Endogenous Partial Insurance and Inequality*

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Abstract

In this paper, we propose a model of endogenous partial insurance and we investigate its implications for macroeconomic outcomes, such as wealth inequality, asset accumulation, interest rate, and consumption smoothing. To this end, we include participation costs to state-contingent asset markets into an otherwise standard Aiyagari (1994) model. We highlight the resulting non-monotonic relationship between wealth and insurance-market participation when insurance is costly. Poor households remain uninsured, middle-class households participate in the insurance market, while rich households decide to self-insure by only purchasing risk-free assets. After theoretically characterizing the endogenous partial equilibrium, we quantify its effect, emphasizing the roles of a participation channel and an interest rate channel.

JEL codes: E21, E44.

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1 Introduction

Recent papers have underscored important stylized facts about the heterogenous degree of risk-sharing and consumption smoothing across US households: using PSID data Guvenen (2007) documents that stockholders smooth less consumption than non-stockholders; similarly, using CEX data Gervais and Klein (2010) find that households with larger financial assets smooth consumption less than households with lower financial assets.[1] These facts are at odds with implications of a standard Aiyagari (1994) model, as the self-insurance channel is not able to capture this observed heterogenous degree of insurance. This caveat couples with other well-known issues of the conventional Aiyagari incomplete market model. First, this model fails to deliver a strong amplification from income to wealth inequality when characterized only by reasonably calibrated income shocks, as summarized in Quadrini and Rios-Rull (2014).[2] Second, standard incomplete market models imply a much larger amount of uninsurable lifetime income risk for individuals than what is estimated in the data, as documented by Guvenen and Smith (2014).

In this paper we first propose a tractable model that generates endogenous partial insurance from a generalization of the standard Aiyagari model and, then, we show that: (i) it amplifies the level of wealth inequality for a given income process with respect to the standard Aiyagari model; (ii) it is able to potentially generate heterogenous degree of consumption smoothing across the wealth distribution in line with the empirical findings of Guvenen (2007) and Gervais and Klein (2010), depending on the assumed asset structure available to agents; and (iii) it generates an aggregate level of insurance that is larger than in Aiyagari (1994) and is closer to the value estimated in Guvenen and Smith (2014).

Our first contribution is to propose a simple model of endogenous partial insurance. In our setting, markets that provide state contingent insurance do exist, but it is costly to access to them. More precisely, in an otherwise standard general equilibrium economy as in Aiyagari (1994), we introduce costs for participating in contingent asset markets.[3] Consequently, households face a trade-off between paying the participation cost and enjoying the gain of consumption smoothing. Conveniently, as two polar cases, our model nests a model with full participation, in which the participation cost is so low that all agents optimally decide to provide insurance to each other, and the standard incomplete market model as in Aiyagari (1994), in which the cost is so high that all agents prefer to accumulate only risk-less assets as consumption buffer

[1] This result is robust to restricting the sample only to working age heads of the household persons, to excluding households living in rural areas, and to excluding self-employed households.

[2] Many authors have extended these models to improve the ability to generate greater wealth inequality. Among these approaches are the addition of special earning risks (Castaneda et al. (2003), Benhabib et al. (2015)), entrepreneurial risks (Quadrini, 2000; Cagetti and De Nardi, 2009; Angeletos, 2007; Buera, 2009)), bequest, human capital, and health risk (De Nardi (2004), Huggett (1996)), stochastic discounting (Krusell and Smith (1998)), and capital income risk (Benhabib et al. (2011)).

[3] The idea that consumption smoothing is costly underpins our approach: being active in financial markets involves monetary costs, broadly defined, such as fees and transactions costs charged by brokers and intermediaries, costs related to information acquisition, and non-monetary costs, such as the opportunity cost of time devoted to find the best portfolio allocation. See Section 6 for further discussion. See also Acemoglu and Zilibotti (1997) for the role of fixed cost on capital accumulation and growth.

and in that case our setting is equivalent to assuming that state-contingent assets do not exist. However, more generally, intermediate levels of participation costs deliver an *endogenous partial-insurance equilibrium*, in which only a fraction of the population decides to insure. We show that under very general conditions on the utility function the individual insurance decision is non-monotone across wealth: poor agents have too few resources to afford paying the participation cost; middle class agents participate in the insurance markets; and richer agents optimally decide only to self-insure as they have already a large quantity of wealth to use as a buffer. This endogenous decision, which produces what we define as *participation channel*, is able to generate consumption smoothing patterns consistent with the findings of Guvenen (2007) and Gervais and Klein (2010).

To provide intuition on the endogenous insurance decision and on the *participation channel*, we first investigate a simpler two-period two-states insurance model similar to the one in Kimball (1990b). We are able to characterize the optimal asset decision conditional on participating or not in the state contingent asset market. We are also able to separate the risk-free asset position in a component that is driven by a consumption smoothing motive (i.e. by an expected future income that is different from the current one) and by a precautionary saving motive (i.e. by different possible realizations of future income). Consider agents that have an instantaneous utility function that features decreasing absolute risk aversion (DARA). When access to the insurance market is costly, agents endogenously decide whether to participate in the state-contingent asset market depending on the level of their wealth. When wealth is large enough, self insuring using only risk-free asset provides adequate insurance against undesired shocks; hence, rich agents are better off by not participating in the insurance market because the cost of paying the fixed cost is larger than the additional benefit brought by state contingent insurance. For lower wealth levels, instead, paying the fixed cost to acquire state contingent assets is optimal, as the gain of insurance exceeds the cost. Importantly, our analysis demonstrates that the heterogeneity of insurance with respect to wealth is a quite general result because, as discussed in Kimball (1990a), commonly used parameterizations of the utility function, such as the constant relative risk aversion utility, display DARA.

We then incorporate the endogenous insurance decision into a standard neoclassical model with idiosyncratic shocks as in Aiyagari (1994). As in the simpler model, we assume that two types of assets are available in the economy: a set of state contingent assets, which can be purchased only by paying a fixed participation cost, and a risk-free asset. Hence, agents first decide whether they want to participate in the state contingent market, and, then, they decide their optimal portfolio. Our first set of results are theoretical. Using a similar approach as in Açıkgöz (2018), we first show that the endogenous partial insurance model can be conveniently written using a recursive formulation, and then we prove the existence of a recursive stationary competitive equilibrium. In this setting, however, there is an additional potential source of multiplicity of equilibria with respect to the standard Aiyagari (1994) model; in that model multiplicity can arise because agents' saving function may react in a non-monotonic way to

changes in the real interest rate, because of the opposite forces generated by the income and substitution effects at an individual level. In the partial insurance model, the multiplicity of equilibria can be also generated by the way endogenous participation feedbacks on asset prices and by the mere presence of additional insurance.[4]

Our second contribution is to quantify the effects of the *endogenous partial-insurance equilibrium* on aggregate inequality, asset prices, and degree of insurance. For this purpose, we consider an income process that follows a seven-state Markov process, obtained by discretizing an AR(1) process using Rouwenhorst (1995)'s method. The underlying process has a persistence equal of 0.958, consistent with Kaplan (2012) and Fève et al. (2017), and a standard deviation of the innovations that is set to match the Gini index of post-tax income in the US of 0.40. Then, we assume an asset structure such that even if agents participate in the state contingent asset market, they are not fully insured. Therefore, part of the risk is exogenously not insurable, while the other part is endogenously insurable. Under this assumption, no matter the level of costs, all agents still have some incentive of accumulating wealth to self insure, although at different degrees across the wealth distribution. This makes the distribution of wealth a smooth function of the level of participation costs and, in this way, we are able to compare the equilibrium outcomes of the model for a large set of costs, from very high to zero. In the paper we consider two types of asset structures: an asymmetric one, for which only some downward risk is insurable, and a symmetric one, for which both upward and downward insurance is available but only to a limited amount of agents; the comparison between these two structures highlights the contribution of different mechanisms of the model. We then compute the equilibrium implications of the model as a function of the participation cost.

When the cost reduces from an arbitrary large level, for which the economy is identical to the Aiyagari (1994) one, to zero we observe the following pattern: (i) not surprisingly participation in the insurance market increases; (ii) the interest rate increases, as higher participation in the insurance market leads to a lower overall demand for assets; (iii) inequality, measured by the wealth Gini index, initially largely increases with participation, and then slightly declines for smaller costs. There are two effects that rationalize this result. First, with endogenous partial insurance middle-class households acquire insurance and, therefore, do not have incentive to accumulate a lot of assets; on the contrary, richest households do not participate in the state contingent asset market and have then a motive to accumulate assets. This *participation channel* contributes to the thickening of the upper tail of the wealth distribution. Second, a general equilibrium effect, for which higher insurance participation leads to lower asset demand and higher interest rate, reinforces the skewness of the wealth distribution. We label this channel as the *interest rate channel*: a higher interest rate penalizes non-insured poor households, which mainly face upward risk and have incentive to dissave, while it benefits richer households as they can enjoy higher returns from their financial wealth.

In the final part of the paper we continue the investigation of the properties of the endoge-

[4]If full insurance were available, absent any participation cost, the equilibrium would be dependent on the assumed initial wealth distribution, as showed by Caselli and Ventura (2000).

nous partial insurance model by analysing its implications for consumption smoothing, wealth concentration, and marginal propensity to consume out of wealth. We focus particularly on how a model of endogenous partial insurance differs from the Aiyagari (1994) model, highlighting the roles of the *participation channel* and *interest rate channel*. It is of particular relevance to consider an alternative asset structure in which few agents can insurance against upward and downward risk. In this case the *interest rate channel* is almost completely muted, and the equilibrium pattern of consumption smoothing across the wealth distribution is therefore almost exclusively due to the *participation channel*; as already discussed, the main outcome is that poor agents do not smooth consumption much, middle class are the most insured agents, while the richest have a lower degree of consumption smoothing compared to the middle class, but, obviously, higher than the poor. Hence, we show that the *participation channel* in an endogenous partial insurance model is able to generate consumption smoothing patterns across the wealth distribution that are drastically different from a standard Aiyagari (1994) model and more in line with the empirical evidence discussed in Guvenen (2007) and Gervais and Klein (2010).

Related literature. In addition to the papers that we have already mentioned, our work expands on several bodies of the literature.

Among the empirical studies conducted on lack of insurance and consumption smoothing as Townsend (1994) and Mace (1991), our work bears similarity to that of Cochrane (1991), and, more recently, Grande and Ventura (2002), who study households' insurance against different types of risk. They show that households are well insured against certain types of risks, such as health problems, but not against other types of risks, such as unemployment (especially involuntary job loss) (see also Blundell et al., 2008).

Our work also relates to the literature linking models of incomplete insurance with empirical evidence as in Broer (2013), who compares the empirical predictions of different incomplete market models with the empirical joint distribution of consumption, income, and wealth; and Krueger and Perri (2005, 2006) or Kaplan and Violante (2010), who assess the degree of insurance beyond self-insurance. In our setting the participation cost modifies the link between income and consumption inequality, through the resulting non-monotone degree of insurance across wealth. Hence, trends in one of these variables are imperfectly transmitted to the other, consistently with the findings in Attanasio et al. (2012) and Aguiar and Bils (2015). The idea to include participation costs to access financial markets has been also investigated by Guvenen (2007) but in a contest with aggregate shocks and with focus on risk sharing among stockholders. On another note our paper is also connected to the paper on global imbalances by Mendoza et al. (2009), where imperfect insurance results from the imperfect verifiability of the income realization. In our case, imperfect insurance results from the endogenous decision not to participate in (potentially imperfect) insurance markets.

Finally, our work links to the literature in finance on limited participation as in Luttmer (1999), Vissing-Jorgensen (2002a) and more recently in Paiella (2007a), Guvenen (2009) or

Attanasio and Paiella (2011a) among others. In these models, the access to the stock market is costly or open only in a subset of periods. Also, even when economists focus on limited asset trading,[5] they generally do not consider frictions related to asset market participation in their models. In connection with this literature, Parker and Vissing-Jorgensen (2009) propose an alternative story so as to why consumption may be more volatile for richer households: these households are holding stocks and stockholders bear more aggregate risks.

The rest of the paper is organized as follows: in Section 2 we present a simple insurance model in order to provide conditions and intuitions for households' insurance decision. In Section 3 we describe the general economic environment. Section 4 provides the theoretical results. Section 5 presents the quantitative findings about participation, wealth inequality, wealth concentration, and consumption smoothing. Section 6 discusses a set of further extensions. Finally, Section 7 provides concluding remarks.

2 A Simple Insurance Model

In this section we present a simple two-period two-state model of endogenous partial insurance, which highlights how participation in the insurance market relates to wealth.

The economy lasts two periods, $t = 0, 1$. The household starts period $t = 0$ with some endowed level of wealth W, or cash-in-hands. In period $t = 1$ the household receives some income that consists of a deterministic component and a stochastic component. The deterministic component is denoted with y. With probability p, the agent has a negative income shock, $-L$, where $L \geq 0$; with probability $1-p$, the household receives a positive income shock, $pL/(1-p)$. We assume that $y < W$ so that the expected level of income is lower than the period-0 cash-in-hands. Also notice, that in expectation the income received is exactly equal to y. We denote with the indicator variable $\mathbb{1}_L$ the realization of the state of nature, so that $\mathbb{1}_L = 1$ if the negative income shocks realize, and $\mathbb{1}_L = 0$ otherwise. The household maximizes the following expected utility function: $\mathbb{E}_0 \left(u(c_0) + u(c_1)\right)$, where $\mathbb{E}_0$ denotes the expectation operator conditional on information available at time 0, and $u(\cdot)$ is the instantaneous utility function. For simplicity, we assume that there is no discounting.

We introduce an endogenous decision of participating in the insurance market in the model. There are two assets available to the agent. At time $t = 0$, the agent can acquire b units of risk-free assets at unit price q_b; this asset repays a unit of consumption at time $t = 1$, regardless the realization of the shock. Also, the agent can acquire a units of a state-contingent asset at unit price q_a; this asset repays a unit of consumption good at time $t = 1$ only if the loss in wealth occurs, i.e. if $\mathbb{1}_L = 1$. Importantly, in order to have access to this state contingent asset, the agent needs to pay a fixed cost κ. The household is not necessarily willing to pay the fixed cost and, hence, we define as $\delta(W, \kappa)$ the choice variable that denotes contingent asset market

[5] This can happen because of lack of commitment (Thomas and Worrall, 1988; Kocherlakota, 1996), trading technologies (Chien et al., 2011) or because of *ad hoc* assumptions as in the incomplete market literature.

participation, given a level of wealth and a participation cost: if the household pays the cost and purchases contingent assets, $\delta(W, \kappa)$ equals 1. Otherwise, it equals 0. Since we are interested in the decision of an individual agent that takes prices as given, we now assume that asset prices are actuarially fair, that is $q_a = p$ and $q_b = 1$.

Let us first derive the problem of an agent that participates in the state contingent asset market, that is when $\delta(W, \kappa) = 1$. In this case, the budget constraints are:

$$c_0 + pa + b + \kappa = W,$$
$$c_1 = y + \mathbb{1}_L(a - L) + (1 - \mathbb{1}_L)\frac{pL}{1-p} + b.$$

FOCs for a and b give:

$$u'(W - pa - b - \kappa) = u'(y + a - L + b), \tag{1}$$
$$u'(W - pa - b - \kappa) = pu'(y + a - L + b) + (1 - p)u'\left(y + b + \frac{pL}{1-p}\right). \tag{2}$$

Combining (1) and (2) we obtain that the optimal contingent asset position in case of participation, denoted with a superscript ^{P}, is: $a^P = \frac{L}{1-p}$, and $b^P = \frac{W-y}{2} - \frac{pL}{1-p} - \frac{\kappa}{2}$.

The portfolio decision of the insured household reveals interesting insights. First, since the state contingent asset insures against downward risk, the agent optimally chooses a positive amount of state contingent assets. Second, the portfolio decision for the risk-free asset incorporates two motives: the first motive is included in the term $\frac{W-y}{2}$ and corresponds to the consumption smoothing incentive; when expected future income is lower than current cash in hands, $y < W$, the agent would like to save to transfer intertemporally consumption. Importantly, this saving motive is present even when there is no uncertainty on future income, that is when $L = 0$. The second motive is included in the term $-\frac{pL}{1-p} - \frac{\kappa}{2}$, and corresponds to the insurance motive. Since the agent has a long position in the state-contingent asset, she has to reduce her risk-free asset position in order to achieve perfect insurance and to smooth the cost of insurance across the two periods. Notice that this insurance motive is present even when expected future income is equal to current cash in hands, that is when $y = W$.

Under the optimal portfolio choice, the agent is able to equate consumption across time and states, and $c_0 = c_1 = \frac{W+y}{2} - \frac{\kappa}{2}$.

The indirect utility of participating is then:

$$V^P(W, \kappa) = 2u\left(\frac{W+y}{2} - \frac{\kappa}{2}\right).$$

Let us now derive the problem of an agent that does not participate in the state contingent

asset market, that is when $\delta(W, \kappa) = 0$. In this case, the budget constraints are:

$$c_0 + b = W,$$
$$c_1 = y + \mathbb{1}_L(-L) + (1 - \mathbb{1}_L)\frac{pL}{1-p} + b.$$

FOC for b gives:

$$u'(W - b) = pu'(y - L + b) + (1 - p)u'\left(y + \frac{pL}{1-p} + b\right) \tag{3}$$

This expression implicitly determines the optimal level of the risk-free asset holding in case of non-participation in the state contingent asset market, $b^N(W)$. We hereby assume, as in Kimball (1990a), that the agent is at such an interior solution. This condition implies a lower bound for the level of wealth, which we denote as $\underline{W}$. We label as *feasible* the values of wealth such that $W \geq \underline{W}$.

We can characterize optimal asset holdings with the following result.

Proposition 1. *Let $u(x)$ be a three-times continuously differentiable utility function, such that $u'(x) > 0$, $u''(x) < 0$, $u'''(x) > 0$ and satisfies the Inada conditions: $\lim_{x\to\infty} u'(x) = 0$, and $\lim_{x\to 0} u'(x) = \infty$. Then, for any feasible level of wealth, i.e. $\forall W \geq \underline{W}$:*

1. *$b^N(W) \geq \frac{W-y}{2}$, with $b^N(W) = \frac{W-y}{2} \iff L = 0$. Also, $\lim_{W\to\infty} b^N(W) = \infty$.*
2. *$\frac{\partial b^N(W)}{\partial W} > 0$ and if $L = 0 \implies \frac{\partial b^N(W)}{\partial W} = \frac{1}{2}$. Also, $\lim_{W\to\infty} \frac{\partial b^N(W)}{\partial W} = \frac{1}{2}$. Hence, $0 < \frac{\partial b^N(W)}{\partial W} \leq \frac{1}{2}$.*
3. *In addition:*
   (a) *$\frac{\partial b^N(W)}{\partial W} = \frac{1}{2}$, $\forall W$, if the utility $u(\cdot)$ displays Constant Absolute Relative Risk Aversion (CARA), i.e. $u''(W) = -zu'(W)$, with z contant.*
   (b) *Otherwise $\frac{\partial b^N(W)}{\partial W} < \frac{1}{2}$, $\forall W$ finite.*

Proof. See Appendix A.2 for the proof. □

Proposition 1 displays the optimal behavior for savings when state contingent asset markets are not available. The first result states that saving do, in general, play a dual role: they guarantee consumption smoothing, so that they are at least weakly greater than $\frac{W-y}{2}$, and they contribute to insurance through a precautionary saving motive. This motive explains the gap between $b^N(W)$ and $\frac{W-y}{2}$. In addition, the proposition highlights two additional important results. First, precautionary saving incentives either move proportionally with wealth or they weaken with wealth. The latter case occurs when the utility function displays Decreasing Absolute Risk Aversion.

We can now compute the indirect utility of non-participating, which is:

$$V^N(W) = u\left(W - b^N(W)\right) + pu(y - L + b^N(W)) + (1 - p)u\left(y + b^N(W) + \frac{pL}{1-p}\right),$$

with $b^N(W)$ implicitly defined by condition (3).

2.1 Gain of Insurance

Let $\mathbb{P}(\kappa)$ be the set of *feasible* wealth levels for which participation in the insurance market is optimal for a given participation cost κ. Formally:

Definition 1. (Participation Set). For a given participation cost κ, for any wealth level in $\mathbb{P}(\kappa)$ insurance market participation is optimal, that is:

$$\mathbb{P}(\kappa) = \left\{W \in [\underline{W}, \infty) : V^P(W, \kappa) > V^N(W)\right\}.$$

Let define the gain of insurance as $G(W, \kappa) = \frac{1}{2}\left(V^P(W, \kappa) - V^N(W)\right)$. It can be rewritten as:

$$G(W, \kappa) = u\left(\frac{W+Y}{2} - \frac{\kappa}{2}\right) - \frac{1}{2}u(W - b^N) - \frac{p}{2}u\left(y + b^N - L\right) - \frac{1-p}{2}u\left(y + b^N + \frac{pL}{1-p}\right). \tag{4}$$

We now restrict our analysis to the case in which $L > 0$, which means that the agent is subject to some income risk. The first set of results concerns the frictionless economy with no costs.

Proposition 2. *(Insurance Incentives without cost) Let $u(x)$ be a three-times continuously differentiable utility function, such that $u'(x) > 0$, $u''(x) < 0$, $u'''(x) > 0$, and satisfies the Inada conditions: $\lim_{x\to\infty} u'(x) = 0$, and $\lim_{x\to 0} u'(x) = \infty$. Then, for any feasible level of wealth, i.e. $\forall W \geq \underline{W}$:*

1. $G(W, 0) > 0$;
2. $\frac{\partial G(W,0)}{\partial W} < 0$.
3. $\lim_{W\to\infty} G(W, 0) = 0$.

Proof. See Appendix A.2 for the proof. □

Proposition 2 shows that, absent any cost, $\kappa = 0$, the agent is always better off by participating in the state contingent asset market, since in this case she can obtain full insurance. In addition, the gain of insurance declines with wealth. The intuition stems from the fact that, when wealth increases, the self-insurance mechanism through precautionary saving using only risk-free assets provides better and better insurance. This result applies to any utility function that is increasing, concave, and whose marginal utility is convex, i.e. $u'''(\cdot) > 0$. This condition means that the utility function displays *prudence*, As discussed in Kimball (1990a), *prudence* measures the strength of the precautionary saving motive, which induces individuals to prepare and forearm themselves against uncertainty they cannot avoid- in contrast to risk aversion, which is how much agents dislike uncertainty and want to avoid it.

We now consider the economy with participation costs.

Proposition 3. *(Insurance Incentive with cost) Let $u(x)$ be a three-times continuously differentiable utility function, such that $u'(x) > 0$, $u''(x) < 0$, $u'''(x) > 0$, satisfies the Inada conditions: $\lim_{x\to\infty} u'(x) = 0$, and $\lim_{x\to 0} u'(x) = \infty$, and features Decreasing Absolute Risk Aversion. Then, for any feasible level of wealth, i.e. $\forall W > \underline{W}$:*

1. *(Existence of Thresholds). Let $\hat{\kappa}$ be the solution of $G(\underline{W}, \hat{\kappa}) = 0$. Then, $\forall \kappa < \hat{\kappa}$, $\exists!$ $\overline{W}(\kappa) > \underline{W}$: $W \in \mathbb{P}(\kappa) \iff \underline{W} < W < \overline{W}(\kappa)$.*

2. *(Comparative static of participation set)*

   - *Participation set coincides with all feasible wealth levels when $\kappa = 0$, that is:*

   $$\mathbb{P}(0) = \{W : W > \underline{W}\}.$$

   - *Participation set is shrinking in participation cost, that is for all $\kappa^1 < \kappa^2$, if $W \in \mathbb{P}(\kappa_2)$ then $W \in \mathbb{P}(\kappa_1)$; hence, $\mathbb{P}(\kappa_2) \subset \mathbb{P}(\kappa_1)$.*
   - *Participation set is empty for any participation cost greater than $\hat{\kappa}$, that is: $\forall \kappa \geq \hat{\kappa}$, $\mathbb{P}(\kappa) = \emptyset$.*

Proof. See Appendix A.3 for the proof. □

This proposition explains a crucial characteristic of the endogenous partial insurance model. When accessing to the insurance market is costly, the agent endogenously decides whether to participate in that state-contingent asset market depending on the level of its wealth. When the wealth level is large enough, $W > \overline{W}(\kappa)$, the agent is better off by not-participating in the insurance market since the cost of paying the fixed cost is larger than the expected benefit of reducing the loss in case of occurrence of the negative shock. For those wealth levels, in fact, $G(W, \kappa) < 0$. This feature of the model with participation cost is displayed in Figure 1, which plots the gain of insurance for an economy with no participation costs (blue solid line) and with a positive participation cost (red dashed line). The shaded area displays the participation set.

In addition, Proposition 1 states that when the cost tends to zero, the participation set corresponds to the entire feasible wealth domain. On the contrary, the participation region disappears when the cost is larger than a certain threshold $\hat{\kappa}$. In this case entering in the insurance market is either infeasible or not beneficial. The necessary condition for the existence of the threshold wealth level is that the utility function features Decreasing Absolute Risk Aversion. Under this assumption, at lower levels of wealth self-insurance does not provide enough insurance and in this case the benefit of full insurance is worth paying the participation cost. On the contrary, for larger levels of wealth, self-insurance is a good enough insurance mechanism and the extra gains for participating in the state-contingent asset markets are not worth paying the participation cost.

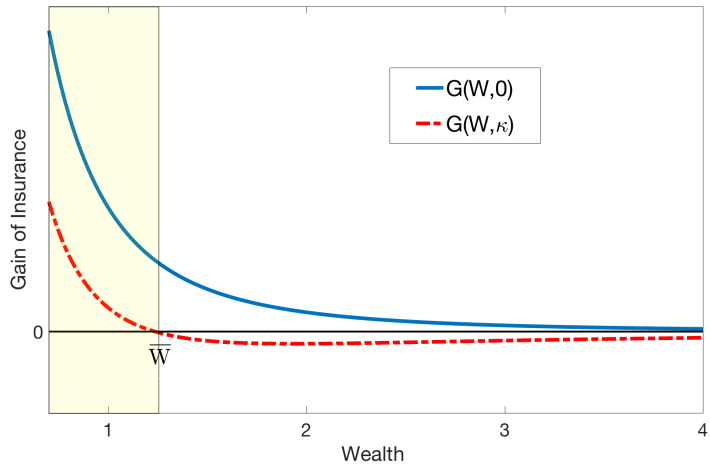


Figure 1 – Gain of Insurance

Note: This graph plots the gain of insurance for an economy with no participation costs (blue solid line) and with a positive participation cost (red dashed line). $\overline{W}$ is the threshold level of wealth for which any agent with wealth lower than that value pays the cost and participates in the contingent asset market and any agent with wealth higher than that value only acquires risk free assets. The shaded area denotes this insurance participation region. We assume a CRRA utility function with risk aversion parameter equal to 2, a loss L equal to 0.3 associated to a probability, p, equal to 0.25, and a participation cost equal to $\kappa = 0.05$.

3 A Model of Endogenous Partial Insurance

In this section we describe the general economic environment. We consider an infinite horizon production economy populated by a continuum of mass 1 of *ex ante* homogenous households. This model follows closely Aiyagari (1994) except for two dimensions: we introduce securities contingent to idiosyncratic states and we simultaneously introduce fixed participation costs for each contingent market. Time is discrete and indexed by $t \in \{0, 1, ...\}$.

Uncertainty and preferences. Each household chooses consumption so as to maximize the following utility: $U = \mathbb{E}_t \sum_t \sum_{y^t} \beta^t u\left(c(y^t)\right)$, where $\mathbb{E}_t$ denotes the expectation operator conditional on information available at time t, $\beta \in (0, 1)$ is the discount factor, y^t denotes the history of labor income exogenous realizations, $c(y^t)$ denotes consumption at date t and u is a strictly increasing and concave function. We assume that u is three times continuously differentiable.

Households inelastically provide labor. At every period they receive a stochastic labor endowment, y_t. Since there is no aggregate uncertainty, this assumption is equivalent to considering that households receive a stochastic good-endowment $\tilde{y}_t = w_t y_t$, where w_t is the wage rate.

We assume that y_t follows a Markov process, which takes values in $Y = \{y_1, ...y_N\}$ and that $\pi(y_j|y_k)$ is the associated transition probability from state k to state j. N, thus, denote the number of states of the income process. We denote by y^t the history of the realizations of the shock, $y^t = \{y_0, y_1, ..., y_t\}$, and by $\Pi(y_k)$ the fraction of households in state k. Total labor endowment is constant and it is the combination of labor provided by households with different

labor shocks, $y_t = y_k$, for $k = 1, .., N$, i.e.:

$$\bar{L} = \sum_{y_t \in Y} \Pi(y_t) y_t.$$

Asset structure. To smooth consumption, households may trade a set of different assets.

First, households can purchase non-contingent bonds. Each of these bonds yields, unconditionally, one unit of goods next period. Let $B(y^t)$ denote the household's position in the risk-free assets and by q_t^f its price. Besides, as in Aiyagari (1994), we impose that this position is bounded below: $B(y^t) \geq -\overline{B}$ where $\overline{B} \geq 0$ is finite.[6]

Second, households can trade contingent assets. Let $\mathcal{A} \subset Y \times Y$ denote the combinations of current and next period income realizations for which contingent assets are available. For each $y_t \in Y$, this also defines a set $\mathcal{A}(y_t)$ of next period states for which households in state y_t can buy contingent assets; the assets available in a given state y_t pay off one unit of consumption good next period only contingently to a specific realization of y_{t+1}. Therefore, the dimension of $\mathcal{A}(y_t)$ indicates also the number of state contingent assets available to an agent with current income y_t. We can then index state contingent assets by $(y_{t+1}|y_t)$. Let $q(y_{t+1}|y_t)$ denote the vector price of such assets, which is of the same dimension of $\mathcal{A}(y_t)$, and let $a(y_{t+1}|y^t)$ denote the vector of asset holdings that insure against next period states, for a household with history of shocks y^t. Note that in our notation contingent asset holding depends on the current state through the history of shock y^t.

As for the risk-free asset, we assume the existence of ad-hoc constraints for the state contingents assets, i.e. $a(y_{t+1}|y^t) \geq \bar{a}$, $\forall y_{t+1} \in \mathcal{A}(y_t)$ and $\forall y_t$, to rule out unbounded positions. Notice that it is sufficient to define $\bar{a}$ in the space $\mathcal{A}$ and that this generic formulation accounts for the case in which constraints are different from asset to asset or in which they coincide with natural limits. At this point there is no need to explicit what these constraints are.[7]

The novelty we introduce in this paper is that purchasing those assets requires paying a fixed fee, κ. Hence, in order to hold any portfolio $\{a(y_{t+1}|y^t)\}_{\forall y_{t+1} \in \mathcal{A}(y_t)}$ of contingent assets, the household has to pay $\sum_{y_{t+1} \in \mathcal{A}(y_t)} q(y_{t+1}|y_t)^\intercal a(y_{t+1}|y^t) + \kappa$, where the superscript $^\intercal$ denotes the transpose operator. Here, for simplicity, we assume that if the agent pays the participation cost she can purchase or sell the preferred quantity of any state contingent asset. One can interpret the cost κ as a generic function, which can incorporates different scenarios. For example, κ could be a constant, could be a function of current income, with the interpretation that it denotes the opportunity cost of time, it could be a function of the state that asset insures, etc...[8] In

[6] We do not provide further foundations for that constraint. It can be exogenous debt limits as in Bewley (1980), natural debt limits as in Aiyagari (1994) or endogenous borrowing constraints as in Zhang (1997) or Abraham and Carceles-Poveda (2010) for such foundations.

[7] As an example, strict short sale constraints on state contingent assets could be rationalized by the existence of competitive contracts between profit-maximizing financial intermediaries and agents that face idiosyncratic income uncertainty, as in Krueger and Uhlig (2006). Finally, notice that the constraints for both the state contingent assets and risk free asset can assume any general form; for example they can be state dependent. For simplicity of notation we do not impose a specific form when presenting the model and we keep their respective notation as $\bar{a}$ and $\bar{B}$, respectively.

[8] An alternative structure of cost would be to pay a fixed cost for purchasing each contingent asset. In this

this section we do not need to specify the exact nature of the function. Finally, one can assume that κ is a pure waste or that it represents pecuniary transaction costs charged by unmodeled financial intermediaries that transform savings into productive capital.

The presence of the fixed cost implies that in each period the household needs to take a discrete decision about whether to participate in the contingent asset market. We denote by $\delta(y^t) \in \{0, 1\}$ the corresponding decision variable, with the following meaning: when $\delta(y^t) = 1$, a household with history y^t decides to enter in the state-contingent asset market and when $\delta(y^t) = 0$, it does not.

In the end, the proceeds of both contingent and risk-less assets are invested in physical capital, whose returns are used to honour assets' payments.

Remark. In the case in which the dimensions of $\mathcal{A}$ are $M \times N$, with $M \geq N - 1$, the set of state contingent assets together with the risk free asset spans the whole space $\mathcal{R}^N$, since in each of the N current state, $M \geq N - 1$ linearly independent state contingent assets are available in addition to the non-contingent asset. Notice that even in this case markets might not be still complete if exogenous constraints on assets are stringent.

Given our setting, a household with a history of shock y^t and a current shock realization y_t faces the following sequence of budget constraints:

$$c(y^t) + q_t^f B(y^t) + \delta(y^t) \left(\sum_{y_{t+1} \in \mathcal{A}(y_t)} q(y_{t+1}|y_t)^\intercal a(y_{t+1}|y^t) + \kappa \right) = B(y^{t-1}) + a(y_t|y^{t-1}) + w_t y_t.$$

Recall that in case of non-participation, $\delta(y^t) = 0$, the household is excluded from the contingent-asset market, and, therefore, in that case $a(y_{t+1}|y^t) = 0$ for all $y_{t+1} \in \mathcal{A}(y_t)$.

Production. As in Aiyagari (1994), we include production in our economy, creating an endogenous net demand for capital. A single representative firm produces using a Cobb-Douglas technology:

$$Y_t = K_t^\alpha L_t^{1-\alpha}, \tag{5}$$

where capital, K_t, and total labor, L_t, are rent from households. Production also implies depreciation of capital at a rate χ. First order conditions for capital and labor are:

$$\alpha \left(\frac{K_t}{L_t} \right)^{\alpha-1} = r_t + \chi, \text{ and } (1-\alpha) \left(\frac{K_t}{L_t} \right)^{\alpha} = w_t,$$

and they determine capital demand, K^d and labor demand, L^d.

Market clearing condition. The asset market-clearing condition pins down aggregate capital, K_{t+1}; if the cost κ is assumed to represent transaction cost to financial intermediaries that

case, households' decides in which state-contingent asset market to enter and, therefore, the participation decision is a set of binary variables.

convert savings and those costs into productive capital, the condition reads:

$$K_{t+1} = \sum_{y^t} \sum_{y_{t+1} \in \mathcal{A}(y_t)} \left[q(y_{t+1}|y_t)^\intercal a(y_{t+1}|y^t)\delta(y^t) + \delta(y^t)\kappa + q_t^f B(y^t) \right],$$

and the goods market-clearing condition pins down aggregate consumption, C_t, as:

$$C_t = \sum_{y^t} c(y^t) = Y_t - K_{t+1} + (1-\chi)K_t.$$

As agents are *ex ante* homogenous, the sum over all histories y^t amounts to sum over all the individuals.[9]

We now move to the definition and characterization of an equilibrium in this economy.

4 Endogenous partial insurance recursive equilibrium

In this section, we define a stationary competitive equilibrium for the endogenous partial insurance model and we show that such an equilibrium exists, no matter the level of cost κ. To this purpose, we first show that we can use standard recursive formulations and we then focus on the definition of a recursive stationary competitive equilibrium. Finally, we also provide some discussion of the potential multiplicity of equilibria and the predictive power of the model.

4.1 Recursive household problems

Let us start by investigating the household consumption-saving decisions in the presence of the discrete decision to participate in the state contingent asset market. We first lay out the sequential formulation of the problem, which naturally follows from the description of the economy in Section 3. In this setting, the problem faced by households is complex: it integrates a double maximization to decide about participation in the contingent asset market and about asset purchases. Formally, this problem can be written as follows:

[9] If instead κ is pure waste the conditions becomes:

$$K_{t+1} = \sum_{y^t} \sum_{y_{t+1} \in \mathcal{A}(y_t)} \left[q(y_{t+1}|y_t)^\intercal a(y_{t+1}|y^t)\delta(y^t) + q_t^f B(y^t) \right],$$

and

$$C_t + \sum_{y^t} \delta(y^t)\kappa = \sum_{y^t} \left[c(y^t) + \delta(y^t)\kappa \right] = Y_t - K_{t+1} + (1-\chi)K_t.$$

Problem 1 (Sequential formulation)**.**

$$\max_{\{\delta(y^t)\}} \quad \max_{\{c(y^t),B(y^t),\{a(y_{t+1}|y^t)\}_{y_{t+1}\in\mathcal{A}(y_t)}\}} \sum_t \sum_{y^t} \beta^t \pi(y^t) u(c(y^t))$$

$$\text{s.t. } c(y^t) + q_t^f B(y^t) + \delta(y^t)\left(\sum_{y_{t+1}\in\mathcal{A}(y_t)} q(y_{t+1}|y_t)^\intercal a(y_{t+1}|y^t) + \kappa\right) = w_t y_t + B(y^{t-1}) + a(y_t|y^{t-1}),$$

$$B(y^t) \geq -\overline{B}, \text{ and} \qquad a(y_{t+1}|y^t) \geq \bar{a} \text{ for all } y_{t+1} \in \mathcal{A}(y_t),$$

$$a(y_{t+1}|y^t) = 0 \text{ for all } y_{t+1} \in \mathcal{A}(y_t), \text{ if } \delta(y^t) = 0.$$

As this problem is to difficult to deal with directly, we want to write it in a recursive form, as in the standard in the literature (see Stokey et al., 1989, among many others). As it will become clear later, we are interested in a stationary recursive equilibrium and therefore, in the following, we threat prices as constant and we omit their dependence to the distributions for convenience of notation. The recursive formulation of the problem is the following:

Problem 2. Given $\{w, q, q^f\}$,

$$V(B, \{a(y)\}, y) = \max_{\delta\in\{0,1\}} \left\{V^{NP}(B, \{a(y)\}, y), V^P(B, \{a(y)\}, y)\right\}, \tag{6}$$

with

$$V^{NP}(B, \{a(y)\}, y) = \max_{B'} \left\{u(c) + \beta \sum_{y'} \pi(y'|y) V(B', \{a'(y')\}, y')\right\}$$
$$\text{s.t. } c + q^f B' \leq wy + B + a(y),$$
$$B' \geq -\overline{B};\ a'(y') = 0 \quad \forall y' \in \mathcal{A}(y).$$

$$V^P(B, \{a(y)\}, y) = \max_{\{a'(y')\}_{y'\in\mathcal{A}(y)}, B'} \left\{u(c) + \beta \sum_{y'} \pi(y'|y) V(B', \{a'(y')\}, y')\right\}$$
$$\text{s.t. } c + \left(\sum_{y'\in\mathcal{A}(y)} q(y'|y)^\intercal a'(y') + \kappa\right) + q^f B' \leq wy + B + a(y),$$
$$B' \geq -\overline{B};\ a'(y') \geq \bar{a} \quad \forall y' \in \mathcal{A}(y).$$

With this notation we emphasize that agents decide on a portfolio of Arrow securities, denoted with $\{a'(y')\}$. Notice that for convenience of notation we have omitted the dependence of contingent asset holding, $a'(y')$, on current state y, since it is clearly a state variable of the problem. Also, only the Arrow security associated with the realized state matters, and, therefore, the state variable for the current value function is $a(y)$. We denote the solution to

this problem by $\{\delta, \{a'(y')\}, B'\} = h(B, \{a(y)\}, y)$.

We make the following two assumptions on the utility function.

Assumption 1. *The utility function $u(c)$ satisfies:* $\lim_{c\to 0} u'(c) = -\infty$ *and* $\lim_{c\to\infty} u'(c) = 0$

Assumption 2. *The utility function $u(c)$ satisfies:* $\liminf_{c\to\infty} -\frac{u''(c)}{u'(c)} = 0$.

Next, we make the following assumption on prices, as in Açıkgöz (2018).

Assumption 3. *The bond price q^f and the wage rate w satisfy: $w > 0$; q^f is finite; and $\beta < q^f$.*

We first show that under these assumptions, the state space is compact and the utility is bounded.

Proposition 4. *Under Assumptions 1, 2, and 3, the state space for the household's problem can be chosen to be compact, i.e. there is exist a finite $\bar{\bar{B}} > 0$ and a finite vector $\bar{\bar{a}} > 0$, such that the optimal choice $B'(B, a, y) < \bar{\bar{B}}$ and $a'(B, a, y) < \bar{\bar{a}}$, for all $y \in Y$, for all $B \geq \bar{B}$, and for all $a \geq \bar{a}$.*

Proof. See the proof in Appendix A.4. □

Compactness of the state space implies that the value function $V : [\bar{B}, \bar{\bar{B}}] \times [\bar{a}, \bar{\bar{a}}] \times Y$ is bounded below and above. We can then apply the standard fixed point argument to the Bellman operator defined over continuous and bounded function. Proposition 11 in Appendix A.5 proves that the value function in equation (6) satisfies the standard properties, which will be used to prove the following result.

Proposition 5. *For given $\{w, q, q^f\}$, there exists a unique value function $V : S \to \mathcal{R}$ and unique policy functions $\delta : S \to \{0, 1\}$, $B' : S \to \mathcal{R}$ and, for all y and all $y' \in \mathcal{A}(y)$, $a'(y') : S \to \mathcal{R}$, that solve Problem 2. This solution coincides with the solution to Problem 1.*

Proof. See the proof in Appendix A.5. □

Insurance decision A novelty of our paper is the presence of the insurance decision by households. They choose δ by comparing the value of participating, V^P, to the value of not participating, V^{NP}. When participating, agents purchase both contingent and non-contingent assets but they also have to pay the fixed cost κ. In the end, their portfolio costs in a given state $y \in Y$:

$$\left(\sum_{y' \in \mathcal{A}(y)} q(y'|y)^\intercal a'(y') + \kappa \right) + q^f B'.$$

In exchange, agents have then a portfolio with state-contingent payoffs that, in our model, provides them with more insurance. In contrast, when not participating, agents only purchase non-contingent assets so that the cost of their portfolio is only $q^f B'$ and its payoffs are non-contingent.

Notice that this choice parallels the one we have presented in Section 2, Proposition 3: agents decide to participate by trading off the gains of insurance with the cost of participation.

4.2 Equilibrium definition

In this subsection, we introduce the definition of a stationary recursive equilibrium. To this purpose, let us first note that agents are indexed by $\{B, \{a\}, y\}$, describing their asset positions as well as their labor endowment. Let define $S = Y \times A$, where A is the set of households' asset positions. As this was clarified with Proposition 4, the sets for asset positions can be taken to be compact. As it is standard in the literature, we can construct the aggregate law of motion.

Define by $\mathcal{P}(Y)$ the power set of Y, and by $\mathcal{B}(A)$ the σ-algebra of A. Let $\mathcal{B}(S) = \mathcal{P}(Y) \times \mathcal{B}(A)$. Finally, define by $\mathcal{M}$ the set of all probability measures on the measurable space $M = (S, \mathcal{B}(S))$. Let define the transition function $Q : S \times \mathcal{B}(S) \to [0, 1]$ by:

$$Q\left((B, \{a\}, y), (\mathcal{A}, \mathcal{P}(Y))\right) = \sum_{y' \in \mathcal{P}(Y)} \begin{cases} \pi(y'|y) & \text{if } (B'(B, \{a\}, y), \{a'(B, \{a\}, y)\}) \in \mathcal{A} \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

for all $(B, \{a\}, y) \in S$, and all $(\mathcal{A}, \mathcal{P}(Y)) \in \mathcal{B}(S)$. Therefore $Q\left((B, \{a\}, y), (\mathcal{A}, \mathcal{P}(Y))\right)$ is the probability that an agent with current assets $(B, \{a\})$ and current income y ends up with assets $(B', \{a'\})$ in $\mathcal{A}$ tomorrow and income $y' \in \mathcal{P}(Y)$ tomorrow.

We can now define a stationary recursive competitive equilibrium.

Definition 2. A *stationary recursive competitive equilibrium* consists of a set of prices (w, q^f, q), value function $V : S \to \mathcal{R}$, policy functions $\delta : S \to \{0, 1\}$, $a'(y') : S \to \mathcal{R}^{dim(\mathcal{A})}$ and $B' : S \to \mathcal{R}$, a probability measure Φ^* such that:

1. V, δ, a' and B' are measurable with respect to $\mathcal{B}(S)$.

2. Given prices (w, q^f, q), the value function $V(B, \{a\}, y)$ and policy functions $\delta(B, \{a\}, y)$, $a'(B, \{a\}, y)$ and $B'(B, \{a\}, y)$ solve the household problem.

3. Given prices (w, q^f, q), the representative firm maximizes profits, i.e., capital demand K^d and labor demand, L^d, satisfies: $\alpha \left(\frac{K^d}{L^d}\right)^{\alpha-1} = r$, and $(1-\alpha) \left(\frac{K^d}{L^d}\right)^{\alpha} = w$, with $r = \frac{1}{q^f} - 1$.

4. Prices (w, q^f, q) clear the markets, i.e.

$$\bar{L} \equiv \sum_{y \in Y} \Pi(y) y = L^d$$
$$K^s \equiv \int \left(q^\intercal a \delta + q^f B + \delta \kappa\right) d\Phi^* = K^d.$$

5. The probability measure Φ^* is invariant with respect to the transition function in equation (7), i.e.: for all $(\mathcal{A}, \mathcal{P}(Y)) \in \mathcal{B}(S)$:

$$\Phi^*(\mathcal{A}, \mathcal{P}(Y)) = \int Q\left((B, \{a\}, y), (\mathcal{A}, \mathcal{P}(Y))\right) d\Phi^*.$$

4.3 Conditions for existence

In this subsection, we characterize the equilibrium and we lay out conditions for its existence. To establish some results, it is useful to define a *stationary recursive equilibrium conditional on insurance decision* as follows:

Definition 3. Given a participation decision $\delta : S \to \{0, 1\}$ and contingent asset $a'(y') : S \to \mathcal{R}^{dim(\mathcal{A})}$, a *stationary recursive competitive equilibrium conditional on insurance decision* δ and a' is a set of prices (w, q^f, q), value function $V : S \to \mathcal{R}$, a policy function $B' : S \to \mathcal{R}$, a probability measure Φ^* such that:

1. V and B' are measurable with respect to $\mathcal{B}(S)$.
2. Given prices (w, q^f, q), and given the functions $\delta(B, \{a\}, y)$ and $a'(B, \{a\}, y)$, the value function $V(B, \{a\}, y)$ and policy function $B'(B, \{a\}, y)$ solve the household problem.
3. Prices (w, q^f, q), value function V, the policy function $B'(B, \{a\}, y)$, and probability measure Φ^* satisfy conditions 3-5 in Definition 2.

This definition is useful to map the endogenous partial insurance equilibria to standard Aiyagari economies. Based on this definition, we can establish the following result regarding the existence of a stationary recursive equilibrium:

Lemma 6. *A value function $V : S \to \mathcal{R}$, policy functions $\delta : S \to \{0, 1\}$, $a'(y') : S \to \mathcal{R}^{dim(\mathcal{A})}$ and $B' : S \to \mathcal{R}$, prices (w, q^f, q), and a probability measure Φ^* constitute a recursive stationary equilibrium when:*

(i) *Given δ and a', the value function V, policy function B', prices (w, q^f, q) and probability measure Φ^* constitute a stationary recursive competitive equilibrium conditional on an insurance decision.*

(ii) *Given prices prices (w, q^f, q), the value function V, policy functions a', B' and δ solve the household recursive problem.*

As a result, finding a stationary recursive equilibrium amounts to find a stationary recursive competitive equilibrium conditional on a insurance decision and then check that this insurance decision is indeed optimal given the other policy functions and the prices resulting from this stationary recursive competitive equilibrium conditional on an insurance decision.

Our first result is about the existence of stationary recursive competitive equilibria conditional on an insurance decision:

Proposition 7. *When the dimension of $\mathcal{A} < N - 1$, for every participation decision $\delta : S \to \mathcal{R}$, there exists at least one stationary recursive equilibrium conditional on an insurance decision δ and a'.*

In any of such equilibrium, the price of the non-contingent asset satisfies $q^f > \beta$ and the prices of contingent assets satisfy, for any $y \in Y$ and any $y' \in \mathcal{A}(y)$, $q(y'|y) = q^f \pi(y'|y)$.

Proof. See Appendix A.6 for the proof. □

The core of the proof relies on the existence result by Açıkgöz (2018) for Aiyagari economies with production. To be able to use this result, we first define a modified income process, resulting from the state contingent asset market participation decision as described in the proof of Proposition 5. We also need that the set of assets cannot allow for full insurance. Under this assumption, the existence of an equilibrium is not a difficult problem.

For a participation cost κ, a stationary recursive equilibrium requires finding a participation decision that: *(i)* is optimal given the participation cost and the level of prices (w, q^f, q); and *(ii)* leads to a *stationary recursive equilibrium given an insurance decision* in which the resulting levels of prices are (w, q^f, q). More formally, Proposition 7 leads to a function $(w, q^f, q) = f(\delta, a')$ and the optimal portfolio decision to a function $\{\delta, a'\} = g((w, q^f, q), \kappa)$ for which (w, q^f, q) are the equilibrium level of prices that are also the fixed point of the following function: $\phi_\kappa(w, q^f, q)) = f(g((w, q^f, q), \kappa)) = (w, q^f, q)$.

Since δ is a discrete variable, we cannot use standard results as the intermediary value theorem or even the more general Kakutani theorem due to lack of continuity. In practice, we obtain such a continuity because the set of points where δ is potentially discontinuous with respect to prices (w, q^f, q) is finite and because, in equilibrium, marginal variations in prices generically do not lead to changes in δ for relevant income and asset levels. This ensures that $\phi_\kappa(w, q^f, q)$ has a closed graph and allows for the use of general results on existence.

Multiplicity of equilibria Let us comment more on the set of equilibria: for a given cost κ, there might be multiple equilibria.[10]

When participation costs are sufficiently high to make insurance never affordable, the economy coincides with an Aiyagari economy and it is known that in that case the equilibrium is not necessarily unique. Light (2017) shows the uniqueness for Aiyagari economies with CRRA preferences, but only with relative risk aversion lower than 1. In contrast, Açıkgöz (2018) provides example where the Aiyagari economy is compatible with multiple equilibria when preferences are CRRA with a relative risk aversion coefficient greater than 1.[11]

When adding participation decisions to insurance markets, we add a potential other source of multiplicity through a feedback between participation decisions and asset prices. Whereas marginal variations of asset prices might lead participation decisions not to change, large variations could well shift participation decisions and lead to another equilibria. More specifically, on the one hand, participation decisions depend on the prevailing interest rate and in particular, if the cost of self-insurance is higher (due to lower risk-free rate or higher price q^f for non-contingent assets), participation to insurance markets may become more attractive. On the other hand, more participation modifies households' savings and thus may reduce the interest rate.

[10] Of course, there are multiple equilibria for different participation costs as we will illustrate in Section 5.

[11] As Açıkgöz (2018) points out, this does not necessarily mean that the Aiyagari model loses its predictive power in this case, as standard calibration procedures usually lead to a single equilibrium.

The mere possibility to have more insurance also contributes to the multiplicity of equilibria. In particular, it is well known that, in the extreme case where the set of contingent assets and borrowing constraints allow for complete markets, the stationary wealth distribution is a function of the initial conditions and, in particular of the initial wealth distribution as pointed out by Caselli and Ventura (2000).

Insurance as a function of participation Finally, let us note that equilibria are quite different with respect to the value of participation costs.

In the case where participation costs are sufficiently high, there is no participation for any level of wealth and income, that is, for all $(B, \{a\}, y) \in S$, $\delta(B, \{a\}, y) = 0$. This happens regardless the price of the asset q^f. The economy coincides then with the standard Aiyagari economy, in which agents smooth consumption only by using non-contingent assets.

In the other extreme case where costs are zero, agents always participate in the available state contingent asset market, that is for all $(B, \{a\}, y) \in S$ such that $\mathcal{A}(y) \neq \emptyset$, $\delta(B, \{a\}, y) = 1$. In this case, the insurance structure is only limited by the exogenous asset structure $\mathcal{A}$.

For intermediate levels of costs, endogenous partial insurance may arise as an equilibrium outcome, that is $\delta(B, \{a\}, y)$ may be equal to 0 for some $(B, \{a\}, y) \in S$ even when $\mathcal{A}(y) \neq \emptyset$ and, for some other $(B, \{a\}, y) \in S$, $\delta(B, \{a\}, y) = 1$.

5 Quantitative Implications of Endogenous Partial Insurance

In this section we present the quantitative implications of the endogenous partial insurance model, with a particular focus on highlighting the strength of the main mechanisms of the model.

5.1 Calibration

5.1.1 Preferences, Technology, and Income Process

The functional form and parameters for the utility function and production function are standard. The utility function is assumed to be CRRA, i.e. $u(c) = c^{1-\sigma}/(1-\sigma)$, with $\sigma = 2$. The discount factor is set at $\beta = 0.96$. Regarding the production technology, the share of capital in the production function is fixed at $\alpha = 0.36$ and the depreciation rate at 0.08.

We discretize the income process by mapping a first order autoregressive process into a first order Markov process with seven states, using Rouwenhorst's method, as assumed in Krueger et al. (2016) for modelling the evolution of income conditional on employment.[12] There are, then, two key parameters: the persistence of the income process, and the standard deviation of its innovations. Following Kaplan (2012) and Fève et al. (2017), we assume that the persistence of income is 0.958. We then adjust the standard deviation of the innovations so that the process generates an income Gini index equal to the one observed in the post-taxed

[12] See Kopecky and Suen (2010) for a detailed description and evaluation of the Rouwenhorst method.

U.S. income data, that is 0.40. This parameterization leads to the following income states, $y \in \{0.14, 0.27, 0.52, 1.00, 1.89, 3.57, 6.75\}$. The resulting transition matrix, whose elements are reported in percent for convenience, is:

$$\pi = \begin{bmatrix} 88.0433 & 11.3314 & 0.6077 & 0.0174 & 0.0003 & 0.0000 & 0.0000 \\ 1.8886 & 88.2458 & 9.4515 & 0.4053 & 0.0087 & 0.0001 & 0.0000 \\ 0.0405 & 3.7806 & 88.3675 & 7.5647 & 0.2432 & 0.0035 & 0.0000 \\ 0.0009 & 0.1216 & 5.6735 & 88.4080 & 5.6735 & 0.1216 & 0.0009 \\ 0.0000 & 0.0035 & 0.2432 & 7.5647 & 88.3675 & 3.7806 & 0.0405 \\ 0.0000 & 0.0001 & 0.0087 & 0.4053 & 9.4515 & 88.2458 & 1.8886 \\ 0.0000 & 0.0000 & 0.0003 & 0.0174 & 0.6077 & 11.3314 & 88.0433 \end{bmatrix}.$$

5.1.2 The Case of Full Insurance Contracts

We now need to specify the asset structure available to the agent. Let us first consider the case in which agents have access to full insurance, which means that $\mathcal{A} > N - 1$, as the set of contingent assets, together with the risk-free asset, allows to span the state space. Then, assuming that borrowing is only limited by the natural debt limits, full insurance is possible. If agents pay the participation cost κ to access insurance markets at any date t, each period consumption equals:

$$c_t(y_0, B_0; \kappa) = r\left(B_0 + \sum_{t \geq 0} \frac{E_0(wy_t)}{(1+r)^t}\right) - \kappa, \tag{8}$$

with y_0 and B_0 denoting, respectively, the initial level of income and the initial level of asset holdings. With full insurance, as pointed out in Caselli and Ventura (2000), consumption, and therefore saving, depends on initial conditions. If agents had access to the full set of contingent assets, would they pay the participation cost to be fully insured? Using equation (8) and the corresponding flow of utility under perfect insurance, $V(y_0, B_0; \kappa) = \frac{1}{1-\beta} u(c(y_0, B_0))$, we can derive, for illustrative purposes, a simple necessary condition for perfect insurance, by comparing the value function for agents deciding not to participate to insurance markets for one period before fully insuring in the rest of the periods (i.e. one-shot deviation from full insurance) to the case in which they decide to get full insurance right away; this comparison is clearly a function of the participation cost κ.[13] This comparison clearly depends on initial conditions and, in particular, on the initial wealth level. Given our theoretical results, we know that, with any positive level of participation costs, there always exists a level of wealth so that agents prefer not to participate to financial markets. Accordingly, using our calibration, Figure 2 plots the level of wealth, as a fraction of average labor income, above which agents are surely not fully insured.

As a consequence, if the initial wealth distribution is not so dispersed, full insurance can

[13] As mentioned, this comparison only yields a necessary condition: if this condition is such that agents opt for full insurance, this does not necessarily mean that an equilibrium will full insurance exist. For example, agents may be better off not participating to insurance markets over multiple periods.

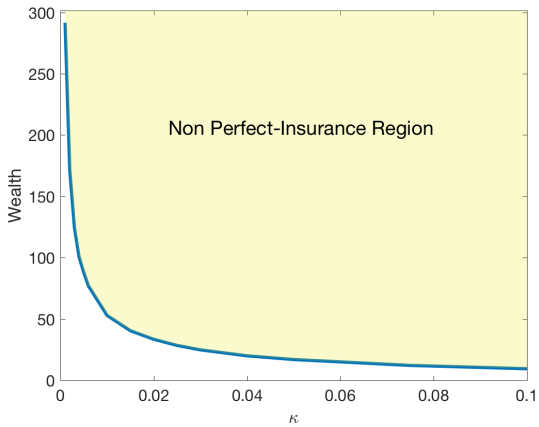


Figure 2 – Wealth threshold (in fraction of average labor income)

Note: the graph plots the threshold level of wealth (measured as wealth to income ratio) above which agents prefer a one-shot deviation from full insurance, as a function of the participation cost κ.

be an equilibrium outcome, and, in this case, the resulting variables of the model are heavily dependent of initial conditions and are very sensitive to the income process. Therefore, in order to better understand the relationship between participation costs and partial insurance, in the rest of the paper we assume asset structures that depart from complete markets, as clearly described in the next section.

Even if based on a necessary condition, Figure 2 illustrates that the threshold level of wealth above which there is only partial insurance could be high, at least for very small participation costs. As a consequence, if the initial wealth distribution is not so dispersed, full insurance can be an equilibrium outcome, and, in this case, the resulting variables of the model are heavily dependent of initial conditions and are very sensitive to the income process. Importantly, such a situation does not arise when markets are incomplete even under participation: in this latter case, as in the Aiyagari model, the remaining uninsurable risk leads the equilibrium wealth distribution to be less dependent on initial conditions.[14]

Therefore, in order to better understand the relationship between participation costs and partial insurance, in the rest of the paper we assume asset structures that depart from complete markets, as described in the next section.

5.1.3 Asset structure and participation cost

We adopt a benchmark structure that allows a clear comparison of the performance of the model for a large range of participation cost, from very high to zero. To do so, in all structures we consider we make full insurance infeasible, which means that even if households pay the participation cost, they can still only insure part of their idiosyncratic risk. As mentioned earlier, this assumption serves two purposes; first, in this case our model is much less sensitive sensitive to the calibration of the income process, which varies a lot in the extant literature,[15]

[14] See Açıkgöz (2018) for a discussion of equilibrium multiplicity in the Aiyagari model.

[15] If full insurance were available, only agents in the middle of the wealth distribution would be completely insured and therefore face no income uncertainty. Hence they would optimally run down their wealth, and they

and, second, even with full participation the model does not degenerate into a complete market model.

We now describe in the details the assumptions for our benchmark asset structure: (i) state contingent assets are available only to insure against downward risk; (ii) for any agent with current income y_i, with $i = \{2, .., 6\}$ the only state contingent asset available is the asset indexed by $(y_{i-1}|y_i)$; (iii) for any agent with the highest current income realization, y_7, the only state contingent asset available is the asset indexed by $(y_5|y_7)$. The first assumption aims to capture properties of healthy or unemployment insurance, which usually allow to receive a payoff only conditionally on the realization of a bad state; the second assumption aims to capture, albeit in a reduced form, the existence of uninsurable entrepreneurial risk, in the same spirit as Quadrini (2000). Hence, more formally, we assume that: $\mathcal{A}(y_i) = \{y_{i-1}\}, \forall i = \{2, ..., 6\}$; $\mathcal{A}(y_7) = \{y_5\}$; and $\mathcal{A}(y_1) = \emptyset$. These assumptions are not crucial for the results, as alternative asset structures in which insurance is for both some upward and downward risk or in which risk is insurable even for higher income states lead to similar implications, as it will be shown in Section 5.6 and 5.7.

Remark. The assumed benchmark asset structure is the result of a clear tradeoff. From one side, risk averse agents are mainly concerned about downward risk and therefore it makes sense not to restrict too much their access to state contingent assets against that type of risk. Given the transition of the Markov process displayed above, it turns out that this asset structure potentially allows insurance for 94 percent of the population that faces downward risk. On the other hand, we also know that there are multiple equilibria due to the possibility of additional insurance.[16] This feature is inconvenient when comparing the properties of the model for different participation costs. In contrast, by leaving some uninsured risk, we allow agents to accumulate wealth, no matter the level of costs

The assumption that agents can insure only against downward risk makes short sale constraints on the contingent assets non binding. In this case, in fact, agents that do participate in the contingent asset market will optimally get a long position in those assets. Hence, in equilibrium, it will be the case that $a'(y') \geq 0$. In addition, as in Aiyagari (1994) we assume that the constraint on non-contingent asset is $\overline{B} = 0$. We will relax this constraint in the robustness section to show that it does not affect significantly the main mechanisms of the model.

Participation cost Finally, we assume that the cost of participation to the state contingent asset market is a fixed proportion of the current income state. This means that an agent with current income state y_i has to pay a cost equal to $\kappa_i = \kappa y_i$ to have access to the assets in

would not be able to transit above the full insurance region. As a result, the rich group might not exist in the invariant distribution unless the poor households can jump to the rich group by having a sufficiently high income shock. As a consequence, slightly different income process could lead to very different equilibria.

[16] If *too much* downward insurance is feasible, when the participation cost is very small the model becomes very close to the full insurance case, for which, as showed by Caselli and Ventura (2000), the equilibrium would be dependent on the assumed initial distribution.

$\mathcal{A}(y_i)$.[17] In the quantitative analysis we will vary the constant κ to investigate the effects of cost of insurance. The mechanisms of the model are unchanged to assuming a constant cost across income realizations, instead. In Section 5.5, we also present the results of our simulations with constant costs.

The last but important ingredient for the calibration is the equilibrium selection procedure and the initial conditions that we use. Our strategy is the following: we start by simulating the economy with a very large cost so that the economy is an Aiyagari one and then we smoothly decrease the participation cost all the way down to 0.[18]

5.2 Participation Cost and Partial Insurance

In Figure 3 we display the percentage of agents in the economy that participate in the contingent asset market, as a function of the parameter κ. When the participation cost is very large, all agents decide to only self-insure. In this case there is no participation in the contingent asset market and the model is equivalent to the Aiyagari's economy. Oppositely, when the cost is zero, everyone participates in the insurance market, since the properties of the utility function imply a strictly positive gain of insurance, as proved in Proposition 2. When the participation cost is at an intermediate level, only a fraction of agents participates in the contingent asset market, and the model delivers endogenous partial insurance. Notice that even a very small cost, for example $\kappa = 0.005$, which is 0.5 percent of income, leads to a partial insurance equilibrium. Not surprisingly the amount of participation is monotonically decreasing with the cost.

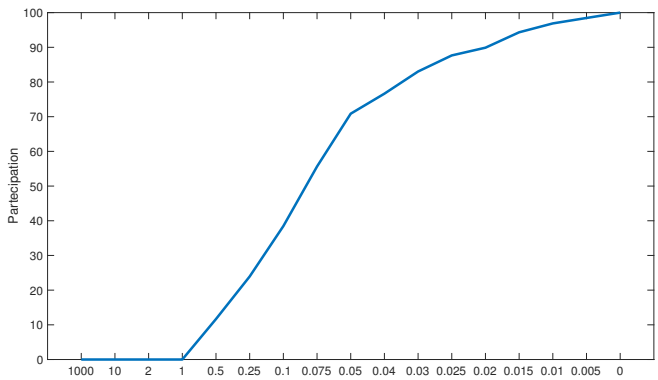


Figure 3 – Participation and Cost

Note: This graph plots the aggregate participation to the state contingent asset market as a function of the cost κ.

Which agents do and do not get insurance with small positive costs? In Figure 4 we answer this question highlighting the main mechanism of the model. Specifically, the figure displays the state contingent asset market participation decision as a solid blue bar, for agents with

[17] As welfare participation costs, instead of monetary costs, are homeomorphic, one could alternatively interpret this specification as opportunity cost of time for agents with different income levels.

[18] More specifically, the sequence of costs that we consider is $\{1000, 10, 2, 1, 0.5, 0.25, 0.10, 0.075, 0.05, 0.04, 0.03, 0.025, 0.020, 0.0150.$ Notice that as Figure 3 displays any cost greater or equal to 1 generates zero participation. Therefore, the fact that the cost sequence is very sparse from 1 to 1000 does not constitute a problem.

different income realization (x-axis) and different wealth (y-axis). The red dot reports the highest level of wealth observed for each income states, which are quite different across the different models.[19] Results are showed for three different costs. In the top panel we fix the cost to a very large number, 1000, so that the economy coincides with the Aiyagari's one. In this case there is no participation. In the bottom panel we fix the cost to zero. In this case there is full participation. Most importantly, in the central panel, we fix the cost to $\kappa = 0.04$, a cost value that, as it will be seen in the next subsection, implies the highest inequality and, therefore, is able to better highlight the mechanism of endogenous partial insurance. In the next section we discuss the quantitative implications of assuming smaller costs. With $\kappa = 0.04$ participation is limited and endogenous. Income poor agents, regardless their wealth, receive a very small flow of resources and for them the cost to participate in the state contingent asset markets is too high. Agents with middle/high income realizations, instead, display heterogenous behavior; if they are wealth rich, that is for any wealth level outside the participation bar, they decide not to participate and to self insure. Otherwise, if they have lower wealth, they do participate. This behavior is consistent with the finding in Proposition 3. Finally, notice that with our benchmark structure, according to which agents with the highest income cannot insure against the most likely downward risk, those agents prefer not to pay the cost and self-insure, since the gain of insuring against a quite unlikely event is lower than the benefit.

While Figure 4 displays one of the main mechanism of the model, i.e. the relationship between endogenous insurance decision and wealth, it does not shed light on the proportion of agents that belong to the insurance region, since that proportion is a function of the stationary joint distribution of income and wealth. Figure 5 clarifies this point by showing the participation share for each income realization, for the cost equal to 0.04; it highlights that the largest fraction of agents have wealth levels that belong to the insurance bar and only a small fraction of very rich agents that do not insure.

5.3 Interest rate, asset accumulation, and inequality

Interest Rate One of the implications of partial insurance concerns the equilibrium interest rate. As we will explain below, the evolution of the interest rate is an important component of one of the main mechanism of the model, since it affects the speed by which agents accumulate assets. Figure 6a plots the equilibrium interest rate as a function of the participation cost. When reducing the participation cost, the risk-free rate increases from Aiyagari economy's value,[20] when there is no participation, to a value very close to the discount rate, when there is full participation.[21] This captures the smoothed evolution of aggregate partial insurance from

[19] We compute the wealth distribution by simulating 10000000 agents.

[20] As pointed out in Aiyagari (1994) and Huggett (1993) when households have only risk-free bonds to self-insure against idiosyncratic shocks, the interest rate paid on these bonds is lower than the interest rate paid when markets are complete. The intuition for this result is simply that high level of interest rates would incentivize households to accumulate an infinite amount of assets, which would allow them to consume infinitely and, of course, to be perfectly insured.

[21] Recall that even with zero cost the model does not generate full insurance, because of the assumed asset structure.

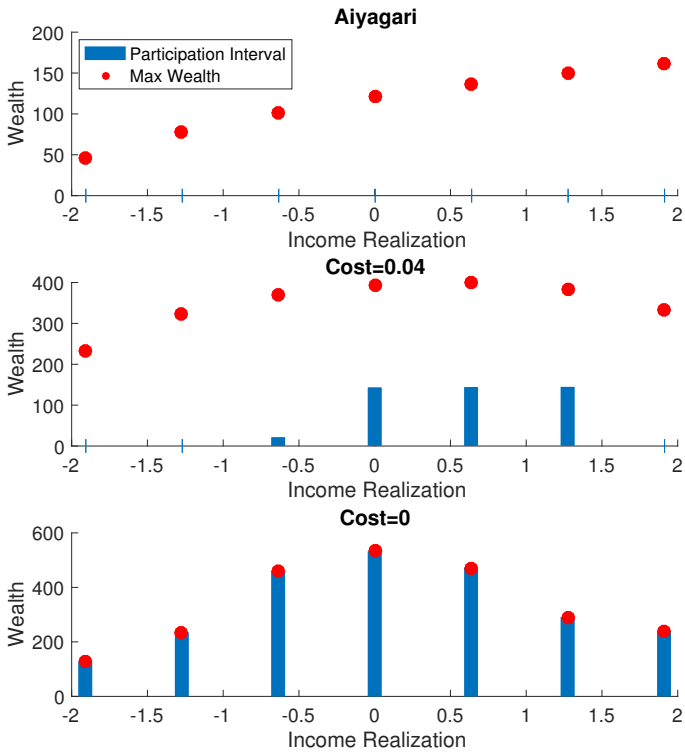


Figure 4 – Participation and Cost

Note: This graph plots participation decisions for agents with different income realizations (x-axis) and different wealth (y-axis). Each blue bar represents participation to the state contingent asset markets. The red dot displays the highest level of wealth observed for each income state. We compute these statistics for three models; the top one is the Aiyagari model, which corresponds to the case of large participation costs; the central one is the model with a participation cost equal to $\kappa = 0.04$; the bottom model assumes zero participation costs, $\kappa = 0$.

self insurance in the Aiyagari case to full participation with zero cost.

When the participation cost is low enough so that some agents acquire contingent assets, the demand for risk free assets declines and the demand for contingent assets increases, as tracked by the solid line (left y-axis) and the dashed line (right y-axis), respectively, in Figure 6b. The resulting overall smaller supply of capital creates an upward force to the interest rate.

Inequality, Lorenz Curve, and Concentration at the top How does the existence of participation costs in contingent asset markets affect the wealth distribution? The answer to this question depends on the interaction between participation costs, income risk, wealth, and interest rate. When there is endogenous partial insurance, two forces operate in different portions of the wealth distribution. On the one hand, insured households do not have incentive to accumulate a lot of assets, since their downward risk is mostly covered by the state contingent asset; we label this effect as *participation channel*. On the other hand, self-insured households, which are the one that have larger wealth, benefit from real interest rates that are higher than in the Aiyagari model and they accumulate more wealth. This force pushes the right tail of the distribution even further to the right; we label this effect as *interest rate channel*. Together,

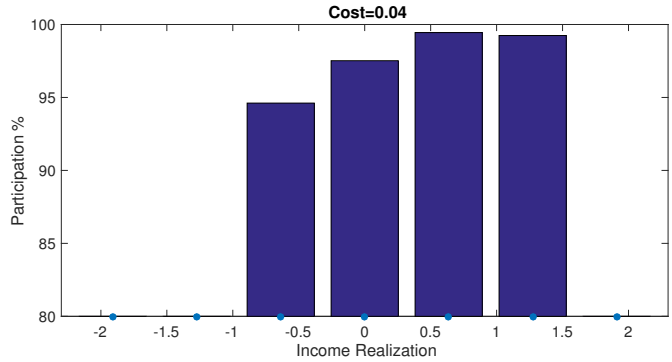
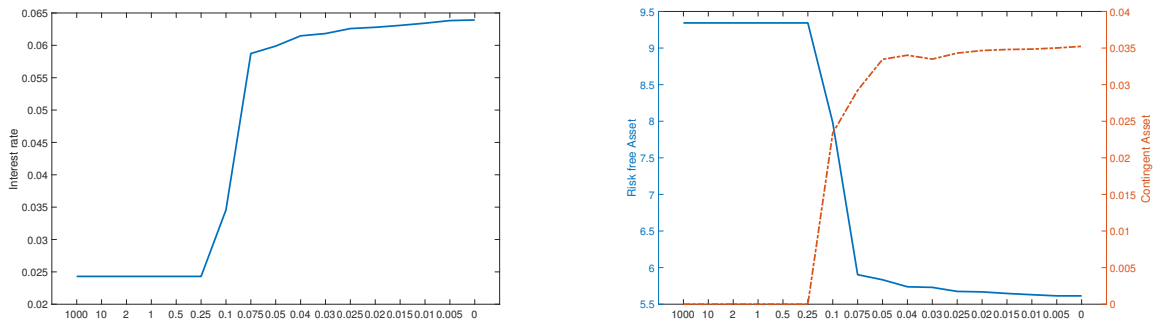


Figure 5 – Participation and Cost

Note: This graph plots the percentage of agents that participate in the state contingent asset market for different income realizations (x-axis) for a participation cost equal to $\kappa = 0.04$. Hence, this figure infers the density of agents that belong in the wealth participation interval displayed in the central panel of Figure 4.



(a) Interest Rate

(b) Assets

Figure 6 – Interest Rate, Assets, and Cost

Note: The left panel plots the equilibrium interest rate for non-contingent assets as a function of the participation cost parameters κ. The right panel plots the equilibrium level of risk free assets (solid line, left y-axis) and of state contingent assets (dashed line, right y-axis).

these two forces contribute to skew the wealth distribution and lead to large wealth inequality.

The combined strength of these two channels is not necessarily monotone with respect to participation cost, as displayed in Figure 7 in which we plot the wealth Gini index as a function of κ. Wealth inequality is relatively low for the Aiyagari model, and it increases quite sharply when participation and the interest rate increase; nevertheless, when participation is already very large and the interest rate is already quite close to the discount rate, additional decline in cost and additional participation reduces inequality. Intuitively when costs are tiny even wealthier agents acquire state contingent assets and, therefore, they have no incentive to accumulate wealth. For our calibration and asset structure, the level of cost that maximizes inequality is $\kappa = 0.04$, which corresponds of an average of cost of 3.70% of average income, i.e. roughly \$2220 per year, assuming an average yearly income of \$60000), once one considers the correspondent equilibrium wage level.

While the cost that maximizes inequality is still quite high, assuming lower costs do not change drastically the quantitative implications of the model, as shown in Table 1. It is nevertheless fair to compare these costs with the estimates found in the literature for financial market

participation. Honka (2014) estimates demand for insurance and quantifies search cost in the US auto insurance industry, finding that the cost of a price search through local agents, mail, and calling centers ranges from \$100 to \$170. Ho et al. (2017) study expansion to the Medicare program in 2006 to estimate search friction in consumer choices; they estimate indicate that removing those frictions could reduce consumer expenditures by around \$330 per enrolee per year. Finally, Lin and Wildenbeest (2019) estimate search friction in the U.S. Medigap insurance market and they find that it ranges from 13 to151\$. Related to a more general definition of financial markets, Vissing-Jorgensen (2002b) estimates the median of per-period cost for stock market participation to be around \$204, in 1982-84 dollars, whose purchasing power corresponds roughly to 480 dollars of year 2017. Attanasio and Paiella (2011b) and Paiella (2007b) estimate per-period participation costs in stock markets and provide a lower bound in units of nondurable consumption, as low as 0.4% and 0.7% of consumption per year, which is roughly in the range 117\$-205\$. Khorunzhina (2013) provide evidence that the average per-period stock market participation cost, measured as a share of income, ranges between 4% and 6% of labor income for non participants. Finally, and on the high end of costs, Kaplan and Violante (2014) use a level of cost up to \$1000 to obtain realistic portfolios for private agents for illiquid assets, which is roughly 2 percent of average annual income. As the goal of this paper is not to estimate a participation cost for state contingent assets market, but to illustrate as clearly as possible the novel mechanism of the participation channel, to keep the exposition clear we will assume a fixed cost equal to 0.04, when needed.

κ	0.04	0.01	0.005
USD equivalent	2220	558	282
Gini index	0.77	0.75	0.75
Interest Rate	6.19%	6.35%	6.39%

Table 1 – Participation Costs

Note: This table presents some summary statistics of the endogenous partial insurance model (i.e. wealth Gini index and interest rate) for three different costs: $\kappa = 0.04$, which is the one that maximizes the wealth Gini index, $\kappa = 0.01$, and $\kappa = 0.005$. The USD equivalent is computed for the average income household with \$60000 and accounting for the equilibrium wage rate in the economy.

The comparison between inequality at zero cost and with a positive costs highlight the importance of the *participation effect* with endogenous partial insurance, which stems from heterogeneity of agents' decision in the economy, from exogenous partial insurance, which stems from simply assuming the non-existence of some assets.

Table 2 reports the different wealth concentration for a model with large cost, so that the economy is identical to the Aiyagari one, for a model with a participation cost equal to 0.04, which is the one that maximizes the Gini index, and for a model with no costs. Endogenous partial insurance generated by $\kappa = 0.04$ increases the share of wealth for the top 1% with respect to the standard incomplete market model and also to a model with full participation, although this share remains smaller than in the data. This means that the larger wealth Gini index in the endogenous partial insurance model mainly comes from the large differences in wealth accumulation across bigger segments of the population, i.e the wealth-poor/middle and

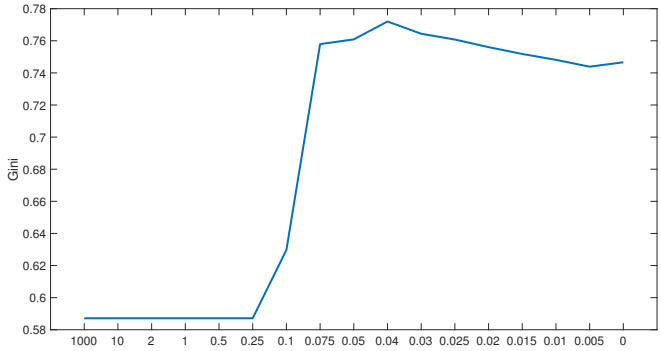


Figure 7 – Participation and Cost

Note: this graph plots the wealth Gini index as a function of the participation cost parameters κ.

middle/large wealth groups. In fact, for the top 5% or even more for the top 10% the wealth concentration is substantially higher in the endogenous partial insurance model. Overall, a model with endogenous partial insurance leads to very high wealth concentration to the top 10, 5, and at a lower degree 1 percent, with respect to the rest of the population. Figure 8 displays the complete wealth distributions, under the form of Lorenz curves, together with the 45 degree line.

	Model			Data
	Aiyagari	Cost =0.04	Zero cost	
Gini index	0.58	0.77	0.74	0.85
Top 1%	6.9%	17.8%	13.6%	34.1%
Top 5%	23.9%	44.1%	39.2%	60.9%
Top 10%	38.2%	60.3%	56.3%	74.4%

Table 2 – Share of Wealth held by percentile

Note: This table presents the concentration of wealth on the top 1,5, an 10 percent of the wealth distribution in the Aiyagari model, endogenous partial insurance model generated by a cost $\kappa = 0.04$, and a model with zero cost. Data values come from Quadrini and Rios-Rull (2014), Table 14.6.

To further point out the relation between wealth, participation, and implied concentration, in Figure 9 we display the average participation rate, in percent, for the different deciles of the wealth distribution. Clearly, the Aiyagari economy (dotted blue line) and the economy with no costs, (dashed red line) represent the two extreme cases, in which nobody and everybody, respectively, participate in the contingent asset market. The pattern for the positive cost (solid black line) highlights the heterogenous behavior of agents with different wealth; consistently with our theoretical results, very few wealthy poor buy state contingent assets; middle class agents consistently participate in the insurance market; and participation reduces for wealthy rich people, as their large amount of wealth and the high interest rate allow them to accumulate a large amount of resources to self-insure, thus avoiding to pay the participation cost.

Degree of partial insurance In their paper Guvenen and Smith (2014) estimate the fraction of partial insurance around 45 percent. In our benchmark model, the aggregate participation level is 86.6 percent. Because the assumed asset structure and the assumed transition matrix of

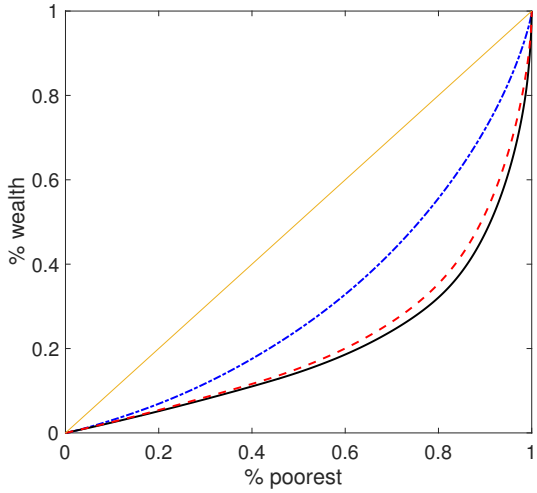


Figure 8 – Lorenz curve

Note: This graph plots the Lorenz curve for the Aiyagari model (dotted blue line), endogenous partial insurance model generated by a cost $\kappa = 0.04$ (solid black line), and a model with zero cost (dashed red line). The solid thin line represents the 45 degree line.

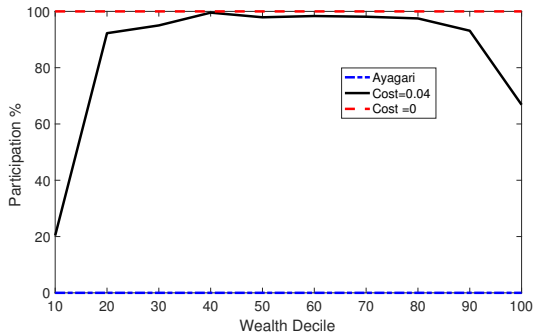


Figure 9 – Wealth and Participation

Note: this graph plots the share of agents that participate in the state contingent asset market in the wealth deciles. We compute these statistics for three models; the Aiyagari model (dotted blue line); the model with a participation cost equal to $\kappa = 0.04$ (solid black line); and a model with zero cost (dashed red line).

income implies that 47 percent of total income risk is insurable,[22] our model leads to a degree of partial insurance of 40.7 percent, a value close to the one estimated in Guvenen and Smith (2014). Our definition of partial insurance can be linked to the one introduced in Guvenen and Smith (2014); however, whereas their form of partial insurance is on the intensive margin - agents can insure a fraction of their income, in our setting partial insurance is on the extensive margin - agents can be insured or not, given the assumed asset structure.

5.4 Consumption smoothing and insurance

What are the implications of the endogenous partial insurance model for consumption smoothing and insurance? In this section we explore this question, particularly pointing out that the predictions of endogenous partial insurance are quite different than the one for the

[22]The assumed exogenous asset structure implies that 0 percent of upward risk and 96 percent of downward risk is insurable, and both upward and downward account for half of the total risk, as implied by the transition matrix π.

Aiyagari economy. We show that the assumed benchmark asset structure is useful to highlight the different roles played by the *participation channel* and by the *interest channel*.

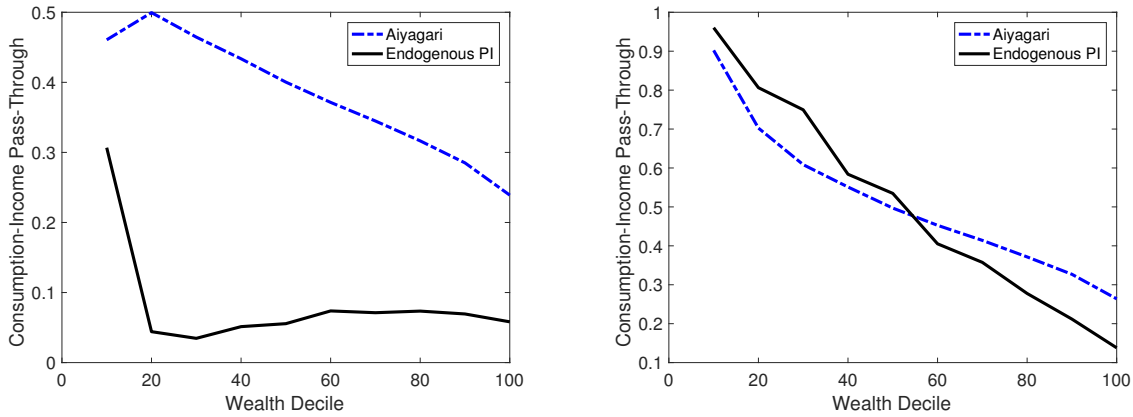
Participation Channel To quantify the role of the participation channel we focus now on negative income shocks, as they are, by assumption, the only insurable shocks. Our theoretical results imply clear predictions about participation: poor households' consumption is particularly sensitive to negative income shocks, as they have too low income to participate in the state contingent asset markets and they might be also borrowing constrained; middle class agents are well insured, as they decide to participate in the contingent asset market; wealthy households' consumption is somewhat still subject to negative income shock as they endogenously decide to self insure. These predictions are indeed born out in our model with partial insurance. Figure 10a displays the model-implied coefficient α_j of the regression:

$$\Delta c_{i,j} = \alpha_j \Delta y_{i,j} \mathbb{1}_{\Delta y_{i,j}<0} + \eta_i, \text{with } j = 1, .., 10, \tag{9}$$

where $\Delta c_{i,j}$ denotes consumption growth of a household i that belongs to the wealth decile j and $\Delta y_{i,j} \mathbb{1}_{\Delta y_{i,j}<0}$ are the negative, and therefore insurable, income shocks. Hence α_j is the consumption-income pass through, conditional on negative income shocks, for wealth decile j. The coefficients are plotted for the two models of interest, the endogenous partial insurance model (solid black line) and the Aiyagari model (dotted blue line). As state contingent assets are available, consumption is overall less sensitive to negative income shocks. However, since the first wealth quantiles are populated by income poor agents that cannot afford to pay the participation cost, the consumption-income pass-through is quite high. Middle-class agents, in the contrary, heavily participate in the insurance market and, as a consequence, their consumption is almost perfectly shielded against negative income shocks. Finally, richer agents are slightly less insured against those shocks as some of them do not participate in the state contingent market; nevertheless, their large amount of financial assets allow them to self-insure quite well.

The interest rate channel Figure 10b plots the regression coefficients α_j when not restricting income shocks to be negative: roughly the bottom half of the wealth distribution is overall more sensitive to income shocks than in the Aiyagari model, while the top half of the distribution is less sensitive. The comparison between the two panels implies that in the endogenous partial insurance equilibrium agents smooth positive income shocks at a lower degree, especially in the lowest part of the wealth distribution. This effect is the result of the *interest rate channel*. In fact, poors' insurance ability is deteriorated by higher interest rates than in the Aiyagari model; in both economies the only tool to insure against upward risk is dissaving, which is much more costly in the endogenous partial insurance model than in the Aiyagari model. That explains why, in the bottom half of the distribution, aggregate consumption-income pass through is larger than in the Aiyagari model; agents in that part of the distribution are more subject to uninsurable upward risk, as they are already in the bottom part of the income distribution, and

for them self-insurance is more costly. On the contrary, the top-half of the distribution is populated by a larger share of agents that face insurable downward risk. Their participation in the state contingent asset market, and the higher return on their saving, enhance their consumption smoothing ability, and, therefore, in that part of the distribution the consumption-income pass through is lower than in the Aiyagari model. The comparison between the two panels of Figure 10 highlight the different roles of the participation and interest rate channels. The former tends to enhance consumption smoothing, while the latter have heterogenous effects on consumption smoothing across the wealth distribution. The overall net effect is a function of the relative strength of the two channels and it heavily depends on the assumed asset structure, as we will point out in Section 5.6.



(a) Negative Income shocks

(b) All Income shocks

Figure 10 – Consumption Income Pass-through

Note: the left panel plots the coefficients α_j for the regression in (9) for the 10 wealth deciles in the Aiyagari model (dashed blue line) and in the endogenous partial insurance model with a positive cost equal to 0.04 (solid black line). The right panel plots the coefficients α_j for the same regression when not conditioning on only negative income shocks.

The distribution of consumption growth We further explore the implications of endogenous partial insurance on the joint distribution of consumption, income and wealth, by plotting in Figure 11 the distribution of consumption growth in the endogenous partial insurance model (solid black line) and the Aiyagari model (dotted blue line). As expected, the participation of the state contingent asset market truncates the distribution of consumption growth in the partial insurance model with respect to the Aiyagari model; although obviously negative consumption growth is observed, because of the agents that do not pay the participation costs and because of our assumption that part of the downward risk is uninsurable, a significant portion of the density of consumption growth moves from the negative side to the positive side. This shift of density is exacerbated by the interest rate channel, as risk-free assets accumulation generates higher return and thus enhances higher consumption. These effects lead to slightly higher standard deviation and skewness of the consumption growth distribution in the partial insurance model.

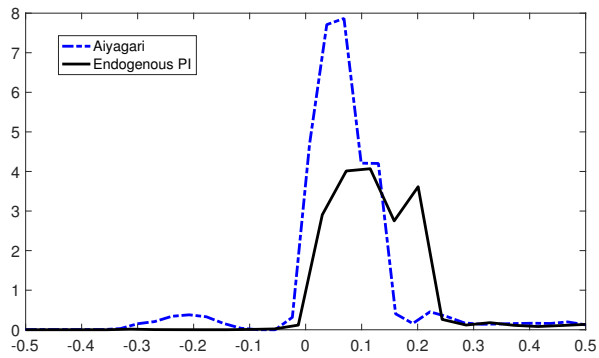


Figure 11 – Consumption Growth

Note: This graph plots the distribution of consumption growth in the Aiyagari model (dashed blue line) and in the endogenous partial insurance model with a positive cost equal to 0.04 (solid black line).

The Marginal Propensity of Consumption out of Wealth How does the endogenous partial insurance model affect the marginal propensity to consume out of wealth? In a recent paper Carroll et al. (2017) point out that standard macroeconomic models that attempt to match wealth inequality imply too low marginal propensity to consume, which, in the empirical literature is estimated to range between 0.2 and 0.6.[23] Standard representative agent models usually deliver a MPC about 0.02-0.04, while heterogenous agents model have potentials to imply a larger MPC when incorporating additional ingredients.[24] In Figure 12 we plot the marginal propensity of consumption out of wealth for the endogenous partial insurance model associated to a participation cost equal to 0.04 (solid black line) and the Aiyagari model (dotted blue line) across the wealth distribution.[25] In aggregate, the implied MPC is higher in the endogenous partial insurance model (0.092) than in the standard Aiyagari model (0.071), to indicate that the mechanisms underlying endogenous partial insurance contributes to generate overall slightly higher MPC. In addition, the difference between the two models is more pronounced in the central part of the wealth distribution, where both the participation and interest rate channels operate; the existence of state contingent assets make participating agents to consume a larger fraction of extra wealth, as they do not need to over-accumulate assets to self-insure. The impact of this channel for MPC depends on the assumed asset structure and on the overall participation.

5.5 The role of participation costs, borrowing constraint

In our benchmark calibration, we have assumed that participation costs are proportional to income and that the borrowing constraint for the non-contingent asset is tight, that is $\bar{B} = 0$. In this subsection, we investigate how our results are modified when relaxing these two assumptions.

[23] See Jappelli and Pistaferri (2010) for a review in this topic.

[24] Kaplan and Violante (2014) and Kaplan et al. (2018) for examples introduce illiquid assets and that generates higher aggregate level of MPC and non-monotone MPC across wealth.

[25] The MPC out of wealth is computed by averaging the numerical derivative of the consumption policy function with respect to total wealth for each wealth decile.

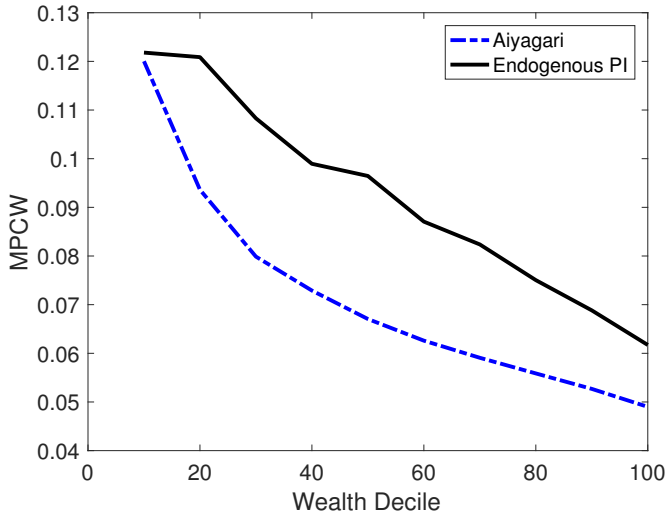


Figure 12 – Marginal propensity of consumption out of wealth

Note: this graph plots the marginal propensity of consumption out of wealth for the Aiyagari model (dashed blue line) and in the endogenous partial insurance model with a positive cost equal to 0.04 (solid black line) across the wealth deciles. The MPC out of wealth is computed by averaging the numerical derivative of the consumption policy function with respect to total wealth for each wealth decile.

Borrowing constraint Let us first consider borrowing constraints that allow for some borrowing ($\bar{B} < 0$). Table 4 reports the level of wealth inequality for different levels of borrowing constraints but the same level of participation cost.

	Aiyagari	Cost =0.04
borrowing limit /average income	Wealth Gini index	
$\bar{B}/avg(wY) = 0$	0.58	0.77
$\bar{B}/avg(wY) = -7\%$	0.58	0.75
$\bar{B}/avg(wY) = -14\%$	0.57	0.73
	Participation	
$\bar{B}/avg(wY) = 0$	0%	77.7%
$\bar{B}/avg(wY) = -7\%$	0%	76.6%
$\bar{B}/avg(wY) = -14\%$	0%	70.4%

Table 3 – The effects of borrowing constraints

Note: This table presents the wealth Gini index (top panel) and aggregate participation (bottom panel) in the Aiyagari model and endogenous partial insurance model generated by a cost $\kappa = 0.04$ when assuming three different borrowing limit in the risk-free asset: no borrowing allowed, a borrowing limit of 7% of income, and a borrowing limit of 14% of income.

The possibility of some borrowing does not alter qualitatively the results, but it has still some quantitative, albeit not drastic, consequences. Even when there is possibility to borrow, wealth inequality in the partial insurance model is quite larger than in the Aiyagari model but it is slightly reduced. Intuitively, in that case the risk free asset becomes a better insurance instrument and that reduces participation in the state contingent assets, which, in turns, diminishes the strength of the participation channel. Nevertheless, the main take away is that the results do not depend significantly by having assumed a zero borrowing constraint in the benchmark specification of the model.

The cost structure In our benchmark calibration, we have assumed that the cost is proportional to income, that is $\kappa(y) = \kappa y$. Let us investigate how the economy is modified when considering constant costs across the income distribution, that assuming $\kappa(y) = \kappa$.

	Aiyagari	Cost =0.04
Cost structure	Gini index	
$\kappa(y) = \kappa y$	0.58	0.77
κ= constant	0.58	0.71
	Participation	
$\kappa(y) = \kappa y$	0%	77.7%
κ= constant	0%	65.3%

Table 4 – The effects of the cost structure

Note: This table presents the wealth Gini index (top panel) and aggregate participation (bottom panel) in the Aiyagari model and endogenous partial insurance model generated by a cost $\kappa = 0.04$ when assuming two different cost structures: participation cost as constant fraction of income $\kappa(y) = \kappa y$, and a constant participation cost.

As Table 4 shows, with constant costs across income levels, the wealth Gini index is lower than with income dependent costs: the main reason is that the participation channel is weaker in the first case. Actually, constant costs are also skewing the cross-section of participation: participation is not only lower but richer households tend to participate more with constant cost and and poorer households will participate less. This finding is intuitive as average income is 1, so households with income below (above) average face a higher (lower) participation cost when it is fixed than when it is proportional to income. Overall, the cost structure affects only quantitatively the results and not qualitatively.

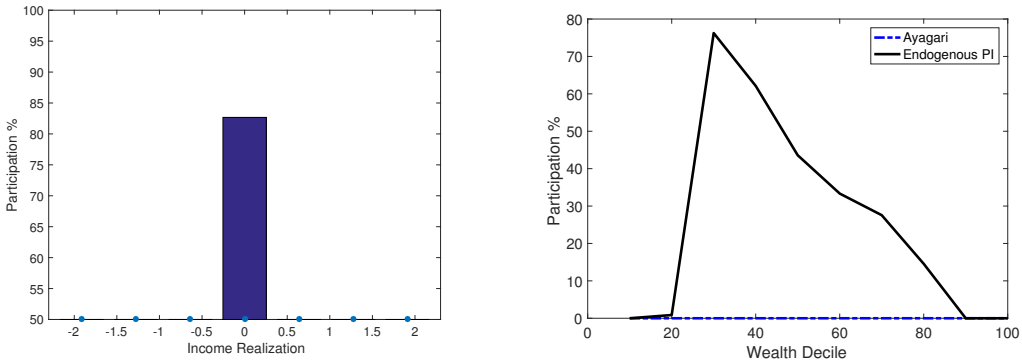
5.6 Alternative Symmetric Asset Structure

The results discussed in this section are general and also apply to symmetric asset structures. As an illustrative example, we now assume an alternative asset structure, labelled as $\mathcal{A}^1$, which, unlike the benchmark case, features symmetry. We assume that only agents with the mean/median income realization, y_4, have access to state contingent assets for the most likely downward and upward risk. State contingent assets do not exists for other states. Hence, formally, $\mathcal{A}^1(y_4) = \{y_3, y_5\}$; and $\mathcal{A}^1(y_i) = \emptyset$, $\forall i \neq 4$. Notice that this asset structure features both downward and upward insurance and, given the assumed income process, it implies that 22 percent of agents have exogenously access to insurance. For the analysis in this subsection we continue to fix the participation cost to $\kappa = 0.04$, for a clear comparison to the benchmark structure. Notice that in this case, as only one income group has access to the state contingent assets, the participation cost proportional to income is equivalent to a constant one. Also, with the possibility of insurance against upward risk, agents might want to go short on the state contingent assets: we allow for this possibility as we set the exogenous borrowing constraint for those assets, $\bar{a}$, to the natural borrowing limit.

As for the benchmark structure, the asset structure $\mathcal{A}^1$ implies a wealth Gini index, equal to 0.64, and an interest rate, equal to 2.75 percent, that are higher than in the Aiyagari model, although by less than in the case of the benchmark structure. The fact that the interest rate is

not so different from the Aiyagari model is particularly useful as this implies that in this case the *interest rate channel* is almost fully muted, and, therefore, this symmetric alternative asset structure allow us to isolate the *participation channel*.

Figure 13 displays the participation decision, for different income levels in Figure 13a, and for different wealth levels in Figure 13b. The left panel shows that, by construction, only agents with median income have access to state contingent assets, and, among them, 82 percent endogenously decide to participate. The right panel helps understanding where the participating agents are located in the wealth distribution. The asymmetry in the graph highlights that agents that decide to not acquire state contingent assets are the richest.



(a) Participation and Income

(b) Participation and Wealth

Figure 13 – Participation with Symmetric Asset Structure

Note: The left panel plots the percentage of agents that participate in the state contingent asset market for different income realizations (x-axis) for the asymmetric alternative asset structure when assuming a cost equal to $\kappa = 0.04$. The right panel plots participation decisions for agents with different income realizations (x-axis) and different wealth (y-axis) Each blue bar represents participation to the state contingent asset markets. The red dot displays the highest level of wealth observed for each income state.

In Figure 14 we report the implied consumption insurance, which are the coefficients α_j of the consumption growth-income growth regression, for different wealth deciles:

$$\Delta c_{i,j} = \alpha_j \Delta y_{i,j} + \eta_i, \text{with } j = 1, .., 10. \tag{10}$$

In the first two quantiles of the wealth distribution, in which there are no insured agents, the symmetric asset structure (solid black line) implies that consumption is more sensitive to income shocks than in the Aiyagari economy, as dissaving is slightly more costly. The difference is therefore only due to the very small *interest rate channel*. In the central part of the distribution, participation is larger, and since participating agents are heavily insured with the assumed alternative asset structure, the consumption-income pass through is very low. Notice that the degree of insurance declines while moving towards the wealthier portion of the distribution, as, there, participation declines. The top of the distribution is only slightly more insured than in the Aiyagari economy, because of, once again, the small interest rate channel that generates higher return for savings.[26]

[26]This particular effect is consistent with Guvenen (2007), which finds that financially wealthy agents do less consumption smoothing that agents with less financial wealth.

In summary, the alternative asset structure generates an aggregate shape of insurance across the wealth distribution that is quite different than the one for the benchmark asset structure, although it highlights exactly the same mechanisms; while in the benchmark asymmetric structure the interest rate channel is quite strong, in the alternative symmetric structure the participation channels largely dominates.

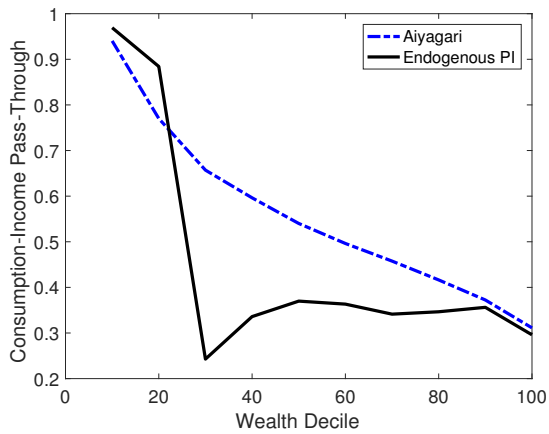


Figure 14 – Insurance with Symmetric Asset Structure

Note: the graph plots the coefficients α_j for the regression in equation (10) for the 10 wealth deciles in the Aiyagari model (dashed blue line) and in the endogenous partial insurance model with a positive cost equal to 0.04 (solid black line) for the alternative asset structure.

5.7 Additional asset structures

We further investigate the role of the insurance structure to show that our results are robust to different assumptions. Specifically, we want to highlight that: (i) the possibility of downward insurance could be extended further with respect to what is assumed in the benchmark structure; and (ii) uninsurable entrepreneurial risk is not a necessary ingredient for obtaining the main results.

Consider the following alternative asset structures.

Alternative Structure 2: more downward insurance The second alternative assumes that all downward risk is insurable to agents, except the uninsurable *entrepreneurial risk*, as we interpreted the transition from income state 7 to income state 6. Hence, formally, $\mathcal{A}^3(y_i) = \{y_1, ..., y_{i-1}\}, \forall i = \{2, ..., 6\}$; $\mathcal{A}^3(y_7) = \{y_1, ..., y_5\}$; and $\mathcal{A}^3(y_1) = \emptyset$.

Alternative Asset Structure 3: no entrepreneurial risk The third alternative assumes that all downward risk is insurable to agents, including the entrepreneurial risk Hence, formally, $\mathcal{A}^4(y_i) = \{y_1, ..., y_{i-1}\}, \forall i = \{2, ..., 7\}$; and $\mathcal{A}^4(y_1) = \emptyset$.

Alternative Asset Structure 4: no insurance at the bottom The forth alternative assumes that it is the lowest states that are less insurable, while the top income states can enjoy downward insurance. Formally: $\mathcal{A}^4(y_i) = \{y_{i-1}\}, \forall i = \{3, ..., 7\}$ and $\mathcal{A}^4(y_1) = \mathcal{A}^4(y_2) = \emptyset$.

	Aiyagari	Cost =0.04				
		Benchmark	Sym.	more downward ins.	no entrepr. risk	no ins. bottom
		$\mathcal{A}$	$\mathcal{A}^1$	$\mathcal{A}^2$	$\mathcal{A}^3$	$\mathcal{A}^4$
Particip. (%)	0	85.9	26.1	87.0	87.9	87.9
Gini index	0.58	0.77	0.65	0.85	0.86	0.81

Table 5 – Implications of Asset structures

Note: This table displays aggregate participation and the wealth Gini index for the Aiyagari model (first column) which is independent on the assumed asset structure as no agents acquire state contingent assets for large enough costs, and in the endogenous partial insurance model when assuming $\kappa = 0.04$, for the benchmark asset structure $\mathcal{A}$, and for the four alternative structures $\mathcal{A}^1$, $\mathcal{A}^2$, $\mathcal{A}^3$, $\mathcal{A}^4$.

As Table 5 displays, the increased inequality in the endogenous partial insurance model with respect to the standard Aiyagari model does not primarily come from the direction of the asymmetry of the insurance structure or from the existence of some uninsurable risk for the highest income agents. At the end, what is more important and drive the results is how the possibility of insurance, and therefore the *participation channel* affects asset accumulation.

6 Further extensions and discussion

Contingent assets in "real life" In our framework, the only exogenous shock is on labor endowment, which, given the nature of the model, is equivalent to labor income. If it is difficult to think about insurance products that help to hedge against wage shock for an employee of a firm, it is more natural to think about insurance products for health shocks or unemployment shocks – in the case where these kinds of shocks can be insured voluntarily.

Note that, interestingly, this possibility of voluntary insurance still remains even in countries where there exist compulsory public insurance. An example is France where health shocks are partially insured through a public system, they can be further insured by an additional private system, for which the individual has some decision power.

Heterogeneity in insurance However, the number of examples where we can clearly identify the distribution of insurance is quite limited. When we are able to observe these insurance decisions, one can observe a lot of heterogeneity in the population, sometimes connected with observable variables as wealth or income. For example, Brown and Finkelstein (2007) document lower private long-term care insurance coverage for poorer households. An additional example is provided by Cole et al. (2009) who show that credit constraints are a key determinant of insurance when studying insurance decision of Indian farmers against rainfall variability.

Interpreting participation costs. A first interpretation of participation costs is a monetary one. These monetary costs arise from financial or insurance intermediaries, possibly related to sunk costs due to an intermediaries' production functions or to screening costs, when agents have to signal their type by willing to pay the fixed costs.[27]

[27]The exact setting leading to this kind of fixed cost would be a dynamic version of Rothschild and Stiglitz (1976).

A second interpretation of participation costs is that they include cognitive costs or shopping-costs: selecting insurance requires time and effort. In line with this approach is the observation by Cole and Shastry (2009) that education is also an important determinant of insurance decisions.

Another alternative form of fixed cost faced by households surfaces when collecting insurance payments when bad shocks occur. Collection requires proofs of damage to address the adverse selection problem. Assuming this alternative form of participation cost would not qualitatively change our results: it would also prevent agents from purchasing insurance against small shocks, and would lead to preferences for purchasing insurance only against large shocks. In this situation, as in our setting, poorer households cannot afford to pay the insurance.

Connection with the lack of commitment model. Another model of limited insurance is the lack of commitment model. In this model, agents' inability to commit to repay limits risk sharing among agents. For example, in its one-sided no commitment version – agents cannot commit to repay their debts but they can save using reliable instruments (see Thomas and Worrall (1988) as an example) –, this inability to commit leads to short-selling or borrowing constraints only prevent households from borrowing against future revenue. In this model, insurance is always possible but its degree can potentially be limited.

This is the main difference with our approach: in our case, lack of insurance is not along the intensive margin but along the extensive one: either some assets are simply not available or agents prefer not to pay the fixed cost to access insurance instruments. In that regard, our approach is closer to Aiyagari (1994).

This difference has important consequences, at least in comparison with the one-sided lack commitment model. Indeed, this model (see Thomas and Worrall (1988) as an example) fails to reproduce lack of downward insurance: short-selling or borrowing constraints only prevent households from borrowing against future revenue and not from accumulating assets for insuring against lower future income.

7 Concluding remarks

In this paper, we study the endogenous partial-insurance equilibrium that characterizes an economy with participation cost in state-contingent asset markets. In this setting households' degree of insurance depends on their wealth. In fact, when preferences feature decreasing absolute risk aversion, the partial-insurance equilibrium is characterized by a set of poor households that are not able to obtain any insurance, by a set of middle-class household that actively participate in the contingent asset market and, hence, are relatively well insured, and, interestingly, by a set of rich households that prefer to self insure by accumulating a large stock of the risk-free assets.

After characterizing the endogenous partial insurance equilibrium from a theoretical point of view, we explore its quantitative implications. We show that in presence of participation costs

such that the equilibrium features partial insurance, our model leads to important implications about wealth inequality, wealth concentration, asset prices, and consumption smoothing. Specifically, when participation costs reduce from an arbitrary large value, such that the economy is equivalent to an Aiyagari (1994) model, to lower values, such that the economy turns into a partial-insurance equilibrium, wealth inequality increases and interest rates rise as a result of a overall lower demand for assets. There are two main channels that rationalize our findings. First, with endogenous partial insurance middle-class households acquire insurance and, therefore, do not have incentive to accumulate a lot of assets; on the contrary, richest households do not participate in the state contingent asset market and have then a motive to accumulate asset. This *participation channel* contributes to the thickening of the upper tail of the wealth distribution. Second, a general equilibrium effect, labelled *interest rate channel* and for which higher insurance participation leads to lower asset demand and higher interest rate, reinforces the skewness of the wealth distribution. The higher interest rate penalizes non-insured poor households, which mainly face upward risk and have incentive to dissave, while it benefits richer agents as they can enjoy higher returns from their financial wealth.

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A Proofs of propositions - Not for publication

A.1 Proof of Proposition 1.

Proof. Proof of 1. Assume $b^N = \frac{W-y}{2}$. The left hand side of equation (3) becomes: $u'\left(\frac{W+y}{2}\right)$. The right hand side becomes: $pu'\left(\frac{W+y}{2} - L\right) + (1-p)u'\left(\frac{W+y}{2} + \frac{pL}{1-p}\right)$. If $L = 0$ the right hand side is equal to $u'\left(\frac{W+y}{2}\right)$ and the equation is satisfied. If $L > 0$, then the right hand side is equal to $\mathbb{E}_0 u'\left(\frac{W+y}{2}\right)$. By assumption, since $u'''(\cdot) > 0$, the marginal utility is convex, and, therefore, $u'(\frac{W+y}{2}) < \mathbb{E}_0 u'\left(\frac{W+y}{2}\right)$. Hence, $b^N = \frac{W-y}{2}$ is not a solution of (3) when $L > 0$. Now, since the left hand side of (3) is monotonically increasing in b, and the right hand side is monotonically decreasing in b, by the intermediate value theorem, there exists $b^N > \frac{W-y}{2}$ that satisfies (3). Since $u'''(\cdot) > 0$ it is trivial to prove the *only if* direction of the statement. Taking the limit of the right hand side of (3) for W going to infinity, we have:

$$\begin{aligned}\lim_{W\to\infty} pu'(y - L + b^N(W)) + (1-p)u'\left(y + \frac{pL}{1-p} + b^N(W)\right) &= \lim_{W\to\infty} pu'(b^N(W)) + (1-p)u'\left(b^N(W)\right) \\ &= \lim_{W\to\infty} u'(b^N(W)),\end{aligned}$$

since y, L, and p are all constant and $b^N(W)$ is increasing in W. Then, by equation (3):

$$\lim_{W\to\infty} u'(W - b^N(W)) = \lim_{W\to\infty} u'(b^N(W)),$$

which implies

$$\lim_{W\to\infty} b^N(W) = \frac{W}{2} = \infty.$$

Proof of 2. Compute $\frac{\partial b^N(W)}{\partial W}$ by applying the implicit function theorem on equation (3). We obtain

$$\frac{\partial b^N(W)}{\partial W} = -\frac{u''(W - b^N)}{-u''(W - b^N) - pu''(y - L + b^N) - (1-p)u''\left(y + \frac{pL}{1-p} + b^N\right)}. \tag{11}$$

By assumption of concavity of the utility function, $u''(\cdot) < 0$ and $\frac{\partial b^N(W)}{\partial W} > 0$. Finally, notice that when $L = 0$, $b^N = \frac{W-y}{2}$, by part 1, and consequently $W - b^N = y + b^N$. In this case $\frac{\partial b^N(W)}{\partial W} = -\frac{u''(W-b^N)}{-2u''(W-b^N)} = \frac{1}{2}$.

In the proof of part 1, we have shown that: $\lim_{W\to\infty} b^N(W) = \frac{W}{2}$. It follows directly that $\lim_{W\to\infty} \frac{\partial b^N(W)}{\partial W} = \frac{1}{2}$. Hence, we have that $\forall W \geq \underline{W}$, $0 < \frac{\partial b^N(W)}{\partial W} \leq \frac{1}{2}$.

Proof of 3. Assuming $L > 0$, then we have the following case:

(a) If the utility is CARA, then by definition $u''(W) = -zu'(W)$. Substituting this equivalence for each term in equation (11) and considering that in equilibrium (3) must hold, the derivative of interest becomes:

$$\frac{\partial b^N(W)}{\partial W} = -\frac{-zu'(W - b^N)}{2zu'(W - b^N)} = \frac{1}{2}.$$

(b) Since $\frac{\partial b^N(W)}{\partial W}$ is increasing and tends to $\frac{1}{2}$ for W going to infinity, it follows that $\forall W \geq \underline{W}$, $\frac{\partial b^N(W)}{\partial W} < \frac{1}{2}$.

□

A.2 Proof of Proposition 2.

Proof. Proof of 1. The gain of insurance with no costs is: $G(W, 0) = u\left(\frac{W+y}{2}\right) - \frac{1}{2}u(W - b^N) - \frac{1}{2}\left[pu\left(y + b^N - L\right) + (1-p)u\left(y + b^N\right.\right.$
By concavity of utility function,

$$\left[pu\left(y + b^N - L\right) + (1-p)u\left(y + b^N + \frac{pL}{1-p}\right)\right] < u(y + b^N),$$

which implies $G(W,0) > u\left(\frac{W+y}{2}\right) - \frac{1}{2}u(W-b^N) - \frac{1}{2}u(y+b^N)$. Since for any feasible W, by Proposition 1, $b^N > \frac{W-y}{2}$,

$$G(W,0) > u\left(\frac{W+y}{2}\right) - \frac{1}{2}u(W-b^N) - \frac{1}{2}u(y+b^N) > u\left(\frac{W+y}{2}\right) - \frac{1}{2}u\left(\frac{W+y}{2}\right) - \frac{1}{2}u\left(\frac{W+y}{2}\right).$$

Hence, $G(W,0) > 0$, for any feasible W.

Proof of (2). Computing derivative $\frac{\partial G(W,0)}{\partial W}$ and using the envelope theorem, we obtain:

$$\frac{\partial G(W,0)}{\partial W} = \frac{1}{2}u'\left(\frac{W+y}{2}\right) - \frac{1}{2}u'(W-b^N).$$

Using Proposition 1, $b^N > \frac{W-y}{2}$ and $W - b^N < \frac{W+y}{2}$. Since by assumption $u'(\cdot)$ is decreasing, $\frac{\partial G(W,0)}{\partial W} < 0$.

Proof of 3. Rewrite

$$\begin{aligned} G(W,0) = \frac{1}{2}\left[u\left(\frac{W+y}{2}\right) - u(W-b^N)\right] + \frac{1}{2}\left[u\left(\frac{W+y}{2}\right) - u\left(y+b^N+\frac{pL}{1-p}\right)\right] + \\ + \frac{p}{2}\left[u\left(y+b^N+\frac{pL}{1-p}\right) - u(y+b^N-L)\right]. \end{aligned}$$

Recall that concavity of $u(\cdot)$ implies that $u(x) - u(x_0) < u'(x_0)(x-x_0)$. Then, bounding above the terms in square brackets we have:

$$\begin{aligned} G(W,0) < \frac{1}{2}u'(W-b^N)\left[-\frac{W-y}{2}+b^N\right] + \frac{1}{2}u'\left(y+b^N+\frac{pL}{1-p}\right)\left[\frac{W-y}{2}-\frac{pL}{1-p}-b^N\right] + \\ + \frac{p}{2}u'(y+b^N-L)\left[\frac{L}{1-p}\right]. \end{aligned}$$

Using (3) to replace $u'(W-b^N)$ and collecting terms, we obtain:

$$\begin{aligned} G(W,0) &< \frac{p}{2}\left[\frac{W-y}{2} - b^N - \frac{L}{1-p}\right]\left[u'\left(y+\frac{pL}{1-p}+b^N\right) - u'\left(y-L+b^N\right)\right] \\ &< \frac{p}{2}\left[b^N - b^N - \frac{L}{1-p}\right]\left[u'\left(y+\frac{pL}{1-p}+b^N\right) - u'\left(y-L+b^N\right)\right], \end{aligned}$$

where in the last inequality we made use of the result in Proposition 1 stating that $\frac{W-y}{2} < b^N$. Hence,

$$G(W,0) < \frac{p}{2}\left[-\frac{L}{1-p}\right]\left[u'\left(y+\frac{pL}{1-p}+b^N\right) - u'\left(y-L+b^N\right)\right].$$

By Inada conditions, $\lim_{x\to\infty} u'(x) = 0$ and since, by Proposition 1 $\lim_{W\to\infty} b^N(W) = \infty$, then $\lim_{W\to\infty} G(W,0) = 0$ □

A.3 Proof of Proposition 3.

Proof. The proof of (1) follows three steps. First, we prove that the function $G(W,\kappa)$ has one and only one minimum at a wealth level W^*. Second, we prove that $G(W^*,\kappa) < 0$. Third, we prove that under the condition for the participation cost $\kappa < \hat{\kappa}$, there exists a unique threshold level $\overline{W}$.

Differentiating definition (4) with respect to W and using the envelope theorem, we obtain:

$$\frac{\partial G(W,\kappa)}{\partial W} = \frac{1}{2}u'\left(\frac{W+y}{2} - \frac{\kappa}{2}\right) - \frac{1}{2}u'(W-b^N)$$

For a wealth level $W^*(\kappa)$ such that $b^N(W^*(\kappa)) = \frac{W-y}{2} + \frac{\kappa}{2}$, we have $\frac{\partial G(W^*(\kappa),\kappa)}{\partial W} = 0$. Since the utility features DARA, then $b^N(W)$ is monotonically increasing, as shown in Proposition 1; then $W^*(\kappa)$ is unique, if it exists. Its existence will be proven in the third step.

To show that the point $W^*(\kappa)$ is a global minimum for the function $G(W,\kappa)$, notice that the term $\left(\frac{W+y}{2} - \frac{\kappa}{2}\right)$ grows with W at the rate $\frac{1}{2}$, while the term $W - b^N$ grows at the rate $\left(1 - \frac{\partial b^N(W)}{\partial W}\right) > \frac{1}{2}$, since the utility

function is DARA. Hence, since $u'(\cdot)$ is decreasing by assumption, for any $W < W^*(\kappa)$, $\frac{\partial G(W,\kappa)}{\partial W} < 0$, and for any $W > W^*(\kappa)$, $\frac{\partial G(W,\kappa)}{\partial W} > 0$. Therefore, the function $G(W,\kappa)$ attains a minimum at $W^*(\kappa)$.

Next, notice that with a similar proof than part 3 of Proposition 2, it holds that $\lim_{W\to\infty} G(W,\kappa) = 0$. It follows that since $G(W,\kappa)$ admits exactly one minimum, $W^*(\kappa)$, since it is decreasing for any $W < W^*(\kappa)$, it is increasing for any $W > W^*(\kappa)$, and it converges to 0 when W goes it ∞, then necessarily $G(W^*(\kappa),\kappa) < 0$. We have proved that the minimum of $G(W,\kappa)$ is negative and that for any $W > W^*(\kappa)$, $G(W,\kappa) < 0$.

As a third step, let $\hat{\kappa}$ be the value of the cost that solves: $G(\underline{W},\hat{\kappa}) = 0$. Recall that, by Proposition 2, $G(\underline{W},0) > 0$. Also, notice that for $G\left(\underline{W}, 2\left[b^N(\underline{W}) - \frac{\underline{W}-y}{2}\right]\right) < 0$, by the condition above. Hence since $G(\underline{W},\kappa)$ is monotonically decreasing in κ, by the intermediate value theorem, $\exists!\ \hat{\kappa} \in \left(0, 2\left[b^N(\underline{W}) - \frac{\underline{W}-y}{2}\right])\right)$: $G(\underline{W},\hat{\kappa}) = 0$. Then, for any κ such that $\kappa < \hat{\kappa}$, then $G(\underline{W},\kappa) > 0$ and $G(W,\kappa)$ reach a negative value at its minimum, W*, which necessarily exists; hence, by the intermediate value theorem, exists a unique $\overline{W}(\kappa) < W^*(\kappa)$ such that $G(\overline{W}(\kappa),\kappa) = 0$.

The proof of (2) comes easily. First, by Proposition 2, for any feasible W, $G(W,0) > 0$. Hence, $\forall W > \underline{W}$, $V^P(W,\kappa) > V^N(W)$ and by definition $\mathbb{P}(0) = \{W : W > \underline{W}\}$.

Second, recall that W^* is determined by the condition: $u'\left(\frac{W^*(\kappa)+y}{2} - \frac{\kappa}{2}\right) - u'(W^*(\kappa) - b^N(W^*(\kappa))) = 0$. Applying the implicit function theorem and the fact that $\frac{W^*(\kappa)+y}{2} - \frac{\kappa}{2} = W^*(\kappa) - b^N(W^*(\kappa))$ and that $\frac{\partial b^N}{\partial W} \leq \frac{1}{2}$, we obtain $\frac{\partial W^*(\kappa)}{\partial \kappa} < 0$. In addition, since also $G(\underline{W},\kappa)$ is decreasing in κ, it must be that $\frac{\partial \overline{W}(\kappa}{\partial \kappa} < 0$. Therefore, $\forall \kappa_2 > \kappa_1$, $\mathbb{P}(\kappa_2) \subset \mathbb{P}(\kappa_1)$.

Finally, as κ increases above $\hat{\kappa}$, $G(\underline{W},\kappa) < 0$, and, therefore, $\forall W > \underline{W}$, $G(W,\kappa) < 0$, since the function $G(W,\kappa)$ starts at negative value, decreases to W^*, and than converges to zero from below from W going to infinity. In this case $\forall W > \underline{W}$, $V^P(W,\kappa) < V^N(W)$ and by definition $\mathbb{P}(\kappa) = \emptyset$. □

A.4 Proof of Proposition 4.

The proof consists of four steps.

In the fist step the key idea is transform our problem in a standard optimal saving/income fluctuation problem with an augmented income process. Consider the budget constraint:

$$c + q^f B' + \delta\left(\sum_{y'\in\mathcal{A}(y)} q(y'|y)^\intercal a'(y') + \kappa\right) = wy + B + a.$$

It can be rewritten as

$$c + q^f B' = \tilde{y}(\delta, y, a'(y')) + B(y), \tag{12}$$

where we have defined the augmented income process $\tilde{y}$ as:

$$\tilde{y}(\delta, y, a'(y')) = wy - \delta\left(\sum_{y'\in\mathcal{A}(y)} q(y'|y)^\intercal a'(y') + \kappa\right) + a.$$

We interpret the function $\tilde{y}(\cdot)$ as an augmented income process, which is function of the exogenous income current realization, y, the participation decision, δ, and of the quantity of state contingent assets purchased, $a'(y')$. The latter two variables are clearly endogenous decisions, but one can think that these decisions, together with future realization of nature, map present augmented income into next period augmented income. Notice that conditional on participation and asset holding decision, which are chosen at time t, the augmented income process is a Markov process. In the rest of the proof we just refer to the augmented income process as $\tilde{y}$, since it is understood that it is a function of endogenous variables.

The second step is to show the following lemma;

Lemma 8. *For any participation decision δ, when contingent asset holdings are bounded, the domain of $\tilde{y}$, which we define as $\tilde{Y}$, is finite.*

Proof. Assume $\delta = 0$. Then $\tilde{y} = wy + a$. In this case $\tilde{y}$ is finite when a is finite.

Now assume $\delta = 1$, the highest value for the augmented income, which we denote by $\tilde{N}$, is

$$\tilde{y}_{\tilde{N}} = wy_N + a - \left(\sum_{y' \in \mathcal{A}(y_N):} q(y'|y_N)^\intercal \bar{a} + \kappa \right).$$

Intuitively, the augmented income is bounded above by the fact that asset holding is bounded below by $\bar{a}$. Similarly, for the worst income realization y_1, we have:

$$wy_1 - \left(\sum_{y' \in \mathcal{A}(y_1)} q(y'|y_1)^\intercal a'(y') + \kappa \right) + a$$

which is bounded when a remains bounded. □

Hence, given the states, the household budget constraint can be rewritten as in (12), in which the augmented income state is finite, for any possible participation decision and asset holding decision. Therefore, as third step, we can apply directly Proposition 4 in Açıkgöz (2018), which proves that the state space for the risk free asset B can be written as a compact state, so that exist a finite $\bar{\bar{B}} > 0$ such that $B'(B, a, y) < \bar{\bar{B}}$ for all $\tilde{y} \in \tilde{Y}$, and for all $B > \bar{\bar{B}}$. Therefore, the following lemma follows directly from our definition of the augmented income process and from Açıkgöz (2018)'s results.

Lemma 9. *If a is bounded, then B is bounded as well.*

The last step is to show that a cannot be unbounded. For this step it is useful to show the following preliminary result:

Lemma 10. *If a is unbounded, then B is unbounded as well.*

Proof. Let us show this by contradiction. Notice that state contingent asset position is bounded below by $\bar{a}$ by assumption. We can bound above the highest possible value for one state contingent asset. Consider an asset that repays only if tomorrow income state is $y' = y_j$. The possible highest labor income is y_N. The non-negativeness of consumption implies that:

$$wy_N - q(y_j|y_N)a'(y_j) - \left(\sum_{y' \neq y_j \in \mathcal{A}(y_N)} q(y'|y_N)^\intercal \bar{a} \right) - \kappa + a - q^f \bar{B} + B \geq 0,$$

which implies that:

$$a'(y_j) \leq \frac{wy_N - \left(\sum_{y' \neq y_j \in \mathcal{A}(y_N)} q(y'|y_N)^\intercal \bar{a} \right) - \kappa + a - q^f \bar{B} + B}{q(y_j|y_N)}$$

Hence, all the state contingent asset are bounded below and above when B is bounded. □

Finally it is easy to show that a cannot be indeed unbounded. In fact, if B were unbounded, given the assumptions on preferences and for any positive κ, there would exist a level of B for which agents are better off not participating and thus $a = 0$ for any of these points. Thus, a cannot be unbounded.

Then the state space for a is bounded and close; therefore, by the Heine-Borel Theorem, it is compact. The same is then obtain from Lemma 9 for B.

A.5 Proof of Proposition 5.

Using Proposition 4, we can restrict our focus on a compact set for assets. As the set of income is also compact and given that V is a continuous function of all these variables, the relevant set of payoffs for the value function is bounded. Thus, we can use standard results for bounded utility functions and the proposition below establishes the existence and the uniqueness of the value function solving Problem 2.

Proposition 11. *The value function V exists and is unique.*

Moreover, the value function V can be obtained by iterations: for any initial value $V' \in \Omega$ and defining the sequence, $V_n = T^n V'$, V_n converges to V.

Proof. This proof extends the proof of Stokey et al. (1989) for discrete variables. It is useful to write the problem in 2 in a more compact way, that is:

$$
\begin{aligned}
V(B, \{a(y)\}, y) = \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta} & \left[u(c) + \beta \sum_{y'} \pi(y'|y) V(B', \{a'(y')\}, y') \right] \\
\text{s.t. } & c + \delta \left(\sum_{y' \in \mathcal{A}(y)} q(y'|y)^{\intercal} a'(y') + \kappa \right) + q^f B' \leq wy + B + a(y), \\
& B' \geq -\overline{B};\ a'(y') \geq \bar{a} \quad \forall y' \in \mathcal{A}(y).
\end{aligned}
$$

For simplicity, we will refer to the the constraints as *cnstr*. Defining T as:

$$
TV = \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta \text{ s.t. } cnstr} \left[u(c) + \beta \sum_{y'} \pi(y'|y) V(B', \{a'(y')\}, y') \right]
$$

it is easy to show that T satisfies Blackwell's conditions. First T is monotonic. For $W \leq V$, we have that :

$$
\begin{aligned}
TW &= \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta \text{ s.t. } cnstr} \left[u(c) + \beta \sum_{y'} \pi(y'|y) W(B', \{a'(y')\}, y') \right] \\
&\leq \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta \text{ s.t. } cnstr} \left[u(c) + \beta \sum_{y'} \pi(y'|y) V(B', \{a'(y')\}, y') \right] \\
&= TV
\end{aligned}
$$

Second T discounts. Let Γ be a positive constant:

$$
\begin{aligned}
T(V + \Gamma) &= \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta \text{ s.t. } cnstr} \left[u(c) + \beta \sum_{y'} \pi(y'|y) \left(V(B', \{a'(y')\}, y') + \Gamma \right) \right] \\
&= \max_{\{a'(y')\}_{y' \in \mathcal{A}(y)}, B', \delta \text{ s.t. } cnstr} \left[u(c) + \beta \left[\Gamma + \sum_{y'} V(B', \{a'(y')\}, y') \right] \right] \\
&= TV + \beta\Gamma
\end{aligned}
$$

We define $X = \{x = \{B', a'(y'), y'\}\}$. Ω denotes the set of functions V such that V is continuous with respect to B and $a(y)$. We need also to prove that:

- Ω with the d_∞ metric is a metric space.
- TV is in the same set as V, which is obvious.

Metric space Let $\{V_n\}$ a Cauchy sequence of Ω. For every $x \in X$, $V_n(x)$ converges to $V(x)$. Let us verify that V is the limit using the d_∞ metric. As $\{V_n\}$ a Cauchy sequence: for some $\epsilon > 0$ and for some $x \in X$, there exists n such that for every p and q satisfying $q \geq p > n$, $|V_p(x), V_q(x)| < \epsilon$. Taking the limit of this expression with respect to q, we obtain that $|V_p(x), V(x)| < \epsilon$. As this is true for every $x \in X$, this implies that $d_\infty(V_p, V)$ converges to 0, which means that V_n converges to V.

Conclusion The requirements of the Contraction Mapping theorem are satisfied. There exists an unique $V \in \Omega$ such that $TV = V$. Furthermore, for any $V' \in \Omega$ and defining $V_1 = TV'$ and, more generally, $V_n = T^n V'$, V_n converges to V. This makes possible iterations on the value function as usual. □

The connexion between being solution to Problem 1 and to Problem 2 easily obtains from standard results, at least in the case of bounded utility function (see Stokey et al., 1989). Indeed, in that case, the discrete participation choice does not prevent $\lim_{n\to\infty}\sum_{t=0}^{n}\beta^t u(c_t)$ to exist (and be finite), which allows to use Theorems 4.2 to 4.5 in chapter 4, thus guaranteeing the equality between the two solutions.

A.6 Proof of Proposition 7.

The first step of the proof is to show that we can use results from Açıkgöz (2018) with the augmented income process as in the proof of Proposition 4. Assumptions 1, 2, 3 guarantee that the assumptions on utility and prices Açıkgöz (2018) are satisfied. In addition, given our assumed production function in equation 5 guarantees that Assumption 5 in Açıkgöz (2018) is also satisfied. We now show that the augmented process $\tilde{y}$, defined in , satisfies the following lemma:

Lemma 12. *Assume that $\pi(y_1|y_1) > 0$, which means that the lowest income state has some persistent. Then, $\pi(\tilde{y}_1|\tilde{y}_1) > 0$, that is the lowest state of the augmented income process is also persistent.*

Proof. Recall that the augmented income process is:

$$\tilde{y} = wy - \delta\left(\sum_{y'} \in \mathcal{A}(y)q(y'|y)^{\intercal}a'(y') + \kappa\right) + a.$$

Assume the current income is at the lowest state, i.e. $y = y_1$. if $\delta_{-1} = 0$ and $\delta = 0$, which means no participation in the previous and current period, than $\tilde{y}_1 = wy_1$ and, by assumption, $\pi(\tilde{y}_1|\tilde{y}_1) > 0$. Now consider the case in which the previous period there was no participation, $\delta_{-1} = 0$, and the agent optimally participates in the current period, $\delta = 1$. In this case, the the augmented income process is: $\tilde{y} = wy_1 - \left(\sum_{y'} \in \mathcal{A}(y)q(y'|y)^{\intercal}a'(y') + \kappa\right)$. The agent has only upside risk and therefore, to smooth consumption she will borrow today to repay debt in the next period that means that $\tilde{y} > wy_1$ in this case. However, since the lowest income state is persistent, there is a sequence of unlucky events for which for several periods the agent will remain in the lowest state, for which he won't get repayment. Notice that this is the only optimal behaviour to smooth consumption, as repayment in the future lowest state, means saving in the current lowest state. In the presence of upside risk and non-zero probability to have higher income, it would contrast with consumption smoothing incentives driven by the concavity of the utility function. If the agent keeps participating and keeps receiving the lowest income his wealth will run down, since in each period the agent has to pay the participation cost. Therefore, eventually the agent either will stop participating, and we go back to the previous case, or he become borrowing constraint at $\bar{B}$ and, if it is optimal to participate in that case, a sequence of low income realization will lead to a constant asset participation decision and therefore to a constant $\tilde{y}_1$. Even in this case, $\tilde{y}_1$ is persistent. These two case are the only relevant cases, since, it is trivial to show that in a sequence of low income realization, when running down assets, either the agent consistently participates or consistently does not participate, once she has reached the borrowing limit. □

A first conclusion is that when the conditions of Lemma 12 are satisfied, $q^f > \beta$, following Proposition 6 in Açıkgöz (2018).

Furthermore, given that net purchases of contingent assets are invested in capital and that capital is supply competitively, we have that, for any $y \in Y$ and for any $y' \in \mathcal{A}(y)$, $q(y'|y) = q^f\pi(y'|y)$. More specifically, the program of firms is:

$$\max\left\{A_tK_t^{\alpha}L_t^{1-\alpha} - w_tL_t - \sum_{y\in Y}\sum_{y'\in\mathcal{A}(y)}\pi(y'|y)\sum_{\{a,B\}\in A}a'(y'|(y,a,B)) - \sum_{y\in Y}\sum_{\{a,B\}\in A}B'(y,a,B)\right\}$$

where $a'(y'|(y,a,B))$ is the amount of contingent asset to state y' purchased when current state variables $\{y,a,B\} \in S$.

The resulting first order conditions are:

$$
\begin{aligned}
&w_t = (1-\alpha)A_t \left(\frac{K_t}{L_t}\right)^{\alpha}, \\
&q^f \alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} = 1, \\
&q(y'|y)\alpha A_t \left(\frac{K_t}{L_t}\right)^{\alpha-1} = \pi(y'|y), \quad \text{for all } y \in Y \text{ and all } y' \in \mathcal{A}(y).
\end{aligned}
$$

In particular, we obtain that $q(y'|y) = q^f \pi(y'|y)$.

Existence of an equilibrium With this definition in hands, we can obtain the following lemma, based on results in the literature:

Lemma 13. *Given an insurance decision $\delta : S \to \{0,1\}$ and $a' : S \to \mathcal{R}^{dim(\mathcal{A})}$, there exists a stationary recursive competitive equilibrium conditional on a insurance decision.*

Proof. Because: (i) the assumed utility function is continuously differentiable, strictly increasing and concave; (ii) it satisfies Assumption 1 and Assumption 2; (iii) the assumed production function in equation (5) is Cobb-Douglas, and (iv) Lemma 12 holds; then, all the assumptions for Theorem 1 in Açıkgöz (2018) are satisfied, and, therefore, there exists a stationary recursive competitive equilibrium conditional on insurance decision. □