Parameter Estimation of Hybrid Fractional-Order Hammerstein-Wiener Box-Jenkins Models Using RIVCF Method

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***Abstract*—This paper proposes an extension of the simplified refined instrumental variable algorithm for the parameter estimation of the stochastic single-input, single-output hybrid fractional-order continuous-time Hammerstein-Wiener Box-Jenkins model. The model parameters are directly estimated from observed input-output data with less constraints such as, that the output static nonlinearity must be invertible. The noise-free model is described by a series of an input static nonlinear sub-model, a fractional-order continuous-time linear model, and then an output static nonlinear sub-model. The two nonlinear sub-models are both given by a sum of the known basis functions. The noise process is described by a Box-Jenkins model. The proposed approach estimates the parameters of the nonlinear and linear sub-models in an iterative manner. In this paper, Monte Carlo simulation analysis shows the proposed algorithm provides accurate and fast converged estimates of the fractional-order Hammerstein-Wiener hybrid Box-Jenkins model.**

***Keywords*-fractional-order systems; Hammerstein-Wiener Box-Jenkins model; parameter estimation; refined instrumental variable algorithm;**

I. Introduction

Fractional calculus has received an increasing attention to model the dynamical behaviour of real systems, processes, and materials [1], [2] as a more suitable mathematical tool than the integer-order calculus. The nonlinear fractional-order models have been used to describe complex nonlinear processes, for example, the nonlinear fractional order Randle's equivalent circuit model for describing the battery system [3]. In control engineering applications, fractional-orders represent extra degrees of freedom and flexibility to increase robustness [4].

A well-known class of the non-linear models is the so-called block-oriented models that comprise different configurations of linear dynamic and static non-linear blocks. The most common structures in this class are cascaded systems with the non-linear sub-system either preceding (Hammerstein model) or following (Wiener model) the linear sub-system. This class of models has been widely employed in practical applications to simplify the complex nonlinear system. The Hammerstein model, for instance, is addressed in robotic therapy for describing the isometric recruitment curve, i.e, the static gain relation between the stimulus activation level and steady-state output torque [5]. In energy storage systems, the battery impedance model has been improved by introducing a Wiener static nonlinearity to the ordinary equivalent circuit model [6], [7]. Hammerstein models have also been employed to represent the air handling unit of large heating ventilation and air conditioning systems [8], where it is principally used to describe the nonlinearity introduced by the valves. The model that uses a nonlinear block both preceding and following a linear dynamic system is called a Hammerstein-Wiener model [9], [10]. The main advantages of the Hammerstein-Wiener class of models are (i) the model transition and stability is mainly characterised by the linear subsystem, (ii) if the inverse of the static nonlinear function exists, the control algorithm can be similarly designed as in the case of linear models.

There is an essential need to parameter estimation when dealing with real-life practical applications. In the discrete-time-domain, the iterative algorithm proposed in [11] is based on accessing the internal signals by using the key term separation principle as a decomposition technique. This algorithm was extended for the case of multi-inputs by Vörös in [12]. The approaches adapted in [11], [12] express the Hammerstein and Wiener models linearly in parameters. The key term separation principle and estimated linear outputs, adopted in [11], [12], are also used in the case of the Wiener model in [13]. The principle drawback of this approach is that the convergence is not guaranteed. Other approaches for discrete iterative methods can be found in [14], [15]. In recent studies in the discrete-time domain, the kernel, Volterra and fractional least mean square algorithms have been applied for estimating the parameters of the Hammerstein models associated with coloured noise process [16]. These approaches managed to estimate the parameters but they required a large number of iterations. The iteration number could be reduced by employing the sliding-window approximation based fractional least mean squares in [17] but still, the iteration number is considerably large. All the aforementioned approaches are in the discrete time domain and are employed for obtaining the continuous-time transfer function of the linear subsystem. A further step is required to convert from the discrete-time to the continuous-time domain and this class of estimation approaches is termed indirect.

The fractional-order continuous-time Hammerstein, Wiener and Hammerstein-Wiener (HFC, WFC and HWFC) models were introduced in [18]. A direct parameter estimation approach from observed input-output data of a stochastic single-input single-output fractional-order continuous-time Hammerstein-Wiener model was also proposed in [18] by extending the iterative simplified refined instrumental variable (SRIV) algorithm. They are termed SRIV for HFC, WFC and HWFC models which are abbreviated as HSRIVCF, WSRIVCF and HWSRIVCF, respectively. The algorithm shows a significant performance but it is limited to white noise. In this paper, an integer-order discrete transfer function is introduced to the HFC, WFC and HWFC models to represent the noise process. This class of noise is known as Box-Jenkins noise. The HWSRIVCF is extended to the refined instrumental variable (RIV), for HWFC models and denoted HWSRIVCF. The extension allows estimating the parameters of the linear sub-model and static nonlinear functions in the model. The approach proposed in this paper reformulates the nonlinear HWFC model to be described by a multi-input, single-output linear fractional-order continuous-time model. The multi-input signals include the outputs of basis functions of the static nonlinear functions whose inputs are the actual input of the static input nonlinear function and the output of the estimated fractional-order continuous-time linear subsystem. The novelty of this paper stems from the use of the simulated linear subsystem output for obtaining the basis functions of the static output nonlinear and extension to the case of fractional-order models with coloured noise described by the Box-Jenkins process.

The paper is structured such that the problem description for a fractional Hammerstein-Wiener model is stated in II. Section. The problem reformulation based on an HWFC model is addressed in III. Section. IV. Section introduces the HWSRIVCF and HWSRIVCF. V. Section evaluates the statistical performance of HWSRIVCF and HWSRIVCF algorithms using a numerical study. Finally, the paper concludes in VI. Section.

II. Problem Description

A fractional-order continuous-time Hammerstein-Wiener model has static input and output (memoryless) nonlinear functions, with an intermediate fractional-order continuous-time subsystem as illustrated in Fig. 1. The HWFC model can be described by the input-output relationship as follows:

$$\begin{aligned}
\bar{u}(t) &= f_u(u(t)) \\
x(t) &= \frac{B(\mathscr{D}^{\beta})}{A(\mathscr{D}^{\alpha})}\bar{u}(t) \\
\bar{x}(t) &= g_x(x(t)) \\
\xi(k) &= \frac{C(q^{-1})}{D(q^{-1})}e(k) \\
y(t_k) &= \bar{x}(t_k) + \xi(k)
\end{aligned} \tag{1}$$

where $u(t)$ and $\bar{u}(t)$ are the input and output of the static input nonlinear function $f_u(u)$. The output of the static input nonlinear function $\bar{u}(t)$ is the input of the fractional-order continuous-time linear subsystem whose output is $x(t)$. $x(t)$ inputs to the static output nonlinear function, denoted $g_x(x)$ which generates $\bar{x}(t)$. The sampled form of $\bar{x}(t)$ at time instance k is denoted $\bar{x}(t_k)$ where $t = k \times T_s$ and T_s is the sampling interval. The fractional-order continuous-time linear subsystem is described by a fractional-order differential equation in a form of the input and output polynomials, denoted $B\left(\mathscr{D}^{\beta}\right)$ and $A\left(\mathscr{D}^{\alpha}\right)$, respectively, and expressed as:

$$\begin{aligned}
A(\mathscr{D}^{\alpha}) &= a_0\mathscr{D}^{\alpha_n} + a_1\mathscr{D}^{\alpha_{n-1}} + \ldots + a_{n-1}\mathscr{D}^{\alpha_1} + a_n \\
B(\mathscr{D}^{\beta}) &= b_0\mathscr{D}^{\beta_m} + b_1\mathscr{D}^{\beta_{m-1}} + \ldots + b_{m-1}\mathscr{D}^{\beta_1} + b_m
\end{aligned} \tag{2}$$

where the coefficients $a_j\,(j = 0,1,\ldots n)$ and $b_j\,(j = 0,1,\ldots m)$ are real constants, $\mathscr{D}^{\alpha}x(t) = \frac{d^{\alpha}x(t)}{dt^{\alpha}}$, $(\alpha_k\,(k = n,n-1,\ldots 1)) \in \mathbb{R}^+$, $(\beta_q\,(q = m,m-1,\ldots 1)) \in \mathbb{R}^+$,$\alpha_n > \alpha_{n-1} \ldots > \alpha_1 > 0$, $\beta_m > \beta_{m-1} \ldots > \beta_1 > 0$ and $\alpha_n > \beta_m$ for physical feasibility. It is assumed that $a_0 = 1$ and the system is commensurate with base-order, denoted α; therefore, $\alpha_i = \alpha \times i$ and $\beta_j = \alpha \times j$. Moreover, it is assumed that the static nonlinear functions are described by a sum of the basis functions and expressed as:

$$\begin{aligned}
\bar{u}(t) &= \sum_{j=1}^{r} \bar{b}_j f_j(u) \\
\bar{x}(t) &= \sum_{i=1}^{l} \bar{a}_i g_i(x)
\end{aligned} \tag{3}$$

where the coefficients $\left\{\bar{a}_i, \bar{b}_j\right\} \in \mathbb{R}$, $(i = 1,\cdots,l),(j = 1,\cdots,r)$. Finally, the last equation in (1) shows $y(t_k)$ is produced by corrupting $\bar{x}(t_k)$, with a discrete noise process, described by $C(q^{-1})$ and $D(q^{-1})$ discrete polynomials and white (zero mean) noise denoted $e(k)$. These polynomial are given as:

$$\begin{aligned}
C\left(q^{-1}\right) &= 1 + c_1q^{-1} + \cdots + c_pq^{-p} \\
D\left(q^{-1}\right) &= 1 + d_1q^{-1} + \cdots + d_vq^{-v}
\end{aligned} \tag{4}$$

where q^{-1} is a backward shift operator and the parameters $c_j\,(j = 0,1,\ldots p)$ and $d_j\,(j = 0,1,\ldots v)$ are real constants.

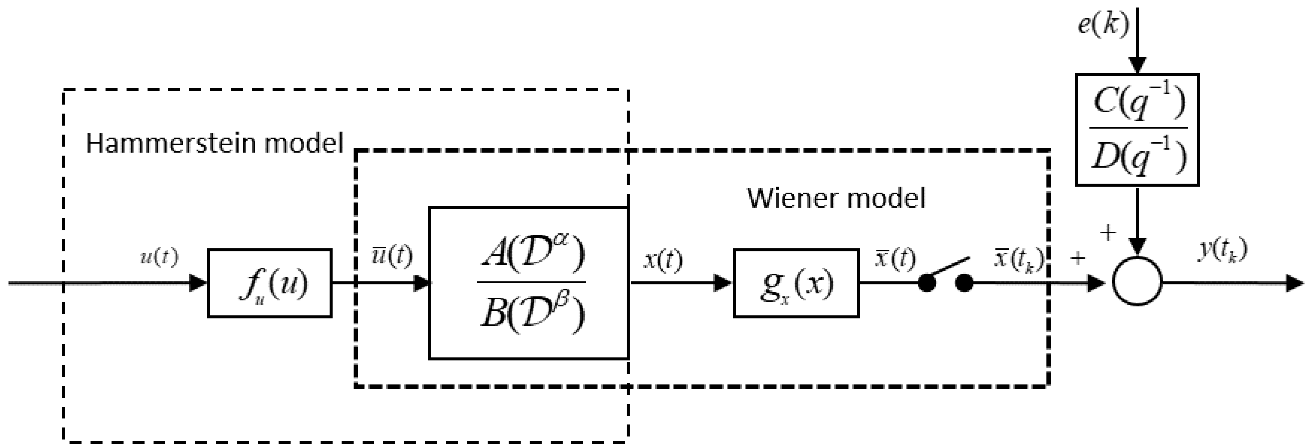

Figure 1: Block diagram of the Hammerstein-Wiener model processes

III. Problem Reformulation

For simplicity's sake, the static nonlinear functions are individually reformulated as two different sub-models with a common linear subsystem. It is then re-coupled to describe the overall system. If the HFC subsystem in (1) is separately treated, it is implied that the HFC model is formed of a cascade of the static input nonlinear function and the linear fractional-time continuous-time subsystem, as shown in the left-hand dotted box in Fig. 1, corresponding to the first two equations in (1). It is assumed that the first parameter of the static input nonlinear function is unity ($\bar{b}_1 = 1$). Since the basis function $f_i(u)$ is assumed to be a priori known and $u(t)$ is measurable, the basis functions can be time-dependent signals and denoted by, for simplicity, $\bar{f}_i(t)$. Under these conditions, the HFC subsystem can be described as a multi-input single-output system:

$$x(t) = \frac{B(\mathscr{D}^\beta)}{A(\mathscr{D}^\alpha)} \left(\bar{f}_1(t) + \sum_{i=2}^{r} \bar{b}_i \bar{f}_i(t) \right) \tag{5}$$

where

$$\bar{f}_i(t) = f_i(u) \tag{6}$$

Both the polynomial $B(\mathscr{D}^\beta)$ and the parameters of the static input nonlinear function can be coupled to yield a vector of the over-parameterised input polynomial $\bar{B}_i(\mathscr{D}^\beta)$ where $\bar{B}_i(\mathscr{D}^\beta) = \bar{b}_i B(\mathscr{D}^\beta)$ and $\bar{B}_1(\mathscr{D}^\beta) = B(\mathscr{D}^\beta)$. Consequently (5) can be re-expressed in vector form as:

$$x(t) = \frac{1}{A(\mathscr{D}^\alpha)} \left[\bar{B}(\mathscr{D}^\beta) \bar{F}(t) \right] \tag{7}$$

where the multi-input polynomial vector $\bar{B}(\mathscr{D}^\beta)$ and input vector $\bar{F}$ are given by:

$$\bar{B}(\mathscr{D}^\beta) = \left[\bar{B}_1(\mathscr{D}^\beta), \quad \bar{B}_2(\mathscr{D}^\beta), \quad \cdots \quad \bar{B}_r(\mathscr{D}^\beta) \right] \tag{8}$$

$$\bar{F}(t) = \left[\bar{f}_1(t), \quad \cdots \quad , \bar{f}_r(t) \right]^T \tag{9}$$

This part illustrates how the WFC subsystem of the HWFC system in (1) is rearranged so that any linear estimator can be employed. If the WFC subsystem of (1) is separately considered, it is realised as a cascade of the linear fractional-order continuous-time model and the static output nonlinear function as shown in the right hand bold dashed box in Fig. 1. The parameter estimation is based on the collected input-output data. The input of the linear fractional-order continuous-time model is considered to be accessible in this section but its output is not accessible. It is assumed that the first basis function of the static output nonlinear function in (3) is linear thus the static output nonlinear function in (3) can be re-described as:

$$\bar{x}(t) = x(t) + \sum_{i=2}^{l} \bar{a}_i g_i(x) \tag{10}$$

where $\bar{a}_1$ is normalised to unity and the basis functions of the static output nonlinear functions $g_i(x)$, in the last term of (10), are considered to be known a priori. Therefore, they can be described by a function of time $\bar{g}_i(t) = g_i(x(t))$ if $x(t)$ is known. Thus, the $\bar{g}_i(t)$ functions are considered as inputs to the system. According to (1), (3) and (10), it is possible to characterise the WFC subsystem as a linear fractional-order continuous-time model with multi-input ($\bar{u}(t)$,$\bar{g}_i(t)$) and single-output $\bar{x}(t)$ as:

$$\bar{x}(t) = \frac{B(\mathscr{D}^\beta)}{A(\mathscr{D}^\alpha)} \bar{u}(t) + \sum_{i=2}^{l} \bar{a}_i \bar{g}_i(t) \tag{11}$$

In this part, both the reformulated forms of HFC in (8) and WFC models in (11) are coupled to re-describe the original HWFC model in (1) in linear in parameter form. They are coupled by a fractional-order continuous-time linear system in one model, representing the noise-free HWFC model and expressed as:

$$\bar{x}(t) = \frac{1}{A(\mathscr{D}^{\alpha})}\bar{B}(\mathscr{D}^{\beta})\bar{F}(t) + \sum_{i=2}^{l}\bar{a}_i\bar{g}_i(t) \tag{12}$$

where the over-parametrised polynomial $\bar{B}(\mathscr{D}^{\beta})$ and vector $\bar{F}(t)$ are given in (8) and (9), and $\bar{g}_i(t) = g_i(x)$. Since measured data is used for parameter estimation, the output of the model in (12) is considered to be corrupted by a noise process. Thus the HWFC model in (12) can be re-expressed as:

$$y(t_k) = \bar{x}(t_k) + \xi(k) \tag{13}$$

where $y(t)$ is the noisy output and $\xi(k)$ represents the noise process.

IV. HWRIVCF ALGORITHMS

Considering $\bar{g}_i(t)$ and $\bar{F}(t)$ to be inputs to the HWFC model, the model may thus be described by a multi-input, single-output linear fractional-order continuous-time model. The error function of (13) is expressed as:

$$\varepsilon_{HW}(t) = y(t) - \left(\frac{1}{A(\mathscr{D}^{\alpha})}\bar{B}(\mathscr{D}^{\beta})\bar{F}(t) + \sum_{i=2}^{l}\bar{a}_i\bar{g}_i(t)\right) \tag{14}$$

where the subscript HW refers to Hammerstein-Wiener.

(14) is reformulated such that the polynomial $A(\mathscr{D}^{\alpha})$ is associated with the noisy output $y(t)$. This leads to the introduction of a filter $\frac{1}{A(\mathscr{D}^{\alpha})}$ in the first and second terms on the right-hand side of (14) for generating the filtered input-output data without filtering the error $\varepsilon_{HW}(t)$. Therefore, (14) can be re-expressed as:

$$\begin{aligned}\varepsilon_{HW}(t) =& A(\mathscr{D}^{\alpha})\frac{1}{A(\mathscr{D}^{\alpha})}y(t) - \\ &\left(\bar{B}(\mathscr{D}^{\beta})\frac{1}{A(\mathscr{D}^{\alpha})}\bar{F}(t) + \sum_{i=2}^{l}\bar{a}_i\bar{g}_i(t)\right)\end{aligned} \tag{15}$$

(15) can be described in a filtered form by a model of the multi-input ($\bar{g}_i$, filtered $\bar{F}$), single filtered output form. Therefore the error function (15) is rearranged and expressed as:

$$\varepsilon_{HW}(t) = A(\mathscr{D}^{\alpha})y_F(t) - \left(\bar{B}(\mathscr{D}^{\beta})\bar{F}_F(t) + \sum_{i=2}^{l}\bar{a}_i\bar{g}_i(t)\right) \tag{16}$$

where the filtered output and the vector of the filtered input are denoted $y_F(t)$ and $\bar{F}_{F,B}(t)$, respectively, and the subscript F indicates the signal is filtered by $\frac{1}{A(\mathscr{D}^{\alpha})}$. The filtered data can be obtained from:

$$\begin{aligned}\bar{F}_F(t) &= \frac{1}{A(\mathscr{D}^{\alpha})}\ \bar{F}(t_k) \\ y_F(t) &= \frac{1}{A(\mathscr{D}^{\alpha})}\ y(t_k)\end{aligned} \tag{17}$$

The initial parameters for simulating the noise-free output $\bar{x}(t)$ can be obtained by applying HWSRIVCF. The error function $\varepsilon_{HW}(t_k) =$ is considered to be the autoregressive–moving-average mode ARMA process such that $\varepsilon_{HW}(t_k) = \xi(k)$; therefore, the assumed white prediction error can be obtained as:

$$\hat{e}(k) = \frac{\hat{D}\left(q^{-1}\right)}{\hat{C}\left(q^{-1}\right)}\varepsilon_{HW}(t_k) \tag{18}$$

In fact, the noise polynomials $\hat{C}\left(q^{-1}\right)$ and $\hat{D}\left(q^{-1}\right)$ are not estimated. However, for simplicity, the integer-order discrete-time noise ARMA process in can be approximated by a higher order AR process with a much larger order of $D\left(q^{-1}\right)$ denominator [19]. Defining:

$$\frac{C\left(q^{-1}\right)}{D\left(q^{-1}\right)}e(k) \approx \frac{1}{\bar{D}\left(q^{-1}\right)}\hat{e}(k) \tag{19}$$

where the order of $\bar{D}\left(q^{-1}\right)$ is selected to be twelve in the illustrative example. (19) leads to rearrange (18) such that:

$$\hat{e}(k) = \bar{D}\left(q^{-1}\right)\varepsilon_{HW}(t_k) \tag{20}$$

The parameters of the $\bar{D}\left(q^{-1}\right)$ polynomial are then estimated based on (20) by using the least squares algorithm. In order to force the coloured error function to be white error, the error function is described as a white prediction error by filtering (16) by $\bar{D}\left(q^{-1}\right)$:

$$\hat{e}(k) = \bar{D}\left(q^{-1}\right)\left(y_F(t_k) - \bar{x}_F(t_k)\right) \tag{21}$$

This leads to obtaining parameters of the most optimal convergence of the multi-input, single-output model. Thus, the pseudo regression form can be deduced based on sampled data and expressed as:

$$\mathscr{D}^{\alpha_n}y_{DF}(t_k) = \varphi_{DF}^T(t_k)\theta + \varepsilon(t_k) \tag{22}$$

where θ and $\varphi_{DF}^T(t_k)$ are given, respectively, as:

$$\begin{aligned}\theta =& [a_1 \quad \cdots \quad a_n \quad \bar{b}_1b_0 \quad \cdots \quad \bar{b}_1b_m \quad \cdots \quad \bar{b}_rb_0 \\ &\cdots \quad \bar{b}_rb_m \quad \bar{a}_2 \quad \cdots \quad \bar{a}_l]^T\end{aligned} \tag{23}$$

$$\begin{aligned}\varphi_{FD}^T(t_k) = [&-\mathscr{D}^{\alpha_{n-1}}y_{FD}(t_k) \quad \cdots \quad -y_{FD}(t_k) \qquad \mathscr{D}^{\beta_m}\bar{f}_{FD,1}(t_k) \quad \cdots \\ &\bar{f}_{FD,1}(t_k) \quad \mathscr{D}^{\beta_m}\bar{f}_{FD,r}(t_k) \quad \cdots \quad \bar{f}_{FD,r}(t_k) \quad \hat{\bar{g}}_{D,2}(t_k), \quad \cdots \quad \hat{\bar{g}}_{D,l}(t_k)]\end{aligned} \tag{24}$$

All filtered terms in (24) can be readily obtained using:

$$\bar{y}_{FD}(t_k) = \bar{D}\left(q^{-1}\right)\bar{y}_F(t_k) \tag{25}$$

There is an issue that $\bar{g}_i(t_k)$ is not accessible, however, $\bar{g}_i(t_k)$ can be simulated. Simulating $\bar{g}_i(t_k)$ requires the $\hat{B}(\mathscr{D}^{\beta})$, $\hat{A}(\mathscr{D}^{\alpha})$ polynomials and the estimated parameters of the static input nonlinear function ($\hat{\bar{b}}_s$) to be available. In this paper, the initial $\hat{B}(\mathscr{D}^{\beta})$, $\hat{A}(\mathscr{D}^{\alpha})$ polynomials are selected according to three main factors which are (i) considering the output steady state of the linear system, (ii) considering whether the linear subsystem is under-damped or over-damped and (iii) the cut-off frequency which can be selected according to the fractional-order state variable filter design [20]. The selection of $\hat{\bar{b}}_s$ does not have a large influence on the estimation. For example in the numerical example in this paper $\hat{\bar{b}}_s$ is selected such as $\hat{\bar{b}}_1 = \hat{\bar{b}}_2 = \hat{\bar{b}}_3 = 1$. An initial estimate of $\hat{A}(\mathscr{D}^{\alpha})$ is used for designing the filter $\frac{1}{\hat{A}(\mathscr{D}^{\alpha})}$. The parameters are then repeatedly estimated at every iteration as indicated by the subscript l, which represents the present iteration index.The HWSRIVCF algorithm is iteratively implemented and summarised as follows:

(i) Compute the multi-input vector using the input static nonlinear function $\bar{F}(t_k)$.

(ii) Simulate the noise-free output $\hat{x}(t)$ using:

$$\hat{x}(t) = \frac{1}{\hat{A}(\mathscr{D}^{\alpha}, \hat{\theta}_{l-1})}\hat{\bar{B}}(\mathscr{D}^{\beta}, \hat{\theta}_{l-1})_i \bar{F}(t_k) \tag{26}$$

where $\hat{x}(t)$ is used as the input to the static output nonlinear function and as the instrumental variable.

(iii) Filter $\hat{x}(t_k)$, $y(t_k)$ and $\bar{F}(t_k)$ to generate their filtered forms with their higher fractional-order derivatives, using:

$$F(\mathscr{D}^{\alpha}) = \frac{1}{\hat{A}\left(\mathscr{D}^{\alpha}, \hat{\theta}_{l-1}\right)} \tag{27}$$

(iv) Generate $\hat{\bar{g}}_i(t_k)$ using $\hat{x}(t_k)$.

(v) Compute $\hat{\varepsilon}_{HW}(t_k)$ using:

$$\hat{\varepsilon}_{HW}(t_k) = y_F(t_k) - \hat{\bar{x}}_F(t_k) \tag{28}$$

The discrete part of the model could be identified by using the higher order AR process with a much larger order of the $D\left(q^{-1}\right)$ denominator polynomial.

(vi) Filter the instrumental variable $\hat{x}(t)$, $\bar{F}_{F,i}(t_k)$, $y_F(t_k)$ and $\hat{\bar{g}}_i(t_k)$ are filtered by $\bar{D}\left(q^{-1}\right)$ as given in (25).

(vii) Obtain the estimated parameters using the instrumental variable least square algorithm:

$$\hat{\theta}_l = \left(\sum_{k=1}^{N} \hat{\varphi}_F(t_k)\,\varphi_F^T(t_k)\right)^{-1} \sum_{k=1}^{N} \hat{\varphi}_F(t_k)\,\mathscr{D}^{\alpha_n} y_F(t_k) \tag{29}$$

where φ_F^T is obtained from (24) and $\hat{\varphi}_F$ is defined as:

$$\begin{aligned}\hat{\varphi}_F^T(t_k) = [&-\mathscr{D}^{\alpha_{n-1}}\hat{x}_{FD}(t_k) \quad \cdots \quad -\hat{x}_{FD}(t_k) \quad \mathscr{D}^{\beta_m}\bar{f}_{FD,1}(t_k) \\ &\cdots \quad \bar{f}_{FD,1}(t_k) \quad \mathscr{D}^{\beta_m}\bar{f}_{FD,r}(t_k) \quad \cdots \quad \bar{f}_{FD,r}(t_k) \quad \hat{\bar{g}}_{D,2}(t_k), \quad \cdots \quad \hat{\bar{g}}_{D,l}(t_k)]\end{aligned} \tag{30}$$

(viii) Repeat steps (i) to (vii) until the sum of the squares of the differences between $\hat{\theta}_{l-1}$and $\hat{\theta}_l$ is significantly small such as 10^{-4} or, for example, five iterations.

Whilst there would appear to be an issue associated with the estimates of the over-parameterised $\bar{B}(\mathscr{D}^{\beta})$ in (8), whereby $\bar{b}_s$ and b_s are combined within one vector, it was shown in [21] that $\bar{b}_s$ can be directly obtained from $B_i(\mathscr{D}^{\beta})$ in (8) by using:

$$\hat{\bar{b}}_i = \frac{1}{m+1}\sum_{k=0}^{m}\frac{\hat{b}_{i,k}}{\hat{b}_{1,k}} \tag{31}$$

where $\hat{b}_{i,k}$ is the estimated form of $b_{i,k} = \bar{b}_i b_k$, given in (23).

The convergence of the refined instrumental variable algorithm was comprehensively analysed for the linear integer-order order model in [22] and the integer-order of Hammerstein-Wiener continuous-time model in [23]. The instrumental variable is the noise-free output, similarly assumed in [23] and the reformulated structure (multi-input and single-output model) in (22) is the same structure as presented in [23]. The difference being that the system here is fractional and multi-input and single-output. This difference does not affect the proof, derived in [23]. Therefore those proofs, given in [23], are used here if consider the algorithm (29) given by (23), (24) and, (30).

and suppose the following assumptions exist:

Assumption 1: The true linear sub-system is asymptotically stable.

Assumption 2: The noise $e(k)$ is white (zero mean) and independent of the system input $u(t)$.

Assumption 3: $u(t)$ is produced such that the data set is sufficiently informative for the identification.

Then the following results are true:

Therefore, the estimates in (23) can be obtained as $N \to \infty$. Further convergence analysis can be found in [23], [22]. The numerical study in this paper empirically shows that the estimates converge to the true parameters within the second to fifth iteration.

Note: It will be noted that the above formulation of the RIVC estimation problem is considerably simplified if it is assumed that the additive noise is white, i.e., $C(q^{-1}) = D(q^{-1}) = 1$. In this case, HWSRIVCF estimation requires the filtering using the continuous-time filter $\frac{1}{A(\mathscr{D}^{\alpha})}$. Consequently, the main steps in the SRIVC algorithm are the same as those in the HWSRIVCF algorithm, except that the noise model estimation and subsequent discrete-time filtering in steps (v) and (vi) of the iterative procedure are no longer required and are neglected.

V. Numerical Study

This section presents a numerical example to evaluate and highlight the performance of the proposed HWSRIVCF algorithm for parameter estimation of an HWFC model. The static input nonlinear function is described by a static polynomial form, i.e. $\bar{f}_i(t) = u^i$ and the output nonlinear function is also represented by a static polynomial function. Thus, the HWFC model is given by:

$$\begin{aligned} \bar{u}(t) &= u(t) + \bar{b}_2 u^2(t) + \bar{b}_3 u^3(t) \\ x(t) &= \frac{b_0}{a_0 \mathscr{D}^{0.5} + a_1} \bar{u}(t) \\ \bar{x}(t) &= x(t) + \bar{a}_2 x^2(t) + \bar{a}_3 x^3(t) \\ y(t_k) &= \bar{x}(t_k) + \frac{1 + 0.3q^{-1}}{1 - 0.6q^{-1}}\, e(t_k) \end{aligned} \tag{32}$$

The sampled input $\bar{u}(t_k)$ and the sampled noisy output $y(t_k)$ are collected and used for parameter estimation.

The system is simulated for 100 (s) with a fixed sampling interval of 10^{-3} (s). The selected Simulink solver is ode4 (Runge-Kutta). The fractional-order integral block is provided by the FOMCON Simulink library with frequency range [0.001 (rad.sec^{-}1); 1000 (rad.sec^{-}1)]. Details about the FOMCON Simulink library can be found in [24]. A square wave signal with a random amplitude is used as an input to the HWFC model.

To evaluate the statistical performance of the proposed approach, a Monte Carlo simulation is performed for 50 runs. The same input signal is used for all runs but the white Gaussian noise is rearranged for different levels whilst keeping a fixed signal to noise ratio (SNR). The noise variance is selected such that the $SNR = 30dB$ and $SNR = 60dB$, where SNR is defined in dB as:

$$SNR = 10 \log \frac{P_{\bar{x}}}{P_e} \tag{33}$$

where $P_{\bar{x}}$ and P_e are the average power of the signals $\bar{x}$ and e, respectively.

It is assumed that the only accessible data is the input $u(t_k)$ and sampled noisy output $y(t_k)$. The system, considered for estimation using the HWSRIVCF algorithm, is a multi-input ($\bar{f}_1(t_k)$, $\bar{f}_2(t_k)$,$\bar{f}_3(t_k)$,$\bar{g}_2(t_k)$, $\bar{g}_3(t_k)$) and single-output $y(t)$ model and expressed as:

$$y(t) = \sum_{i=1}^{3} \frac{\bar{b}_i B\left(\mathscr{D}^{\beta}\right)}{A\left(\mathscr{D}^{\alpha}\right)} \bar{f}_i(t_k) + \bar{a}_2 \hat{\bar{g}}_2(t) + \bar{a}_3 \hat{\bar{g}}_3(t) + \varepsilon(t) \tag{34}$$

where $\hat{\bar{g}}_i(t) = \hat{x}^i(t)$ but here $\hat{x}(t) = \sum_{i=1}^{3} \frac{\hat{\bar{b}}_i \hat{B}(\mathscr{D}^{\beta},\hat{\theta})}{\hat{A}(\mathscr{D}^{\alpha},\hat{\theta})} \bar{f}_i(t_k)$ and $\bar{f}_i(t_k) = u^i(t_k)$, and $\varepsilon(t)$ is the modelling error.

The initial input and output fractional-order linear polynomials are selected such that all roots $s_i^{0.5}$ are located at 1, hence $\hat{B}(\mathscr{D}^{\beta}, \hat{\theta}_0) = 1$ and $\hat{A}(\mathscr{D}^{\alpha}, \hat{\theta}_0) = \mathscr{D}^{0.5} + 1$. The parameters of the static input nonlinear function are selected to be unity so that $\bar{b}_1 = \bar{b}_2 = \bar{b}_3 = 1$.

The results obtained from the Monte Carlo simulation analysis are presented in mean and standard deviations in Table I. Table I demonstrates that the obtained results match the theory beyond the HWSRIVCF algorithm where it gives unbiased estimates of the HWFC model parameters. Although the noise is relatively high, at the level of 30dB, the proposed algorithm converges.This means, in the case of the lower noise level (higher SNR), the mean value of the estimates converge more towards the true values and the standard deviations also decrease. Thus, the SNR significantly affects the convergence of the parameters to their true values.

Although the noise is rather high, at the level of $30dB$, the proposed method converges. The standard deviations, obtained by HWSRIVCF are always larger than those obtained by HWRIVCF but they are still within reasonable limits, as expected. This is caused by the HWSRIVCF method being designed for an OE model estimation scenario instead of Box-Jenkins noise model scenario. This further increases the motivation of using HWSRIVCF method when used in practise. The standard deviations and the mean values obtained both approaches are always correlated to the level of the measured noise.

The obtained results indicate that the parameter estimation errors associated with the static nonlinear functions are lower when compared to the parameter estimation errors associated with the dynamic process. It should be emphasised that the parameter estimation errors of the static and dynamic submodels of the overall Hammerstein-Wiener model are related via the error covariance matrix of the estimator (29). Nevertheless, the simulation results do indicate that the presented HWSRIVCF is statistically more efficient when estimating the static part of the overall Hammerstein-Wiener model. This observation is expected since the demand on having persistently exciting system input is somewhat more relaxed when estimating static functions as oppose to dynamical processes.

The error between the estimated and the actual outputs of the input and output static nonlinear functions is small despite the existence of standard deviations of the estimates.

Table I: Monte Carlo simulation results of parameter estimation of the HWCF system where $a_1 = \bar{a}_1 = \bar{b}_1$ =1

SNR	Algorithm		$a_0 = 0.5$	$b_0 = 1$	$\bar{a}_2 = 0.01$	$\bar{a}_3 = 0.3$	$\bar{b}_2 = 0.02$	$\bar{b}_3 = 0.25$
30dB	HWSRIVCF	mean	0.4923	1.0102	0.0115	0.2912	0.023	0.2602
		std	0.0070	0.0272	0.0070	0.0120	0.0043	0.0020
	HWRIVCF	mean	0.4975	1.0031	0.0108	0.2989	0.0021	0.2522
		std	0.0020	0.0108	0.0043	0.0060	0.0009	0.0017
60dB	HWSRIVCF	mean	0.4946	1.0088	0.0109	0.2974	0.0210	0.2531
		std	0.0038	0.0097	0.0052	0.0081	0.0003	0.0015
	HWRIVCF	mean	0.4992	1.0016	0.0102	0.2996	0.0202	0.2506
		std	0.0420	0.0530	0.0011	0.0032	0.0002	0.0008

VI. Conclusions

Fractional-order continuous-time nonlinear Wiener and Hammerstein-Wiener (WFC and HWFC,respectively) models have been widely used for system identification and control applications. In this work, the HWFC model structure is characterized by a cascade connection of nonlinear static functions that transform the input and output signals of a fractional-order continuous-time dynamic model. The static nonlinear functions are represented by a sum of basis functions. Prompted by the advantages WFC and HWFC models, this paper has shown the extension of the simplified refined instrumental variable algorithm to the refined instrumental variable for the HWFC models. This extension is for handling the colored noise in a form of Box-Jenkins model.

The refined instrumental variable approach was developed for continuous-time integer-order linear models parameter estimation. It has shown in this paper how the algorithm is formulated and developed within the context of fractional-order Hammerstein-Wiener model. This is achieved by describing the fractional-order continuous-time Hammerstein-Wiener model as a mluti-input single-out form. The major advantage of the proposed approach is that the output static nonlinear function does not need to be invertible. Note, that the proposed HWRIVCF/HWSRIVCF algorithms are also applicable to HFC and WFC model structures as a special case and that these differ only in the type of assumed noise model. Following the instrumental variable approach, the initialization process does not have a large impact on the number of iterations required for convergence. The effectiveness of the proposed approach has been evaluated by a numerical study via a Monte Carlo simulation study.

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References

[1] R. E. Gutiérrez, J. M. Rosário, and J. Tenreiro Machado, "Fractional order calculus: basic concepts and engineering applications," *Mathematical Problems in Engineering*, vol. 2010, 2010.

[2] W. Allafi, I. Zajic, and K. J. Burnham, *Identification of Fractional Order Models: Application to 1D Solid Diffusion System Model of Lithium Ion Cell*. Cham: Springer International Publishing, 2015, pp. 63–68.

[3] S. Buller, M. Thele, E. Karden, and R. W. D. Doncker, "Impedance-based non-linear dynamic battery modeling for automotive applications," *Journal of Power Sources*, vol. 113, no. 2, pp. 422 – 430, 2003, proceedings of the International Conference on Lead-Acid Batteries, {LABAT} '02.

[4] J. Sabatier, P. Lanusse, P. Melchior, and A. Oustaloup, *Fractional Order Differentiation and Robust Control Design: CRONE, H-infinity and Motion Control*. Springer, 2015, vol. 10.

[5] W. Xu, B. Chu, and R. E. Rogers, "Iterative learning control for robotic-assisted upper limb stroke rehabilitation in the presence of muscle fatigue," *Control Engineering Practice*, vol. 31, pp. 63–64, 2014.

[6] W. Widanage, A. Barai, G. Chouchelamane, K. Uddin, A. McGordon, J. Marco, and P. Jenning, "Design and use of multisine signals for li-ion battery equivalent circuit modelling. part 2: Model estimation," *Journal of Power Sources*, vol. 324, pp. 61 – 69, 2016.

[7] W. Allafi, K. Uddin, C. Zhang, R. M. R. A. Sha, and J. Marco, "On-line scheme for parameter estimation of nonlinear lithium ion battery equivalent circuit models using the simplified refined instrumental variable method for a modified wiener continuous-time model," *Applied Energy*, vol. 204, pp. 497 – 508, 2017.

[8] I. Zajic, T. Larkowski, M. Sumislawska, K. J. Burnham, and D. Hill, "Modelling of an air handling unit: A hammerstein-bilinear model identification approach," in *2011 21st International Conference on Systems Engineering*, Aug 2011, pp. 59–63.

[9] A. Wills, T. B. Schön, L. Ljung, and B. Ninness, "Identification of hammerstein–wiener models," *Automatica*, vol. 49, no. 1, pp. 70–81, 2013.

[10] J. Schoukens, J. G. Nemeth, P. Crama, Y. Rolain, and R. Pintelon, "Fast approximate identification of nonlinear systems," *Automatica*, vol. 39, no. 7, pp. 1267–1274, 2003.

[11] J. Vörös, "An iterative method for hammerstein-wiener systems parameter identification," *Journal of electrical engineering*, vol. 55, no. 11-12, pp. 328–331, 2004.

[12] J. Vörös, "Parameter identification of wiener systems with multisegment piecewise-linear nonlinearities," *Systems and Control Letters*, vol. 56, no. 2, pp. 99 – 105, 2007.

[13] L. Zhou, X. Li, and F. Pan, "Least-squares-based iterative identification algorithm for wiener nonlinear systems," *Journal of Applied Mathematics*, vol. 2013, 2013.

[14] D. Wang and F. Ding, "Least squares based and gradient based iterative identification for wiener nonlinear systems," *Signal Processing*, vol. 91, no. 5, pp. 1182 –1189, 2011.

[15] H.-T. Chen, S.-H. Hwang, and C.-T. Chang, "Iterative identification of continuous-time hammerstein and wiener systems using a two-stage estimation algorithm," *Industrial & Engineering Chemistry Research*, vol. 48, no. 3, pp. 1495–1510, 2009.

[16] N. Chaudhary and M. Raja, "Identification of hammerstein nonlinear armax systems using nonlinear adaptive algorithms," *Nonlinear Dynamics*, vol. 79, no. 2, pp. 1385–1397, 2015.

[17] M. Aslam, I. Chaudhary, and Z. Raja, "A sliding-window approximation-based fractional adaptive strategy for hammerstein nonlinear armax systems," *Nonlinear Dynamics*, vol. 87, no. 1, pp. 519–533, 2017.

[18] W. Allafi, "Parameter estimation of the fractional-order hammerstein-wiener model using simplified refined instrumental variable fractional-order continuous time," *IET Control Theory & Applications*, July 2017.

[19] T. Söderström and P. Stoica, *System identification*. Prentice-Hall, Inc., 1988.

[20] O. Cois, A. Oustaloup, T. Poinot, and J.-L. Battaglia, "Fractional state variable filter for system identification by fractional model," in *Control Conference (ECC), 2001 European*. IEEE, 2001, pp. 2481–2486.

[21] V. Laurain, M. Gilson, H. Garnier, and P. Young, "Refined instrumental variable methods for identification of hammerstein continuous-time box–jenkins models," in *47th IEEE Conference on Decision and Control, CDC'08*, 2008, pp. 210–216.

[22] X. Liu, J. Wang, and W. X. Zheng, "Convergence analysis of refined instrumental variable method for continuous-time system identification," *IET Control Theory Applications*, vol. 5, no. 7, pp. 868–877, May 2011.

[23] B. Ni, M. Gilson, and H. Garnier, "Refined instrumental variable method for hammerstein-wiener continuous-time model identification," *IET Control Theory Applications*, vol. 7, no. 9, pp. 1276–1286, June 2013.

[24] A. Tepljakov, E. Petlenkov, and J. Belikov, "Fomcon: Fractional-order modeling and control toolbox for matlab," in *Proc. 18th Int Mixed Design Conference of Integrated Circuits and Systems*, 2011, pp. 684–689.