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# THE BALMER SPECTRUM OF RATIONAL EQUIVARIANT COHOMOLOGY THEORIES

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ABSTRACT. The category of rational  $G$ -equivariant cohomology theories for a compact Lie group  $G$  is the homotopy category of rational  $G$ -spectra and therefore tensor-triangulated. We show that its Balmer spectrum is the set of conjugacy classes of closed subgroups of  $G$ , with the topology corresponding to the topological poset of [7]. This is used to classify the collections of subgroups arising as the geometric isotropy of finite  $G$ -spectra. The ingredients for this classification are (i) the algebraic model of free spectra of the author and B.Shipley [17], (ii) the Localization Theorem of Borel-Hsiang-Quillen and (iii) tom Dieck's calculation of the rational Burnside ring [4].

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## 1. INTRODUCTION

1.A. **Context.** This paper relates the general structure of the category of  $G$ -equivariant cohomology theories for a compact Lie group  $G$  to the structure of the Lie group  $G$ . To start with, the category is tensor-triangulated, since it is the homotopy category of the monoidal

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model category of  $G$ -spectra. To see the broad features of this category we restrict attention to cohomology theories whose values are rational vector spaces: a  $G$ -equivariant cohomology theory with values in rational vector spaces is represented by a  $G$ -spectrum with rational homotopy groups.

The crudest structural features of this category are reflected in the localizing subcategories and the thick subcategories of compact objects. These in turn are based on an understanding of the Balmer spectrum. We show that the Balmer primes are in bijective correspondence with conjugacy classes of closed subgroups of  $G$ , and we identify both the containments amongst them and the topology. These correspond precisely to the structures identified in [7].

It turns out that to prove these results we only need an understanding of free spectra for various groups, as described in [17]. Accordingly the arguments presented here are clearer and more elementary than those presented in [15] for tori, and simultaneously give results for all compact Lie groups  $G$ . The present paper renders [15] obsolete.

In retrospect this result gives an intrinsic justification for the form of the full algebraic models of [9, 11, 12, 18, 9, 10], and suggest a revisionist approach to the general project of giving an algebraic model for rational  $G$ -spectra.

**1.B. Tensor triangulated categories.** We recall some standard terminology from the study of tensor-triangulated categories (tt-categories) and the basic definitions from [1].

If  $\mathbb{C}$  is a tensor triangulated category, an object  $T$  is called *small*, *finite* or *compact* if for any set of objects  $Y_i$ , the natural map

$$\bigoplus_i [T, Y_i] \xrightarrow{\cong} [T, \bigvee_i Y_i]$$

is an isomorphism (where  $[A, B]$  denotes the  $\mathcal{C}$ -morphisms from  $A$  to  $B$ ). We write  $\mathbb{C}^c$  for the tensor triangulated subcategory of compact objects. The word ‘compact’ has become unavoidable by virtue of the superscript  $c$  that it engendered, but we will often use the word ‘finite’ since it is suggestive of a finite  $CW$ -complex.

We say that a full subcategory  $A$  of  $\mathbb{C}$  is *thick* if it is closed under completing triangles and taking retracts. It is *localizing* if it is closed under completing triangles and taking arbitrary coproducts (it is then automatically closed under retracts as well). We say that  $A$  is an *ideal* if it is closed under triangles and tensoring with an arbitrary element.

For a general subcategory  $B$  we write  $\text{Thick}(B)$  for the thick subcategory generated by  $B$  and  $\text{Thick}_\otimes(B)$  for the thick tensor ideal generated by  $B$ . The latter depends on the ambient category, and we will only write  $\text{Thick}_\otimes(B)$  in the category  $\mathbb{C}^c$  of compact objects (so  $B$  is compact, and only tensor products with compact objects are permitted). We write  $\text{Loc}(B)$  for the localizing subcategory generated by  $B$ , and  $\text{Loc}_\otimes(B)$  for the localizing tensor ideal generated by  $B$ ; because an infinite coproduct of compact objects will usually not be compact, this only makes sense for the full category  $\mathbb{C}$  and tensor products with arbitrary objects of  $\mathbb{C}$  are permitted.

We will generally be interested in thick and localizing tensor ideals, because without closure under tensor products the structure is hard to understand. We will give an example to illustrate this in a special case in Section 11.

**Definition 1.1.** A *prime ideal* in a tensor triangulated category is a thick proper tensor ideal  $\wp$  with the property that  $a \otimes b \in \wp$  implies that  $a$  or  $b$  is in  $\wp$ .

The *Balmer spectrum* of a tensor-triangulated category  $\mathbb{C}$  is the set of prime tensor ideals of the triangulated category of compact objects:

$$\mathrm{Spc}(\mathbb{C}) = \{\wp \subseteq \mathbb{C}^c \mid \wp \text{ is prime}\}.$$

We may use this to define the *support* of a compact object:

$$\mathrm{supp}(X) = \{\wp \in \mathrm{Spc}(\mathbb{C}) \mid X \notin \wp\}.$$

This in turn lets us define the Zariski topology on  $\mathrm{Spc}(\mathbb{C})$  as generated by the closed sets  $\mathrm{supp}(X)$  as  $X$  runs through compact objects of  $\mathbb{C}$ .

**Example 1.2.** The motivating example is that if  $\mathbb{C} = D(R)$  is the derived category of a commutative Noetherian ring  $R$  then there is a natural homeomorphism

$$\mathrm{Spec}(R) \xrightarrow{\cong} \mathrm{Spc}(D(R))$$

where the classical algebraic prime  $\wp_a$  corresponds to the Balmer prime  $\wp_b = \{M \mid M_{\wp_a} \simeq 0\}$ . To avoid disorientation it is essential to emphasize that this is order-reversing, so that maximal algebraic primes correspond to minimal Balmer primes; either way these are the closed points.

**1.C. Transformation groups.** Our classification is in effect in terms of traditional invariants of transformation groups, namely fixed points and Borel cohomology.

If  $A$  is a based  $G$ -space and  $K$  is a subgroup of  $G$ , the fixed point space  $A^K$  admits an action of the Weyl group  $W_G(K) = N_G(K)/K$  of  $K$ . We will make constant use of the extension of this functor to  $G$ -spectra, which is the  $K$ -geometric fixed point functor  $\Phi^K$ . It is an extension in the sense that  $\Phi^K(\Sigma^\infty A) \simeq \Sigma^\infty(A^K)$ . We will generally omit notation for the suspension spectrum, and accordingly write  $\Phi^K A$  for the fixed point space as well as the associated suspension spectrum.

The functor  $\Phi^K$  has other familiar properties in that it is a tensor triangulated functor: it preserves triangles and  $\Phi^K(X \wedge Y) \simeq \Phi^K X \wedge \Phi^K Y$ .

This is used to define the *geometric isotropy*<sup>1</sup> of a  $G$ -spectrum  $X$ :

$$\mathcal{I}_g(X) = \{K \mid \Phi^K X \not\approx_1 0\}$$

is the collection of closed subgroups  $K$  for which the geometric fixed points  $\Phi^K X$  are non-equivariantly essential.

The geometric isotropy is an excellent way to organize our understanding of  $G$ -spectra. In particular, a (homotopically) *free*  $G$ -spectrum is one which is either contractible or has geometric isotropy  $\{1\}$ .

We are especially interested in the category of rational equivariant cohomology theories. Each rational equivariant cohomology theory  $E_G^*(\cdot)$  is represented in the sense that there is a rational  $G$ -spectrum  $E$  so that for any based  $G$ -space  $X$ ,

$$E_G^*(X) = [X, E]_G^*.$$

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<sup>1</sup> In [5] and the author's subsequent work this was called *stable isotropy* to distinguish it from the usual unstable notion. The corresponding notion for categorical fixed points does not seem to be useful, so this caused no confusion.

The name of 'geometric isotropy' from [20] seems to have acquired currency, and the symmetry between 'stable' and 'unstable' does not seem sufficient to overturn this advantage.

More precisely, there is a stable symmetric monoidal model category of rational  $G$ -spectra. Its homotopy category **G-spectra** is tensor triangulated, and equivalent to the category of rational equivariant cohomology theories and stable natural transformations. We will work throughout at the level of tensor triangulated categories.

1.D. **The Balmer spectrum.** There are some obvious primes in the category of  $G$ -spectra: for any closed subgroup  $K$  of  $G$ , we take

$$\wp_K = \{X \mid \Phi^K X \simeq_1 0\}.$$

To see this is prime we note that  $0$  is a prime in homotopy category of finite rational spectra (since that is equivalent to the derived category of  $\mathbb{Q}$ -modules) and

$$\wp_K = (\Phi^K)^*((0)) \text{ where } \Phi^K : \mathbf{G-spectra} \rightarrow \mathbf{spectra}.$$

This gives a prime for each closed subgroups  $K$  of  $G$ , and conjugate subgroups give the same primes. In fact this gives all primes, and we may describe the containments between them.

We say that  $L$  is *cotoral* in  $K$  if  $L$  is a normal subgroup of  $K$  and  $K/L$  is a torus.

**Theorem 1.3.** *The Balmer spectrum of prime thick tensor ideals in the category of finite rational  $G$ -spectra is in bijective correspondence to the closed subgroups of  $G$ . Containment corresponds to cotoral inclusion:*

$$\wp_K \subseteq \wp_H \text{ if and only if } K \text{ is conjugate to a subgroup cotoral in } H.$$

The Zariski topology of  $\mathrm{Spc}(\mathbf{G-spectra})$  is the Zariski topology on the  $f$ -topology from [7].

The  $f$ -topology and the Zariski topology it generates will be explained in Section 10, where the proof will also be completed. For now we just remark that the fact that  $\wp_K \subseteq \wp_H$  if  $K$  is cotoral in  $H$  comes from the classical Borel-Hsiang-Quillen Localization Theorem. The reverse implication comes from tom Dieck's calculation of the rational Burnside ring. It is a remarkable vindication of the Balmer spectrum that it captures the space of subgroups and cotoral inclusions, and even the  $f$ -topology of [7]. This can be put down to the fact that both the Balmer spectrum and the analysis of rational  $G$ -spectra are principally based on the Localization theorem and the calculation of the rational Burnside ring.

**Remark 1.4.** In the light of Theorem 1.3, for any finite spectrum  $X$ , the support in the sense of Balmer for this set of primes coincides with the geometric isotropy:

$$\mathrm{supp}(X) = \{H \mid X \notin \wp_H\} = \{H \mid \Phi^H X \not\simeq_1 0\} = \mathcal{I}_g(X).$$

1.E. **Classification of thick tensor ideals.** Continuing with finite spectra, we classify the finitely generated thick tensor ideals in  $G$ -spectra.

**Theorem 1.5.** (i) *If  $X$  is a finite rational  $G$ -spectrum then  $\mathcal{I}_g(X)$  is closed under passage to cotoral subgroups and its space of cotorally maximal elements is open and compact in the  $f$ -topology.*

(ii) *Any set of subgroups which is closed under cotoral specialization and whose set of cotorally maximal elements is open and compact in the  $f$ -topology occurs as  $\mathcal{I}_g(X)$  for some finite rational  $G$ -spectrum  $X$ .*

(iii) *If  $X$  and  $Y$  are finite rational  $G$ -spectra with  $\mathcal{I}_g(Y) \subseteq \mathcal{I}_g(X)$  then  $Y$  is in the thick tensor ideal generated by  $X$ .*

One of the things coming out of this is the importance of ‘basic cells’. First, we note that rationally the *natural* cells  $G/K_+$  are often decomposable, and are finite wedges of certain *basic* cells ([11], see Subsection 6.D). In one sense these are embodiments of the topology on the space of conjugacy classes, and they can be used as the basis for a theory of cell complexes and weak equivalences.

**1.F. Classification of localizing tensor ideals.** In fact the strategy of proof is to begin by considering infinite spectra and then deduce how finite spectra behave, using Thomason’s Localization Theorem (recorded here as 8.1).

**Theorem 1.6.** *The localizing tensor ideals of  $\mathbf{G}$ -spectra are in bijective correspondence with arbitrary collections of conjugacy classes of closed subgroups of  $G$ . The localizing tensor ideal corresponding to a collection  $\mathcal{H}$  of subgroups closed under conjugacy is*

$$\mathbf{G}\text{-spectra}\langle\mathcal{H}\rangle = \{X \mid \mathcal{I}_g(X) \subseteq \mathcal{H}\}.$$

For each  $K$  we may consider the localizing subcategory

$$\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle = \{X \mid \mathcal{I}_g(X) \subseteq (K)_G\}$$

of  $G$ -spectra ‘at  $K$ ’. Theorem 1.6 shows that  $\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$  is minimal in the sense that it is generated by any non-trivial object. This statement follows from an understanding of free  $W_G(K)$ -spectra via [17], and is the key ingredient in proving Theorem 1.6. Indeed, it is also the main step in identifying the Balmer spectrum as a set, and in the classification of thick tensor ideals of finite spectra.

**1.G. Notation.** We will tend to let subgroups follow the alphabet, so that

$$G \supseteq H \supseteq K \supseteq L.$$

and the trivial group is denoted 1. We write  $(H)_G$  for the  $G$ -conjugacy class of  $H$  and we write  $L \subseteq_G K$  if  $L$  is  $G$ -conjugate to a subgroup of  $K$ .

We write  $\text{Sub}(G)$  for the set of closed subgroups of  $G$ , and  $\text{Sub}(G)/G$  for the set of conjugacy classes. We write  $\mathcal{F}G \subseteq \text{Sub}(G)$  for the set of subgroups of finite index in their normalizer, and  $\Phi G = \mathcal{F}G/G$  for the corresponding set of conjugacy classes.

Given a partially ordered set (*poset*)  $X$  and a subset  $A \subseteq X$ , we write  $\Lambda_{\leq}(A) = \{b \in X \mid b \leq a \text{ for some } a \in A\}$  for the downward closure of  $A$ .

We will consider two partial orderings on the set of closed subgroups of a compact Lie group  $G$ . We indicate containment of subgroups by  $K \subseteq H$ , and refer to this as the *classical* ordering. Accordingly  $\Lambda_{cl}(K)$  consists of all closed subgroups of  $K$ . More often we consider conjugacy classes of subgroups and the ordering induced by subconjugacy. The more significant ordering in this paper is that of *cotoral inclusion*. We write  $K \leq H$  if  $K$  is normal in  $H$  and  $H/K$  is a torus. The importance of this ordering arises from the Localization Theorem. The set  $\Lambda_{ct}(K)$  consists of all closed subgroups cotoral in  $K$ . More often we consider conjugacy classes of subgroups and the ordering induced by being cotoral up to conjugacy.

All homology and cohomology will be reduced and have rational coefficients. All spectra will be rational.

A  $G$ -equivalence of  $G$ -spectra will be denoted  $X \simeq Y$ . A non-equivariant equivalence of  $G$ -spectra will be denoted  $X \simeq_1 Y$  for emphasis.

## Part 1. Infinite spectra

### 2. ISOTROPY SEPARATION

The purpose of this section is to recall some well known facts about universal spaces in equivariant topology: the spaces with single subgroup isotropy are basic building blocks. This corresponds to the way that to each prime in commutative algebra there is an indecomposable injective, and that all modules can be constructed from them.

**Definition 2.1.** Writing  $\{\subseteq_G K\}$  for the family of subgroups subconjugate to  $K$  and similarly for proper subgroups, for any subgroup  $K$  of  $G$  we may define the  $G$ -space  $E\langle(K)_G\rangle$  by the cofibre sequence

$$E\{\subsetneq_G K\}_+ \longrightarrow E\{\subseteq_G K\}_+ \longrightarrow E\langle(K)_G\rangle$$

It is clear from the definition that the geometric isotropy of  $E\langle(K)_G\rangle$  is precisely the conjugacy class  $(K)_G$  and that  $\Phi^K E\langle(K)_G\rangle \simeq_1 S^0$ . Where the context makes clear the ambient group we will write  $E\langle K\rangle = E\langle(K)_G\rangle$ .

**Lemma 2.2.** *The  $G$ -spectra  $E\langle K\rangle$  as  $K$  runs through closed subgroups of  $G$  generate the category of  $G$ -spectra as a localizing tensor ideal.*

**Proof:** We consider the collection  $\mathcal{CF}$  of families  $\mathcal{F}$  so that  $E\mathcal{F}_+$  lies in  $\text{Loc}(E\langle K\rangle \mid K \subseteq G)$ , ordered by inclusion. We show that  $\mathcal{F} = \text{All}$  is in  $\mathcal{CF}$ , so that  $S^0 = E\text{All}_+$  can be built from  $E\langle K\rangle$ , and hence the other cells can be obtained by using smash products.

Any increasing chain  $\{\mathcal{F}_\alpha\}_\alpha$  has an upper bound. Indeed, a functorial construction of the universal space (such as the bar construction) shows there is a functor from families to  $G$ -spaces, and hence the increasing chain gives a strict diagram, and by considering fixed points we see

$$E\left(\bigcup_{\alpha} \mathcal{F}_\alpha\right)_+ = \text{holim}_{\rightarrow \alpha} [(E\mathcal{F}_\alpha)_+].$$

Since there are countably many conjugacy classes, we may assume the chain is sequential and hence the homotopy colimit is in the localizing subcategory of its terms.

Finally we suppose that  $\mathcal{F}$  is maximal in  $\mathcal{CF}$ . If it is not  $\text{All}$ , since descending chains of subgroups are finite, we may find a subgroup  $K$  which not in  $\mathcal{F}$  but with all subgroups in  $\mathcal{F}$ . Then we have a cofibre sequence

$$E\mathcal{F}_+ \longrightarrow E\mathcal{F} \cup \{(K)_G\}_+ \longrightarrow E\langle(K)_G\rangle,$$

which contradicts maximality. □

### 3. FREE $G$ -SPECTRA

We have defined the category of free  $G$ -spectra to be the localizing subcategory of **G-spectra** consisting of the spectra with geometric isotropy contained in  $\{1\}$ . This coincides with the localizing subcategory generated by  $G_+$ . It is also equivalent to the homotopy category of several well known model categories.

It was proved in [17] that a model category for free  $G$ -spectra is Quillen equivalent to an algebraic model. Writing  $G_e$  for the identity component of  $G$  and  $G_d = G/G_e = \pi_0(G)$  for

the discrete quotient, we see that  $G_d$  acts on  $G_e$  and hence we may form the skew group ring  $H^*(BG_e)[G_d]$ . This gives rise to a Quillen equivalence between free  $G$ -spectra and an algebraic model. This in turn induces an equivalence

$$\mathbf{free}\text{-}G\text{-spectra} \simeq D(\text{tors-}H^*(BG_e)[G_d]\text{-mod}).$$

of triangulated categories.

Note that in concrete terms it means we have an efficient method of calculation: for any free  $G$ -spectra  $X$  and  $Y$ , there is an Adams spectral sequence

$$E_2^{*,*} = \text{Ext}_{H^*(BG_e)}^{*,*}(H_*^{G_e}(X), H_*^{G_e}(X))^{G_d} \Rightarrow [X, Y]_*^G.$$

However for our purposes we only need a structural consequence.

**Theorem 3.1.** *The category*

$$D(\text{tors-}H^*(BG_e)[G_d]\text{-mod}).$$

*is generated as a localizing tensor ideal by any non-trivial element.*

**Proof:** The statement that the derived category of DG-modules over a graded polynomial ring is generated by any non-trivial element is well known as part of the classification of localizing subcategories of modules over the polynomial ring.

Suppose then that  $M$  is a non-trivial element of  $D(\text{tors-}H^*(BG_e)[G_d]\text{-mod})$ . We form  $M[G_d] := \mathbb{Q}[G_d] \otimes M$  and its fixed point set  $M' := M[G_d]^{G_d}$ . From the algebraic result, this generates all torsion modules over  $H^*(BG_e)$ . Hence  $M[G_d] \cong M'[G_d]$  constructs all torsion modules of the form  $N'[G_d]$  for a DG-torsion  $H^*(BG_e)$ -module  $N'$ . However any torsion  $H^*(BG_e)[G_d]$ -module  $N$  is a retract of  $N'[G_d]$  where  $N' = N[G_d]^{G_d}$ . Hence  $M$  generates the whole category as a localizing tensor ideal.  $\square$

**Corollary 3.2.** *For any compact Lie group  $G$ , the category of free rational  $G$ -spectra is generated as a localizing tensor ideal by any non-trivial element.*

**Proof:** Since the equivalence of [17] was not shown to be monoidal, a few words of proof are required.

If  $X$  is a non-trivial free  $G$ -spectrum then so is  $(G/G_e)_+ \wedge X$ . This has homotopy which is free over  $\mathbb{Q}[G_d]$ . Applying the theorem we note that the corresponding DG-torsion- $H^*(BG_e)[G_d]$ -module generates all such modules as a triangulated category. It follows that any free  $G$ -spectrum of the form  $(G/G_e)_+ \wedge Y$  is in the localizing tensor ideal generated by  $X$ . Since  $Y$  is a retract of  $(G/G_e)_+ \wedge Y$ , this completes the proof.  $\square$

#### 4. LOCALIZING TENSOR IDEALS

Recall that we write **G-spectra** for the homotopy category of rational  $G$ -spectra and **G-spectra**( $\mathbf{K}$ ) for the category of rational  $G$ -spectra which are either contractible or have geometric isotropy conjugate to  $K$ . We will show that **G-spectra**( $\mathbf{K}$ ) is generated as a localizing tensor ideal by any non-trivial element.

4.A. **Spectra over a normal subgroup.** It is well known that if  $\Gamma$  is a compact Lie group with normal subgroup  $\Delta$  then geometric fixed points induce an equivalence

$$\Gamma\text{-spectra } \langle \supseteq \Delta \rangle \simeq \Gamma/\Delta\text{-spectra}$$

of tensor triangulated categories. The first category consists of  $\Gamma$ -spectra with geometric isotropy consisting of subgroups containing  $N$  (sometimes called  $\Gamma$ -spectra ‘over  $\Delta$ ’).

Given a closed subgroup  $K$  of  $G$ , we apply this when  $\Gamma = N_G(K)$ ,  $\Delta = K$ . In particular we have an isomorphism

$$\Phi^K : [X, Y]^{N_G(K)} \xrightarrow{\cong} [\Phi^K X, \Phi^K Y]^{W_G(K)}$$

whenever the geometric isotropy of  $Y$  lies in  $\langle \supseteq K \rangle$ . This applies integrally.

4.B. **Restriction.** Suppose  $G \supseteq H \supseteq K$ . If we have a  $G$ -spectrum  $X$ , we may restrict the  $G$ -conjugacy class  $(K)_G$  to a collection of subgroups of  $H$ . This may well be bigger than  $(K)_H$  and the subgroups of the  $G$ -conjugacy class  $(K)_G$  which lie inside  $H$  may break into several  $H$ -conjugacy classes.

**Example 4.1.** If  $G = SO(3)$  then all elements of order 2 are conjugate, and we may take  $K$  to be generated by a half turn around the  $z$ -axis. Now take  $H = O(2)$  to be the normalizer of  $K$ . The involutions in  $O(2)$  fall into two conjugacy classes: the rotations of order 2 (actually a singleton) and the reflections (forming a space homeomorphic to a circle). Accordingly, when we restrict a spectrum  $X$  with geometric isotropy  $H$  to an  $N$ -spectrum, the geometric isotropy will no longer be a single conjugacy class.

The example is typical.

**Lemma 4.2.** *There is a finite decomposition*

$$(K)_G \cap \text{Sub}(H) = \coprod_{i=0}^n (K_i)_H,$$

where  $K_i = K^{\gamma_i}$  for  $\gamma_i \in G$  and  $\gamma_0 = e$ .

**Proof:** It is clear that  $(K)_G$  breaks into a disjoint union of  $H$ -conjugacy classes, each of which is a closed subset of the space of subgroups with the Hausdorff metric topology. By the Montgomery-Zippin Theorem (‘close-means-subconjugate’) they are also open, so there are finitely many.  $\square$

**Remark 4.3.** This corresponds to the fact that

$$\Phi^K(G_+ \wedge_H Y) \simeq \bigvee_{i=0}^n \gamma_i \Phi^{K_i} Y.$$

Now we specialize to  $H = N = N_G(K)$ .

**Lemma 4.4.** *There is an open and closed subset  $U \subseteq \Phi N$  so that  $K \in \Lambda_{ct}(U)$  but  $K_i \notin \Lambda_{ct}(U)$  for  $i \neq 0$ .*

**Proof:** It suffices to prove that if  $K' \subseteq N$  is a  $G$ -conjugate of  $K$  distinct from  $K$ , then there is no subgroup of  $N$  in which both  $K$  and  $K'$  are cotal. It suffices to show that if the image  $K'/K \cap K'$  of  $K'$  in  $W = N/K$  is a torus then it is trivial.

For this, we note that since the maximal tori of  $K$  and  $K'$  are subtori of  $N$  which are  $G$ -conjugate, they are also  $N$ -conjugate [14, 13.4], so we may suppose that  $K \cap K'$  is their common maximal torus. Accordingly if  $K'/K \cap K'$  is a torus, it must be trivial.  $\square$

Now choose an open and closed subspace  $U$  as in Lemma 4.4 and let  $e_K^G \in A(N)$  denote the corresponding idempotent.

**Lemma 4.5.** *For  $G$ -spectra  $X$  and  $Y$  with geometric isotropy  $K$ , restriction to  $N_G(K)$  induces an isomorphism*

$$[X, Y]^G \xrightarrow{\cong} [e_K^G X, e_K^G Y]^N$$

**Proof:** The restriction

$$[X, Y]^G \xrightarrow{\cong} [X, Y]^{N_G(K)}$$

is induced by the projection map  $G_+ \wedge_N e_K^G S^0 \rightarrow G/G_+$ , which is an equivalence in geometric  $K$ -fixed points by construction.  $\square$

**4.C. From  $G$ -spectra at  $K$ .** Assembling the above information we may understand maps of  $G$ -spectra with geometric isotropy  $K$  in terms of their geometric  $K$ -fixed points. For brevity we write  $N = N_G(K)$  and  $W = W_G(K)$ , and we consider the two maps

$$[X, Y]^G \xrightarrow{\text{res}_N^G} [X, Y]^N \xrightarrow{\Phi^K} [X, Y]^W.$$

**Lemma 4.6.** *For  $G$ -spectra  $X$  and  $Y$  with geometric isotropy  $K$ , restriction to  $N_G(K)$  and passage to geometric  $K$ -fixed points induce isomorphisms*

$$[X, Y]^G \xrightarrow{\cong} [e_K^G X, e_K^G Y]^N \xrightarrow{\cong} [\Phi^K X, \Phi^K Y]^{W_G(K)}$$

**Proof:** The first isomorphism is Lemma 4.5 and the second is the fact from Subsection 4.A.  $\square$

**Corollary 4.7.** *The category  $\mathbf{G}\text{-spectra}\langle \mathbf{K} \rangle$  is a minimal localizing subcategory of the category of  $G$ -spectra in the sense that it is generated by any non-trivial element.*

**Proof:** By Theorem 3.2 the category of  $\mathbf{N}\text{-spectra}\langle \mathbf{K} \rangle \simeq \mathbf{free}\text{-}\mathbf{W}\text{-spectra}$  is a minimal localizing subcategory of  $N$ -spectra.

Now if  $X$  is a non-trivial  $G$ -spectrum with geometric isotropy  $K$  then  $e_K^G X$  is non-trivial as an  $N$ -spectrum and therefore generates all  $\mathbf{N}\text{-spectra}\langle \mathbf{K} \rangle$ . We may coinduce this construction to see that  $F_N(G_+, e_K^G X)$  builds any spectrum of the form  $F_N(G_+, Y)$  for  $Y$  in  $\mathbf{N}\text{-spectra}\langle \mathbf{K} \rangle$ .

If  $\mathcal{F}$  is any family, the collection of  $\mathcal{F}$ -spectra is generated as a localizing category by the cells  $G/L_+$  with  $L$  in  $\mathcal{F}$ . It is also generated as a localizing category by the duals  $DG/L_+$  for  $L \in \mathcal{F}$ . Accordingly, if  $\mathcal{F}$  consists of subconjugates of  $N$ , the  $\mathcal{F}$ -spectra are built by objects  $F_N(G_+, Z)$ . Now if  $Z = Z' \wedge E\langle K \rangle_N$  we note

$$F_N(G_+, Z) \simeq E\langle (K)_G \rangle \wedge F_N(G_+, Z').$$

Since any object of  $\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$  is of the form  $E\langle(K)_G\rangle \wedge Y$  with  $Y$  built using subconjugates of  $K$ , so we conclude that  $\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$  is generated by the objects  $F_N(G_+, Y)$  for  $Y$  in  $\mathbf{N}\text{-spectra}\langle\mathbf{K}\rangle$ .  $\square$

**Theorem 4.8.** *The localizing tensor ideals of  $G$ -spectra are precisely the unions of the single geometric isotropy category spectra  $\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$ .*

For any  $G$ -spectrum  $X$ ,

$$\text{Loc}_\otimes(X) = \sum_{K \in \mathcal{I}_g(X)} \mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$$

**Proof:** The  $G$ -spectrum  $E\langle K\rangle$  has geometric isotropy precisely  $(K)_G$ . Thus if  $K \in \mathcal{I}_g(X)$  we find an object  $E\langle K\rangle \wedge X \in \text{Loc}_\otimes(X)$  with geometric isotropy precisely  $K$ . Hence by the Minimality Theorem 4.7,  $\text{Loc}_\otimes(X)$  contains  $\mathbf{G}\text{-spectra}\langle\mathbf{K}\rangle$ .

It follows from Lemma 2.2 that  $X$  lies in the localizing tensor ideal generated by  $E\langle K\rangle \wedge X$  for all  $K$ .  $\square$

## Part 2. Finite spectra

### 5. THE LOCALIZATION THEOREM

The main ingredient in understanding containment of primes is the Localization Theorem.

We revisit some basic facts from transformation groups in our language. The basic tool is Borel cohomology,  $H_G^*(X) := H^*(EG \times_G X, EG \times_G pt) = H^*(EG_+ \wedge_G X)$ .

The idea is that for finite spectra, geometric isotropy is determined by Borel cohomology. It then follows from the Localization Theorem that the geometric isotropy is closed under passage to cotoral subgroups.

**Lemma 5.1.** *If  $K$  is connected and  $X$  is a finite  $K$ -CW-complex, then  $X$  is non-equivariantly contractible if and only if  $H_K^*(X) = 0$ .*

**Proof:** First, if  $X$  is simply connected, the Hurewicz theorem shows  $X \simeq *$  if and only if  $H_*(X) = 0$ . This is equivalent to  $H^*(X) = 0$ .

Next, we have a fibration  $X \rightarrow EK \times_K X \rightarrow BK$  so the Serre spectral sequence shows that if  $H^*(X) = 0$  then also  $H_K^*(X) = 0$ . Conversely, since  $K$  is connected the Eilenberg-Moore theorem gives

$$C^*(X) \simeq C^*(EK \times_K X) \otimes_{C^*(BK)} \mathbb{Q},$$

(where the tensor product is derived, and the cochains are unreduced). This shows that  $H_K^*(X) = 0$  implies  $H^*(X) = 0$ .  $\square$

It follows that  $\mathcal{I}_g(X)$  can be detected from Borel cohomology of fixed points.

**Corollary 5.2.** *If  $X$  is finite,  $K \in \mathcal{I}_g(X)$  if and only if  $H_{W_G^c(K)}^*(\Phi^K X) \neq 0$ .*  $\square$

For us the fundamental fact is the following consequence of the Localization Theorem.

**Proposition 5.3.** *If  $X$  is finite then  $\mathcal{I}_g(X)$  is closed under passage to cotoral subgroups.*

**Proof:** The Localization Theorem states that if  $T$  is a torus and  $Y$  is a finite  $T$ -CW-complex then

$$H_T^*(Y) \longrightarrow H_T^*(\Phi^T Y) = H^*(BK) \otimes H^*(\Phi^T Y)$$

becomes an isomorphism when the multiplicatively closed set  $\mathcal{E}_T = \{e(W) \mid W^T = 0\}$  of Euler classes  $e(W) \in H^{|W|}(BT)$  is inverted. The proof uses the fact that the  $T$ -space

$$S^{\infty V(T)} = \bigcup_{W^T=0} S^W,$$

has  $H$ -fixed points  $S^0$  if  $H = T$  and is contractible otherwise. Accordingly, we have a  $T$ -equivalence

$$Y \wedge S^{\infty V(T)} \simeq \Phi^T Y \wedge S^{\infty V(T)}.$$

It follows that if  $H^*(\Phi^T Y) \neq 0$  then also  $H_T^*(Y) \neq 0$ .

Now suppose  $H \in \mathcal{I}_g(X)$ . Let  $Y = \Phi^K X$  and let  $T = H/K$ . The hypothesis states  $\Phi^T Y = \Phi^{H/K} \Phi^K X \not\cong *$  so that  $H_T^*(\Phi^T Y) \neq 0$ . By the localization theorem it follows that  $H_T^*(Y) \neq 0$ , and from Lemma 5.2  $\Phi^K X = Y \neq 0$ . Hence also  $K \in \mathcal{I}_g(X)$ .  $\square$

The Localization Theorem explains the importance of the cotoral ordering. The first half is as follows.

**Corollary 5.4.** *If  $K \leq H$  then  $\wp_K \subseteq \wp_H$ .*  $\square$

We will see in Lemma 7.4 below that the reverse implication also holds.

## 6. BURNSIDE RINGS AND BASIC CELLS

A distinctive feature of working rationally is that there are usually many idempotents in the rational Burnside ring. We follow through the implications of this for cell structures.

Integrally, the relevant cells are homogeneous spaces  $G/K_+$ , and the relevant ordering of subgroups is classical containment. Rationally, the splitting of Burnside rings means that the relevant cells are certain basic cells  $\sigma_{K,U}$  (a retract of  $G/K_+$  introduced below) and the relevant ordering of subgroups is cotoral inclusion.

We begin by running through one approach to *classical* equivariant cell complexes, and then introduce basic cells and follow the same pattern to give the rational analysis in terms of *basic* cells.

**6.A. Unstable classical recollections.** Classically, we are used to the idea that based  $G$ -spaces  $P$  are formed from cells  $G/K_+$ . The classical unstable isotropy is defined by

$$\mathcal{I}'_{un}(P) = \{K \mid \Phi^K P \neq *\};$$

it is not homotopy invariant, but it does have the obvious property that it is closed under passage to subgroups.

It is natural to move to a homotopy invariant notion

$$\mathcal{I}_{un}(P) = \{K \mid \Phi^K P \not\cong *\}.$$

This notion fits well with the cells we use, since

$$\mathcal{I}_{un}(G/K_+) = \{L \mid L \subseteq_G K\} = \Lambda_{cl}((K)_G).$$

Note that we are linking notions of cell and isotropy with a partial order on subgroups.

The homotopy invariant version of unstable isotropy may not be closed under passage to subgroups, so that we only know that  $X$  is equivalent to a complex constructed from cells  $G/K_+$  with  $K \in \Lambda_{cl}\mathcal{I}_{un}(T)$ , where  $\Lambda_{cl}$  indicates that we take the closure under the classical order (i.e., under containment). This can be proved by killing homotopy groups, or by the method described in the next subsection for the stable situation.

**6.B. Stable classical recollections.** Moving to the stable world, for a  $G$ -spectrum  $X$  we have the *geometric* (or *stable*) isotropy

$$\mathcal{I}_g(X) = \{K \mid \Phi^K X \not\cong_1 0\},$$

homotopy invariant by definition. Evidently since geometric fixed points extend ordinary fixed points on spaces,

$$\mathcal{I}_{un}(P) \supseteq \mathcal{I}_g(\Sigma^\infty P),$$

and if  $P$  can be constructed from cells in  $A$  then  $\Sigma^\infty P$  can be constructed from stable cells in  $A$ .

The attraction of geometric isotropy and geometric fixed points arises from the fact that their properties are familiar from the category of based spaces. Perhaps the most important instance is that of the Geometric Fixed Point Whitehead Theorem, and we state it here because it is fundamental to our approach. The result is well known to all users of geometric fixed points, and is usually deduced using isotropy separation to see that an equivalence in all geometric fixed points is an equivalence in all categorical fixed points.

**Lemma 6.1.** (Geometric Fixed Point Whitehead Theorem) *A map  $f : X \rightarrow Y$  of  $G$ -spectra is an equivalence if  $\Phi^K f : \Phi^K X \rightarrow \Phi^K Y$  is a non-equivariant equivalence for all closed subgroups  $K$ .  $\square$*

Next, we note that

$$\mathcal{I}_g(G/K_+) = \Lambda_{cl}((K)_G).$$

Any  $G$ -spectrum  $X$  can be constructed from stable cells  $G/K_+$  with  $K \in \Lambda_{cl}\mathcal{I}_g(X)$ . One way to prove this is to construct a filtration analogous to the (thickened) fixed point filtration of a space. Simplifying this, if  $\mathcal{F} = \Lambda_{cl}\mathcal{I}_g(X)$  then  $X \wedge \tilde{E}\mathcal{F}$  has trivial geometric fixed points (and is thus contractible by the Geometric Fixed Point Whitehead Theorem). Now  $X \wedge E\mathcal{F}_+$  may be constructed from cells  $G/K_+$  for  $K \in \mathcal{F}$ . In effect we use the result for the special case  $E\mathcal{F}_+$  together with the fact that  $G/H_+ \wedge G/K_+$  can be constructed from cells  $G/K'_+$  with  $K' \subseteq K$ .

We now see how we can take advantage of the additional flexibility of working rationally.

**6.C. The Burnside ring.** We recall tom Dieck's determination of the rational Burnside ring [4]. To any self-map  $f : S^0 \rightarrow S^0$  we may associate the *mark*

$$m(f) : \text{Sub}(G) \rightarrow \mathbb{Q}$$

defined by

$$m(f)(K) = \deg(\Phi^K f : S^0 \rightarrow S^0).$$

Obviously this is constant on conjugacy classes so we can view it as a function on  $\text{Sub}(G)/G$ . More significantly, by the Localization Theorem if  $L$  is cotalal in  $K$  we have  $m(f)(L) = m(f)(K)$ . Accordingly  $m(f)$  is determined by its restriction to the space of cotalally maximal conjugacy classes. Since every infinite compact Lie group has a non-trivial torus,  $\max_{ct}(\text{Sub}(G)/G)$  is the space  $\Phi G$  of conjugacy classes of subgroups of finite index in their normalizers. We may topologize  $\Phi G$  as the quotient space of a space of closed subgroups with the Hausdorff metric topology, and as such  $m(f)$  is continuous. The mark homomorphism

$$m : [S^0, S^0]^G \xrightarrow{\cong} C(\Phi G, \mathbb{Q})$$

is an isomorphism of rings.

When  $G$  is the product of a torus and a finite group  $Q$  this means  $A(G) \cong \prod_{(\overline{K})} \mathbb{Q}$ , with the product over conjugacy classes of subgroups  $\overline{K}$  of  $\overline{G} = Q$ . Accordingly we obtain one primitive idempotent  $e_{\overline{L}}$  for each conjugacy class of subgroups of  $Q$ .

In general  $\Phi G$  is compact, Hausdorff and totally disconnected. Its topology has a basis of open and closed sets  $U$ . Idempotents of  $A(G)$  correspond to open and closed subsets: to each  $U$  we write  $e_U$  for the idempotent with  $U$  as support.

**6.D. Basic cells.** The classical cell  $G/K_+$  is  $S^0$  induced up from  $K$ , so if the  $K$ -equivariant sphere decomposes, so does  $G/K_+$ .

The building blocks are thus the *basic cells*

$$\sigma_{K,U} := G_+ \wedge_K e_U^K S^0,$$

where  $U$  is an open and closed neighbourhood of  $K$  in  $\Phi K$ .

**Remark 6.2.** If  $K$  is isolated in  $\Phi K$ , we write  $\sigma_K = \sigma_{K,\{K\}}$ . When  $G$  is a torus, all subgroups are of this type, and in general all subgroups of a maximal torus are of this type.

We develop cell structures based on basic cells. The advantage is that the geometric isotropy of  $\sigma_{K,U}$  is smaller than that of  $G/K_+$ . The advantage is clearest when  $K$  is isolated in  $\Phi K$ , but in general we can achieve similar results by letting  $U$  range over smaller and smaller neighbourhoods of  $K$ .

**Lemma 6.3.** *The geometric isotropy of  $\sigma_{K,U}$  consists of all subgroups  $G$ -conjugate to a subgroup cotalal in an element of  $U$ :*

$$\mathcal{I}_g(\sigma_{K,U}) = \Lambda_{ct}(U_G).$$

**Proof:** If  $L \in U$  then  $\Phi^L e_U S^0 \simeq_1 S^0$ , and  $Y$  is a  $K$ -equivariant retract of  $G_+ \wedge_K Y$ . Hence  $L \in \mathcal{I}_g(\sigma_{K,U})$ . By Proposition 5.3,  $\Lambda_{ct}(U) \subseteq \mathcal{I}_g(\sigma_{K,U})$ .

Conversely, we show  $\mathcal{I}_g(\sigma_{K,U}) \subseteq \Lambda_{ct}(U_G)$ . Certainly  $\mathcal{I}_g(\sigma_{K,U}) \subseteq \mathcal{I}_g(G/K_+) \subseteq \Lambda_{cl}((K)_G)$ . Next, if  $L$  is a subgroup of  $K$  not  $G$ -conjugate to a subgroup cotalal in a subgroup in  $U$  then there is an idempotent  $e$  orthogonal to  $e_U$ . Indeed, we may suppose  $L$  is of finite index in its normalizer (or else we replace it by the inverse image of the maximal torus in  $W_G(L)$  in  $N_G(L)$ ). By hypothesis  $(L)_G \cap U = \emptyset$ , and hence there is an open and closed subset  $V$  containing it and still disjoint from  $U$ . Now  $\Phi^L(G_+ \wedge_K Y) \simeq \Phi^L(G_+ \wedge_K e_V Y)$  and  $e_V e_U S^0 \simeq 0$ .  $\square$

## 7. APPLICATIONS OF BASIC CELLS

With a little additional work on the topology of the space of subgroups, basic cells are extremely valuable.

**7.A. The  $f$ -topology.** We recall the  $f$ -topology on  $\text{Sub}(G)$  from [7]. We write  $d$  for the Hausdorff metric on the space of closed subgroups, and if  $H$  is a closed subgroup of  $G$  we consider

$$O(H, \epsilon) = \{K \subseteq H \mid |W_H(K)| < \infty, d(H, K) < \epsilon\}$$

(i.e. only considering subgroups of finite index in their normalizers). A base of neighbourhoods of a subgroup  $H$  in  $\text{Sub}(G)$  consists of the sets

$$O(H, A, \epsilon) = \bigcup_{a \in A} O(H, \epsilon)^a$$

where  $A$  runs through neighbourhoods of the identity in  $G$  and  $\epsilon > 0$ . This topology induces the quotient topology on the space  $\text{Sub}(G)/G$  of conjugacy classes, which we again call the  $f$ -topology.

**Lemma 7.1.** *The image  $U_H^\epsilon$  of  $O(H, \epsilon)$  in  $\Phi H$  is open and closed. The set of maximal elements in  $\mathcal{I}_g(\sigma_{H, U_H^\epsilon})$  is the image of  $O(H, \epsilon)$  in  $\text{Sub}(G)/G$ .  $\square$*

The following characterization of the topology may be helpful.

**Lemma 7.2.** *The topology on  $\text{Sub}(G)/G$  is the quotient topology for the maps  $\Phi H \rightarrow \text{Sub}(G)/G$  as  $H$  runs through closed subgroups of  $G$ : a set is open in  $\text{Sub}(G)/G$  if and only if its pullback to  $\Phi H$  is open for all  $H$ .*

**Proof:** Lemma 8.6 (b) of [7] shows that the maps  $\Phi H \rightarrow \text{Sub}(G)$  are continuous. It remains to show that if  $U$  is a subset of  $\text{Sub}(G)/G$  which has the property  $U \cap \Phi H$  is open for all  $H$  then  $U$  is open. If  $(K)_G \in U$  then the fact that  $U \cap \Phi K$  is open shows it contains a neighbourhood  $U_K^\epsilon$  of  $(K)_H$  for some  $\epsilon > 0$ . The image of this in  $\text{Sub}(G)/G$  is open in the  $f$ -topology.  $\square$

The collection of all closed subgroups of  $G$  is a poset under cotal inclusion.

**Proposition 7.3.** [7, 8.7, 8.8] *Giving  $\text{Sub}(G)$  the  $f$ -topology and topologizing the space  $\text{Sub}_1(G)$  of cotal inclusions as a subspace of  $\text{Sub}(G) \times \text{Sub}(G)$ , we obtain a topological category. The source and target maps are open maps.  $\square$*

Note that for any set  $A$  of subgroups,  $\Lambda_{ct}(A) = s(t^{-1}(A))$ , so that if  $A$  is open so is  $\Lambda_{ct}(A)$ .

**7.B. Basic cells and primes.** The geometric isotropy of the basic cells determines the cotal order.

**Corollary 7.4.** *If  $\wp_L \subseteq \wp_K$  then  $L$  is conjugate to a subgroup cotal in  $K$ .*

**Proof:** If  $L$  is not cotalal we will show that there is a clopen neighbourhood  $U_K$  of  $K$  in  $\Phi K$  so that  $\sigma_{K,U_K} \in \wp_L \setminus \wp_K$ .

First note that  $s^{-1}(L)$  is closed. Its intersection with  $\{L\} \times \Phi K$  is compact, and so  $t(s^{-1}(L) \cap \{L\} \times \Phi K)$  is closed. Since it does not contain  $K$  there is a neighbourhood  $U_K$  of  $K$  in  $\Phi K$  so that  $(L)_G \cap \Lambda_{ct}(U_G) = \emptyset$  and hence  $\sigma_{K,U} \in \wp_L$ , and since  $K \in U_K$  we see  $\sigma_{K,U_K} \notin \wp_K$ .  $\square$

These basic cells show that cotalally unrelated subsets can be separated. We will not use the following result elsewhere (which may excuse the forward reference in the proof).

**Corollary 7.5.** *If  $K_1$  and  $K_2$  are two subgroups cotalally unrelated (in the sense that  $\Lambda_{ct}((K_1)_G) \cap \Lambda_{ct}((K_2)_G) = \emptyset$ ), then there are finite complexes  $X_1$  and  $X_2$  with  $K_i \in \mathcal{I}_g(X_i)$  and  $\mathcal{I}_g(X_1) \cap \mathcal{I}_g(X_2) = \emptyset$*

**Proof:** We will take  $X_1 = \sigma_{K_1,U_1}$  and  $X = \sigma_{K_2,U_2}$ . It remains to show we can choose neighbourhoods  $U_1, U_2$  of  $K_1, K_2$  that are also unrelated.

Consider the set  $V = \mathcal{I}_g(\sigma_{K_1,U_1} \wedge \sigma_{K_2,U_2}) = \Lambda_{ct}(K_1, U_1) \cap \Lambda_{ct}(K_2, U_2)$ . We will show in Theorem 9.4 below that  $A = \max_{ct} V$  is open and compact. As above  $s^{-1}A$  intersects  $\Phi K_i$  in a compact set, and so  $t(s^{-1}A \cap \Phi K_i)$  is closed and does not contain  $K_i$ . Accordingly we can choose neighbourhoods  $U_i$  of  $K_i$  not meeting it.  $\square$

**7.C. Basic detection.** Basic cells play a comparable role to classical cells in that they generate the category and detect equivalences. The smaller isotropy means that we can make stronger statements. Note that the lemma holds however small the neighbourhoods  $U_K$  are.

**Lemma 7.6.** *If we choose an open and closed neighbourhood  $U_K$  of  $K$  in  $\Phi K$  for each conjugacy class of closed subgroups of  $G$  then the natural cell  $G/K_+$  is built from basic cells  $\sigma_{L,U_L}$  with  $L \subseteq K$ .*

*Accordingly, the category of rational  $G$ -spectra is generated by the basic cells  $\sigma_{K,U_K}$ .*

**Proof:** It suffices to show that the classical cells  $G/K_+$  are built from the basic cells. The proof is by induction on  $K$  (i.e., we work with the poset of all subgroups ordered by inclusion, and note that there are no infinite decreasing chains).

Since  $G/1_+ = \sigma_1$ , the induction begins. Now suppose  $K$  is non-trivial and that  $G/L_+$  is built from basic cells for proper subgroups  $L$  of  $K$ . Now  $G/K_+$  is a sum of  $\sigma_{K,U_K}$  and the spectra  $G_+ \wedge_K e_{K,U'_K} S^0$  where  $U'_K$  is the complement of  $U_K$ . Since  $K \notin \mathcal{I}_g(\sigma_{K,U'_K})$ , we find  $\sigma_{K,U'_K}$  is built from cells  $G/L_+$  where  $L$  is a proper subgroup of  $K$ . By induction it is thus built from a  $\sigma_{L,U_L}$ .  $\square$

There is a useful criterion for vanishing of homotopy in terms of geometric isotropy.

**Lemma 7.7.** (i) *If  $\Lambda_{cl}(K) \cap \mathcal{I}_g(X) = \emptyset$  then  $[G/K_+, X]_*^G = 0$ .*

(ii) *If  $\Lambda_{ct}(K, U) \cap \mathcal{I}_g(X) = \emptyset$  then  $[\sigma_{K,U}, X]_*^G = 0$*

**Proof:** The first statement is immediate from the Geometric Fixed Point Whitehead Theorem, since  $\mathcal{I}_g(\text{res}_K^G X) = \emptyset$ .

For the second, we note

$$[\sigma_{K,U}, X]_*^G = [e_{K,U}S^0, X]_*^K = [e_{K,U}S^0, e_{K,U}X]_*^K.$$

By hypothesis  $\mathcal{I}_g(e_{K,U}X) = \Lambda_{ct}(K, U) \cap \mathcal{I}_g(\text{res}_K^G X) = \emptyset$ , so that  $e_{K,U}X \simeq_K 0$  by the Geometric Fixed Point Whitehead Theorem.  $\square$

A slightly refined version of the Whitehead Theorem holds in the rational context: to establish an equivalence we need only check on basic  $K$ -homotopy groups for  $K$  in the cotal closure of the geometric isotropy rather than for the full classical closure.

**Proposition 7.8.** *Suppose that  $\mathcal{I}_g(X), \mathcal{I}_g(Y) \subseteq \mathcal{K}$  for a set  $\mathcal{K}$  of subgroups, and that for some open neighbourhoods  $U_K$ , the map  $f : X \rightarrow Y$  induces an isomorphism of  $[\sigma_{K,U_K}, \cdot]_*^G$  for all  $K \in \mathcal{K}$ . Then  $f$  is an equivalence.*

**Proof:** Taking  $Z$  to be the mapping cone of  $f$ , it suffices to show that if  $\mathcal{I}_g(Z) \subseteq \mathcal{K}$  and  $[\sigma_{K,U_K}, Z]_*^G = 0$  for all  $K \in \mathcal{K}$  then  $Z \simeq 0$ . We will show that in fact  $\mathcal{I}_g(Z) = \emptyset$ . If not, there is a minimal counterexample,  $K \in \mathcal{I}_g(Z)$ . We then have

$$0 = [\sigma_{K,U_K}, Z]_*^G = [e_{U_K}S^0, Z]_*^K.$$

There is a  $K$ -equivariant cofibre sequence

$$E\mathcal{P}_+ \wedge Z \rightarrow Z \rightarrow \tilde{E}\mathcal{P} \wedge Z,$$

and it suffices to argue  $[e_{U_K}S^0, E\mathcal{P}_+ \wedge Z]_*^K = 0$  since then we have

$$0 = [e_{U_K}S^0, Z]_*^K = [e_{U_K}S^0, \tilde{E}\mathcal{P} \wedge Z]_*^K = [S^0, \Phi^K Z],$$

where the last equality is because  $K \in U_K$ .

We have

$$[e_{U_K}S^0, E\mathcal{P}_+ \wedge Z]_*^K = [e_{U_K}, e_{U_K}E\mathcal{P}_+ \wedge Z]_*^K$$

and  $e_{U_K}E\mathcal{P}_+ \wedge Z \simeq 0$  by minimality of  $K$ .  $\square$

**7.D. Basic structures.** When we work rationally, classical containment of subgroups is replaced by cotal inclusion. Cells  $G/K_+$  are now often decomposable, and we have basic cells  $\sigma_{K,U_K}$ . If  $K$  is a torus then  $\sigma_K = G/K_+$ , but if  $K$  is not a torus then there is a base of neighbourhoods of  $K$  for any one of which we have a proper inclusion

$$\mathcal{I}_g(\sigma_{K,U_K}) = \Lambda_{ct}(K, U_K) \subset \Lambda_{cl}(K) = \mathcal{I}_g(G/K_+).$$

**Lemma 7.9.** *Any  $G$ -spectrum  $X$  can be constructed from basic cells  $\sigma_{K,U_K}$  with  $K$  in  $\Lambda_{ct}(\mathcal{I}_g(X))$ .*

**Proof:** Take  $\mathcal{K} = \Lambda_{ct}\mathcal{I}_g(X)$ . We may construct a map  $p : P \rightarrow X$  so that  $P$  is a wedge of suspensions of basic cells  $\sigma_{K,U_K}$  for  $K \in \mathcal{K}$ , and so that  $p_*$  is surjective on  $[\sigma_{K,U_K}, \cdot]_*^G$  for all  $K \in \mathcal{K}$ . Iterating this, we form a diagram

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & P_0 & & P_1 & & P_2 & & \end{array}$$

We take  $X_\infty = \operatorname{holim}_{\rightarrow_s} X_s$ , and note that since  $\sigma_{K,U_K}$  is small for each  $K$ , it follows  $[\sigma_{K,U_K}, X_\infty]_*^G = 0$  for  $K \in \mathcal{K}$ . Since  $\mathcal{I}_g(X_\infty) \subseteq \Lambda_{ct} \mathcal{I}_g(X) = \mathcal{K}$ , it follows from Proposition 7.8 that  $X_\infty \simeq 0$ . Arguing with the dual tower, we see  $X$  can be constructed from cells  $\sigma_{K,U_K}$ : indeed, we define  $X^s$  by the cofibre sequence

$$X^s \longrightarrow X \longrightarrow X_s.$$

By definition  $X^0 \simeq *$ , and we have

$$\Sigma^{-1}P_s \longrightarrow X^s \longrightarrow X^{s+1}.$$

Again  $X^\infty = \lim_{\rightarrow_s} X^s$ , and since  $X_\infty \simeq 0$ , we see  $X^\infty \simeq X$ . □

**Remark 7.10.** One can imagine other proofs. One is to construct an analogue of the map  $E\mathcal{F}_+ \longrightarrow S^0$  for a family  $\mathcal{F}$ . This is a spectrum  $E\langle \mathcal{K} \rangle$  with geometric isotropy  $\mathcal{K}$  and a map  $E\langle \mathcal{K} \rangle \longrightarrow S^0$  which is an equivalence in geometric  $K$ -fixed points for all  $K \in \mathcal{K}$  (one construction follows from the results below). One then mimics the rest of the proof in the classical case.

It then follows that  $X \simeq X \wedge E\langle \mathcal{K} \rangle$ . Now we construct  $E\langle \mathcal{K} \rangle$  out of basic cells  $\sigma_K$  with  $K \in \mathcal{K}$ , and claim that  $G/H_+ \wedge \sigma_K$  can be constructed from basic cells  $\sigma_{K'}$  for  $K'$  cotoral in  $K$ .

The two natural notions of finiteness for rational spectra coincide.

**Lemma 7.11.** *A rational  $G$ -spectrum is constructed from finitely many basic cells  $\sigma_{K,U_K}$  if and only if it is constructed from finitely many classical cells  $G/K_+$ .*

**Proof:** Since  $\sigma_{K,U_K}$  is a retract of  $G/K_+$ , a basic-finite complex is a finite complex. The standard cell  $G/K_+$  is a basic-finite complex (it is built by basic cells using Lemma 7.6, and then  $G/K_+$  is a retract of a finite basic complex using smallness). Accordingly, any classical-finite complex is basic-finite. □

## 8. THE CLASSIFICATION OF PRIMES

We wish to deduce the classification of prime ideals from our classification of localizing tensor ideals. We pause to introduce the underpinning principle.

**8.A. The Thomason Localization Theorem.** The reason that it is a viable strategy to first classify localizing subcategories of infinite objects and then deduce classifications of finite objects is Thomason's Localization Theorem.

The proof is quite formal, but it is an extremely powerful general principle. It is well known in various forms, but we want a version for tensor ideals in an abstract setting; this is entirely in the spirit of [24] and I am grateful to G. Stevenson for the following proof showing that it follows directly from the results there.

**Theorem 8.1.** (Thomason's Localization Theorem [22, 24]) *(i) If  $\mathbb{C}$  is a triangulated category generated by compact objects and  $A$  is a set of small objects then*

$$\operatorname{Loc}(A) \cap \mathbb{C}^c = \operatorname{Thick}(A).$$

(ii) If  $\mathbb{C}$  is a tensor triangulated category generated by compact objects and  $A$  is a set of small objects then

$$\text{Loc}_{\otimes}(A) \cap \mathbb{C}^c = \text{Thick}_{\otimes}(A).$$

**Remark 8.2.** We emphasize that in Part (ii),  $\text{Thick}_{\otimes}(A)$  only permits tensoring with compact objects, whilst  $\text{Loc}_{\otimes}(A)$  permits tensoring with arbitrary objects.

**Proof:** Part (i) is [22, 2.1.3].

We deduce Part (ii) from Part (i). It suffices to show  $\text{Loc}_{\otimes}(A) = \text{Loc}(\text{Thick}_{\otimes}(A))$  since then Part (i) shows

$$\text{Loc}_{\otimes}(A) \cap \mathbb{C}^c = \text{Loc}(\text{Thick}_{\otimes}(A)) \cap \mathbb{C}^c = \text{Thick}_{\otimes}(A).$$

It is clear that  $\text{Loc}_{\otimes}(A) \supseteq \text{Loc}(\text{Thick}_{\otimes}(A))$ , so it suffices to show that  $\text{Loc}(\text{Thick}_{\otimes}(A))$  is an ideal. For this we calculate

$$\begin{aligned} \text{Loc}(\text{Thick}_{\otimes}(A)) &= \text{Loc}(\text{Thick}(\{x \otimes a \mid x \in \mathbb{C}^c, a \in A\})) \text{ [24, Lemma 3.9]} \\ &= \text{Loc}(\{x \otimes a \mid x \in \mathbb{C}^c, a \in A\}) \\ &= \text{Loc}(\mathbb{C} \otimes \text{Loc}(A)) \text{ [24, Lemma 3.11]} \end{aligned}$$

Evidently  $\mathbb{C} \otimes \text{Loc}(A)$  is closed under tensoring, so this is an ideal by [24, Lemma 3.9].  $\square$

**8.B. The primes.** We finally want to prove that we have found all the primes.

**Lemma 8.3.** *If  $\wp$  is a prime and  $\wp = \bigcap_{L \in A} \wp_L$  then  $A$  has a unique minimal element  $L$  and  $\wp = \wp_L$ .*

**Proof:** If not we can choose a subgroup  $L$  in  $A$  which is not redundant. Thus

$$\wp = \wp_L \cap \bigcap_{K \in A \setminus \{L\}} \wp_K,$$

and since  $\wp_L$  is not redundant, we may choose  $X_L \in \wp_L \setminus \wp$  and  $Y_L \in \bigcap_{K \in A \setminus \{L\}} \wp_K \setminus \wp$ . This contradicts the fact that  $\wp$  is prime since  $X_L \wedge Y_L \in \wp$  but  $X_L \notin \wp$  and  $Y_L \notin \wp$ .  $\square$

**Theorem 8.4.** *The prime tensor ideals of compact  $G$ -spectra are precisely those of the form*

$$\wp_K = \{X \in \mathbf{G}\text{-spectra}^c \mid \Phi^K X \simeq_1 0\}$$

*for some closed subgroup  $K$ . The geometric isotropy of  $\wp_K$  consists of subgroups  $H$  in which  $K$  is not cotoral up to conjugacy*

$$\mathcal{I}_g(\wp_K) = \text{All} \setminus V(K).$$

**Proof:** If  $G$  is non-trivial the collection of contractible spectra is not prime since if  $F_1$  and  $F_2$  are non-conjugate finite subgroups the smash product of  $\sigma_{F_1} \wedge \sigma_{F_2}$  is contractible.

From Theorem 4.8, if  $\wp$  is prime then

$$\text{Loc}_{\otimes}(\wp) = \sum_{K \in \mathcal{I}_g(\wp)} \mathbf{G}\text{-spectra}\langle \mathbf{K} \rangle.$$

By Thomason's Localization Theorem 8.1, if  $Y \in \text{Loc}_{\otimes}(\wp)$  is compact then  $Y \in \text{Thick}_{\otimes}(\wp) = \wp$ . Hence

$$\wp = \{X \in \mathbf{G}\text{-spectra}^c \mid \mathcal{I}_g(X) \subseteq \mathcal{I}_g(\wp)\}.$$

Now consider the complement of  $\mathcal{I}_g(\varphi)$ . If it has a single cotorally minimal conjugacy class  $K$  then  $\varphi = \varphi_K$ .

If  $K_1, K_2$  are cotorally minimal in the complement of  $\mathcal{I}_g(\varphi)$  then if  $K_1$  and  $K_2$  are not conjugate, we may choose open and closed neighbourhoods  $U_i$  of  $K_i$  in  $\Phi K_i$ . Therefore  $\sigma_{K_1, U_1} \wedge \sigma_{K_2, U_2} \in \varphi$  but  $\sigma_{K_1, U_1} \notin \varphi$ .  $\square$

## 9. GEOMETRIC ISOTROPY OF FINITE SPECTRA

In this section we will give the classification of the collections of subgroups occurring as the geometric isotropy of a finite  $G$ -spectrum. Once again we apply Thomason's Localization Theorem to deduce facts about thick tensor ideals of finite spectra from facts about localizing tensor ideals of arbitrary spectra.

**9.A. Thick tensor ideals of finite spectra.** The geometric isotropy coincides with the support, and the point of support is that it determines the thick tensor ideals.

**Theorem 9.1.** *If  $X$  and  $Y$  are finite  $G$ -spectra then if  $\mathcal{I}_g(Y) \subseteq \mathcal{I}_g(X)$  then  $Y \in \text{Thick}_{\otimes}(X)$ .*

**Proof:** By Theorem 4.8  $\text{Loc}_{\otimes}(X) = \sum_{K \in \mathcal{I}_g(X)} \mathbf{G}\text{-spectra}\langle \mathbf{K} \rangle$ ,  $Y \in \text{Loc}_{\otimes}(X)$ . By Thomason's Localization Theorem 8.1, since  $Y$  is finite this means  $Y \in \text{Thick}_{\otimes}(X)$ .  $\square$

The more interesting question is which sets occur as  $\mathcal{I}_g(X)$  for finite  $G$ -spectra  $X$ . Since the geometric isotropy is closed under cotoral specialization by Proposition 5.3, the set is determined by its set  $\max_{ct}(\mathcal{I}_g(X))$  of cotorally maximal subgroups.

For tori, the classification was given in [15] (it will also follow from Theorem 9.4 below).

**Example 9.2.** If  $G$  is a torus then the sets  $\mathcal{I}_g(X)$  for finite  $X$  are precisely those with finitely many cotorally maximal elements: the sets of the form  $\Lambda_{ct}(A)$  where  $A$  is a finite set of cotorally unrelated subgroups.

However, for more general subgroups the statement will inevitably be more complicated.

**Example 9.3.** If  $G = O(2)$  then for any  $n$  the set  $U_n$  consisting of  $O(2)$  and the dihedral subgroups of order  $\geq 2n$  is clopen in  $\Phi O(2)$  we see  $\max_{ct}(\mathcal{I}_g(\sigma_{O(2), U_n})) = (O(2), U_n)$ .

**9.B. Characterization of thick tensor ideals.** The following finiteness theorem is fundamental.

**Theorem 9.4.** *If  $X$  is a finite spectrum then  $\max_{ct}(\mathcal{I}_g(X))$  is an open compact set in the  $f$ -topology. All open, compact and cotorally undrelated sets occur as  $\max_{ct}(\mathcal{I}_g(X))$  for some finite  $G$ -spectrum  $X$ .*

**Proof:** First we show that  $\max_{ct}(\mathcal{I}_g(X))$  is open. Suppose that  $K \in \mathcal{I}_g(X)$ . If  $K$  has no neighbourhood in  $\mathcal{I}_g(X)$  we can find a sequence of subgroups  $L_i$  tending to  $K$  with  $L_i \notin \mathcal{I}_g(X)$ , and by the Montgomery-Zippin theorem we may suppose  $L_i$  is a subgroup of  $K$ . It suffices to view  $X$  as a  $K$ -spectrum, and by Illman's Theorem that too is finite. By the Freudenthal Suspension theorem, we may suppose  $X$  it is the suspension spectrum of a finite  $K$ -CW-complex  $Z$ . There are finitely many non- $K$ -fixed cells, so for  $i$  sufficiently large, the subgroups  $L_i$  are not conjugates to subgroups of them and  $\Phi^{L_i} Z = \Phi^K Z$ .

Next we show that  $\max_{ct}(\mathcal{I}_g(X))$  is compact. For each  $K \in \mathcal{I}_g(X)$  we choose a clopen neighbourhood  $U_K$  of  $K$  in  $\Phi K$ . Note that and since  $\mathcal{I}_g(X)$  is open we may suppose  $U_K \subseteq \mathcal{I}_g(X)$ . By Theorem 4.8,  $X \in \text{Loc}_{\otimes}\{\sigma_{K,U_K} \mid K \in \mathcal{I}_g(X)\}$ , and since  $X$  is small, there are finitely many subgroups  $K_1, \dots, K_N$  so that  $X$  is finitely built by  $\sigma_{K_1, U_1}, \dots, \sigma_{K_N, U_N}$ . Considering the dual process,  $X$  can be converted to a point by using these same cells. Since  $\max_{ct}(\sigma_{K,U_K}) = (K, U_K)$  the only way that an element of  $\max_{ct}(\mathcal{I}_g(X))$  can be removed is if it lies in some  $U_i$ , so that  $U_1, \dots, U_N$  is a finite cover of  $\max_{ct}(\mathcal{I}_g(X))$ .

Finally, we show that all compact, open and cotorally unrelated subsets occur as the cotorally maximal elements in the geometric isotropy of a wedge of basic cells. All compact open subsets of  $\text{Sub}(G)/G$  are finite unions of sets  $(K, U_K)$ . We claim the same is true for those which are cotorally unrelated. Such sets are realizable as the geometric isotropy of  $\bigvee_K \sigma_{K,U_K}$ . It remains to observe that if there are any cotal relations among the sets  $(K, U_K)$  then we may replace them by a smaller collection without cotal relations.

Indeed, we need only note that for any  $K, L$  the set  $(L, U_L) \setminus \Lambda_{ct}(K, U_K)$  is a finite union of sets  $(M, U_M)$ . This follows since  $\Phi L \cap \Lambda_{ct}(K, U_K)$  is an open set. This follows from Lemma 7.2.  $\square$

Some may prefer the following reformulation. A thick tensor ideal is called *finitely generated* if it is generated by a finite number of small spectra (or equivalently, by just one).

**Corollary 9.5.** *The finitely generated thick tensor ideals of finite rational  $G$ -spectra are precisely the fibres of geometric isotropy*

$$\mathcal{I}_g : \text{finite-rational-}G\text{-spectra} \longrightarrow \mathcal{P}(\text{Sub}(G));$$

*the image consists of collections of subgroups which are closed under cotal specialization, and whose cotorally maximal parts are open and compact.*  $\square$

## 10. ZARISKI TOPOLOGIES

We have shown that the map

$$\text{Sub}(G)/G \xrightarrow{\cong} \text{Spc}(G\text{-spectra})$$

associating a closed subgroup  $K$  of  $G$  with the Balmer prime  $\wp_K$  is a bijection. Furthermore we have shown that this is an isomorphism of posets in the sense that  $\wp_L \subseteq \wp_K$  if and only if a conjugate of  $L$  is cotal in  $K$ . The purpose of this section is to show that topological structures correspond.

**10.A. The Zariski topology.** According to [1, Definition 2.1], the Zariski topology on  $\text{Spc}(G\text{-spectra})$  has closed sets

$$Z(\mathfrak{X}) = \{\wp \mid \wp \cap \mathfrak{X} = \emptyset\} = \bigcap_{X \in \mathfrak{X}} Z(X)$$

where  $\mathfrak{X}$  runs through collections of finite rational  $G$ -spectra. Since  $X \in \wp_K$  if and only if  $K \notin \mathcal{I}_g(X)$ , under the correspondence between primes and subgroups, we have

$$Z(X) = \mathcal{I}_g(X),$$

and the topology is generated by the geometric isotropy of finite spectra.

We may recover the poset structure on  $\mathrm{Spc}(G\text{-spectra})$  from the topology since the closure of a prime consists of all primes contained in it:

$$\overline{\{\wp\}} = \{\mathfrak{q} \mid \mathfrak{q} \subseteq \wp\}.$$

We are rather used to the Zariski topology in the prime spectrum of a Noetherian ring, where a collection  $A$  of maximal ideals is closed if and only if  $A$  is finite (as forced by the fact that maximal ideals are closed). However  $\mathrm{Spc}(G\text{-spectra})$  usually has the property that many other subsets are also closed. We will see that this can be viewed as saying that the set of primes is a topological poset, with the poset encoding the closures of points (a standard construction for  $T_0$ -spaces), and the much coarser topology viewing points at the ends of non-trivial morphisms as ‘far apart’.

**Remark 10.1.** We may always view the Zariski topology on the Balmer spectrum as the  $f$ -topology on a poset.

The closures of points form a poset since the Zariski topology is  $T_0$ . The  $f$ -topology is defined to be generated by the closed sets  $\max_{ct} V$  where  $V$  runs through the generating closed sets  $\mathrm{supp}(X)$ . These are generators since  $\max_{ct}(V \cup W) \supseteq \max_{ct}(V) \cup \max_{ct}(W)$ .

We may recover the original Zariski topology by taking specialization closures. Again the images of generators are generators since  $\Lambda(A \cup B) \supseteq \Lambda(A) \cup \Lambda(B)$ .

**10.B. The Zariski  $f$ -topology.** We can also generate a Zariski  $f$ -topology (or  $zf$ -topology) on  $\mathrm{Sub}(G)/G$  by combining the  $f$ -topology and the partial order. We define the sets  $\Lambda_{ct}(V)$  to be  $zf$ -closed whenever  $V$  an  $f$ -closed set, and then consider the topology they generate.

We note that the set  $\Lambda_{ct}(A)$  consists of all sources of arrows arriving in  $A$ , so that  $\Lambda_{ct}(A) = s(t^{-1}(A))$ . Since  $s$  is an open map, it is  $f$ -open if  $A$  is  $f$ -open.

**10.C. The homeomorphism.**

**Theorem 10.2.** *Associating a closed subgroup  $K$  to a prime  $\wp_K$  induces a homeomorphism*

$$(\mathrm{Sub}(G)/G, zf) \cong (\mathrm{Spc}(G\text{-spectra}), \text{Zariski}).$$

**Proof:** The Zariski topology on  $\mathrm{Sub}(G)/G$  is generated by  $\Lambda_{ct}(K, U_K) = \mathcal{I}_g(\sigma_{K, U_K})$ , so that the Zariski topology on  $\mathrm{Spc}(G\text{-spectra})$  is at least as fine. It remains to show that for any finite spectrum  $X$ , the geometric isotropy  $\mathcal{I}_g(X)$  is already in the topology generated by the basic cells. This was the main part of Theorem 9.4.  $\square$

## 11. SEMIFREE $\mathbb{T}$ -SPECTRA

The point of this section is to show that it is much harder to classify thick subcategories than thick tensor-ideals. It will suffice to look at semifree  $\mathbb{T}$ -spectra for the circle group  $\mathbb{T}$  i.e., those  $\mathbb{T}$ -spectra with  $\mathcal{I}_g(X) \subseteq \{1, \mathbb{T}\}$ . The model for these [9] is sufficiently simple that we may be explicit.

**11.A. The model of semifree  $\mathbb{T}$ -spectra.** The model  $\mathcal{A}_{sf}(\mathbb{T})$  of semifree  $\mathbb{T}$ -spectra can be obtained from the model  $\mathcal{A}(\mathbb{T})$  of all  $\mathbb{T}$ -spectra by restriction, but it is easier to repeat the construction from scratch. In fact  $\mathcal{A}_{sf}(\mathbb{T})$  is the abelian category of objects  $\beta : N \rightarrow \mathbb{Q}[c, c^{-1}] \otimes V$  where  $N$  is a  $\mathbb{Q}[c]$ -module,  $V$  is a graded  $\mathbb{Q}$ -vector space and  $\beta$  is the  $\mathbb{Q}[c]$

map inverting  $c$ . In effect, we have the  $\mathbb{Q}[c]$ -module  $N$ , together with a chosen ‘basis’  $V$ . Morphisms are commutative squares

$$\begin{array}{ccc} M & \xrightarrow{\theta} & N \\ \downarrow & & \downarrow \\ \mathbb{Q}[c, c^{-1}] \otimes U & \xrightarrow{1 \otimes \phi} & \mathbb{Q}[c, c^{-1}] \otimes V \end{array}$$

The category  $\mathcal{A}_{sf}(\mathbb{T})$  is of injective dimension 1, and the ring  $\mathbb{Q}[c]$  is evenly graded, so every object of  $d\mathcal{A}_{sf}(\mathbb{T})$  is formal, and we will identify semifree  $\mathbb{T}$ -spectra  $X$  (or objects of  $d\mathcal{A}_{sf}(\mathbb{T})$ ) with their image  $\pi_*^A(X)$  in the abelian category  $\mathcal{A}_{sf}(\mathbb{T})$ .

The fact we are talking about *ideals* is essential for Corollary 9.5. If we consider semifree  $G$ -spectra when  $G$  is the circle then there are just two thick tensor ideals of finite spectra

- free spectra (with geometric isotropy 1, generated by  $G_+$ )
- all spectra (with geometric isotropy  $\{1, G\}$ , generated by  $S^0$ ).

On the other hand, the thick subcategory generated by  $S^0$  (without the ideal property) does not contain  $G_+$ , and we will make it explicit. The classification of thick subcategories in general seems complicated, and we do not give a complete answer.

**11.B. Wide spheres.** The small objects with  $\beta$  injective are the objects  $X = (\beta : N \rightarrow \mathbb{Q}[c, c^{-1}] \otimes V)$  with  $\beta$  injective,  $V$  finite dimensional and  $N$  bounded above; these objects are called *wide spheres* [9].

We note that  $\mathbb{Q}[c, c^{-1}] \otimes V$  is the same in each even degree and the same in each odd degree. We therefore let

$$|V|_0 = \bigoplus_k V_{2k} \text{ and } |V|_1 = \bigoplus_k V_{2k+1}.$$

We will fix isomorphisms

$$|V|_0 \cong (\mathbb{Q}[c, c^{-1}] \otimes V)_0 \text{ and } |V|_1 \cong (\mathbb{Q}[c, c^{-1}] \otimes V)_1,$$

and then use multiplication by powers of  $c$  to identify other graded pieces of  $\mathbb{Q}[c, c^{-1}] \otimes V$  with the appropriate one.

We will want to think of stepping down the degrees in steps of 2, so we take

$$|V|_{\geq 2k} = \bigoplus_{n \geq k} V_{2n} \text{ and } |V|_{\geq 2k+1} = \bigoplus_{n \geq k} V_{2n+1}$$

for the parts of  $V$  above a certain point, but in the same parity.

Similarly, we move  $N_{2k}$  into degree 0 by multiplication by  $c^k$ :

$$\overline{N}_{2k} := c^k N_{2k} \subseteq |V|_0 \text{ and } \overline{N}_{2k+1} := c^k N_{2k+1} \subseteq |V|_1.$$

Having established the framework, we will restrict the discussion to the even part, leaving the reader to make the odd case explicit.

If  $X$  is nonzero in even degrees, since  $X$  is small there is a highest degree  $2a - 2$  in which  $N$  is non-zero, and since  $N[1/c] = \mathbb{Q}[c, c^{-1}] \otimes V$  there is highest degree  $2b$  in which  $N$  coincides with  $|V|_0$ . Accordingly, we have a finite filtration

$$0 = \overline{N}_{2a} \subseteq \overline{N}_{2a-2} \subseteq \cdots \subseteq \overline{N}_4 \subseteq \overline{N}_2 \subseteq \cdots \subseteq \overline{N}_{2b} = |V|_0.$$

We wish to consider two increasing filtrations on  $|V|_0$

$$\cdots \subseteq |V|_{\geq 2k+2} \subseteq |V|_{\geq 2k} \subseteq |V|_{\geq 2k-2} \subseteq \cdots \subseteq |V|_0$$

and

$$\cdots \subseteq \overline{N}_{2k+2} \subseteq \overline{N}_{\geq 2k} \subseteq \overline{N}_{\geq 2k-2} \subseteq \cdots \subseteq |V|_0.$$

**11.C. Two conditions on wide spheres.** In crude terms, we will show the thick subcategory generated by  $S^0$  consists of objects so that (a) the dimensions of the spaces in the  $V$ - and  $N$ -filtrations agree and (b) the  $V$  filtration is slower than the  $cN$  filtration. The purpose of this subsection is to introduce the two conditions.

**Condition 11.1.** We say that a wide sphere is *untwisted* if it satisfies the following two conditions

- (1)  $\dim(\overline{N}_i) = \dim(|V|_{\geq i})$  for all  $i$
- (2)  $V \cap cN = 0$

We will be showing that these characterize the thick subcategory generated by  $S^0$ . We must at least show that the conditions are inherited by retracts, and this verification will lead us to some useful introductory discussion.

**Lemma 11.2.** *Condition 11.1 is closed under passage to retracts.*

**Proof:** It is immediate that Condition 11.1 (ii) is inherited by retracts. We also note that Condition 11.1 (ii) implies one of the inequalities for Condition 11.1 (i):

$$\dim(\overline{N}_i) \geq \dim(|V|_{\geq i}).$$

Now suppose  $X = X' \oplus X''$  and that  $X$  satisfies Condition 11.1. As observed already,  $X'$  and  $X''$  both satisfy the second condition, and hence both satisfy the first condition with  $=$  replaced by  $\geq$ . With lower case letters denoting dimensions of vector spaces (for example  $n_a = \dim(N_a)$ ), this means we have a pair of increasing sequences

$$0 = n'_a, n'_{a-2}, n'_{a-4}, \cdots \text{ and } 0 = v'_{\geq a}, v'_{\geq a-2}, v'_{\geq a-4} \cdots$$

reaching  $v'$  and a pair of increasing sequences

$$0 = n''_a, n''_{a-2}, n''_{a-4}, \cdots \text{ and } 0 = v''_{\geq a}, v''_{\geq a-2}, v''_{\geq a-4} \cdots$$

reaching  $v''$ . Since  $X$  satisfies Condition 11.1(i), the sum of the first pair and the second pair give two equal sequences (i.e., the sequence  $n'_i + n''_i = n_i$  and the sequence  $v'_{\geq i} + v''_{\geq i} = v_{\geq i}$  are equal). Thus if one pair deviates from equality in the positive direction, the other deviates in the negative direction. Since Condition 11.1(ii) shows there is no negative deviation, we must have equality for both pairs throughout.  $\square$

It is useful to be able to consider the changes of dimension and form the generating function. In fact to any wide sphere, we may associate to it two Laurent polynomials

- The geometric  $T$ -fixed point polynomial

$$p_{\mathbb{T}}(t) = \sum_i \dim_{\mathbb{Q}}(V_i) t^i$$

- The 1-Borel jump polynomial

$$p_1(t) = \sum_i \dim_{\mathbb{Q}}(N_i/cN_{i+2})t^i$$

Condition 11.1(i) is then equivalent to the condition

$$p_{\mathbb{T}}(t) = p_1(t).$$

**Remark 11.3.** We note that Condition 11.1(i) is not closed under passage to retracts. Indeed,  $S^z \vee S^{2-z}$  satisfies the first condition with  $p_1(t) = p_{\mathbb{T}}(t) = t^2 + 1$ . However  $S^z$  (with  $p_{\mathbb{T}}(t) = 1$  and  $p_1(T) = t^2$ ) and  $S^{2-z}$  (with  $p_{\mathbb{T}}(t) = t^2$  and  $p_1(T) = 1$ ) do not.

11.D. **Attaching a  $\mathbb{T}$ -fixed sphere.** To start with,  $S^0 = (\mathbb{Q}[c] \rightarrow \mathbb{Q}[c, c^{-1}] \otimes \mathbb{Q})$  and then direct sums of these model wedges of  $\mathbb{T}$ -fixed spheres with  $N = \mathbb{Q}[c] \otimes V$ , and of course it is easy to see that Condition 11.1 holds for these.

However  $N$  does not always sit so simply inside  $\mathbb{Q}[c, c^{-1}] \otimes V$  for the objects built from  $S^0$ . We may see this in a simple example.

**Example 11.4.** Up to equivalence there are precisely three wide spheres with  $p_1(t) = p_{\mathbb{T}}(t) = 1 + t^2$ . Evidently in all cases  $V = \mathbb{Q} \oplus \Sigma^2 \mathbb{Q}$ ,  $\overline{N}_{2k} = 0$  for  $k \geq 4$  and  $\overline{N}_{2k} = |V|$  for  $k \leq 0$ . The only question is how the 1-dimensional space  $\overline{N}_2$  sits inside  $|V| = V_0 \oplus V_2$ . The three cases are  $N_2 = V_0$  (which is  $S^z \vee S^{2-z}$ ),  $N_2 = V_2$  (which is  $S^0 \vee S^2$ ), and the third case (giving just one isomorphism type) in which  $N_2$  is a 1-dimensional subspace not equal to  $V_0$  or  $V_2$ .

We note that the third example is the mapping cone  $M_f$  for any non-trivial map  $f : S^1 \rightarrow S^0$  (in the semifree category, there is only one up to multiplication by a non-zero scalar). In this case up to isomorphism,  $N_2$  is generated by  $c^{-1} \otimes \iota_0 + c^0 \otimes \iota_2$ .

We observe then that the second and third of these three are in  $\text{Thick}(S^0)$ , and we see that the first does not satisfy Condition 11.1 (ii).

**Lemma 11.5.** *Given a cofibre sequence,*

$$S^n \xrightarrow{f} X \rightarrow Y,$$

*if  $X$  is a wide sphere then so is  $Y$  and if  $X$  in addition satisfies Condition 11.1 then so does  $Y$ .*

**Proof:** Suppose first that  $X$  is entirely in one parity. Without loss of generality, we may suppose  $X$  is in even degrees.

If  $n$  is odd then  $\pi_*^{\mathcal{A}}(S^n)$  is purely odd and we have a short exact sequence

$$0 \rightarrow \pi_*^{\mathcal{A}}(X) \rightarrow \pi_*^{\mathcal{A}}(Y) \rightarrow \pi_*^{\mathcal{A}}(S^{n+1}) \rightarrow 0.$$

It follows that  $Y$  is a wide sphere (i.e., the basing map is injective). The condition on dimensions is immediate, since this is split as vector spaces. For the second condition, we know that any element  $(v, \lambda \iota) \in V_Y \cap cN_Y$  with  $v \in V_X$  must have  $\lambda \neq 0$  since  $X$  satisfies the condition. However  $\lambda \iota \notin c\mathbb{Q}[c]$ . Altogether,  $Y$  satisfies Condition 11.1.

Alternatively, suppose  $n$  is even. To calculate  $[S^n, X]_*^{\mathbb{T}}$  we take an injective resolution of  $X$ . We argue that this takes the form

$$0 \rightarrow X \rightarrow e(V) \rightarrow f(\Sigma^2 V \otimes k[c]^{\vee}) \rightarrow 0.$$

To start with, since  $X$  is a wide sphere,  $X$  embeds in  $e(V)$ . The cokernel is zero at  $\mathbb{T}$  and hence of the form  $f(T)$  for some torsion  $\mathbb{Q}[c]$ -module  $T$ . At 1 the cokernel is  $(\mathbb{Q}[c, c^{-1}] \otimes V)/N$ ;

since this is divisible it is a sum of copies of  $\mathbb{Q}[c]^\vee$ . Finally, in view of Condition 11.1 (i)  $T = \Sigma^2 V \otimes \mathbb{Q}[c]^\vee$  as claimed.

Now we may use the Adams spectral sequence to see that  $[S^n, X]_0^\mathbb{T} = V_n$ . If the original map  $f$  is trivial, then  $\pi_*^A(Y) = \pi_*^A(X) \oplus \pi_*^A(S^{n+1})$ , and the result is again clear. Otherwise we have a diagram

$$\begin{array}{ccc} S^n & \xrightarrow{f} & X \\ & & \\ \Sigma^n \mathbb{Q}[c] & \xrightarrow{\theta} & N \\ \downarrow & & \downarrow \\ \mathbb{Q}[c, c^{-1}] \otimes \Sigma^n \mathbb{Q} & \xrightarrow{1 \otimes \phi} & \mathbb{Q}[c, c^{-1}] \otimes V \end{array}$$

This shows that since  $\phi$  is mono then also  $\theta$  is mono and furthermore, by Condition 11.1(ii),  $\theta$  is the inclusion of a summand. It follows that the map  $f$  is split. Indeed, splittings of  $\phi$  are given by codimension 1 free summands  $N'$  of  $N$ . This automatically has  $\overline{N}'$  of codimension 1 in  $|V|$ . We need only choose  $N'$  so that  $\overline{N}'$  avoids  $\theta(\Sigma^n \mathbb{Q})$ . This gives a compatible splitting of  $\phi$ . It follows that in fact  $Y$  is a retract of  $X$ , and the result follows from Lemma 11.2.

Finally if  $X$  has components in both even and odd degrees, then  $X \simeq X_{ev} \vee X_{od}$  and we may argue as follows. Without loss of generality we suppose  $n$  is even. If  $f$  maps purely into  $X_{ev}$  or purely into  $X_{od}$  the other factor is irrelevant and the above argument deals with this case. Otherwise  $f$  has components mapping into both  $X_{ev}$  and  $X_{od}$ . The above argument shows that  $\pi_*^A(Y)$  is a retract in even degrees and it is unaltered in odd degrees.  $\square$

**11.E. Spectra built from  $\mathbb{T}$ -fixed spheres.** We have now done the main work and can identify the thick subcategory generated by  $S^0$ .

**Corollary 11.6.** *The thick subcategory generated by  $S^0$  consists of wide spheres satisfying Condition 11.1.*

**Proof:** First, we observe that the thick subcategory  $\text{Thick}(S^0)$  can be constructed by alternating the attachment of  $\mathbb{T}$ -fixed spheres and taking retracts; the fact that any element of  $\text{Thick}(S^0)$  is a wide sphere satisfying Condition 11.1 then follows from Lemmas 11.2 and 11.5. The point is that we must show that if we construct  $Z$  using a cofibre sequence  $X \rightarrow Y \rightarrow Z$  with  $X, Y$  in the thick subcategory then  $Z$  may be constructed from  $X$  by using the two processes. Formally, we are applying induction on the number of cells, so we may suppose  $Y$  is constructed from the two processes. If  $X$  is formed by attaching spheres, we may form  $Z$  from  $Y$  by attaching the corresponding spheres. If  $X$  is a retract of  $X'$  formed from spheres then  $f : X \rightarrow Y$  extends over  $X' = X \vee X''$  by using 0 on the second factor and then  $Z$  is a retract of  $Z'$ .

Now we show that any wide sphere satisfying Condition 11.1 is in the thick subcategory generated by  $S^0$ . We argue by induction on the dimension of  $|V|$ . The result is obvious if  $V = 0$ . Suppose that  $X$  is a wide sphere satisfying the given condition and that the result is proved when the geometric  $\mathbb{T}$ -fixed points have lower dimension.

Note that if  $t^n$  is the smallest degree in which  $p_{\mathbb{T}}(t)$  is non-zero we may choose a vector  $v \in V_n \setminus \overline{N}_{n+2}$ . Accordingly  $X$  has a direct summand  $\mathbb{Q}[c] \otimes v \rightarrow \mathbb{Q}[c, c^{-1}] \otimes v$ , which

corresponds to a map  $S^n \rightarrow X$ . Since  $v \notin \overline{N}_{n+2}$ , the quotient  $Y$  again has injective basing map and obviously satisfies the polynomial condition. Since  $n$  is the smallest degree in which  $V$  is non-zero,  $v$ , the direct summand  $\mathbb{Q} \cdot v$  may be removed from  $|V|$  without affecting the filtration condition. By induction  $Y \in \text{Thick}(S^0)$ , and hence  $X \in \text{Thick}(S^0)$  as required.  $\square$

**11.F. Spectra built from representation spheres.** Since smashing with any sphere  $S^{kz}$  is invertible, this allows us to deduce the thick subcategory generated by any sphere.

**Corollary 11.7.** *The thick subcategory generated by  $S^{kz}$  consists of wide spheres which are  $k$ -twisted in the sense that*

- (1)  $p_1(t) = t^{-2k} p_{\mathbb{T}}(t)$ .
- (2)  $V \cap c^{k+1} N = 0$

**Proof:** This is immediate from Corollary 11.6 and the observations

$$V(S^{kz} \wedge X) = V(X) \text{ and } N(S^{kz} \wedge X) = c^{-k} N(X).$$

$\square$

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