RESOLUTION AND COMPLETION OF ALGEBRAIC VARIETIES

by

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INTRODUCTION

We state here the principal definitions and results, and present some of the motivation and philosophy that could otherwise be hidden by the details of lemma and proof. But first we relate the history of the most important of the ideas and results upon which we depend.

For the convenience of the reader we list here the headings under which the topics discussed fall. The first four headings are historical, the next four discuss and state the new definitions and results, and the last two are concerned with unanswered questions.

ABELIAN DIFFERENTIALS

The study of algebraic curves as a topic in differential and integral calculus, that started around the end of the eighteenth century with the discovery of the remarkable properties of elliptic integrals, singled out certain differentials. Suppose that y is given implicitly as a function of x by an algebraic equation $f(x, y) = 0$. To find the poles, if any, and thus the convergence or otherwise of an integral such as

$$\int g(x, y) \, dx$$

where g is a rational function of x and y, it can be helpful to use the technique known as "integration by substitution" or "reparameterization". The problem is this - for any given value of x there may be an infinities and zeros involved in the corresponding values of y and g . Their net effect depends upon how the geometry of the curve X defined by the equation $f = 0$ interacts with the integrand. This can be subtle. But if, perhaps after reparameterization, the point (x, y) is a nonsingular point of X then the associated poles of the integral are relatively transparent.

Abel and Jacobi discovered many remarkable properties of those integrals that are without poles, even at the "line at infinity" attached to the (x, y) - plane. Such an integral is called "abelian" or "of the first kind", as is the differential form that constitutes the integrand. Later, Riemann developed the theory to encompass differentials whose poles were bounded.

The act of integration can be performed not only upon simple differentials but also upon the tensor product of two or more simple differentials. The poles of such compound r-fold differentials are studied in the same way. The collection of all abelian differentials, compound and simple, forms what is known as the canonical ring of the curve. The canonical ring of a variety of any dimension consists of the r-fold differentials of top degree. It is important because it is an invariant not just of the variety, but of its field of rational functions. Weierstrass showed that for curves this ring was finitely generated. He also studied the image of the associated map to projective space.

The extension of these results from curves to varieties of higher dimension has been one of the great tasks and accomplishments of twentieth century algebraic geometry. For example, in 1962 Zariski and Mumford demonstrated that the canonical ring of a surface is finitely generated. There is much activity now devoted to the same question for threefolds, and some partial results.

RESOLUTION OF SINGULARITIES

Newton was an early student of singularities. Solving the equation

$$f(x, y) = 0$$

for y as a function of x is straightforward if the partial derivative in the x direction of f does not vanish. At those points at which it does, Newton showed in his 1676 letter to Oldenburg how to calculate from the co-efficients of the polynomial f an expansion for y as a sum of multiples of fractional powers of x. Later, this technique was refined by Puiseux and applied to calculate the intersection multiplicities required by Bezout's theorem. The characteristic pairs of a branch of a plane curve singularity are equivalent to a knowledge of the fractional exponents that arise in the Newton-Puiseux expansion. They also determine the complex topological link of the critical point.

Apart from the study of plane curve singularities and certain other special cases, little new was done in resolution until the beginning of this century. The reason for this is doubtless the lack of a sufficient reason for tackling what has turned out to be a major project. Resolution, like unique factorization, is an algebraic problem whose solution depends upon concepts and algorithms. But it is a far deeper result whose proof seems to necessarily involve a complex net of inductions upon subtle integer invariants of singular points.

First Jung, and then Walker, tackled the resolution of surface singularities - but their methods seem to be removed from the main thrust of modern development. What is important is that in 1939 Zariski gave a clear and evidently rigorous proof of the resolution of surface singularities. Subsequently, he proved the existence of local uniformization in all dimensions [1940] - from which it is possible to compute the canonical ring - and the resolution of three-folds [1944]. There matters stood until Hironaka proved resolution in all dimensions [1964].

The results of the preceeding paragraph apply only to varieties over a ground field of characteristic zero. New difficulties arise in finite characteristic. Abhyankar has solved them for all but a few surfaces and threefolds.

It is now well established that resolution should proceed through the performance of a succession of monoidal transformations, the centers of which are determined by the geometry, and in particular the discrete invariants, of the singular space. It is also clear that for the inductive methods used to work, one must prove not just resolution of abstract varieties, or even embedded resolution, but that it is always possible to "resolve" an ideal on a variety into the ideal of a normal crossings divisor.

When resolving a curve, the center of every monoidal transformation is a point, and if the chosen point lies outside the singular locus, then the transformation is trivial. Thus, there are no real decisions to make about the location of centers. All that needs to be done is to show that the process terminates. There is only one possible rule to follow. In higher dimension, care must be exercised while choosing the center. Moreover, "the" resolution is far from unique.

For curves, there is no free choice, while in higher dimensions many of the abundance of rules lead nowhere, or at least have not been proven to terminate in a resolution. All the same, the necessity that exists for curves still exists in higher dimension, but it takes a different form. It is true, for example, that any differential regular on a resolution must be abelian, and so the places that appear on any resolution must be sufficient to determine whether or not a differential is abelian. We develop this observation by showing that, given a singular variety, certain places must appear on its resolution. (To be precise, I should say its the resolution of its singular locus to a normal crossings divisor. Such is called a ncd resolution). These places are called essential.

I do not know how important this realm of necessity is to the study of singular varieties and their resolutions.

COMPUTATION IN ALGEBRAIC GEOMETRY

Outstanding though they are, the results of Hironaka and Abhyankar are theoretical. There is not, to my knowledge, but one example of a singular variety resolved according to the algorithms, other than plane curves. This is a general failing. There are few explicit examples of computation in the literature of resolution.

Serre's papers on coherent sheaves provide a major example of how, in algebraic geometry, computational methods can be applied to produce theoretical results that can, in turn, be applied to facilitate further computations. For example, sheaf cohomolgy is used to solve the Yang-Mills soliton equations. The application of Euclid's algorithm to prove the first properties of natural numbers is simpler example of the immense value, even when considering abstract questions, of possessing effective computational methods.

An algebraic variety, say over an algebraically closed field, is an infinite object when considered as the collection of its points. As such, it is not fit to compute with, either by man or by machine. What is required is a finite but faithful representation of the object being considered. A principal requirement is that those quantities that especially concern us can be readily computed from the representation. Computer science uses the term "data-structure" to describe such a representation. The use of fans (in the sense of Demazure and myself) provide such a data-structure for torus embeddings.

The opportunity provided by torus embeddings and fans has largely been ignored. The little that has been done is confined to providing examples and occasional proofs in papers that are predominantly theoretical and general. No doubt this is due to the large amount of arithmetic and bookkeeping involved in even slight examples conjoined with a widespread reluctance and perhaps even inability amongst mathematicians generally to use automatic computing machinery in their work. My personal experience is that the amount of labor required to turn a computer into an object as easy to use as a pocket calculator but dedicated to some simple task is quite considerable and hard to justify, if it is to have but little use.

To show that the canonical ring of an algebraic variety is finitely generated is an outstanding problem whose solution will surely proceed through the use of methods that are computational.

HISTORY OF TORUS EMBEDDINGS

A torus embedding is defined to be a separated normal algebraic variety M containing an algebraic torus T as a dense open subset in such a manner that the birational action of the torus on the variety is in fact biregular. Such varieties are stratified by the orbits of the action of the torus. Since their introduction in 1970 by

Demazure, the reason for studying torus embeddings has changed several times, as has the combinatorial apparatus used to describe them.

Demazure's concern was with subgroups of the Cremona group - the group of automorphisms of the field of rational functions in several variables over a field. He proved that every such group of maximal rank was in fact the group of biregular automorphims of some complete and non-singular torus embedding. (Enriques had proved the corresponding result for two variables. The surface torus embeddings were rational scrolls).

Demazure showed that each non-singular torus embedding determined a simplicial complex Σ whose vertices are points of the lattice $\Lambda^{\vee}$ of one parameter subgroups of the given torus. This complex lies in $\Lambda^{\vee}$ in such a manner that each interior of the cone on the convex hull of a simplex σ of Σ does not intersect any other such. Moreover, every integral point in the rational span of a simplex σ is already in the integral span. Demazure introduced the term "fan" to describe such complexes, and showed that every fan in $\Lambda^{\vee}$ gives rise to a nonsingular torus embedding. He then posed the problem of extending these results to general, possibly singular, torus embeddings.

This was done by Kempf, Mumford, Knudsen, and Saint-Donat. Their preoccupation was with the proof of what is known as the semi-stable reduction theorem, which required them to consider the process of inducing a torus embedding M' from M by taking a finite cover T' —> T of the original torus. To them, M and M' to are in some sense equivalent, and so they introduced a combinatorial description common to both. They used the notion of a finite rational partial polyhedral decomposition of the space $\check{\Lambda}_Q = \check{\Lambda}\otimes_Z Q$ into cones. (If σ is as simplex of the fan Σ of a (nonsingular) torus embedding then to it would correspond the interior of the cone generated by the convex hull of σ.

Kempf et al gave the combinatorial conditions necessary and sufffficient for a finite rational partial polyhedral decomposition to define a torus embedding. Two other results are of importance to us. They developed a theory that applied to stratified varieties that locally were analytically isomorphic to a torus embedding. The obvious example of such a variety - a nonsingular variety M equipped with a normal crossings divisor D - was for some reason not considered. Their theory of toroidal embeddings was the first extension of the combinatorial apparatus of torus embeddings to a more general context.

They also computed the cohomology of invertible sheaves - i.e. linear systems - on complete torus embeddings. This must have eased the next development, which was the study of hypersurfaces in torus embeddings.

But we will discuss Danilov's contribution first. He gave a systematic and clearly written expose of the theory, making it accessible to mathematicians outside of algebraic geometry. He computed both the Chow ring and the homology ring of complete torus embeddings, and showed that they were equal. He introduced the notation Λ, $\Lambda^{\vee}$ for characters and one parameter subgroups - Demazure had used M, M^{*}; Kempf et al, M and N. Finally, he revived the word "fan", applying it however to the object previously described by the large and clumsy phrase "finite rational partial polyhedral decomposition". (It could be said however that the object, being itself large and clumsy, deserved its elephantine appellation).

Arnol'd and many others discovered the wealth of geometry latent in isolated critical points of smooth functions. It was noticed that for many interesting functions f, co-ordinates could be chosen at the critical point so that both the topology and the discrete geometric invariants depended only upon the Newton polyhedron of the Taylor expansion of f . It was a happy co-incidence that the theory of torus embeddings provides an admirable tool for studying such critical points. For example, an embedded resolution of the level surface

$$X = \{ p \mid f(p) = 0 \}$$

can be explicitly constructed by producing from the Newton polyhedron a suitable fan, and thereby defining a torus embedding. Hypersurfaces that can be resolved in this way are called nondegenerate.

Some invariants of nondegenerate hypersurfaces, and even complete intersections, have been calculated. In particular, Khovanskii has shown that the first of the plurigenera is equal to the number of monomials that lie strictly inside the Newton polyhedron. This generalises the classical formula

$$g = (d - 1)(d - 2)$$

for the genus of a nonsingular plane curve of degree d. His results can be thought of as being a partial generalization of the formulae for the genus of singular plane curves. Much remains unknown. Given their importance, it is embarassing that we can compute neither the monodromy group nor the number of moduli for an arbitary nondegenerate critical point of a function f from its Newton polyhedron.

The theory of torus embeddings has a fundamental limitation. It is not always possible, given a hypersurface, to choose co-ordinates so that it is non-degenerate. For example, such co-ordinates exist for an analytically irreducible plane curve singularity just in case it has but one characteristic pair. If it has several, more powerful techniques are needed to describe the resolution.

As we have seen, the primary focus of the theory of torus embeddings is changing. The group action has become incidental - it is now merely a convenient means of stratifying the variety and a helpful device in the proofs. Instead, what seems to be vital is that it provides a large family of birationally equivalent varieties, each member of which can be explictly described using notions that are fundamentally combinatorial and so - at least in principle - amenable to calculation. To go beyond the fundamental limitation, one must add to the family.

The theory of torus embeddings when considered abstractly has three essential features that it should pass on to its descendants.

* All varieties in the family have a common open set which is affine and whose co-ordinate ring is given explicitly. Each function in the co-ordinate ring has a normal form.
* There is provided a family of places at infinity. The order of vanishing of a function along such a place is readily computable from the normal form.
* It is known which of the complexes, whose vertices lie in the family of places, can arise as the fan of a (partial) completion of the common open subset.

The theory of torus embeddings is but one way of satisfying the above requirements. It is in fact the embryo of a more general form.

PLACES

Along with the rise of schemes in algebraic geometry there was a neglect of valuative techniques. Except for Abhyankar's work in finite characteristic, little has been done since Zariski's contributions in the 1940's. The field K = k(X) of rational functions of an algebraic variety X has many valuations. Especially important are those are defined, obtained, or characterized by geometric means. It is well known that if X′ is a, say, nonsingular model for X and $D' \subset X'$ is an irreducible smooth subvariety of codimension one then by assigning to each rational function f in K its order of vanishing along D we obtain a valuation ν of K. This valuation is discrete and of rank one. We re-introduce the now all but forgotten term "place" to describe those valuations that can be gotten in this way. The pair $D' \subset X'$ is called a model for ν, and is said to realize ν (as the place associated to a codimension one subvariety). I do not know if every discrete rank one valuation is a place.

We give [p26-42] some useful definitions and results concerning places for which no suitable reference exists. Given some model X for a field K of rational functions, we classify the places of K according to how they 'lie over' X. The center D_ν of ν on X is the closure of the image of D′ under the rational map X′ —> X. It does not depend upon the model chosen for ν. It may be the empty set, in which case we say that the place is infinite, or lies at infinity. A variety X is complete just in case all places of its rational function field are finite.

If X is say normal then every finite place falls into at least one of the classes :-

exceptional : D_ν has codimension at least two

singular : D_ν lies in the singular locus of X

common : D_ν has codimension one and is not contained in the singular locus of X

If v is a common place then upon removing Sing D_ν and Sing X from the pair $D_\nu \subset X$ we obtain a model for ν. If v is common, we will also say that it appears on X (as the place associated to a codimension one irreducible subvariety of X).

The algebraic notion of place is related to the geometric notion of monoidal transformation.

<u>Realization Theorem</u> [p39,40] Suppose that X is nonsingular. Every finite place ν determines an alternating sequence

$$\begin{tikzcd}
X_n - \mathrm{Sing}\, D_{n,\nu} \arrow[r, phantom, "\subset"] & X_n \arrow[dl] & \\
X_{n-1} - \mathrm{Sing}\, D_{n-1,\nu} \arrow[r, phantom, "\subset"] & X_{n-1} \arrow[dl, dotted] & \\
X - \mathrm{Sing}\, D_{\nu} \arrow[r, phantom, "\subset"] & X = X_0 &
\end{tikzcd}$$

of inclusions and monoidal transformations. This sequence becomes stable ($X_n = X_{n+1}$ for n large enough). It terminates in a model for ν.

The proof of this result is interesting. Because ν has a geometrical realization, we can define the order of vanishing $\nu(\omega)$ of any differential ω along ν. Using this we define for every finite place of X its discrepancy $\mathrm{dis}_X(\nu)$, which is a non-negative integer that, roughly speaking, measures how far is from appearing on X. This number drops under each of the above monoidal transformations and is zero just in case ν appears on X. The theorem is almost certainly true even if X is singular, but in that case to show that the discrepancy drops requires results concerning Kahler differentials on singular varieties that are not readily to hand.

Conversely, every finite sequence of inclusions and monoidal transformations as above - subject to the condition that each center be the closure of the image of the one above it - is the sequence associated to the place for which the exceptional divisor of the last monoidal transformation provides a model. Thus, we see that the concept of a finite place is not far removed from that of a sequence of monoidal transformations.

SINGULAR VARIETIES

If the differential ω is regular at the smooth points of a possibly singular variety X then we say that ω is regular on X. If ω is regular on some (equivalently every) resolution of X then it is customary to say that ω is abelian or of the first kind. We provide a quantitative refinement of this notion by defining the order $\mathrm{ord}_X(\omega)$ to be the smallest value of $\nu(\omega)$ as ν runs over the singular places of X. By using the notion of logarithmic differentials, we define the logarithmic order in the same way. A singular place ν for which the equation

$$\nu(\omega) = \mathrm{ord}_X(\omega)$$

holds is said to be minimal (with respect to ω). We study this equation using the realization theorem. From this we obtain results concerning the geometric structure of the resolution, if such exists. The method also applies to completions and torus embeddings.

In what follows, the case of X a curve is an exception. This is because the resolution of a curve is unique. Indeed, Dedekind and Weber used the set of places to constitue a nonsingular model. We assume that X has dimension at least two for the remainder of the introduction. (The next result is false for X a curve and the one following is trivial).

<u>Theorem</u> [p65,66] If w is an r-fold differential regular on X then
i) $\mathrm{ord}_X(\omega) \geq -r$ or $\mathrm{ord}_X(\omega) = -\infty$
ii) $\mathrm{ord}_X(\omega;\ \log) \geq 0$ or $\mathrm{ord}_X(\omega;\ \log) = -\infty$

Theorem [p51,68] Suppose that $X' \longrightarrow X$ is a ncd resolution of X with exceptional divsor E $= E_1 \cup \ldots \cup E_r$. Suppose also that ω is an r-fold differential regular on X. Then
i) if $ord_X(\omega)$ is non-negative then it is equal to $\min\{\nu_{E_i}(\omega) \mid i = 1\ldots r\}$
ii) provided values less than zero are counted as $-\infty$, then $\min\{\nu_{E_i}(\omega;\log) \mid i = 1..r\}$ is equal to $ord_X(\omega;\log)$.

In other words, given any regular differential ω, any ncd resolution contains a minimal place - subject to suitable conventions about negative infinity. In fact, the order of a differential can be computed from an ncd resolution. (The study of more general resolutions becomes involved). If ω is of top degree, more can be said.

Theorem [p70,73] Suppose that ω is an r-fold differential of top degree regular on X and that $X' \longrightarrow X$ is a ncd resolution of X with exceptional divisor E. Then
i) if $ord_X(\omega) > -r$ then every ω minimal place appears on X' as a component of E.
ii) if $ord_X(\omega) > -r$ then every -minimal place can be realized by making a toral transformation of the pair $E \subset X'$.

From case (i) of the result we obtain a number of places essential to the resolution of the singularity to a normal crossings divisor. It is clear from the examples I have studied that further results will be required to obtain all the essential places of even quite simple singular varieties. We will now describe the concept of a toral transformation.

TORAL MODIFICATIONS

It is surprising that the problem of places minimal with respect to a differential gives rise to a theory that has much in common with the theory of torus embeddings. The combinatorial apparatus of fans etc provides a means of describing those sequences of monoidal transformations and birational modifications that can be derived from a normal crosssings divisor. The pre-existing theory of toroidal embeddings provides a solution of sorts to this problem - but its continual comparison to torus embeddings is irrelevant, confusing and restrictive.

> <u>Definition</u> [p92] Suppopse that on the smooth variety M there lies a normal crossings divisor D. A monoidal transformation M' —> M is said to be toral if its center is a component of the intersection of some of the components of D. In this case the reduced total transform D' of D is also a normal crossings divisor.

A succession

$$M_{(n)} \longrightarrow M_{(n-1)} \longrightarrow \ldots M_{(0)} = M$$

of monoidal transformations is said to be toral if for each i the transformation $M_{(i+1)} \longrightarrow M_{(i)}$ is toral with respect to the reduced total transform $D_{(i)}$ of D.

One can find, at least locally, a differential ω of top degree whose divisor on M is -D. (For torus embeddings such a differential exists for the whole space). Loosely speaking, the principal result is that the toral theory is the study of the solution for ν of the equation $\nu(\omega) = -1$. Although useful, such a statement is not precise and so cannot be a theorem.

<u>Theorem</u> [p98] A finite place ν of M is toral just in case $\nu(\omega) = -1$.

<u>Corollary</u> Suppose that $M' \longrightarrow M$ is a toral transformation. Then every toral place of M is a toral place of M' and vice versa.

To a normal crossings divsor D on a nonsingular variety M we associate [p117] a simplicial complex $\Sigma(M)$ that is, for torus embeddings, practically equivalent to that of Demazure. We call it the fan of D on M. Its vertices are the places associated to the components of D. A set σ of places is said to belong to Σ if the intersection of the corresponding components of D is nonempty. It is not at all clear what the definition should be in the case when the divisor D does not have normal crossings.

We show [p104] that to each toral place there corresponds:

a) a simplex σ of the fan Σ of D on M.

b) to each vertex of σ, a strictly positive integer.

c) to each vertex of Σ not in σ, the integer zero.

and moreover

d) these integers taken together are without commmon factor.

To each assignment satisfying the above four conditions and a choice of a component of the intersection of the components of D that correspond to σ, there corresponds a unique toral place. To simplify we assume that any such intersection has but one component.

The objects defined by the first three conditions are called weights, while those that also satisfy condition (d) are called simple. Because of our assumption, to each toral place ν there corresponds a simple weight $\alpha(\nu)$, and to each simple α a toral place ν_α. We use the notation $\Lambda^\vee(M)$ to denote the space of weights on M. To each weight α there corresponds a simplex σ_α of Σ. Those weights whose associated simplex is σ constitute the open chamber associated to σ.

We can enlarge the class of varieties being considered.

> <u>Definition</u> [p110] Suppose that M′ —> M is a smooth birational modification, and that the reduced total transform D′ of D has normal crossings. If the place associated to the components of D′ are toral then we say that M′ —> M is a toral modification.

We show that if M′ —> M is a toral modification then the weight space $\Lambda^\vee(M')$ can be thought of as a subset of $\Lambda^\vee(M)$, and that each open chamber of the former lies in a unique open chamber of the latter. Thus, $\Lambda^\vee(M')$ can be thought of as being a subdivision of the division of $\Lambda^\vee(M)$ into chambers. If M′ ⟶ M is proper then the two weight space have the same points, but are subject to a different division into open chambers.

From algebraic geometry we deduce conditions that the fan Σ' of a toral modification $M' \longrightarrow M$ must satisfy. They are analogous to those used by Demazure, Kempf, et al for torus embeddings. From a fan Σ' satisfying these conditions a toral modification $M(\Sigma') \longrightarrow M$ that contains M' as an open set can be constructed. By applying the same arguments to torus embeddings we obtain

> Theorem [p127] Suppose that $M \supset T$ is a non-singular completion such that $D = M - T$ is a normal crossings divisor. Suppose also that the divisor (ω) of ω is $-D$. Then M is a torus embedding.

ORDER AND HEART

we return to the theory of torus embeddings to calculate the order of differentials. Because for any toral place ν the order of ω along ν is -1, it follows that the order of some differential $g\omega$ of top degree is simply $\nu(g) - 1$. Now suppose that X is a nondegenerate hypersurface defined by the vanishing of the function f, and that f does not divide g. If ν is the extension of some place ν' of X then the equation

$$\nu'(\ \mathrm{Res}(g\omega\ /f)\ ) = \nu(g) - \nu(f) - 1$$

will enable us to calculate the order on X of any differential of top degree. If I is an integral point lying in the interior of the Newton polyhedron Δ of f then the order of the corresponding differential is non-negative, and the minimal places correspond to integral boundary points of a dual object that is known in the theory of convex polytopes as the polarization of Δ about I. It is shown that two nondegenerate plane curves and also the surface E_8 defined by the equation

$$x^2 + y^3 + z^5 = 0$$

admits a resolution, all of whose places are minimal.

For a nondegenerate hypersurface, it is fairly straightforward to compute from its Newton polyhedron the order of a differential of top degree, and the corresponding minimal places. Khovanskii showed that if f is the local equation of the hypersurface and g is some function regular on the torus, not divisible by f, then the Poincare residue of $g\,\omega\ /\ f$ will be of the first kind on X provided the Newton polyhedron of g is strictly contained in that of f. For r-fold differentials of top degree more care is needed. Using the integral structure of the lattice Λ we define [p146] for each polyhedron Δ with integral vertices an interior $H(\Delta)$which we call the heart. The vertices of the heart are rational, but may not be integral. It has a finite number of faces. If f and g are as above then the Poincare residue of the differential

$$g\,\omega^{\otimes r}\ /\ f^{r}$$

is of the first kind just in case the Newton polyhedron of g is contained in the r-fold sum of $H(\Delta)$ with itself. From this we show [p147] that for nondegenerate affine hypersurafces the ring of all differentials of top degree of the first kind is finitely generated. Although the Poincare residue map is surjective locally, it is not so globally. To obtain the canonical ring of nondegenerate projective hypersurfaces using these methods, one must first deal with this cohomological phenomena, of which examples are provided [p149].

TORUS EMBEDDINGS PROBLEMS

The theory of torus embeddings and minimal places provide some insight into aspects of the construction, resolution, and geometry of algebraic varieties. Less is known than is unknown. Many problems wait to be solved. Here are six.

* Determine the places essential to the resolution of E_8.
* How many minimal embedded resolutions of E_8 are there? (There should be only a finite number).
* Suppose that $M \supset T$ is a normal and complete variety such that the differential ω has a simple pole along each codimension one component of $M - T$. Must M be a torus embedding?
* Compute the plurigenera of a nondegenerate hypersurface X from its Newton polyhedron Δ. In particular, show that the canonical ring is finitely generated.

* Construct formal algorithms - i.e. computer programs - for computing a resolution and the geometric invariants of nondegenerate hypersurfaces.

* Using a machine search, systematically explore Newton polyhedra for interesting examples of varieties. It would be nice to have a large collection of three-folds lying between rational and general type.

TORAL THEORY PROBLEMS

I have already indicated that the use of torus embeddings does not apply to many examples. We ought to be able to compute with more than torus embeddings. That we can substitute for the group action the structured family of places minimal with respect to a suitable differential indicates that the definition we introduce here is sound.

Definition Suppose that D is a reduced and effective divisor on a nonsingular variety M. A sequence $M_{(n)} \dashrightarrow M_{(n-1)} \dashrightarrow \ldots M_{(0)} = M$ of monoidal transformations is said to be toral with respect to D is at each stage the center of the monoidal transformation $M_{(i+1)} \dashrightarrow M_{(i)}$ is, locally, a transverse intersection of components of the reduced total transform $D_{(i)}$

of D on $M_{(i)}$. The places associated to the components of $D_{(n)}$ are called toral. If M admits a toral transformation $M_{(n)} \dashrightarrow M$ for which $D_{(n)}$ is a normal crossings divisor then $D \subset M$ is said to be self-resolving.

A broad definition such as the above, needs applications to test it to show its usefulness. The following statement is almost surely true. An example is given at the end of the thesis.

<u>Assertion</u> Suppose that X_n is an analytically irreducible plane curve singularity possessing n characteristic pairs. Then co-ordinate axes H and plane curves X_j possessing j characteristic pairs can be chosen so that for j equal to 1 through to n, the resolution of X_j is toral with respect to $D \cup X_1 \cup \ldots \cup X_{j-1}$.

A second possible application is to the resolution of generic hypersurfaces in finite characteristic. It has been known for some time that if a finite set Δ of monomials is chosen then the hypersurface X defined by the vanishing of the polynomial

$$f = \Sigma \lambda_I x^I, I \in \Delta$$

is non-degenerate provided the co-efficients λ_I are chosen generically from a ground field of characteristic zero. In finite characteristic the Bertini theorem upon which this result depends fails.

In characteristic zero it is possible to find a single completion M of T that will resolve any generic element of the system. This will not always be possible in finite characteristic. Suppose we take a completion M of T that would resolve the hypersurface if it were non-degenerate. Upon the smooth variety M, Δ will define a linear system - not necessarily complete - without fixed points. In finite characteristic it is possible that each element of the system have a nonempty singular locus which moves. Its location wil depend upon the element chosen of the system. In such case, it will be necessary to introduce auxiliary hypersurfaces in order to resolve X.

SUMMARY

Little is known concerning the resolution of the singular locus of an algebraic variety, apart from that they exist when the ground field has characteristic zero and in some other cases. We obtain results concerning the geometric structure of the resolution, if such exists, of any given singularity. Our results depend upon some preliminary material in Chapter I, and are valid for any characteristic.

The main technique used is to study the vanishing locus of differential forms and its behavior under any given monoidal transformation. This is done in Chapter II. From this we obtain a range of results. Inter alia we show that certain places must appear in the resolution of any given singular locus to a divisor. Our results do not exhaust the power of the technique.

In Chapter III we study certain birational modifications, called toral, that can be defined from a normal crossings divisor. Singularities that can be resolved by such modifications are used to give examples for the results of Chapter II. Also, we prove a result concerning the completion of algebraic varieties. Suppose that the complete and nonsingular variety M contains an algebraic torus T as the complement of a normal crossings divisor D. Then the birational action of T on M is biregular just in case some differential of top degree regular on T has a pole of order one along each component of D.

The results of Chapter III can be thought of as belonging to the theory of torus embeddings. However, they are a special case of an as a yet unwritten theory of general toral modifications, the definition of which appears in the Introduction, and an example of which concludes the work. This general theory offers a chance to prove more generally valuable results that are presently known only in special cases, such as the finite generation of the canonical ring.

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While completing this work, my father, Maurice Fine, died after a long illness. When I was young, he communicated to me his own love of mathematics, and much

of what he knew concerning functions of a single complex variable. Later, he helped fund my studies at Warwick University while I was without state support. He shared with me the excitement of mathematical discovery. If he were alive and well, this work would have pleased him very much.

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INTRODUCTION

From the beginning, differentials appear in the study of the singularities of an algebraic variety. For example, the singular locus of a hypersurface $X \subset M$ with local equation $f = 0$ is defined by the vanishing of the differential df. This work exploits a more subtle phenomena. Suppose that $X' \to X$ is a birational modification and also that the differential ω is regular on X (i.e., regular on the nonsingular locus). If X is nonsingular then ω is regular on X'. Moreover, if ω is of top degree then it vanishes on the exceptional locus $E \subset X'$ of the morphism. Such statements can fail to be true if X is singular. By studying the behavior of ω on a representative class of modifications $X' \to X$ we obtain results concerning the resolutions of X, if such exist (see Ch. II §1).

Suppose that X is embedded in an ambient nonsingular space M. By a work of Hironaka we know that if the ground field has characteristic zero then it is possible to resolve X by performing a sequence $M' \to M$ of monoidal transformations on M at nonsingular centers. A single monoidal transformation is a simple object. It is a little more complicated than its center, which can be defined locally by setting suitable local parameters equal to zero. But a sequence of more than two or three such transformations can be most involved.

In this work we use two, related methods for studying sequences of monoidal transformations. The first is via the concept of a place. A place is any valuation ν of the field K of rational functions of M which gives the order of vanishing along some codimension one subvariety D of some model M' of K. To each finite place of M there corresponds a finite sequence $M_{(n)} \to M_{(n-1)} \to \cdots \to M_{(0)} = M$ of monoidal transformation of a particular type, and vice versa. The restriction on the transformation is that at each stage $M_{(i+1)} \to M_{(i)}$ the center of the next transformation should dominate the center of the last (Ch. I, §4).

The second approach imposes restrictions upon the centers in a different way. We give here the general definition of a toral transformation.

<u>Definition</u>. Suppose that D is a reduced and effective divisor on the nonsingular variety M. A sequence $M_{(n)} \to M_{(n-1)} \to \cdots \to M_{(0)} = M$ of monoidal transformations is *toral* with respect to D if at each stage the center of the monoidal transformation $M_{(i+1)} \to M_{(i)}$ is, locally, a transverse intersection of components of the reduced total transform $D_{(i)}$ of D on $M_{(i)}$. The places associated to the components of $D_{(n)}$ are called toral.

If D is assumed to be a normal crossings divisor then the resulting theory is very similar to the pre-existing theory of torus embeddings. The former theory can be developed using concepts and insights derived from the latter. Moreover, there exists locally a differential ω of top degree such that the toral places are precisely those finite places along which ω has a pole of order one. Except for a few examples, the general theory of toral modifications is unexplored (Ch. III, §1, §2, §5.3).

Conversely, the singularities of a variety X can be used to produce some places of X. Suppose that the differential ω is regular on X. The smallest value $\mathrm{ord}_X(\omega)$ of $\nu(\omega)$ as ν varies over all singular places of X is called the order of ω; the ν that achieve the minimum are called minimal with respect to ω. Here are some results. Suppose that ω is an n-fold differential of top degree. Its order is either $-\infty$, or it is at least $-r$. If equal to $-r$ and the singular locus of X admits a resolution $X' \to X$ to a normal crossings divisor E then the minimal places are obtained by performing monoidal transformations on X' that are toral with respect to E. If larger than $-r$ then the minimal places appear on any resolution $X' \to X$ of X to a normal crossings divisor. (Such places are called essential.) Somewhat similar results hold if ω is not of top degree, although they are not so striking. The same techniques are applied also to embedded resolution and logarithmic order (Ch. II, §1-4).

The minimal and essential places capture at least some of the geometric structure of a given singularity.

How much we cannot yet say, for the results are not definitive and the range of examples is not representative. Presently, examples are limited to those hypersurface singularities that can be resolved by a modification toral with respect to a normal crossings divisor. For such singularities, the Newton polyhedron contains a good part of the geometry. Some examples appear in Chapter III §5.

The methods and results used are valid without hypothesis on the characteristic of the ground field. The pathologies of finite characteristic are exemplified by the properties of differentiation. It may be that this work can be used to give insight into the resolution of singularities over a ground field of finite characteristic.

Various results and questions concerning the completion of nonsingular algebraic varieties arise. If D is a normal crossings divisor then its fan Σ is defined as a simplicial complex whose vertices are the places corresponding to the components of D (Ch. III, §2.3). It is convenient to assume that any intersection of components of D is connected. To each completion M of the nonsingular variety U by a normal crossings divisor D, a fan Σ is associated.

The first problem is this. Suppose that σ is a finite set of places of U. Does it appear in the fan of some completion of U? The question arises from the study of minimal and essential places (Ch. II, §3.6).

Now suppose that M and M' are completions of U whose fans are equal. Is it true that the natural birational map M --→ M' is biregular? Solutions to these problems are provided in the case where all the places of D are toral with respect to a given normal crossings divisor (Ch. III, §2).

Of the many birational invariants of an algebraic variety X, its canonical ring is perhaps the most important. We compute the local version of this ring for those hypersurfaces $X \subset \mathbb{A}^n$ that may be resolved by a modification toral with respect to the co-ordinate hyperplanes. It follows immediately that in this case the ring is finitely generated. We indicate the difficulties that arise when the same procedure is applied to a hypersurface of $\mathbb{P}^n$. Almost certainly, these difficulties can be overcome (Ch. III, §4).

In part, this work is intended to pioneer a general theory of toral modifications. The value of such a theory resides largely in its ability to enlarge our understanding and knowledge of algebraic varieties. It is optimistic to expect such a theory to solve the problem of the finite generation of the canonical ring. But not, I believe, foolish.

CHAPTER I : FUNDAMENTAL CONCEPTS

In order to establish notation and conventions we review in the first three sections of this Chapter the results and definitions we will need from the foundations of the subject. In the final section we discuss places and prove a number of results for which no suitable reference exists.

§1. VARIETIES AND EXPONENTS

1. Affine varieties

Throughout k will denote an algebraically closed field, of any characteristic. An algebra A over k that is finitely generated and without divisors of zero is said to be a *co-ordinate ring.* Suppose that $f_1, \ldots, f_n$ generate A over k. Let $I \subset k[T_1, \ldots, T_n]$ be the ideal of relations between the f_i. Let X denote the set of points in k^n at which every element of I vanishes. We say that X is an *affine algebraic variety* and that $A = k[X]$ is its co-ordinate ring. Clearly, every element f of k[X] determines a map $f: X \to k$. If we chose $g_1, \ldots, g_m$ to generate A then we would obtain a subset X' of k^m. But because X and X' have the same - not isomorphic but same - co-ordinate ring we say that they are equal - not isomorphic.

The n-fold product k^n can be given the structure of a vector space. An affine algebraic variety X is said to be *affine n-space* $\mathbb{A}^n$ if its co-ordinate ring k[X] is isomorphic to $k[T_1,\ldots,T_n]$. The variety $\mathbb{A}^n$ has no natural vector space structure. Any other affine variety X can be thought of as a closed subset $X \subset \mathbb{A}^n$ for n large enough.

Elements of k[X] will be called *regular functions*. Polynomials are elements of $k[T_1,\ldots,T_n]$. A regular map $f:X \to Y$ between affine algebraic varieties is a mapping from the points of X to the points of Y that respects regular functions. Equivalently, it is a k-algebra homomorphism $k[Y] \to k[X]$.

If f is a regular function on X then V(f) if the closed subset consisting of all points of X at which f vanishes. $V(f_1,\ldots,f_n)$ is defined similarly.

A *rational map* $f:X \dashrightarrow Y$ between two algebraic varieties is a regular map $f':U' \to Y$ where U' is an open subset of X. The rational map $f'':U'' \to Y$ is said to be equal to f' if f' and f'' agree on $U' \cap U''$. The largest open subset U on which a rational map $f:X \dashrightarrow Y$ is regular is called the *domain of definition* of f. The complement Z of U is called the *indeterminacy locus*.

A rational map $f:X \dashrightarrow k$ is called a rational function. The collection of all rational functions forms the *field* k(X) *of rational functions*. It is merely the field of fractions of $k[X]$. Its transcendence degree d is the *dimension* of X.

The *local ring* at a point P of X is the subring of $k(X)$ consisting of all rational functions $f: X \dashrightarrow k$ *regular* at P. In other words, that P belongs to the domain of definition U of f.

to an affine variety

Clearly a rational map $f: X \dashrightarrow Y$ is simply a homomorphism $k[Y] \to k(X)$. If $k(Y)$ and $k(X)$ are isomorphic via f, we say that f is a birational isomorphism. Usually, we will start with a field K, finitely generated over k, and consider different varieties X, X' etc. all of whose co-ordinate rings are K. In other words $k[X]$ is a subring of K etc. Varieties denoted by the same letter will be birationally isomorphic, and will be referred to as different *models* for the given field of rational functions. Often, we will call a **regular** map $X' \to X$ a *modification* **or birational modification of X.**

A variety is said to be normal if all its local rings are integrally closed. Every variety X has a normalization $\bar{X}$. The singular locus of normal varieties has codimension at least two. For details concerning this, see [35, Ch. II §5]

A warning about notation. A *closed set* always means a set that is locally defined by the vanishing of some finite number of regular functions. Its complement is called an *open set*. A variety is almost always irreducible, unless it is a subvariety, in which case it may very well be reducible.

2. Projective varieties

It has been known for several centuries that many properties of affine space $\mathbb{A}^n$ and its subvarieties are best understood by first adding the hyperplane at infinity. More precisely, the point of *projective* space $\mathbb{P}^n$ are the lines through the origin of a vector space V over k of dimension n+1. Suppose that H is a hyperplane in V not passing through the origin. In a natural way we associate to each set X of points in $\mathbb{P}^n$ a set $X \cap H$ of points in H. Now, H is in a natural manner an affine algebraic variety. We say that X is a *projective algebraic variety* if $X \cap H$ is an affine variety for every hyperplane H.

Moreover, $X \cap H$ can be thought of as a subset at X in a natural manner. We thus arrive at the notion of an *open affine cover*.

The *homogeneous co-ordinate ring* of $\mathbb{P}^n$ is the ring k[V] graded by the condition that linear functions $f: V \to k$ have degree one. We have no need for this notion.

By a *quasi-projective* variety we mean an open subset of a projective variety. By making use of affine open covers, we can extend the notions previously defined to this wider class of varieties.

Theorem 1. Suppose that X is a projective variety. Then for any variety Y the projection $Y \times X \to Y$ carries closed set to closed sets.

The proof of this theorem appears in [35, Ch. I §5.2].

Definition. A map $\pi : Y \to X$ is *projective* if we may write it locally as $Y \to \mathbb{P}^n \times X \to X$ where the first map is the inclusion of a closed set, and the second the natural projection.

3. Abstract varieties

This notion is not yet forty years old. It is motivated in part by the 'atlas of charts' approach to differential manifolds. Its main utility is that it frees us from the necessity of embedding our varieties in a projective space of sufficiently high dimension before we may perform quite natural geometric constructions. The price that is to be paid, however, is that we need to check that the resulting variety is *separated*. This can be burdensome at times. We will not give the appropriate definitions and results here. The interested reader will find them in [35, Ch. VI].

We will need, however the following.

Definition. A variety X is *complete* if, for any Y the projection $p : Y \times X \to Y$ carries closed sets to closed sets.

A map $X' \to X$ is *proper* if for every variety Y the natural map $p:X' \times Y \to X \times Y$ carries closed sets to closed sets. Clearly, projective varieties are complete and projective maps proper.

The next result shows that these notions are not so far from that of being projective.

Chow's Lemma (i) Suppose that X is a separated variety. Then there is a projective birational modification $X' \to X$ and a projective variety $\bar{X}'$ such that $X' \dashrightarrow \bar{X}'$ is regular, being the inclusion of an open set.

(ii) Suppose that $X' \to X$ is a birational modification. Then there is a projective modification $X'' \to X'$ and a projective modification $\bar{X}'' \to X$ such that $X'' \dashrightarrow \bar{X}''$ is regular, being the inclusion of an open set.

This is not the standard form of Chow's lemma, but it is exactly that which we will need in §4.3. The proof is a straightforward combination of the constructions used in [35, Ch.VI 2.1 and Exercise 7] and [16, p.107], and so is left to the reader.

4. Divisors

Suppose that X is a, say, nonsingular algebraic variety and that $X' \to X$ is a surjective modification. Of all the subvarieties D of X, those of codimension one have a certain stability. We think of X' and X as being two

largest
different partial completions of their common open set U. If D_o is a closed subset of U, we will use D to denote its closure in X and D' its closure in X'. We will say that D' is the *strict transform* the closed subset D of X. This definition is sound so long as D meets U. That D **has** codimension one guarantees this, while if D has codimension two or more the *Proj* construction [16, Ch. II §7] enables us to 'blow up' D to a closed subset of codimension one.

This situation is dual to that of points. Suppose that $X' \to X$ is a surjective birational map and that X is, say, non-singular. To each point P' of X' corresponds a unique point P of X. To each closed subset D of X whose codimension is one, there corresponds, as we have just seen, a D' of X'.

<u>Definition</u>. Suppose that X is an algebraic variety. An irreducible subvariety D of codimension one is called an *irreducible* or *prime divisor*. A *divisor* is a formal sum over $\mathbf{Z}$ of prime divisors. Its *support* is the union of the prime divisors whose co-efficient is not zero. A divisor is *effective* if no negative co-efficients appear, and it is *reduced* if only 0 and 1 appear. The *reduction* of an effective divisor is also thought of as its support.

If SingX has codimension at least two then we can associate - see §4.1 - a divisor to every rational function f of k(X). Such divisors are *principal*. *Locally principal* means exactly what it says. These are also known as Cartier divisors, while the original notion are also known as Weil divisors. If X is as above - say X is normal - the Weil divisors modulo principal divisors forms the (divisor) *class group* Cl(X) while Cartier modulo principal is the *Picard group* Pic (X).

We will see in §2.2 that if X is non-singular then every divisor is locally principal. The proof of this result depends on a theorem of Gauss. He showed that $\mathrm{Cl}(\mathbb{A}^n) = 0$ or, in his own language, that each polynomial $f \in k[T_1,\ldots,T_n]$ has a unique factorization into prime factors (up to multiplication by units and order, of course).

Finally, suppose that D is a reduced effective divisor of X and that $\phi: X' \to X$ is a birational modification. The *reduced total transform* of D is the reduced effective divisor whose support is $\phi^{-1}(D)$. This definition makes sense only if $\phi^{-1}(D)$ actually is a union of irreducible components of codimension one. We will see in §3.2 that this is so if X is non-singular, or even if every divisor D has a multiple rD that is locally principal.

5. Exponents

We develop here notation concerning repeated multiplication. Suppose that $u_1,\dots,u_n$ is a sequence of elements of a field. If $i_1,\dots,i_n$ is a sequence of integers, we will denote the product

$$u_1^{i_1},\dots,u_j^{i_j},\dots,u_n^{i_n}$$

by u^I, where u stands for $(u_1,\dots,u_n)$ and I stands for $(i_1,\dots,i_n)$. In this context we will call I variously a *multi-index*, an *exponent*, or, even an *index*, while the product u^I is called a monomial in u. If I is a subset of $\{1,\dots,n\}$ we will use u^I to denote the product of those u_j for which $j \in I$. In this context, ^ over a symbol means that it is omitted from the expression. For example $\hat{2}.3.5 = 3.5 = 15$.

We will develop this notation. If $u = (u_1,\dots,u_n)$ is a sequence of n field elements and C is an $n \times m$ matrix then we use u^C to denote the sequence of m functions $(u^{C_1},\dots,u^{C_m})$ where C_i is the i^{th} column of C, thought of as an exponent. If the matrices C and D may be multiplied together then the identity

$$(u^C)^D = u^{(CD)}$$

follows from the usual properties of multiplication.

In fact, we have that

$$\prod u_i (\Sigma\ C^i_j\ D^j_k) = \prod (\prod u_i C^i_j)\ D^j_k$$

where we follow the Einstein convention of summing or multiplying over the repeated indices.

Transformations of the type described here were introduced into algebraic geometry by Zariski [38].

6. Simple weights

In our notation, an exponent corresponds to a column vector and the sequence u of functions to a row vector. We will call a row vector $\alpha = (\alpha_1,\ldots,\alpha_n)$ of integers a *weight* or *weighting*, and αI will be called the *degree* of I with respect to α.

The weight α will be said to be *simple* if its elements $\alpha_1,\ldots,\alpha_n$ have no common factor. A set $\alpha^1,\ldots,\alpha^m$ of weights is said to be simple if, whenever the equation

$$\lambda_1\alpha^1 + \ldots \lambda_m\alpha^m \in \mathbf{Z}^n \qquad \lambda_i \in \mathbb{Q} \qquad (1)$$

is satisfied, the co-efficients λ_i lie in $\mathbf{Z}$. This agrees with the previous definition. The reader is asked to check that the pair (1,2) and (2,1) of weights is not simple. Clearly, a simple set of weights is linearly independent and so part of a basis for $\mathbb{Q}^n$.

Theorem 1 A collection $\alpha^1,\ldots,\alpha^m$ of weights is simple just in case it is part of a basis $\alpha^1,\ldots,\alpha^n$ of the free abelian group $\mathbb{Z}^n$.

Proof. If $\alpha^1,\ldots,\alpha^m$ is simple then the quotient $G = \mathbb{Z}^n/\langle\alpha^1,\ldots,\alpha^m\rangle$ is an abelian group without torsion. We can, if $m < n$, find a simple element α of G, which is represented by an element α^{m+1} of $\mathbb{Z}^n$. We can apply this process to $G' = G/\langle\alpha\rangle$ and thus obtain by repetition the desired basis. The converse is trivial.

It is in practice laborious to produce such an extension for a simple set of weights. A careful analysis of the proof that we give should convince the reader of this fact. We will use the theorem in the following form.

Corollary. Suppose that $\alpha^1,\ldots,\alpha^m$ is simple. Then there are integer matrices A and B which satisfy

$$AB = BA = \text{identity matrix}$$

and also that the i^{th} row of A is α^i.

The proof of this result is simply the application of Cramer's rule to the matrix A obtained by extending α^i to a basis. Again, we remark that the labour involved in such a computation can be prodigious.

§2. DIFFERENTIAL FORMS

1. Definitions

In differential geometry one starts with the notion of the derivative of a function, and uses it to define the ring of differentiable functions. In algebraic geometry, we start with the ring of regular or polynomial functions and define from it the notion of derivative [16, Ch. II §8].

Definition. A map $d:A \to D$ from a k-algebra A to an A module D is called a *derivation* if it satisfies the product formula

$$d(fg) = f\,dg + g\,df.$$

and $dc = 0$ for c any element of k.

Of all the derivations that the ring A possesses, one is distinguished. Through it any other derivation will factor.

Definition. The module $\Omega^1(A)$ of *Kähler differentials* of the ring A is defined in the following manner. In the ring $A \otimes_k A$ take the ideal I generated by all elements of the form $1 \otimes a - a \otimes 1$. The quotient I/I^2 has the structure of an A module, and it is called $\Omega^1(A)$. The (*Kähler*) *derivative* df of the element f of A is the residue of $1 \otimes a - a \otimes 1$ in $\Omega^1(A)$.

If A is the co-ordinate ring of an affine algebraic variety X we will write $\Omega^1[X]$ in the place of $\Omega^1(A)$, while if A is $k(X)$ we will write $\Omega^1(X)$.

It is easily seen that this construction respects localization and so, in particular, $\Omega^1[X]$ can be thought of as a subset of $\Omega^1(X)$.

If X is not an affine variety a little more care needs to be taken in the definition of $\Omega^1[X]$. In that case, we say that $\Omega^1[X]$ consists of those elements ω of $\Omega^1(X)$ that lie in $\Omega^1[U]$ for every affine open subset U of X.

An element ω of $\Omega^1(X)$ is called a *rational differential form.* If ω lies in $\Omega^1[U]$ we say that it is *Kähler* on U. If ω is a rational differential, it can be written as a sum of products of the form $f_i\, dg_i$ and so ω is Kähler on the open subset U of X on which the functions f_i and g_i are regular.

The differential previously defined have degree one. For higher differential forms we need the Grassman or exterior algebra. Its definition and properties as found in, say [16, p.127] are assumed. By $\Omega^n(A)$ we mean the n^{th} exterior power of $\Omega^1(A)$. We can, moreover, form tensor powers of differential forms. By a differential ω of *type* $(r_1,\ldots,r_s)$ we mean an element of

$$\Omega^{r_1}(A) \otimes_{k(X)} \cdots \otimes_{k(X)} \Omega^{r_s}(A)$$

and we will say that an element ω of $\Omega^n(A)$ is *simple,* and of degree n. The preceding remarks apply to these more general differentials also.

The space module $\Omega^n(X)$ is a vector space over the field k(X). The next result, which is [35, Ch. III §4.4, Theorem 3], gives its dimension.

<u>Theorem 1</u>. If d is the dimension of X then $\Omega^n(X)$ has dimension over k(X) equal to the binomial co-efficient $\binom{d}{n}$.

Thus, $\Omega^n(X)$ has dimension zero if $n > d$, and one if $n = d$. A differential of type $(d,\ldots,d)$ will be said to be an r-fold *differential* of *top degree*, if r is the number of repetitions of d. A one-fold differential of top degree is called, of course, a simple differential of top degree.

If $f_1,\ldots,f_n$ is a sequence of functions and I is a subset of $\{1,\ldots,n\}$ then by df_I we will mean the differential of degree #I obtained by taking, in the natural order, the exterior product of the functions f_i, for $i \in I$. The reader is asked to provide, if wished, similar conventions for differentials of more complicated type. In the sequel, differentials of top degree are often used. By df_I, without reference to I, we will mean $df_1 \wedge \ldots \wedge df_n$.

If X is an affine variety whose co-ordinate ring is generated over k by $f_1,\ldots,f_n$ then $\Omega^1[X]$ is spanned by the differentials $df_1,\ldots,df_n$ and every differential ω satisfies an equation

$$\omega = \Sigma\, \omega_I\, df_I$$

where the summation is over appropriate indices I, and the co-efficient functions ω_I are regular on X.

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$$\omega = \Sigma\, \omega_I\, df_I$$

where the summation is over appropriate indices I, and the co-efficient functions ω_I are regular on X.

2. Simple points

A point P on an algebraic variety X is said to be simple if the tangent space to X at P is a vector space whose dimension is the same as that of X [35, Ch. 2 §1.2]. In fact, a point P is simple just in case it has an affine open neighbourhood U such that there are functions $u_1, \ldots, u_d$ ($d = \dim X$) regular on U whose derivatives $du_1, \ldots, du_d$ span $\Omega^1[U]$ over $k[U]$. Such a sequence of functions is called a *system of local parameters* for U. If f is a function regular on such a U then

$$df = \Sigma\, f_i\, du_i$$

for unique co-efficients f_i regular on U. The f_i are called the partial derivatives of f with respect to $u_1, \ldots, u_d$. If f is given a formal power series expansion then the f_i are calculated in the usual manner.

More generally, if ω is a rational differential then the equation

$$\omega = \Sigma\, \omega_I\, du_I$$

has a unique solution for co-efficients ω_I and ω is Kähler on U just in case the ω_I are all regular on U. For a proof of this result, see [35, Ch. III §4]. For a proof of the next, see [35, Ch. II §1.4].

Theorem 2. The simple points of an (irreducible) variety X form a dense open subset.

We will use Sing X to denote the closed subset of singular (i.e. not simple) points. If X is normal then Sing X has codimension at least two. If Sing X is empty we say that X is non-singular. We will usually use the letter M to denote a non-singular variety. Such varieties enjoy many geometric properties. Some are stated below, and in §3.

Theorem 3. Suppose that M is non-singular.

(i) If $X \subset M$ is irreducible and of codimension one then on each set U of some affine open cover of M there is a regular function f such that $X = V(f)$. Such an f is called a *local equation* for X on U. Sing X is defined locally by the equations $df = 0$ and $f = 0$.

(ii) Suppose that N is a non-singular subvariety of codimension n. Then on each U of some cover there are local parameters $u_1,\ldots,u_d$ such that $u_1 = \ldots = u_n = 0$ are equations defining N.

The proof of this result can be found in [35, Ch. II §3].

We use the notion of a transversal intersection of subvarieties - a definition can be found in [35, Ch. II §2.1]. A divisor D on a smooth variety M is said to have normal **crossings if the components of its support Supp D are transverse.** Equivalent to this is the condition that on each open subset U of some open cover of M there are local parameters $u_1,\ldots,u_d$ such that $u^I = 0$ is a local equation for Supp D.

3. Poincaré residue

The Poincaré residue homomorphism is a generalization to higher dimension of the Cauchy residue of a contour integral in the theory of functions of a single complex variable. In the latter theory one takes, for instance, a function $f(z)$ with a simple pole at the origin and applies to it the operator $\int_C dz$ where C is a circular path around the origin. The conclusion is that

$$g(0)\ 2\pi\sqrt{-1} = \int_C f(z)\, dz$$

where $g(z) = zf(z)$.

In defining the Poincaré residue, we make a number of changes:

- the definition will be purely algebraic
- we will 'integrate' a differential form ω, not a function, which is of top degree
- the differential ω will have a simple pole along a subvariety X of codimension one
- the residue will be a differential form of top degree on X.

The definition will first be made in a special case and then extended.

Theorem 4. Suppose that $X \subset U$ is a non-singular subvariety of the smooth variety U. Suppose also that $f = 0$ is a local equation for X and that ω is a simple differential of top degree such that $f\omega$ is regular on M.

Consider the equation

$$\omega = \frac{df}{f} \wedge \eta. \qquad (2)$$

Then (i) All solutions to (2) have the same restriction to X.

(ii) The equation (2) has a solution.

<u>Definition</u>. The restriction to X of a solution η of (2) is called the Poincaré residue Res(ω) of ω on X.

<u>Theorem 4</u> (Cont.)

(iii) Res(ω) is Kähler on X

(iv) Res(ω) depends only on X and ω.

Proof. (i) If η and η' are two solutions then their difference $\bar{\eta}$ will satisfy the equation $df \wedge \bar{\eta} = 0$. As X is non-singular there is, at least locally, a system $u_1, \ldots, u_d$ of local parameters for U such that $u_1 = f$. Clearly, $\bar{\eta}$ necessarily lies in the ideal of the Grassmann algebra generated by df. So the restriction of $\bar{\eta}$ to X is zero.

(ii) Using the above system of local parameters we can write ω as $\frac{1}{f}\, \omega_o\, df \wedge du_2 \wedge \ldots \wedge du_d$ where ω_o is a function regular on U. Then the differential (-1) times $\omega_o\, du_2 \wedge \ldots \wedge du_d$ will satisfy (2). More generally, if $u_1, \ldots, u_d$ are local parameters and $df = \Sigma\, f_i\, du_i$ then we can solve the equation with the differential

$$\omega_o \left(\frac{du_1 \wedge \ldots \widehat{du_i} \ldots du_d}{f_i}\right) (-1)^{i-1} \qquad (3)$$

so long as f_i does not vanish on X. As X is non-singular, at none of its points do all the f_i vanish.

It is clear from the proof of (ii) that (iii) is satisfied.

(iv) The condition on ω clearly does not depend upon the choice of local equation f for X. Any other local equation $\bar{f}'$ is equal to vf for some regular function v not vanishing on X.

We will use the notation of (ii) and formula (3) for η. The partial derivatives $\bar{f}_i$ of $\bar{f}$ satisfy the equation

$$\bar{f}_i = v_i f + v f_i$$

while the co-efficient ω_o satisfies

$$\bar{\omega}_o = \omega_o v$$

and so the difference between the two expressions (3) for η is that one has

$$\bar{\omega}_o/\bar{f}_i = \omega_o v/(v_i f + vf_i)$$

in the place of ω_o/f_i. But as f vanishes on X, the two expressions are equal on X.

We can now conclude the definition. If $X \subset M$ is a irreducible and of codimension one, we can find an open subset U of M on which X is non-singular. If we use U to obtain a differential Res(ω) on X, it is clear from the theorem that the choice of U is irrelevant. However, the argument shows only that Res(ω) is Kähler at the non-singular points of X.

Definition. A differential ω on a variety X is *regular* if it is Kähler at the non-singular points of X.

The next result can be proved by use of local parameters to express differentials.

Theorem 5. Suppose that ω is a rational differential of the smooth variety X. Then the set Z of points at which ω is not Kähler is closed, and each of its components has codimension one.

4. Comments

The behaviour of differential forms on a singular space is delicate. Already we have the concepts of Kähler, regular, and rational.

In Chapter II we introduce two new concepts, those of logarithmic and of the first kind. The following inclusions exist.

Kähler $\subset$ logarithmic $\subset$ first kind $\subset$ regular $\subset$ rational

The conditions of Chapter II respect proper birational modifications. In other words, if $X' \to X$ is such then ω is logarithmic or the first kind on X just in case it is on X'.

The same is not true for Kähler and regular.

§3. NON-SINGULAR VARIETIES

1. Rational maps

Theorem 1. Suppose that X is a non-singular variety and that $\phi: X \dashrightarrow \mathbb{P}^n$ is a rational map. Then there is a set Z of codimension at least two such that ϕ is regular on X-Z. (The proof appears in [35, Ch. II §3.1] and depends upon the fact that in the local rings of X factorization is unique. That X maps to $\mathbb{P}^n$ is not important.)

Corollary. Suppose that X is non-singular and $\phi: X \dashrightarrow Y$ is a rational map and Y is complete. Then Z as above exists.

Proof. We may suppose by Chow's lemma that $Y \subset \mathbb{P}^n$. Let $\psi: X-Z \to \mathbb{P}^n$ be given by the theorem. Necessarily, the image of ψ lies in Y.

2. Birational morphisms

Suppose that $\psi: X' \to X$ is a birational morphism. Then there are open subsets U' and U of X' and X that are isomorphic via ϕ. The complement E in X' of the largest such U' is called the *exceptional locus* of $X' \to X$. Suppose that P' is a point of X' and that $P = \phi(P')$ is the corresponding point of X. Always, the local ring of P is contained in that of P'. Moreover, P' is in E just in case these rings are unequal. Thus, E consists of all points P' of X' such that the rational map $\phi^{-1}: X \dashrightarrow X'$ is not defined at P.

If X is non-singular then more can be said.

Theorem 2. Suppose that $\phi: X' \to X$ is a birational map and X is non-singular. Then E is a union of irreducible subvarieties of codimension one, and $\phi(E)$ has dimension two or more. In other words, E is contracted by ϕ.

Proof. Suppose that $P' \in E$ and $P = \phi(P')$. As the local rings of P and P' are different, we can find a function f regular at P' but not at P. Moreover, we can insist that $f(P') = 0$. But the map is birational so $f = g/h$ for functions g and h regular at P. Because P is a simple point we can choose g and h so that the varieties $V(g)$ and $V(h)$ have no components in common. Now consider the equations $g = h = 0$. On X they define a subvariety of codimension two. But on X', $fh = g$ and so they define a subvariety of codimension one. This concludes the proof.

We do not need to assume that X is non-singular. It is enough that every element f of $k(X)$ have a power f^r such that $f^r = g/h$ for functions g and h regular at P for which the subvarieties $V(g)$ and $V(h)$ have no components in common. This is true iff the local class group at every point P of X is a torsion group. Equivalently, if every Weil divisor has a multiple which is locally principal. The theorem is thus valid under this weaker hypothesis. This result tells us something about small resolutions (Chapter II §1.2) and the like.

3. Monoidal transformations

The simplest method of obtaining a birational morphism with an exceptional locus is to perform what is known as a monoidal transformation. It is also known as blowing up [16], [32] and as the σ-process [35]. **Twice (p34, 70) we use the more general Proj construction [16, II§7].**

Definition. Suppose that $u_1,\ldots,u_n$ are part of a system of local parameters on a smooth affine variety M and that the equations $u_1 = \ldots = u_n = 0$ define an irreducible non-empty subvariety D. The *monoidal transformation* $M' \to M$ *centered at* D is the variety obtained by patching together the affine varieties M_i' whose co-ordinate rings are provided by the equation

$$k[M_i'] = k[M][u_1/u_i,\ldots,u_n/u_i]$$

along their common open subsets M_{ij}'.

The properties of this construction are discussed in [35, Ch. VI §2.2]. Firstly, the construction of $M' \to M$ depends only upon D, and not at all on the choice of the u_i. However, D must be non-singular if we are to find such u_i. Thus, given a non-singular $D \subset M$ we can find an open cover of M such that on each open set U there are local parameters $u_1,\ldots,u_n$ defining D. Each open set has associated to the modifications U_i', which can be patched to give U'. Then the U' themselves can be patched together to give the desired modification $M' \to M$. As before, this is called the monoidal transformation of M along D, and does not depend upon the choices made for the cover and the local parameters.

Abstract varieties were introduced into algebraic geometry by Weil [42] so that he could perform constructions without the cumbersome necessity of first embedding all varieties into projective space of sufficiently high dimension. He wished to be able to construct abelian varieties. He later showed that all abelian varieties are indeed projective [16, p.105]. It is in fact true that if M is projective then so is M'. The easiest way to obtain this result is to use the cohomology theory of coherent sheaves [41].

The reader will appreciate that an explicit description of a sequence of monoidal transformations can become most complicated. The theory of toral modifications, say of a normal crossings divisor D on M, is an elegant solution to the problem of explicit description. The reader will see in Chapter III §2.4 that the local definition of such modifications generalizes that of a monoidal transformation.

Suppose that $\sigma: M' \to M$ is a monoidal transformation whose center is D. The fiber $\sigma^{-1}(D)$ has codimension one. It is in fact a projective space bundle over D [35,loc cit] and the varieties $M' - \sigma^{-1}(D)$ and M-D are isomorphic. If D is already of codimension one then M' equals M.

Differential forms have their own distinct law of transformation under a monoidal transformation. For example suppose that $\tilde{\mathbb{A}}^2 \to \mathbb{A}^2$ is the monoidal transformation centered at a point P. If $k[x,y]$ is $k[\mathbb{A}^2]$ and the equations $x = y = 0$ define P then $\tilde{\mathbb{A}}^2$ has an affine cover, to which are associated the co-ordinate rings

$$k[x,u] \qquad\qquad k[v,y]$$

subject to the equations

$$xu = y \qquad uv = 1 \qquad yv = x$$

and the exceptional divisor E has equation

$$x = 0 \qquad\qquad y = 0$$

on the open sets of the cover.

The differential $dx \wedge dy$ is a generator for $\Omega^2[\mathbb{A}^2]$. As a differential of $k[x,u]$ it undergoes the transformation

$$\begin{aligned} dx \wedge dy &= dx \wedge d(xu) \\ &= dx \wedge (xdu + udx) \\ &= x\, dx \wedge du \end{aligned}$$

which vanishes along E.

The preceding observation is a simple point. We will, in Chapter II, use a generalization to obtain results concerning the resolution of singular points. In Chapter III we use the same ideas to study some sequences of monoidal transformations.

§4. PLACES

1. Definition and examples

If f is a non-zero function regular on an algebraic variety X then the closed subset $V(f)$ defined by the vanishing of f is the union of a finite number of subvarieties whose codimension is one. Conversely, given an irreducible subvariety $D \subset X$ of codimension one, we can compute the order of vanishing of a regular function f along D [35, Ch. III.§1.1]. This we do under the assumption that X is nonsingular.

By choosing a suitable affine open subset U of X we may assume that D is defined by a single equation $t = 0$. Suppose that the function f is regular on U. Then the *order of vanishing* of f along D is defined to be the largest value of n for which t^n divides f. It is denoted by $\nu_D(f)$. This order is infinite just in case f is identically zero.

It is straightforward to verify that $\nu_D(f)$ depends only upon D and f, and not upon the choice of U or t. Moreover, because

$$\nu_D(fg) = \nu_D(f) + \nu_D(g)$$

for f and g regular on U, we may extend ν_D to any rational function $h = f/g$ in a unique manner by using the formula

$$\nu_D(h) = \nu_D(f) - \nu_D(g).$$

If not every point of D is singular, we may suppose that D is non-singular by choosing a suitable open subset of X.

<u>Definition</u>. A mapping ν that assigns to each non-zero rational function $f \in K$ an integer $\nu(f)$ is said to be a *place* of K if there is a non-singular model U for K on which lies a non-singular codimension one subvariety D such that $\nu(f)$ is equal to $\nu_D(f)$ for every f. The integer $\nu(f)$ is called the *order* (of vanishing) of f along the place ν, and the pair U,D is called a *model* for the place.

If K is k(X) we will say that ν is a place of X.

<u>Theorem 1</u>. Suppose that ν is a place of K. Then

(i) $\nu(t) = 1$ for some $t \in K$

(ii) $\nu(fg) = \nu(f) + \nu(g)$

(iii) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$

(iv) $\nu(f) = 0$ for $f \in k - \{0\}$

(v) $\nu(f) = \infty$ iff $f = 0$.

This result proved in [35, loc. cit]. If ν assigns an integer $\nu(f)$ to every element of K in such a manner that (i) - (v) are satisfied then ν is what is known as a rank one discrete valuation of K. It seems probable that any such valuation is in fact a place. As we will not need such a result, I have made no attempt to prove it. A somewhat similar result is obtained in [38].

<u>Example 1</u>. Suppose that X is normal and that D is an irreducible subvariety of codimension one. As X is

normal, SingX has codimension two or more and so D is not contained in SingX. Write U = X - SingX - SingD. Then U, D $\cap$ U is a model for a place ν_D that measures the order of vanishing along D.

<u>Example 2</u>. Suppose that X is $\mathbb{A}^n$

(a) By a theorem of Gauss k[X] is a unique factorization domain and so any codimension one subvariety D is defined by a single equation f = 0. If $g \in k[X]$ then $\nu_D(g)$ is simply the number of times that f divides g.

(b) If we define $\nu(f)$ to be -deg f for $f \in k[X]$ then we obtain a place, which corresponds to the hyperplane at infinity of $\mathbb{P}^n$.

<u>Example 3</u>. Suppose that K is of transcendence degree one. Then there is a unique non-singular complete model X for K. The places of K correspond exactly to the points of X [35, Ch. II §4.5].

<u>Example 4</u>. Let $M' \to M$ be the monoidal transformation of $\mathbb{A}^2$ centered at the origin. Associated to the exceptional divisor E is a place ν_E. If $f = \Sigma\,\lambda_{ij}\, x^i\, y^j$ is a polynomial then $\nu_E(f) = \min\,\{i+j \mid \lambda_{ij} \neq 0\}$.

<u>Example 5</u>. Suppose that $M' \to M$ is a birational regular map between two smooth varieties. Then the exceptional locus E is a union of irreducible codimension one subvarieties E_i. To each E_i corresponds a place ν_i.

2. Place associated to a weight

Suppose that the non-singular irreducible subvariety D_o is the transverse intersection of the non-singular irreducible hypersurfaces $D_1,\ldots,D_n$. A place is associated to the exceptional divisor E of the monoidal transformation centered at D_o. We provide here a generalization of this modification. In Chapter III §2.3-2.6 we develop the construction introduced here to obtain a generalization of the notion of a monoidal transformation.

Definition. A sequence $\alpha = (\alpha_1,\ldots,\alpha_n)$ of positive integers without a common factor is called a *simple weight*.

Definition. Suppose that M is affine and that each hypersurface D_i is defined by a single equation $u_i = 0$. The *modification determined by the simple weight* α is the affine variety $M_\alpha \to M$ for which

$$k[M_\alpha] = k[M][u^I \mid \alpha I \geq 0]$$

is the co-ordinate ring.

It is clear that this construction depends only upon $D_1,\ldots,D_n$ and α and that moreover it respects localization and so it is in fact global. Results will be proved using the local form of the construction. The following result shows that the definition is sound.

Lemma. The ring $k[M_\alpha]$ is finitely generated.

Proof. We can find matrices A and B, all of whose entries are integers, which satisfy the equations

$$AB = BA = \text{identity}$$

and moreover α is the first row of A.

We write $U = u^B$. Thus $U^A = U^{BA} = u$. The monomial u^I is equal to U^J where $J = AI$. It follows that $k[M_\alpha]$ is generated over $k[M]$ by those monomials U^J for which the first component j_1 of J is non-negative. This concludes the proof.

Lemma. M_α is non-singular.

Proof. We may suppose that $u_1,\ldots,u_n$, $v_{n+1},\ldots,v_d$ forms a system of local parameters on M. We shall show that $U_1,\ldots,U_n$, $v_{n+1},\ldots,v_d$ forms a system of local parameters on M_α. In other words, if f is a regular function on M_α then the equation

$$df = \Sigma\, f_i\, dU_i + \Sigma\, f_i\, dv_i \qquad (4)$$

has a solution, where the functions f_i, are regular on M_α.

As $u_j = U^J$ where J is the j^{th} column of A, we see that du_j can be expressed in the form (4). Moreover, if f is regular on M then

$$df = \Sigma\, f_i\, du_i + \Sigma\, f_i\, du_i$$

for functions f_i regular on M and so *a fortiori* regular on M_α. Thus df for any f regular on M can be expressed in the form (4).

Any function f regular on M_α is a sum of products of the form $f_J\, U^J$ and so it follows immediately that df can be expressed in the form (4).

We turn now to the geometry of the map $\pi: M_\alpha \to M$. We will use D_α to denote the closed subset of M_α defined by the equation $U_1 = 0$. As U_1 is part of a system of local parameters, D_α is non-singular. It is clear that $M_\alpha - D_\alpha$ and $M - D_1 \cup \dots \cup D_n$ are isomorphic, for both varieties have $k[M][u^I]$ as their co-ordinate ring.

From the equation $U^A = u$ we see that U_1 divides u_i exactly α_i times in $k[M_\alpha]$ and moreover that $\pi^{-1}(D_o)$ is D_α.

Lemma. D_α is irreducible.

Proof. There is a natural map ϕ from $k[D_o] \otimes_k k[u^I | \alpha I = 0]$ to $k[D_\alpha]$. It is enough to show that ϕ is an isomorphism. Each element f of $k[M_\alpha]$ is a sum of elements of the form $f_o G + U_1 H$ where $f_o \in k[M]$ and $G \in k[u^I | \alpha I = 0]$ and $H \in k[M_\alpha]$. Thus, the residue of f in $k[D_\alpha]$ is a sum of elements of the form $\phi(f' \otimes G)$ where f' is the residue of f_o in $k[D_o]$ and so ϕ is onto.

As the two rings have the same dimension, the kernel of ϕ must be zero and so ϕ is an isomorphism.

Definition. The place for which M_α, D_α is a model is called the *place associated to the simple weight* α, and it will be denoted by ν_α. Clearly, $\nu_\alpha(u^I) = \alpha I$.

If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a sequence of non-negative integers without common factor, we will call it also a simple weight. By M_α, D_α, and ν_α we will mean the result of applying the construction already defined to the weight α'

obtained by omitting those α_i that are zero from α, and those D_i for which α_i is zero from $D_1,\ldots,D_n$.

3. Center of a place

We can associate to each place ν of $k(X)$ a possibly empty subvariety D of X. But before we can do this we need a result concerning models.

Theorem 2. Suppose that U_i, D_i $i = 1,2$ are two models for this place ν. Then the open subset U_1' of U_1 on which the rational map $U_1 \dashrightarrow U_2$ is regular is also a model for ν. Moverover, U_1 and U_2 contain isomorphic open subsets $\tilde{U}_i$ that are models for ν.

Proof. We can suppose that U_2 is affine and that $\bar{U}_2 \subset \mathbb{P}^n$ is a projective completion. The rational map $\phi: U_1 \dashrightarrow \bar{U}_2$ is regular away from a closed subset Z of codimension two and so $U_1 - Z$ is a model for ν.

We will next show that $\phi(D_1)$ is not contained in $\bar{U}_2 - U_2$. If it were then we could find a linear rational function f on $\mathbb{P}^n$ which vanishes on an open dense subset of $\phi(D_1)$ but not on any such subset of D_2. From this it would follow that $\nu_{D_1}(f) > 0$ and $\nu_{D_2}(f) = 0$. But U_i, D_i were assumed to be models for the same place ν. This shows that U_1' is a model for ν.

Now let $\tilde{U}_1$ be the open subset of U_1 obtained by removing from U_1' the exceptional divisor E of $U_1' \to U_2$.

If D_1 were a component of E then there would be a function f regular on U_2 that was zero on $\phi(D_1)$ but not on D_2. But this is not possible. As $\tilde{U}_1 \to U_2$ is an isomorphism onto its image, we have found suitable isomorphic open subsets.

The reader is asked to verify that the definition following does not depend upon the model chosen by using the preceding theorem.

Definition. Suppose that ν is a place of k(X) and that U, D is a model for ν.

If the rational map $\phi: U \dashrightarrow X$ is regular at no point of D we say that ν is an *infinite* place of X. Otherwise we say that ν is finite. The closure D_ν of $\phi(D)$ is called the *center* of ν on X. If D_ν has codimension two or more we say that ν is an *exceptional* place. If D_ν has codimension one and is not contained in SingX then $U = X - \mathrm{Sing}X - \mathrm{Sing}D_\nu$ is a model for ν. (see below). We will in this case say that ν is a *common* place of X, or that ν *appears* on X (as the place associated to a codimension one subvariety). If $X' \to X$ is a birational modification such that the place ν appears on X', we say that $X' \to X$ *realizes* ν (as the place associated etc.).

Theorem 3. Suppose that ν appears on X, and that D is its center. Then ν is equal to the place ν_D associated to D.

Proof. We can suppose that X is non-singular. We can find a model $U \to X$ for ν whose exceptional locus E has no components other than D_ν. But D_ν cannot be exceptional as its dimension is equal to that of its image D. Thus, U is an isomorphism onto its image and the result is proven.

Example 1. Suppose that X is normal. Then every place is either exceptional, common, or infinite.

Example 2. Suppose that D is an irreducible subvariety of X. Let $\phi : X' \to X$ be the modification obtained by first removing SingD from X then monoidally transforming along the remaining points of D and, finally, normalizing the resulting variety. The reader is asked to verify that every component E of $\phi^{-1}(D)$ has codimension one and the associated place ν_E has center D. Thus, any irreducible subvariety in the center of some place.

Example 3. The place ν_α defined in §4.2 is finite and has center D_o.

Theorem 4. (We use the notation of §4.2). Suppose that ν is a finite place of M. Write $\alpha' = (\nu(D_1),\ldots,\nu(D_n))$. Suppose $\alpha' \neq 0$ and use α to denote the simple weight that satisfies $n\alpha = \alpha'$ for some positive integer n. Then ν is a finite place of M_α.

Proof. We may suppose that M is affine and that D_i has equation $u_i = 0$. We can find a smooth modification $U \to M$ on which ν appears and for which D_ν is the only component of the exceptional locus. The reader is asked to verify that every function regular on M_α is regular on U. It then follows that the rational map $U \to M_\alpha$ is regular and so ν is a finite place of M_α.

Definition. Suppose that ν is a finite place of X and Y is a subvariety of X. The smallest value of $\nu(f)$ as f ranges over the rational functions of X regular at some point of D_ν and some point of Y that vanish along Y is called *the order of* Y *along* ν and is denoted by $\nu(Y)$. This quantity is non-negative, and is positive iff $D_\nu \subset Y$.

We will prove a result that characterizes complete varieties and proper birational maps in terms of finite places.

Theorem 5. (a) A birational morphism $X' \to X$ is proper iff X and X' have the same finite places.

(b) A variety X is complete iff every place of k(X) is finite.

Proof. We will use Chow's Lemma to reduce the result to the quasiprojective case.

(a) Suppose that $X' \to X$ is a proper birational morphism, and that ν is a finite place of X. Then there is a map $\phi: U \to X$ where U is a model for ν. Let Γ be the graph of ϕ in $U \times X$ and let Γ' be the closure of the graph of $\phi': U \dashrightarrow X'$. As $X' \to X$ was assumed to be proper the map $\Gamma' \to \Gamma$ induced by $U \times X' \to U \times X$ is surjective. The graph Γ is isomorphic to U, which is non-singular, and so we may apply [35,p265] to conclude that the rational map $\Gamma \dashrightarrow \Gamma'$ is defined away from a subvariety Z of codimension two or more. Thus, the composite map $U \to \Gamma \dashrightarrow \Gamma' \to X'$ shows that ν is a finite place of X'. That every finite place of X' is a finite place of X is trivial.

If $X' \to X$ is a not necessarily proper modification, we apply Chow's Lemma. There is a proper modification $X'' \to X'$ and a projective modification $\bar{X}'' \to X$ such that X'' is an open subset of $\bar{X}''$, all varieties being birationally isomorphic. Moreover, $X' \to X$ is proper just in case X'' is equal to $\bar{X}''$.

From the preceding argument we see that X'' and X' have the same collection of finite places, and similarly for $\bar{X}''$ and X. It is clear that X'' and $\bar{X}''$ have the same collection of finite places iff they are equal.

(b) By Chow's Lemma there is a proper modification $X' \to X$ and a projective variety $\bar{X}'$ such that X' is an open subset of $\bar{X}'$, and moreover X' and $\bar{X}'$ are equal iff X is complete. By part (a), X and X' have the same collection of finite places. By §3.1,Th 1, every rational map $U \to \bar{X}'$ is defined away from a set Z of codimension two or more, if U is smooth. So every place of $k(X)$ is finite on $k(\bar{X}')$. As before, X' and $\bar{X}'$ have the same collection of finite places just in case they are equal.

4. Discrepancy and realization

We introduce some definitions concerning the geometry and algebra of a model for a place. We then introduce a quantity which is a measure of how far a finite place is from appearing, and use it to prove a result concerning the realization of a finite place.

A function f is said to be *regular* along the place ν if $\nu(f)$ is non-negative. It is easily seen that f is regular along ν iff every model U for ν contains an open subset U', also a model for ν, on which f is regular. If $\nu(t) = 1$ then there is a model U, D for ν, for which $t = 0$ is the equation of D. Accordingly, we will say that t is a *local equation* for ν if $\nu(t) = 1$.

The functions $u_1,\dots,u_d$ are said to be *local parameters* for ν if there is a model U, D for which $u_1 = 0$ is the equation of D and $u_1,\dots,u_d$ is a system of local parameters on U.

The reader is asked to verify that $u_1,\dots,u_d$ forms a system of local parameters for ν just in case $\nu(u_1) = 1$, $\nu(u_i) = 0$ $i \geq 2$, and $du_1,\dots,du_j$ is a basis for $\Omega'(K)$.

Now suppose that ω is a simple differential of top degree and that $u_1,\dots,u_d$ is a system of local parameters for ν. The equation

$$\omega = \omega_o \, du_1 \wedge \dots \wedge du_n$$

has a unique solution for ω_o a rational function. The reader is asked to check that the quantity $\nu(\omega_o)$ does not depend upon the choice of local parameters. It is called the *order of vanishing* of ω along ν and is denoted by $\nu(\omega)$. It is immediate that $\nu(f\omega) = \nu(f) + \nu(\omega)$.

Definition. Suppose that ν is a finite place of X. The smallest value of $\nu(\omega)$ as ω varies over all simple differentials of top degree Kähler at some point P of D_ν is called the *discrepancy* of ν over X. We will denote this number by $\mathrm{dis}_X(\nu)$. It is never negative.

Example 1. If X is non-singular and $u_1,\ldots,u_d$ are local parameters on X then $\mathrm{dis}_X(\nu)$ is equal to $\nu(du_1 \wedge \ldots \wedge du_d)$.

Example 2. Suppose that ν is the place associated to the exceptional divisor of a monoidal transformation $M' \to M$ whose center D has codimension c. Then $\mathrm{dis}_M(\nu) = c-1$, a result that is easily seen by describing M' in terms of suitable local parameters on M.

Example 3. Suppose that $X \subset \mathbb{A}^2$ is defined by the equation $x^2 = y^3$. The map $\phi:\mathbb{A}^1 \to \mathbb{A}^2$, $t \to (t^3,t^2)$ provides a resolution. Let ν be the place whose center is the origin. Then it is enough to calculate $\nu(dx')$ and $\nu(dy')$ where x' is the residue of x on X etc. As $\phi^*(x) = t^3$, $\phi^*(dx') = 3t^2dt$. Similarly $\phi^*(dy') = 2t\,dt$. Thus, $\mathrm{dis}_X(\nu)$ is one, unless the characteristic is two, in which case it too is two.

Example 4. If $X \subset \mathbb{A}^2$ is the curve defined by $x^2 = y^2 + x^3$ and the characteristic is unequal to two then both of the two places centered at the origin have discrepancy zero. (If the characteristic is two then the curve is isomorphic to that of the previous example).

Notation

Suppose that M is non-singular and that ν is a finite place. For the next few pages we will use $M' \to M$ to denote the birational modification of M obtained by first removing Sing D_ν and then monoidally transforming along the remaining points of D_ν. As $M' \to M$ - Sing D_ν is projective, ν is a finite place of M'. Moreover, $M = M'$ just in case M is a model for ν. We will use $M^{(n)}$ to denote the n-fold iterate of this process.

Theorem 6. $\mathrm{dis}_{M'}(\nu) \le \mathrm{dis}_M(\nu) - \mathrm{cd}(D_\nu) + 1$.

Proof. We can suppose that D_ν is non-singular and that on M we have local parameters $u_1,\ldots,u_d$ such that $u_1 = \ldots u_n = 0$ are the equations defining D_ν, where $n = \mathrm{cd}(D_\nu)$. We can also suppose that ν is finite on the open subset M_1' of M'. The functions

$$u_1,\ u_2/u_1,\ldots,u_n/u_1,\ u_{n+1},\ldots,u_d \qquad (5)$$

are local parameters on M_1' and the exceptional divisor E has equation $u_1 = 0$.

The differential $\omega = du_1 \wedge \ldots \wedge du_d$ written in terms of the local parameters (5) is $u_1^{n-1}\omega'$ where

$$\omega' = du_1 \wedge d(u_2/u_1) \wedge \ldots d(u_n/u_1) \wedge du_{n+1} \ldots \wedge du_d.$$

As the center of ν on M' is contained in E we see that $\nu(u_1) \ge 1$. Because $\mathrm{dis}_{M'}(\nu) = \nu(\omega')$ the result follows.

<u>Corollary 1</u>. The place ν appears on M iff

$$\mathrm{dis}_M(\nu) = 0.$$

Proof. If ν appears then the discrepancy is zero. If ν does not appear then $\mathrm{cd}(D_\nu)$ is two or more. As $\mathrm{dis}_{M'}(\nu)$ is non-negative the result follows.

<u>Corollary 2</u>. $\mathrm{dis}_{M'}(\nu) < \mathrm{dis}_M(\nu)$ unless ν appears on M.

<u>Theorem 7</u>. Suppose that ν is a finite place of the smooth variety M. If $n = \mathrm{dis}_M(\nu) + 1$ then $M^{(n)}$ is a model for ν. Moreover, if $m \geq n$ then $M^{(n)}$ and $M^{(m)}$ are isomorphic.

Proof. The second assertion is an immediate consequence of the first. As $\mathrm{dis}_M(\nu) > \mathrm{dis}_{M^{(1)}}(\nu) > \ldots$ until ν appears, the place ν must appear on $M^{(n-1)}$. Then $M^{(n)}$ is a model.

<u>Definition</u>. Suppose that ν is a finite place of the smooth variety M. Then the modification $M^{(n)} \to M$, for n large, enough is called the *modification of* M *determined by* ν, and it is denoted by $M_{(\nu)} \to M$. It is a *realization of* ν *as a finite place* of M.

5. <u>Comments</u>

Dedekind and Weber gave an algebraic treatment of results of Riemann on algebraic curves [8, p.58-73]. In doing this they used methods developed by Kummer in his work on algebraic numbers. They also introduced the notion of a place of an algebraic curve. This concept is a special case of the more general concept of a valuation of a field - a concept useful not only in algebraic geometry [38] but also in the study of algebraic numbers [34] and of ideals in commutative rings [25].

In the introductory texts by Lefschetz [29] and Walker [36] the term place was used to mean a formal analytic parameterization of an algebraic curve, subject to a suitable notion of equivalence. As such, the notion was linked to the theory of Puiseux expansions and Newton's polygon.

Zariski used the general notion of a valuation in his work on local uniformization and resolution of singularities [40]. It is noteworthy that such an application of the general concept was contrary to Krull's expectations - see Zariski's preface in [40]. Zariski's paper [38] on local uniformization contains, *inter alia*, a clear outline of the general theory of valuations.

Lang, in his introduction text [27], uses the word place to stand for a concept equivalent to that of a general valuation of a field. Since then the term has passed into disuse. For example, it is not used in the introductory texts of Hartshorne [16], Mumford [32], and Shafarevich [16]. That Hironaka had in the intervening years proved resolution of singularities (in characteristic zero) [17] explains this, at least in part. Hironaka's proof involved a complicated net of inductions, and extensive use of concepts in local algebra. As a consequence, Zariski's earlier work in this area and the methods that he used were thought of as being of less interest.

The concept of place that I have introduced is, in the language of the general theory of valuations, a rank one discrete valuation associated to a codimension one subvariety of a non-singular model of the field in question. Such valuations enjoy many useful and interesting geometric properties and so deserve a shorter title. I feel that it is proper to retrieve the word place to describe these valuations.

The application of either a place or a valuation ν to a differential form ω does not appear in any of the standard references [9], [25], [34]. The word discrepancy was first used by Reid [33], to refer to a related concept.

Theorem 7, concerning the realization of a finite place of a non-singular variety, is used time and time again in the sequel. It would be useful to have a similar result for singular varieties.

CHAPTER II : ORDER AND MINIMAL PLACES

In this chapter we will study the behaviour of differential forms ω regular at the smooth points of an irreducible variety X. When X is singular our main concern will be the behaviour of ω on birational modifications $X' \to X$ and particularly on a resolution of X, if such exists. When X is non-singular our main concern is with the behaviour of ω on completions $\overline{X}$ of X, especially in the case where $\overline{X}$ is smooth and $\overline{X} - X$ is a normal crossings divisor.

In Chapter I we encountered Kähler and regular differentials. In this chapter we introduce additional concepts concerning the behaviour of differentials.

§1. DIFFERENTIALS OF THE FIRST KIND

1. Definition and first properties

Definition. A differential one form ω is *regular* along the place ν if it is a sum of products of the form $f\,dg$ where both f and g are regular along ν. A differential ω is *regular* along ν if it is a sum of products of one forms regular along ν.

Theorem 1. Suppose that ν is a finite place of X and that the differential ω is Kähler on X. Then ω is regular along ν.

Proof. We can suppose that X is affine, and that ω is a sum of products $f_I dg_I$ where the functions f_I and g_i are regular on X. As ν is finite the functions are regular along ν also.

Corollary. Suppose M is smooth and ω is regular on M. Then ω is regular along any finite place of M. If M is complete, ω is regular along every place ν of M.

Proof. As M is smooth, every regular differential is Kähler. If M is complete, every place is finite.

The corollary may not hold for singular varieties. For example, if X is the image of the map $\phi(t) = (t^2, t^3)$ from $\mathbb{A}^1$ to $\mathbb{A}^2$ then the differential dt/t is regular at all smooth points of X, but not at the origin of $\mathbb{A}^1$.

Suppose that U is a model for the place ν and that $u_1, \ldots, u_d$ forms a system of local parameters. Every differential one form ω has a unique representation $\omega = \sum \omega_i \, du_i$.

Theorem 2. The differential $\omega = \sum \omega_i \, du_i$ is regular along ν just in case the co-efficients ω_i are regular along ν.

Proof. Clearly, if the co-efficients are regular along ν then so is ω.

Conversely, suppose f and g are regular along ν. Then the open subset U' of U obtained by removing from U the polar locus of g is also a model for ν. As the function g is regular on U', in the expansion $dg = \sum g_i\, du_i$ the partial derivatives g_i are regular along on U'. And so when we write $f\,dg$ as $\sum \omega_i\, du_i$, the co-efficients are regular along ν.

Corollary 1. The differential $\omega = \sum \omega_I\, du_I$ is regular along ν just in case the co-efficients ω_I are regular along ν.

Corollary 2. The differential ω is regular along ν just in case the open subset U' of U obtained by removing from U the points at which ω is not regular is still a model for ν.

Corollary 3. The differential ω is regular on X just in case it is regular along every place ν appearing on X.

Proof. We can assume that X is non-singular. Suppose ω is not regular. By Theorem 5 of Ch. 1, §2.3 the points at which ω is not regular forms a divisor. Let D be a component of this divisor and ν the associated place. If ω were regular along ν then by Corollary 2 it would be regular at some points of D.

The converse follows from the corollary to Theorem 1.

Corollary 4. Suppose that M is a non-singular complete model for the field K and that M' is some other model. Every differential regular ω on M is regular on M'.

Proof. By Corollary 3 we need only show that ω is regular along all places appearing on M'. But this follows from the corollary to Theorem 1.

Definition. A differential ω of the field K is said to be *everywhere regular* if it is regular along every place ν of K.

Corollary 5. If the differential ω is regular on a non-singular complete model M of K, then it is everywhere regular.

The product $\omega \otimes \omega'$ of two differentials ω and ω' of top degree, with ω r-fold and ω' r'-fold, is a r + r' fold differential of top degree. If ω and ω' are everywhere regular then so is $\omega \otimes \omega'$. Thus the everywhere regular differentials of top degree form a ring, which is called the canonical ring of the field K. This ring, whose dimension is at most one more than that of K, is very important in the classification of varieties up to birational isomorphism. If K has dimension one or two, the canonical ring is finitely generated [39]. In higher dimensions the question is unanswered.

Abel first defined such differentials in his studies of integrals $\int g\,dx$ where the integrand $g\,dx$ is a rational function of x and y and where y is given implicitly as a function of x by a polynomial equation $f(x,y) = 0$. For example, $\int \sqrt{x^2 + 1}\,dx$. If $\int g\,dx$ satisfied certain conditions, he said that it was the first kind. In our terminology, the differential $g\,dx$ is an everywhere regular differential of the field of rational functions associated to the curve X defined by $f(x,y) = 0$. Abel showed that the space of such differentials was of finite dimension. This dimension is the geometric genus of X. A modern exposition of Abel's work is contained in [14].

<u>Definition</u>. A finite place ν of the variety X is said to be *singular* if its center is contained in the singular locus of X.

A differential ω, regular on X, is said to be *of the first kind* if it is regular along all singular places of X.

If M is non-singular then it has no singular places and so ω is of the first kind iff it is regular on M. In this case, ω is then regular along any finite place ν of M. Now suppose that X is singular. If the differential ω is of the first kind and ν is a finite place of X then either ν is singular, in which case ω is regular along ν; or ν is a finite place of the smooth locus of X. The differential ω is regular on this locus, by assumption, and so again, ω is regular along ν. This proves part of the next result. The remainder follows immediately from Theorem 2, Corollary 3.

<u>Theorem 3</u>. The differential ω is of the first kind on X iff it is regular along every finite place of X.

Corollary. Suppose X is complete. Then the differential ω is of the first kind iff it is everywhere regular.

Even though the variety X may have no differentials that are everywhere regular, on affine varieties there are many differentials of the first kind. For example, if the functions f and g are regular on X then $f\,dg$ is of the first kind. More generally, any differential ω Kähler on X is of the first kind. However, the following example shows that not every differential of the first kind is Kähler on X.

Suppose that X be the plane curve defined by the equation $y^2 = x^3$. It has a birational parametrization $x = t^2$ and $y = t^3$. The differential dt is of the first kind but is not Kähler on X.

Theorem 4. Suppose $X' \to X$ is a resolution of X. Then the differential ω is of the first kind iff it is regular on X'.

Proof. If ω is of the first kind it is regular along every finite place of X. As every place appearing in X' is finite, ω is regular on X'.

Conversely, suppose ω is regular on X'. As $X' \to X$ is proper, any finite place ν of X is a finite place of X'. As X' is non-singular, ω is regular along ν.

Corollary. Suppose $X' \to X$ is a resolution whose exceptional places are $\nu_1,\ldots,\nu_k$. Then a differential ω regular on X is of the first kind just in case it is regular along the places $\nu_1,\ldots,\nu_k$.

If the variety X admits a resolution then there is a finite collection $\nu_1,\ldots,\nu_k$ of places such that any differential ω regular on X which is also regular along $\nu_1,\ldots,\nu_k$ is regular on any other singular place ν of X. To find such a finite collection of places is a weakened form of the problem of resolution of singularities. At the end of this section we will associate to X a collection of places, which is finite if the singularity admits a resolution.

2. Small resolutions

The exceptional locus E of a resolution $X' \to X$ need not be a divisor. An extreme case is where every component of E has codimension at least two. In this case we say that the exceptional locus of the resolution is *small*. We will given an example of a small resolution.

Let (r,s,t,u) be co-ordinates on $\mathbb{A}^4$ and let X be defined by the equation rt = su. It is easily seen that the origin is the only singular point of X. Now suppose that (v:w) are homogeneous co-ordinates for $\mathbb{P}^1$ and that X' is the subvariety of $\mathbb{A}^4 \times \mathbb{P}^1$ defined by the equations vt = wu and vs = wr in addition to rt = su. Let π be the natural projection from X' to X.

First we show that X' is irreducible and non-singular. On the portion of $\mathbb{A}^4 \times \mathbb{P}^1$ upon which v is non-zero, the equations of X' are

$$t = \frac{w}{v} \cdot u \qquad s = \frac{w}{v} \cdot r \qquad rt = su$$

and as the third equation is a consequence of the first two the result follows. (The case of w non-zero is exactly similar.)

If the point $P = (r,s,t,u)$ is a non-singular point of X then one of the co-ordinates, say r, is non-zero. In this case the point $P' = (r,s,t,u) \times (r:s)$ is the unique point of $\pi^{-1}(P)$. This shows that $X' \to X$ is a resolution.

The exceptional locus of π is $(0,0,0,0) \times \mathbb{P}^1$.

Further results concerning small resolutions may be found in [28].

3. Order of differentials

We have just seen that a differential ω regular on X must satisfy additional conditions in order to be of the first kind, and that if $X' \to X$ is a resolution then these conditions are finite in number. More exactly, it is enough that ω be regular along the exceptional places of X'.

Contrarywise, a differential ω regular on X imposes conditions that a birational map $X' \to X$ must satisfy if it is to be a resolution. For example, if ω is not of the first kind but is regular on X' then X' is not a resolution of X.

The quantity $\mathrm{ord}_X(\omega)$, which we are about to define, refines the notion of ω being of the first kind. We will prove in this section some initial results. In §3 we will make a deeper study of the properties of this quantity.

Definition. Suppose that ν is a place and that ω is a differential form. The largest value of n for which $t^{-n}\omega$ is regular along ν, where t is a local equation for ν, is said to be the *order of* ω *along* ν and is denoted by $\nu(\omega)$.

The differential $t^{-n}\omega$ is regular along ν just in case it is regular on some model U of ν. This shows that $\nu(\omega)$ does not depend upon the choice of t, and also that $\nu(\omega)$ is non-negative iff ω is regular along ν.

Definition. If ω is a differential regular on X then the smallest value of $\nu(\omega)$ as ν runs through the singular places of X, is called the *order of* ω with respect to X and will be denoted by $\mathrm{ord}_X(\omega)$.

If $\nu(\omega)$ is without a lower bound we write $\mathrm{ord}_X(\omega) = -\infty$ while if X has no singular places we write $\mathrm{ord}_X(\omega) = +\infty$.

If the differential ω is regular on X and $X' \to X$ is a resolution then the behaviour of ω on X' is sufficient to determine whether or not ω is of the first kind. A similar result holds for the order of ω. Clearly, ω is of the first kind iff $\mathrm{ord}_X(\omega)$ is non-negative.

Theorem 5. Suppose that the differential ω is regular on X and that $X' \to X$ is a resolution whose exceptional locus E is a divisor.

If $\mathrm{ord}_X(\omega)$ is negative then $\nu(\omega)$ is negative for some exceptional place ν of X'. **(i.e. ν appears on X' but not X).**

If $\mathrm{ord}_X(\omega)$ is non-negative then for some exceptional place ν of X', the quantities $\nu(\omega)$ and ord (ω) are equal.

Proof. If $\mathrm{ord}_X(\omega)$ is negative then ω is not of the first kind and so fails to be regular along some exceptional place ν of X'.

Now suppose that $\mathrm{ord}_X(\omega)$ is non-negative. Let n denote the smallest value of $\nu'(\omega)$ for ν' an exceptional place of X'. Clearly, n is non-negative. It is enough to show that if ν is a singular place of X then $\nu(\omega) \geq n$.

If ν is a singular place its center on X' lies in E. As E was assumed to be a divisor, D_ν lies in some component $D_{\nu'}$ of E. We can then find an open subset U of X upon which ν and ν' are finite and upon which $D_{\nu'}$ has local equation t. As $\nu'(\omega)$ is at least n, the differential $t^{-n}\omega$ is regular on U, and so regular along ν.

As D_ν is contained in $D_{\nu'}$, the local equation t vanishes on D_ν. So $\nu(t)$ is at least one and $\nu(\omega) = \nu(t^n \cdot t^{-n}\omega)$ is at least n.

Definition. Suppose that the differential ω is regular on X. A singular place ν is said to be *minimal* with respect to ω if the quantities $\nu(\omega)$ and $\mathrm{ord}_X(\omega)$ are equal.

Corollary. If ω is of the first kind on X then every resolution $X' \to X$ whose exceptional locus is a divisor contains a minimal place. Unless X is non-singular.

If ω is of top degree we can prove a stronger result.

4. Essential places

Theorem 6. Suppose that ω is a differential of top degree that is of the first kind on X. Then every place ν minimal with respect to ω appears in any resolution $X' \to X$ whose exceptional locus E is a divisor.

Proof. As in the proof of Theorem 5, we use n to denote the smallest value of $\nu'(\omega)$ as ν' ranges over the exceptional places of $X' \to X$. It is enough to show that if ν is any singular place not appearing on X', then $\nu(\omega)$ is strictly greater than n.

As before, the center D_ν lies in some component $D_{\nu'}$ of E. We find an open subset U of X' on which $D_{\nu'}$ has local equation $t = 0$, and on which ν is finite. Then the differential $t^{-n}\omega$ is regular on U and $\nu(t)$ is at least one.

As the place ν is exceptional on U, its discrepancy is strictly positive and so $\nu(t^{-n}\omega) > 0$.

Therefore

$$\nu(\omega) = \nu(t^n) + \nu(t^{-n}\omega) > n.$$

<u>Definition</u>. A singular place ν that appears in every resolution $X' \to X$ whose exceptional locus is a divisor is *essential* to the resolution of the singular locus of X to a divisor, or more briefly, *essential* to the resolution of X.

If Sing X does not admit a resolution to a divisor then every singular place is essential.

Suppose X and ω are as in the theorem.

<u>Corollary 1</u>. Every minimal place is essential.

<u>Corollary 2</u>. If X admits a resolution $X' \to X$ then the number of minimal places is finite.

Proof. If the exceptional locus E of X' is a divisor then the result is immediate. Otherwise, the result follows by using the sequence of modifications of X' determined by a singular place ν.

The argument used in the proof of the theorem shows that if ν is minimal then it is realized by performing just one monoidal transformation on a subset U of X'. It is now enough to prove the next result.

Lemma. Suppose that U is smooth and that $D' \subset D$ are two irreducible subvarieties, and that ν' and ν are the places realized by monoidal transforming at the smooth locus of D and D' respectively. Suppose in addition that ω is a differential of top degree regular on U.

Then $\nu(\omega) < \nu'(\omega)$ unless D is equal to D'.

Proof. Let I be the ideal of D and J the ideal of all regular functions whose divisor on U is at least as large as that of ω. Suppose n is the largest integer such that $I^n \supset J$. A computation in local parameters will show that $\nu(\omega) = n + \mathrm{dis}_U(\nu)$.

A similar result holds for D' and so the result follows.

It is not convenient to give examples for this theorem here. In Chapter 3 we will introduce some constructions that enable us to give an explicit description of the resolution of certain singularities. In Ch. 3 §4 we will give some examples.

In this section we assumed that ω was a differential regular on a possibly singular variety, and we then examined the behaviour of ω on birational modifications $X' \to X$. Much of the results remain true if instead we assume ω to be regular on a smooth variety M and then studied its behaviour on completions and partial completions $\overline{M}$.

Of course, to insist that ω be regular on M and every completion $\overline{M}$ is to insist that ω be everywhere regular. Such differentials may not exist. For example, if M is birational to $\mathbb{P}^n$, no such forms exist whereas in studying singularities we can suppose that X is affine, in which the Kähler differentials are of the first kind.

In the next section we consider a class of differentials which seem to be more appropriate to the study of completions of smooth varieties.

It is well known that curves have a unique resolution. It follows from the factorization theorem for birational maps between surfaces [35, Ch IV §3.4] that surface singularities have a minimal resolution. Thus, every place appearing on the minimal resolution is essential. The first result concerning essential places in higher dimension is due to Reid [33, Lemma 2.3].

§2. LOGARITHMIC DIFFERENTIALS

1. Definition and examples

Suppose that f is a differentiable function. According to the chain rule, the derivative of the composite function $\ell n(f)$ is df/f. This derivative is a measure of the rate of change of f, as compared to the magnitude of f itself. (Growth is exponential when this derivative is constant.)

Even though the logarithm function does not exist for arbitrary fields, it is still possible to define the logarithmic derivative.

Definition. If f is a non-zero element of the field K, the differential form df/f is the *logarithmic derivative* of f.

Example 1. It is characteristic of differentiation that $d(fg) = f\,dg + g\,df$. If both sides are divided by fg the equation

$$\frac{d(fg)}{fg} = \frac{df}{f} + \frac{dg}{g}$$

is obtained. Thus, the logarithmic derivative of a product is the sum of the logarithmic derivatives of the factors.

Example 2. The logarithmic derivative of f^n is the logarithmic derivative of f, multiplied by n.

Example 3. Suppose that both f and f^{-1} are regular on X. Then the logarithmic derivative df/f is regular on X.

<u>Example 4</u>. Suppose $f = ug^n$ where u, u^{-1} and g are regular on X. Then

$$\frac{df}{f} = \frac{du}{u} + n\,\frac{dg}{g}$$

which is not regular along the zero locus of g, unless the characteristic divides n.

<u>Example 5</u>. If g and h multiply to give one, their logarithmic derivatives add to give zero.

<u>Definition</u>. A differential one form ω is *logarithmic* along the place ν if it can be expressed as a sum of products of the type $f\,dg/g$ with f regular along ν.

<u>Example</u>. Suppose that the differential one form ω is regular along ν. We can express ω as a sum of products of type $f\,dg$ with both f and g regular along ν. As $f\,dg$ is equal to $fg(dg/g)$ we see that ω is logarithmic along ν.

<u>Theorem 1</u>. Suppose that $u_1,\ldots,u_d$ are local parameters for a model U of ν, and that ω is a differential one form. Then

(i) ω has a unique representation

$$\omega = \sum \omega_i \frac{du_i}{u_i} \qquad (1)$$

(ii) ω is logarithmic along ν just in case all the co-efficients ω_i are regular along ν.

Proof. Part (i) is merely a reformulation of Ch. 1 §2.1, Theorem 1 .

If the co-efficients ω_i are regular along ν then ω is obviously logarithmic along ν. It remains to be shown that if f is regular along ν, then the co-efficients ω_i in the expansion

$$f \frac{dg}{g} = \sum \omega_i \frac{du_i}{u_i}$$

are regular along ν. We may as well suppose that f = 1.

If we write g as $u_1^n h$ where $n = \nu(g)$ then both h and 1/h are regular along ν. Then the subset U' of U obtained by removing the zero and polar locii of h is still a model for ν.

Then

$$\frac{dg}{g} = n \frac{du_1}{u_1} + \frac{dh}{h}$$

and the first summand is clearly of the correct form. As dh/h is regular on U', the co-efficients h_i in the expansion

$$\frac{dh}{h} = \sum h_i \, du_i$$

are regular along ν. This concludes the proof.

Corollary 1. The differential form $\omega = \sum \omega_i \, du_i$ is logarithmic along ν iff $\nu(\omega_1) \geq -1$ and $\nu(\omega_i) \geq 0$ for $i \neq 1$.

Corollary 2. Suppose that U is a model for ν and that the differential ω is regular on $U - D_\nu$ and logarithmic along ν. Then ω is logarithmic along any finite place ν' of U.

Proof. The co-efficients ω_i in (1) are regular along ν because ω is logarithmic. They are regular on $U - D_\nu$ because ω is regular. Being regular along every place appearing on U, the co-efficients are regular on U. As ν' is finite, the co-efficients are regular on ν' also. Now ω is patently logarithmic along ν'.

Definition. A differential ω is *logarithmic* along ν if it is a sum of products of differential one forms logarithmic along ν.

Corollary 3. The differential $\omega = \sum \omega_I \, du_I/u^I$ is logarithmic along ν iff the co-efficients ω_I are regular along ν.

Corollary 4. The differential $\omega = \sum \omega_I \, du_I$ is logarithmic along ν iff the order of the pole possessed by ω_I is not larger than the number of times du_1 occurs in du_I.

In particular, an r-fold differential ω of top degree is logarithmic along ν iff $\nu(\omega) \geq -r$.

2. Logarithmic at infinity

For a differential form ω to be regular along every finite place of a smooth variety M it is enough that it be regular along every place that appears on M. If we insist only that ω be logarithmic the situation is somewhat different. Consider, for example, the differential two-form $\omega = dx \wedge dy/xy(x + y)$ on the plane $\mathbb{A}^2$. It is regular and so logarithmic along every place appearing on $\mathbb{A}^2$ except the co-ordinate axes and the line $x + y = 0$.

Along the x axis we write ω as $\frac{1}{(x + y)} \frac{dx}{x} \wedge \frac{dy}{y}$ and as $(x + y)$ is a unit along the place ν corresponding to the x axis ω is logarithmic along ν. The same representation of ω shows that it is logarithmic along the y axis. To show that ω is regular along $(x + y) = 0$ we write ω as $\frac{1}{x} \frac{d(x + y)}{x + y} \wedge \frac{dy}{y}$.

Now suppose that ν_E is the place that is realized by making a monoidal transformation centered at the origin. We have seen that $\nu_E(dx \wedge dy) = 1$. Clearly $\nu_E(x) = \nu_E(y) = \nu_E(x + y) = 1$. And so $\nu_E(\omega) = 1 - 1 - 1 - 1 = -2$. According to Theorem 1, Corollary 4, ω is not logarithmic along ν_E.

The difficulties presented by this example do not arise if we suppose that the locus along which ω is not regular is a normal crossings divisor.

<u>Theorem 2</u>. Suppose that D is a normal crossings divisor on a smooth variety M and that the differential form ω is regular on M - D and logarithmic along the places associated to D. Then ω is logarithmic along any finite place ν of M.

Proof. The differential form ω has a unique representation $\omega = \sum \omega_I \frac{du_I}{u^I}$. As ω is regular on M - D, the co-efficients ω_I are regular on M - D. The differential ω was assumed logarithmic along each component D_i of D and so by Theorem 1 the co-efficients ω_I are regular along the places ν_i associated to the D_i. The functions ω_I being regular along every place appearing on M, are regular on M itself. As ν is a finite place of M the co-efficients are regular along ν. Now ω is clearly logarithmic along ν.

Definition. A differential ω regular on a smooth variety M is *logarithmic at infinity* if it is logarithmic along every infinite place of M.

Corollary. Suppose $\overline{M}$ is a smooth completion of M such that $D = \overline{M} - M$ is a normal crossings divisor. A differential ω regular on M is logarithmic at infinity iff it is logarithmic along the places associated to D.

3. A computation of discrepancy

We introduced, in Chapter I, § 4. , a place ν_α associated to a connected transverse intersection $D_1 \cap \dots \cap D_n$ of divisors and a sequence $\alpha = (\alpha_1,\dots,\alpha_n)$ of positive integers. It is now convenient to compute the discrepancy of ν_α.

Theorem 3. $\mathrm{dis}_M(\nu_\alpha) = \sum \alpha_i - 1$.

Proof. We will conserve the notation of Chapter I, § 4.2 We have local parameters $u_1,\dots,u_n$, $v_{n+1},\dots,v_d$ on M and local parameters $U_1,\dots,U_n$, $v_{n+1},\dots,v_d$ on M_α. The parameters u and U are related by the equations

$$u^B = U \qquad\qquad U^A = u$$

where the matrix A has determinant ±1, its first row is α, and B is the inverse of A.

It follows that

$$u_j = U^{j\text{th}} \text{ column of } A$$

and so by the properties of the logarithmic derivative

$$\frac{du_j}{u_j} = \sum_j \alpha_j^i \frac{dU_j}{U_j}$$

whence

$$\frac{du_I}{u^I} = |A| \frac{dU_I}{U^I}$$

by the properties of the Grassman algebra.

It now follows that

$$\frac{du_I}{u^I} \wedge \frac{dv_J}{v_J} = |A| \frac{dU_I}{U_I} \wedge \frac{dv_J}{v_J}$$

and so $\nu_\alpha(\omega) = \nu(u^I) - \nu(U^I)$ which is equal to $\sum \alpha_i - 1$.

4. Comments

The recent study of logarithmic differential forms was initiated by Deligne, in his extension of Hodge theory to singular and non-complete algebraic varieties X. To do this he used differentials ω having a local expression

$$\sum \omega_I \frac{dz_I}{z_I}$$

on some fixed completion or resolution X', the co-efficients ω_I being C^∞.

Whether or not a differential ω regular on X has such an expression on X' depends, in general, upon the choice of X'. By using Hironaka's results concerning the resolution of complex algebraic varieties, Deligne was able to show that the differentials ω induce a structure, which he called mixed Hodge structure, on the cohomology of X, which does not depend upon the choice of X'.

Iitaka showed that if the co-efficients ω_I are holomorphic functions then ω being of the required type does not depend upon the choice of X'. (These differentials are called logarithmic in this section). His argument again depended upon Hironaka's results and so was valid only in characteristic zero. Later, D. Wright realized that a modification of the argument would work in finite characteristic. This appears in the paper [21] of Kambayashi, and provides an alternative proof of Theorem 2.

Miyanishi and others have developed a classification theory for non-complete algebraic varieties that is parallel to that of Enriques and Kodaira for complete surfaces. The theory is a classification up to proper birational morphisms, and logarithmic differential forms take the place of regular differentials.

The cancellation problem of Zariski asks if the algebraic variety X satisfies $\mathbb{A}^1 \times X \cong \mathbb{A}^1 \times \mathbb{A}^2$ then must X be isomorphic to $\mathbb{A}^2$. Kambayashi, Fujita and others have used the theory of logarithmic differentials to solve this and related problems.

For further information, the reader is advised to consult the appropriate references in the bibliography.

§3. ORDER

1. Definition

Suppose that the differential form ω is regular on the singular algebraic variety X. We are concerned to investigate the behavior of ω along the singular places of X. Already we have defined the order ord(ω) as the smallest value of $\nu(\omega)$ as ν varies over all singular places. We will make a similar definition for logarithmic differentials.

Definition. Suppose that ω is a differential form and that t is a local equation for the place ν. The largest value of n for which $t^{-n}\omega$ is logarithmic along ν is called the *logarithmic order* of ω along ν and is denoted by $\nu(\omega; \log)$.

It is clear that the definition does not depend upon the choice of t. Also clear, as regular forms are logarithmic, is that $\nu(\omega) \leq \nu(\omega; \log)$.

Theorem 1. Suppose that $u_1, \ldots, u_d$ are local parameters for a model of the place ν. Then the order of the differential $\sum\omega_I du_I$ and the logarithmic order of $\sum\omega_I du_I/u^I$ are the same. The common value is the smallest of the numbers $\nu(\omega_I)$.

Proof. The differentials are respectively regular and logarithmic along ν just in case the coefficient functions ω_I are. (§1.1 Theorem 2, Corollary 1 and §2.1 Theorem 1, Corollary 2.)

Corollary. If ω is a differential of type $(r_1,\ldots,r_s)$ then the inequality

$$\nu(\omega) \leq \nu(\omega;\ \log) \leq \nu(\omega) + s$$

holds. Moreover, if ω is r-fold and of top degree then $\nu(\omega;\ \log) = \nu(\omega) + r$.

Example. Suppose that u and v are local parameters on a surface and that ν is the place associated to the curve $u = 0$. Then

$$\nu(u) = 1 \qquad \nu(v) = 0$$

$$\nu(du) = 0 \qquad \nu(dv) = 0$$

$$\nu(du;\ \log) = 1 \qquad \nu(dv;\ \log) = 0$$

$$\nu(du \wedge dv) = 0 \qquad \nu(du \wedge dv;\ \log) = 1$$

The regular order of a differential has the useful property that, for any given differential ω and any given model X the equation $\nu(\omega) = 0$ holds for all but a finite number of the places ν appearing on X. Logarithmic order does not have the corresponding property.

If, preserving the notation of the previous example, ν_c denotes the place associated to the curve $u - c = 0$ then

$$du = (u - c)\,\frac{d(u - c)}{(u - c)}$$

for any constant c and so $\nu_c(du;\ \log) = 1$.

However, when associated already to any model there is already a finite collection of places that are special then logarithmic order has sharper properties than regular order. This occurs primarily in studying completions and resolutions, in which case the special places are those at infinity and those that are singular respectively.

Definition. Suppose that the differential ω is regular on a variety X. Then the *logarithmic order* $\mathrm{ord}_X(\omega;\log)$ is the smallest value of $\nu(\omega;\log)$ as ν ranges over all singular places of X. As before, we write $\mathrm{ord}_X(\omega;\log) = -\infty$ and $+\infty$ respectively if $\nu(\omega;\log)$ is without lower bound and X is non-singular respectively.

The reader is asked to formulate the corresponding definition that is appropriate to the study of completions of non-singular varieties.

2. Value of logarithmic order

Logarithmic order satisfies the following result. Regular order has no similar property, except of course in top degree.

Theorem 2. Suppose that ω is regular on X and that $\nu(\omega;\log)$ is negative for some singular place ν. Then $\mathrm{ord}_X(\omega;\log) = -\infty$, unless X is a curve.

The proof of this result is not short. It depends upon the construction of a sequence ν_n of singular places with $\nu_n(\omega;\ \log)$ decreasing without bound.

Lemma. Suppose that ω is a differential of type $(r_1,\ldots,r_s)$. Then the smallest value of $\nu(\omega\wedge\eta;\ \log)$ as η ranges over **differentials logarithmic along ν of** *complementary type* -- i.e., of type $(d-r_1,\ldots,d-r_s)$ is equal to $\nu(\omega;\ \log)$.

Proof. This result, and the analogous result for regular order, follow immediately from Theorem 1.

We will now prove Theorem 2. Suppose that $\nu(\omega;\ \log) < 0$ and that $U \to X$ is a model for ν. According to the Lemma there is a differential η of complementary type which is logarithmic along U and for which $\nu(\omega\wedge\eta;\ \log) < 0$. By replacing U by a suitable open subset we may assume that η is regular on $U - D_\nu$, and that $\omega \wedge \eta$ has no poles or zeroes on U other than D_ν.

Now let $u_1, \ldots, u_d$ be local parameters on U such that the equations $u_1 = 0$ and $u_2 = 0$ have a solution. Let ν_n be the place ν_α defined in Ch 1 §4.2 for $\alpha = (n,1,0\ldots0)$. According to §2.3 Theorem 3, $\mathrm{dis}_u(\nu) = n$ while $\nu_n(u_1) = n$ and $\nu_n(u_2) = 1$ by construction.

We will now calculate $\nu_n(\omega\wedge\eta;\ \log)$. By assumption, $\omega \wedge \eta$ can be written as $u_1^{-m}\,(du_I/u^I)^{\otimes s}$ multiplied by a unit, where m is at least one. It is easy to see that $\nu_n(du_I/u^I;\ \log) = 0$ and so $\nu_n(\omega\wedge\eta;\ \log) = -mn$.

The differential η is, by §2.1 Theorem 1 Corollary 2, logarithmic along ν_n. As ν_n is, by construction, a singular place of X, the result follows from the Lemma. The quantity $\nu_n(\omega;\ \log)$ can be no larger than $\nu_n(\omega\wedge\eta;\ \log)$. This completes the proof.

3. Logarithmic order and resolution

Definition. Suppose that the differential ω is regular on X. If ω is logarithmic along every singular place ν of X, we say that ω is *logarithmic* on X.

Clearly, ω is logarithmic iff $\mathrm{ord}_X(\omega;\ \log) \geq 0$.

Theorem 3. Suppose that ω is regular on X and that $X' \to X$ is a resolution of Sing X to a normal crossings divisor E. Then

(i) ω is logarithmic iff it is logarithmic along every component of E. Moreover, in this case

(ii) $\mathrm{ord}_X(\omega;\ \log) = \min\{\nu(\omega;\ \log) \mid \nu$ is associated to a component of $E\}$.

Proof. Part (i) follows from §2.2 Theorem 2. Part (ii) is proved in exactly the same way as §1.3 Theorem 5.

<u>Definition</u>. Suppose that ω is logarithmic on the variety X. A singular place ν is said to be *log-minimal* with respect to ω, or where confusion is not possible minimal, if $\nu(\omega;\ \log) = \mathrm{ord}_X(\omega;\ \log)$.

<u>Corollary</u>. If ω is logarithmic and $X' \to X$ resolves Sing X to a normal crossings divisor, then X' contains a minimal place.

The comments towards the end of §1, by and large, hold true for logarithmic order. In particular, stronger results can be proven if ω is assumed to be of top degree.

4. <u>Minimal Centers</u>

We saw in §1.4 that if ω is a differential of top degree and is of the first kind on X then the equation

$$\nu(\omega) = \mathrm{ord}_X(\omega)$$

has only a finite number of solutions for ν a singular place of X. Assuming that X has a resolution.

If, however, ω is only logarithmic it is possible that the equation

$$\nu(\omega;\ \log) = \mathrm{ord}_X(\omega;\ \log) \tag{2}$$

may have an infinity of solutions for ν a singular place. However, the infinity is countable and the number of solutions with any given discrepancy is finite.

We will here and in §3.5 obtain some results concerning the equation (2). These results will be used in Chapter III to develop the theory of toral modifications.

Definition. Suppose that $X' \to X$ is a smooth modification with exceptional locus E. Suppose also that ω is a differential logarithmic on X. A subvariety D of E is said to be *log-minimal* with respect to ω if the place ν realized by choosing D for the center of a monoidal transformation is log-minimal with respect to ω. Where confusion will not result, we will say simply that D is minimal.

Clearly, a similar definition can be made for regular forms. If the differential ω is such that no smooth modification $X' \to X$ contains a minimal center, then every minimal place is essential. This is the logic of the argument we used in §1.4 to show the existence of essential places.

The location of log-minimal centers is easy to describe if ω is of top degree.

Theorem 4. Suppose that $X' \to X$ is a smooth modification for which the reduced transform E of Sing X is a normal crossings divisor.

Suppose also that ω is an r-fold differential of top degree logarithmic on X.

(i) If $\mathrm{ord}_X(\omega; \log) \geq 1$ then X' has no minimal centers.

(ii) Suppose $\mathrm{ord}_X(\omega; \log) = 0$. Let $E_1 \ldots E_m$ denote the components of E corresponding to the minimal places appearing on X'. Every minimal center is a component D of some intersection $\bigcap\{E_i \mid i \in I\}$, $I \subset \{1,\ldots,m\}$ of components of E.

(iii) Such a component D is indeed minimal if it does not lie in any component of the zero locus, as a regular form, of ω.

Proof. We will use the fact that, as ω is of top degree, $\nu(\omega; \log) = \nu(\omega) + r$.

Let D be some subvariety whose codimension c is two or more and let ν be the place realized through making a monoidal transformation with center D. The discrepancy $\mathrm{dis}_{X'}(\nu)$ equals $c - 1$. Because E has normal crossings, D can lie in at most c components of E. If E' is a component containing D then $\nu(E') = 1$.

Let n denote the logarithmic order of ω. Thus, the divisor of ω, as a **rational** differential, on X' is at worst $(n - r)E$.

Suppose also that ω is an r-fold differential of top degree logarithmic on X.

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Proof. We will use the fact that, as ω is of top degree, $\nu(\omega; \log) = \nu(\omega) + r$.

Let D be some subvariety whose codimension c is two or more and let ν be the place realized through making a monoidal transformation with center D. The discrepancy $\mathrm{dis}_{X'}(\nu)$ equals $c - 1$. Because E has normal crossings, D can lie in at most c components of E. If E' is a component containing D then $\nu(E') = 1$.

Let n denote the logarithmic order of ω. Thus, the divisor of ω, as a **rational** differential, on X' is at worst $(n - r)E$.

(i) We are assuming that n is strictly positive.

$$\begin{aligned} \nu(\omega; \log) &= \nu(\omega) + r \\ &\geq r\, \mathrm{dis}_{X'}(\nu) + \nu((n - r)E) + r \\ &\geq r(c - 1) + c(n - r) + r = cn \end{aligned}$$

As c is at least two, this proves the result.

(ii) and (iii) The preceding inequality is valid for n = 0. We want to know when equality holds.

It is clear that if one of the following occurs then strict inequality holds.

a) The codimension of D is strictly greater than the number of components E' of E containing D.

b) There is some component E' of E containing D along which the logarithmic order of ω is not zero.

c) There is some prime divisor F, necessarily not a component of E, along which ω, as a regular differential, has a zero and which contains D.

It is also clear that if none of the above hold then ν is minimal. This concludes the proof.

Corollary. Suppose that the differential ω is of top degree and logarithmic on X. If $\mathrm{ord}_X(\omega; \log) > 0$ then the minimal places are essential to the resolution of Sing X to a normal crossings divisor.

5. Minimal Places

The previous result concerning the location of minimal centers, together with the sequence of modifications determined by a place, give to us very precise results concerning minimal places.

Notation. Suppose that ω is an r-fold differential of top degree logarithmic on X, and that $\mathrm{ord}_X(\omega; \log) = 0$. Suppose also that $X' \to X$ is a smooth modification of X whose exceptional locus E' is a normal crossings divisor. We will use $\widetilde{X'}$ to denote the open subset of X' obtained by removing from X' all components of E' whose associated places are not minimal, and the zero divisor of ω as a regular differental. Let $\widetilde{E'}$ denote the exceptional divisor of $\widetilde{X'}$.

Theorem 5. Suppose that $X' \to X$ and ω are as above. Then minimal centers correspond to components of intersections of components of $\widetilde{E'}$.

(i) Suppose X" is the monoidal transformation of X' at a minimal center. Then $\widetilde{X''}$ is equal to the monoidal transformation of $\widetilde{X'}$ at the corresponding center. Again, the minimal centers of X'' correspond to components of intersections of components of $\tilde{E}''$.

(ii) Suppose that ν is a minimal place finite on X'. Then the sequence of modifications determined by ν is a sequence of monoidal transformations at minimal centers.

Proof. The first assertion concerning minimal centers is simply Theorem 4 in a different form.

(i) Clearly, the appropriate monoidal transformation of $\tilde{X}'$ is a subset of X''. That it is indeed $\tilde{X}''$ follows by applying Theorem 4 to each exceptional divisor of X''.

(ii) It is enough, by induction, to show that the center D_ν' of ν on X' is minimal. So let us suppose that it is not. By removing a closed subset from X' we may suppose that D_ν' is nonsingular and that it intersects in a transverse manner those components of E which do not contain D_ν'.

Now let $X'' \to X'$ be the monoidal transformation of X' at center D_ν'. The exceptional locus E'' of $X'' \to X$ is a normal crossings divisor, because we first removed the "bad" points of D_ν'. Moreover, as D_ν' was assumed not to be minimal the differential ω has a zero, as a logarithmic differential, along the exceptional locus E_1'' of $X'' \to X'$.

The remainder of the argument is routine. We may find an open subset U of X" on which ν is finite and upon which local parameters $u_1, \ldots, u_d$ define E", with u_1 defining E_1''. Then ω can be written in the form

$$\omega = u_1 \cdot f \cdot (du_I/u^I)^{\otimes r}$$

where f is regular on U.

By §2.2 Theorem 2, $(du_I/u^I)^{\otimes r}$ is logarithmic along ν. Moreover, f is regular along ν. Finally, the sequence of modifications determined by a finite place has the property that the center of the next modification lies in the exceptional divisor of the last. Thus, $D_\nu'' \subset E''$, and so u_1 has a zero along ν. Thus, $\nu(\omega;\ \log) \geq 1$.

This concludes the proof.

<u>Corollary</u>. Suppose that X admits a resolution, and that ω is a differential of top degree whose logarithmic order is zero. Then the equation

$$\nu(\omega;\ \log) = 0$$

has at most a countable number of solutions for ν a singular place of X. Moreover, the number of solutions with $\mathrm{dis}_X(\nu) \leq n$ is finite, for every value of n.

As for r-fold differentials of top degree the logarithmic and regular orders are related by the equation $\nu(\omega; \log) = \nu(\omega) + r$, we could have formulated the arguments and results here in terms of regular order alone. However, if ω is not of top degree then the logarithmic and regular orders are not simply related, the one to the other. It seems that the study of logarithmic order will provide the best results. Theorem 2 supports this belief.

6. Comments

Suppose that ω is an r-fold differential of top degree regular on X, and of logarithmic order zero. Every resolution $X' \to X$ of Sing X to a normal crossings divisor must realize a minimal place. Moreover, it is clear that these minimal places ν whose discrepancy is no larger than the discrepancy of any other minimal place must appear in X'. For the minimal places exceptional on X' have a discrepancy over X'.

This argument, although crude, shows that some of the minimal places are essential. To produce an effective criterion to determine which of the minimal places appearing on X' are inessential is, apparently, difficult. A minimal place ν is inessential if we can find a resolution $\overline{X} \to X$ upon which lies a minimal center D for which the corresponding minimal place is ν. Moreover, the minimal places associated to $\overline{E}$

that contain D must have discrepancy strictly less than that of ν. Thus, there is only a finite number of posibilities for the minimal places ν_i corresponding to the components $\overline{E}_i$ of $\overline{E}$ whose intersection is D.

We are thus led to consider the following question.

Problem. Suppose that $\{\nu_1,\ldots,\nu_n\} = \sigma$ is a finite collection of exceptional places of X. Is there a smooth modification $\overline{X} \to X$ whose exceptional locus $\overline{E}$ is a normal crossings divisor, has components $\overline{E}_1, \ldots, \overline{E}_n$ whose intersection D is nonempty, and for which ν_i is the place associated to $\overline{E}_i$?

Of course, if X is a surface then the results concerning factorization of regular birational maps and removal of the locus of indeterminacy [35, Chapter IV §3] allow us to obtain a complete answer to this question.

Some first results concerning this problem in higher dimension appear in Chapter III §2. Using the definition of §2.3 of that chapter, the problem is to determine whether or not σ is a simplex of the fan of some $\overline{X}$. The result we have -- Chapter III §2.3 Theorem 5 -- is that if the places of σ are of a particular kind -- toral -- then σ can so appear just in case the set $\{\alpha^1,\ldots,\alpha^n\}$ of associated weights is simple.

By using the general theory of toral modifications as outlined in the Introduction, I hope to obtain some further results concerning this problem.

§4. EMBEDDED RESOLUTION

1. Definitions

The methods introduced earlier in this chapter can be used to obtain results concerning the resolution of a singular subvariety X embedded in a nonsingular ambient space M.

We will confine ourselves here to a special case. *We assume that* $X \subset M$ *is a hypersurface having local equation* $f = 0$. *We assume also that* ω *is an r-fold differential of top degree of* $k(M)$, *and that* ω *is regular on* $M - X$ *and logarithmic along* X.

These assumptions are to hold until the end of this section, where we will indicate the changes and additions that may be required in order that a more general theory may be obtained. I intend to accomplish this development in a later work.

Before we proceed, it is useful to introduce some definitions and notations concerning the geometry of $X \subset M$.

Suppose that $M' \to M$ is a nonsingular modification of M which is an isomorphism away from Sing X. We will, for brevity, call $M' \to M$ a *modification* of M. The reduced total transform of X is denoted by X' and the strict transform by st X'. The exceptional divisor of $M' \to M$ is denoted by E'. Thus, $E' \cup \mathrm{st}\, X' = X'$.

<u>Definition</u>. A proper modification $M' \to M$ is said to *resolve* X if the total transform X' is a normal crossings divisor and the strict transform st X is nonsingular.

The last condition is imposed because we do not want to consider the identity map $\mathbb{A}^2 \to \mathbb{A}^2$ to be a resolution of the curve $X \subset \mathbb{A}^2$ given by $xy = 0$.

<u>Definition</u>. A place ν of M is *singular* if its center lies in Sing X.

If $M \to M'$ resolves X then st $X' \to X$ is a resolution of Sing X to a normal crossings divisor. All the earlier definitions and results will apply to st $X' \to X$. We will use $\bar{\nu}$, $\bar{\nu}_i$, etc. to denote places of X. There is, it seems, no direct and immediate relationship between singular places of M and singular places of X.

However, suppose that $M' \to M$ is a smooth modification of M for which the total transform X' of X is a normal crossings divisor. Suppose also that a given singular place $\bar{\nu}$ of X is realized as a divisor $\bar{D}$ on st X'. The center $\bar{D}$ lies in E' and so there is a unique component E of E' for which $\bar{D}$ is a component of $E \cap$ st X'. The place ν of M associated to E will be called the *extension* of $\bar{\nu}$ to M determined by M', or when confusion will not result, simply as the extension of $\bar{\nu}$.

Suppose that $M'' \to M'$ is the monoidal transformation whose center is $\bar{D}$. As the codimension of $\bar{D}$ in st X' is one, the induced map st $X'' \to$ st X' is an isomorphism.

However, the extension of $\overline{\nu}$ determined by M" differs from that determined by M'. In the former case, it is the exceptional place of $M'' \to M'$.

A component $\overline{D}$ of the intersection st $X' \cap E'$ will be called a *marginal center*.

2. Poincaré Residue

The key to understand the relation between the embedded theory and the abstract theory is the Poincaré Residue homomorphism. The residue Res(ω) is a differential of top degree regular on X. By studying the residue map, we can determine the conditions that are most sensibly imposed on ω.

For the convenience of the reader we recall some properties of the residue map.

Suppose that $u_1,\ldots,u_d$ are local parameters on some open subset U which has a nonempty intersection with the hypersurface X. Suppose also that X has local equation $\overline{f} = 0$ on X.

We can write

$$d\overline{f} = \overline{f}_1\, du_1 + \ldots + \overline{f}_d\, du_d$$

where the partial derivatives $\overline{f}_i$ are regular on U. Moreover, as X has nonsingular points at least one of the f_i does not vanish identically on X.

The differential ω can be written as

$$\omega = \omega_0 (du_I)^{\otimes r}/\overline{f}^r$$

for a suitable coefficient function ω_0. We assume that ω_0 does not have a pole along X.

Then the residue Res(ω) is the restriction to X of the differential

$$\omega_0 \left[\frac{du_1 \wedge \ldots \widehat{du_i} \ldots \wedge du_d}{\bar{f}_i} \right]^{\otimes r} (-1)^{r(i-1)}$$

where, as usual, ^ means that du_i is to be omitted.

We saw in Chapter I § 2.3 that Res(ω) depends only upon ω and X.

3. Order

The motivation for the definition of order that is to follow is provided in part by the next result.

Theorem 1. Suppose that $M' \to M$ is a modification for which X' is a normal crossings divisor. Suppose also that $\bar{\nu}$ is a singular place appearing on st X' and that ν is its extension to M.

Then $\nu(\omega) \leq \bar{\nu}(\mathrm{Res}(\omega))$.

Proof. We will use the notation introduced earlier. We can find an open subset U of M' on which E' has local equation $u_1 = 0$ and for which ν is the only finite exceptional place.

On U the strict transform st X' has local equation $\bar{f} = 0$ where

$$\bar{f} = u_1^n f$$

and $n = \nu(f)$.

On U the differential ω can be written as

$$\omega = u_1^m \bar{\omega}/\bar{f}^r$$

where $m = \nu(\omega)$ and $\bar{\omega}$ is regular on U. $\mathrm{Res}(\bar{\omega}/\bar{f}^r)$ is regular along $\bar{\nu}$ and as $\bar{\nu}(u_1) = 1$ the result follows.

We cannot write equality because it is possible that $\bar{\omega}$ vanish along $\bar{D}$ even though it does not vanish along E'. This cannot occur if ω is without zeroes on M - X.

<u>Corollary 1</u>. The quantities $\nu(\omega)$ and $\nu(\mathrm{Res}(\omega))$ are unequal iff the strict transform of the divisor of the zeroes of ω on M contains $\bar{D}$.

The place ν is not the only possible extension of $\bar{\nu}$ to M.

<u>Corollary 2</u>. After repeatedly performing a monoidal transformation at the marginal center $\bar{D}$ associated to $\bar{\nu}$ we obtain an extension ν of $\bar{\nu}$ for which

$$\nu(\omega) = \bar{\nu}(\mathrm{Res}(\omega))$$

Proof. On U, we can find local parameters $u_1,\ldots,u_d$ for which $u_1 = 0$ is a local equation for st X' and $u_2 = 0$ is a local equation for E'. Let $w_i = 0$ be a local equation for the strict transform of the divisor of the zeroes of ω on the space M_i' obtained by performing i monoidal transformations.

The places ν_i obtained by performing the indicated sequence of monoidal transformations are those associated to the weights $(i,1,0,\ldots,0)$, as defined in Chapter I §4.2.

Let $\overline{D} = \overline{D}_0, \overline{D}_1, \ldots$ denote the marginal centers on the M_i'. It follows from the criterion of Corollary 1 that $\nu_i(\omega) = \overline{\nu}(\mathrm{Res}(\omega))$ just in case $\nu_{i+1}(w_i) = 0$. To show that this is ultimately the case we use the formal power series expansion of $w = w_0$.

We can write

$$w = \sum \lambda_{ij} u_1^i u_2^j$$

where λ_{ij} is a power series in $u_3, \ldots, u_d$. We assumed that w does not vanish on E' so $\lambda_{i,0}$ is nonzero for some value of i. Let m be the smallest such value.

Now let a/b be the smallest fraction of the form j/(m - i) where the co-efficient λ_{ij} is nonzero. See Figure 1. We suppose that a/b is written in lowest terms.

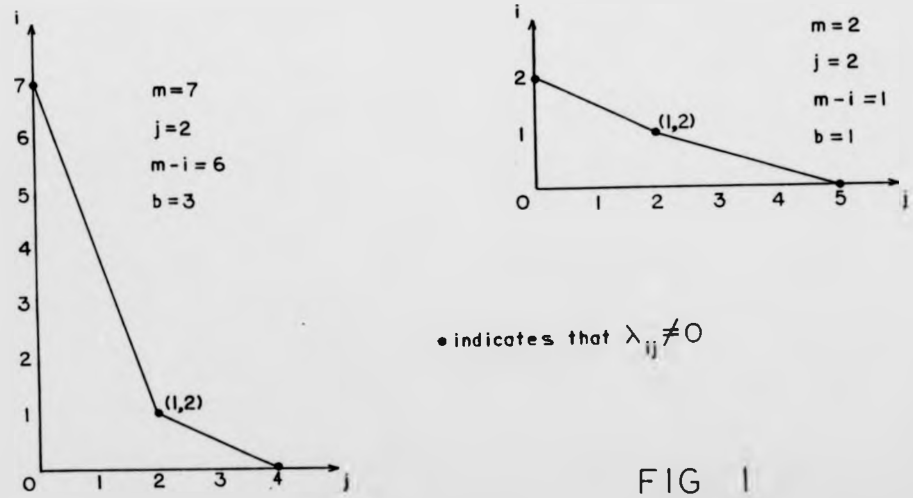


FIG 1

The reader is asked to verify that b is the number of monoidal transformations required. The proof here is similar to the proof in [32, § 8B] of the existence of an embedded resolution for singular plane curves.

Definition. The order $\mathrm{ord}_X(\omega)$ of the differential ω is the smallest value of $\nu(\omega)$ as ν varies over the singular places of M. If there is no lower bound we write $\mathrm{ord}_X(\omega) = -\infty$. If there are no singular places, $\mathrm{ord}_X(\omega) = \infty$.

Corollary 3. If every singular place $\bar{\nu}$ of X admits an extension ν to a place of M then

$$\mathrm{ord}_X(\mathrm{Res}\ (\omega)) \geq \mathrm{ord}_X(\omega).$$

We cannot assert equality because there may be some singular place ν of M which is not the extension of some singular place $\bar{\nu}$ of X.

If the ground field is of characteristic zero then we may use Hironaka's resolution theorem [17]. Suppose that $M' \to M$ resolves X. If $\bar{\nu}$ is a singular place of X then it determines a sequence of modifications on st X' leading to its realization. If we use these centers to modify the ambient space M' also then we are led to realize an extension ν of $\bar{\nu}$.

It ought to be possible, by using a variation of the realization theorem, that every singular place $\bar{\nu}$ of X admits an extension to M.

Definition. The differential ω is said to be *of the first kind* if $\mathrm{ord}_X(\omega) \geq 0$.

The differential ω is said to be *logarithmic* if $\mathrm{ord}_X(\omega) \geq -r$.

Clearly, if ω is of the first kind then so is $\mathrm{Res}\,(\omega)$ etc., if every singular place of X admits an extension to M.

4. Minimal places

As before, we can obtain results concerning minimal centers and minimal places. The novelty is provided by the existence of marginal centers.

Theorem 2. Suppose that $\mathrm{ord}_X(\omega) < -r$. Then $\mathrm{ord}_X(\omega) = -\infty$.

Proof. Suppose that $\nu(\omega) < -r$ for the singular place ν of M. Then the modification $M_{(\nu)} \to M$ determined by ν has an exceptional divisor E along which ω is worse than logarithmic. We now apply the construction used in the proof of §3.2, Theorem 2.

Definition. A singular place ν of M is said to be *minimal* with respect to ω if $\nu(\omega) = \mathrm{ord}_X(\omega)$.

Suppose that $M' \to M$ is a modification of M. A subvariety D of M' is said to be *minimal* with respect to ω if the place ν realized by choosing D as the center of a monoidal transformation is minimal.

The location of minimal centers depends upon $\mathrm{ord}_X(\omega)$.

<u>Theorem 3</u>. Suppose that $\mathrm{ord}_X(\omega) = -r$. Suppose also that $M' \to M$ is a modification of M for which the total transform X' of X is a normal crossings divisor, and that the strict transform $\mathrm{st}\, X$ is non-singular.

Let $\tilde{M}'$ denote the open subset of M' obtained by removing

a) the strict transform of the zero locus of ω.

b) those components of E' along which ω is not minimal.

c) those components of $\mathrm{st}\, X'$ along which the logarithmic order of ω is not zero.

We use $\tilde{X}'$ to denote the total transform of X on $\tilde{M}'$. Then

i) minimal centers correspond precisely to components of intersections of components of $\tilde{X}'$.

ii) every minimal place ν finite on M' can be realized by performing a succession of monoidal transformations at minimal centers.

<u>Theorem 4</u>. Suppose that $\mathrm{ord}_X(\omega) > -r$. Let $M' \to M$ and $\tilde{M}'$ be as in Theorem 3. Then

i) minimal centers correspond precisely to marginal centers $\overline{D}$ of $\tilde{M}'$.

ii) every minimal place ν finite on M' can be realized by performing a succession of monoidal transformations of minimal centers.

<u>Corollary</u>. If X admits an embedded resolution $M' \to M$ then $\mathrm{ord}_X(\omega)$, if finite, is the smallest value of $\nu(\omega)$ as ν ranges over the singular places of M'.

The proof of Theorems 3 and 4 is completely similar to that of § 3.5, Theorem 5.

5. Minimal places (continued)

The remarks made at the end of § 3 apply here also. In particular, a solution of the problem in § 3.6 would be most useful. We can, however, obtain an interesting result concerning the structure of minimal places using elementary methods.

We will assume that $X \subset M$ admits an embedded resolution, and that $\mathrm{ord}_X(\omega) > -r$. Even though the number of minimal places is quite possibly infinite, there will be only a finite number possessing any given discrepancy. Also clear is that only a finite number of them have any given value for $\nu(f)$ where f is the local equation for X.

Suppose now that $M' \to M$ resolves X. The reader is asked to verify that if $M'' \to M'$ is a monoidal transformation then M' and M'' have the same number of centers minimal with respect to ω. Recall that by Theorem 4, all such centers are marginal. From this fact it follows that the collection

$$C_n = \{\nu \mid \nu \text{ minimal for } \omega \text{ and } \nu(f) = n\}$$

of places has a constant number of elements for n large enough. This number is the number of minimal centers on M'.

The resolution $M' \to M$ can be used to induce an additional structure on the collections C_n. Each place ν_n of C_n that does not appear on M' has a center D'_{ν_n}, which is a minimal center. To each such place ν_n there is a unique place $\nu_{n+1} \in C_{n+1}$ which has the same center on M' as ν_n. Moreover, ν_n and ν_{n+1} are both extensions of the same place $\bar{\nu}$ of X. We will show that this structure, for n large enough, does not depend upon the choice of M'.

Theorem 5. If n is large enough, there is only one place $\nu_n \in C_n$ that can be an extension of the place $\bar{\nu}$ of X; where $\bar{\nu}$ corresponds to some given marginal center $\bar{D}$ of a resolution $M' \to M$.

The proof of this result depends upon a lemma, the proof of which is left to the reader. It is similar to the proof of Corollary 2 to Theorem 1. It is a computation of order of vanishing.

Lemma. Suppose that $\nu_n, \nu_{n+1}, \ldots$ is the sequence of elements of $C_n, C_{n+1}, \ldots$ that is obtained from a resolution $M' \to M$ by successively monoidally transforming at some fixed marginal center $\bar{D}$ associated to the place $\bar{\nu}$ of X.

Then

$$\lim_{i \to \infty} \nu_i(g) = \bar{\nu}(g) \qquad (3)$$

for any function g regular on M. The convergence is not uniform.

Outline of a Proof of Theorem 5. Let $\bar{\nu}^i$ be the places of X corresponding to the centers $\bar{D}$ of M' **minimal** and marginal to X'. for ~~ω~~. As these places are distinct we can find functions

$g_1,\ldots,g_r$ regular on M and not vanishing on X such that the sequences $\bar\nu^i(g_1),\ldots,\bar\nu^i(g_r)$ are distinct for distinct values of i. Each $\bar\nu^i$ induces, through M', a sequence ν^i_j, of places of M.

It follows from the lemma that $\nu^i_j(g_k) = \bar\nu^i(g_k)$ for all values of i and k, so long as i is larger than some number N.

Now suppose that $\tilde{M}' \to M$ is some other resolution and that $\tilde{\bar\nu}^i$ are the places of X corresponding to the marginal centers that are minimal for ω. These places have extensions in C_n for n large enough. It follows from the choice of $g_1,\ldots,g_r$ that if ν_n and ν_{n+1} are two extensions for the same place $\tilde{\bar\nu}^i$ then $\nu_n(g_j) = \nu_{n+1}(g_j)$. It then follows from the Lemma that each $\tilde{\bar\nu}^i$ is equal to some $\bar\nu^j$, as both satisfy the equation (3) for the same sequence ν_i of places. The theorem now follows, the extension ν_n of $\bar\nu$ is the unique place in C_n that satisfies $\nu_n(g_i) = \bar\nu(g_i)$.

It follows from the proof of this theorem that the places $\bar\nu$ corresponding to the marginal centers $\bar{D}$ do not depend upon the choice of the resolution M' → M. But this we knew already, for such places are minimal with respect to the differential Res (ω) of X.

6. Comments

We now outline the changes that should be made in order to obtain a more general theory. The assumptions concerning ω and X are no longer in force.

If ω' is an r-fold differential of top degree regular on M and $X \subset M$ is a subvariety then results can be obtained by considering the quantity

$$n(\nu) = \nu(\omega') - \nu(X)$$

for all singular places ν of M. When X is a hypersurface with local equation $f = 0$ then the two theories are congruent once we write $\omega' = f^r\omega$.

Now suppose that X is a hypersurface and that ω is a differential not of top degree. There is a residue map for differentials in this case, but to state it one needs the concept of the *weight* [6] of a logarithmic differential form. We will say no more concerning this, in this work.

CHAPTER III : TORAL MODIFICATIONS

The class of birational modifications introduced in this chapter is large enough to provide an interesting range of examples for the results of Chapter II. A single monoidal transformation can be easily described by using a suitable system of local parameters, as can the modifications discussed here. These modifications are far from being completely general.

We conclude the chapter with an example of a plane curve singularity that cannot be resolved by the modifications of this chapter.

§1. TORAL TRANSFORMATIONS

1. Definition and examples

Definition. Suppose that on the smooth variety M there lies a normal crossings divisor D. A monoidal transformation $M' \to M$ is said to be *toral* if its center is a component of the intersection of some components of D. In this case the reduced total transform D' of D is also a normal crossings divisor.

A succession $M_{(n)} \to M_{(n-1)} \to \dots M_{(0)} = M$ of monoidal transformations is said to be *toral* if for each i the transformation $M_{(i+1)} \to M_{(i)}$ is toral with respect to the reduced total transform $D_{(i)}$ of D.

Example 1. Suppose that $M_{(0)}$ is $\mathbb{A}^2$ equipped with the two co-ordinates. The only toral monoidal transformation $M_{(1)} \to M_{(0)}$ has for its center the origin of $\mathbb{A}^2$. The resulting divisor $D_{(1)}$ has three components -- the strict transforms of the co-ordinate axes and the exceptional divisor. The first two components intersect the third, but not each other. Thus, there are two possibilities for $M_{(2)}$.

Example 2. If $M_{(0)}$ is either $\mathbb{P}^d$ or $\mathbb{A}^d$ then every component of $D_{(n)}$ is birational to $\mathbb{A}^{n-1}$. This shows that toral transformations are not completely general.

Example 3. If $\mathbb{A}^3$ is monoidally transformed with center the origin then the resulting exceptional locus is isomorphic to $\mathbb{P}^2$. Suppose we now transform again, choosing as center a curve C in $\mathbb{P}^2$. The reader is invited to verify that co-ordinate hyperplanes can be chosen on $\mathbb{A}^3$ so that the resulting transformation is toral just in case C is a line on $\mathbb{P}^2$.

Example 4. Suppose that $\mathbb{P}^{3'} \to \mathbb{P}^3$ is a monoidal transformation whose center is a smooth curve C. For this transformation to be toral, there must be two smooth hypersurfaces intersecting transversally, which have C as a component of their intersection. This is a genuine problem, even though a local solution is trivial.

Example 5. Suppose that $X \subset M$ is a hypersurface and that $X' \subset M'$ is an embedded resolution. Suppose also that ω is an r-fold differential of top degree regular on $M - X$ with $\mathrm{ord}_X(\omega) = -r$. Then the sequence of modifications required for the realization of an ω-minimal place ν is a toral transformation of M' with respect to the normal crossings divisor X'.

2. Transformations of the plane

The toral modifications of $\mathbb{A}^2$ have a particularly simple description, which will be generalized later.

Suppose that $\mathbb{A}^{2'} \to \mathbb{A}^2$ is a toral transformation. Our first concern is with the total transform D_i' of the co-ordinate axes D_i, $i = 1, 2$. We can write

$$D_i' = \Sigma\, \alpha_i(\nu) D_\nu$$

where the sum is over the places ν associated to D'. In this way we associate a pair of non-negative numbers $\alpha(\nu) = (\alpha_1(\nu), \alpha_2(\nu))$ to each component of D'.

Next we associate to each toral transformation $M_{(n)} \to M_{(0)}$ of $\mathbb{A}^2$ a graph $G_{(n)}$ whose vertices are the pairs $\alpha(\nu)$ corresponding to the components of $D_{(n)}$. Two vertices are joined by an edge if the corresponding components of D' intersect. By induction, it is clear that two components of $D_{(n)}$ can intersect in at most one point. We will embed the vertices of the graph in $\mathbb{N} \times \mathbb{N}$. Here are two examples.

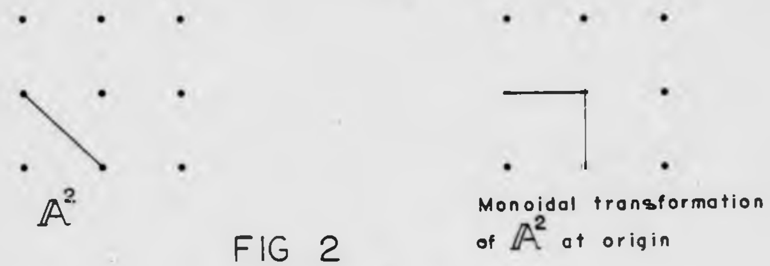


FIG 2

Proposition 1. Suppose that $M_{(n+1)} \to M_{(n)}$ is obtained by choosing as center the point corresponding to the edge (a,b) - (c,d) of $G_{(n)}$. Then the graph $G_{(n+1)}$ is obtained from $G_{(n)}$ by replacing that edge with (a,b) - (a + c, b + d) - (c,d).

The proof of this result is left as an exercise. Here are some examples (the center of the transformation is indicated by a double edge).

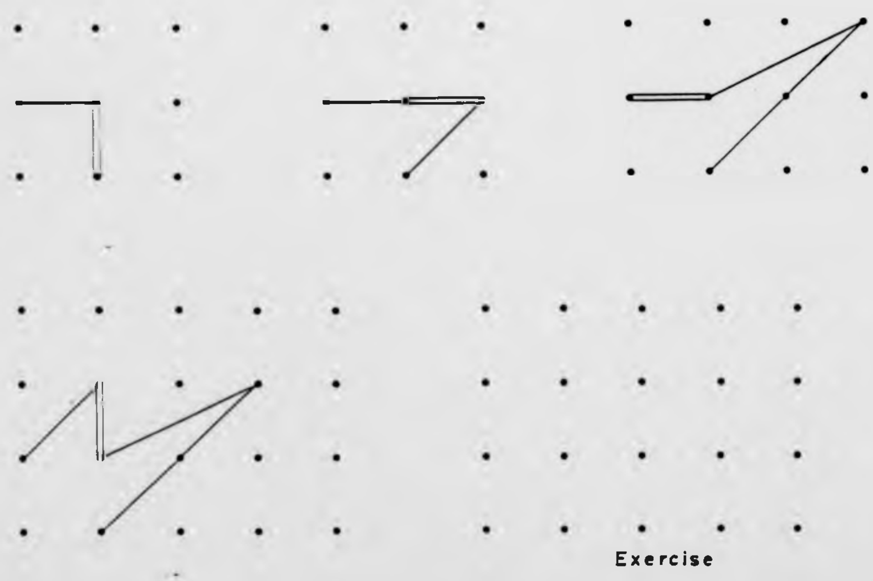


FIG 3

As the determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \begin{vmatrix} a & b \\ a+c & b+d \end{vmatrix} \quad \begin{vmatrix} a+c & b+d \\ c & d \end{vmatrix}$$

are equal to each other, it follows by induction on n that if (a,b) and (c,d) are two adjacent vertices of $G_{(n)}$ then

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ is } \pm 1.$$

It also follows by induction that if

$$(1,0) - (a_1,b_1) - \ldots - (a_n,b_n) - (0,1)$$

is the graph of $G_{(n)}$ then the ratio $(a_i:b_i)$ is an increasing function of i.

<u>Proposition 2.</u> The pair of non-negative integers (a,b) is a vertex of some $G_{(n)}$ just in case a and b are coprime.

Proof. Clearly, the condition is necessary.

To prove sufficiency, we need to construct a sequence of toral monoidal transformations. Suppose the pair is (A,B). At each stage there will be a unique edge (a,b) - (c,d) such that

$$\frac{a}{b} < \frac{A}{B} < \frac{c}{d},$$

unless (A,B) is a multiple of some point of the graph. This edge is associated to the center of the monoidal transformation. We need to show that the process terminates, with the point (A,B) appearing on the graph.

Before proceeding to the proof of this result, I will give an example. To realize (3,5) we should make the following sequence of transformations.

$$(1,0) - (0,1)$$

$$(1,0) - (1,1) - (0,1)$$

$$(1,0) - (1,1) - (1,2) - (0,1)$$

$$(1,0) - (1,1) - (2,3) - (1,2) - (0,1)$$

$$(1,0) - (1,1) - (2,3) - (3,5) - (1,2) - (0,1).$$

The positive quantity

$$(bA - aB) + (cB - dA) \qquad (1)$$

is a measure of how far (A,B) is from being realized. If, after making the transformation, we have

$$\frac{a+c}{b+d} < \frac{A}{B} < \frac{c}{d}$$

then the corresponding quantity is

$$[(b+d)A - (a+c)B] + (cB - dA) \qquad (2)$$

and it is easily seen that (2) is strictly less than (1).

In this way we see the process terminates. As A and B were supposed co-prime, the result follows.

3. Toral places

Definition. Suppose that M is a smooth variety upon which lies a normal crossings divisor D. A finite place ν of M is *toral* if for some toral transformation $M' \to M$, it is associated to some component of D'.

Theorem 1. Suppose that ω is a simple differential of top degree whose divisor (ω) on M is $-D$.

(i) If $M' \to M$ is a toral transformation then the divisor of ω on M' is $-D'$.

(ii) A finite place ν of M is toral just in case $\nu(\omega) = -1$.

Proof. The arguments required for the proof of this result have been used already. They appear in the discussion of minimal centers (Ch II§3.4)

These arguments also demonstrate the next result.

Corollary 1. Suppose that ν is a toral place of M. Then the modification $M_{(\nu)} \to M$ determined by ν is a toral transformation.

Corollary 2. Suppose that $M' \to M$ is a toral transformation. Then every toral place of M is a toral place of M' and vice versa.

Proof. We may suppose that M is affine and that D is defined by local parameters $u_1, \ldots, u_d$. A place ν of M is finite iff it is a finite place of M'. It is toral if $\nu(\omega) = -1$ where $\omega = du_I/u^I$.

If M is a surface, it should be possible to give a direct proof of Corollary 2. But if M is of higher dimension such a proof would become complicated by many details.

We now continue with our discussion of $\mathbb{A}^2$.

<u>Proposition 3.</u> Suppose ν and ν' are two toral places of $\mathbb{A}^2$ and $\alpha(\nu) = \alpha(\nu')$. Then $\nu = \nu'$.

Proof. Suppose $M_{(n)}$ is a model for ν. By Corollary 2, ν' is a toral place of $M_{(n)}$. From what we know of the graph $G_{(n)}$, it is impossible that ν' be an exceptional place of $M_{(n)}$. Thus, ν' must appear on $M_{(n)}$. The place ν is the only possibility.

We now give a different proof of Proposition 2. As A and B are coprime we can solve the equation

$$bA - aB = 1 \tag{3}$$

with positive integers a and b.

Let x and y be co-ordinates for $\mathbb{A}^2$. Write

$$u = \frac{x^b}{y^a} \qquad v = \frac{y^A}{x^B}.$$

The equations

$$x = u^A v^a \qquad y = u^B v^b \tag{4}$$

follow from (3). They show that if M is the affine variety whose co-ordinate ring is $k[u,v]$, then M is birational to $\mathbb{A}^2$ and that the natural map $M \to \mathbb{A}^2$ is regular.

Let ν be the place on M associated to the line $u = 0$. It follows from (4) that $\nu(x) = A$ and $\nu(y) = B$.

By Corollary 2, it is enough to show

$$v\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = -1.$$

From (4) we have

$$\frac{dx}{x} = A\frac{du}{u} + a\frac{dv}{v}$$

$$\frac{dy}{y} = B\frac{du}{u} + b\frac{dv}{v}$$

and so the result follows from (3).

4. Weights

The results previously obtained for $\mathbb{A}^2$ hold in a more general setting. In order that they may be stated and proved in the simplest form, we will assume until (i.e. up to p127) the end of the next section that the normal crossings divisor $D = D_1 \cup \ldots \cup D_k$ lying on the smooth variety M has connected intersections. In other words, that for any subset $\sigma \subset \{1,\ldots,k\}$ the intersection $D_\sigma = \bigcap\{D_i \mid i \in \sigma\}$ has at most one component. By induction it is immediately verified that if $M' \to M$ is a toral transformation then M' also has connected intersections.

Often, we will use the symbols D_i, ν_i, and i in the place of each other.

<u>Definition</u>. Suppose ν is a finite place of M. The *weight* $\alpha(\nu)$, α *associated to* ν is the sequence $(\alpha_1(\nu) \ldots \alpha_k(\nu))$ of non-negative integers defined by $\alpha_i(\nu) = \nu(D_i)$.

<u>Example 1</u>. If the center D_ν of ν lies in D_i then $\alpha_i(\nu) > 0$, and conversely.

<u>Example 2</u>. If ν_i is the place associated to D_i then $\alpha_j(\nu_i) = \delta_{ij}$.

<u>Example 3</u>. If ν is a finite place of M - D then $\alpha(\nu) = 0$.

The value of $\alpha(\nu)$ is not arbitrary, for every finite place ν has a non-empty center D_ν.

<u>Definition</u>. Suppose that σ is a subset of the places $\nu_1, \ldots, \nu_k$ associated to the components $D_1, \ldots, D_k$ of the normal crossings divisor D lying on the smooth variety M. If the intersection

$$D_\sigma = \bigcap \{D_\nu \mid \nu \in \sigma\}$$

is non-empty, we say that σ is a *simplex* of M.

Recall that we are assuming that D_σ has at most one component. Clearly, any subset of a simplex is also a simplex. By convention, we will write $D_\emptyset = M$.

<u>Example</u>. Suppose that σ is a simplex of M and that the place ν_E is realized by making a monoidal transformation whose center is D_σ . Then $\alpha_i(\nu_E) = 1$ if $\nu_i \in \sigma$ and is zero otherwise. Moreover, ν_E has discrepancy $\#\sigma - 1$.

<u>Theorem 2</u>. Suppose that ν is a finite place of M and α is its weight. Then

$$\sigma(\nu) = \{\nu_i \mid \alpha_i \neq 0\}$$

is a simplex of M. Moreover, $D_\nu \subset D_\sigma$.

Proof. If $\alpha_i \neq 0$ then $D_\nu \subset D_i$. Thus, D_ν is contained in $\bigcap\{D_i \mid \alpha_i \neq 0\}$ which is equal to $D_{\sigma(\nu)}$. As ν was supposed finite, $\sigma(\nu)$ is a simplex.

<u>Definition</u>. The *support* Supp(α) of an element $\alpha \in \mathbb{N}^k$ is the set $\{i \mid \alpha_i \neq 0\}$. If Supp($\alpha$) is a simplex, we write $\sigma(\alpha)$ instead, and say that α is *admissable*, or that it is a *weight*.

The set $\Lambda^\vee(M)$ of all admissable weights is called the *weight space* of M. The sum $\alpha + \beta$ of two weights is also a weight just in case $\sigma(\alpha) \cup \sigma(\beta)$ is a simplex, in which case we say that the addition is *admissable*.

<u>Proposition</u>. Suppose that ν is a finite place of M and that α is the weight associated to ν. (See page 101.) Then α is admissable in the sense of the Definition 1.

Proof. According to Theorem 2, Supp(α) is a simplex.

<u>Definition</u>. Suppose that $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a sequence of non-negative rational numbers, and that Supp(α), which is defined as before, is a simplex. In that case we say that α is a *rational weight*. The associated space of weights is denoted by $\Lambda^\vee_{\mathbb{Q}}(M)$.

The set $|\sigma|$ of rational weights α whose support $\sigma(\alpha)$ is equal to σ is called the *open chamber* of $\Lambda^\vee_{\mathbb{Q}}(M)$ associated to σ. Each weight α lies in a single open chamber $|\sigma(\alpha)|$. Thus the open chambers form a partition of rational weight space. We will, however, use the word "division" in the place of "partition." This is because the effect of a toral monoidal transformation on weight space is, as we shall see, analoguous to the barycentric subdivision of a geometric simplex.

The closure $\mathrm{Cl}|\sigma|$ of $|\sigma|$ is called the *closed chamber* associated to σ. It consists of all weights α whose simplex $\sigma(\alpha)$ is contained in σ.

If we wish to emphasize that the weight α lies in $\Lambda^\vee(M)$ rather than $\Lambda^\vee_{\mathbb{Q}}(M)$ we will say that α is an *integral* weight.

Suppose $\alpha^1, \ldots, \alpha^n$ are weights such that $\sigma(\alpha') \cup \ldots \cup \sigma(\alpha^n)$ is a simplex σ. We say in that case the sum

$$\lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n \qquad (5)$$

is *admissable*, provided that the λ_i are non-negative. If moreover, the α^i are integral weights and (5) is an integral weight only when the λ_i are integers, we say that $\{\alpha^1,\ldots,\alpha^n\}$ is a *simple* set of weights. This conforms to the definition in Chapter I §1.

Example. An integral weight α is *simple* if it is not a multiple of some other integral weight.

5. Models for toral places

In Chapter I §4.2 we associated to a normal crossings divisor D and a simple weight α a modification $M_\alpha \to M$ and a place ν_α. We also showed that if ν is a finite place of M with $\alpha(\nu) = n\alpha$ then ν is a finite place of M_α. This demonstrates part (ii) of the next result.

Theorem 3.

(i) The place ν_α is toral.

(ii) Suppose that ν is a finite place of M and that $\alpha(\nu) = n\alpha$ for the simple weight α. Then ν is a finite place of M_α.

(iii) Suppose in addition that ν is toral. Then $\nu = \nu_\alpha$ and so $n = 1$ and $\alpha(\nu)$ is simple.

Proof. In proving the remainder of the theorem we may assume that M is affine, and that upon M the local parameters $u_1, \ldots, u_d$ define D. Write $\omega = du_I/u^I$. Chapter II § 2.3 **shows** that $\nu_\alpha(\omega) = -1$ and so by Theorem 1, ν_α is toral.

Now suppose that the place ν is toral. By part (ii), ν is a finite place of M_α. Again by Theorem 1, ν is a toral place of M_α. But the divisor D_α on M_α has but one component and so the places ν_α and ν must be equal.

Corollary 1. Toral places ν of M and simple weights α of $\Lambda^\vee(M)$ are in perfect correspondence via

$$\alpha \mapsto \nu_\alpha, \quad \nu \mapsto \alpha(\nu) .$$

Corollary 2. Suppose that ν is a toral place and α the corresponding simple weight. The simplex $\sigma(\nu)$ was defined (Theorem 2) to be $\sigma(\alpha)$ and so the two are equal.

Moreover, the center D_ν of ν on M and the intersection D_σ are equal, where $\sigma = \sigma(\alpha)$.

Proof. As ν is a toral place, D_ν is an intersection of components of D. But D_σ is the intersection of all the components which contain D_ν.

Corollary 3. Suppose $M' \to M$ is a toral transformation and that $D_1' \cap \ldots \cap D_n' = D_\sigma'$ is a non-empty intersection of some components of D'. Let ν denote the toral place realized by choosing the intersection D_σ' as the center for a monoidal transformation.

Then

$$\alpha(\nu) = \alpha^1 + \ldots + \alpha^n$$

where α^i is the weight associated to the place ν_i associated to the component D_i' of D'.

Proof. It follows from the definition of $\alpha^i(\nu)$ that the expression

$$\Sigma \alpha_j^i \, D_i'$$

gives the total transform on M' of the divisor D_j on M. (The sum is over all components of D'.) It then follows that the multiplicity $\alpha_j(\nu)$ of D_j along ν is obtained by adding those α_j^i for which $D_\sigma' \subset D_i'$. This proves the result.

Corollary 4. Suppose that ν is a toral place and α is the associated weight. The weight α lies in a unique open chamber $|\sigma|$. Then $D_\nu = D_\sigma$.

Proof. The place ν is realized by a sequence $M_{(\nu)} \to M$ of toral transformations, the center for the next lying in the exceptional divisor of the last. A straightforward argument by induction, using Corollary 3, shows that any place ν so obtained satisfies

$$\alpha(\nu) = \lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n$$

where the λ_i are strictly positive integers and $\alpha^1, \ldots, \alpha^n$ are the weights associated to the places of σ.

It is sometimes convenient to use the symbol α instead of the symbol ν for the associated place. We use this convention in the next two examples.

Example 1. The plane $\mathbb{A}^2$ has simple weights $\alpha = (3,5)$ $\beta = (2,3)$. Suppose there is a toral transformation for which $\sigma = \{(3,5),(2,3)\}$ is a simplex. Then the place ν realized by choosing D_σ as a center is $(5,8)$

Now suppose that $\{(3,4),(5,6)\} = \sigma$ is a simplex for some toral transformation of $\mathbb{A}^2$. Then the place so realized is $(8,10)$. Bur places correspond only to simple weights and so our initial assumption is false. This observation will be extended in the next section.

Example 2. Suppose that $\alpha = (2,3,5)$ and that $M_{(\nu)} \to \mathbb{A}^3$ is the sequence of monoidal transformations determined by the associated place. At each stage we will use Corollary 4 to determine the location of the center, and will use Corollary 3 to determine the resulting exceptional place. In this way we can obtain a description of $M_{(\nu)}$.

This example is similar to the discussion of the $\mathbb{A}^2$ in §1.4. However, in order that pictures may be drawn, we will use barycentric co-ordinates to represent a simple weight as a point on a 2-simplex.

At each stage, a simplex of the toral transformation corresponds to the vertices of a simplex of the subdivided simplex. Figure 4 describes barycentric co-ordinates. Figure 5 describes the realisation of (2,3,5)

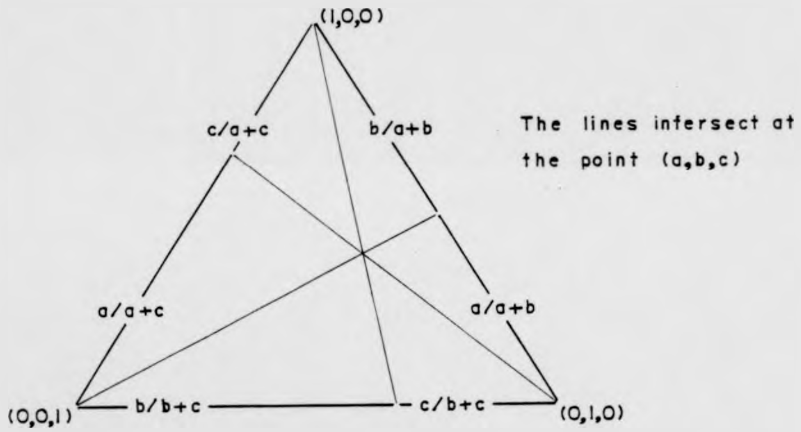


FIG 4 BARYCENTRIC CO-ORDINATES

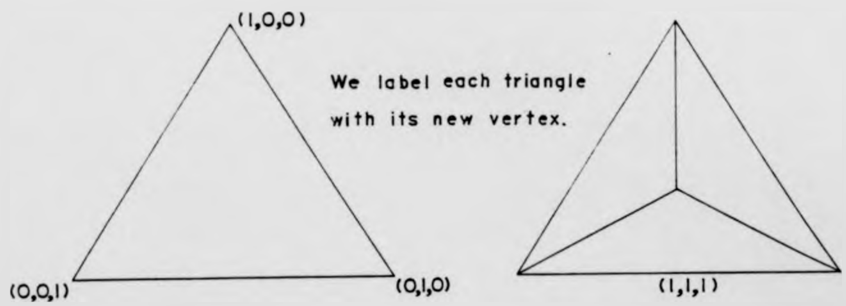
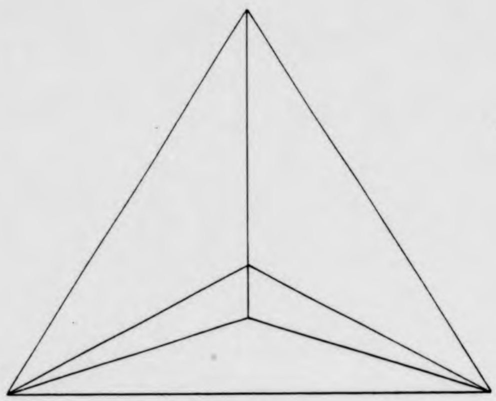
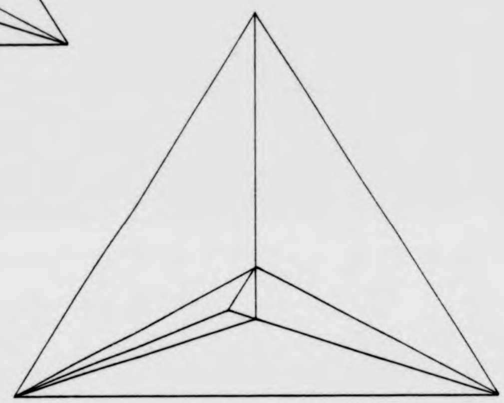


FIG 5

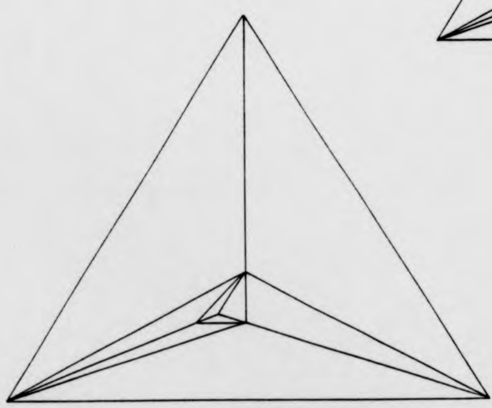
FIG 5 (CONT)



(1,2,2)



(2,3,4)



(2,3,5)

§2. TORAL MODIFICATIONS

1. Definition and examples

It is convenient to have available a class of varieties somewhat larger than that provided by toral transformations.

Definition. Suppose that $M' \to M$ is a smooth birational modification, and that the reduced total transform D' of D has normal crossings. If the places associated to the components of D' are toral then we say that $M' \to M$ is a *toral modification.*

Example 1. If $M' \to M$ is a toral transformation then it is also a toral modification.

Example 2. If $M' \to M$ is a toral modification and U is an open subset of M' then $U \to M$ is also a toral modification.

Example 3. The modification $M_\alpha \to M$ is toral.

Example 4. Suppose that $M' \to M$ is a proper toral modification of the surface M. Because M is a surface, $M' \to M$ factors as a sequence of monoidal transformations [35, Ch IV §3.4 and Ch VI §2.3 Remark 1], and as every exceptional place is toral, this sequence of transformations is toral. Thus, $M' \to M$ is a toral transformation.

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The next result is an extension of Theorem 1, §1.3

<u>Theorem 1</u>. Suppose that $M' \to M$ is a toral modification. Then every toral place of M' is a toral place of M. Moreover, we may regard the weight space of M' as a subset of the weight space of M.

Suppose in addition that $M' \to M$ is proper. Then M' and M have the same toral places. Moreover, we may regard the respective weight spaces as having the same elements.

Proof. The assertion concerning places is local on M. Suppose that M is affine and upon it the local parameters $u_1, \ldots, u_d$ define D. We write $\omega = du_I/u^I$ and proceed to use Theorem 1 of §1.

As the places of D' are toral the divisor of ω on M' is $-D'$. Thus, if ν is any toral place of M', $\nu(\omega) = -1$. Every place finite on M' is finite on M and so ν is a toral place of M.

Now suppose also that $M' \to M$ is proper. If ν is a toral place of M then $\nu(\omega) = -1$ and, as $M' \to M$ is proper, ν is a finite place of M'. So ν is a toral place of M'.

We saw in Theorem 3 Corollary 1 of §1.5 that toral places and simple weight correspond exactly. This induces a mapping from the simple weights of $\Lambda^{\vee}_{\mathbb{Q}}(M')$ to the simple weights of $\Lambda^{\vee}_{\mathbb{Q}}(M)$. To extend this mapping to rational weights we note that if β' is a non-zero rational weight of M' then the equation $\lambda\beta' = \alpha'$ has a unique solution for α' a simple weight. If the weight α' maps to α then we insist that β' map to β, where β solves the equation $\lambda\beta = \alpha$.

The mapping is clearly injective, and bijective if $M' \to M$ is proper.

Henceforth, we will think of the one weight space as being a subset of the other.

2. Subdivision of weight space

Even though the respective weight spaces of a proper toral modification $M' \to M$ may be identified as sets, they do not have the same decomposition into chambers. However, the decompositions are compatable. We will see later that a proper toral modification $M' \to M$ corresponds exactly to a subdivision of $\Lambda^{\vee}_{\mathbb{Q}}(M)$. This subdivision must satisfy properties that will be stated later.

Theorem 2. Suppose that $M' \to M$ is a toral modification. Then every open chamber $|\sigma'|$ of M' lies in a unique open chamber $|\sigma|$ of M.

Proof. As $\Lambda^{\vee}_{\mathbb{Q}}(M)$ is a disjoint union of open chambers, we need only show the existence of such an open chamber.

Let $\alpha^1, \ldots, \alpha^n$ be the simple weights corresponding to the places $\nu_1, \ldots, \nu_n$ of σ'. To each index i there corresponds a divisor D_i' on M' which has an image D_i on M. We can think of D_i as the center, on M, of the place ν_i and so, by Corollary 4 to Theorem 3 of §1.5, D_i is the intersection D_{σ_i} corresponding to the simplex σ_i in whose open chamber α^i lies.

We may now prove the result. It is enough to show that the union

$$\sigma_1 \cup \ldots \cup \sigma_n = \sigma \qquad (6)$$

is a simplex of M. As σ' is a simplex of M', the D_i' have nonempty intersection. As the image in M of D_i' is D_{σ_i}', these also have a nonempty intersection. So σ is a simplex.

Weight space is a disjoint union of open chambers.

Definition. A *subdivision* of a rational weight space $\Lambda^\vee_{\mathbb{Q}}(M)$ is a disjoint collection of subsets $|\sigma'|$ of $\Lambda^\vee_{\mathbb{Q}}(M)$ such that each $|\sigma'|$ lies in some open chamber of M.

If every point of $\Lambda^\vee_{\mathbb{Q}}(M)$ lies in some $|\sigma'|$ we say that the subdivision is *complete*.

Corollary 1. Suppose $M' \to M$ is a toral modification. Then $\Lambda^\vee_{\mathbb{Q}}(M')$ is a subdivision of $\Lambda^\vee_{\mathbb{Q}}(M)$. If the modification is proper then the subdivision is complete.

<u>Corollary 2</u>. Every closed chamber $Cl|\sigma'|$ of M' lies in some closed chamber $Cl|\sigma|$ of M.

Proof. We take for σ the simplex given by the Theorem. We now need to show that if $\overset{c}{\sigma}'$ is a subset of σ' then the corresponding simplex $\overset{c}{\sigma}$ provided by the Theorem is a subset of σ. But this is obvious, for $\overset{\circ}{\sigma}$ is just some sub-union of (6).

<u>Corollary 3</u>. Suppose that $M' \to M$ is a toral modification and that the sum $\lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n$ of rational weights of M' is admissable. Then it is admissable on M also.

Proof. The sum is admissable just in case the λ_i are non-negative and the α^i lie in a single closed chamber.

To conclude this discussion, we will show that admissable addition is compatable with the inclusion of weight spaces.

<u>Theorem 3</u>. Suppose that $M' \to M$ is a toral modification and that $\lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n$ is admissable on M'. Let α' denote the answer as computed in $\Lambda^\vee_{\mathbb{Q}}(M')$ and α the answer when computed in $\Lambda^\vee_{\mathbb{Q}}(M)$. Then $\alpha = \alpha'$.

Proof. We will reduce the problem to a special case, to which we apply algebraic geometry. Addition is associative and respects multiplication by scalars.

So we need only consider the case $\lambda\alpha + \mu\beta$. We may write α as $\sum \lambda_i \alpha^i$ where the α^i correspond to places that appear on M' and β similarly. Moreover, the summation

$$\lambda(\sum \lambda_i \alpha^i) + \mu(\sum \mu_i \beta^i)$$

is still admissable. And so we are reduced to the case $\lambda\alpha + \mu\beta$ where α and β correspond to places that appear on M'. If $\alpha = \beta$ the result is trivial. So suppose $\alpha \neq \beta$.

As addition respects multiplication by scalars we may suppose that λ and μ are co-prime positive integers A and B. The sum $A\alpha + B\beta$ being admissable, the divisors D_α' and D_β' on M' intersect.

We will first prove the result for A = B = 1. In that case the sum $\alpha + \beta$ corresponds, on M', to the place ν realized by making a monoidal transformation at the center $D_\alpha \cap D_\beta$. It is clear that the place ν has weight, on M, that is given by the sum on M of α and β.

To prove the result for general A and B we make a sequence of monoidal transformations on M' to realize $A\alpha + B\beta$ and argue inductively. That such a sequence exists is a consequence of the proof of Proposition 2, §1.2.

Before we can prove the next result we will need some terminology. Suppose $M' \to M$ is a toral modification and that

$$D' = D_1' \cup \ldots \cup D_n'$$

$$D = D_1 \cup \ldots \cup D_m.$$

A weight α of M' was defined to be an element $(\alpha_1', \ldots, \alpha_n')$ of $\mathbb{Z}^n$. But we think of α as a weight of M and in this case it is an element $(\alpha_1, \ldots, \alpha_m)$ of $\mathbb{Z}^m$. The n-tuple and the m-tuple reprsent the same weight and, if α is a simple weight, they represent the same place ν_α.

Definition. Suppose that α is a weight of M' and of M. Then the numbers

$$(\alpha_1', \ldots, \alpha_n') \qquad (\alpha_1, \ldots, \alpha_k)$$

are the *co-ordinates* of α on M' and M respectively. The numbers α_i' and α_i are, respectively, the multiplicities along ν_α of D_i' and D_i respectively.

Theorem 4. Suppose that $M' \to M$ is a toral modification and that σ is a simplex of M'. Then the set of elements of σ, thought of as elements of $\mathbb{Z}^k$, is simple.

Proof. Suppose that $\alpha^1, \ldots, \alpha^n$ are the vertices of σ. It is enough to show that if $\lambda_1, \ldots, \lambda_n$ are positive rational numbers such that

$$\alpha = \lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n \qquad (7)$$

tion and that

$$D' = D_1' \cup \ldots \cup D_n'$$

$$D = D_1 \cup \ldots \cup D_m.$$

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$$\alpha = \lambda_1\alpha^1 + \ldots + \lambda_n\alpha^n \qquad (7)$$

is integral, then the λ_i are integers. We can, by Theorem 3, compute this sum on M' if we so wish.

But on M' the co-ordinate of α^i is of the form $(0,\ldots,0,1,0,\ldots,0)$ and so the result is now obvious.

Remark. The result can be established also by the following argument. Let α be given by (7), and let $n\alpha$ be the smallest multiple of α such that the co-efficients $n\lambda_i$ are all integers. Then $n\alpha$ is a simple weight of M' and so corresponds to a toral place ν of M'. We show that the co-ordinates of ν on M are given by $n\alpha$. As ν is also a toral place of M, its co-ordinates on M form a simple k-tuple and so $n = 1$.

3. Fans

The next concept is useful in understanding the subdivision of $\check{\Lambda}_{\mathbb{Q}}(M)$ induced by a toral modification $M' \to M$.

Definition. Suppose D is a normal crossings divisor with connected intersections on a smooth variety M. The *fan* $\Sigma(M)$ is a simplicial complex whose vertices are the places associated to the components of D. A set σ of places **belongs to** the complex if the corresponding components of D have a non-empty intersection.

This definition is not adequate if D does not have connected intersections.

Example 1. $\Sigma(\mathbb{A}^2) = \{\emptyset,\{\nu_{(1,0)}\},\{\nu_{(0,1)}\},\{\nu_{(1,0)},\nu_{(0,1)}\}\}$

Example 2. Suppose $M' \to M$ is a toral transformation of $\mathbb{A}^2$. In §1.2 we associated to M' a graph G'. A graph can be thought of as a collection of vertices, together with a collection of pairs of vertices, which we will call edges. Using this interpretation and making obvious identifications

$$G' \cup \{\emptyset\} = \Sigma(\mathbb{A}^2).$$

Example 3. The fan Σ of $\mathbb{A}^n$ has n vertices. Every subset σ of vertices appears as a simplex of Σ .

Example 4. The fan Σ of $\mathbb{P}^{n+1}$ has n + 1 vertices. Every proper subset of vertices appears as a simplex of Σ .

Theorem 5. Suppose that $M' \to M$ is a toral modification. We will think of the vertices of $\Sigma' = \Sigma(M')$ as being simple weights of M. Then

(i) every simplex σ' of Σ' lies in some closed chamber of $\Lambda^{\vee}_{\mathbb{Q}}(M)$.

(ii) every simplex σ' is simple

(iii) the open chambers $|\sigma'|$ associated to the simplices of Σ' are disjoint. Moreover, if $M \to M'$ is proper

(iv) the subdivision of $\Lambda^{\vee}_{\mathbb{Q}}(M)$ into open chambers induced by Σ' is compete.

Conversely, suppose that Σ' is a finite simplicial complex of simple weights that satisfies (i), (ii), and (iii). Then there is a toral modification $M(\Sigma')$ whose fan is Σ', such that for any other toral modification M' with $\Sigma(M') = \Sigma'$, the rational map $M' \dashrightarrow M(\Sigma')$ is regular, and is an isomorphism onto its image. Clearly, $M(\Sigma')$ is unique, if it exists.

Moreover, if (iv) holds then the modification $M(\Sigma') \to M$ is proper.

Proof. Assume that $M' \to M$ is a toral modification.

(i) follows immediately from Theorem 2, Corollary 2;

(ii) is Theorem 4;

(iii) follows from §1.5, Theorem 3, Corollary 2. If two distinct open chambers of the form $|\sigma'|$ intersected then there would be a simple weight α in their intersection. It would then follow that the corresponding place ν would have two distinct centers on M'. But this is absurd.

(iv) follows from Theorem 1 of §1.1 over M_Σ.

The proof of the converse occupies the remainder of this section. We first define a modification $M_\sigma \to M$ for each simplex σ of Σ'. Next we show that we can patch the M_σ together into the modification $M(\Sigma')$. Finally, we demonstrate that $M(\Sigma')$ has the required properties.

4. Modification associated to a simplex

The construction described here is a variant of the modification $M_\alpha \to M$ constructed in Chapter I §4. . We first suppose that M is affine and on it are local parameters $u_1, \ldots, u_d$ which define D. If $\sigma = (\alpha^1, \ldots, \alpha^n)$ is a simple set of weights we define M_σ to be the affine variety whose co-ordinate ring is given by

$$k[M_\sigma] = k[M][u^I \mid \alpha^i(I) \geq 0 \quad i = 1 \ldots n].$$

As before, we can find matrices A and B, all of whose entries are integers, that satisfy the equations

$$AB = BA = \text{identity}$$

and in addition α^i is the i^{th} row of A.

We write $U = u^B$ and thus $U^A = u$. For any multi-index I,

$$u^I = u^{BAI} = U^{AI}$$

and as the first n rows of A are α^i $i = 1, \ldots, n$, we see that $\alpha^i(I) \geq 0$ just in case the first n elements of AI are non-negative. We have thus proven the next result.

Theorem 6. $k[M_\sigma] = k[M][U^J \mid J_i \geq 0 \quad i = 1, \ldots n]$.

Corollary 1. The ring $k[M_\sigma]$ is generated over $k[M]$ by $U_1, \ldots, U_d; U_{d+1}^{-1}, \ldots, U_d$

Corollary 2. If σ' is a subset of σ then M_{σ}' is a principal open subset of M_σ.

Proof. It is enough to prove the result for $\sigma - \sigma'$ having α^i as its only element. In that case

$$k[M_{\sigma}'] = k[M_\sigma][U_i^{-1}]$$

which proves the result.

Theorem 7. The variety M_σ is smooth, with local parameters $U_1, \ldots, U_d$.

Proof. If J denotes the j^{th} column of A then $U^J = u_j$ and so du_j can be written in the form

$$du_j = \Sigma(u_j)_i dU_i$$

where the co-efficients $(u_j)_i$ are regular on M_σ.

If f is a function regular on M then $df = \Sigma f_j du_j$ for functions f_j regular on M_σ and so

$$df = \Sigma f_i dU_i$$

for functions f_i regular on M_σ.

It now follows immediately that if $g = \Sigma g_J U^J$ is some function regular on M_σ then the functions g_i defined by the equation

$$dg = \sum g_i dU_i$$

are regular on M_σ also. This concludes the proof.

<u>Corollary 1</u>. The modification $M_\sigma \to M$ is toral.

Proof. The divisor D on M has equation $u^I = 0$ where $I = (1,\ldots,1)$ and so the total transform of D on M' has equation $U^{AI} = 0$. As the i^{th} row of A is α^i we see that the i^{th} component of AI is $\alpha_1^i + \ldots + \alpha_d^i$ which is strictly positive. Thus, the reduced total transform D' has equation $U^I = 0$. As the U_i form a system of local parameters, D' has normal crossings.

It follows from Theorem 6, Corollary 2 that if M' is obtained from M_σ by removing all components of D' except that associated to U_i, then $M' = M_{\alpha^i}$. It now follows from §1.5 Theorem 3 that the associated place is toral.

<u>Corollary 2</u>. If α is a simple weight lying in the closed chamber $Cl|\sigma|$ then the corresponding place ν is finite on M_σ.

Proof. As the modification $M_\sigma \to M$ is toral, the weight space of M_σ is a subset of that of M. It is clear that $Cl|\sigma|$ is the appropriate subset. The result now follows.

Even though M_σ was defined locally, it is clear that it depends only upon the divisor defined by the local parameters. Thus, the construction is in fact global.

5. Patching and Separation

We will show that if Σ' satisfies condition (iii) of **Theorem 5 §2.3** then the associated M_σ may be patched together to give a separated variety. The patching data is provided in the following manner. If σ and σ' are two simplices of Σ' then we identify M_σ and $M_{\sigma'}$ along their common open subset $M_{\sigma\cap\sigma'}$. It is trivial to verify that the identifications satisfy the patching conditions. Separation is harder. We may assume that M is affine with local parameters $u_1, \ldots, u_d$, etc.

Theorem 8. The space $M(\Sigma')$ obtained by patching the M_σ together is separated if $|\sigma'|$ the open chambers associated to σ' are disjoint.

Proof. We can assume that M is affine etc. According to [35, Ch V §4.3] it is enough to show that the monomials of $k[M_\sigma]$ and $k[M_{\sigma'}]$ together **generate** the monomials of $k[M_{\sigma\cap\sigma'}]$. The proof of this result in [7, p. 557] I do not understand. The proof is left to the reader in [22, p. 24]. A proof is given in [4, §5.4] but it depends upon an assertion that is not justified. The author claims that there is a hyperplane H in $\mathbb{Q}^d$ which contains $\sigma \cap \sigma'$ and the remaining points of σ and σ' lie on opposite sides of H.

An immediate consequence of this assertion is that there is some exponent I for which $\alpha I = 0$ on $\sigma \cap \sigma'$ and for the remaining vertices of σ and σ' respectively I is positive and negative respectively. It now follows that if u^J lies in $k[M_{\sigma\cap\sigma'}]$ then u^{J+nI}, u^{J-nI} lie in $k[M_\sigma]$ and $k[M_{\sigma'}]$ respectively, for n large enough. Peter McMullen has provided me with a proof of the assertion in a personal communication.

6. Completion of Proof

First, we will prove that if Σ' is the fan of M' then the rational map $M' \dashrightarrow M(\Sigma')$ is regular. If σ is a simplex of Σ' use M_σ' to denote the open subset of M' obtained by removing all components of D' except those associated to σ. As every place appearing on M_σ' appears on M_σ, every function regular on M_σ is regular on M_σ'. As M_σ is affine this shows that the map $M_\sigma' \to M_\sigma$ is regular. Moreover, the map cannot have an exceptional place and so is an isomorphism onto its image. As the subsets M_σ' cover M' the map $M' \dashrightarrow M(\Sigma')$ is regular, and can be thought of as the inclusion of an open set.

Next, we will show that if Σ' induces a complete subdivision of the weight space $\Lambda^\vee_{\mathbb{Q}}(M)$ then $M(\Sigma')$ is complete. By the criterion of Chapter I §4.3 it is enough to show that every place ν finite on M is finite on $M(\Sigma')$. But, if α is the simple weight associated to ν then, by §1.5 Theorem 3, ν is a finite place of M_α. But M_α is a toral modification of M_σ, where σ is the simplex of Σ' whose open chamber contains α.

7. Comments

Demazure was led to define and study non-singular torus embeddings in his paper [7] on subgroups of the Cremona group. (A *torus embedding* is a variety X containing a torus $T \subset X$ as an open subset in such a way that the birational action of T on X is biregular). Kempf et al [22] extended the results to cover not only varieties $X \supset T$ with normal singularities but also varieties that were only locally isomorphic to torus embeddings. The major result in this work is the proof of what is known as the semi-stable reduction theorem.

Because they wished to perform a succession of cyclic covers, Kempf et al changed the definition of a fan. Demazure uses a simplicial complex as the fan, they use a collection of cones. (We call these cones chambers). Subsequent authors have continued to use the Kempf et al definition. Our definition is the same as that of Demazure, once one associates to each simple one parameter subgroup a place.

A number of Russian authors - see §3.1 - used the techniques of torus embeddings to construct explicit resolutions for various singularities. Hitherto, Hironaka's resolution theorem [17] was somewhat abstract. Explicit descriptions of the resolution of particular singularities were uncommon.

This was no doubt due to the large amount of labor involved in such a calculation. In this context, the fan should be thought of as a combinatorial device that enables one to perform a large number of monoidal transformations. Work of Hironaka, for example [18], is closely related to this point of view.

That the differential form ω in many ways controls the structure of torus embeddings and their monoidal transformations is a consequence of the theory of minimal places.

The general definition of a toral modification given in the Introduction has several sources

- the wish to compute the essential places of a hypersurface $X \subset \mathbb{A}^n$ that is nondegenerate in the sense of §3.1.
- the fact that the theory of torus embeddings cannot resolve an analytically irreducible plane curve singularity $X \subset \mathbb{A}^2$ unless it has only one characteristic pair.
- the wish to extend to more general hypersurfaces results that are known for nondegenerate hypersurfaces. I have in mind the existence of the *relative canonical model* [33, Problem 6.3].
- the belief that much of the theory developed here will extend in a useful manner to a more general context.

We will now state and outline the proof of the main result of this section in the language of torus embeddings.

<u>Theorem 9</u>. Suppose that $M \supset T$ is a nonsingular completion of the torus T such that $D = M - T$ is a normal crossings divisor. Suppose also that the divisor of the differential $\omega = (dx/x) \wedge \ldots \wedge (dz/z)$ is $-D$. Then M is a torus embedding.

Proof. Solutions of the equation $\nu(\omega) = -1$ correspond to simple elements of the lattice $\Lambda^{\vee}$ dual to the lattice Λ of monomials. This can be proven by taking projective space as a completion of T. We need to show that the fan S of D satisfies the conditions (ii) and (iii) of Theorem 5, and also that S induces a complete division of $\Lambda^{\vee}_{\mathbb{Q}}$.

As there will only be one toral place with any given weight, it follows that D has connected intersections, and also that (iii) is satisfied. If a simplex σ were not simple then one could obtain a toral place whose associated weight α - which lies in $|\sigma|$ - is not simple. This shows (ii). As M is complete, every place is finite, and so the division must be complete.

We will now state and prove the main result of this section in the language of torus embeddings.

Theorem 9. Suppose that $M \supset T$ is a non-singular completion such that $D = M - T$ is a normal crossings divisor. Suppose also that the divisor (ω) of ω is $-D$. Then M is a torus embedding.

Proof. We consider the fan Σ as a subset of the lattice $\Lambda^{\vee}$ dual to the lattice Λ of monomials. We need first to show that Σ satisfies conditions (ii) and (iii) of Theorem 5, and also that Σ induces a complete division of $\Lambda^{\vee}_{\mathbb{Q}}$. But if (ii) or (iii) fails then a suitable sequence of monoidal transformations produces a contradiction. If the decomposition is not complete, then neither is Σ. The reader is asked to verify that the map $M \dashrightarrow M(\Sigma)$ is an isomorphism, being regular and having no exceptional divisors.

§3. NONDEGENERATE HYPERSURFACES

1. Definition and comments

Definition. Suppose that X is a hypersurface on the smooth variety M upon which lies a normal crossings divisor D. If there is a proper toral modification $M' \to M$ such that the union of the strict transform st X of X' with D' is a normal crossings divisor, then we say that X is *non-degenerate*, with respect to D, and that $M' \to M$ *resolves* X.

For X to be non-degenerate it is not enough that we can find a modification $M' \to M$ that provides an embedded resolution of X. We insist also that the total transform of X intersect the strict transform of D transversally. In this section we state a criterion for X to be non-degenerate. For such X we will supply conditions that $M' \to M$ must satisfy if it is to resolve X. Examples are provided in §5 of this Chapter.

The idea of applying the methods of torus embeddings to the study of hypersurface singularities and their resolutions originated from the Russian school of geometry. From Arnol'ds classification theory of degenerate critical points [2] came many computational problems, of which I will mention but two. The one is to compute the Milnor number μ, and the other is to compute the monodromy, or at least its zeta function, for any given critical point of a holomorphic function.

It so happens that critical points of sufficiently low modality have a normal form in which the corresponding hypersurface is non-degenerate. As the classification of Arnol'd is only explicitly computed for modality at most two, **almost** all critical points appearing in Arnol'ds papers are non-degenerate.

Thus, to compute an invariant of such a singularity it is enough to express the invariant as some quantity which can be derived from a knowledge of the resolution. The first result of this kind that I am aware of is due to Kushnirenko [26], who showed how to compute the Milnor number. Later, Varchenko [37], drawing upon work of A'Campo [1], was able to compute the zeta function of monodromy for such singularities. The paper of A'Campo shows that the zeta function could be computed from a knowledge of the total transform of an embedded resolution of the hypersurface. It is the earliest non-trivial example of a result which shows that the geometry of a resolution of a singularity is far from arbitrary.

Since then there have been many papers concerned with non-degenerate hypersurfaces in torus embeddings.

A particularly interesting example of a non-degenerate hypersurface is the classical representation of an Enriques surface as a quartic in $\mathbb{P}^3$ with ordinary singularities along the edges of a tetrahedron [14, p. 632].

2. Newton polyhedron

Suppose that M is smooth and that on it lies a normal crossings divisor D and a hypersurface X. We will associate to X a polyhedron $\Delta(X)$ which lies in the dual of the weight space of M. From this polyhedron we will be able to compute $\nu(X)$ for any toral place ν. Moreover, if X is non-degenerate then the polyhedron $\Delta(X)$ can be used to determine whether or not a proper toral modification $M(\Sigma')$ resolves X. This will be done in §3.3.

First, we will calculate $\nu(X)$ in local parameters. Suppose that $u_1, \ldots, u_d$ provide local parameters and that ν is some toral place, to which the simple weight α corresponds. Let A denote the matrix with inverse B such that $U = u^B$ provides a system of parameters for M_α. At least locally on M, X has a local equation $f = 0$, and we may give f a formal power series expansion

$$f = \Sigma f_I u^I$$

for suitable co-efficients f_I. In terms of the local parameters U we have

$$f = \sum f_I U^{AI}$$

and $\nu(X)$ is the largest exponent n such that U_1^n divides f. For example, $\nu(U^{AI})$ is equal to the first co-ordinate of AI. But that is just αI. It is now clear that $\nu(f)$ is equal to the smallest value of αI, as I ranges over those exponents for which f_I is non-zero. We will call the set Supp(f) of such **exponents** the *support* of the power series expansion of f.

Theorem 1. Suppose the toral place ν and the simple weight α correspond to each other. If the hypersurface X has local equation f = 0 then $\nu(X) = \min\{\alpha I \mid I \in \mathrm{Supp}\ f\}$.

Whether or not a particular co-efficient in the power series expansion of a local equation is zero will depend, in general, upon the choice of local equation and of local parameters. However, the following definition captures the essential content of the support but depends neither upon the choice made of equations, nor of local parameters.

Definition. An exponent I lies in the *Newton polyhedron* $\Delta(X)$ if $\nu(u^I)$ is at least as large as $\nu(X)$ for every toral place ν. The Newton polyhedron $\Delta(X)$ is the convex hull of its exponents.

The *Newton polyhedron* $\Delta(f)$ is the convex hull of the elements of the form $I + P$ where $I \in \mathrm{Supp}(f)$ and P is an exponent for which u^P is regular on M.

We will need some definitions concerning convex sets. Suppose that Δ is a convex set in one space and that α is an element of the dual space. We are not assuming that Δ is compact, but we will have Δ closed. We will use $\alpha\Delta$ to denote the smallest value of αv for $v \in \Delta$. If there is no such value, we write $\alpha\Delta = -\infty$. Notice that $(\alpha + \beta)\Delta$ need not equal $\alpha\Delta + \beta\Delta$.

The collection of those points $v \in \Delta$ that satisfy $\alpha v = \alpha\Delta$ is convex, and is called the *face* Δ^α defined by α. If Δ^α is a single point $\{v\}$, we say that v is a vertex of Δ. For example, every boundary point is a vertex of a disc. The convex set Δ is a face of itself, being Δ^α for α the linear function that is identically zero.

The faces Δ^α and Δ^β can be the same for α and β distinct. However, if $\Delta^\alpha = \Delta^\beta$ then they are both equal to Δ^γ for γ a convex combination of α and β. We will use δ as a symbol to denote a face of Δ.

If Δ^α has codimension one in the ambient space then we say that Δ^α is a ***facet*** of Δ. If Δ^α is a facet and $\Delta^\alpha = \Delta^\beta$ then one of α and β is a scalar multiple of the other.

Theorem 2. (i) $\Delta(f)$ has only a finite number of vertices;

(ii) there is a finite number of simple weights α_i which define $\Delta(f)$; i.e.,

$$\Delta(f) = \{v \mid \alpha_i v \geq \alpha\Delta(f) \mid i = 1,\ldots,N\}$$

(iii) $\Delta(f) = \Delta(X)$.

Proof. (i) is proved in [18, p. 277]. For the convenience of the reader, we sketch the proof here. If $d = 1$ then the result is trivial. Suppose now that $d = 2$. Of all exponents (a,b) appearing in Supp(f) we can find two, (a_0,b_0) and (a_1,b_1) such that a_0 and b_1 are as small as possible. It is now clear that any other vertex must lie in the triangle whose vertices are (a_0,b_0), (a_1,b_1) and (a_0,b_0). And so the number of such is finite. As a consequence, the number of faces is finite.

In dimension three or more there is the possibility of a large number of non-compact faces. We say that a face Δ^α is *parallel* to a co-ordinate plane P if Δ^α contains an open subset of some translate of P. But the argument used for $d = 2$ shows that there is only a finite number of faces of Δ parallel to any given co-ordinate plane P whose codimension is two. If $d = 3$, we take a point from each of these faces and use them, as we used (a_0,b_0) and (a_1,b_1) for $d = 2$, to put a bound on the number of vertices.

In this way an inductive proof of the result may be constructed.

(ii) is an immediate consequence of (i). The weights α_i can be chosen to be simple because the vertices of $\Delta(f)$ are rational.

(iii) It is an immediate consequence of the definitions that $\Delta(f) \subset \Delta(X)$. Moreover, if $I \in \Delta(X)$ then $\alpha I \geq \nu(X) = \nu(f) = \alpha\Delta(f)$ for all simple weights α. And so the result follows from (ii). This concludes the proof.

The preceding discussion was based upon local parameters. But the properties and constructions depend only upon D, and not on the choice of parameters. There are no obstructions to making a global construction, in which an exponent $I = (i_1,\ldots,i_n)$ is replaced by a formal sum $i_1D_1 + \ldots + i_nD_n$ of components of D. However, in contrast to the construction of weight space, we allow inadmissable sums. I will neither develop here the properties of the global construction, nor study the effect of a toral modification upon the Newton polyhedron. In his paper [18] Hironaka say that resolution can be thought of as "sharpening" the Newton polyhedron.

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3. Nondegeneracy criterion

Definition. Suppose that $X \subset M$ is a hypersurface and that α is a simple weight. If st $X \subset M_\alpha$ is transversal to D_α, we say that X is *transversal* to α.

The proof of the next result is spread over the remainder of this section.

Theorem 3. A hypersurface $X \subset M$ is nondegenerate just in case it is nonsingular away from D and transversal to every simple weight α.

We will show here that the condition is necessary. Suppose that the toral modification $M' \to M$ provides a resolution of X. Each simple weight α determines a sequence $M'_{(\alpha)} \to M'$ of monoidal transformations. At each stage the center is a component of an intersection of components of the normal crossings divisor st $X \cup D'$ and so $M'_{(\alpha)} \to M$ resolves X. As M_α is an open subset of $M'_{(\alpha)}$, X is transversal to α. As $M' \to M$ is an isomorphism away from D, X is nonsingular away from D.

Conversely, suppose that X is nonsingular **away from D** and transversal to every α. In §3.4 we give necessary and sufficient conditions on Σ for $M(\Sigma) \to M$ to resolve X. In §3.5 we show that it is possible to construct such a Σ.

We give now a criterion, due to Khovanskii [23] for $X \subset \mathbb{A}^n$ to be transversal to α.

Definition. Suppose that $f = \sum \lambda_I U^I$ is a polynomial and that α is a weight. The sum

$$f^\alpha = \sum_{\alpha I = m} \lambda_I u^I \quad (m = \nu_\alpha(f))$$

of those monomials of f whose multiplicity along ν_α is equal to that of f is called the α-leading term of f.

Example. The (1,1) leading term of $x^2 + xy^2 + y^3$ is x^2. The (3,2) leading term is $x^2 + y^3$. The (2,1) leading term is y^3. More generally, if $2m > 3n$ then the (m,n) leading term is y^3. See figure 6.

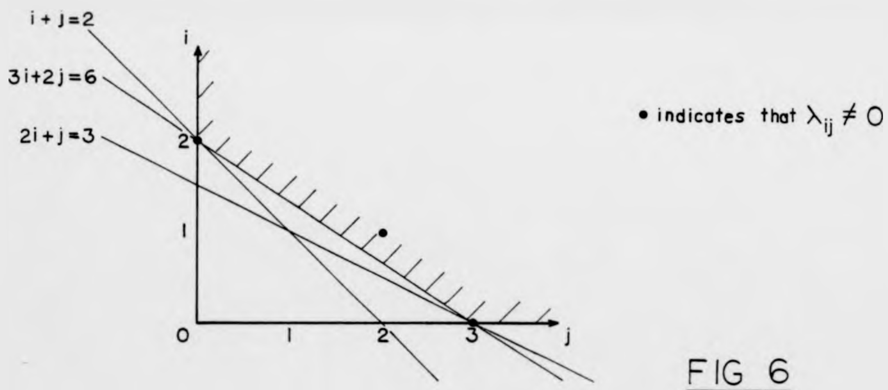

FIG 6

Theorem 4. The hypersurface $V(f) \subset \mathbb{A}^n$ is transversal to α just in case $V(f^\alpha) \subset A^n$ is nonsingular away from D.

Proof. Suppose that $U_1, \ldots, U_n$ are local parameters for M_α and that U_1 and f' are local equations for D_α and st X in M_α. We can write

$$f' = g + U_1 h \quad h \in k[M\ ]$$

where $g = U_1^{-m} f^{\alpha}$ is a polynomial in $U_2,\ldots,U_n$. $V(f)$ is transversal to α iff at every point of $V(U_1,f')$ the differential df' is not a multiple, possibly zero, of dU_1. As $V(U_1,f')$ equals $V(U_1,g)$ and as

$$df' = dg + U_1 dh + h dU_1$$

it follows that $V(f)$ is transversal to α just in case $V(g)$ is transversal to D_{α}. As $g \in k[U_2,\ldots,U_m]$ this last condition is equivlent to $V(g)$ being nonsingular away from D_{α}. But away from D_{α} and D respectively, $V(g)$ and $V(f^{\alpha})$ are equal. This concludes the proof.

4. Condition on fan

Definition. To each face δ of Δ there corresponds an *open chamber* $|\delta|$ of weight space, consisting of all α that satisfy the equation $\Delta^{\alpha} = \delta$. In this way Δ induces a decomposition of weight space.

If $P_1,\ldots,P_n$ are the vertices of δ and $Q_1,\ldots,Q_m$ are the remaining vertices of Δ then it is easily seen that

$$|\delta| = \{\alpha \,|\, \alpha P_1 = \ldots = \alpha P_n < \alpha Q_i, \quad i = 1,\ldots,m\}$$

and so the *closure* is given by

$$\mathrm{Cl}\ |\delta| = \{\alpha \,|\, \alpha P_1 = \ldots = \alpha P_n \leq \alpha Q_i, \quad i = 1,\ldots,m\}$$

and thus

$$\mathrm{Cl}\ |\delta| = \{\alpha \,|\, \delta \subset \Delta^{\alpha}\}.$$

The complement of $|\delta|$ in Cl $|\delta|$ is the *boundary* Bdy $|\delta|$ of $|\delta|$.

Lemma. (i) Suppose that P is a point of Δ. Then $\alpha P \geq \alpha\Delta$, with equality iff $P \in \Delta^{\alpha}$.

(ii) $\alpha,\ldots,\beta$ lie in a common closed chamber Cl $|\delta|$ of Δ iff

$$(\alpha + \ldots + \beta)\Delta = \alpha\Delta + \ldots + \beta\Delta.$$

Proof. (i) is an immediate consequence of the definitions of $\alpha\Delta$ and Δ^{α} on p. 132.

(ii) There is a vertex P of such that

$$\begin{aligned}(\alpha + \ldots + \beta)\Delta &= (\alpha + \ldots + \beta)P \\ &= \alpha P + \ldots + \beta P \\ &\geq \alpha\Delta + \ldots + \beta\Delta\end{aligned}$$

and by (i) equality holds iff $P \in \Delta^{\alpha}, \ldots, P \in \Delta^{\beta}$. If equality holds then $\alpha,\ldots,\beta \in$ Cl $|P|$. Conversely, if $\alpha,\ldots,\beta \in$ Cl $|\delta|$ and P is a vertex of δ then $P \in \Delta^{\alpha},\ldots$ $P \in \Delta^{\beta}$ and so equality holds.

Theorem 5. Suppose that the hypersurface $X \subset M$ is transversal to every α and nonsingular away from D. A proper toral modification $M' \to M$ resolves X just in case the decomposition of weight space induced by its fan Σ is a subdivision of that induced by the Newton polyhedron Δ of X.

Proof. Suppose that $\sigma = \{\alpha,\ldots,\beta\}$ is a simplex of Σ. Write $\gamma = \alpha + \ldots + \beta$. It is the weight corresponding to the place realized by monoidally transforming M' along $D_\alpha \cap \ldots \cap D_\beta$. If M' M resolves X then the multiplicity $\nu_\gamma(X)$ is equal to the sum of the multiplicities $\nu_\alpha(X), \ldots, \nu_\beta(X)$. In terms of Δ we have

$$\gamma\Delta = \alpha\Delta + \ldots + \beta\Delta$$

and so by part (ii) of the lemma, the vertices $\alpha,\ldots,\beta$ of σ lie in a single closed chamber Cl $|\delta|$ of Δ. If δ' is the largest face of Δ with this property then $|\sigma| \subset |\delta'|$.

Conversely, suppose that X is transversal etc., it is enough to show that st $X \subset M_\gamma$ is transversal to D at every point of $D_\alpha \cap \ldots \cap D_\beta$. We can suppose that $u_1,\ldots,u_n$ are local parameters on M_σ and that $u_1,\ldots,u_m$ provide local equations for $D_\alpha,\ldots,D_\beta$. Suppose that $f = 0$ is a local equation for st X. Upon writing $f = \sum f_i du_i$ the condition that X be transversal becomes that the system

$$f = u_1 = \ldots = u_m = f_{m+1} = \ldots = f_n = 0 \qquad (8)$$

of equations has no solution.

On M_γ the functions

$$U_1 = u_1; \quad U_2 = u_2/u_1,\ldots \quad U_m = u_m/u_1;$$

$$U_{m+1} = u_{m+1},\ldots \quad U_n = u_n$$

constitute a system of local parameters. Because $\gamma\Delta = \alpha\Delta + \ldots + \beta\Delta$, f is a local equation for st $X \subset M_\gamma$. A simple calculation shows that

$$df = (f_1 + f_2U_2 + \dots + f_mU_m)dU_1$$
$$+ f_2U_1dU_2 + \dots + f_mU_1dU_m$$
$$+ f_{m+1}dU_{m+1} + \dots + f_ndU_n$$

and we assumed that X is transversal to γ the system of equations

$$f = U_1 = f_2U_2 = \dots = f_mU_1 = f_{m+1} = \dots = f_n = 0$$

has no solution on M_γ. From this it follows immediately that (8) has no solution. This concludes the proof.

5. Construction of fan

Suppose that $|\Delta|$ is the division of weight space corresponding to a hypersurface $X \subset M$ that is transversal to every etc. To complete the proof of Theorem 3 it is necessary to construct a fan Σ that induces a subdivision of $|\Delta|$. The quickest way to show that such a fan exists is to construct a subdivision of $|\Delta|$, to which a fan Σ corresponds.

The reader will find in [4, §8] a procedure for constructing such a subdivision. One first divides $|\Delta|$ into simplicial chambers. Each simplicial chamber is then further divided until it is of the form $|\sigma|$, for σ a simple collection of weights. This construction provides, in the interpretation of [4], a resolution for the singular toric modification $M(\Delta) \to M$.

This construction is not completely satisfactory. It is a general principle, by now well established, that singularities should be resolved by making a sequence of monoidal transformations. There is no good reason why nondegenerate singularities should be treated in a different manner. Also, the procedure of [4, §8] is computationally awkward. It does not seem to be amenable to mechanical calculations.

The author's attempts to find an algorithm that will construct a suitable Σ by making a sequence of monoidal transformations have not succeeded. A solution to the following special case would yield a general algorithm.

Problem. Suppose that σ is a simple set of weights and $|\delta|$ is the intersection of $|\sigma|$ with a rational linear subspace of weight space. It is required to produce a fan Σ corresponding to a sequence of monoidal transformations such that $|\delta|$ is a union of chambers of Σ.

Example 1. If $|\delta|$ has dimension one then the realization of the unique simple weight α in $|\delta|$ will provide such a fan.

Example 2. See figure 7. Here $|\sigma|$ has dimension 3 and $|\delta|$ dimension 2. Necessarily, the simple weights α and β corresponding to the boundary of $|\delta|$ will appear as vertices in any suitable Σ. Moreover, for the configuration drawn, any monoidal transformation will produce another simple weight that also must appear as a vertex of Σ.

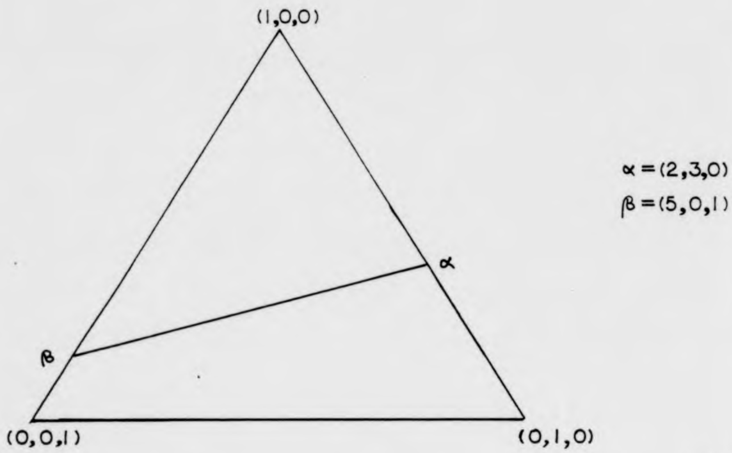

FIG 7

§4. DIFFERENTIALS OF TOP DEGREE

In this section we are concerned with the order of differentials ω of M, of top degree, with respect to a nondegenerate hypersurface $X \subset M$. Throughout we will use ω to denote an r-fold differential of top degree regular on M - X. For simplicity we will suppose that X is a hypersurface in $\mathbb{A}^n$ with local equation f = 0 and Newton polyhedron Δ. We will let $u_1,\ldots,u_n$ be local parameters on $\mathbb{A}^n$.

1. Order.

Recall from Ch. II §4, that $\mathrm{ord}_X(\omega)$ is the smallest value of $\nu(\omega)$ as ν runs over all the sungular places of M. Recall also that it is enough to consider only those ν appearing in a resolution $M' \to M$ of X. This is subject to the provision that if $\nu(\omega)$ is less than -r, then it counts as $-\infty$.

Definition. A simple weight α is *singular* if its center $D_\alpha \subset M$ lies in Sing X. If the center is a divisor, we say that α *appears* on M.

We will now obtain a formula for $\nu_\alpha(\omega)$. First, we write ω in the form

$$\omega = (1/f)^r \omega_0 (du_I/u^I)^r \tag{9}$$

for a suitable co-efficient function ω_0. As ω was assumed regular away from X, so is ω_0; and moreover it has a zero of order at least r along the co-ordinate hyperplanes. For ω to be logarithmic along X it is necessary and sufficient that ω_0 be regular along X.

Thus, the differential ω is regular on M - X and logarithmic along X just in case the function ω_0 defined by (9) is regular on M and moreover vanishes to order r along the co-ordinate hyperplanes.

Theorem 1. Suppose that ω_0 has Newton polyhedron $\Delta(\omega_0)$. Then the expression

$$\alpha\Delta(\omega_0) - r\alpha\Delta - r \tag{10}$$

is equal to $\nu_\alpha(\omega)$.

Proof. From the equations

$$\nu_\alpha(\omega_0) = \alpha\Delta(\omega_0) \quad \omega_\alpha(f) = \alpha\Delta \quad \nu_\alpha(du_I/u^I) = -1$$

the result follows immediately.

It is convenient to introduce a factor of r into the first term of (10).

Definition. The *Newton polyhedron* $\Delta(\omega)$ of the differential ω consists of all rational exponents I such that rI lies in $\Delta(\omega_0)$. In other words, $\Delta(\omega) = \Delta(\omega_0)/r$. It need not have integral vertices.

Corollary 1. $\nu_\alpha(\omega) = r[\alpha\Delta(\omega) - \alpha\Delta - 1]$.

Corollary 2. The order $\mathrm{ord}_x(\omega)$ is the minimum of $r[\alpha\Delta(\omega) - \alpha\Delta - 1]$ over all singular weights α. (This is subject of course to the provision that an order less than $-r$ counts as $-\infty$.)

It is sufficient in Corollary 2 to consider only those α appearing in a resolution $M' \to M$. However, in some cases we can obtain results without computing a resolution.

Theorem 2. Suppose that ω is a simple differential. Then ω is of the first kind just in case $\Delta(\omega)$ lies in the interior of Δ.

Proof. First, we need some facts. Suppose that α is the simple weight corresponding to a facet of Δ. It is easily seen that either α is singular, or that it appears on M. As ω is simple, $\Delta(\omega)$ has integral vertices. Finally, we need that ω is of the first kind if and only if $\nu_\alpha(\omega)$ is nonnegative for every simple weight α.

If ω is of the first kind then

$$\nu_\alpha(\omega) = \alpha\Delta(\omega) - \alpha\Delta - 1 \geq 0 \tag{11}$$

for every α corresponding to a facet of Δ and so $\Delta(\omega)$ is in the interior of Δ. Conversely, if $\Delta(\omega)$ is in the interior of Δ then $\alpha\Delta(\omega) > \alpha\Delta$ for every α. As $\alpha\Delta(\omega)$ is an integer the inequality (11) follows.

The result is not valid if ω is not simple. However, the next result holds for r-fold differentials. Its proof is similar.

Theorem 3. A differential ω is logarithmic just in case $\Delta(\omega)$ is contained in Δ and $\alpha\Delta(\omega) \geq 1$ for the α that appear on M.

2. Heart

Instead of considering the order of some given differential, we can look at the differentials whose order is at least some given quantity. But first we will divide by r.

Definition. The quantity $\mathrm{ord}_x(\omega)/r$ is called the normalized order of the r-fold differential ω. Clearly, if less than -1 then it is $-\infty$.

The reader can check that those differentials whose normalized order is at least some given quantity λ form a graded algebra. If $\lambda = 0$ we have differentials of the first kind, while if $\lambda = -1$ we have the logarithmic differentials. To compute this algebra we need another polytope.

Definition. Suppose that λ is at least -1. The λ-*heart* $H_\lambda(\Delta)$ of the Newton polyhedron Δ consists of those rational exponents I for which

(i) $\alpha I - \alpha\Delta - 1 \geq \lambda$ for every singular α, and

(ii) $\alpha I - \alpha\Delta - 1 \geq 0$ for the α that appear on M.

If $\lambda = 0$ we will write $H(\Delta)$ and call it the heart of Δ.

The next result follows immediately from Theorem 1.

Theorem 4. A differential ω has normalized order at least λ just in case $\Delta(\omega) \subset H_\lambda(\Delta)$.

Theorem 5. $H_\lambda(\Delta)$ has a finite number of faces. If λ is rational, it has rational vertices.

Proof. To compute $\mathrm{ord}_X(\omega)$ it is enough to look at $\nu_\alpha(\omega)$ as α runs over the singular places of a resolution. Thus, $H_\lambda(\Delta)$ is defined by a finite number of inequalities, and so has a finite number of faces. If λ is rational then so are the defining inequalities, and thus the vertices are also.

3. Finite generation.

Theorem 6. Suppose that λ is at least -1 and is rational. If $X \subset \mathbb{A}^n$ is a nondegenerate hypersurface then the algebra of differentials of order at least λ is finitely generated.

Proof. The differential

$$(1/f)^r u^J (du_I/u^I)^r$$

lies in the algebra just in case J lies in $rH_\lambda(\Delta)$. From this we see that the algebra is isomorphic to

$$k[u^J t^r \mid r \geq 0,\ J \in rH_\lambda(\Delta),\ r \text{ and } J \text{ integral}]$$

which is the semigroup ring of the semigroup

$$\{(J,r) \mid r \geq 0,\ J \in rH_\lambda(\Delta),\ r \text{ and } J \text{ integral}\}.$$

This semigroup consists of all integral points that satisfy a finite number of homogeneous rational inequalities. It is well known -- for instance [4, §1] -- that such semigroups are finitely generated. Hence, the algebra also is finitely generated.

There seem to be two reasons why the algebra is not generated by its elements of degree one. The first is that the vertices of $H_\lambda(\Delta)$ may not be integral. If J is a vertex of $H_\lambda(\Delta)$ and r the smallest positive integer such that rJ is integral then the differential corresponding to J does not lie in the subalgebra generated by elements of degree less than r.

The second reason is more subtle. The *affine span* of a face δ of $H_\lambda(\Delta)$ consists of all points of the form

$$\Sigma \lambda_i I_i, \Sigma \lambda_i = 1, I_i \in \delta$$

while for the *affine integral span* we insist that λ_i and I_i be integral. If there is a face δ of $H_\lambda(\Delta)$ whose affine span has integral points that do not lie in the affine integral span then the algebra is not generated in degree one.

4. Comments.

The first results of the type, in this section are due to Khovanskii [23, Theorem 1]. He showed that the arithmetic genus of a nondegenerate hypersurface $X \subset \mathbb{P}^n$ is equal to the number of integral points interior to its Newton polyhedron Δ. He also derives a related formula for a complete intersection. However, his methods fail to compute the higher genera.

We can define $H(\Delta)$ in exactly the same way for projective hypersurfaces. Then, via Poincare residue, there corresponds to each integral point of $rH(\Delta)$ an r-fold differential of top degree of X that is everywhere regular. It can happen, for $r > 1$, that these differentials are linearly dependent. Also, we may not get all such differentials in this way. These phenomena are at root cohomological.

Plane curves provide the simplest examples of this inadequacy of $H(\Delta)$. We use the Riemann-Roch theorem to compute plurigenera. See figure 8.

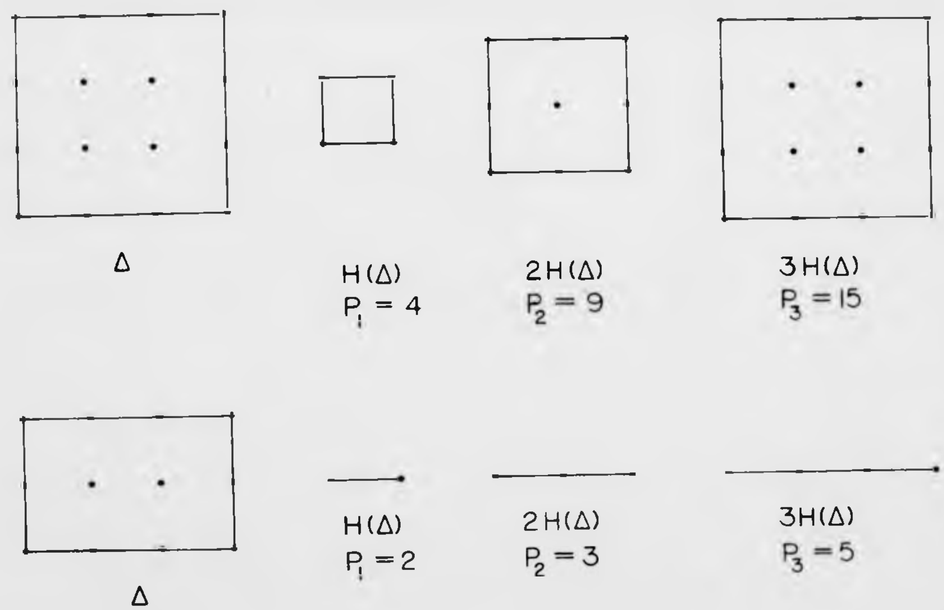

FIG 8

§5 EXAMPLES

In this section we provide few examples that illustrate some of the concepts developed earlier. There are two examples of nondegenerate plane curves, and an example of a nondegenerate surface. Finally, we give an example of a plane curve whose resolution requires the general concept of a toral transformation, as provided in the Introduction.

It seems that to compute $H(\Delta)$ one must first compute a resolution. This requires a great many computations. Until a mechanical procedure is developed for performing such computations, the amount of labor contained in an example will be large.

1. Two nondegenerate curves

The Newton polyhedron of the curve $x^3 + y^5 = 0$ has but one compact facet, corresponding to the weight (5,3). To realize (5,3) and thereby resolve the curve we make the following sequence of transformations.

$$(1,0) - (0,1)$$
$$(1,0) - (1,1) - (0,1)$$
$$(1,0) - (2,1) - (1,1) - (0,1)$$
$$(1,0) - (2,1) - (3,2) - (1,1) - (0,1)$$
$$(1,0) - (2,1) - (5,3) - (3,2) - (1,1) - (0,1)$$

We will compute $H(\Delta)$. The reader is asked to verify that it consists of those fractional exponents (i,j) that satisfy the following system of inequalities.

$$
\begin{array}{rl}
i \geq 1 & (1,0) \\
2i + j \geq 6 & (2,1) \\
5i + 3j \geq 16 & (5,3) \\
3i + 2j \geq 10 & (3,2) \\
i + j \geq 4 & (1,1) \\
j \geq 1 & (0,1)
\end{array}
$$

It is obvious that inequality (5,3) follows from (3,2) and (2,1) taken together. Similarly, (3,2) follows from (1,1) and (2,1). Thus, of the six inequalities, only the first two and the last two are needed to define $H(\Delta)$.

To find the vertices of $H(\Delta)$ it is necessary to solve some equations.

$$\left.\begin{array}{r} i = 1 \\ 2i + j = 6 \end{array}\right\} \text{ has solution } (1,4)$$

$$\left.\begin{array}{r} 2i + j = 6 \\ i + j = 4 \end{array}\right\} \text{ has solution } (2,2)$$

$$\left.\begin{array}{r} i + j = 4 \\ j = 1 \end{array}\right\} \text{ has solution } (3,1)$$

The reader is asked to check that the differential corresponding to each of the vertices has order zero. Also to be verified is that each exceptional place of the resolution is minimal for at least one of these differentials. Figure 9 refers to this example.

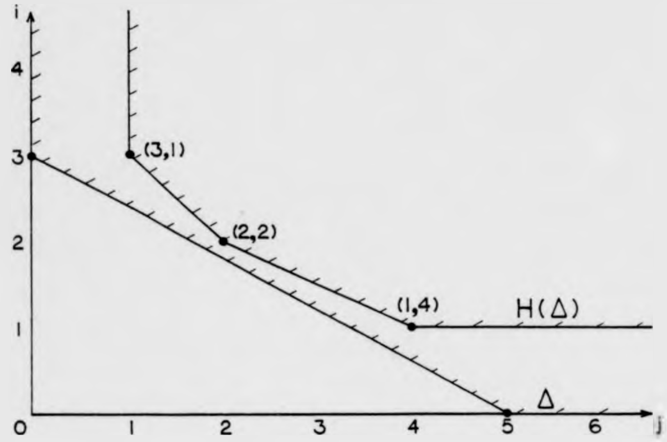

FIG 9

We will perform a similar collection of calculations for the plane curve $x^7 + y^{19} = 0$. The following sequence of monoidal transformations realizes the place (19,7) and thereby provides a resolution.

$(1,0) = (0,1)$

$(1,0) = (1,1) - (0,1)$

$(1,0) = (2,1) - (1,1) - (0,1)$

$(1,0) - (3,1) = (2,1) - (1,1) - (0,1)$

$(1,0) - (3,1) = (5,2) - (2,1) - (1,1) - (0,1)$

$(1,0) - (3,1) = (8,3) - (5,2) - \dots$

$(1,0) - (3,1) - (11,4) = (8,3) - (5,2) - \dots$

$(1,0) - (3,1) - (11,4) - (19,7) - (8,3) - (5,2) - \dots$

We now write down the system of inequalities that defines $H(\Delta)$.

$$
\begin{array}{rll}
i \geq 1 & \qquad\qquad & (1,0) \\
3i + j \geq 20 & & (3,1) \\
11i + 4j \geq 77 & & (11,4) \\
19i + 7j \geq 134 & & (19,7) \\
8i + 3j \geq 57 & & (8,3) \\
5i + 2j \geq 36 & & (5,2) \\
2i + j \geq 15 & & (2,1) \\
i + j \geq 8 & & (1,1) \\
j \geq 1 & & (0,1)
\end{array}
$$

The reader is asked to verify these redundancies in the system of inequalities.

(11,4) follows from (3,1) and (8,3)

(19,7) follows from (11,4) and (8,3)

(5,2) follows from (8,3) and (2,1)

(1,1) follows from (2,1) and (0,1)

To find the vertices of $H(\Delta)$ we must solve some equations.

$$\left.\begin{array}{r} i = 1 \\ 3i + j = 20 \end{array}\right\} \text{ has solution } (1,17)$$

$$\left.\begin{array}{r} 3i + j = 20 \\ 8i + 3j = 57 \end{array}\right\} \text{ has solution } (3,11)$$

$$\left.\begin{array}{r} 8i + 3j = 57 \\ 2i + i = 15 \end{array}\right\} \text{ has solution } (6,3)$$

$$\left.\begin{array}{r} 2i + j = 15 \\ j = 1 \end{array}\right\} \text{ has solution } (7,1)$$

As before, the reader is asked to check that the differential corresponding to each of the vertices listed above has order zero. Also to be verified is that each exceptional place of the resolution is minimal for at least one of these differentials.

These last two properties hold for any nondegenerate plane curve. The first property fails in higher dimension -- the singularity $rt = su$ provides the simplest counterexample. It is likely that every nondenerate surface has a toral resolution, all of whose exceptional places are minimal with respect to some differential of top degree. There is no evidence either for or against the truth of the statement in higher dimensions.

2. A nondegenerate surface

The surface $x^2 + y^3 + z^5 = 0$ is nondegenerate. It has a long and distinguished history. For various reasons it is known as E_8. Its Newton polyhedron has but one compact facet, which corresponds to the weight $(3 \times 5,\ 5 \times 2,\ 2 \times 3)$. From the inequality

$$3 \times 5 + 5 \times 2 + 2 \times 3 > 2 \times 3 \times 5$$

it follows that the point $(1,1,1)$ lies in the interior of Δ. Consequently, $H(\Delta)$ is defined by the equations

$$i \geq 1 \qquad j \geq 1 \qquad k \geq 1$$

The reader is asked to check that Figure 10 represents a resolution of E_8. Also to be checked is that the differential corresponding to $(1,1,1)$ has order zero along each of the exceptional places.

The fan comes from a sequence of monoidal transformations, with exceptional places D to R. The table gives the discrepancy of each place, and the center immediately antecedent to its realisation.

A	(1,0,0)	0	-
B	(0,1,0)	0	-
C	(0,0,1)	0	-
D	(1,1,1)	2	ABC
E	(2,1,1)	3	AD
F	(2,2,1)	4	BE
G	(3,2,1)	5	AF
H	(3,2,2)	6	DE
I	(4,3,2)	8	EF
J	(5,3,2)	9	EG
K	(5,4,2)	10	FG
L	(6,4,3)	12	EI
M	(7,5,3)	14	GI
N	(8,5,3)	15	GJ
O	(9,6,4)	18	IJ
P	(10,7,4)	20	GM
Q	(12,8,5)	24	JM
R	(15,10,6)	30	MN

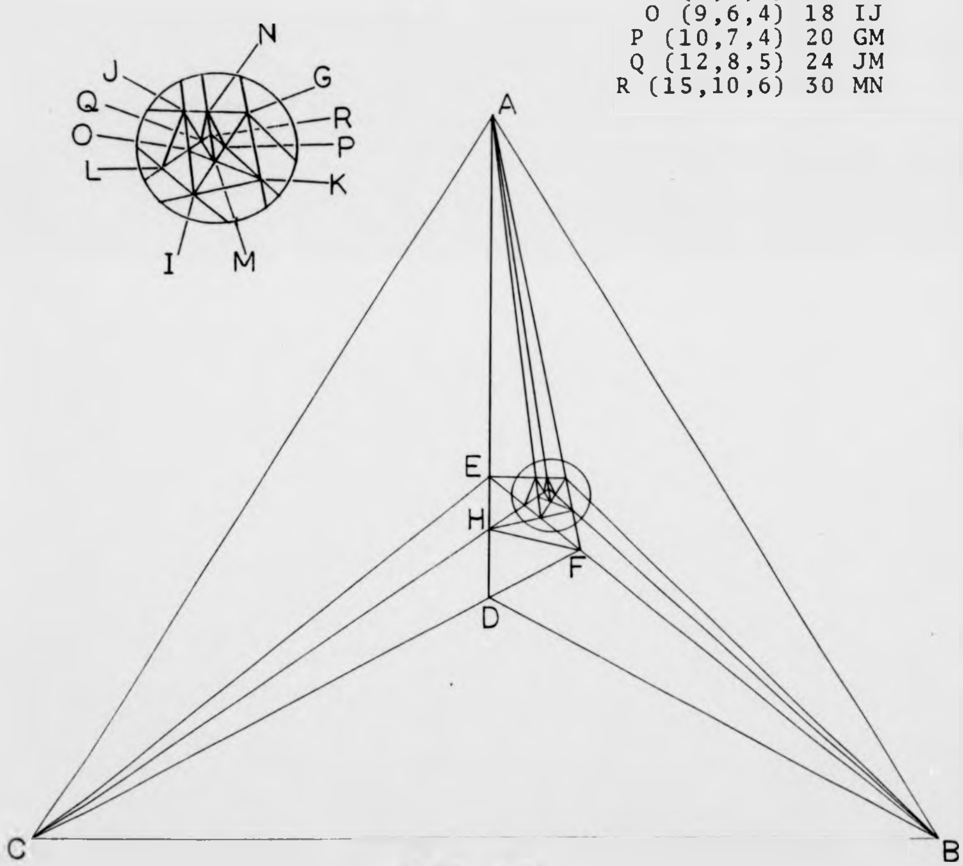

FIG 10

It is well known that plane curves admit an absolutely minimal embedded resolution. The next three statements are plausible, and constitute an extension of this result to a higher dimension.

(a) If $M' \to M$ provides a resolution of E_8 then each of the exceptional places of Figure 10 appears on M'.

(b) If $M' \to M$ is a smooth proper modification where exceptional places are precisely those of Figure 10 then $M' \to M$ is toral and provides a resolution of E_8.

(c) If $\tilde{M} \to M$ provides a resolution of E_8 then it factors $\tilde{M} \to M' \to M$ through one of the modifications $M' \to M$ of (a).

The techniques required to prove such results do not yet exist. A solution to the problem on page 77 would probably settle (a) and (b). The methods in this work will prove (a) if $M' \to M$ is assumed toral.

3. <u>A degenerate curve</u>

We will indicate how it is possible to resolve the curve X_2 defined by the equation

$$(x^2 + y^3)^2 + x^5 = 0$$

by a sequence of monoidal transformations, toral according to the general definition given in the Introduction. Notice that the term x^5 lies in the interior of the Newton polyhedron Δ of X_2.

There are many ways of seeing that X_2 is degenerate with respect to any normal crossing divisor on the plane. For example the resolution $M' \to M$ is obtained by repeatedly transforming at the singular point(s) of X_2 and its strict transforms. If the modification is toral then the exceptional locus E of $M' \to M$ has a fan Σ with at most two ends. (An end is a vertex connected by an edge to only one other vertex). The sequence of transformations we are about to describe consists of monoidal transformations at the singular locus of X_2 and its strict transforms, but the fan of its exceptional locus has three ends.

We will take for the divisor D the two co-ordinate axes together with the curve X_1 defined by the equation $x^2 + y^3 = 0$. As X_1 is nondegenerate with respect to the co-ordinate axes we can and will resolve it by monoidal transformations toral with respect to the co-ordinate axes.

(0,1) = (1,0)

(0,1) - (1,1) = (1,0)

(0,1) - (1,1) = (2,1) - (1,0)

(0,1) - (1,1) - (3,2) - (2,1) - (1,0)

The strict transform of X_1 passes through the component $D_{(3,2)}$ of the exceptional locus. The intersection lies on the affine variety corresponding to the edge (1,1) - (3,2). On this affine variety there are local parameters u and v, which are related to x and y by the equations

$$x = uv^3 \qquad y = uv^2$$
$$u = x^{-2}y^3 \qquad v = xy^{-1}.$$

From these equations, the equations

$$\begin{aligned} x^2 + y^3 &= u^2v^6 + u^3v^6 \\ &= u^2v^6(1 + u) \end{aligned}$$

and

$$\begin{aligned} (x^2 + y^3)^2 + x^5 &= u^4v^{12}(1 + u) + u^5v^{15} \\ &= u^4v^{12}[(1 + u)^2 + uv^3] \end{aligned}$$

follow. Thus, the divisor $D_{(3,2)}$ and the strict transforms of X_1 and X_2 have local equations

$$v = 0 \qquad 1+u = 0 \qquad (1+u)^2 + uv^3 = 0$$

respectively. Defining new local parameters by $U = 1+u$ and $V = v$ these equations become

$$V = 0 \qquad U = 0 \qquad U^2 + UV^3 - V^3 = 0.$$

The reader is asked to check that the strict transform of X_2 is nondegenerate with respect to $D_{(3,2)}$ and st X_1. Thus, the resolution can be completed by making further toral transformations. This is left to the reader.

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