu = \int_B \hat{T}(e^{-\sigma})1_A\,d\mu.$$

The conclusion follows from Proposition 7.1 by the continuous time version of Karamata's Tauberian Theorem [11, Theorem 1.7.1]. ∎

**Lemma 7.2** *Assume the setting of Theorem 2.6 with $\beta \in (0,1]$. Let $z : [0,\infty) \to [0,\infty)$ be integrable. Then*

$$\liminf_{t\to\infty} m(t) \int_0^t z(t-y)\, dU_{A.B}(y) \le d_\beta \mu(A)\mu(B) \int_0^\infty z\, dx.$$

**Proof** This is proved in the same way as [20, Lemma 9] using Theorem 2.6. ∎

**Corollary 7.3** *Assume the setting of Theorem 2.6 with $\beta \in (0,1]$. Then for any $h > 0$,*

$$\liminf_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) \le d_\beta \mu(A)\mu(B)h.$$

**Proof** Let $z = 1_{[0,h]}$. By Lemma 7.2,

$$\begin{aligned}\liminf_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) &= \liminf_{t\to\infty} m(t+h) \int_0^{t+h} z(t+h-y)\, dU_{A.B}(y) \\ &\le d_\beta \mu(A)\mu(B) \int_0^\infty z\, dx = d_\beta \mu(A)\mu(B)h,\end{aligned}$$

as required. ∎

## 7.2 Lower bound for $\liminf$

We require the following local limit theorem. Recall that $c_\beta = i\int_0^\infty e^{-i\sigma}\sigma^{-\beta}\, d\sigma$. Let $q_\beta(t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ibt} e^{-c_\beta |b|^\beta}\, db$.

**Lemma 7.4** *Assume the setting of Theorem 2.4 with $\beta \in (0,1)$. Let $d_n > 0$ be an increasing sequence with $d_n \to \infty$ such that $n\mu(\tau > d_n) = n\ell(d_n)d_n^{-\beta} \to 1$, as $n \to \infty$. Then for any $h > 0$ there exists $e_n > 0$ with $\lim_{n\to\infty} e_n = 0$ such that for all $t > 0$, $n \ge 1$,*

$$\Big|\mu(y \in A \cap F^{-n}B : \tau_n(y) \in [t, t+h]) - \frac{h}{d_n} q_\beta(t/d_n)\mu(A)\mu(B)\Big| \le \frac{e_n}{d_n}.$$

The proof of Lemma 7.4 combines results from Section 5 with arguments from [38] and is given for completeness in Appendix A. (A related argument [3, Theorem 6.3] based on [12] gives a similar conclusion but without the error term. As pointed out in [20], the full result requires proceeding as in [38].)

**Corollary 7.5** *Assume the setting of Theorem 2.4. Then for any $h > 0$,*

$$\liminf_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) \ge d_\beta \mu(A)\mu(B)h.$$

**Proof** Let $m \ge k \ge 0$. By (2.1) and Lemma 7.4,

$$\begin{aligned} U_{A,B}(t+h) - U_{A,B}(t) &\ge \sum_{n=k}^{m} \mu(y \in A \cap F^{-n}B : \tau_n(y) \in [t, t+h]) \\ &= \sum_{n=k}^{m} \frac{h}{d_n} q_\beta(t/d_n)\mu(A)\mu(B) + E_{k,m}, \end{aligned}$$

where $E_{k,m} = \sum_{n=k}^{m} e_n/d_n$.

Let $\kappa \in (1, 1/\beta)$. Then $d_n^{-1} = O(n^{-\kappa})$ and $E_{k,m} = O(\sup_{n\ge k} |e_n|) \to 0$ as $k \to \infty$.

Choosing $k = [C_1 t^\beta/\ell(t)]$ and $m = [C_2 t^\beta/\ell(t)]$, for fixed $C_2 > C_1 > 0$ and arguing word for word as in [20, Proof of eq. (7.2)], we obtain

$$\liminf_{t\to\infty} m(t)(U_{A,B}(t+h) - U_{A,B}(t)) \ge \mu(A)\mu(B) \int_{C_1}^{C_2} x^{-1/\beta} q_\beta(x^{-1/\beta})\, dx.$$

Now let $C_1 \to 0$ and $C_2 \to \infty$ and use that $\int_0^\infty x^{-1/\beta} q_\beta(x^{-1/\beta})\, dx = d_\beta$. ∎

# 8 General class of observables

In this section, we extend mixing for semiflows, Corollary 3.1, to cover more general classes of observables. As well as being of interest in its own right, this is useful for the extension to flows in Section 9. Throughout, we suppose that we are in the setting of Corollary 3.1; in particular $\beta \in (\frac{1}{2}, 1]$ and (H) holds.

From now on we suppose that $Y$ is a metric space with Borel probability measure $\mu$ and that $F$ and $\tau$ are almost everywhere continuous. Let $\mathcal{C}$ be a collection of measurable subsets $A \subset Y$ with $1_A \in \mathcal{B}$ such that

(i) $\mu(\partial A) = 0$ for all $A \in \mathcal{C}$,

(ii) $A_1 \cap A_2 \in \mathcal{C}$ for all $A_1, A_2 \in \mathcal{C}$,

(iii) $\mathcal{C}$ is a basis for the topology on $Y$.

In practice, we can often take $\mathcal{C}$ to consist of all measurable sets $A \subset Y$ with $1_A \in \mathcal{B}$ and $\mu(\partial A) = 0$. This is the case for the examples in Section 10.

**Proposition 8.1** *Let $\mathcal{C}' = \{A \times [a_1, a_2] \subset Y^\tau : A \in \mathcal{C}\}$. Let $\mathcal{D}$ be the ring generated by $\mathcal{C}'$. Choose a sequence $H_n \in \mathcal{D}$, $n \ge 1$, such that $H_n \subset H_{n+1}$ and $\mu(Y^\tau \setminus \bigcup_n H_n) = 0$. Let $v : Y^\tau \to \mathbb{R}$ be bounded, almost every continuous, and supported in $H_n$ for some $n$, and let $w \in L^1(Y^\tau)$. Then*

$$\lim_{t\to\infty} m(t) \int_{Y^\tau} v\, w \circ F_t\, d\mu^\tau = d_\beta \int_{Y^\tau} v\, d\mu^\tau \int_{Y^\tau} w\, d\mu^\tau. \tag{8.1}$$

**Proof** It is immediate that conditions (i)–(iii) for $\mathcal{C}$ are inherited by the collection $\mathcal{C}'$ of subsets of $Y^\tau$. In addition

(iv) $\mu(A) < \infty$ for $A \in \mathcal{C}'$, and there exist $A_1, A_2, \ldots \in \mathcal{C}'$ such that $\mu(Y \setminus \bigcup A_n) = 0$.

These are the conditions listed in [29, pages 434–435].

By Corollary 3.1,

$$\lim_{t\to\infty} m(t)\mu^\tau(A \cap F_t^{-1}B) = d_\beta \mu^\tau(A)\mu^\tau(B), \tag{8.2}$$

for all $A \in \mathcal{C}'$ and all measurable rectangles $B \subset Y^\tau$. The argument now proceeds as in [29] with obvious modifications since only $A$ is restricted to lie in $\mathcal{C}'$. (In [29], $A$ and $B$ both lie in $\mathcal{C}'$ leading to additional restrictions on $w$.)

By [29, page 435], property (8.2) extends first to all $A \in \mathcal{D}$, and second to all measurable subsets $A \subset Y^\tau$ measurable such that $\mu(\partial A) = 0$ and $A \subset H_n$ for some $n$. The result now follows from [29, lower half of page 435] (approximating $v$ by step functions involving admissible sets $A$, and approximating $w$ by simple functions). ∎

One possible choice for the sequence $H_n$ is the following:

**Corollary 8.2** *Define $H_n = \{(y,u) \in Y \times [0,\infty) : \tau(y) - n \le u \le \tau(y)\}$, $n \ge 1$. Then* (8.1) *holds for all bounded and almost every continuous functions $v : Y^\tau \to \mathbb{R}$ supported in $H_n$ for some $n$, and all $w \in L^1(Y^\tau)$.*

**Proof** Let $\mathcal{C}'' = \mathcal{C}' \cup \{E_n,\, n \ge 1\}$ where $\mathcal{C}'$ is the collection of rectangles in Proposition 8.1 and $E_n = \bigcup_{j=1}^n F_j^{-1}(Y \times [0,1])$. Let $\mathcal{I} = \{C \cap E_n : C \in \mathcal{C}',\, n \ge 1\}$ and define $\mathcal{C}''' = \mathcal{C}'' \cup \mathcal{I}$. Then $\mathcal{C}'''$ is closed under finite intersections, and hence conditions (i)–(iv) are satisfied by the collection $\mathcal{C}'''$. We claim that property (8.2) holds for all $A \in \mathcal{C}'''$. Certainly, the sets $E_n$ lie in the ring generated by $\mathcal{C}'''$, and $H_n \subset E_n$, so the conclusion follows from [29] with $E_n$ playing the role of $H_n$.

It remains to verify the claim. By Corollary 3.1, property (8.2) holds for all $A \in \mathcal{C}'$. By Remark 3.2, this holds also for the sets $E_n$. Finally, if $I \in \mathcal{I}$, then $I$ is contained in one of the rectangles in $\mathcal{C}'$ and $\mu^\tau(\partial I) = 0$. Hence $1_I$ is a bounded and almost everywhere continuous function supported in a rectangle in $\mathcal{C}'$. The claim follows from Proposition 8.1. ∎

# 9 Mixing for infinite measure flows

In this section, we show how mixing for semiflows extends to mixing for flows.

## 9.1 Assumptions and disintegration

We suppose throughout that $F_t : Y^\tau \to Y^\tau$ is a suspension semiflow over a map $F : Y \to Y$ with nonintegrable almost every continuous roof function $\tau : Y \to \mathbb{R}^+$ satisfying $\operatorname{ess\,inf} \tau > 1$ and $\mu(\tau > t) = \ell(t)t^{-\beta}$, $\beta \in (\frac{1}{2}, 1]$, and we assume that (H) holds. We also assume that there exists a collection $\mathcal{C}$ of subsets of $Y$ as in Section 8.

Let $X = Y \times N$ where $Y$ and $N$ are bounded metric space. Let $f(y, z) = (Fy, G(y, z))$ where $F : Y \to Y$ and $G : Y \times N \to N$ are continuous almost everywhere.

Define $\tau : X \to \mathbb{R}^+$ by setting $\tau(y, z) = \tau(y)$ and define the suspension $X^\tau = \{(x, u) \in X \times \mathbb{R} : 0 \le u \le \tau(x)\}/\sim$ where $(x, \tau(x)) \sim (fx, 0)$. The suspension flow $f_t : X^\tau \to X^\tau$ is given by $f_t(x, u) = (x, u + t)$ computed modulo identifications, with ergodic invariant measure $\mu_X^\tau = \mu_X \times$ Lebesgue.

Under two additional assumptions (F1) and (F2) below, we show in Theorem 9.5 that Corollary 3.1 for the semiflow $F_t$ applies equally to the flow $f_t$.

First, we assume contractivity along $N$:

**(F1)** $\lim_{n\to\infty} d(f^n(y, z), f^n(y, z')) = 0$ for all $z, z' \in N$ uniformly in $y \in Y$.

Let $\pi : X \to Y$ be the projection $\pi(y, z) = y$. This defines a semiconjugacy between $f$ and $F$. There exists a unique $f$-invariant ergodic probability measure $\mu_X$ on $X$ such that $\pi_*\mu_X = \mu$, see for instance [9, Section 6].

Recall that $R$ denotes the transfer operator for $F : Y \to Y$.

**Proposition 9.1** *Fix $z_0 \in N$. Suppose $v \in C^0(X)$. Then the limit*

$$\eta_y(v) = \lim_{n\to\infty} (R^n v_n)(y), \qquad v_n(y) = v \circ f^n(y, z_0),$$

*exists for almost every $y \in Y$ and defines a probability measure supported on $\pi^{-1}(y)$. Moreover $y \mapsto \eta_y(v) = \int_{\pi^{-1}(y)} v\, d\eta_y$ is integrable and $\int_X v\, d\mu_X = \int_Y \int_{\pi^{-1}(y)} v\, d\eta_y\, d\mu(y)$.*

**Proof** See for instance [14, Proposition 3]. ∎

**Remark 9.2** The proof of [14, Proposition 3] shows that the sequence $R^n v_n$ is Cauchy in $L^\infty(Y)$. If the metric on $Y$ can be chosen so that $R^n v_n$ is continuous for each $n$, then $\bar{v} \in C^0(Y)$. (In fact, it can often be shown that $\bar{v}$ is Hölder when $v$ is Hölder [14].)

Note that $X^\tau = Y^\tau \times N$. Given $v \in C^0(X^\tau)$, define

$$\bar{v} : Y^\tau \to \mathbb{R}, \qquad \bar{v}(y, u) = \int_{x\in\pi^{-1}(y)} v(x, u)\, d\eta_y(x).$$

Then

$$\int_{X^\tau} v \, d\mu_X^\tau = \int_{Y^\tau} \bar{v}(y,u) \, d\mu^\tau(y,u).$$

We require the additional assumption:

**(F2)** The function $\bar{v} : Y^\tau \to \mathbb{R}$ is almost everywhere continuous.

**Remark 9.3** If $v$ is uniformly continuous, then for any $\epsilon > 0$ there exists $\delta < 0$ such that $|\bar{v}(y,u) - \bar{v}(y,u')| < \epsilon$ for all $(y,u),(y,u') \in Y^\tau$ with $|u-u'| < \delta$. This combined with Remark 9.2 shows that condition (F2) is easily satisfied in practice for a large class of observables $v \in C^0(X^\tau)$.

**Remark 9.4** The set up in this section (skew product $X = Y \times N$, roof function $\tau$ constant in the $N$ direction) is not very restrictive. Suppose that $T_t : M \to M$ is a smooth flow defined on a Riemannian manifold $M$ and that $\Lambda$ is a partially hyperbolic attractor, so there exists a continuous $DT_t$-invariant splitting $T_\Lambda M = E^s \oplus E^{cu}$ where $E^s$ is uniformly contracting and dominates $E^{cu}$. By [7, Proposition 3.2, Theorem 4.2], the stable bundle $E^s$ extends to a neighbourhood $U$ of $\Lambda$ and integrates to a $T_t$-invariant collection $\mathcal{W}^s$ of stable leaves that topologically foliate $U$.

This means that we can choose a topological submanifold $X \subset M$ that is a cross-section to the flow $T_t$ formed as a union of stable leaves, and automatically the roof function $\tau$ is constant along stable leaves. (This construction has been widely used recently [5, 6, 8, 10].) Assuming for convenience the existence of a global chart for $\mathcal{W}^s$, we obtain a Poincaré map $f : X \to X$ where $X = Y \times N$ with $N$ playing the role of the stable direction. Moreover, $f$ has the desired skew product form $f(y,z) = (Fy, G(y,z))$, where $F : Y \to Y$ is defined by quotienting along the stable leaves, and condition (F1) is automatically satisfied. Also (F2) holds by Remark 9.2. Hence our set up holds in its entirety provided $F : Y \to Y$ and $\tau : Y \to \mathbb{Z}^+$ satisfy the required properties.

## 9.2 The mixing result

Choose a sequence $H_n$, $n \geq 1$, of subsets of $Y^\tau$ as in Proposition 8.1.

**Theorem 9.5** *Suppose that $\mu(\tau > n) = \ell(n) n^{-\beta}$ where $\beta \in (\frac{1}{2}, 1]$. Let $v \in C^0(X^\tau)$ be supported in $C \times N$ where $C$ is a closed subset of* $\operatorname{Int} H_n$ *for some $n \geq 1$. Let $w \in C^0(X^\tau)$ be uniformly continuous and supported on a set of finite measure. Assume that (H), (F1) and (F2) hold. Then*

$$\lim_{t\to\infty} m(t) \int_{X^\tau} v \, w \circ f_t \, d\mu_X^\tau = d_\beta \int_{X^\tau} v \, d\mu_X^\tau \int_{X^\tau} w \, d\mu_X^\tau.$$

**Proof** Following [10], we define $w_s : Y^\tau \to \mathbb{R}$, $s > 0$, by setting

$$w_s(y,u) = \overline{w \circ f_s} = \int_{x\in\pi^{-1}(y)} w \circ f_s(x,u)\, d\eta_y(x).$$

Note that $\int_{Y^\tau} |w_s|\, d\mu^\tau \le \int_{X^\tau} |w| \circ f_s\, d\mu_X^\tau = \int_{X^\tau} |w|\, d\mu_X^\tau$ so $w_s \in L^1(Y^\tau)$ for all $s$.

The semiconjugacy $\pi : X \to Y$ extends to a measure-preserving semiconjugacy $\pi^\tau : X^\tau \to Y^\tau$, $\pi^\tau(x,u) = (\pi x, u)$. Write $m(t) \int_{X^\tau} v\, w \circ f_t\, d\mu_X^\tau = I_1(s,t) + I_2(s,t)$ where

$$I_1(s,t) = m(t) \int_{X^\tau} v\, w_s \circ \pi^\tau \circ f_{t-s}\, d\mu_X^\tau,$$
$$I_2(s,t) = m(t) \int_{X^\tau} v\, (w \circ f_s - w_s \circ \pi^\tau) \circ f_{t-s}\, d\mu_X^\tau.$$

For $t > s$,

$$I_1(s,t) = m(t) \int_{X^\tau} v\, w_s \circ F_{t-s} \circ \pi^\tau\, d\mu_X^\tau = m(t) \int_{Y^\tau} \bar{v}\, w_s \circ F_{t-s}\, d\mu^\tau.$$

Since $\bar{v}$ is bounded and almost every continuous, supported in $H_n$, and $w_s \in L^1(Y^\tau)$, it follows from Proposition 8.1 that for all $s > 0$,

$$\lim_{t\to\infty} I_1(s,t) = d_\beta \int_{Y^\tau} \bar{v}\, d\mu^\tau \int_{Y^\tau} w_s\, d\mu^\tau = d_\beta \int_{X^\tau} v\, d\mu_X^\tau \int_{X^\tau} w\, d\mu_X^\tau.$$

Choose $\psi : Y^\tau \to [0,1]$ continuous such that $\operatorname{supp} v \subset \operatorname{supp} \psi \times N \subset H_n \times N$. Define

$$D_s : Y^\tau \to \mathbb{R}, \qquad D_s(y,u) = \operatorname{diam} w \circ f_s((\pi^\tau)^{-1}(y,u)).$$

Note that $|D_s| \le 2|w|_\infty$ and $\mu^\tau(\operatorname{supp} D_s) \le \mu_X^\tau(f_s^{-1} \operatorname{supp} w) = \mu_X^\tau(\operatorname{supp} w) < \infty$, so $D_s \in L^1(Y^\tau)$. Also, $|w \circ f_s(x,u) - w_s \circ \pi^\tau(x,u)| \le D_s \circ \pi^\tau(x,u)$. Hence for $t > s$,

$$|I_2(s,t)| \le |v|_\infty\, m(t) \int_{X^\tau} \psi \circ \pi^\tau\, D_s \circ \pi^\tau \circ f_{t-s}\, d\mu_X^\tau = |v|_\infty\, m(t) \int_{Y^\tau} \psi\, D_s \circ F_{t-s}\, d\mu_Y^\tau.$$

Since $\psi \in C^0(Y^\tau)$ is supported in $H_n$ and $D_s \in L^1(Y^\tau)$, it again follows from Proposition 8.1 that for all $s > 0$,

$$\limsup_{t\to\infty} I_2(s,t) \le |v|_\infty\, d_\beta \int_{Y^\tau} \psi\, d\mu^\tau \int_{Y^\tau} D_s\, d\mu^\tau.$$

By uniform continuity of $w$ and (F1), $\lim_{s\to\infty} |D_s|_\infty = 0$. Hence $|D_s|_1 \le |D_s|_\infty\, \mu^\tau(\operatorname{supp} D_s) \le |D_s|_\infty\, \mu_X^\tau(\operatorname{supp} w) \to 0$ as $s \to \infty$. This combined with the estimates for $I_1$ and $I_2$ yields the desired result. ∎

# 10 Examples

In this section, we demonstrate how the methods in this paper apply to nonMarkovian intermittent flows and suspensions over unimodal maps with nonintegrable roof functions.

## 10.1 NonMarkovian intermittent flows

We consider the case of intermittent semiflows. The corresponding results for flows follow immediately as discussed in Remarks 9.4 and 10.3.

Consider the map $f : [0,1] \to [0,1]$ given by

$$f(x) = x(1 + c_1 x^{1/\beta}) \bmod 1 \quad \text{where } \beta \in (\tfrac{1}{2}, 1],\ c_1 > 0.$$

This is an example of an AFN map [44], namely a nonuniformly expanding one-dimensional map with at most countably (in this case finitely) many branches with finite images and satisfying Adler's distortion condition $\sup |f''|/|f'|^2 < \infty$. Up to scaling, there is a unique absolutely continuous invariant measure $\mu_0$. The measure $\mu_0$ is infinite and the density has a singularity at the neutral fixed point 0.

Let $\tau_0 : [0,1] \to [1,\infty)$ be a roof function of bounded variation and Hölder continuous, and let $f_t$ denote the suspension semiflow on $[0,1]^{\tau_0}$ with invariant measure $\mu_0^{\tau_0} = \mu_0 \times \text{Lebesgue}$. Note that there is now a neutral periodic orbit of period $\tau_0(0)$.

In [13], under a Dolgopyat-type condition on $\tau_0$ and for sufficiently regular observables $v$ and $w$ supported away from the neutral periodic orbit, we proved a mixing result with rates and higher order asymptotics. Here we obtain the mixing result without requiring the Dolgopyat-type condition or high regularity for the observables. It suffices that $f_t$ has two periodic orbits (other than the neutral periodic orbit) whose periods have irrational ratio. Define $m(t) = \begin{cases} \log t & \beta = 1 \\ t^{1-\beta} & \beta \in (\frac{1}{2}, 1) \end{cases}$. We show that

$$\lim_{t\to\infty} m(t) \int v\, w \circ f_t \, d\mu_0^{\tau_0} = \text{const} \int v \, d\mu_0^{\tau_0} \int w \, d\mu_0^{\tau_0}, \tag{10.1}$$

where the constant depends only on $g$. Here, $v$ is any continuous function supported away from the neutral periodic orbit and $w$ is any integrable function.

**Remark 10.1** We have restricted to the case $\beta \in (\frac{1}{2}, 1]$ since this is required for Corollary 3.1. However, Corollary 3.5 and Corollary 3.3 hold for all $\beta \in (0, 1]$.

If $c_1$ is a positive integer, then $f$ is Markov and is a special case of the class of maps considered by [41]. In this case, it suffices that $\tau_0$ is Hölder continuous. Moreover, it follows from [18] that the mixing result (10.1) holds for all $\beta \le 1$. When $c_1$ is not an integer, $f$ is not Markov and [18] does not apply, as far as we can tell, regardless of the value of $\beta$.

The first step is to pass from the original suspension semiflow on $[0,1]^{\tau_0}$ to a suspension of the form $Y^\tau$ where $(Y,\mu)$ is a probability space and $\tau$ is an nonintegrable roof function.

We take $Y$ to be the interval of domain of the rightmost branch of $g$. Define the first return map $F = f^\sigma : Y \to Y$ where $\sigma = \min\{n \ge 1 : f^n y \in Y\}$. Then $\mu = (\mu_0|Y)/\mu_0(Y)$ is an absolutely continuous invariant probability measure for $F$. Define the induced roof function $\tau \to \mathbb{R}^+$ given by $\tau(y) = \sum_{\ell=0}^{\sigma(y)-1} \tau_0(f^\ell y)$. Let $F_t : Y^\tau \to Y^\tau$ be the corresponding suspension semiflow with infinite invariant measure $\mu^\tau$.

Since $\tau_0$ is Hölder, it is standard that $\mu(\tau > t) \sim ct^{-\beta}$ for some $c > 0$ (see for example [13, Proposition 9.1]).

**Proposition 10.2** *Suppose that $f_t$ has two periodic orbits (other than the neutral one) whose periods have irrational ratio. Then hypothesis (H) holds with $\mathcal{B} = \mathrm{BV}$ being the space of bounded variation functions on $Y$, with norm $\|v\|_{\mathrm{BV}} = |v|_1 + \operatorname{Var} v$.*

**Proof** Hypotheses (H)(i,iii) are verified in [13, Proposition 9.2]. Also, hypothesis (H)(ii) is verified in [13, Proposition 9.2] for $s \in \overline{\mathbb{H}} \cap B_\delta(0)$.

To complete the verification of (H)(ii), we proceed as follows. Since the density $d\mu/d\,\mathrm{Leb}$ lies in BV and is bounded above and below, it suffices to work with the non-normalised transfer operator $\hat P(ib)v = P(e^{ib\tau}v)$ where $\int_Y Pv\, w\, d\,\mathrm{Leb} = \int_Y v\, w \circ F\, d\,\mathrm{Leb}$.

Let $\lambda = \inf g|_Y > 1$. Fix $L > 0$. It suffices to show that there exists a constant $C'$ such that

$$\|\hat P(ib)^n v\|_{\mathrm{BV}} \le C' n |v|_1 + C' n \lambda^{-n} \operatorname{Var} v,$$

for all $|b| \le L$, $n \ge 1$, $v \in \mathrm{BV}$.

Let $n \ge 1$ and let $\{I\}$ be the partition of domains of branches for $F^n$. There is a constant $C_0$ independent of $n$ such that $\sup_I 1/(F^n)' \le C_0 \operatorname{diam} I$ for all $I$. Also $F' \ge \lambda$, so $|1/(F^n)'| \le 1/\lambda^n$ for all $n$.

Write

$$\hat P(ib)^n v = \sum_I \{\zeta_n\, e^{ib\tau_n} v\} \circ \psi_I\, 1_{F^n I},$$

where $\zeta_n = 1/(F^n)'$, $\psi_I$ is the inverse branch $(F^n|_I)^{-1}$, and $\tau_n = \sum_{j=0}^{n-1} \tau \circ F^j$ (not to be confused with $\tau_0$). We have the standard estimate

$$\begin{aligned} |\hat P(ib)^n v|_1 \le |\hat P(ib)^n v|_\infty &\le \sum_I \sup{}_I(\zeta_n |v|) \le \sum_I \sup{}_I \zeta_n (\inf{}_I |v| + \operatorname{Var}_I v) \\ &\le \sum_I \sup{}_I \zeta_n (\operatorname{diam} I)^{-1} \int_I |v| \; + \; \sum_I \lambda^{-n} \operatorname{Var}_I v \le C_0 |v|_1 + \lambda^{-n} \operatorname{Var} v. \end{aligned}$$

Next,

$$\begin{aligned}\operatorname{Var}(\hat{P}(ib)^n v) &\le \sum_I \operatorname{Var}_I(\zeta_n\, e^{ib\tau_n} v) + 2\sum_I \sup_I(\zeta_n|v|) \\ &\le \sum_I \operatorname{Var}_I(\zeta_n v) + \sum_I \sup_I(\zeta_n|v|) \operatorname{Var}_I e^{ib\tau_n} + 2C_0|v|_1 + 2\lambda^{-n}\operatorname{Var} v.\end{aligned}$$

A standard argument shows that

$$\sum_I \operatorname{Var}_I(\zeta_n v) \le C_1|v|_1 + \lambda^{-n}\operatorname{Var} v,$$

where $C_1 = \sup_n |(F^n)''/[(F^n)']^2|$. Also,

$$\operatorname{Var}_I e^{ib\tau_n} \le |b| \operatorname{Var}_I \tau_n \le L\sum_{j=0}^{n-1} \operatorname{Var}_I(\tau\circ F^j) = L\sum_{j=0}^{n-1} \operatorname{Var}_{F^jI}\tau.$$

Let $a$ be the domain of a branch for $F$. Then $\tau|_a = \sum_{\ell=0}^{\sigma(a)-1} \tau_0\circ f^\ell$. Since the images $f^\ell a$ are disjoint for $\ell < \sigma(a)$, it follows that $\operatorname{Var}_a \tau \le \operatorname{Var}\tau_0$. But $F^jI$ lies in such a domain $a$, so $\operatorname{Var}_{F^jI}\tau \le \operatorname{Var}\tau_0$ and it follows that $\operatorname{Var}_I e^{ib\tau_n} \le Ln\operatorname{Var}\tau_0$. Hence

$$\sum_I \sup_I(\zeta_n|v|) \operatorname{Var}_I e^{ib\tau_n} \le Ln\operatorname{Var}\tau_0 \sum_I \sup_I(\zeta_n|v|) \le Ln\operatorname{Var}\tau_0\,(C_0|v|_1 + \lambda^{-n}\operatorname{Var} v).$$

Combining these estimates we have shown that $\|\hat{P}(ib)^n v\|_{\mathrm{BV}} \le (3C_0 + C_1 + C_0L\operatorname{Var}\tau_0)n|v|_1 + (4 + L\operatorname{Var}\tau_0)n\lambda^{-n}\operatorname{Var} v$ as required.

Passing to the $L^2$ adjoint of $\hat{R}(ib)$, to verify (H)(iv) it is equivalent to rule out the possibility that there exists $b \ne 0$ and a BV eigenfunction $v : Y \to S^1$ such that $e^{ib\tau} v\circ F = v$. Suppose that $y \in Y$ is a periodic point of period $k$ for $F$. Now, BV functions have one-sided limits, and $F$ is orientation preserving, so $v(y+) = v(F^k(y+))$. Substituting into the equation $e^{ib\tau_k} v\circ F^k = v$ we obtain $e^{ibq} = 1$ where $q = \tau_k(y+)$ is the period of the corresponding periodic orbit for $f_t$. This is impossible under the periodic orbit assumption, so the BV eigenfunction $v$ cannot exist. ∎

It follows from Corollary 8.2 that mixing for $F_t$ holds for all continuous $v$ supported in $H_n = \{(y,u) \in Y\times[0,\infty) : \tau(y) - n \le u \le \tau(y)\}$ for some $n \ge 1$, and all $w \in L^1(Y^\tau)$.

The projection $\pi : Y^\tau \to [0,1]^{\tau_0}$, $\pi(y,u) = g_u(y,0)$, defines a measure-preserving semiconjugacy from $F_t : Y^\tau \to Y^\tau$ to $f_t : [0,1]^{\tau_0} \to [0,1]^{\tau_0}$. Let $v, w : [0,1]^{\tau_0} \to \mathbb{R}$ be observables where $v$ is bounded and almost everywhere continuous supported away from the neutral periodic orbit, and $w$ is integrable. Define the lifted observables $v = v\circ\pi$, $w = w\circ\pi : Y^\tau \to \mathbb{R}$. Then $v\circ\pi$ is a bounded and almost everywhere continuous function supported in $H_n$ for some $n$, and $w\circ\pi$ is integrable. This completes the proof of (10.1).

**Remark 10.3** Combining this example with Remark 9.4 leads to examples of partially hyperbolic intermittent flows preserving an infinite measure. See [32, 33] for similar examples in the discrete time invertible setting. In addition to extending to continuous time, our examples are an improvement over those in [32, 33] as far as mixing is concerned, since we require no assumptions on smoothness of foliations (in contrast to [32]) or Markov structure (in contrast to [33]).

## 10.2 Suspensions over unimodal maps

We consider a class of examples studied in [13, Example 1.2]. Again, the emphasis is on mixing rather than mixing rates, with significantly relaxed hypotheses on the roof function and the observables.

Let $f : [0,1] \to [0,1]$ be a $C^2$ unimodal map with unique non-flat critical point $x_0 \in (0,1)$. We suppose further that $f$ is *Collet-Eckmann* [16]: there are constants $C > 0$, $\lambda_{\rm CE} > 1$ such that $|(f^n)'(fx_0)| \geq C\lambda_{\rm CE}^n$ for all $n \geq 1$. It follows [25] that there is a unique acip $\mu_0$ that is mixing up to a finite cycle. We restrict to the case when $\mu_0$ is mixing. Finally, we suppose that $x_0$ satisfies *slow recurrence* in the sense that $\lim_{n\to\infty} n^{-1} \log|f^n x_0 - x_0| = 0$.

Consider a roof function $\tau_0 : [0,1] \to \mathbb{R}^+$ of the form $\tau_0 = g(x)|x - x_0|^{-1/\beta}$ where $\beta \in (\frac{1}{2}, 1)$ and $g : [0,1] \to (1,\infty)$ is differentiable, and form the suspension semiflow $f_t : [0,1]^{\tau_0} \to [0,1]^{\tau_0}$. We suppose as usual that $f_t$ has two periodic orbits whose periods have irrational ratio. (This holds for typical choices of $g$.) Under these assumptions, we show that

$$\lim_{t\to\infty} t^{1-\beta} \int v\, w \circ f_t \, d\mu_0^{\tau_0} = \text{const}\, t^{-(1-\beta)} \int v \, d\mu_0^{\tau_0} \int w \, d\mu_0^{\tau_0}.$$

Here, $v$ is any continuous function supported in $[0,1] \times [0,1]$ and $w$ is any integrable function.

The proof proceeds as in [13, Section 10]. We sketch the main ingredients. By [13, Lemma 10.2(a)], $\mu_0(\tau_0 > t) = ct^{-\beta} + O(t^{-2\beta})$ where the constant $c > 0$ is given explicitly. By [43], $f : [0,1] \to [0,1]$ is modelled by a Young tower $F : Y \to Y$ where $Y$ is a tower with exponential tails over a suitable inducing set $Z \subset [0,1]$. The roof function $\tau_0$ lifts to a roof function $\tau : Y \to \mathbb{R}^+$ satisfying $\mu(\tau > t) = ct^{-\beta} + O(t^{-2\beta})$ where $\mu$ is the SRB measure on $Y$.

It remains to verify hypothesis (A). In [13, Section 8.1], a new function space $\mathcal{B}$ is defined for Young towers with exponential tails, and hypothesis (A)(i,ii) are verified. This relies on a technical condition called (H3) in [13] which is verified in [13, Lemma 10.3]. (The Lasota-Yorke inequality (A)(ii) is proved in [13, Theorem B.2] for $s \in \overline{\mathbb{H}} \cap B_1(0)$ but holds equally for $s \in \overline{\mathbb{H}} \cap B_L(0)$ for any $L > 0$.) By [13, Proposition 8.6 and Lemma 10.4], hypothesis (A)(iii) is satisfied. Finally, hypothesis (A)(iv) is immediate from the quasicompactness assumptions (A)(i,ii) and the assumption about periodic orbits for $f_t$.

# A Proof of the local limit theorem

In this section, we give the proof of Lemma 7.4. Let

$$\mu_n(I) = \mu(y \in A \cap F^{-n}B : \tau_n(y) \in I).$$

Define

$$V_n(t,h,a) = \int_{-\infty}^{\infty} K_a(t-t')\mu_n([d_n t', d_n(t'+h)])\,dt',$$

where

$$K_a(x) = a^{-1}K(a^{-1}x), \qquad K(x) = \tfrac{1}{2\pi}\sin^2 \tfrac{1}{2}x.$$

**Lemma A.1** *Let $L > 0$. Then*

$$V_n(t,h,a) = h\big\{q_\beta(t)\mu(A)\mu(B) + e(n,h,a,t)\big\} \quad \textit{for } a \ge (Ld_n)^{-1},$$

*where $e(n,h,a,t) \to 0$ as $n \to \infty$, $h \to 0$ and $a \to 0$, uniformly in $t \in \mathbb{R}$.*

**Proof** In fact, we show that

$$|V_n(t,h,a) - hq_\beta(t)\mu(A)\mu(B)| \le \text{const.}\, h\{e_1(n) + e_2(h) + e_3(a)\}$$

where $\lim_{n\to\infty} e_1(n) = \lim_{h\to 0} e_2(h) = \lim_{a\to 0} e_3(a) = 0$.

As in Section 5, we write $\hat{R}(ib) = \lambda(ib)P(ib) + \tilde{Q}(b)$ for $|b| \le \delta$, where $\tilde{Q}(b) = R(ib)Q(ib)$. Then

$$\hat{R}(ib)^n = \lambda(ib)^n P(0) + \lambda(ib)^n(P(ib) - P(0)) + \tilde{Q}(b)^n. \tag{A.1}$$

Moreover, there exist constants $C > 0$, $\gamma > 1-\beta$, $\alpha_1 \in (0,1)$, where

$$\|P(ib) - P(0)\|_{\mathcal{B}\to L^1} \le C|b|^\gamma, \quad \|\tilde{Q}(b)^n\|_{\mathcal{B}} \le C\alpha_1^n, \quad \text{for all } |b| \le \delta,\ n \ge 1. \tag{A.2}$$

Also, by (H), we can choose $C > 0$, $\alpha_1 \in (0,1)$ so that

$$\|\hat{R}(ib)^n\|_{\mathcal{B}} \le C\alpha_1^n \quad \text{for all } b \in [\delta, L],\ n \ge 1, \tag{A.3}$$

(see, for instance, [28, Corollary 2, part 2]).

By Corollary 5.4(a), $1 - \lambda(ib) \sim c_\beta \ell(1/|b|)b^\beta$. Hence

$$\lambda(ib) \sim e^{-c_\beta \ell(1/|b|)|b|^\beta} \text{ as } b \to 0, \qquad \lim_{n\to\infty} \lambda(id_n^{-1}b)^n = e^{-c_\beta|b|^\beta}. \tag{A.4}$$

By (A.4) and Potter's bounds, there exists $C_1, C_2 > 0$ and $\beta' \in (0,\beta)$ such that

$$|\lambda(id_n^{-1}b)|^n \le C_1 e^{-C_2|b|^{\beta'}} \quad \text{for all } |b| \le \delta d_n,\ n \ge 1. \tag{A.5}$$

Let $k(b) = \begin{cases} (1-|b|), & |b|<1 \\ 0, & |b|\ge 1 \end{cases}$. Then $k(ab) = \int_{\mathbb{R}} e^{ixb} K_a(x)\,dx$. We compute that

$$\begin{aligned} V_n(t,h,a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ib(t-t')}k(ab)\,db \int_{A\cap F^{-n}B} 1_{\{\tau_n\in[d_nt',d_n(t'+h)]\}}\,d\mu\,dt' \\ &= \frac{1}{2\pi}\int_{|b|\le a^{-1}} e^{-ibt}k(ab)\int_{A\cap F^{-n}B}\int_{d_n^{-1}\tau_n-h}^{d_n^{-1}\tau_n} e^{ibt'}\,dt'\,d\mu\,db \\ &= \frac{1}{2\pi}\int_{|b|\le a^{-1}} e^{-itb}k(ab)\,(1-e^{-ihb})\,(ib)^{-1}\int_{A\cap F^{-n}B} e^{id_n^{-1}b\tau_n}\,d\mu\,db \\ &= \frac{h}{2\pi}\int_{|b|\le a^{-1}} e^{-itb}g(b,h,a)\int_B \hat{R}(id_n^{-1}b)^n 1_A\,d\mu\,db, \end{aligned}$$

where $g(b,h,a) = k(ab)\,(1-e^{-ihb})\,(ihb)^{-1}$.

Note that $|g(b,h,a)|\le 1$. Using (A.3) and that $a\ge (Ld_n)^{-1}$,

$$\begin{aligned} \Big|\int_{\delta d_n\le|b|\le a^{-1}} e^{-itb}g(b,h,a)\int_B \hat{R}(id_n^{-1}b)^n 1_A\,d\mu\,db\Big| &\le \|1_A\|_{\mathcal{B}}\int_{\delta d_n\le|b|\le Ld_n}\|\hat{R}(id_n^{-1}b)^n\|_{\mathcal{B}}\,db \\ &= \|1_A\|_{\mathcal{B}}\,d_n\int_{\delta\le|b|\le L}\|\hat{R}(ib)^n\|_{\mathcal{B}}\,db \le C\|1_A\|_{\mathcal{B}}\,d_n\alpha_1^n. \end{aligned}$$

Hence this term can be incorporated into $e_1(n)$.

It remains to analyse

$$\frac{h}{2\pi}\int_{|b|\le\delta d_n} e^{-itb}g(b,h,a)\int_B \hat{R}(id_n^{-1}b)^n 1_A\,d\mu\,db = \frac{h}{2\pi}(I_1+I_2+I_3),$$

where by (A.1),

$$\begin{aligned} I_1 &= \int_{|b|\le\delta d_n} e^{-itb}g(b,h,a)\int_B \lambda(id_n^{-1}b)^n P(0)1_A\,d\mu\,db, \\ I_2 &= \int_{|b|\le\delta d_n} e^{-itb}g(b,h,a)\int_B \lambda(id_n^{-1}b)^n (P(id_n^{-1}b)-P(0))1_A\,d\mu\,db, \\ I_3 &= \int_{|b|\le\delta d_n} e^{-itb}g(b,h,a)\int_B \tilde{Q}(d_n^{-1}b)^n 1_A\,d\mu\,db. \end{aligned}$$

By (A.2) and (A.5),

$$|I_2| \le \int_{|b|\le\delta d_n} C_1e^{-C_2|b|^{\beta'}}C|d_n^{-1}b|^{\gamma}\|1_A\|_{\mathcal{B}}\,db \le CC_1\|1_A\|_{\mathcal{B}}\,d_n^{-\gamma}\int_{-\infty}^{\infty}|b|^{\gamma}e^{-C_2|b|^{\beta'}}\,db \ll d_n^{-\gamma},$$

and

$$|I_3| \le d_n\int_{|b|\le\delta} C\alpha_1^n\|1_A\|_{\mathcal{B}}\,db \ll d_n\alpha_1^n.$$

Again, these terms can be incorporated into $e_1(n)$.

This leaves the term $I_1 = I_1'\mu(A)\mu(B)$ where $I_1' = \int_{|b|\le\delta d_n} e^{-itb}g(b,h,a)\lambda(id_n^{-1}b)^n\,db$. Write $I_1' = J_1 + J_2 + J_3$ where

$$J_1 = \int_{|b|\le\delta d_n} e^{-itb}k(ab)\big\{(1-e^{-ihb})(ihb)^{-1}-1\big\}\lambda(id_n^{-1}b)^n\,db,$$
$$J_2 = \int_{|b|\le\delta d_n} e^{-itb}(k(ab)-1)\lambda(id_n^{-1}b)^n\,db,$$
$$J_3 = \int_{|b|\le\delta d_n} e^{-itb}\lambda(id_n^{-1}b)^n\,db.$$

Since $|(1-e^{-ihb})(ihb)^{-1}-1| \le \frac{1}{2}h|b|$ it follows from (A.5) that

$$|J_1| \le h\int_{-\infty}^{\infty} C_1e^{-C_2|b|^{\beta'}}|b|\,db \ll h.$$

Also,

$$|J_2| \le \int_{-\infty}^{\infty} |k(ab)-1|C_1e^{-C_2|b|^{\beta'}}\,db,$$

which converges to zero by the dominated convergence theorem as $a\to 0$. These are the sole contributions to $e_2$ and $e_3$ respectively.

Finally,

$$|J_3 - 2\pi q_\beta(t)| \le \int_{|b|\le\delta d_n} |\lambda(id_n^{-1}b)^n - e^{-c_\beta|b|^\beta}|\,db + \int_{|b|\ge\delta d_n} e^{-c_\beta|b|^\beta}\,db,$$

which converges to zero by (A.4), (A.5) and the dominated convergence theorem as $n\to\infty$. ∎

**Lemma A.2** *Let $\epsilon > 0$ and $L > 0$. There exists $n_0 \ge 1$ and $h_0 > 0$ such that*

$$h(q_\beta(t)\mu(A)\mu(B)-\epsilon) \le \mu_n([d_nt, d_n(t+h)]) \le h(q_\beta(t)\mu(A)\mu(B)+\epsilon),$$

*for all $n\ge n_0$, $h\in[(Ld_n)^{-1}, h_0]$, $t\in\mathbb{R}$.*

**Proof** Let $\tilde q_\beta = q_\beta\mu(A)\mu(B)$. Since $q_\beta$ is the Fourier transform of an $L^1$ function, $\tilde q_\beta$ is uniformly continuous and bounded. Let $q_\infty = |\tilde q_\beta|_\infty$ and choose $h_1\in(0,1)$ such that $|\tilde q_\beta(t) - \tilde q_\beta(t')| \le \frac{1}{4}\epsilon$ whenever $|t-t'|\le h_1$.

For $\epsilon_1 > 0$, set $\epsilon_2 = \int_{|x|>1/\epsilon_1} K(x)\,dx$. We choose $\epsilon_1 \in (0,\frac{1}{6})$ sufficiently small that

$$(q_\infty + 2\epsilon_1 q_\infty + \tfrac{1}{2}\epsilon)(1-\epsilon_2)^{-1} - q_\infty \le \epsilon, \qquad 2\epsilon_1 q_\infty + \epsilon_2(q_\infty+\epsilon) \le \tfrac{1}{2}\epsilon. \tag{A.6}$$

By Lemma A.1, there exists $n_0 \geq 1$ and $h_0 \in (0, h_1)$ such that for all $n \geq n_0$, $h \in [(Ld_n)^{-1}, h_0]$, $t \in \mathbb{R}$,

$$V_n(t - \epsilon_1 h, h(1+2\epsilon_1), \epsilon_1^2 h) \leq h(1+2\epsilon_1)\tilde{q}_\beta(t-\epsilon_1 h) + \tfrac{1}{6}\epsilon h \qquad \text{(A.7)}$$
$$\leq h(1+2\epsilon_1)(\tilde{q}_\beta(t) + \tfrac{1}{4}\epsilon) + \tfrac{1}{6}\epsilon h \leq h(\tilde{q}_\beta(t) + 2\epsilon_1 q_\infty + \tfrac{1}{2}\epsilon),$$

where we used the constraint $\epsilon_1 \leq \frac{1}{6}$. Also, we can ensure that

$$V_n(t + \epsilon_1 h, h(1-2\epsilon_1), \epsilon_1^2 h) \geq h(1-2\epsilon_1)\tilde{q}_\beta(t+\epsilon_1 h) - \tfrac{1}{4}\epsilon h \qquad \text{(A.8)}$$
$$\geq h(1-2\epsilon_1)(\tilde{q}_\beta(t) - \tfrac{1}{4}\epsilon) - \tfrac{1}{4}\epsilon h \geq h(\tilde{q}_\beta(t) - 2\epsilon_1 q_\infty - \tfrac{1}{2}\epsilon).$$

Now, for $|t'| \leq \epsilon_1 h$,

$$\mu_n([d_n(t+\epsilon_1 h - t'), d_n(t - \epsilon_1 h - t' + h]) \leq \mu_n([d_n t, d_n(t+h)])$$
$$\leq \mu_n([d_n(t-\epsilon_1 h - t'), d_n(t+\epsilon_1 h - t' + h]).$$

Also $\int_{-\infty}^{\infty} K\,dx = 1$, so

$$1 - \epsilon_2 = \int_{|x|\leq 1/\epsilon_1} K(x)\,dx = \epsilon_1^2 h \int_{|x|\leq 1/\epsilon_1} K_{\epsilon_1^2 h}(\epsilon_1^2 h x)\,dx = \int_{|x|\leq \epsilon_1 h} K_{\epsilon_1^2 h}(x)\,dx.$$

Hence

$$\begin{aligned}
&V_n(t-\epsilon_1 h, h(1+2\epsilon_1), \epsilon_1^2 h)\\
&\quad = \textstyle\int_{-\infty}^{\infty} K_{\epsilon_1^2 h}(t')\mu_n([d_n(t-\epsilon_1 h - t'), d_n(t+\epsilon_1 h - t' + h)])\,dt'\\
&\quad \geq \textstyle\int_{|t'|\leq \epsilon_1 h} K_{\epsilon_1^2 h}(t')\mu_n([d_n(t-\epsilon_1 h - t'), d_n(t+\epsilon_1 h - t' + h)])\,dt'\\
&\quad \geq \textstyle\int_{|t'|\leq \epsilon_1 h} K_{\epsilon_1^2 h}(t')\mu_n([d_n t, d_n(t+h)])\,dt' = (1-\epsilon_2)\mu_n([d_n t, d_n(t+h)]).
\end{aligned}$$

By (A.6) and (A.7),

$$\mu_n([d_n t, d_n(t+h)]) \leq (1-\epsilon_2)^{-1} V_n(t - \epsilon_1 h, h(1+2\epsilon_1), \epsilon_1^2 h)$$
$$\leq h(\tilde{q}_\beta(t) + 2\epsilon_1 q_\infty + \tfrac{1}{2}\epsilon)(1-\epsilon_2)^{-1} \leq h(\tilde{q}_\beta(t) + \epsilon).$$

Arguing similarly, and exploiting the last estimate for $\mu_n([d_n t, d_n(t+h)])$,

$$\begin{aligned}
&V_n(t+\epsilon_1 h, h(1-2\epsilon_1), \epsilon_1^2 h)\\
&\quad \leq \textstyle\int_{|t'|\leq \epsilon_1 h} K_{\epsilon_1^2 h}(t')\mu_n([d_n(t+\epsilon_1 h - t'), d_n(t-\epsilon_1 h - t' + h)])\,dt'\\
&\qquad\qquad + \int_{|t'|\geq \epsilon_1 h} K_{\epsilon_1^2 h}(t') h(q_\infty + \epsilon)\,dt'\\
&\quad \leq \mu_n([d_n t, d_n(t+h)]) + \epsilon_2 h(q_\infty + \epsilon).
\end{aligned}$$

By (A.6) and (A.8),

$$\mu_n([d_n t, d_n(t+h)]) \geq V_n(t+\epsilon_1 h, h(1-2\epsilon_1), \epsilon_1^2 h) - \epsilon_2 h(q_\infty + \epsilon)$$
$$\geq h((\tilde{q}_\beta(t) - 2\epsilon_1 q_\infty - \tfrac{1}{2}\epsilon - \epsilon_2(q_\infty + \epsilon)) \geq h(\tilde{q}_\beta(t) - \epsilon).$$

This completes the proof. ∎

**Proof of Lemma 7.4** After a change of variables, Lemma A.2 reads as follows:

Let $\epsilon > 0$ and $L > 0$. There exists $n_0 \geq 1$ and $h_0 > 0$ such that

$$\sup_{t\in\mathbb{R}} d_n \Big|\mu_n([t, t+h]) - \frac{h}{d_n} q_\beta(d_n^{-1}t)\mu(A)\mu(B)\Big| \leq h\epsilon, \tag{A.9}$$

for all $n \geq n_0$, $h \in [L^{-1}, d_n h_0]$.

Fix $h > 0$ and define $e_n = \sup_{t\in\mathbb{R}} d_n \big|\mu_n([t, t+h]) - \frac{h}{d_n} q_\beta(d_n^{-1}t)\mu(A)\mu(B)\big|$. We must show that $\lim_{n\to\infty} e_n = 0$.

Let $L = 1/h$. By (A.9), for any $\epsilon > 0$ there exists $n_0 \geq 1$, $h_0 > 0$, such that $e_n \leq h\epsilon$ for all $n \geq n_0$ subject to the constraint $d_n h_0 \geq h$. Since $d_n \to \infty$, there exists $n_1 \geq n_0$ such that $d_n h_0 \geq h$ for all $n \geq n_1$. Hence $e_n \leq h\epsilon$ for all $n \geq n_1$ as required. ∎

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