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Essays on Endogenous Formation of Bilateral Partnerships

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Acknowledgement

To my father,
Νίκων

Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

A constant theme in the endogenous network formation literature has been the tension between what is privately optimal for the agents who create a network, and what maximises social welfare. When the network structure affects, and is affected, by subsequent agent behaviour, it is also pertinent to ask whether the overall network and behaviour outcome can agree with society's interests. We explore these questions by critically reviewing seminal papers in the literature of static and dynamic network formation, highlighting negative results, and investigating the sources of inefficiencies. We then present two models featuring an endogenous partnership formation stage and a subsequent endogenous non-cooperative effort provision stage. In both models, effort provision actions feature strategic complementarity, and agents face a negative externality from the links of partners. Partnerships are non-exclusive and agents face either an indirect or a direct cost of effort provision. In the first model, agents are ex ante homogeneous, whereas in the second model agents have heterogeneous productivity. For various general families of production functions, we pinpoint the efficient linking and effort provision strategy profile and compare it with the set of stable networks. Even though in both models the game is one-shot and the agents are myopic, we prove that the efficient network structure and effort provision profile will always be sustained as an equilibrium of the overall game. For homogeneity, we directly contrast our positive results with the negative results of the Jackson and Wolinsky (1996) [31] Co-author model. We, moreover, prove that, for some families of production functions, the efficient network is the unique stable network. For heterogeneity, we additionally perform comparative statics in order to observe changes in relative specialisation to the high-type partnership, as the relative productivity ratio and the degree of concavity vary. Overall, modelling the interplay between link formation decisions and endogenous effort provision allows us to reach positive results where stability and efficiency are reconciled.

Chapter 1

Endogenous network formation and the tension between private and social welfare

1.1 Introduction

Bilateral partnerships are ubiquitous in economic and social life. In such partnerships, the pair of self-interested agents can represent single individuals, firms, countries, or any other autonomous institutions and organisations of a given form. The overall structure of these partnerships can then be formally modelled as a network of inter-connected agents, where a link exists between two agents only if they are partners. [1] The study of the endogenous formation of cooperation structures started with Myerson (1977) [35] and Aumann and Myerson (1988) [1] but remained at a relatively early stage, until it was rapidly accelerated in the last twenty years. The development of tractable theoretical models of network formation has, as noted by Jackson (2016) [27], recently accelerated once more, driven largely by the need for rich yet tractable models that can be applied in the empirical analysis of

[1] In what follows, whenever we refer to a network, unless otherwise specified, we will mean an undirected network where links need to be formed by mutual consent. On the contrary, in directed networks links are formed if at least one party wishes to do so.

real world networks.[2]

A constant theme, ever since the initial emergence of this literature, has been the potential tension in the endogenous network formation process between what is privately optimal for the agents who make the decisions that create the network, and what would be welfare maximising from a societal perspective.[3] Models of endogenous networks have been examining whether Adam Smith's 'invisible hand' could be extended to the network formation process: Can the self-interested agents produce network outcomes that coincide with what is beneficial for society as a whole?

Moreover, in situations where the network structure affects, and is in turn affected by, subsequent behaviour within the network, we need to ask whether there can be circumstances under which the overall network and behaviour outcome agrees with society's interests. As noted in Jackson (2016) [27], models of endogenous network formation need to capture the 'co-evolution' of partnerships and behaviours but this is an area that has not been studied to a great extent, even though the scope for applications is immense. A key example is the co-determination of financial investments and the regime that monitors them. More generally, in social and economic interactions relationships are formed for a reason, or, in fact, to serve potentially multiple interdependent functions. In such cases, the behaviour and outcomes in the network affect back the incentives of strategic agents to maintain or sever their network relationships. Such co-dependencies can be crucial for understanding networks that function, for instance, as structures for cooperation, coordination or intermediation, and for analysing policy interventions in such complex environments. If, on the contrary, as noted by Vega-Redondo (2016) [38], the co-determination of links and actions is ignored then models can lead to misleading results and offer a limited or mistaken understanding of real-world network relationships.

Overall, whether the answers to the previous questions are positive or negative has obvious consequences for policy making. In the case of negative results, showing a misalignment between private and social incentives and optimal outcomes, we would need to measure the extent of the mismatch and examine the potential for policy interventions. More importantly,

[2]The initial and more recent contributions on network formation and other areas of network research are surveyed among others in Jackson (2005a) [22], Jackson (2008) [24], Jackson (2014) [26], Bloch and Dutta (2011) [3], Goyal (2007) [13], Goyal (2016) [14].

[3]See for example Jackson (2005a) [22], Jackson (2005b) [23].

the sources of the observed inefficiencies would need to be determined. As pointed out in Dutta and Jackson (2003) [9], this is no trivial task. For individual models the tension can be attributed to externalities of linking ignored by private players, or to private concerns over bargaining power when the allocation of value is also endogenously determined. However, a general characterisation of the determinants of alignment or divergence of social and private incentives so far does not exist.

Furthermore, in real world networks agents exhibit heterogeneous characteristics that affect their behaviour. The differences in agents' ex ante features can lead to differences in their strategic behaviour and, overall, to different network positions and payoffs. Even more importantly, this heterogeneity has the potential to become a further source of tension between the network outcomes that arise and what society would wish to happen. Although the danger of lack of tractability is clear, it is of interest to try and develop models that capture ex ante agent heterogeneity, determine its effects on network formation and network interaction incentives, and relate it to ex post agent heterogeneity in the network.

In order to attempt to answer some of these questions set above, the first task is to define a suitable measure of social efficiency. In the subsection that follows, we present and compare some of the efficiency concepts that have been used in the literature on network formation models. Secondly, various authors have used different processes of network formation and different concepts of stability or equilibrium in order to characterise the result of agents' self-interested behaviour. We will present most of the stability concepts that are relevant for network formation models and explain the differences in the underlying linking processes.

Finally, we will turn to our main area of interest: presenting some characteristic endogenous network formation models and their results regarding the tension between stability and efficiency. Our motivation will be to try and uncover the differences in the model setup that can lead to significant differences in predictions about efficiency. We will also pay particular attention to dynamic models of network formation. For these models, we will ask whether relaxing the static character of network formation can repair the mismatch between stable and efficient network outcomes. Lastly, we will relate the concepts and models presented in this chapter to the models that of subsequent chapters.

1.2 Measuring social efficiency

Before proceeding to compare the endogenously derived networks of any formation process with what would be optimal from the point of view of society as a whole, we need to carefully define what social efficiency is. Given the specific assumptions of a model, there are two obvious ways of defining efficiency in networks. The first is the familiar notion of Pareto efficiency, applied in a network setup.

Consider a network represented by a graph (N, g), where $N = \{1, \ldots, n\}$ denotes the set of agents in the network and g the binary matrix of relationships between them. In particular, $g_{ij} = g_{ji} = 1$ if agents i, j are linked and $g_{ij} = g_{ji} = 0$ otherwise[4]. We denote by G the set of all possible networks that can be formed by the agents in N. [5] A *path* in a network g between nodes i and j will be a sequence of connected nodes $i_1 i_2 i_3 \ldots i_{K-1} i_K$ such that $g_{i_k i_{k+1}} = 1$ for each $k \in [1, \ldots, K-1]$ with $i_1 = i$ and $i_K = j$ and such that all nodes in the sequence are distinct. The network (N, g) will then be *connected* if for each $i, j \in N$ there exists a path in (N, g) between i and j. We next define a *component* of (N, g) as a distinct maximal connected subgraph of the network:

Definition 1. *A component of a network (N, g) is a nonempty subnetwork (N', g') such that $\emptyset \neq N' \subset N$, $g' \subset g$, (N', g') is connected, and if $i \in N'$ and $g_{ij} = 1$ then $j \in N'$ and $g'_{ij} = 1$.*

Assume a value function v that specifies the total value generated by the network. Assume further that there is a fixed exogenous allocation rule Y that determines the payoff of each agent corresponding to the value v created by each network in G. Then a network will be Pareto efficient given the set of agents, the allocation rule and the value function, if there is no alternative network that can provide equal or higher payoff for all agents and strictly higher payoff for at least one of them:

Definition 2. *A network g is Pareto efficient in G for (v, Y) if there exists no other network $g' \in G$ such that $Y_i(g', v) \geq Y_i(g, v)$ $\forall i \in N$ with $Y_i(g', v) > Y_i(g, v)$ for some $i \in N$.*

[4] Using the convention $g_{ii} = 0$.

[5] When N is considered fixed or given, we will often refer to the network as g.

Pareto efficiency is a weak concept as it does not demand total welfare maximisation. In order to see that, we next define the most frequently used concept of strong efficiency, following Jackson and Wolinsky (1996) [31], and then proceed to a comparison of the two concepts.

A network will be (strongly) efficient, given some value function if there is no other network that can give a higher total value. While there always exists a (strongly) efficient network, it does not need to be unique; it is possible that the maximum value can be produced by more than one networks.

Definition 3. *A network g is (strongly) efficient in G for v if there exists no other network $g' \in G$ such that $v(g') > v(g)$.*

We notice that the exogenous allocation rule Y does not affect whether a network is efficient or not from the perspective of society. This provides a straightforward intuition for the plausibility of a conflict between stability and efficiency: agents, in the absence of value transfers, benefit only from the value allocated to them by Y and not from the value v that is generated by the whole network.

Strong efficiency is, therefore, much more demanding than Pareto efficiency: a network is (strongly) efficient only if it is Pareto efficient, for a given value function v, *irrespective* of the allocation rule Y. We proceed with a simple example, following in intuition the co-author model of Jackson and Wolinsky (1996) [31], through which this difference in the two notions of efficiency and the fact that Pareto efficiency is weaker will be made manifest.

Example 1. *Consider a population of four identical agents defining the set of feasible networks G. Two agents i, j are neighbours only if a direct link exist between them by mutual consent i.e. if $g_{ij} = g_{ji} = 1$. Let d_i denote the number of neighbours of agent $i \in N = \{1, 2, 3, 4\}$ and N_i the set of neighbours of agent i. Assume that agents possess an effort endowment of one unit each, which they allocate equally among all of their partners. If an agent has no partners then assume that her effort cannot be productively employed, and hence she makes a payoff of zero. Assume that the value from a partnership between two agents i, j is given by the sum of the efforts $(\frac{1}{d_i}, \frac{1}{d_j})$ exerted by each of them plus the product of the two efforts. This formulation allows for some synergy in the production of network value. Therefore,*

$$v_{ij} = \frac{1}{d_i} + \frac{1}{d_j} + \frac{1}{d_i}\frac{1}{d_j}. \tag{1.1}$$

Then the total value created by the network will be equal to the sum of the value of all partnerships:

$$v(g) = \sum_{i \in N, j \in N_i, j > i} [\frac{1}{d_i} + \frac{1}{d_j} + \frac{1}{d_i}\frac{1}{d_j}]. \tag{1.2}$$

Assume, finally, that the allocation rule is fixed and specifies that each agent i receives a half of the value created by each of their partnerships. So,

$$Y_i(g) = \frac{1}{2} \sum_{j \in N_i} [\frac{1}{d_i} + \frac{1}{d_j} + \frac{1}{d_i}\frac{1}{d_j}]. \tag{1.3}$$

Consider now the following two networks in G: network g_p of two isolated pairs of agents, and network g' with a line of three agents and one isolated agent: take for instance the 132 *line with agent* 3 *in the centre and agent* 4 *isolated. Then,*

$$\begin{aligned} v(g_p) &= 6 \\ v(g') &= 4 \\ Y_i(g_p) &= 1.5, \ \textit{for each } i \in N \\ Y_i(g') &= 1, \ \textit{for } i = 1, 2 \\ Y_3(g') &= 2 \end{aligned} \tag{1.4}$$

It follows from Jackson and Wolinsky (1996) that g_p is strongly efficient. On the other hand, g' is not strongly efficient since $v(g_p) > v(g')$. However, g' is Pareto efficient.

These efficiency concepts are suitable for models where no transfers of value between players are allowed. A transfer between two players is called direct if the two players are connected and indirect if the two players are not connected in the network. In the case where any transfer of value is allowed among players, as in Bloch and Jackson (2007) [5], strong efficiency and Pareto efficiency become equivalent. This is intuitive as an efficient network g can become, if suitable indirect transfers are provided, preferable to all agents and strictly preferable for at least one, than any inefficient network g' that is Pareto efficient without transfers.

In order to illustrate this, we can look at network g' from the above example; the isolated pairs network creates a higher total value of 6. All

agents prefer moving to g_p from g' with agent 4 strictly preferring it, if indirect transfers are made so that e.g. agents $1, 3, 2$ are allocated the same value as in g' and agent 4 receives the remaining $6 - 4 = 2$ instead of zero in g'. We, therefore, conclude that Pareto efficiency, though weak in general, is perfectly suitable for models where any reallocation of value is permissible.

Finally, following Jackson (2005a) [22], we present the concept of *constrained efficiency*, which is suitable for models where reallocations of value among agents are possible but such transfers need to satisfy certain constraints. More specifically, instead of allowing for all or no reallocations, as in strong efficiency and Pareto efficiency respectively, reallocations of value are allowed as long as they are *anonymous* and *component balanced*.[6].

An allocation is anonymous if it does not depend on the identity of the players occupying the various nodes; if players are relabelled, the allocation must change with the labels:

Definition 4. *Consider a permutation $\pi : N \to N$ and, for any $g \in G$, let $g^\pi = \{\{\pi(i), \pi(j)\} | g_{ij} = 1\}$. Define $v^\pi(g) = v(g^\pi)$. An allocation rule Y is anonymous if, for any value function $v \in V$, network $g \in G$, and permutation of the set of players π, $Y_{\pi_i}(g^\pi, v^\pi) = Y_i(g, v)$.*

An allocation is component balanced if the total value allocated in every component (N', g') of network (N, g) is equal to the total value created by that component. [7]

Definition 5. *An allocation rule Y is component balanced if $\sum_{i \in N'} Y_i(g, v) = v(g')$ for each v, $g \in G$, and component (N', g') of (N, g).*

In order to illustrate these restrictions, note, in the context of Example 1.2.1, that the transfer specified is not anonymous: agent 3's identity matters as she needs to be allocated a higher share than all the others to agree to move to g_p. But an anonymous and component balanced allocation rule will give 1.5 to each agent in an isolated pair. Therefore, agent 3, who receives 2 in g', will be strictly worse-off in g_p. We hence see that g_p does not dominate g' in terms of constrained efficiency.

Definition 6. *A network g will be constrained efficient relative to value function v if and only if it is Pareto efficient relative to v and Y, for ev-*

[6]The following definitions are based on Jackson (2008) [24]

[7]We restrict attention here to component additive allocations, where the value created by each component is independent of how other components are organised.

ery allocation rule Y that is anonymous and component balanced. In other words, there exists no $g' \in G$ and anonymous and component balanced Y such that $Y_i(g', v) \geq Y_i(g, v)$ $\forall i \in N$ with $Y_i(g', v) > Y_i(g, v)$ for at least one $i \in N$.

This is, therefore, an intermediate concept of efficiency that falls between Pareto efficiency and strong efficiency. In particular, Jackson (2005a) [22] points out that for a component balanced and anonymous allocation rule Y, efficient networks are a subset of constrained efficient networks, which are, in turn, a subset of Pareto efficient networks with no transfers permitted. Existence of an efficient network thus guarantees the existence of a constrained efficient and a Pareto efficient network in G, for any value function and allocation rule. Note, finally, that models have made use of different concepts of constrained efficiency by demanding a set of constraints other than anonymity and component balance.

1.3 Stability in static models of endogenous network formation

The natural next step in our analysis is to consider the various stability and equilibrium definitions that have been used in the literature of endogenous network formation in order to describe the outcome of the network formation process.[8] This section will present the concepts of Nash stability, as in Myerson (1977) [35], Pairwise Nash stability and Pairwise stability, following Jackson and Wolinsky (1996) [31], Bilateral Equilibrium as in Goyal and Vega-Redondo (2007) [19], and Strong equilibrium, as discussed in Dutta and Mutuswami (1997) [10] and Jackson and van den Nouweland (2005) [28]. We will restrict attention to stability and equilibrium concepts without value transfers between agents.

Starting with Nash stability, Myerson (1991) [35] models the bilateral link formation process as a non-cooperative simultaneous-move game. A strategy for a player $i \in N = \{1, \ldots, n\}$ is an announcement of intended links $s_i \in S_i = \{0, 1\}^{n-1}$, where $s_{ij} = 1$ or $s_{ij} = 0$ if agent i wants to form a link with j or not, respectively. Links need mutual consent in order to be

[8] For a comprehensive analysis and comparison of some of the numerous concepts see Bloch and Jackson (2006) [4].

formed i.e. ij will be formed only if $s_{ij} = s_{ji} = 1$. Therefore, a strategy profile s for all players induces a network $g(s)$ and individual payoffs $\Pi_i(g(s))$ for the players. [9] Then a network $g(s)$ is Nash stable if and only if there is no unilateral deviation in the linking strategy of any player that would lead them to a strictly higher payoff:

Definition 7. *A strategy profile s of linking announcements is a Nash Equilibrium of the linking game iff $\Pi_i(g(s)) \geq \Pi_i(g(s'_i, s_{-i}))$ for all $i \in N$, $s'_i \in S_i$. The network $g(s)$ is then Nash stable.*

There is broad consensus in the network formation literature that this is too weak an equilibrium concept and unsuitable for the study of undirected link formation. It allows for too many equilibria networks with undesirable properties. The key example is the empty network which is always Nash stable as the agents cannot coordinate their linking announcements in order to form a new link and are merely allowed to unilaterally deviate in the form of link cutting.

Some of these problems are remedied by the refinement of Pairwise Nash Equilibrium. Agents are now allowed to deviate in pairs in order to form a new link, or to unilaterally deviate by cutting as many of their existing links as they wish.[10]

Definition 8. *A strategy profile s is a Pairwise Nash Equilibrium of the simultaneous-move linking game iff (a) $\Pi_i(g(s)) \geq \Pi_i(g(s'_i, s_{-i}))$ for all $i \in N$, $s'_i \in S_i$, and (b) for any i such that $\Pi_i(g(s)+ij) > \Pi_i(g(s)) \Rightarrow \Pi_j(g(s)+ij) < \Pi_j(g(s))$. Network $g(s)$ is then called Pairwise Nash stable.*

A related concept is that of Pairwise stability, employed by Jackson and Wolinsky (1996) [31]. Agents are allowed to deviate in pairs to form a new link, or to unilaterally deviate by cutting any one link. A network will now be Pairwise stable if neither of these types of deviations can be strictly profitable for both link-forming agents or for the link-cutting agents, respectively. We, therefore, directly observe that the set of Pairwise Nash stable networks is a refinement on the set of Pairwise stable networks as, in the former case, agents are permitted more types of deviations.

[9] We will denote by $g+ij$ the new network formed by g with the addition of a new link ij, and by $g-ij$ the network formed by g with the omission of existing link ij.

[10] For a formal study of this equilibrium concept see for example Calvó-Armengol and Ilkilic (2009) [7].

Definition 9. *A network g is Pairwise stable with respect to a payoff function Π if: (a) for all $i \in N$ and links $ij \in g$, $\Pi_i(g) \geq \Pi_i(g - ij)$, and (b) for all $ij \notin g$, if $\Pi_i(g + ij) > \Pi_i(g)$ then $\Pi_j(g + ij) < \Pi_j(g)$.*

We note that, contrary to Myerson's Nash stability, Pairwise stability and Pairwise Nash stability do not correspond to a purely non-cooperative game. They are instead allowing for 'intuitive' deviations by a pair of agents, since link formation requires, by definition, mutual consent and hence some form of bilateral cooperation. Pairwise stability is an attractive concept for use in many applications due to its tractability and good predictive power. However, it can also be considered too weak as it allows for a very limited set of deviations. Therefore, networks that are Pairwise stable may not be stable against richer deviations, one example already discussed being the simultaneous deletion of multiple links by an agent.[11]

The next equilibrium concept we will discuss is the Bilateral Equilibrium of Goyal and Vega-Redondo (2007) [19]. This can be described as a further refinement on the Pairwise Nash Equilibrium concept. Agents are now allowed to unilaterally deviate by cutting as many links as they wish, and to *bilaterally deviate* by forming a link between them and/or deleting any combination of their links that they wish. Intuitively, on top of any unilateral deviations involving link cutting, agents are additionally allowed all possible deviations in a coalition of size two, in full recognition of the cooperative nature of undirected linking. In a Bilateral Equilibrium network, none of these deviations can be profitable for all agents involved.

Definition 10. *A strategy profile s is a Bilateral Equilibrium if the following conditions hold: (a) For any $i \in N$ and every $s_i, s_i' \in S_i$, $\Pi_i(s) \geq \Pi_i(s_i', s_{-i})$, and (b) for any pair of players $i, j \in N$ and every strategy pair (s_i, s_j), $\Pi_i(s_i', s_j', s_{-i-j}) > \Pi_i(s) \Rightarrow \Pi_j(s_i', s_j', s_{-i-j}) < \Pi_i(s)$.*

This equilibrium concept is appealing in situations where links are costly and hence agents have limitations, via a fixed endowment or a convex cost of linking, as to how many links they will create. In such setups, Pairwise stability notions can characterise as stable, networks that would not survive a bilateral deviation where agents, at the same time as forming the link

[11] In Bloch and Jackson (2007) [5], it is in fact pointed out that the set of Pairwise Nash equilibria is the intersection of Nash equilibria of Myerson's game and Pairwise stable networks.

between them, can cut a number of other links in order to free up endowment or balance out the cost of the additional link formed.

Finally, following Jackson and van den Nouweland (2005) [28], we look at an even stronger stability concept, Strong stability, where coalitions of any size are allowed to deviate by forming links between them and/or severing any of their links. A network will then be Strongly stable if, for any possible coalitional deviation, there is always at least one member of the coalition who would block it because they end up with a strictly smaller share than before.

Definition 11. *A network g is Strongly stable with respect to an allocation rule Y and a value function v if for any subset of agents $C \subseteq N$, g' that is obtainable from g via deviations by C, and $i \in C$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in C$ such that $Y_j(g', v) < Y_j(g, v)$.*

This definition of Strong stability is slightly stronger than that originally introduced by Dutta and Mutuswami (1997) [10]: here a coalition of agents is allowed to deviate if some among them are strictly better off and all others are weakly better off. On the contrary, in Dutta and Mutuswami (1997) [10], all members of a coalition need to be strictly better off for a deviation to proceed. The above definition, therefore, implies Bilateral stability (and Pairwise stability) if we look at coalitions of size two; the set of Strongly stable networks will be a subset of the set of Bilateral equilibrium networks. However, demanding that a network survives from any possible deviation of any coalition of agents in $C \subseteq N$ can prove to be so restrictive that no Strongly stable network exists. Moreover, stability concepts allowing for coalitions of a greater size than two to deviate, move even further away from the setup of a non-cooperative game without any corresponding justification in the linking process.

Finally, as pointed out in Jackson (2005) [22], all of the above stability and equilibrium concepts are not only static but also fully myopic. In particular, myopic individuals do not predict or take into account how the others may react to their (unilateral or bilateral) deviations. For example, a bilateral deviation of a pair of agents where they form the link between them and cut several links with others, may result in further deviations by other agents, until a stable network is reached and no further deviations are profitable. While the bilateral deviation might have been deemed strictly

profitable, it can well be that in the final network, after all reactions of other players to it have taken place, the pair becomes strictly worse off than in the original network.

1.4 The tension between stability and efficiency

In this section, we turn to the analysis of some endogenous network formation models, selected to reflect the motivation and interest of this and the remaining two chapters. [12] Our focus in what follows will be on inefficiencies in the endogenously determined networks.

The first models in this literature were explicitly preoccupied with the tension between stability and efficiency and with potential mechanisms to restore efficiency by aligning social with private incentives. The recent literature, on the other hand, has been focusing on the interplay and co-evolution of links and actions. Overall efficiency can then be examined in a richer setup where the network is built for an explicit function, and any inefficiencies can be attributed to a variety of sources. It will be of particular interest to see to what extent observed inefficiencies can be attributed to agent myopia, and whether, and to what extent, dynamic evolution of the network can restore efficiency.

1.4.1 A seminal model on the tension between stability and efficiency

We start with the seminal contribution of Jackson and Wolinsky (1996) [31] which was among the first to illustrate the mismatch between stability and efficiency when agents are free to shape the network based on their private incentives. The authors employ the concepts of strong efficiency and pairwise stability. They present two stylized models for which they characterise the stable and efficient networks, and show that the set of stable networks and the set of efficient networks do not always intersect. Then, returning to a more general model setup, they show that there are network value functions for which no anonymous and component balanced allocation

[12] For more exhaustive reviews of the older and more recent literature in endogenous network formation, look for example at Bloch and Dutta (2011) [3], Dutta and Jackson (2003) [9], Goyal (2016) [14], Jackson (2005) [22], Jackson (2008) [24], Jackson (2014) [26], Jackson (2005) [23], Jackson (2011) [25], Vannetelbosch and Mauleon (2016) [36] and Vega-Redondo (2016) [38].

rule can support the strongly efficient network as a pairwise stable network of the game. Therefore, there is tension between stability and efficiency that is, moreover, not easily resolved for general classes of value functions and allocation rules.

The first stylised model that the authors present is the Connections model: agents link with others for the purpose of social communication. Social communication can be direct, between neighbours, or indirect, between neighbours of neighbours etc., however, the value of communication decays as the distance between the agents in the network increases. Communication linking also entails a cost that the agent takes into account when determining her linking strategy. An agent pays only for forming direct links but can then enjoy the benefit of indirect links with no additional cost. So, a link between i and j also brings benefits to any neighbour of i who is not directly connected to j. So,this is a model of positive externalities. The authors show that, for a symmetric version of the model, the unique strongly efficient network in the connections model is either the complete graph, or a star encompassing everyone, or the empty network.[13] Intuitively, which network is the efficient one will depend on how high the cost of direct linking is compared to the decay in the value of communication from indirect links.

Next, the authors show that in the symmetric connections model, where each agent's allocated value is the utility that she receives from the communication network, a pairwise stable network has at most one (non-empty) component. In particular, for a small linking cost, the unique pairwise stable network is the complete network. For intermediate cost values, a star encompassing all players is pairwise stable but not necessarily the unique pairwise stable network. Finally, for high cost values, any pairwise stable network which is non-empty is such that each player has at least two links. Therefore, while it is possible to achieve efficiency for sufficiently low cost values, inefficient stable networks may well arise for intermediate and high linking cost values.

The second stylised model that is presented is the Co-author model: agents can be thought of as researchers whose productivity is a function of the number of their co-authors. Then each link can be interpreted as a

[13] A network is complete if all possible links are formed. It is empty if no links are formed between agents. It has a star structure if a set of agents, the core or centre, is connected with everyone else and the remaining agents, the periphery, is only connected with the agents in the centre.

(mutually agreed) research collaboration. There is no direct cost of link formation but each agent has a fixed endowment to distribute, assume equally, among their collaborators. Therefore, the more links the agent has, the less she is going to offer to each of her co-authors. Moreover, the more fellow collaborators each of an agent's own collaborators has, the less she is going to benefit from that link. Each new link formed, therefore, reduces the value of all existing links. So, this is a model of negative externalities. Each collaboration product is defined as the mathematical sum plus the mathematical product of the two collaborators' efforts. The productivity of a player is then determined by the sum of research products from all their collaborations.

The authors show that, under these assumptions, if the number of agents N is even, then the strongly efficient network consists of $N/2$ separate pairs. However, a pairwise stable network can be partitioned into fully connected components, each of which has a different number of members. Therefore, the stable and efficient networks do not coincide; the stable network will be over-connected from a social perspective. This inefficiency is attributed to the fact that the self-interested agents do not fully internalise the negative externality of additional links to the value created by existing links.

Finally, the authors turn to a more general version of their model, which encompasses both stylised models presented above. They prove that, for three or more agents, there is no allocation rule Y that is anonymous and component balanced and which can support, for each network value function v, at least one strongly efficient graph as pairwise stable. The authors explain that this negative result is not due to non-existence; there can always be found an anonymous and component balanced allocation rule for which a pairwise stable network does exist. However, such a rule will always have the property that, for some value functions, the pairwise stable networks that it supports are all inefficient.

In Jackson (2005) [22], it is, furthermore, shown that there does not exist any component balanced and anonymous allocation rule such that, for every value function, there exists a constrained efficient network that is pairwise stable. This result further strengthens the above incompatibility result of Jackson and Wolinksy (1996) [31] by extending it to the less strict case of constrained efficiency.

Jackson and Wolinsky (1996) [31] prove that stability and efficiency can

be reconciled if the requirement for component balance is dropped. If, on the other hand, anonymity is dropped instead, Dutta and Mutuswami (1997) [10] show that there exists a component balanced allocation rule Y such that the intersection of efficient and pairwise stable networks is non-empty. Moreover, it is shown that Y is anonymous, in addition to component balanced, for some networks in this set. These results are making use of the concept of strong stability, however, they can be shown to extend to the case of pairwise stability as well. Therefore, we conclude that if one is willing to drop component balance, efficiency and stability are no longer incompatible. Moreover, if one is willing to drop anonymity, then again stability and efficiency can be reconciled. Finally, for some value functions, neither component balance nor anonymity needs to be sacrificed.

Nevertheless, as Jackson (2011) [25] points out, it is striking that, even when agents have full information and the ability to reallocate value up to some, not too strict, constraints of component balance and anonymity, the conflict between stability and efficiency still persists. In particular, this fact contradicts the spirit of the 'Coase theorem': One would expect that with full information and the opportunity to make value transfers, fully efficient outcomes would be always obtained. This has not always been true, however, for the multi-agent endogenous network formation literature.

Bloch and Jackson (2007) [5] return to the problem of reconciling efficiency with stability by examining the use of transfers. They investigate different setups based on the types of transfers that are permissible. More specifically, they contrast the implications of the following transfer regimes: (i) transfers can be made only between directly connected agents or between indirectly connected agents as well; (ii) transfers to a link can be agreed contingent on only that link being formed or on the entire formed network; (iii) players can pay other players to induce them to refrain from forming links or no such payments are allowed.

The authors find that, in the case where only directly linked agents are allowed to make transfers to each other, efficient networks can be, but will not always be, supported in equilibrium. Even if indirect transfers are allowed, in order to guarantee that efficient networks form, players need to moreover be able to make those transfers contingent on the entire network. The intuition is that there are multiple potential sources of inefficiencies that need to be dealt with: indirect payments are used to deal with positive

externalities of linking but network contingent transfers can also be needed to deal with the combinatorial nature of network formation. Finally, in order to deal with the inefficiency stemming from negative externalities in network formation, like in the case of the co-author model of Jackson and Wolinsky (1996) [31], players need to be able to pay other players to induce them not to form additional links.

Finally, other authors have examined whether (at least part of) the conflict between stability and efficiency can be attributed to the fact that agents are myopic. Grandjean, Mauleon, and Vannetelbosch (2011) [20] look at the Jackson and Wolinsky (1996) [31] model when agents are instead assumed to be farsighted: when contemplating a deviation they take into account the full series of subsequent deviations that can result out of it. In this context, a set of networks G' is defined as *Pairwise farsightedly stable* (i) if all possible farsighted pairwise deviations from any network $g \in G'$ to a network not in G' are deterred by the threat of ending worse off or equally well off; (ii) if there exists a farsighted improving path from any network outside the set G' leading to some network within G';[14] and (iii) if there is no proper subset of G' satisfying the first two conditions. The authors show that even farsightedness is not able to eliminate the conflict between stability and strong efficiency for intermediate levels of link formation cost. However, it is shown that farsightedness does reduce the discrepancy between pairwise stable and efficient networks when the cost of linking takes high enough values.

1.4.2 Dynamic network formation models

Faced with the incompatibility of efficiency and stability exposed by the Jackson and Wolinsky (1996) [31] model, a stream of literature has naturally examined whether this negative result could be attributed to the static nature of the network formation game. The first model to study this was by Watts (2001) [39], who explicitly presents a dynamic version of the Jackson and Wolinksy (1996) [31] Connections model.

More specifically, Watts(2001) [39] proposes a dynamic model where agents are allowed to cut or form links. The process starts with the empty network and then, as time goes by, random pairs of agents meet and decide whether to form a link between them or to unilaterally sever links, in or-

[14] Assume we start at a network outside G'. A farsighted improving path is a series of profitable deviations by farsighted agents that ultimately leads us to a network in G'.

der to (myopically) maximise current period payoffs. In each period, a link is randomly chosen to be updated with uniform probability. In particular, in an extension of pairwise-stability, when forming a link agents are at the same time allowed to simultaneously cut any number of their existing links with mutual agreement. In this setup, a network will be stable if no single player wants to sever any one direct link, and no pair of players wants to form the link between them, with the possibility of simultaneously cutting any number of their existing links.

Note that this stability concept is still weaker than Bilateral Equilibrium. It cannot, however, be compared in a straightforward way with pairwise Nash stability because, in the latter, players are not allowed to form a link and cut links simultaneously but they are allowed to unilaterally sever any number of links instead of just one.

Payoffs are specified as in the Connections model of Jackson and Wolinsky (1996) [31], which we reviewed earlier. Results are, therefore, similarly driven by the relationship between the size of benefits from indirect links and the size of the cost of direct link formation. In particular, the author verifies that the Jackson and Wolinsky (1996) [31] stability and efficiency results carry through for the static version of her model.

The author then moves on to determine to which network structures the dynamic network formation process will converge, asking whether the process can converge to the efficient network. She proves that the dynamic process does not always converge to the efficient star network structure. In such cases, it will either converge to another inefficient stable network or move in cycles, visiting the same series of networks in a specific order. For certain parameter values such cycles can be ruled out but the possibility of convergence to an inefficient network still remains.

In particular, the dynamic formation process with myopic agents will be path-dependent: if the benefit from maintaining an indirect link of length two is greater than the net benefit from maintaining a direct link, then the efficient network will only form if the order in which agents meet takes a particular pattern. Finally, as the number of agents in the network increases, meeting in the specific pattern or path necessary for convergence to the efficient star network will be less likely. The possibility of convergence to an inefficient network will then be even higher.

A natural extension of the above model would be to the case where agents

are not fully myopic in their strategic decision making. Dutta et al. (2005) [8] present a dynamic model of network formation where agents have some degree of foresight i.e. when deciding on a current action, they evaluate its effect on their entire discounted stream of payoffs.

Their dynamic link updating process is as follows: at any date a pair of agents is randomly chosen and allowed to unilaterally break any link they already have, and bilaterally form the link between them. This corresponds to a limited form of cooperation, where a coalition of size two is allowed to jointly deviate by forming the link among its members and/or deleting existing links with non-coalition members. Payoffs for that date are instantly realised and the whole process is then repeated. An equilibrium network formation process will be, in this context, a strategy profile for the dynamic game such that no active pair at any state can benefit either from unilateral link cutting or from bilateral link formation. When contemplating such deviations, active agents will be farsighted i.e. for a given value function they will evaluate the entire stream of profits that will accrue based on their actions and the consequences of their actions for all dates in the future. Note that the myopic case, examined by Watts (2001) [39] above, can be obtained as the special case when agents are perfectly impatient, with a discount factor equal to zero.

The authors define efficient networks using the definition of strong efficiency and then specify different ways in which a dynamic network formation process can yield these efficient outcomes; what they call different concepts of 'absorption'. Namely, the efficient network will be strongly absorbing when the network formation process reaches it, regardless of the network we begin with. It will instead simply be a stationary network if there is no guarantee that it will be reached but, if reached, there will be no further deviations.

Given the above model setup, the authors show that there are valuation structures in which no equilibrium strategy profile can sustain an efficient network. They then proceed to determine conditions on the valuation structure such that an efficient network will be strongly absorbing. They find that, for valuation functions that satisfy link monotonicity[15], the strongly

[15] A valuation function satisfies *link monotonicity*, if an individual's payoff is increasing in her number of links. A valuation function satisfies *increasing returns to link creation*, if it satisfies link monotonicity in a subcollection of components, with the additional requirement that aggregate value also increases over this subcollection. Formal definitions can

efficient complete network will be strongly absorbing at some strategy profile for all discount rates larger than zero, i.e. whenever agents are not fully myopic. In addition, in cases where the valuation function satisfies increasing returns to linking and the allocation rule is the component-wise egalitarian rule, the complete network will be strongly absorbing at some pure strategy equilibrium profile, provided that the common discount rate is sufficiently large.

However, a direct consequence of the above results is that the efficient network will not be strongly absorbing *at all equilibria* even if all the conditions we pose on the valuation function, the allocation rule and the discount factor are met. This leads to the conclusion that the tension between stability and efficiency cannot be fully resolved by a dynamic network formation setup, even if agents are allowed to be farsighted.

1.4.3 Endogenous network formation and the static and dynamic interplay between links and actions

In a recent review of the literature in network formation, Vega-Redondo (2016) [38] stresses the importance of creating models that incorporate both strategic linking and strategic behaviour in the formed network. In many real life situations, agents have control of both their structure of interactions and the behaviour that they exhibit in them. Therefore, there is a great interest in models that can account for both of these dimensions and shed light in the 'co-evolution' of links and actions.

Even though endogenising both the network and agents' behaviour in it increases the analytical challenges of a model, Vega-Redondo argues that such models are indispensable for the study of situations as broad as those of coordination, cooperation, intermediation, bargaining, local public good provision, learning, and conflict in networks.

In the remaining section, we will review a selection of such models. We will focus on the different methods for the co-determination of equilibrium networks and embedded behaviour, and on the assessment of the efficiency of such outcomes. We will also investigate the sources of inefficiencies when they are predicted by the models. More specifically, we will look at some characteristic models, both of a static and a dynamic setup, which fall into

be found in Dutta et al.(2005) [8]. The component-wise egalitarian rule allocates equal shares to all agents in any given component.

the broad category of endogenous networks with endogenous actions featuring by strategic complementarity.

We begin with a model of a coordination game played in an endogenous directed network proposed by Goyal and Vega-Redondo (2005) [18]. In the basic model, links are costly and one-sided so that a player can unilaterally choose their partners and then play a coordination game with all of them. There is a constraint on the action space demanding that the same action be played in all coordination games by the same agent. The authors look at the stable networks that can result out of the strategic cost-benefit behaviour of agents, and at whether the efficient coordination outcome can arise, both in a static and a dynamic setup.

Agents unilaterally choose their location in the network, which then determines the set of direct partners with whom they will play a 2×2 symmetric coordination game with common action sets $A = \{\alpha, \beta\}$. There are two, Pareto ranked, Nash equilibria of the one-shot simultaneous move game with coordination, $\{(\alpha, \alpha), (\beta, \beta)\}$, which result in coordination payoffs (d, d) and (b, b), respectively. Payoffs for the outcome (α, β), where agents fail to coordinate, are (e, f). The coordination game is described by the following overall relationship between payoff values: $d > f$, $b > e$, $d > b$, $d+e < b+f$. The state (α, α) is the Pareto optimal one since $d > b$, but choosing β is the risk dominant action for players, since $b + f > d + e$ i.e. the average payoff from β is higher than that from playing α. Therefore, there is a conflict between risk dominance and efficiency. Links are costly, with cost $c > 0$, and, in the basic model, one-sided, so that the linking game is a fully non-cooperative game. Since each player is obliged to choose the same action in the games played with all of their neighbours, strategic behaviour will also be influenced by the structure of the network.

Starting with a static analysis of the model, the authors show that network structure and coordination results are driven by the cost of link formation c. In particular, if the cost is low, with $c < e$, then players have incentives to link with everyone, irrespective of the actions played by others, hence the complete network is obtained. The complete network is also obtained if costs are high enough, with $d > c > b$, because in that case everyone linked must be choosing the efficient action α so linking has a guaranteed high payoff. However, for intermediate cost values, a wider range of outcomes can arise in equilibrium; the complete network or a network of two

distinct complete components are both possible equilibria. This stems from the fact that, intuitively, for intermediate cost values, linking is profitable only when all others choose the same action and coordination is achieved.

In similar spirit, the equilibrium results in the coordination game are also driven by the size of linking costs, and for intermediate cost values a wide range of outcomes can arise in equilibrium. These include social conformity to the efficient or to the inefficient action, as well as action heterogeneity.

As a result of the above, for intermediate cost values, it is possible for the equilibrium to feature neither the complete network nor the efficient coordination outcome. Therefore, in such cases, equilibrium selection is crucial. In order to try and achieve this, the authors look at a dynamic version of the model where agents are allowed to adjust their links. Once more, the level of the link formation cost remains a key factor that drives results, however, much sharper equilibrium predictions can now be obtained.

In particular, for intermediate costs, the authors show that the complete network will be stable in the long run, and that, in that case, social conformism will also arise. However, if the cost of link formation is high enough, agents will all conform to the socially inefficient risk-dominant action. It is only for low enough cost values that the efficient coordination outcome can be obtained. Therefore, allowing for dynamic adjustment in linking, although helpful for equilibrium selection and sharper predictions on the overall network structure and behaviour, is not always enough to eliminate inefficiencies. It is still possible that agents coordinate in the Pareto dominated outcome.

These results carry through when the model is extended to a setup where linking takes place by mutual consent and the linking cost is equally divided. For the static model, it is shown that the complete network is the unique non-empty strict Nash network, and that social conformism always takes place. However, both the Pareto optimal and the Pareto dominated coordination outcome can still arise in equilibrium. For the dynamic version of this model, it is again the case that a threshold level of linking cost exists: For all cost levels above this threshold the efficient outcome is the unique equilibrium outcome. On the contrary, for all cost levels below the threshold, the inefficient outcome becomes the unique equilibrium.

This negative result echoes those of Jackson and Watts (2002) [30]. They present a dynamic model where agents play a coordination game with their

neighbours but are also able to choose who these neighbours are by being periodically allowed to add or cut links. The authors come up with a multiplicity of stable outcomes. These crucially include networks where the equilibria of the coordination game are neither efficient nor risk-dominant.

We, finally, briefly turn to two models of peer effects. In these models, links and effort decisions are made in a context of local complementarities in effort levels and positive local externalities.[16] Hiller (2012) [21] in his working paper presents a simple model of undirected link formation. An agent's optimal effort provision decision depends on the structure of the network but also on the effort provision decisions of the other agents, giving rise to strategic complementarity. A broad family of payoff functions is used which feature individual payoffs that are convex in the effort levels of direct neighbours. It is shown that pairwise Nash stable networks display either a complete, an empty or a core-periphery structure. Although no direct comparison of equilibrium results with social efficiency is made, it is clear that depending on which of the three equilibria is selected, stability and efficiency may not agree.

KG'Anig, Tessone, and Zenou (2012) [33], on the other hand, present a dynamic model of peer effects with a linear-quadratic payoff specification. They look at a dynamic process of link formation where, in each period, agents play the following two-stage game: in stage one, each agent chooses an effort level considering the network as fixed; in the second stage the network structure can be updated by a randomly selected agent who is allowed to create a costless new link. Looking at the stable networks of this dynamic link formation process with endogenous actions, they show that these networks will feature 'nestedness' i.e. that the network structure will be such that the set of neighbours of each agent is a subset of the set of neighbours of each agent with a higher degree. This family of networks obviously contains the core-periphery family of networks. Therefore, we see that dynamic models are once more able to give a sharper prediction regarding the shape of stable networks that are possible to arise.

[16] Local complementarities in effort levels correspond to the case where a higher effort by a partner induces an agent to further increase their own effort. Positive local externalities of linking refer to the case where an agent is better off the more connected their partners are.

1.5 Outline and contribution

In the two chapters that follow, our fundamental contribution will be succeeding in reconciling efficiency with stability in the context of two models of endogenous link formation and endogenous actions. Both models will be dynamic to the extent that the one-shot game played will be a game with two sequential stages: an endogenous link formation stage where links are two-sided and hence partial cooperation for pairs of agents is permitted; and an endogenous non-cooperative decision-making stage where agents choose their effort provision after observing the entire network that has formed in stage one.

In both models, the effort provision actions of agents will feature strategic complementarity. Each bilateral partnership will result in some production that uses the two efforts as inputs. This production will then be fully enjoyed by the two partners according to an egalitarian rule but will be perfectly excludable from anyone else. There is, moreover, a negative externality to an agent from the other links of their partners. Partnerships are non-exclusive and the agents either have a fixed effort endowment to allocate, hence introducing an indirect cost of effort provision, or face a direct convex cost of effort. In the first model of Chapter 2, agents will be ex ante homogeneous in all features, whereas in the second model of Chapter 3 agents have heterogeneous productivity.

We are going to use the definition of strong efficiency presented earlier and pinpoint, for general families of production functions, the efficient linking and effort provision strategy profile, in order to compare it with the stable networks obtained. Stability will be defined using a variant of the Bilateral Equilibrium concept previously presented in this Chapter, in order to incorporate a Nash Equilibrium effort provision profile for the second stage of the game. Agents will use backwards induction when deciding on their linking profiles, anticipating that linking deviations will lead to a new network where effort provision in the second stage will be determined by a new Nash Equilibrium. However, when agents consider unilateral or bilateral deviations in the linking stage, they will take as given the linking decisions of all non-deviating agents.

Even though in both models the two-stage game that agents play is one-shot and agents are myopic, we are able to prove that the efficient network

structure and effort provision profile will, in all cases, be sustained as an equilibrium of the overall two-stage game. This will be shown both for a model with an arbitrary number of homogeneous agents and various families of production functions, and for a model with four heterogeneous agents and concave net production. For the homogeneous model, we will be able to directly contrast our positive results with the negative results of the seminal Jackson and Wolinsky (1996) [31] Co-author model. In particular, for the production function of their model as well as for some more general families of productions functions, we are able to prove that the efficient network is the unique stable network and, therefore, social efficiency and stability always agree.

We believe that these results stem from the fact that, firstly, our equilibrium concept allows for all intuitive deviations in a bilateral link formation model. Since agents form links with mutual consent, it is intuitive to allow them to deviate in pairs in the link formation stage of the game. Secondly and most importantly, both of our models include a distinct endogenous effort provision stage. It is this explicit modelling of the interplay between link formation decisions and endogenous effort provision actions that allows us to reach these important positive results and contribute to a long stream of literature in endogenous network formation by showing that stability and efficiency can be reconciled.

Chapter 2

Endogenous Formation of Bilateral Partnerships with Homogeneous Types

2.1 Introduction

In this section we study a game with a distinct link formation stage and a subsequent link-specific effort provision stage, where investment incentives depend on the network structure that has arisen from the first stage. We extend the Bilateral Equilibrium stability concept used in Goyal Vega-Redondo (2007) [19] to a two-stage game in order to incorporate a Nash Equilibrium in effort allocations in the second stage of the game. We prove that, even though inefficient equilibria may exist, the efficient network is always an equilibrium, under both strict concavity and strict convexity of synergistic production. We are hence able to revisit the seminal Co-author model of Jackson and Wolinsky (1996)[31]. In that paper, as well as in an important string of subsequent literature, an important tension was shown between the endogenously determined and the socially optimal network. More specifically, it was shown that the pairwise stable networks would be more connected than efficiency demands, and this was attributed by the authors to the negative externalities arising from link formation. On the contrary, in what follows, we are able to reconcile efficiency with stability, both for a production function corresponding to that model and for more general families of production functions. As we will see, this will be achieved

by endogenising the agents' link-specific effort provision decisions, and being able to compare the set of efficient outcomes with the set of equilibrium outcomes of the resulting two-stage game.

2.2 The Model

We consider a finite population of homogeneous agents $N = \{1, 2, ..., n\}$. The agents' interaction is modelled as the following two-stage game:

The first stage is the Myerson linking game described in Chapter 1. So, all agents *simultaneously* announce the set of agents with whom they want to form bilateral partnerships. Let $s_i \in \{0, 1\}^{n-1}$ denote the set of agents with whom i wants to form a link, the interpretation being that $s_{ij} = 1$ denotes that i wants to form a link with j. A partnership between i and j forms iff both the agents want to form the link or partnership; that is, if $s_{ij} = s_{ji} = 1$. We assume that there is a "small" cost $\bar{\mu} > 0$ which has to be borne by both agents if a link is formed. In particular, for this chapter, we assume $\bar{\mu} < 1$. We use $\mu(g)$ to denote the total cost of forming the links in any network g. So, using $g_{ij} = 1$ if $ij \in g$ and $g_{ij} = 0$ otherwise, for $i, j \in N$ and $i \neq j$:

$$\mu(g) = 2\bar{\mu} \sum_{ij} g_{ij} \tag{2.1}$$

Each action profile $s = (s_1, ..., s_n)$ will, therefore, induce a network $g(s)$. We denote by G the set of all possible networks that can be formed by the n players. Let $d_i(g)$ denote the degree of i in the network g, while $N_i(g)$ is the set of her neighbours in g. A *component* of any graph g is a subgraph such that all nodes in the subgraph are connected by a path, while no nodes outside the subgraph are connected to nodes in the subgraph. Components will be denoted by q and the set of nodes (or players) in q will be denoted by $N(q)$.

At the end of the first stage, players observe the network which has formed.

In the second stage, agents simultaneously announce their effort allocation decisions after observing network $g(s)$. Each agent i has an endowment of *one* unit of effort, and has to decide how to allocate this effort endowment across partnerships. For any player i, let $E_i = \{e_i \in R_+^{n-1} | \sum_{j \in N-\{i\}} e_{ij} \leq 1\}$, with $e_{ij} = 0$ whenever $g_{ij} = 0$. This introduces an indirect cost of effort,

as the more effort is exerted in one partnership the less remaining effort the agent has to exert elsewhere. It also implies a negative externality to each agent from the other partnerships that their own partners have. Let $E = E_1 \times E_2 \times \ldots \times E_n$. A feasible strategy for player i in the second stage of the game is, therefore, a mapping $m_i : G \to E_i$. Whenever there is no confusion about the network which has formed in the first stage, we will simply represent effort allocation decisions by e_i, e_i', etc. instead of $m_i(g), m_i'(g)$ and so on.

Suppose the network g has formed in the first stage, and $m(g) = (e_1, \ldots e_n)$ $\in E$. Each partnership $ij \in g$ results in the production of some output according to the following production function:

$$F(e_{ij}, e_{ji}) = e_{ij} + e_{ji} + f(e_{ij}, e_{ji}) \tag{2.2}$$

where $f(e_{ij}, e_{ji}) = 0$ if either $e_{ij} = 0$ or $e_{ji} = 0$. $f(e_{ij}, e_{ji})$ is a strictly increasing function whenever $e_{ij} > 0, e_{ji} > 0$. The term $f(e_{ij}, e_{ji})$ represents the synergy between the two members of the partnership ij. The assumptions on f imply that the synergy is generated only when both partners put in positive effort into the partnership. We will call a partnership 'active' whenever $e_{ij} > 0, e_{ji} > 0$, i.e. whenever the synergy is positive.

This is a generalisation of the Jackson and Wolinsky (1996) [31] model of co-authorship in two respects. First, in the Jackson and Wolinsky model, effort choice is not endogenous. Each individual chooses an equal amount of effort on each of her links, so that $e_{ij} = \frac{1}{d_i(g)}$ for all $j \in N_i(g)$. Given this specification, their model's linear term in the production function is the same as ours. Second, their synergy term between i and j is simply the product $e_{ij}e_{ji}$, whereas we allow for a more general specification. Our main contribution is to show that for some general specifications of the synergy term (which include the case presented by Jackson and Wolinsky), there will always be an equilibrium of the overall game which supports the efficient outcome. This comes into direct contrast with their seminal inefficiency result and we believe that this difference is due to the endogenous specification of effort choice.[1]

After the two-stage game is played, payoffs are realised for all agents. We

[1] Of course, since effort choice is not endogenous in the Jackson and Wolinsky co-author model, their model is an one-stage model where individuals only choose partners knowing that each individual will equalise effort across partnerships.

assume, similarly to the Jackson-Wolinsky model, that there is an egalitarian rule according to which each partner's utility/payoff from a partnership is equal to the total production of the partnership. In other words, it is assumed that the product of each partnership is a non-rival public good for the two partners but perfectly excludable from everyone else, amounting to no spill-overs across partnerships. An agent's payoff will, therefore, be the sum of the product of all of her partnerships, net of the total cost of link formation she incurs:

$$\Pi_i(s,m) = \sum_{j \in N-\{i\}} F(e_{ij}, e_{ji}) - d_i\bar{\mu} \text{ where } m(g(s)) = e \tag{2.3}$$

We note here that this production function specification includes, through the additive term, the potential for free-riding on the other partner's effort, since one agent receives the non-synergistic product of the other agent even if she does not exert much effort herself in that partnership. However, it also allows for incentives to specialise in a partnership as agents will allocate their effort in order to maximise the total synergistic product they receive from their partnerships.

We next introduce the definition of an equilibrium of the two-stage game:

Definition 12. *The pair of strategy profiles (s^*, m^*) is a Subgame Perfect Bilateral Equilibrium (SPBE) iff:*

(i) For all $i \in N$, for all m_i, and for all s,

$$\Pi_i(s, m^*) \geq \Pi_i(s, m_i, m^*_{-i}). \tag{2.4}$$

(ii) For all $i \in N$, for all s_i,

$$\Pi_i(s^*, m^*) \geq \Pi_i(s_i, s^*_{-i}, m^*). \tag{2.5}$$

(iii) For all pairs of players $\{i,j\} \in N$, for all (s_i, s_j):

$$\Pi_i(s^*, m^*) \geq \Pi_i(s_i, s_j, s^*_{-ij}, m^*) \tag{2.6}$$

or,

$$\Pi_j(s^*, m^*) \geq \Pi_j(s_i, s_j, s^*_{-ij}, m^*). \tag{2.7}$$

As in any subgame perfect equilibrium of a two-stage game, players anticipate that an equilibrium will be played in the second stage of the game. The equilibrium payoffs of the second stage game become the payoffs of the first stage game. Equation 2.4 imposes the requirement that the second stage strategy profile m^* must be a Nash Equilibrium of the second stage game at *all* possible subgames; that is no matter which graph is formed in the first stage, players' choice of effort allocation must be a Nash Equilibrium in that subgame. Equation 2.5 imposes the requirement that no individual has a strictly profitable unilateral deviation in the first stage, assuming that players will use m^* in the second stage. However, Nash Equilibrium has a well-known problem in any network formation game - the empty graph can always be supported as a Nash Equilibrium since bilateral consent is required to form a link. Following much of the literature on network formation, we allow pairs of players to deviate together in the first stage in order to correct for this problem. Equations 2.6 and 2.7 impose the requirement that no pair of individuals should be able to deviate jointly with both of them becoming strictly better off.

2.3 Efficiency and Subgame Perfect Bilateral Equilibria

In this section, we discuss the extent to which Subgame Perfect Bilateral Equilibria can support the efficient outcome(s) of the overall game under various assumptions on the synergy function f.

We first start with a lemma which will be useful subsequently and is also of some independent interest. The lemma shows that every second stage game is actually a *potential* game.[2]

We remind the reader that a normal form game (N, S, Π) is an exact potential game if there is a function $P : S \to R$ such that for all $s \in S$, for all $i \in N$, for all $s'_i \in S_i$,

$$P(s) - P(s'_i, s_{-i}) = \Pi_i(s) - \Pi_i(s'_i, s_{-i}) \tag{2.8}$$

Lemma 1. *For all $g \in G$, the second stage game is an exact potential game.*

[2]See Monderer and Shapley (1996) [34].

Proof. Fix any g. Consider the following candidate potential function:

$$P(g,e) = \sum_{i \in N} \sum_{j>i, j \in N_i(g)} F(e_{ij}, e_{ji}) \tag{2.9}$$

To check that this is indeed a potential, consider strategy profiles m and (m_i', m_{-i}). Let $m(g) = e$ and $m_i'(g) = e_i'$. Then,

$$\begin{aligned} \Pi_i(g, m(g)) - \Pi_i(g, m_i'(g), m_{-i}(g)) &= \sum_{j \in N_i(g)} \left[F(e_{ij}, e_{ji}) - F(e_{ij}', e_{ji}) \right] \\ &= P(g,e) - P(g,e') \end{aligned} \tag{2.10}$$

This shows that P is a potential function and establishes the lemma. □

We use the following definition of efficiency:

Definition 13. *A pair of strategy profiles (s,m) is efficient iff for all $g' \in G$, for all $e' \in E$,*

$$\begin{aligned} \sum_{i \in N} \sum_{j \in N_i(g(s))} F(e_{ij}, e_{ji}) &\geq \sum_{i \in N} \sum_{j \in N_i(g')} F(e_{ij}', e_{ji}') \text{ if } \mu(g) \leq \mu(g') \\ &> \sum_{i \in N} \sum_{j \in N_i(g')} F(e_{ij}', e_{ji}') \text{otherwise.} \end{aligned} \tag{2.11}$$

So, (s,m) is efficient if it maximises *total* output produced across all possible networks and all possible feasible allocations of effort in the second stage. This is, of course, stronger than the notion of Pareto efficiency, but is in line with the literature on networks starting from Jackson and Wolinsky (1996) [31]. The definition of efficiency given here assumes that the cost of link formation is very small. Thus, if $\mu(g) > \mu(g')$, then even a small difference in second-stage output is sufficient to outweigh the cost difference in network formation.

Notice that a pair (g,e) can be efficient only if the vector of effort allocation decisions e maximises output *given* that the network g has formed. Given the form of the potential function P, it is clear that e must then maximise the value of the potential at g. Moreover, we know that if a strategy profile is a pure strategy Nash Equilibrium of a potential game, it is a

stationary point of the potential. [3]

Since the nature of the results will depend on the specification of the synergy function f, it will be convenient to conduct the discussion separately for convex and concave synergy functions.

2.3.1 The Strictly Convex Case

For this section, we suppose that the synergy function f is strictly convex in the effort inputs:

Assumption 2.1: Let the synergy function be $f(e_{ij}, e_{ji}) = z(e_{ij})z(e_{ji})$ with $z(.)$ being increasing and strictly convex.

In this case, we prove that the efficient network is supported as a Subgame Perfect Bilateral Equilibrium. The following theorem provides a formal statement.

Theorem 1. *Let f satisfy Assumption 2.1. Then,*

(i) If n is even, the unique efficient pair of strategy profiles (s, m) must be such that $g(s)$ consists of $\frac{n}{2}$ components, each component containing a pair of individuals. Moreover, $m(g(s)) = e$ with $e_{ij} = 1$ iff $ij \in g(s)$.

(ii) If n is odd, the unique efficient pair of strategy profiles (s, m) must be such that $g(s)$ consists of $\frac{n-3}{2}$ components each containing a pair of individuals, and another minimally connected component $q = \{ij, jk\}$. Moreover, $m(g(s)) = e$ is chosen so that for all $i \in N$, $e_{ij} = 1$ for some $j \in N_i(g)$. [4]

(iii) In all cases above, the efficient pair of strategies is a Subgame Perfect Bilateral Equilibrium.

(iv) Moreover, there cannot be a Subgame Perfect Bilateral Equilibrium which is not efficient.

Proof. Suppose for some $i \in N$ and some feasible effort allocation vector, $\mid J = \{j \mid e_{ij} > 0\} \mid > 1$, $J \subseteq N_i$, $\sum_{j \in J} e_{ij} \leq 1$, and assume for simplicity

[3] An interior Nash Equilibrium will be, for a general f, a stationary point of the potential. Also, there may be other Nash equilibria which are at the boundary of the strategy space, with some neighbours setting $e_{ij} = e_{ji} = 0$.

[4] This is straightforward for all i if $d_i(g) = 1$. For $j \in N(q)$ for whom $d_j(g) = 2$, choose any $l \in N_j(g)$ and set $e_{jl} = 1$.

that $e_{ji} = \overline{e}$ for all $j \in J$. Consider a feasible reallocation of effort such that, for some $j^* \in J$, $e'_{ij^*} = 1$, and $e'_{ij} = 0$ for all other $j \in J$ $j \neq j^*$. Then, from strict convexity of f, it follows that

$$f(1, \overline{e}) > \sum_{j \in J} f(e_{ij}, \overline{e}). \tag{2.12}$$

This immediately establishes that total synergy output is maximised if each i sets $e_{ij} = 1$ for some j where $e_{ji} = 1$. This is possible for all i iff n is even. So, (i) follows since the total cost of link formation is also minimised at the isolated pairs network.

Now, suppose n is odd. It is again easy to check that total synergy output is maximised when pairs of i, j put $e_{ij} = e_{ji} = 1$. This is possible only for $\frac{n-1}{2}$ pairs. Suppose $\bar{\mu} < 1$. Consider the component q where $d_j(q) = 2$. Suppose $e_{jk} = e_{kj} = 1$, $e_{ij} = 1, e_{ji} = 0$. Then, i and j produce no synergistic output but i contributes output of 1 unit. If $\bar{\mu} < 1$, it is efficient for i and j to form the partnership. Hence, (ii) is true.

Suppose (s^*, m^*) is an efficient pair of strategies. If n is even,

$$\Pi_i(s^*, m^*) = 2 + f(1, 1) - \bar{\mu} \text{ for all } i \in N \tag{2.13}$$

It is easy to check that no pair of individuals can deviate and get a strictly higher payoff.

Suppose n is odd. Again, $(n - 2)$ individuals attain the payoff specified in equation 2.13. One individual has two links i.e. is the centre of a line where both partners exert all effort to her but she exerts all effort in one of the two partnerships. Therefore, the central agent attains the payoff:

$$\Pi_i(s^*, m^*) = 3 + f(1, 1) - 2\bar{\mu} > 2 + f(1, 1) - \bar{\mu} \tag{2.14}$$

the second inequality following from $\bar{\mu} < 1$. The remaining individual obtains the payoff $1 - \bar{\mu} > 0$ for $\bar{\mu} < 1$. This agent would profit from a deviation to an isolated pair but there is no other agent who would become strictly better off from deviating with him. This establishes (iii).

To complete the proof of the theorem, suppose (s, m) is a Subgame Perfect Bilateral Equilibrium pair of strategies but is not efficient.

Let g be the network formed in the first stage of the game. We recall that the second stage game is a potential game. Moreover, since f is strictly

convex, the potential is also strictly convex. If g consists of components each containing pairs (i, j), it is straightforward that the unique second stage payoff is maximised at the efficient effort level where $e_{ij} = e_{ji} = 1$.

Suppose now that g is such that for at least two agents i, j

$$\Pi_k(s, m) < 2 + f(1, 1) - \bar{\mu} \text{ for } k = i, j \tag{2.15}$$

Then, i and j can deviate in the first stage, form the bilateral partnership and achieve the payoff given in equation 2.13.

So, at most one agent can fail to attain this payoff.

Suppose that n is even. Note that whatever the structure of g, the upper bound on total synergy output is $\frac{n}{2} f(1, 1)$. Moreover, convexity of f also implies that each i will want to put $e_{ij} = 1$ for some $j \in N_i(g)$ such that $e_{ji} > 0$. It is now straightforward to check that (iv) is true for n even.

Now suppose n is odd. The same argument as before establishes that $\frac{n-3}{2}$ bilateral partnerships must form. This leaves some set $\{i, j, k\}$ of agents. Suppose k is isolated. Our specification of the production function implies that k cannot produce anything unless he is in a partnership (even if his partner does not eventually contribute any effort in the partnership). So, k will form a partnership with either i or j.

But, even when no agent is isolated, convexity implies that the unique Nash Equilibrium must involve specialisation of effort.

This completes the proof of the theorem.

□

This theorem provides a contrast to the incompatibility of efficiency and stability in the co-author model in Jackson and Wolinsky (1996) [31]. Given the difference in the frameworks, it is worth asking whether the difference in results is due to the fact that here individuals can choose effort endogenously or whether the difference can be explained by strict convexity of the synergy function. The results in this section and the next ones suggest that, while the form of the synergy function may play an important role, not allowing the individuals to choose effort allocations is also crucial in precipitating the incompatibility results.

2.3.2 The Jackson and Wolinsky (1996) Co-author model

In this section, we consider the special case where the synergy function has the following form:

$$f(e_{ij}, e_{ji}) = e_{ij}e_{ji} \tag{2.16}$$

In this case, the marginal product of own effort is a linear function of the effort of the other partner. This is of course the case studied by Jackson and Wolinsky (1996) [31] and is particularly interesting in view of the difference in results. We assume in this section that n is even. The following theorem formally states that the network where all agents are connected in pair components, with each agent hence exerting all of their effort endowment in their unique partnership, is the unique efficient and the unique Subgame Perfect Bilateral Equilibrium network.

Theorem 2. *Suppose n is even and the synergy function is $f(e_{ij}, e_{ji}) = e_{ij}e_{ji}$. Then,*

(i) Any efficient pair of strategies (s, m) must be of the form where $g(s)$ consists of $\frac{n}{2}$ components, each component containing a pair i, j and with $e_{ij} = e_{ji} = 1$.

(ii) Any efficient pair of strategies can be supported as a Subgame Perfect Bilateral Equilibrium.

(iii) Moreover, there cannot be a Subgame Perfect Bilateral Equilibrium which is not efficient.

Proof. The proof of (i) is available in Jackson and Wolinsky [31]. So, we prove only (ii) and (iii).

We, first, show (ii) i.e. that the efficient network g of $\frac{n}{2}$ pair components with $e_{ij} = e_{ji} = 1$ is a Subgame Perfect Bilateral Equilibrium.

Consider any efficient profile (s, m) with $g = g(s)$. Suppose $ij \in g$. Clearly, neither i nor j has any unilateral deviation which is profitable. So, consider a bilateral deviation by say i and k where $ik \notin g$. Let $g' = g + ik$ be the new network and consider the component q consisting of i, j, k and l where $ij, kl \in g$.

In the second stage of the game, consider any allocation in g' with $0 < e_{ik} < 1$, $0 < e_{ki} < 1$, hence $0 < e_{ij} < 1$, $0 < e_{kl} < 1$. But we know that $e_{ji} = e_{lk} = 1$. Therefore, agent i has the incentive to unilaterally deviate

by decreasing effort e_{ik} and increasing e_{ij} in order to increase total synergy. These deviation incentives exist until $e_{ij} = 1, e_{ik} = 0$. The same holds for agent k. Therefore, no allocation with positive effort by i, k in all links can be a Nash Equilibrium. Since agents will expect this when making link decisions, the ik link will not be formed. Therefore, the efficient network of pair components is a Subgame Perfect Bilateral Equilibrium.

We next show that the efficient network g is the unique Subgame Perfect Bilateral Equilibrium. The proof of this is very similar to the corresponding statement in the previous theorem.

First, note that any pair of agents i, j can deviate in the first stage of the game, form the bilateral partnership and obtain payoff equal to the output:

$$\Pi = 1 + 1 + e_{ij}e_{ji} - \bar{\mu} = 3 - \bar{\mu} \tag{2.17}$$

So, if (s, m) is to be a Subgame Perfect Bilateral Equilibrium, at most one agent can get a strictly lower payoff. Now take any equilibrium (s, m). Let g be the network formed through s. If g has a component with an odd number of nodes, then there must be at least *two* such components since n is even. It is easy to check that at lest one player gets strictly less payoff than Π in an odd component. But, then there will be two such players since there are at least two such components.

Hence, g can have only even components. Let h be a component of g with $N(h)$ being even. Given $f(e_{ij}, e_{ji}) = e_{ij}e_{ji}$, synergy output in h is maximised when $N(h)$ is partitioned into pairs $\{i, j\}$ with $e_{ij} = e_{ji} = 1$. Note that if h is not a pair, then at least two agents have at least two links, and so pay a linking cost of at least $2\bar{\mu}$. So, the net payoff of at least two agents must be less than Π and hence g cannot be a Subgame Perfect Bilateral Equilibrium. □

This result demonstrates the difference brought about by making the choice of effort endogenous, since the form of the synergy function is identical to that used by Jackson and Wolinsky (1996) [31] in their co-author model. In the latter, i and k gain by forming the link between themselves because they are *committed* to devoting half of their effort to this link when they have only two neighbours. This commitment arises because the individuals cannot choose their allocation of effort, since it is exogenously specified that all individuals divide their endowment equally across all neighbours. However,

equal division of effort is not necessarily a Nash Equilibrium. The proof of the above theorem rests on the simple observation that if j devotes 1 unit to the link with i, and k only 1/2, then i will want to put more effort on the link with j.

2.3.3 The Strictly Concave Case

We next study the case when the synergy function f is strictly concave in the effort inputs. It then follows that the production function F will also be strictly concave.

Assumption 2.2: The synergy function $f(e_{ij}, e_{ji}) = z(e_{ij})z(e_{ji})$ is increasing in each argument, differentiable and strictly concave.

Under this assumption, the following theorem formally states and proves that the unique efficient network, consisting of the complete network with equal effort division among links, is also a Subgame Perfect Bilateral Equilibrium network. This is not, however, unique as there are other, inefficient Subgame Perfect Equilibrium networks.

Theorem 3. *Let F be defined as above and μ be sufficiently small.*

(i) The complete network with equal effort allocation (g_c, e^*) is the unique efficient network.

(ii) The efficient network is supported as a Subgame Perfect Bilateral Equilibrium.

(iii) If $n > 2$, there can be other inefficient Subgame Perfect Bilateral Equilibria.

Proof. Assume that the link formation cost μ is small enough that the saving on the cost from forming less than $n-1$ links does never compensate for the loss in total output or total individual output.

Also, assume $n > 2$ since (i) and (ii) are are obviously true when $n = 2$.

(i) Consider any pair (g, e) where $g \subset g_c$. First, note that there is an effort allocation vector $\tilde{e}$ in g_c such that $\tilde{e}_{ij} = e_{ij}$ for all $ij \in g$. This implies that total output must be maximised in g_c.

Second, we show that total output in g_c is maximised at e^*.

Let e^{**} be the efficient effort allocation vector in g_c. We establish that $e^* = e^{**}$ in two steps.

(a) If e^{**} is interior (that is $e^{**}_{ij} > 0$ for all ij), then $e^{**} = e^*$.

(b) e^{**} must be interior.

To establish (a), note that for all i, $\frac{\partial f(e_{ij},e_{ji}}{\partial e_{ij}} = \frac{\partial f(e_{ik},e_{ki}}{\partial e_{ik}}$ for all $j,k \neq i$ only at e^*. Since the equality of i's marginal productivity across links must be a necessary condition for efficiency in the interior, this establishes (a).

Part (b) follows straightaway from the assumption of strict concavity. For suppose $e_{ij} = e_{ji} = 1$. If there is another pair k,l such that $e_{kl} = e_{lk} = 1$, then choose e' such that each of $\{i,j,k,l\}$ puts in effort $1/2$ on two of the links amongst themselves, such that each of the four edges has effort pairs $(1/2, 1/2)$. Since

$$2f(1/2, 1/2) > f(1,1), \tag{2.18}$$

e cannot be efficient. If only i,j put effort exclusively on one link, then from arguments in part (a) above, all others (and there must be at least three others), equalise effort across all of the remaining $(n-3)$ links. Again, there is a feasible reallocation of effort which increases total output due to the strict concavity of f.

This establishes (i).

(ii) Since the second-period game is a strictly concave potential game, the potential must be maximised at e^*. This must then be a Nash Equilibrium of the second period game given that g_c has formed.

In order to show that there is some (g_c, m^*) with $m^*(g_c) = e^*$ that is a Subgame Perfect Bilateral Equilibrium, we need to show that possible deviations in the first period cannot be profitable.

It is easy to show that unilateral link deletion cannot be profitable for any individual. Suppose i,j deviate and cut some links so that the network g forms. Consider e such that

$$\begin{aligned} e_{kl} &= \frac{1}{n-3} \text{ for all } k,l \notin \{i,j\} \\ e_{pq} &= 1 \text{ for } pq \in \{ij, ji\} \end{aligned} \tag{2.19}$$

Then, e is a Nash Equilibrium, given that g has formed, since no unilateral deviation can increase synergy output. Noting that, from strict concavity,

$$f(1,1) < (n-1)f(\frac{1}{n-1}, \frac{1}{n-1}) \tag{2.20}$$

it is now obvious that (g_c, m^*) is a Subgame Perfect Bilateral Equilibrium;

each deviating pair i, j assumes that the second period equilibrium will be of the form e.

(iii) Inefficient equilibria can be supported as Subgame Perfect Bilateral Equilibria. Consider a pair components network g for n even, with $ij \notin g$. This network results in a total production of $\frac{n(n-1)}{2} F(\frac{1}{n-1}, \frac{1}{n-1})$, which is inefficient. Consider a bilateral deviation where the two agents i, j form a new link. In the new network, there is a (second period) Nash Equilibrium in which $e_{ij} = e_{ji} = 0$. So, the first period bilateral deviation by i, j is not profitable and hence g is a Subgame Perfect Bilateral Equilibrium.

□

In conclusion, even in the existence of inefficient equilibrium networks, the uniquely efficient network and effort provision is always supported in a Subgame Perfect Bilateral Equilibrium of the game. Overall, in this chapter, we have shown, not only that efficiency is always achievable as an equilibrium outcome, but that, for a fairly large family of production functions, stability and efficiency always agree.

Chapter 3

Endogenous Formation of Bilateral Partnerships with Heterogeneous Types

3.1 Introduction

In this section, we look at the case of heterogeneous agent productivity. It is easy to see that, as in Chapter 2, the efficient network is a Subgame Perfect Bilateral Equilibrium. The main purpose of this chapter is to perform comparative statics in order to observe changes in relative specialisation to the high-type partnership, as the relative productivity ratio and the degree of concavity vary. Although we focus on a model with four agents, it is still very difficult to get analytically tractable closed form solutions. That is why we take recourse to simulations that allow us, however, to draw some intuitive conclusions as to agents equilibrium behaviour.

3.2 The Model

We consider a population of four agents $N = \{1, 2, 3, 4\}$. Two agents have a high productivity ($t_i = h$) and two have a low productivity ($t_i = l$), with productivity levels being perfectly observable.

In the first stage of the game, the agents play a simultaneous-move linking-game, identical to the one of Chapter 2, where they form bilateral but non-exclusive partnerships, bearing a small symmetric cost per

link $\bar{\mu}$. Each action profile $s = (s_1, ..., s_4)$ will, therefore, induce a network $g(s) = (g_{ij})_{i,j \in N}$. Keeping the same notation as in Chapter 2 unless otherwise specified, we denote by $G(2,2)$ the set of all possible networks that can be formed by $n_h = 2$ high and $n_l = 2$ low productivity players.

In the second stage, after observing network $g(s)$, agents simultaneously announce their effort provision decisions. An action for player i in the second stage of the game will, therefore, be an effort provision decision $e_i(s)$ among all of i's partnerships, with $e_{ij}(s) \in [0, +\infty)$. Effort allocation decisions, once made, are perfectly observable by all.

Agents face an endogenous quadratic cost of exerting effort: $c_i = \frac{(\sum_{j=1}^{N} e_{ij})^2}{2}$. This captures the fact that exerting an additional unit of effort will have an increasing additional cost, irrespective of the partnership to which this effort is allocated. The cost function is the same for low and high productivity types.

Each partnership $\{ij\}$ results in the production of some output according to the following synergy production function:

$$f(\overline{e}_{ij}, \overline{e}_{ji}) = f(\overline{e}_{ij}\overline{e}_{ji}) = (\overline{e}_{ij}\overline{e}_{ji})^{\gamma} = (t_i e_{ij} t_j e_{ji})^{\gamma}, \tag{3.1}$$

where $\overline{e}$ gives effort in efficiency units and $\gamma > 0$.

We assume that the agents enjoy the full product of each partnership. Therefore, payoffs for each agent in the two-stage game, ignoring the cost of link-formation, will be equal to the sum of the product of each partnership ij, minus the cost of effort provision among all of their partnerships:

$$\Pi_i(\overline{e}_i, \sum_{j \in N-\{i\}} \overline{e}_j) = \sum_{j \in N-\{i\}} (\overline{e}_{ij}\overline{e}_{ji})^{\gamma} - \frac{(\sum_{j \in N-\{i\}} e_{ij})^2}{2}. \tag{3.2}$$

We use the same equilibrium concept used in Chapter 2: A Subgame Perfect Bilateral Equilibrium of the two-stage game will prescribe for each player $i \in N$: (i) a linking strategy s_i, with strategy profile s defining network g, and (ii) a Nash Equilibrium effort strategy $m_i(g)$ in the second stage of the game for any feasible network $g \in G$. At a Subgame Perfect Bilateral Equilibrium, no type of player will have a strictly profitable unilateral deviation in the effort provision stage of any network that can be formed. Moreover, no type of player will be able to strictly increase payoffs by deviating unilaterally in the first stage, assuming that players will play Nash

Equilibrium efforts in the second stage. Finally, no pair of players, consisting of any combination of types, will be able to deviate jointly and be strictly better off.

Before proceeding, however, we will present some benchmark cases that provide some initial intuition related to the more complex comparative statics that will follow in the subsequent sections. As a first benchmark case, and in order to highlight the role of cost convexity in the co-determination of effort provision, consider the complete network where all links are formed and assume instead that the cost was linear, e.g. taking the form $c_i = \sum_{j \in N-\{i\}} \delta e_{ij}$. In this case, the marginal cost of effort provision in each link is constant and equal to δ. Therefore, the effort exerted in each partnership will not be affected by the effort that the agent exerts elsewhere in the network. Agent i, for $\gamma < \frac{1}{2}$, will hence simply choose a high enough effort e_{ij} so that the marginal product of the ij partnership $\gamma e_{ij}^{\gamma-1}(t_i t_j e_{ji})^{\gamma}$ is equated with the fixed marginal cost δ.

But as the fixed marginal cost is the same for all agents, this means that marginal products of a link will also be equal:

$$\gamma e_{ij}^{\gamma-1}(t_i t_j e_{ji})^{\gamma} = \delta = \gamma e_{ji}^{\gamma-1}(t_i t_j e_{ij})^{\gamma} \Rightarrow e_{ij} = e_{ji} \tag{3.3}$$

Therefore, we get that effort provision levels in a partnership will be equal for the two agents, irrespective of their types.

Considering now that there are three types of partnerships as $t_i \in \{h, l\}$, the above means that for linear cost we can easily derive closed form solutions for the three effort levels $\{e_{hh}, e_{ll}, \tilde{e} = e_{lh} = e_{hl}\}$ exerted in the partnerships between two high, two low, and one high and one low type, respectively. In particular, we derive:

$$\{e_{hh} = (\frac{\delta}{\gamma} h^{-2\gamma})^{\frac{1}{2\gamma-1}}, e_{ll} = (\frac{\delta}{\gamma} l^{-2\gamma})^{\frac{1}{2\gamma-1}}, \tilde{e} = (\frac{\delta}{\gamma}(hl)^{-\gamma})^{\frac{1}{2\gamma-1}}\} \tag{3.4}$$

In order to perform comparative statics, an easy manipulation gives:

$$\left\{\tilde{e} = \frac{\gamma}{\delta}^{\frac{1}{1-2\gamma}}(hl)^{\frac{\gamma}{1-2\gamma}}, e_{hh} = \frac{\gamma}{\delta}^{\frac{1}{1-2\gamma}}(h)^{\frac{2\gamma}{1-2\gamma}}, e_{ll} = \frac{\gamma}{\delta}^{\frac{1}{1-2\gamma}}(l)^{\frac{2\gamma}{1-2\gamma}}\right\} \tag{3.5}$$

It is then straightforward to see that an increase in the fixed marginal cost δ will cause all Nash Equilibrium effort provision levels to drop. We

can also observe that the effort exerted by the two agents in a partnership will be higher, the higher the product of their types, i.e. $e_{hh} > \tilde{e} > e_{ll}$, and that these differences become larger, the larger the ratio of productivity of the high type over that of the low type. For example, $\frac{\tilde{e}}{e_{hh}} = (\frac{l}{h})^{\frac{\gamma}{1-2\gamma}} < 1$ for $l < h$ and $\frac{\gamma}{1-2\gamma} > 0$, with the ratio becoming smaller the smaller $\frac{l}{h}$ is. Finally, the effect of an increase in the concavity of the production function, i.e. a drop in γ, on the equilibrium effort levels can be positive or negative depending on the exact values of the parameters.

In order to get some insight on the role of concavity, assume for simplicity that $h > l = 1$ and $\delta = 1$, while we assume still that $\gamma < \frac{1}{2}$. Then the equilibrium effort levels are given by the simplified expressions:

$$\left\{e_{ll} = \gamma^{\frac{1}{1-2\gamma}}, \tilde{e} = \gamma^{\frac{1}{1-2\gamma}} h^{\frac{\gamma}{1-2\gamma}} = e_{ll} h^{\frac{\gamma}{1-2\gamma}}, e_{hh} = \gamma^{\frac{1}{1-2\gamma}} h^{\frac{2\gamma}{1-2\gamma}} = e_{ll} h^{\frac{2\gamma}{1-2\gamma}}\right\} \tag{3.6}$$

We notice that $e_{hh}, \tilde{e}$ are functions of e_{ll} so we start with an examination[1] of $\frac{\partial e_{ll}}{\partial \gamma}$. Using the formula for the derivative of a function of γ in the power of another function of γ, we get:

$$\frac{\partial e_{ll}}{\partial \gamma} = \frac{\gamma^{\frac{1}{1-2\gamma}}}{1-2\gamma}\left[\frac{2\ln\gamma}{1-2\gamma} + \frac{1}{\gamma}\right] \tag{3.7}$$

We can confirm that this derivative has a root $\gamma^* \in (0, \frac{1}{2})$ such that for all $\gamma > \gamma^*$, $\frac{\partial e_{ll}}{\partial \gamma} < 0$ and for all $\gamma < \gamma^*$, $\frac{\partial e_{ll}}{\partial \gamma} > 0$. This means that for concave enough production functions (small enough γ), the derivative is negative; hence a further increase in the degree of concavity by a drop in γ will cause the exerted effort in the ll partnership to increase in equilibrium. On the contrary, for low enough degrees of concavity (large enough γ), the derivative is positive; hence an increase in the degree of concavity by a fall in γ will cause the exerted effort in the ll partnership to fall in equilibrium.

The intuition for this result is the following: whenever the production function becomes more concave, the l-agent has an incentive to drop their exerted effort level. But a drop in the effort of one partner will further incentivise the other to drop their exerted effort further in equilibrium, negatively affecting payoffs. For high enough degrees of concavity, the incentives

[1] In this and all subsequent sections, Wolfram Alpha has been used whenever necessary for calculations and for the production of graphs.

to prevent one's partners from decreasing the effort they exert are dominant, while for low enough degrees of concavity incentives to drop own effort levels due to the increase in concavity dominate instead.

Note, finally, that for a low enough γ, the derivatives of $\tilde{e}$ and e_{hh} also become negative, but the thresholds for this are lower than γ^*, with the threshold for e_{hh} to become negative being the lowest.

As a second benchmark case, assume the effort provision cost is quadratic but the synergy function has $\gamma = 1$, so that the marginal product of effort in any link is linear in the exerted effort of the other partner and equal for i to $b_i b_j e_{ji}$. Then the first order condition for any agents in any connected network is:

$$b_i b_j e_{ji} = \sum_{j \in N-i} e_{ij} \tag{3.8}$$

We can show that under this specification there does not always exist an interior Nash Equilibrium such that agents of symmetric type and network position have symmetric effort provision strategies. As the simplest example, consider the isolated pairs. Taking first order conditions and imposing ex post symmetry, we get $e_{hh} = h^2 e_{hh}$ and $e_{ll} = l^2 e_{ll}$. This is only feasible for $h = l = 1$ i.e. for homogeneous types.

Third, we investigate the case where the production function is of Cobb-Douglas form with constant returns to scale i.e. when $\gamma = \frac{1}{2}$. In this case, we show that there is not a unique Nash Equilibrium for the complete network. Taking first order conditions and imposing ex post symmetry for each type, normalising $h > l = 1$, we get:

$$e_{hh} + 2e_{hl} = \gamma h^{\gamma} e_{hl}^{\gamma-1} e_{lh}^{\gamma} = \gamma h^{2\gamma} e_{hh}^{2\gamma-1} \tag{3.9}$$

$$e_{ll} + 2e_{lh} = \gamma h^{\gamma} e_{lh}^{\gamma-1} e_{hl}^{\gamma} = \gamma e_{ll}^{2\gamma-1} \tag{3.10}$$

For $\gamma = \frac{1}{2}$ these give:

$$e_{lh} = h e_{hl} \tag{3.11}$$

$$\gamma h = e_{hh} + 2e_{hl} \tag{3.12}$$

$$\gamma = e_{ll} + 2e_{lh} \tag{3.13}$$

This is a system of three constraints with four unknowns. It does not have a unique solution but infinitely many: choose any e_{hl}, e_{lh} such that $e_{lh} =$

$he_{hl} \geq 0$, $e_{hl} \leq h\gamma$, $e_{lh} \leq \gamma$. Then there are e_{hh}, e_{ll} that satisfy the above, with $e_{hh} = h\gamma - 2e_{hl} > e_{ll} = \gamma - 2he_{hl}$.

As a fourth special case, we investigate what happens for $\frac{1}{2} < \gamma < 1$ i.e when the production function is not strictly concave but is a Cobb-Douglas with increasing returns to scale. For the first order conditions of the high type, using ex-post symmetry and normalising $h > l = 1$, in the complete network, we get:

$$\frac{\partial(\gamma h^{2\gamma} e_{hh}^{2\gamma-1})}{\partial e_{hh}} = \gamma h^{2\gamma}(2\gamma - 1)e_{hh}^{2(\gamma-1)} > 0. \tag{3.14}$$

This means that if the two high types could coordinate and increase efforts together, the marginal product of the hh link would keep increasing so full specialisation would be optimal. Therefore, a bilateral deviation to the hh isolated pair would be strictly profitable. We can, moreover, show that the Nash Equilibrium effort level in the hh pair would be higher than the equilibrium effort level in an hl isolated pair:

$$e_{hh}^p = (\gamma h^{2\gamma})^{\frac{1}{2(1-\gamma)}} > e_{lh}^p = e_{hl}^p = (\gamma h^{\gamma})^{\frac{1}{2(\gamma-1)}} \tag{3.15}$$

But, starting from any other network, a pair of agents will always have incentives to bilaterally deviate and fully specialise in the link between them. Therefore, we can show that the unique Subgame Perfect Bilateral Equilibrium will be the hh ll isolated pairs network with $e_{hh}^p = (\gamma h^{2\gamma})^{\frac{1}{2(1-\gamma)}}, e_{ll}^p = \gamma^{\frac{1}{2(1-\gamma)}}$.

Finally, for $\gamma > 1$, it is straightforward to see that the marginal product from any link keeps increasing in own effort so it will always exceed the marginal cost:

$$\frac{\partial(\gamma e_{ij}^{\gamma-1}(t_i t_j e_{ji})^{\gamma})}{\partial e_{ij}} = \gamma(\gamma - 1)e_{ij}^{\gamma-2}(t_i t_j e_{ji})^{\gamma} > 0. \tag{3.16}$$

Therefore, in this case, no interior solution exists for any network as all players will exert infinite effort in order to maximise profits.

We will, from now on, focus on the case of strict concavity of f where $\gamma < \frac{1}{2}$ and convex costs as described above. We will investigate whether, and under what conditions, links between high and low productivity agents can arise in equilibrium for the two-stage game. We will also investigate

the determinants of effort provision levels and perform some comparative statics. In particular, we will try to uncover how productivity heterogeneity and concavity affect the effort specialisation incentives of agents in their high, relatively to their, low productivity partners.

3.3 The Second-stage Effort Provision Game

We show that the second stage game admits a strictly concave potential. Therefore, for each network formed in the first stage of the game, there is an interior Nash Equilibrium in effort in the second stage. Moreover, this will be unique for reasons which are exactly the same as in Chapter 2. We proceed to show that the complete network is then the unique efficient network. We, finally, characterise Nash Equilibrium effort provision levels for the complete network and the pairs network, and perform comparative statics on the productivity heterogeneity and the concavity parameters.

3.3.1 Efficiency

In order to examine efficiency of Nash Equilibrium effort provision, we will use the same in essence definition of efficiency that was used in Chapter 2 - namely (g, e) is efficient if the total output minus the cost of effort[2] and cost of network formation is maximised.

We begin our analysis by proving that the complete information second stage game is an exact potential game, as defined in Chapter 2.

Lemma 2. *For all $g \in G$, the second stage game under perfect information is an exact potential game.*

Proof. Fix any g. Consider the following candidate potential function:

$$P(g, e) = \sum_{i \in N} \left[\sum_{j > i, j \in N_i} f(\bar{e_{ij}}, \bar{e_{ji}}) - \frac{(\sum_{j \in N - \{i\}} e_{ij})^2}{2} \right] \tag{3.17}$$

To check that this is indeed a potential, consider strategy profiles m and

[2]In Chapter 2, there was no cost of effort since everyone had a fixed endowment.

(m'_i, m_{-i}). Let $m(g) = e$ and $m'_i(g) = e'_i$. Then,

$$\Pi_i(g, m(g)) - \Pi_i(g, m'_i(g), m_{-i}(g)) = \sum_{j \in N-\{i\}} \left[f(\bar{e}_{ij}, \bar{e}_{ji}) - f(\bar{e'}_{ij}, \bar{e}_{ji}) \right]$$
$$- \left[\frac{(\sum_{j \in N-\{i\}} e_{ij})^2}{2} - \frac{(\sum_{j=1}^{N} e'_{ij})^2}{2} \right] = P(g, e) - P(g, e') \tag{3.18}$$

This shows that P is a potential for this game and establishes the lemma.

□

We state without proof the following theorem, which proves that there is a unique efficient network that consists of the complete network in its unique interior Nash equilibrium in effort.[3]

Theorem 4. *Consider the complete information exact potential game and let $\bar{\mu}$ be sufficiently small in the first stage linking game. Then,*

(i) There is a unique interior Nash Equilibrium e^* in the complete network g_c where players allocate effort to equalise marginal synergy of links with marginal cost of effort.

(ii) The network (g_c, e^*) is the unique efficient network.

3.3.2 Effort provision in the Complete network with concave production and quadratic cost

In this section, we analyse how the Nash Equilibrium of the complete network changes as we increase the productivity ratio $b := h/l = h$ for l normalised to 1, and the degree of concavity $\gamma < \frac{1}{2}$. In the unique interior Nash Equilibrium of the complete network with 2 high and 2 low types, each type will play a symmetric strategy by choosing effort levels in each link such that the marginal product of own effort in each of their links is equal, and equal to the marginal cost. The Nash Equilibrium will, therefore, be described by four effort levels $(e_{hl}, e_{hh}, e_{lh}, e_{ll})$ [4] which solve the following system of equations:

[3] The proof is omitted because it is almost identical to that of the corresponding theorem in Chapter 2.

[4] For homogeneous types, $h = l = 1$ and in the unique interior Nash Equilibrium $e_i^{NE} = (\frac{\gamma}{|N|-1})^{\frac{1}{2(1-\gamma)}}$ $\forall i \in N$, i.e. increasing in γ and decreasing in N.

$$2e_{hl} + e_{hh} = \gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1} \tag{3.19}$$

$$2e_{hl} + e_{hh} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} \tag{3.20}$$

$$2e_{lh} + e_{ll} = \gamma b^{\gamma} e_{hl}^{\gamma} e_{lh}^{\gamma-1} \tag{3.21}$$

$$2e_{lh} + e_{ll} = \gamma e_{ll}^{2\gamma-1} \tag{3.22}$$

Dividing (3.19) by (3.21), and manipulating (3.20) and (3.22), we get:

$$\frac{2e_{hl} + e_{hh}}{2e_{lh} + e_{ll}} = \frac{\gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1}}{\gamma b^{\gamma} e_{hl}^{\gamma} e_{lh}^{\gamma-1}} = \frac{e_{lh}}{e_{hl}} \tag{3.23}$$

$$2e_{hl} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} - e_{hh} \Rightarrow e_{hl} = \frac{1}{2} e_{hh} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1] \tag{3.24}$$

$$2e_{lh} = \gamma e_{ll}^{2\gamma-1} - e_{ll} \Rightarrow e_{lh} = \frac{1}{2} e_{ll} [\gamma e_{ll}^{2(\gamma-1)} - 1] \tag{3.25}$$

Using (3.24) and (3.25), equation (3.23) becomes:

$$\frac{\gamma b^{2\gamma} e_{hh}^{2\gamma-1} - e_{hh} + e_{hh}}{\gamma e_{ll}^{2\gamma-1} - e_{ll} + e_{ll}} = \frac{\frac{1}{2} e_{ll} [\gamma e_{ll}^{2(\gamma-1)} - 1]}{\frac{1}{2} e_{hh} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]} \Rightarrow$$

$$\frac{b^{2\gamma} e_{hh}^{2\gamma}}{e_{ll}^{2\gamma}} = \frac{\gamma e_{ll}^{2(\gamma-1)} - 1}{\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1} \Rightarrow$$

$$e_{hh}^{2\gamma} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1] = (\frac{1}{b})^{2\gamma} e_{ll}^{2\gamma} [\gamma e_{ll}^{2(\gamma-1)} - 1] \tag{3.26}$$

Combining (3.19) and (3.20), we get:

$$\gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} \Rightarrow e_{lh}^{\gamma} e_{hl}^{\gamma-1} = b^{\gamma} e_{hh}^{2\gamma-1} \tag{3.27}$$

which, using (3.24) and (3.25), gives:

$$b^{\gamma} e_{hh}^{2\gamma-1} = (\frac{1}{2} e_{hh} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1])^{\gamma-1} (\frac{1}{2} e_{ll} [\gamma e_{ll}^{2(\gamma-1)} - 1])^{\gamma} \Rightarrow$$

$$e_{hh}^{2\gamma-1} = (\frac{1}{b})^{\gamma} (\frac{1}{2})^{\gamma+\gamma-1} e_{hh}^{\gamma-1} e_{ll}^{\gamma} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{\gamma-1} [\gamma e_{ll}^{2(\gamma-1)} - 1]^{\gamma} \Rightarrow$$

$$\frac{e_{hh}^{\gamma}}{[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{\gamma-1}} = (\frac{1}{b})^{\gamma} (\frac{1}{2})^{2\gamma-1} e_{ll}^{\gamma} [\gamma e_{ll}^{2(\gamma-1)} - 1]^{\gamma} \Rightarrow$$

$$\frac{e_{hh}^{2\gamma}}{[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2(\gamma-1)}} = (\frac{1}{b})^{2\gamma}(\frac{1}{2})^{2(2\gamma-1)} e_{ll}^{2\gamma}[\gamma e_{ll}^{2(\gamma-1)} - 1])^{2\gamma}. \tag{3.28}$$

Equations (3.26) and (3.28) then provide a 2×2 system in e_{hh}, e_{ll}. Combining them, by dividing (3.26) with (3.28), we get:

$$\frac{e_{hh}^{2\gamma}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1][\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2(\gamma-1)}}{e_{hh}^{2\gamma}} = (\frac{1}{2})^{2(1-2\gamma)}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma} \Rightarrow$$

$$[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2\gamma-1} = (\frac{1}{2})^{2(1-2\gamma)}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma} \Rightarrow$$

$$\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1 = \left[(\frac{1}{2})^{2(1-2\gamma)}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma}\right]^{1/(2\gamma-1)}$$

$$\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1 = [2^2[\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1}] \tag{3.29}$$

Substituting (3.29) back into (3.26), we get e_{hh} as a function of e_{ll}:

$$e_{hh}^{2\gamma}[2^2[\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1}] = (\frac{1}{b})^{2\gamma} e_{ll}^{2\gamma}[\gamma e_{ll}^{2(\gamma-1)} - 1] \Rightarrow$$

$$e_{hh} = (\frac{1}{2})^{\frac{1}{\gamma}} \frac{1}{b} e_{ll}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{\frac{1}{\gamma}} \tag{3.30}$$

Finally, using constraint (3.30) to substitute for e_{hh} in constraint (3.29), we get an expression only in e_{ll}:

$$\gamma b^{2\gamma}\left[\frac{1}{b}(\frac{1}{2})^{\frac{1}{\gamma}} e_{ll}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{\frac{1}{\gamma}}\right]^{2(\gamma-1)} - 1 = [2^2[\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1}] \Rightarrow$$

$$1 = 2^2[\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1}[\gamma b^2 2^{\frac{2(1-2\gamma)}{\gamma}} e_{ll}^{2(\gamma-1)}[\gamma e_{ll}^{2(\gamma-1)} - 1]^{\frac{3\gamma-2}{\gamma}} - 1] \tag{3.31}$$

In order to simplify [5] constraint (3.31), we can set $y := \gamma e_{ll}^{2(\gamma-1)} - 1$ to get:

$$\frac{1}{4} = \left[b^2 2^{\frac{2(1-2\gamma)}{\gamma}}(y+1)y^{\frac{2(\gamma-1)}{\gamma}} - \frac{1}{y}\right] \tag{3.32}$$

or

$$1 = \left[b^2 2^{\frac{2(1-\gamma)}{\gamma}} \frac{y+1}{y^{\frac{2(1-\gamma)}{\gamma}}} - \frac{4}{y}\right] \tag{3.33}$$

[5] Wolfram Mathematica has been used in this section.

Notice that constraint (3.32) is a polynomial equation with unknown power $\alpha := \frac{2(1-\gamma)}{\gamma}$, $\gamma < 1/2$. There is, therefore, no general solution that would provide us with a closed form solution for e_{ll}. What we can do instead is form the implicit derivative $\frac{\partial y}{\partial b}$ and study its sign for various values of the concavity parameter γ.

From equations (3.24), (3.25) and (3.29) above, we can make the following interesting observation for the equilibrium effort ratios $\frac{e_{lh}}{e_{ll}}$ and $\frac{e_{hh}}{e_{hl}}$:

$$\frac{e_{lh}}{e_{ll}} = \frac{1}{2}[\gamma e_{ll}^{2(\gamma-1)} - 1] = \frac{y}{2} \tag{3.34}$$

$$\frac{e_{hh}}{e_{hl}} = \frac{2}{[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]} = \frac{2}{[2^2[\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1}]} = \frac{y}{2} \tag{3.35}$$

Therefore, we have shown the following:

Proposition 1. *In the unique interior Nash Equilibrium of the complete network, the relative effort specialisation ratios between high and low type neighbours will be equal for the two types:*

$$\frac{e_{lh}}{e_{ll}} = \frac{e_{hh}}{e_{hl}} \tag{3.36}$$

As b changes, notice that the ratios will be changing in equilibrium according to:

$$\frac{\partial(\frac{e_{lh}}{e_{ll}})}{\partial b} = \frac{\partial(\frac{e_{hh}}{e_{hl}})}{\partial b} = \frac{1}{2}\frac{\partial y}{\partial b}. \tag{3.37}$$

Although it is impossible to get a closed-form solution for y, we can use (3.33) to evaluate the derivatives of y with respect to b and γ. Since (y, b) are positive for $\gamma < \frac{1}{2}$, $\alpha := \frac{2(1-\gamma)}{\gamma} > 2$, (3.33) can be put in the following useful alternative form:

$$(y+4)y^{\alpha} = 2^{\alpha} b^2 y(y+1) \tag{3.38}$$

The derivatives then are:

$$\frac{\partial y(b,\alpha)}{\partial b} = \frac{2^{1+\alpha} b y^2 (1+y)}{2^{\alpha}\alpha b^2 y - 2^{\alpha} b^2 y^2 + 2^{\alpha}\alpha b^2 y^2 - 4y^{\alpha}} \tag{3.39}$$

and

$$\frac{\partial y(b,\alpha)}{\partial \alpha} = \frac{2^{\alpha} b^2 y^2 (1+y)(\ln(2) - \ln(y))}{2^{\alpha}\alpha b^2 y - 2^{\alpha} b^2 y^2 + 2^{\alpha}\alpha b^2 y^2 - 4y^{\alpha}} \tag{3.40}$$

with $\frac{\partial \frac{e_{hh}}{e_{hl}}}{\partial b} = \frac{1}{2}\frac{\partial y(b,\alpha)}{\partial b}$ and $\frac{\partial \frac{e_{hh}}{e_{hl}}}{\partial \gamma} = \frac{1}{2}\frac{\partial y}{\partial \gamma} = \frac{1}{2}\frac{\partial y}{\partial \alpha}\frac{\partial \alpha}{\partial \gamma} = \frac{-1}{\gamma^2}\frac{\partial y}{\partial \alpha}$.

The numerator of $\frac{\partial y(b,\alpha)}{\partial b}$ is clearly positive for all values of (b, α) while the numerator of $\frac{\partial y(b,\alpha)}{\partial \alpha}$ will be positive for $y > 2$, negative for $y < 2$ and zero for $y = 2$. [6]

The denominator of both implicit partial derivatives is the same and can be simplified to $2^\alpha b^2(\alpha y - y^2 + \alpha y^2) - 4y^\alpha$. However, no general conclusion can be made about its sign as it seems dependent on the values of b, α and y i.e. equilibrium e_{ll}.

We can, nevertheless, investigate the change in the effort ratios (captured by the change in $y = 2\frac{e_{hh}}{e_{hl}} = 2\frac{e_{lh}}{e_{ll}}$), using the constraint: $(y+4)y^\alpha = 2^\alpha b^2 y(y+1)$, by fixing b and taking various values for the concavity parameter $\gamma < \frac{1}{2}$, and by fixing γ and taking various values for the productivity ratio parameter b.

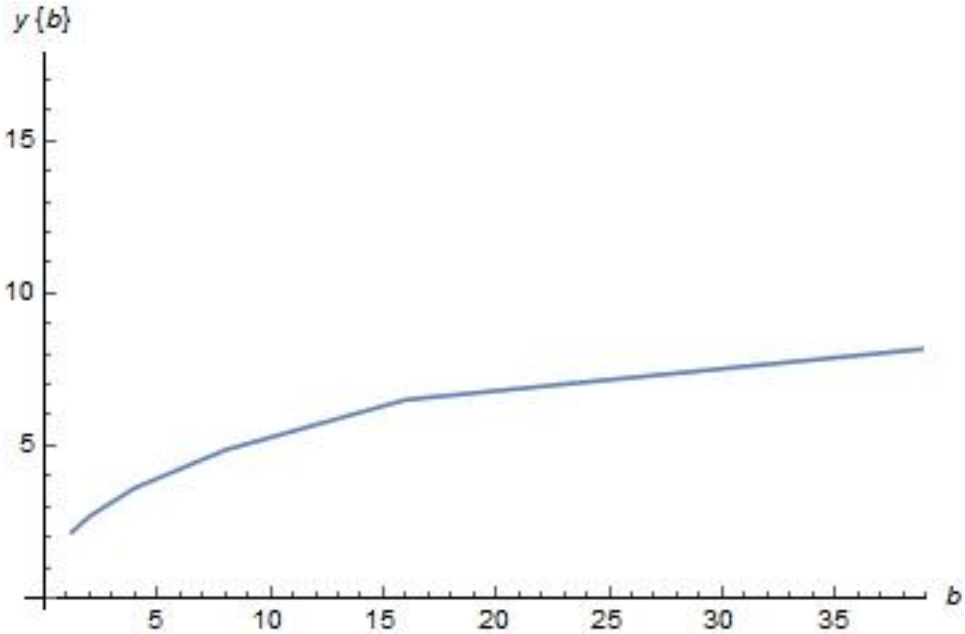

Figure 3.1: $\gamma = 0.25$

(i) $\gamma = 1/4, b \in \{1.2, 2, 4, 8, 16, 160\}$

For $\gamma = 1/4$, $\alpha = \frac{2(1-\gamma)}{\gamma} = 6$ hence $(y+4)y^6 = 2^6 b^2 y(y+1)$ and $\frac{\partial y(b)}{\partial b} = \frac{-(128by(1+y))}{64b^2(1+2y)-y^5(24+7y)}$.

[6] $y = 2$ corresponds to all effort levels being equal in the Nash Equilibrium and can only intuitively result for $b = 1$.

The unique positive real solution is: for $b = 1.2$ $y = 2.16254$; for $b = 2$ $y \approx 2.69143$; for $b = 4$ $y \approx 3.61901$; for $b = 8$ $y \approx 4.85915$; for $b = 16$ $y \approx 6.51174$; and for $b = 160$ $y \approx 16.9617$.

Evaluating the implicit derivative for some of these values, we get that $\frac{-(128by(1+y))}{64b^2(1+2y)-y^5(24+7y)}$ is approximately equal to: 0.566146 for $(b = 2, y = 2.7)$; 0.385122 for $(b = 4, y = 3.62)$; 0.173025 for $(b = 16, y = 6.5)$; 0.0436069 for $(b = 160, y = 16.96)$; and 0.000693128 for $(b = 160,000)$.

(ii) $\gamma = 1/8, b \in \{1.2, 2, 4, 8, 16, 160\}$

For $\gamma = 1/8$, $\alpha = \frac{2(1-\gamma)}{\gamma} = 14$ hence $(y+4)y^{14} = 2^{14}b^2y(y+1)$ and $\frac{\partial y(b)}{\partial b} = \frac{-(32768by(1+y))}{16384b^2(1+2y)-y^13(56+15y)}$.

The unique positive real solution is: for $b = 1.2$ $y \approx 2.05841$; for $b = 2$ $y \approx 2.23131$; for $b = 4$ $y \approx 2.48933$; for $b = 8$ $y \approx 2.77711$; for $b = 16$ $y \approx 3.09802$; and for $b = 160$ $y \approx 4.45284$.

Evaluating the implicit derivative for some of these values, we get that $\frac{-(32768by(1+y))}{16384b^2(1+2y)-y^13(56+15y)}$ is approximately equal to: 0.177532 for $b = 2$; 0.0302784 for $b = 16$; and 0.00441718 for $b = 160$.

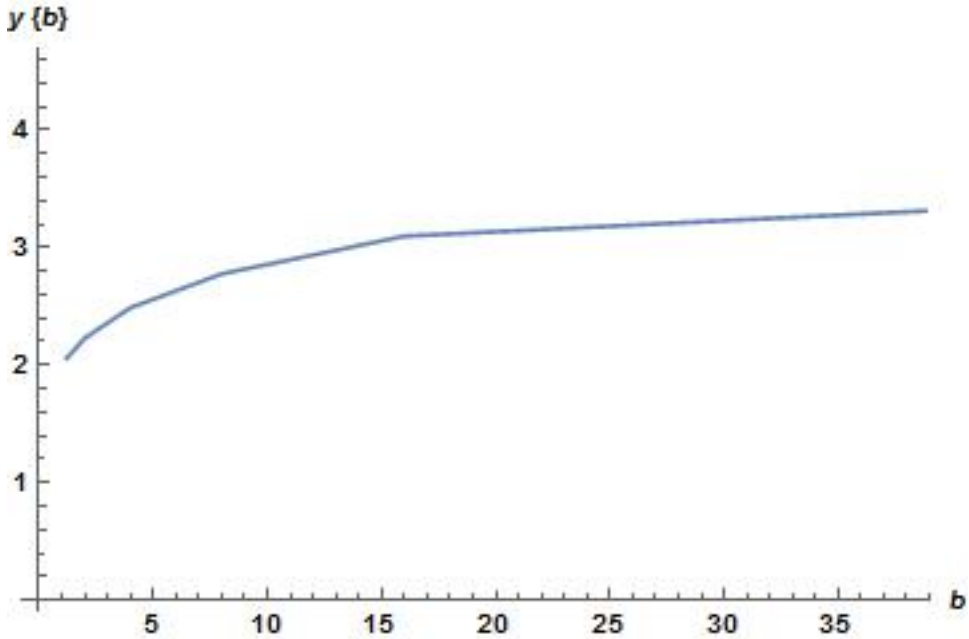

Figure 3.2: $\gamma = 0.125$

(iii) $\gamma = 1/32, b \in \{1.2, 2, 4, 8, 16, 160\}$

For $\gamma = 1/32$, $\alpha = \frac{2(1-\gamma)}{\gamma} = 62$ hence $(y+4)y^{62} = 2^{62}b^2y(y+1)$.

The unique positive real solution is: for $b = 1.2$ $y \approx 2.01206$; for $b = 2$ $y \approx 2.04623$; for $b = 4$ $y \approx 2.09352$; for $b = 8$ $y \approx 2.14191$; for $b = 16$ $y \approx 2.19142$; and for $b = 160$ $y \approx 2.36424$.

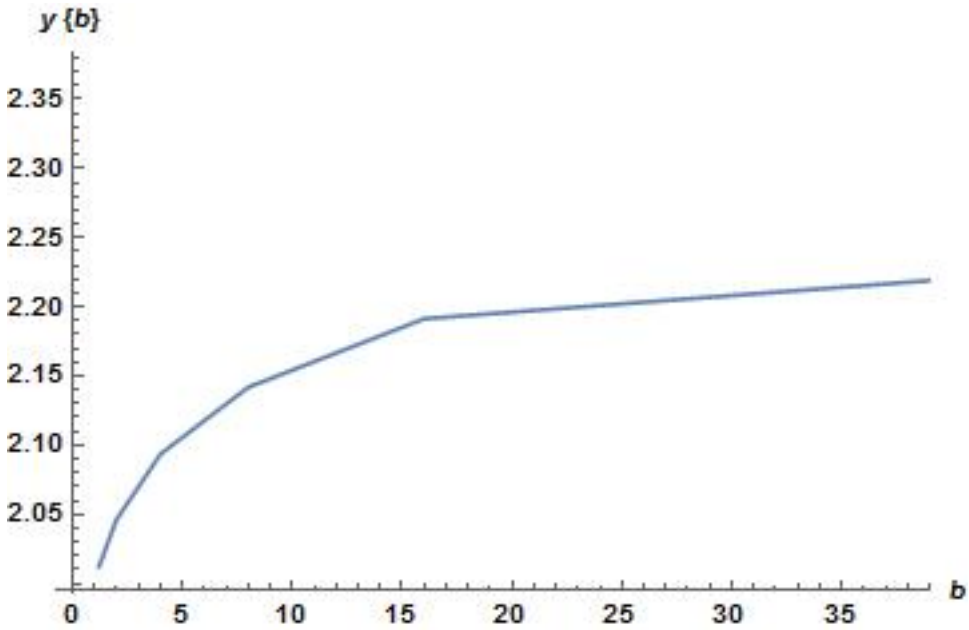

Figure 3.3: $\gamma = 0.03125$

We can also observe that the absolute value of the effort ratios is falling, for the same productivity heterogeneity parameter b, as the production function becomes more concave i.e. for lower γ. This should become more evident by studying the effort ratios and the derivative with respect to α, for example, for $b = 2$ and $\gamma \in \{\frac{2}{5}, \frac{1}{8}, \frac{1}{40}\}$. By evaluating the implicit derivative of y with respect to α for $b = 2$, we get that it is equal to -2.19431 for $\alpha = 3$, -0.0182697 for $\alpha = 14$, and -0.000478098 for $\alpha = 78$.

We next produce a similar series of diagrams, by taking the value of b as fixed and varying the degree of concavity γ.

(iv) $b = 2, \gamma \in \{0.25, 0.125, 0.3125, 0.025\}$

Fix $b = 2$. Then, the unique positive real solution is: for $\gamma = 1/4$, $\alpha = 6$, $y \approx 4.5589$, for $\gamma = 1/8$, $\alpha = 14$, $y \approx 2.23131$; for $\gamma = 1/32$, $\alpha = 62$, $y \approx 2.04623$; for $\gamma = 1/40$, $\alpha = 78$, $y \approx 2.03649$.

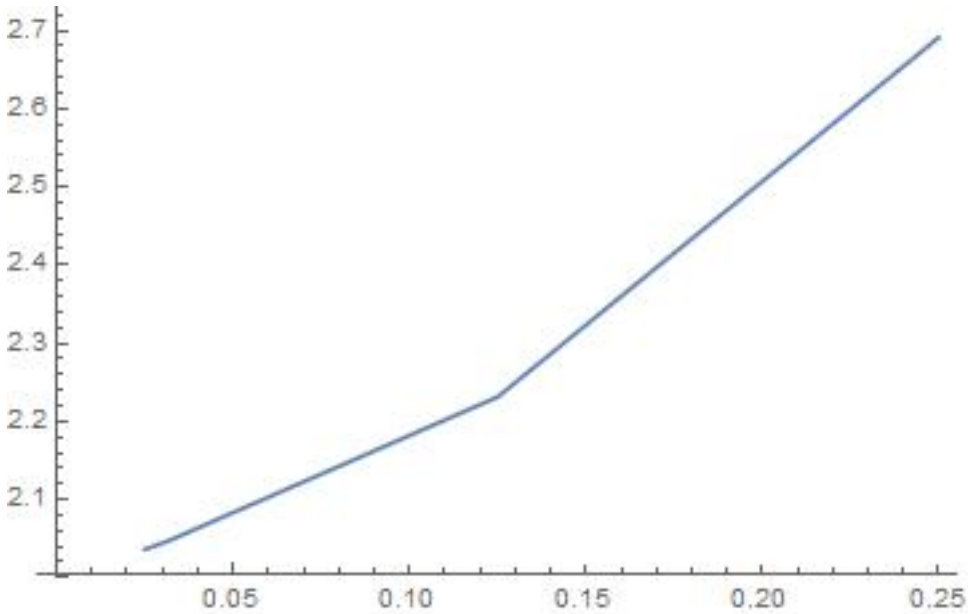

Figure 3.4: $b = 2$

(v) $b = 2, \gamma \in \{0.25, 0.125, 0.3125, 0.025\}$

Fix $b = 8$. Then, the unique positive real solution is: for $\gamma = 1/4$, $\alpha = 6$, $y \approx 4.85915$, for $\gamma = 1/8$, $\alpha = 14$, $y \approx 2.77711$; for $\gamma = 1/32$, $\alpha = 62$, $y \approx 2.14191$; for $\gamma = 1/40$, $\alpha = 78$, $y \approx 2.11149$.

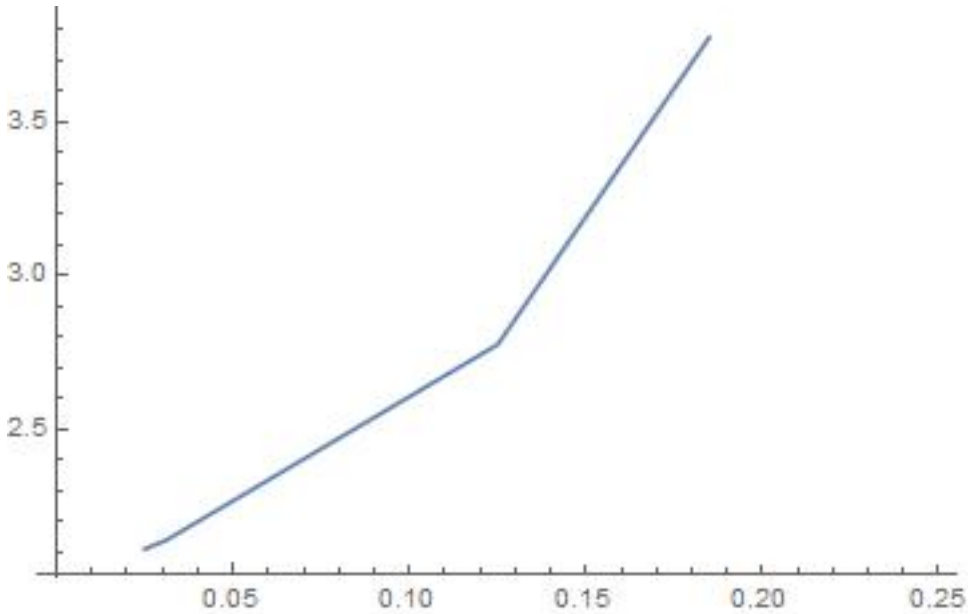

Figure 3.5: $b = 8$

(vi) $b = 2, \gamma \in \{0.25, 0.125, 0.3125, 0.025\}$

Fix $b = 160$. Then, the unique positive real solution is: for $\gamma = 1/4$, $\alpha = 6$, $y \approx 16.9617$, for $\gamma = 1/8$, $\alpha = 14$, $y \approx 4.45284$; for $\gamma = 1/32$, $\alpha = 62$, $y \approx 2.36424$; for $\gamma = 1/40$, $\alpha = 78$, $y \approx 2.28312$.

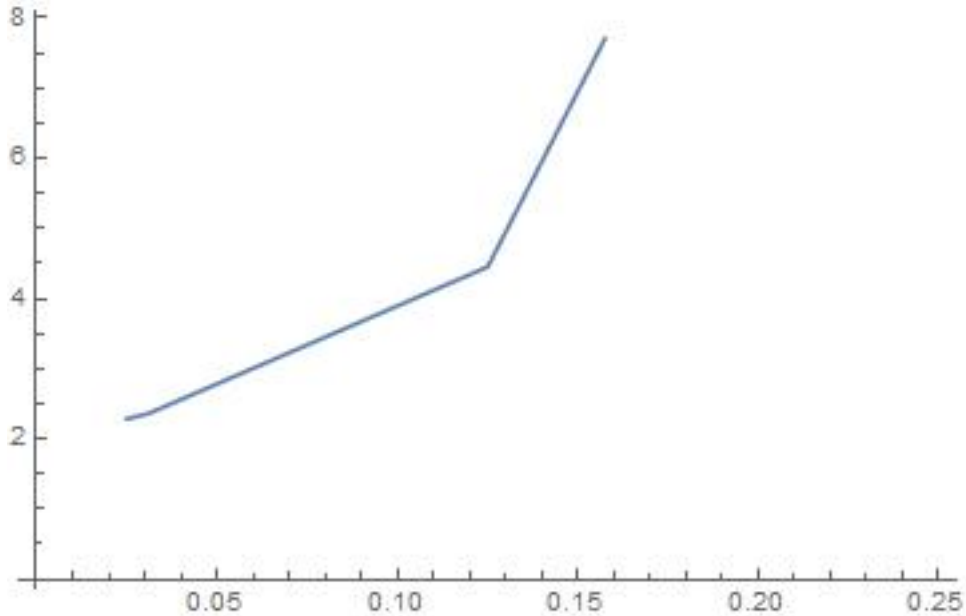

Figure 3.6: $b = 160$

Therefore, the numerical approximation results of this section suggest that $y(b)$ is an increasing, concave function. We can hence make the following statement:

Statement 1. *(i) The equilibrium high-over-low effort ratio is increasing as the productivity ratio $h/l = b$ increases. However, the rate of increase is falling as b increases.*

(ii) For fixed productivity ratio b, the equilibrium high-over-low effort ratio is falling as the production function becomes more concave i.e. as γ falls. This rate of decrease falls as the production function becomes more concave, and approaches zero for highly concave production functions.

Finally, with regards to total effort as b changes, for the total effort of the low type TE_l we get from (3.22) that:

$$\frac{\partial TE_l}{\partial b} = \gamma(2\gamma - 1)e_{ll}^{2(\gamma-1)}\frac{\partial e_{ll}}{\partial b} \tag{3.41}$$

so, as b changes, TE_l and e_{ll} will change in opposite directions, since $(2\gamma - 1) < 0$ and all other terms are positive. Moreover, from the definition of y we get:

$$\frac{\partial y}{\partial b} = \gamma 2(\gamma - 1)e_{ll}^{2\gamma-3}\frac{\partial e_{ll}}{\partial b} \tag{3.42}$$

which shows that y and e_{ll} change in opposite directions as b changes. Therefore, we reach the following conclusion:

Statement 2. *As the productivity ratio b increases, y increases so e_{ll} falls but the Nash Equilibrium total effort of the low type TE_l will increase overall: $\frac{\partial TE_l}{\partial b} > 0$. Therefore, Nash Equilibrium e_{lh} increases as b increases.*

However, from (3.20) and (3.30), we can easily conclude that the derivative of total effort of the high type with respect to b is not necessarily monotonic. As b increases, it is intuitive that e_{hh} increases but e_{hl} and TE_h may fall as the high-type needs to exert less real effort to be as efficient as before, achieving the same results with a lower cost.

The intuition of this section is extendable to larger networks e.g. the complete network consisting of equal and even numbers of high and low productivity agents.[7]

[7]For linear cost, it is easy to show that the Nash Equilibrium high type effort specialisation ratio is increasing in high type productivity b, tends to one as $\gamma \to 0$ and tends to infinity as $\gamma \to \frac{1}{2}$.

3.4 The Linking Game

In this section, we turn to the first stage linking game. We prove existence and investigate uniqueness of a Subgame Perfect Bilateral Equilibrium under our assumptions for concave production and quadratic cost of effort. In particular, we show that the efficient network and effort allocation strategy profile are a Subgame Perfect Bilateral Equilibrium of the game. [8]

3.4.1 Existence and Efficiency of Subgame Perfect Bilateral Equilibria

We are going to show that the complete network g^c with the efficient effort allocation profile is a Subgame Perfect Bilateral Equilibrium of the game by showing that no unilateral or bilateral deviation is strictly profitable for any type(s) of players.

First, we consider an auxiliary network, the circular network where the high and low types are connected and each has one *hl* link. We will refer to this as the *hhll* circle g^{cir} and show that a bilateral deviation to a pair is not profitable for either the high or the low types. In appendix A, we are able to derive closed form solutions for the *hhll* circle, which we then use for the proofs of the following propositions.

We here provide a graph of the hhll circle and the associated Nash Equilibrium effort levels.

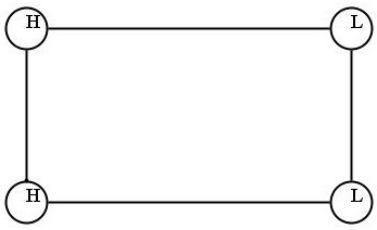

Figure 3.7: *hhll* circle

Using $y = [\gamma e_{ll}^{2(\gamma-1)} - 1] = b^{\frac{2\gamma}{2-3\gamma}}$, the Nash Equilibrium effort levels $e_{hh}, e_{hl}, e_{lh}, e_{ll}$ are:

$$e_{ll} = \gamma^{\frac{1}{2(1-\gamma)}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \tag{3.43}$$

[8] For concave production and linear cost, it is trivial to show the unique Subgame Perfect Bilateral Equilibrium network is the efficient network.

$$e_{lh} = ye_{ll} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{2\gamma}{2-3\gamma}} (1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \tag{3.44}$$

$$e_{hh} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{3\gamma}{2-3\gamma}} (1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \tag{3.45}$$

$$e_{hl} = \frac{e_{hh}}{y} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{\gamma}{2-3\gamma}} (1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \tag{3.46}$$

Proposition 2. *A deviation from the hhll circle to the hh pair is not profitable.*

Proof. See Appendix B. □

Proposition 3. *A deviation from the hhll circle to the ll pair is not profitable.*

Proof. See Appendix C. □

The following then directly follows:

Corollary 1. *A deviation from the hhll circle to the hl pair is not profitable for the high type.*

We, next, use the above two propositions to show that a deviation to the *hh* pair or the *ll* pair is not profitable when starting from the complete network g^c.

Proposition 4. *A deviation from the complete network to the hh pair or the ll pair is not profitable.*

Proof. It suffices to show that the high types and the low types make even higher payoffs in the complete network than in the *hhll* circle.

Consider an effort allocation $\tilde{e}$ in the complete network where $e^c_{hh} = e^{cir}_{hh}$, $e^c_{ll} = e^{cir}_{ll}$ and $e^c_{hl} = \frac{e^{cir}_{hl}}{2}$, $e^c_{lh} = \frac{e^{cir}_{lh}}{2}$ so that total effort and hence total cost is the same for both types between the two networks.

Total production, however, strictly increases due to the concavity of the production function:

$$f(e_{hl}, e_{lh}) = f\left(2\frac{e_{hl}}{2}, 2\frac{e_{lh}}{2}\right) = 2^{2\gamma} f\left(\frac{e_{hl}}{2}, \frac{e_{lh}}{2}\right) < 2f\left(\frac{e_{hl}}{2}, \frac{e_{lh}}{2}\right) \tag{3.47}$$

for $\gamma < \frac{1}{2}$. Therefore, total network payoff is strictly higher for the complete network under $\tilde{e}$ than in the Nash Equilibrium of the *hhll* circle.

Due to the unique global maximum of the potential, we, moreover, know that the Nash Equilibrium allocation e_c^* of the complete network will result in an even higher total network payoff than allocation $\tilde{e}$:

$$\sum_{i \in N} \Pi_i^c(e_{NE}^c) > \sum_{i \in N} \Pi_i^c(\tilde{e}). \tag{3.48}$$

It remains to be shown that individual payoffs, both for the high and the low types, are strictly higher in (g^c, e_{NE}^c) than in $g^c, \tilde{e}$ i.e. that:

$$\Pi_h^c(e_{NE}^c) > \Pi_h^c(\tilde{e}) > \Pi_h^{cir} > \Pi_h^{pair} \tag{3.49}$$

$$\Pi_l^c(e_{NE}^c) > \Pi_l^c(\tilde{e}) > \Pi_l^{cir} > \Pi_l^{pair} \tag{3.50}$$

Using the first order conditions of the *hhll* circle, it is clear that allocation $\tilde{e}$ is not a Nash Equilibrium allocation in the complete network: The marginal cost of the high agent, for example, will be equal for $\tilde{e}$ in the complete network and the Nash Equilibrium allocation e_{cir}^* of the circle. This is, in turn, equal to the Nash Equilibrium marginal products for the high type from the *hh* and *hl* links in the circle. Since the allocated efforts in the *hh* link are the same under $\tilde{e}$ in g^c and under the Nash Equilibrium of g^{cir}, these marginal products will also be equal. However, the marginal product of the *hl* link in g^c under $\tilde{e}$ will be lower, since both effort levels have been halved. This is shown in the following:

$$\begin{aligned} MC_h^c(\tilde{e}) &= MC_h^{cir}(e_{cir}^*) = e_{hl} + e_{hh} = MP_{hl}^{cir}(e_{cir}^*) = \gamma b^\gamma e_{lh}^\gamma e_{hl}^{\gamma-1} \\ &= MP_{hh}^{cir}(e_{cir}^*) = MP_{hh}^c(\tilde{e}) = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} \\ &< MP_{hl}^c(\tilde{e}) = \gamma b^\gamma (\frac{e_{lh}}{2})^\gamma (\frac{e_{hl}}{2})^{\gamma-1} = \gamma b^\gamma 2^{1-2\gamma} e_{lh}^\gamma e_{hl}^{\gamma-1} \end{aligned} \tag{3.51}$$

since $\gamma < \frac{1}{2}$.

Therefore, to sum this up, in the complete network, the high type under allocation $\tilde{e}$ has:

$$MC_h^c(\tilde{e}) = MP_{hh}^c(\tilde{e}) < MP_{hl}^c(\tilde{e}) \tag{3.52}$$

with the left hand side decreasing in e_{hh} and the right hand side decreasing

in e_{hl}. Similarly, for the low type:

$$MC_l^c(\tilde{e}) = MP_{ll}^c(\tilde{e}) < MP_{lh}^c(\tilde{e}) \tag{3.53}$$

Therefore, a unilateral deviation from $\tilde{e}$ is strictly profitable where the high types increase e_{hl} and drop e_{hh}, which causes an increase in MP_{hh}^c and a fall in MP_{lh}^c. Similarly, the low types have an incentive to unilaterally deviate by increasing e_{lh} and dropping e_{ll}.

From the definition of the production function, efforts are strategic complements for all partners: $\frac{\partial^2 \Pi_i}{\partial e_{ij} \partial e_{ji}} > 0$ for all types. Therefore, unilateral deviations will reinforce each other and hence will all move towards the same direction; an initial increase of e_{hl} will be met with an increase of e_{lh} and so on.

Unilateral deviation incentives are exhausted when the marginal products of all links of each type become equal and equal to the agent's marginal cost of effort provision. This happens at the unique interior equilibrium of g^c.

We, therefore, conclude that at the Nash Equilibrium of the complete network, the effort allocations satisfy:

$$e_{hl}^{NE} > \tilde{e}_{hl}, e_{lh}^{NE} > \tilde{e}_{lh}. \tag{3.54}$$

But this means that both the set of the low-types L and the set of the high-types H receive more effort than in $\tilde{e}$ under the Nash Equilibrium of g^c.

Therefore, both the high and the low types are strictly better off in the Nash Equilibrium of the complete network. □

Proposition 5. *Any other unilateral or bilateral deviation from the complete network by the high types is payoff-dominated by the hhll circle or the hh pair.*

Proof. We consider all remaining bilateral deviations of the hight types, omitting the trivial case where any agent is isolated in the post-deviation network.

The remaining deviations for the high types, where the hh link is retained, are: (i) h_1 cuts one hl link, h_2 cuts two hl links; (ii) h_1 cuts zero links, h_2 cuts two hl links (iii) h_1 cuts zero links, h_2 cuts one hl link.

(i) h_1 cuts one hl link, h_2 cuts two hl links:

This creates network $3D$ (see Figure 3.8). We will show that h_2 will be worse-off than in the hh isolated pair, which we have shown to be worse than g^c for all types.

It suffices to show that for any post-deviation network effort allocation e' with $e'_{h_2h_1} = e^{pair}_{hh}$, $e'_{h_1h_2} < e^{pair}_{hh}$.

Assume not i.e. that $e'_{h_1h_2} \geq e^{pair}_{hh} = e'_{h_2h_1}$ and $e'_{h_1l_1} > 0$. But then, for agent h_1, $MC'_{h_1} = e'_{h_1h_2} + e'_{h_1l_1} > MC^{pair}_{h_1} = MP^{pair}_{hh} \geq MP'_{h_1h_2}$, where $MP'_{h_1h_2}$ is the marginal product of agent h_1 from the h_1h_2 link. This is clearly not a Nash Equilibrium in the post-deviation network: h_1 needs to offer lower effort $e'_{h_1h_2} < e^{pair}_{hh}$ to maximise payoffs, unless $e_{h_2h_1}$ increases.

Therefore, for agent h_2 to receive effort $\hat{e}_{h_1h_2} = e^{pair}_{hh}$ in the post-deviation network, they have to offer higher effort $\hat{e}_{h_2h_1} > e^{pair}_{hh}$. Consider now this allocation $\hat{e}$ in the hh pair. It is clear that h_2 is strictly worse-off, as they exert more effort but receive the same effort compared with the Nash Equilibrium of the isolated hh pair.

We conclude that agent h_2 will be strictly worse off in the post-deviation network.

(ii) h_1 cuts zero links, h_2 cuts two hl links:

This creates network $4F$. Similarly to (i), we conclude via contradiction that h_2 will be worse-off in the post-deviation network than in the hh pair: In order to keep receiving the same effort as in the hh pair, they would have to exert more effort themselves.

(iii) h_1 cuts zero links, h_2 cuts one hl link:

This creates network $5A$. We show that agent h_2 is worse-off than in the $hhll$ circle $4A$, which we have shown to be worse than g^c for both types.

Assume otherwise, namely that h_2 is equally well off in $5A$ as in $4A$ by receiving and exerting the same effort level in each link. But for h_1 (similarly for l_2) $MP_{h_1h_2}$ remains the same so MC_{h_1} needs to remain the same. So for any allocation in $5A$ with $e_{h_1l_2}, e_{l_2h_1} > 0$, h_1 needs to lower $e_{h_1l_1}$. Assume such an allocation exists that is a Nash

Equilibrium of $5A$ and call it e'. Then an allocation $\hat{e}$ which places the same effort levels as e' in the links of the $hhll$ circle $4A$ will be a Nash Equilibrium in $4A$. This contradicts uniqueness of the interior Nash Equilibrium in $4A$.

Therefore, in order to receive the same effort from his links as in $4A$, h_2 would need to exert more effort in his links in $5A$ than in $4A$.

A similar argument shows that h_2 cannot receive higher effort in $5A$ than in $4A$ without exerting more to such a level that his total payoff is again lower than in $4A$.

We conclude that agent h_2 is strictly worse-off in $5a$ than in $4a$ so also worse than in g^c.

Consider, finally, any h-type deviation where the hh link is cut: If only the hh link is cut we are in $5C$. The network is symmetric for each type and the l-type set retains the same number of links and receives higher effort from the h-types in the Nash Equilibrium of $5C$. Therefore, both low types are better off which means that both high types are strictly worse off, since via efficiency of the full network total payoffs fall.

The remaining cases are: (i)h_1, h_2 cut the hh link and h_1 cuts one hl link. Then we are in $4D$ and h_1 is worse off than in the hl isolated pair, which is worse than the hh isolated pair; (ii) h_1, h_2 cut the hh link and one hl link each forming the $hllh$ line $3B$. They are then worse off than in the hl pair, which is worse than the hh pair; (iii) h_1, h_2 cut the hh link and one hl link each forming an l-centred star $3F$: they are again worse off than in the hl pair, which is worse than the hh pair.

□

Proposition 6. *: Any other unilateral deviation and any bilateral deviation from the complete network by the two low types or by one high and one low type are not strictly profitable.*

Proof. The proof follows identical steps as the proof of Proposition 3.5 and is omitted. □

Combining results from the previous propositions, we have, therefore, shown that:

Theorem 5. *For a concave production function and a quadratic cost function, the efficient network* (g^c, e^{NE}) *is a Subgame Perfect Bilateral Equilibrium.*

Proof. The proof follows directly from Theorem 4 and Propositions 3.4.1-3.4.5 □

Finally, we show that the efficient network is not the unique Subgame Perfect Bilateral Equilibrium network.

Proposition 7. *The efficient network* g^c, e^{NE} *is not the unique Subgame Perfect Bilateral Equilibrium.*

Proof. Consider any other network $\tilde{g} \in G$. If a single agent or a pair of agents considering a deviation expect that a border Nash Equilibrium will be played in the post-deviation network, with e.g. zero effort exerted in the new link formed, then $(\tilde{g}, \tilde{e}^{NE})$ is another Subgame Perfect Bilateral Equilibrium of the game. □

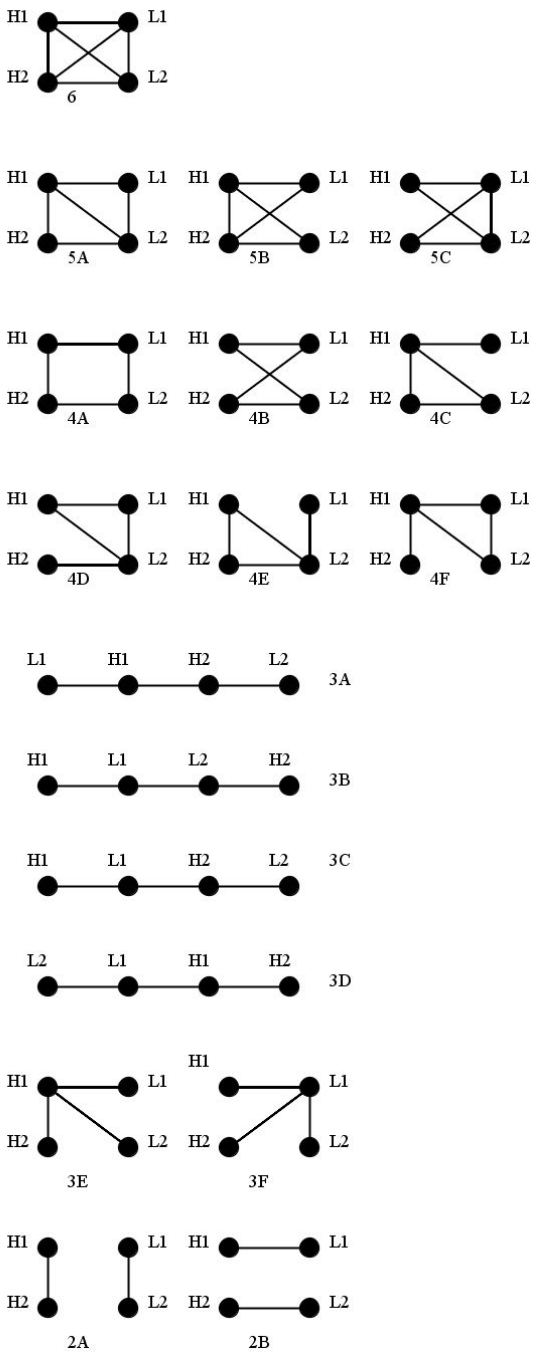

Figure 3.8: List of Connected graphs

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Appendices

Appendix A

Effort provision in the *hhll* Circular network

In this section, we repeat the analysis of the complete network for the *hhll* circle. In the unique Nash Equilibrium of the *hhll* circle with two low and two high types, each type will play a symmetric strategy, by choosing the effort levels for each link such that the marginal products of own effort for all links are equal and equal to the marginal cost. The Nash Equilibrium will, therefore, be described, as in the complete network, by four effort levels $(e_{hl}, e_{hh}, e_{lh}, e_{ll})$ which solve the following system:

$$e_{hl} + e_{hh} = \gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1} \tag{A.1}$$

$$e_{hl} + e_{hh} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} \tag{A.2}$$

$$e_{lh} + e_{ll} = \gamma b^{\gamma} e_{hl}^{\gamma} e_{lh}^{\gamma-1} \tag{A.3}$$

$$e_{lh} + e_{ll} = \gamma e_{ll}^{2\gamma-1}, \tag{A.4}$$

where we have again normalised $l = 1$ and set $h = b > 1$.

Dividing (A.1) with (A.3), and manipulating (A.2) and (A.4), we get:

$$\frac{e_{hl} + e_{hh}}{e_{lh} + e_{ll}} = \frac{\gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1}}{\gamma b^{\gamma} e_{hl}^{\gamma} e_{lh}^{\gamma-1}} = \frac{e_{lh}}{e_{hl}} \tag{A.5}$$

$$e_{hl} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} - e_{hh} \Rightarrow e_{hl} = e_{hh}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1] \tag{A.6}$$

$$e_{lh} = e_{ll}^{2\gamma-1} - e_{ll} \Rightarrow e_{lh} = e_{ll}[\gamma e_{ll}^{2(\gamma-1)} - 1] \tag{A.7}$$

Using (A.6) and (A.7), (A.5) becomes:

$$\frac{\gamma b^{2\gamma} e_{hh}^{2\gamma-1} - e_{hh} + e_{hh}}{\gamma e_{ll}^{2\gamma-1} - e_{ll} + e_{ll}} = \frac{e_{ll}[\gamma e_{ll}^{2(\gamma-1)} - 1]}{e_{hh}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]} \Rightarrow$$

$$\frac{b^{2\gamma} e_{hh}^{2\gamma}}{e_{ll}^{2\gamma}} = \frac{\gamma e_{ll}^{2(\gamma-1)} - 1}{\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1} \Rightarrow$$

$$e_{hh}^{2\gamma}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1] = (\frac{1}{b})^{2\gamma} e_{ll}^{2\gamma}[\gamma e_{ll}^{2(\gamma-1)} - 1] \tag{A.8}$$

Combining (A.1) and (A.2), we get:

$$\gamma b^{\gamma} e_{lh}^{\gamma} e_{hl}^{\gamma-1} = \gamma b^{2\gamma} e_{hh}^{2\gamma-1} \Rightarrow$$

$$e_{lh}^{\gamma} e_{hl}^{\gamma-1} = b^{\gamma} e_{hh}^{2\gamma-1} \tag{A.9}$$

which, using (A.6) and (A.7), gives:

$$b^{\gamma} e_{hh}^{2\gamma-1} = (e_{hh}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1])^{\gamma-1} (e_{ll}[\gamma e_{ll}^{2(\gamma-1)} - 1])^{\gamma} \Rightarrow$$

$$e_{hh}^{2\gamma-1} = (\frac{1}{b})^{\gamma} (e_{hh}^{\gamma-1} e_{ll}^{\gamma} [\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{\gamma-1} [\gamma e_{ll}^{2(\gamma-1)} - 1]^{\gamma} \Rightarrow$$

$$\frac{e_{hh}^{\gamma}}{[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{\gamma-1}} = (\frac{1}{b})^{\gamma} e_{ll}^{\gamma} [\gamma e_{ll}^{2(\gamma-1)} - 1]^{\gamma}$$

and by squaring both sides, we get:

$$\frac{e_{hh}^{2\gamma}}{[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2(\gamma-1)}} = (\frac{1}{b})^{2\gamma} e_{ll}^{2\gamma} [\gamma e_{ll}^{2(\gamma-1)} - 1])^{2\gamma} \tag{A.10}$$

Constraints (A.8) and (A.10) then provide a 2×2 system in e_{hh}, e_{ll}. Combining them by dividing (A.8) with (A.10) we get:

$$\frac{e_{hh}^{2\gamma}[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1][\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2(\gamma-1)}}{e_{hh}^{2\gamma}} = [\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma} \Rightarrow$$

$$[\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1]^{2\gamma-1} = [\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma} \Rightarrow$$

$$\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1 = [[\gamma e_{ll}^{2(\gamma-1)} - 1]^{1-2\gamma}]^{1/(2\gamma-1)}$$

$$\gamma b^{2\gamma} e_{hh}^{2(\gamma-1)} - 1 = [\gamma e_{ll}^{2(\gamma-1)} - 1]^{-1} \tag{A.11}$$

Substituting (A.11) back into (A.8), we get e_{hh} as a function of e_{ll}:

$$e_{hh}^{2\gamma}[[\gamma e_{ll}^{2(\gamma-1)}-1]^{-1}]=(\frac{1}{b})^{2\gamma}e_{ll}^{2\gamma}[\gamma e_{ll}^{2(\gamma-1)}-1]\Rightarrow$$

$$e_{hh}=\frac{1}{b}e_{ll}[\gamma e_{ll}^{2(\gamma-1)}-1]^{\frac{1}{\gamma}} \tag{A.12}$$

Finally, using (A.12) to substitute for e_{hh} in (A.11), we get an expression only in e_{ll}:

$$\gamma b^{2\gamma}[\frac{1}{b}(e_{ll}[\gamma e_{ll}^{2(\gamma-1)}-1]^{\frac{1}{\gamma}}]^{2(\gamma-1)}-1=[[\gamma e_{ll}^{2(\gamma-1)}-1]^{-1}]\Rightarrow$$

$$1=[\gamma e_{ll}^{2(\gamma-1)}-1]^{-1}[\gamma b^2 e_{ll}^{2(\gamma-1)}[\gamma e_{ll}^{2(\gamma-1)}-1]^{\frac{3\gamma-2}{\gamma}}-1] \tag{A.13}$$

In order to simplify (A.13), we can set $y:=\gamma e_{ll}^{2(\gamma-1)}-1$ to get:

$$1=\left[b^2(y+1)y^{\frac{2(\gamma-1)}{\gamma}}-\frac{1}{y}\right] \tag{A.14}$$

or

$$y=b^{\frac{2\gamma}{2-3\gamma}} \tag{A.15}$$

with $\frac{\partial y}{\partial b}=\frac{2\gamma}{2-3\gamma}b^{\frac{5\gamma-2}{2-3\gamma}}>0$ and $\frac{\partial y}{\partial b}=\frac{4b^{\frac{2a}{2-3a}}lnb}{(2-3a)^2}>0$ for $b>1$.

From (A.15), we can then solve directly for e_{ll}:

$$y=\gamma e_{ll}^{2(\gamma-1)}-1=b^{\frac{2\gamma}{2-3\gamma}}\Rightarrow e_{ll}=\gamma^{\frac{1}{2(1-\gamma)}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{(2(1-\gamma))}} \tag{A.16}$$

and use this to find the Nash Equilibrium values of e_{hh}, e_{hl}, e_{lh}.

More interestingly, from constraints (A.6), (A.7) and (A.11) above, we observe that the equilibrium effort ratios $\frac{e_{lh}}{e_{ll}}$ and $\frac{e_{hh}}{e_{hl}}$ will, similarly to the complete network, be equal:

$$\frac{e_{lh}}{e_{ll}}=[\gamma e_{ll}^{2(\gamma-1)}-1]=y=\frac{e_{hh}}{e_{hl}} \tag{A.17}$$

hence, $\frac{\partial\frac{e_{lh}}{e_{ll}}}{\partial b}=\frac{\partial y}{\partial b}=\frac{\partial\frac{e_{hh}}{e_{hl}}}{\partial b}>0$ and $\frac{\partial\frac{e_{lh}}{e_{ll}}}{\partial\gamma}=\frac{\partial y}{\partial\gamma}=\frac{\partial\frac{e_{hh}}{e_{hl}}}{\partial\gamma}>0$.

We can, therefore, conclude that in the unique interior Nash Equilibrium of the *hhll* circle, the relative effort specialisation in the high type neighbour will be equal for the two types:

$$\frac{e_{lh}}{e_{ll}}=\frac{e_{hh}}{e_{hl}} \tag{A.18}$$

and will be increasing as productivity heterogeneity increases, or as the production function becomes less concave.

This confirms the intuition and approximation results of the diagrams for the complete network, for a simpler symmetric network where closed-form solutions can be obtained.

Using $y = [\gamma e_{ll}^{2(\gamma-1)} - 1] = b^{\frac{2\gamma}{2-3\gamma}}$, we can derive the exact Nash Equilibrium effort levels $e_{hh}, e_{hl}, e_{lh}, e_{ll}$ as follows:

$$
\begin{aligned}
e_{ll} &= \gamma^{\frac{1}{2(1-\gamma)}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \\
e_{lh} &= y e_{ll} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{2\gamma}{2-3\gamma}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \\
e_{hh} &= b^{-1}\gamma^{\frac{1}{2(1-\gamma)}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} b^{\frac{2}{2-3\gamma}} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{3\gamma}{2-3\gamma}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}} \\
e_{hl} &= e_{hh} y^{-1} = \gamma^{\frac{1}{2(1-\gamma)}} b^{\frac{\gamma}{2-3\gamma}}(1 + b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-1}{2(1-\gamma)}}
\end{aligned}
\tag{A.19}
$$

As the difference in productivity between types increases, it can easily be shown that e_{ll} falls and e_{hh}, e_{lh} increase, while e_{hl} is not necessarily monotonic.

Appendix B

Proof of Proposition 3.4

In an isolated hh pair, both agents choose effort to maximise payoffs. Using symmetry, we then get:

$$b^{2\gamma}\gamma e_h^{\gamma-1}e_h^{\gamma} = 2\frac{e_h}{2} \Rightarrow e_h^{2\gamma-2} = \frac{1}{\gamma b^{2\gamma}} \Rightarrow$$

$$e_h = (\frac{1}{\gamma b^{2\gamma}})^{\frac{1}{2(\gamma-1)}} \Rightarrow e_h = (\gamma b^{2\gamma})^{\frac{1}{2(1-\gamma)}} \tag{B.1}$$

Therefore, the Nash Equilibrium payoff for each high type is:

$$\Pi_h^{hh} = b^{2\gamma}(\gamma b^{2\gamma})^{\frac{2\gamma}{2(1-\gamma)}} - \frac{(\gamma b^{2\gamma})^{\frac{2}{2(1-\gamma)}}}{2} = \frac{(2-\gamma)}{2}\gamma^{\frac{\gamma}{1-\gamma}}b^{\frac{2\gamma}{1-\gamma}} \tag{B.2}$$

Using the $hhll$ circle's Nash Equilibrium effort levels of section 3.4, we can calculate the Nash Equilibrium payoff of the high type:

$$\Pi_h^c = b^{2\gamma}e_{hh}^{2\gamma} + b^{\gamma}e_{hl}^{\gamma}e_{lh}^{\gamma} - \frac{(e_{hh}+e_{hl})^2}{2}$$

$$= \gamma^{\frac{\gamma}{1-\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-\gamma}{1-\gamma}}[b^{\frac{4\gamma}{2-3\gamma}} + b^{\frac{2\gamma}{2-3\gamma}}] - \frac{1}{2}\gamma^{\frac{1}{1-\gamma}}b^{\frac{2\gamma}{2-3\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} \tag{B.3}$$

Therefore, the difference in payoffs after a bilateral deviation from the $hhll$ circle to the isolated hh pair is:

$$\Delta\Pi_h = \frac{(2-\gamma)}{2}\gamma^{\frac{\gamma}{1-\gamma}}b^{\frac{2\gamma}{1-\gamma}} - [\gamma^{\frac{\gamma}{1-\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-\gamma}{1-\gamma}}[b^{\frac{4\gamma}{2-3\gamma}} + b^{\frac{2\gamma}{2-3\gamma}}] - \frac{1}{2}\gamma^{\frac{1}{1-\gamma}}b^{\frac{2\gamma}{2-3\gamma}}$$

$$(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}}] = \frac{(2-\gamma)}{2}\gamma^{\frac{\gamma}{1-\gamma}}b^{\frac{2\gamma}{1-\gamma}} - [\gamma^{\frac{\gamma}{1-\gamma}}b^{\frac{2\gamma}{2-3\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-\gamma}{1-\gamma}}$$

$$[b^{\frac{2\gamma}{2-3\gamma}}+1-\frac{1}{2}\gamma(1+b^{\frac{2\gamma}{2-3\gamma}})]] = \gamma^{\frac{\gamma}{1-\gamma}}(\frac{2-\gamma}{2})[b^{\frac{2\gamma}{1-\gamma}}-b^{\frac{2\gamma}{2-3\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}}] \quad \text{(B.4)}$$

Since the first and the second terms of the above product are positive for any $0 < \gamma < \frac{1}{2}$, it suffices to show that the third term in the brackets is negative:

$$b^{\frac{2\gamma}{1-\gamma}} - b^{\frac{2\gamma}{2-3\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} < 0 \Rightarrow$$

$$b^{\frac{2\gamma}{2-3\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} > b^{\frac{2\gamma}{1-\gamma}} \Rightarrow$$

$$b^{(\frac{2\gamma}{2-3\gamma}-\frac{2\gamma}{1-\gamma})}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} > 1 \Rightarrow$$

$$b^{\frac{2\gamma(2\gamma-1)}{(2-3\gamma)(1-\gamma)}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} > 1 \Rightarrow$$

$$(b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-(1-2\gamma)}{1-\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} > 1 \Rightarrow$$

$$\left[\frac{1+b^{\frac{2\gamma}{2-3\gamma}}}{b^{\frac{2\gamma}{2-3\gamma}}}\right]^{\frac{1-2\gamma}{1-\gamma}} > 1 \quad \text{(B.5)}$$

which holds for any $b > 1$, $\gamma < \frac{1}{2}$, since then the fraction in the brackets is greater than one and the power it is set to is greater than zero.

We have, therefore, shown that a bilateral deviation from the *hhll* circle to the isolated *hh* pair is not strictly profitable for the high types.

Appendix C

Proof of Proposition 3.5

In an isolated ll pair, both agents choose effort to maximise payoffs. Using symmetry, we then get:

$$\gamma e_l^{2\gamma-1} = 2\frac{e_l}{2} \Rightarrow e_l^{2\gamma-2} = \frac{1}{\gamma}$$

$$\Rightarrow e_l = (\frac{1}{\gamma})^{\frac{1}{2(\gamma-1)}} \Rightarrow e_l = \gamma^{\frac{1}{2(1-\gamma)}} \tag{C.1}$$

Therefore, the Nash Equilibrium payoff for each high type is:

$$\Pi_l^{ll} = \gamma^{\frac{2\gamma}{2(1-\gamma)}} - \frac{\gamma^{\frac{2}{2(1-\gamma)}}}{2} = \frac{(2-\gamma)}{2}\gamma^{\frac{\gamma}{1-\gamma}} \tag{C.2}$$

Using the $hhll$ circle's Nash Equilibrium effort levels of section 3.4, we can next calculate the Nash Equilibrium payoff of the low type:

$$\Pi_l^c = e_{ll}^{2\gamma} + b^\gamma e_{hl}^\gamma e_{lh}^\gamma - \frac{(e_{ll}+e_{lh})^2}{2}$$

$$= \gamma^{\frac{\gamma}{1-\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{-\gamma}{1-\gamma}}[1+b^{\frac{2\gamma}{2-3\gamma}}] - \frac{1}{2}\gamma^{\frac{1}{1-\gamma}}(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}} \tag{C.3}$$

Therefore, the difference in payoffs after a bilateral deviation from the $hhll$ circle to the isolated hh pair can be shown to be:

$$\Delta\Pi_l = \gamma^{\frac{\gamma}{1-\gamma}}(\frac{2-\gamma}{2})[1-(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}}] \tag{C.4}$$

Since the first and the second terms of the above product are positive for any $0 < \gamma < \frac{1}{2}$, it suffices to show that the third term in the brackets is

negative:

$$1-(1+b^{\frac{2\gamma}{2-3\gamma}})^{\frac{1-2\gamma}{1-\gamma}}<0 \Rightarrow$$

$$1^{\frac{1-\gamma}{1-2\gamma}}<1+b^{\frac{2\gamma}{2-3\gamma}} \tag{C.5}$$

which holds for any $b>1$, $0<\gamma<\frac{1}{2}$.

We have, therefore, shown that a bilateral deviation from the *hhll* circle to the isolated *ll* pair is not strictly profitable for the low types.