Optimal equity auctions with two-dimensional types*

Tingjun Liu
Faculty of Business and Economics
The University of Hong Kong
tjliu@hku.hk

Dan Bernhardt
University of Illinois, danber@illinois.edu
University of Warwick, m.d.bernhardt@warwick.ac.uk

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Abstract

We analyze the design and performance of equity auctions when bidder’s valuations and opportunity costs are private information, distributed according to an arbitrary joint density that can differ across bidders. We identify, for any incentive compatible mechanism, an equivalent single-dimensional representation for uncertainty. We then characterize the revenue-maximizing and surplus-maximizing equity mechanisms, and compare revenues in optimal equity and cash auctions. Unlike in cash auctions, the adverse selection arising from bidders’ two-dimensional private information in equity auctions can lead to a global violation of the regularity condition, which represents a maximal mismatch between incentive compatibility and maximization of revenue or surplus. Such mismatch can lead a seller to exclude bidders and demand a bidder-specific stake from a non-excluded bidder; or separate bidders into groups, and sell to them sequentially.

Keywords: Dimensionality reduction; Adverse selection; Global violation of regularity

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1 Introduction

A central premise underlying the use of auctions is that bidders have private information that affects both their payoffs and a seller’s revenue. In practice, bidder information is often multi-dimensional. Concretely, a bidder may have private information about both the gross revenues that it can generate with an auctioned asset and its opportunity cost. For instance, an acquiring firm in a corporate takeover may be privately informed about its synergy with the target and its own standalone value; and in project-rights auctions bidders are often privately informed about the opportunity costs associated with alternative projects to pursue.\footnote{Examples include auctions to sell oil or timber leases (e.g., Hendricks and Porter 1988, Paarsch 1997), auctions for highway building contracts; DKS and Skrzypacz (2013) provide other examples.}

In auctions where bidders pay with cash, if the asset for sale is indivisible, the sole determinant of a bidder’s strategy and seller revenues is the bidder’s net valuation, i.e., the difference between gross revenues and costs.\footnote{With divisible assets, if a bidder’s opportunity cost of acquiring a quantity is not proportional to its gross valuation of the quantity, the two dimensions of uncertainty need not collapse to one dimension.} However, when bidders pay with securities that tie payments to the revenues generated (Hansen 1985; DeMarzo, Kremer and Skrzypacz, DKS 2005), because the winner retains only a share of revenues but incurs all opportunity costs, the revenue-cost composition affects bidding strategies. Bidders with different revenue-cost compositions may select the same bid, even though the bid’s value to a seller can vary. Thus, the multi-dimensional informational structure affects seller revenue in security-bid auctions.

Illustrating the potential consequences, Che and Kim (2010) show that when a bidder’s opportunity cost rises deterministically with the expected total cash flow sufficiently quickly, an extreme form of adverse selection arises in standard second-price security-bid auctions: bidders with higher NPVs bid less, resulting in low seller revenues. This suggests that securities auctions should be designed with care when opportunity costs are private information.

We derive the optimal equity auction design when bidders’ values and opportunity costs are private information.\footnote{Equity auctions are the most common form of security auctions: Andrade et al. (2001) report that 58% of mergers and acquisitions are paid entirely in equity. Faccio and Masulis (2005) and Eckbo et al. (2015) document reasons underlying the exclusive use of equity.} We allow for an arbitrary joint density over the expected cash flows \(u_i\) and opportunity cost \(x_i\) of a bidder \(i\), requiring only independence across bidders, mild continuity conditions on the density, and a compact, connected support. Thus, the extent
of adverse selection is arbitrary, and can vary across bidders.

We first find a transformation of a bidder’s objective that simplifies the resulting envelope condition.\textsuperscript{4} We show that in any incentive-compatible mechanism, an agent’s preferences are determined only by $r_i \equiv \frac{\omega_i}{v_i}$, the ratio of his opportunity cost to expected project revenue. This ratio represents the equity stake the agent needs to break even. While an agent’s preferences are determined only by this one-dimensional summary of his two-dimensional type, it does not guarantee that two type pairs with the same break-even equity stake are pooled in equilibrium. Nonetheless, a novel application of the envelope theorem reduces the design problem to a one-dimensional problem. It follows that almost all types of a bidder with the same $r_i$ have the same equilibrium probability of winning and the same expected equity share retained—even if project revenues net of opportunity costs are very different for these types.

These results yield a general characterization of any incentive-compatible equity mechanism that we use to analyze revenue-maximizing and surplus-maximizing mechanisms. When a seller seeks to maximize revenues, we use this one-dimensional representation to express a bidder’s virtual valuation as a function of $r_i$ only, denoted $\phi(r_i)$. Different bidder types with the same $r_i$ have the same virtual valuation, which depends on the aggregate properties of all types with that $r_i$. Expected seller revenues decompose into the sum of a component that reflects the virtual valuation of the winning bidder and a component that reflects the rents obtained by a bidder type that needs the highest equity share to break even.

Under the regularity condition that $\phi_i(r_i)$ decreases in $r_i$, incentive compatibility and virtual valuations move in the same direction. We show that the resulting optimal selling mechanism has the features that (1) a bidder with the highest virtual valuation wins whenever his virtual valuation exceeds the seller’s valuation, and (2) a bidder type with the highest break-even equity stake extracts no rents. When the design problem is not regular, a concavification argument similar to that in Myerson (1981) is used to solve for the optimal mechanism. In the optimal design, the winning bidder has the highest adjusted virtual valuation, and a bidder type with the highest break-even equity stake still earns no rents.

\textsuperscript{4}The original, un-transformed objective contains a quantity that is the product of a term that depends on a bidder’s private type, and two terms that are endogenously determined in equilibrium—the equity share $q_i$ and the probability of winning $G_i$. The product of these three terms complicates the envelope condition. We transform the objective function so that the term that depends on a bidder’s private type is multiplied by only one endogenously-determined term, resulting in a tractable envelope condition.
With two-dimensional uncertainty, the regularity condition in equity auctions becomes demanding. Virtual valuations measure available rents, and thus tend to grow with a bidder’s net valuation \( v_i - x_i \). In cash auctions, the relevant bidder type is \( v_i \), so the regularity condition, which mandates that virtual valuations increase with bidder type, holds under mild assumptions—a monotone hazard condition on the distribution of types suffices. In contrast, in equity auctions, the relevant bidder type is \( r_i \), and a lower \( r_i \) need not imply a higher net valuation. This is true even when \( x_i \) depends deterministically on \( v_i \), and more so, due to aggregation, when \( x_i \) and \( v_i \) are distributed on a two-dimensional space. Indeed, the regularity condition can be violated at every \( r_i \): virtual valuations can increase in \( r_i \), going in the opposite direction of incentive compatibility over the entire domain. Global violations imply a maximal mismatch between revenue maximization and incentive compatibility. Revenue maximization requires the asset to be allocated to bidders with higher virtual valuations—who have higher break-even equity stakes when the regularity condition is globally violated—but incentive compatibility demands that bidder types with lower break-even equity requirements be weakly more likely to win.

If the regularity condition for a bidder is globally violated, the optimal design pools all types so that the winning probability does not vary with type. If such violation occurs for all bidders, it is optimal to identify the bidders with the highest (constant) adjusted virtual valuation and sell to one of them when that adjusted virtual valuation exceeds the asset’s value to the seller, demanding the highest share that this bidder would cede to regardless of his type. Hence, if bidders are ex-ante identical and the regularity condition is globally violated, then even with multiple bidders, the optimal mechanism is a take-it-or-leave-it offer—expected seller revenue in the optimal design is unaffected by the number of bidders.\footnote{In contrast, in standard auction formats expected revenues fall with the number of bidders: the greater is competition, the lower is the winner’s break-even equity stake; and if regularity is globally violated, this lower break-even stake corresponds to a lower virtual valuation, and hence lower revenue.}

This feature contrasts with optimal cash auctions. Consistent with this design feature, Boone and Mulherin (2007) find that mergers and acquisitions involving equity are twice as likely as pure cash acquisitions to have a single bidder.\footnote{Computed from data in their Table IV.} If, alternatively, the regularity condition for some, but not all, bidders is globally violated, one can implement the optimal mech-
anism by first conducting an auction among bidders for whom the regularity condition is not globally violated, setting a reserve price that reflects the value of selling to bidders for whom the regularity condition is globally violated. A seller only turns to bidders exhibiting extreme adverse selection when the virtual valuations of the other bidders are not high enough.

Our analysis provides insights into the forces affecting revenue comparisons of optimal equity and optimal cash auctions. When uncertainty solely concerns valuations, Hansen (1985) shows that equity auctions generate higher revenues because equity bids tie payments to bidder types. When bidders have two-dimensional private information, additional forces come into play. First, with severe adverse selection, where the regularity condition is largely violated, revenues in optimal equity auctions rise minimally with the number of bidders. This favors cash if there are enough bidders. Second, the distribution of net valuations in the one-dimensional representation for equity mechanisms second-order stochastically dominates that for the original two-dimensional distribution (and hence for cash auctions). With few bidders, this favors equity because it lets a seller set the reserve price more efficiently, reducing the risk of no sale. In contrast, with many bidders, what matters primarily is the upper tail of the distribution, especially with substantial two-dimensional uncertainty. Reflecting the combined effects, optimal equity auctions generate more revenues than cash auctions if there are few bidders or moderate levels of two-dimensional uncertainty, while cash auctions do better with extensive two-dimensional uncertainty and many bidders.

Our dimensionality reduction result extends to mechanisms that maximize expected social surplus, where we identify a ‘surplus valuation’ that drives the optimal design. This analysis reveals how equity auctions are inefficient relative to cash auctions when bidders have two-dimensional types. With cash, a second-price auction ensures an efficient allocation, and any pooling is surplus-decreasing. In sharp contrast, in equity auctions, pooling can be surplus-maximizing, just as it can be revenue-maximizing. Indeed, when adverse selection is severe, disregarding all bids and selling to any bidder with the highest adjusted surplus valuation maximizes surplus. This contrast with cash auctions reinforces how in equity auctions with severe adverse selection, incentive compatibility requires “inferior” types (in terms of either virtual or surplus valuations) to win with a weakly higher probability. Thus, a mechanism designer can do no better than randomize.
**Literature Review.** The study of equity auctions begins with Hansen (1985). Like Hansen (1985), subsequent papers (e.g., DKS, Che and Kim (2010), Deb and Mishra (2014) or Liu (2016)) assume one-dimensional information for bidders. DKS analyze optimal selling mechanisms in standard auction formats when bidders have the same opportunity cost and select bids from a class of ordered securities. They show that steeper securities yield higher seller revenues. Che and Kim (2010) introduce bidders whose opportunity costs rise deterministically with their valuations and show that severe adverse selection can arise that causes steeper securities to generate less revenues in standard auctions. They also illustrate how the monotonicity of break-even stakes relates to the extent of adverse selection in equity auctions.

In contrast, we consider two-dimensional private information for bidders. We identify a way to reduce the dimensionality that we use to characterize incentive-compatible equity mechanisms, and to identify revenue- and surplus-maximizing mechanisms. We can recover Che and Kim’s (2010) framework when the distribution of types has a positive mass only on a one-dimensional subset of the two-dimensional space. However, one cannot just myopically use a one-dimensional formulation: not all two-dimensional distributions reduce to a setting in which \( x_i \) is a deterministic function of \( v_i \); and even if \( x_i \) is deterministically and monotonically related to \( v_i \), our dimensionality reduction approach is still needed when the relevant bidder type \( r_i \) evolves non-monotonically. Moreover, for a given two-dimensional distribution, equity auctions typically reduce to a different one-dimensional distribution than that for cash auctions. We derive a stochastic dominance relationship between the two distributions, and show how the implications for revenue comparisons of optimal equity and cash auctions depend on both the extent of two-dimensional uncertainty and the number of bidders.

Ekmekci, Kos and Vohra (2016) consider the problem of selling a firm to a single buyer who is privately informed about post-sale cash flows and the benefits of control. The offer consists of a menu of tuples of cash-equity mixes, and the bidder must obtain a minimum 50% stake to gain control. They provide sufficient conditions for the optimal mechanism to take the form of a take-it or leave-it offer for either 50% of the firm, or for all shares. The minimum

\[ \text{minimum} \]
stake introduces a discontinuity in buyer payoffs, which Ekmekci et al. address by characterizing incentive-compatible mechanisms via the exclusion boundary, separating types who obtain controlling stakes from those who do not. In contrast, we examine auctions in which bidders offer equities for full ownership and characterize the optimal auction for multiple buyers.

The study of mechanism design problems when agents have type-dependent opportunity costs (outside options) was initiated in Jullien (2000) and Rochet and Stole (2002), who show that an interior type can have zero surplus. In their models, the value of the opportunity cost is independent of the allocated quantity. In our single unit auction setting, however, this value depends on the allocated quantity: a bidder wins and incurs an opportunity cost, or loses and incurs no cost; and if allocations are probabilistic, the expected opportunity cost is proportional to the expected allocation. More fundamentally, equity complicates design, because the monetary value of payments depends on a bidder’s private information about valuations, and private information about opportunity costs introduces further complications.

Lastly, Deneckere and Severinov (2017) consider an agent whose gross utility $u(q, \alpha, \theta)$ depends on quantity $q$ and privately-known type $(\alpha, \theta)$. His net utility in a direct mechanism is

$$\pi = u(q(\hat{\alpha}, \hat{\theta}), \alpha, \theta) - t(\hat{\alpha}, \hat{\theta}), \quad (1)$$

where $(\hat{\alpha}, \hat{\theta})$ is the reported type, and $t(\hat{\alpha}, \hat{\theta})$ is the principal’s revenue. They identify conditions on preferences such that if $q_1 > 0$ is optimal for a type $(\alpha_1, \theta_1)$, and

$$u_q(q_1, \alpha_2, \theta_2) = u_q(q_1, \alpha_1, \theta_1) \quad (2)$$

for any $(\alpha_2, \theta_2)$ with $\alpha_2 < \alpha_1$, then $(\alpha_2, \theta_2)$’s optimal choice is also $q_1$, reducing the dimensionality. Their set-up relates to a cash-only version of our setting. To see why, let $(\alpha, \theta)$ be the bidder’s gross valuation and opportunity cost, let $q$ be the expected probability of winning, and let $t$ be the expected cash payment (unconditional on winning). Then $u(q, \alpha, \theta) = q(\alpha - \theta)$, and (2) becomes $\alpha_2 - \theta_2 = \alpha_1 - \theta_1$. That is, the relevant bidder type is the net valuation. Equity payments materially alter the framework: net utility becomes

$$\pi = q(\hat{\alpha}, \hat{\theta})e(\hat{\alpha}, \hat{\theta})\alpha - q(\hat{\alpha}, \hat{\theta})\theta, \quad (3)$$

and seller revenue is $q(\hat{\alpha}, \hat{\theta})(1 - e(\hat{\alpha}, \hat{\theta}))\alpha$, where $q(\hat{\alpha}, \hat{\theta})$ and $e(\hat{\alpha}, \hat{\theta})$ are the winning probability and the bidder’s equity share. Equation (3) cannot be cast in the form of (1), and the form of seller revenue differs, hinging on both the bidder’s reported and true (private) type.
These differences highlight how mechanism design involving equity payments with two-dimensional types differ from existing studies of mechanism design with cash transfers, underscoring that different approaches for dimensionality reduction are needed.

2 The Model

$n \geq 1$ risk-neutral bidders compete to acquire an asset/project, over which control rights are indivisible. Bidder $i$ ($i = 1, \ldots, n$) is privately informed about both the expected value $v_i > 0$ of the cash flows that the asset would generate if he wins, and his opportunity cost $x_i \in (0, v_i)$ of acquiring the asset. The seller has a publicly-known reservation value $v_s$ of retaining the asset. Thus, the expected social surplus if $i$ wins is $v_i - x_i - v_s$. For each bidder $i$, $v_i$ and $x_i$ are jointly distributed according to a strictly positive, continuous pdf $f_i(v_i, x_i)$ over a compact and connected set $S_i$. The densities $f_i$ can differ across bidders, and $v_i$ and $s_i$ are independently distributed across bidders. We assume $\max_i (v_i, x_i) \in S_i$, $v_i - x_i - v_s > 0$, i.e., there are potential gains to trade. We use $t_i \equiv (x_i, v_i)$ to denote bidder $i$’s type, $f(t) \equiv \Pi_{i=1}^n f_i(t_i)$ to denote the joint density of $t \equiv (t_1, t_2, \ldots, t_n)$, and $f_{-i}(t_{-i}) \equiv \Pi_{k \neq i} f_k(t_k)$ to denote the joint density of the types of bidders other than $i$, where $t_{-i} \equiv (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$.

We consider direct-revelation mechanisms in which the winner pays with equities. Let $Q_i(t'_i, t_{-i}) \in [0, 1]$ be the equity share of the project’s cash flows that bidder $i$ retains when he reports being type $t'_i$ and other bidders report $t_{-i}$. Let $W_i(t'_i, t_{-i})$ be the probability that bidder $i$ wins, and let $W_0(t'_i, t_{-i})$ be the probability that no bidder wins (i.e., the seller retains the asset). We require $\Sigma_{j=0}^n W_j(t) = 1$ for all $t$. We define $G_i(t'_i)$ to be the probability bidder $i$ wins when $i$ reports $t'_i$ and all other bidders report truthfully:

$$G_i(t'_i) = \int W_i(t'_i, t_{-i}) f_{-i}(t_{-i}) \, dt_{-i}. \quad (4)$$

We define $q_i(t'_i)$ to be the expected equity share that bidder $i$ retains conditional on winning by reporting type $t'_i$ when all others report truthfully,

$$q_i(t'_i) G_i(t'_i) = \int Q_i(t'_i, t_{-i}) W_i(t'_i, t_{-i}) f_{-i}(t_{-i}) \, dt_{-i}. \quad (5)$$

If bidder $i$ has type $t_i$ but reports $t'_i$, and other bidders report truthfully, his expected profit is:

$$h_i(t_i, t'_i) = (v_i q_i(t'_i) - x_i) G_i(t'_i). \quad (6)$$
Equilibrium requires that the incentive compatibility condition hold,

$$ t_i = \arg \max_{t'_i} h_i (t_i, t'_i) . \quad (7) $$

The seller’s expected revenue is the sum of the winner’s payments plus the reservation value when the seller retains the asset:

$$ \pi_s = \int \left( \sum_{i=1}^{n} W_i (t) (1 - Q_i (t)) v_i + W_0 (t) v_s \right) f (t) \, dt. \quad (8) $$

Our objective is to identify the mechanism that maximizes expected revenue (8) subject to incentive compatibility (7) and individual rationality, i.e., that $h_i(t_i, t'_i) \geq 0$ for all $i$ and $t_i$.

We start by examining the properties of any incentive-compatible mechanism. Observe that bidder $i$’s objective function (6) contains a term $v_i q_i (t'_i) G_i (t'_i)$, where $v_i$ depends on the bidder’s private type, and both $q_i (\cdot)$ and $G_i (\cdot)$ are endogenously determined in equilibrium. This complicates the envelope condition vis à vis cash auctions, where the term in a bidder’s objective function that depends on the bidder’s private type is multiplied only by one endogenous term, $G_i (\cdot)$. That is, $q_i = 1$ in cash auctions.

Our first step is to simplify the envelope condition by transforming a bidder’s objective function. We make a simple but fundamental observation: in equation (7), bidder $i$’s gross expected valuation $v_i$ is in his information set, so it can be treated as a constant in his optimization problem. This lets us rescale (6), bidder $i$’s expected profit when he has type $t_i$, but reports $t'_i$ and all other bidders report truthfully. We define

$$ m_i (t_i, t'_i) \equiv \frac{h_i (t_i, t'_i)}{v_i} = (q_i (t'_i) - r_i) G_i (t'_i), \quad (9) $$

$$ r_i \equiv \frac{x_i}{v_i} \quad (11) $$

is the fraction of equity a bidder must retain to break even. Equivalently, $r_i$ is the fractional cost bidder $i$ incurs to generate a unit cash flow. We can express incentive compatibility (7) as

$$ m_i (t_i, t_i) = \max_{t'_i} (q_i (t'_i) - r_i) G_i (t'_i). \quad (12) $$
Equation (12) reveals the advantage of the transformation: in the argument for optimization, the term that depends on a bidder’s private information is multiplied by only $G_i$. This simplifies the resulting envelope condition relative to working directly with (7).

Our second step is to reduce the dimensionality to a single dimension. The right-hand side of equation (12) only depends on a bidder’s type according to $r_i = x_i/v_i$. This means that $m_i(t_i, t_i)$ must be a function of $r_i$ only. Hence, any two type pairs $t'_i = (x'_i, v'_i)$ and $t''_i = (x''_i, v''_i)$ such that $\frac{x'_i}{v'_i} = \frac{x''_i}{v''_i} = r'_i$ should have the same $m_i$:

$$m_i(t'_i, t'_i) = m_i(t''_i, t''_i),$$

i.e.,

$$(q_i(t'_i) - r'_i) G_i(t'_i) = (q''_i(t_i) - r'_i) G''_i(t_i).$$

Equation (14) relates the winning and allocation rules, $G_i$ and $q_i$, of the two types. However, it does not imply that

$$G_i(t'_i) = G_i(t''_i) \quad \text{and} \quad q_i(t'_i) = q_i(t''_i),$$

because $G''_i(t_i)$ and $q_i(t''_i)$ can differ from $G'_i(t_i)$ and $q_i(t'_i)$, and yet (14) still holds. That is, due to the two-dimensional nature of the set $S_i$, an infinite number of type pairs $(x_i, v_i)$ typically correspond to any given $r_i$, and (15) need not hold for all such pairs. We next use the envelope condition to derive the stronger result that (15) holds in “almost all cases”.

To proceed, we project every point of $S_i$ onto $r_i$, and define $\underline{r}_i$ and $\bar{r}_i$ to be respectively the minimum and maximum values of $r_i$. To facilitate the analysis, we add the mild structure that there is no atom at any value of $x_i/v_i = r_i$. Thus, $\underline{r}_i < \bar{r}_i$. Define $\tilde{f}_i(r_i)$ to be the associated probability density function over $r_i$, which is strictly positive for $r_i \in (\underline{r}_i, \bar{r}_i)$.

By (13), without loss of generality we define

$$M_i(r_i) \equiv m_i(t_i, t_i).$$

By (12), $M_i(r_i)$ is the maximum of a family of affine functions and hence is convex. Because $M_i(r_i)$ is also bounded and hence absolutely continuous, it is differentiable almost everywhere in the interior of its domain. Define $R_i$ to be the set of all $r_i \in (\underline{r}_i, \bar{r}_i)$ at which $M_i(r_i)$

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8This ensures that bidder types where $M_i(r_i)$ (see equation (16)) is non-differentiable have zero mass.
is differentiable, which includes all points in \((r_i, \bar{r}_i)\) except those of measure zero. In light
of the no-atom condition, we ignore the measure zero set of points at which \(M_i(r_i)\) is non-
differentiable in any analysis that involves integration. By (12) and the envelope theorem,
at every point \(t_i = (v_i, x_i) \in S_i\) with \(x_i/v_i \in R_i\), we obtain the simple relation:
\[
\frac{dM_i(r_i)}{dr_i} = -G_i(t_i).
\]
Thus, if \(\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i\), then because \(G_i(t_i')\) and \(G_i(t_i'')\) both equal \(-\frac{dM_i(r_i)}{dr_i}|_{r=r_i'}\), we have:

**Lemma 1** In any incentive-compatible equity mechanism, (15) holds if \(\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i\).

We now derive the necessary and sufficient conditions for incentive compatibility and
individual rationality.

**Theorem 1 (necessary and sufficient conditions for IC and IR)** An equity mechanism is incentive compatible and individually rational if and only if, for all \(i\):

(i) If \(t_i' = (v_i', x_i')\) and \(t_i'' = (v_i'', x_i'')\) satisfy \(\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''}\), then \(m_i(t_i', t_i') = m_i(t_i'', t_i'')\). That is, \(m_i(t_i, t_i)\) is a function of \(r_i\) only.

(ii) The function \(M_i(r_i)\) defined in (16) is weakly convex over \(r_i \in [r_i, \bar{r}_i]\).

(iii) If \(x_i/v_i \in R_i\), then (17) holds. Further, (15) holds for any type pairs \(t_i'\) and \(t_i''\) such that \(\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \in R_i\).

(iv) The winning probability is non-increasing in \(r_i\) for all \(i\): for any type pairs \(t_i' = (x_i', v_i')\) and \(t_i'' = (x_i'', v_i'')\) if \(\frac{x_i'}{v_i'} > \frac{x_i''}{v_i''}\), then \(G_i(t_i') \leq G_i(t_i'')\).

(v) \(M_i(\bar{r}_i) \geq 0\).

Part (iv) follows from the incentive compatibility condition. It reflects the intuition that \(r_i\) represents the fractional cost of generating one unit of revenue, and that a bidder with a smaller fractional cost is more competitive. Part (iii) shows that (15) holds if \(x_i/v_i \in R_i\). However, for type pairs \(t_i'\) and \(t_i''\) such that \(\frac{x_i'}{v_i'} = \frac{x_i''}{v_i''} \notin R_i\), (15) need not hold. That is, at \(r_i\) where \(M_i(r_i)\) is not differentiable, \(G_i(t_i')\) and \(G_i(t_i'')\) can differ. Part (iv) bounds the amount by which they can differ: any \(G_i(t_i')\) must lie between \(\lim_{r \to r_i^-} \frac{dM_i(r)}{dr} \) and \(\lim_{r \to r_i^+} \frac{dM_i(r)}{dr}\).
We next derive an expression for expected revenue in any incentive compatible mechanism. Abusing notation slightly, for all \( t_i \in S_i \) such that \( r_i \in R \), we use (17) to write \( G_i(t_i) \) as \( G_i(r_i) \). Because an absolutely continuous function is the definite integral of its derivative,

\[
M_i(r_i) = M_i(\bar{r}_i) + \int_{r_i}^{\bar{r}_i} G_i(r) \, dr.
\]

(18)

By (18), \( M_i(r_i) \) is non-increasing in \( r_i = \frac{x_i}{v_i} \), as is \( G_i(r) \).

To facilitate the analysis, for a given \( r_i \), we define

\[
a_i(r) \equiv \lim_{\epsilon \to 0} E[v_i | r_i \leq x_i/v_i \leq r_i + \epsilon],
\]

(19)

where the expectation is taken over \( S_i \). Here, \( a_i(r) \) is the expected value of \( v_i \) conditional on \( x_i/v_i \) being in an “\( \epsilon \)” neighborhood of \( r \). Note that this expectation is not the same as \( E[v_i | x_i/v_i = r] \) because (19) implies a weighting that is non-uniform for different type pairs with the same \( r \). The lemma below shows how to do this weighting in the two-dimensional case where the probability density \( f_i(x_i, v_i) \) is finite, and Example 1 provides an illustration. Example 2 will show how to do the weighting in the limiting case where the two-dimensional distribution degenerates to a single dimension, so that \( f_i(x_i, v_i) \) is infinite.

**Lemma 2** For each \( R \geq 0 \), define \( \rho_i(R) \equiv f_i \left( x_i = \frac{r_i}{\sqrt{1+r^2}} R, v_i = \frac{1}{\sqrt{1+r^2}} R \right) \), where \( f_i(x_i, v_i) \) is the probability density for \((x_i, v_i) \in S_i\), and \( f_i(v_i, x_i) = 0 \) for \((v_i, x_i) \notin S_i\). Then

\[
a_i(r) = \frac{1}{\sqrt{1+r^2}} \frac{\int_0^R R^2 \rho(R) \, dR}{\int_0^R R \rho(R) \, dR}.
\]

We now use standard mechanism-design techniques to decompose expected seller revenue.

**Definition 1** Bidder i’s virtual valuation at \( r_i \in [r_i, \bar{r}_i] \) in equity auctions is

\[
\phi_i(r_i) \equiv (1 - r_i) a_i(r_i) - \frac{\int_{r_i}^{\bar{r}_i} a_i(r) \tilde{f}_i(r) \, dr}{\tilde{f}_i(r_i)}.
\]

(20)

**Theorem 2** *(Revenue decomposition)* In any incentive-compatible mechanism of equity auctions, the seller’s expected revenue (8) decomposes:

\[
\pi_s = \pi_{s,a} + \pi_{s,b},
\]

(21)
where

\[ \pi_{s,a} \equiv -\sum_{i=1}^{n} M_i(\tilde{r}_i) \int_{\Sigma_i} a_i(r) \tilde{f}_i(r) \, dr \]  

(22)

and

\[ \pi_{s,b} \equiv \int_{S_1 \times \ldots \times S_n} \left( \sum_{i=1}^{n} W_i(t) \phi_i(r_i) + W_0(t) v_s \right) f(t) \, dt_1 \ldots dt_n. \]  

(23)

\( \pi_{s,a} \) is bounded by bidders’ rationality constraints: because \( M_i(\tilde{r}_i) \geq 0 \), its maximum possible value is zero, which obtains if \( M_i(\tilde{r}_i) = 0 \) for all \( i \). \( \pi_{s,b} \) is maximized by allocating the asset to the bidder with maximal virtual valuation \( \phi_i(x_i) \) if that \( \phi_i(x_i) \) exceeds the seller’s valuation. It is instructive to use (19) and \( r_i = v_i/x_i \) to rewrite the first term of \( \phi_i(x_i) \) in (20) as

\[ (1 - r_i) a_i(r_i) \equiv \lim_{\epsilon \to 0} E[v_i - x_i \mid r \leq x_i/v_i \leq r + \epsilon]. \]  

(24)

Hence, this term is the bidder’s expected net valuation (conditional on \( x_i/v_i \) being in an “\( \epsilon \)” neighborhood of \( r \)), similar to the first term in the virtual valuation for a cash auction. The second term of the virtual valuation reflects the difference that the relevant bidder type is the break-even stake in equity auctions whereas it is the net valuation in cash auctions.

Theorem 2 implies a dimensionality reduction result: a seller’s revenue from any given mechanism for any two-dimensional distribution over \( (x_i, v_i) \) equals that for a one-dimensional distribution where \( x_i \) and \( v_i \) are parameterized in terms of \( r_i \):

**Corollary 1 (One-dimensional representation)** A seller’s revenue in a given mechanism for any two-dimensional distribution of \( (x_i, v_i) \) is the same as that for the one-dimensional distribution in which for each bidder \( i \), all points \( (x_i, v_i) \) associated with a given \( r_i \) are replaced by the single point \( (r_i a_i(r_i), a_i(r_i)) \), and \( r_i \) is distributed according to \( \tilde{f}_i(r_i) \).

Corollary 1 contains subtle implications. First, in our one-dimensional representation, only one type pair \( (x_i(r_i), v_i(r_i)) \) corresponds to any \( r_i \). However, this does not imply that \( v_i \) is a deterministic function of \( x_i \) (Example 1 will illustrate this). Thus, it is not without loss of generality to consider only settings in which \( x_i \) and \( v_i \) are deterministically related. Second, if we start with a one-dimensional problem, our one-dimensional representation result is still needed if multiple type pairs correspond to a given \( r_i \) (Example 2 illustrates
Moreover, dimensionality reduction in cash auctions typically results in a different one-dimensional distribution as bidder types aggregate via net valuation, so if one wishes to compare equity and cash auctions, one needs to begin with the higher dimensional primitives.

The following proposition highlights this difference in the dimensionality reduction.

**Proposition 1**  The distribution of net valuations $v_i - x_i$ for the one-dimensional distribution in Corollary 1 second-order stochastically dominates the distribution of net valuations for the original two-dimensional distribution.

Proposition 1 has implications for revenue comparisons of equity and cash auctions that we later explore. We next illustrate how to use the one-dimensional representation.

**Example 1** (Skrzypacz (2013)). $n$ ex-ante identical bidders bid with equity. Bidder $i$ expects revenue $v_i = x_i + y_i$, where the net valuation, $y_i$, and opportunity cost, $x_i$, are each independently and uniformly distributed on $[0, 1]$.

We have $r_i = 0$ and $\bar{r}_i = 1$. In the appendix, we obtain the form of $a_i(r_i)$ and show that a seller’s expected revenue from any mechanism given the two-dimensional distribution of the example is the same as it would be were $x_i$ and $v_i$ parameterized (via $r_i$) as

$$\left(x_i, v_i\right) = \left(\frac{2r_i}{3 \max\{r_i, 1 - r_i\}}, \frac{2}{3 \max\{r_i, 1 - r_i\}}\right),$$

(25)

where $r_i$ is distributed according to

$$\tilde{f}_i(r_i) = \frac{1}{2} \left(1 + \left(\min\left\{\frac{1 - r_i}{r_i}, \frac{r_i}{1 - r_i}\right\}\right)^2\right) / \left(r_i^2 + (1 - r_i)^2\right).$$

(26)

In equation (25), the relationship between $x_i$ and $v_i$ is $\triangleright$-shaped: there are two $x_i$ corresponding to each $v_i \in \left(\frac{2}{3}, \frac{4}{3}\right)$. Nonetheless, there is only one type pair associated with each $r_i$, which is what our one-dimensional representation result details.

### 3 Optimal Equity Mechanisms

An optimal equity mechanism maximizes a seller’s expected revenue subject to the incentive compatibility and individual rationality conditions. We first derive optimal equity mechanisms given the standard regularity condition in Assumption 1:
Assumption 1 The design problem is regular: \( \phi_i(r_i) \) strictly decreases over \([\underline{r}_i, \bar{r}_i]\) for all \(i\).

Proposition 2 Under Assumption 1, a direct-revelation mechanism is optimal if and only if

(i) the winning rule \( W_i(t_i, t_{-i}) \) depends on \( t_i \) only through \( r_i \), satisfying

\[
W_i(t) = \begin{cases} 
1 & \text{if } \phi_i(r_i) > \max \left\{ \max_{j \neq i} \{ \phi_j(r_j) \}, v_s \right\} \\
0 & \text{if } \phi_i(r_i) < \max \left\{ \max_{j \neq i} \{ \phi_j(r_j) \}, v_s \right\}
\end{cases},
\]

where \( \phi_i(r_i) \) is the virtual valuation in (20), for all \( i \) and \( t \), ties are broken arbitrarily.

(ii) the equity-retention rule \( q_i(t_i) \) depends on \( t_i \) only through \( r_i \), via

\[
(q_i(r_i) - r_i) G_i(r_i) = \int_{r_i}^{\bar{r}_i} G_i(r) \, dr, \quad \text{for all } i \text{ and } r_i.
\]

Corollary 2 Under Assumption 1, an optimal equity mechanism is given by: the winning rule in (i) of Proposition 2, and the following equity-retention rule for the winning bidder,

\[
Q_i(t) = \phi_i^{-1} \left( \max \left\{ \max_{j \neq i} \{ \phi_j(r_j) \}, v_s \right\} \right).
\]

\(Q_i(t)\) is the maximum value of the break-even \( r_i \) corresponding to a virtual valuation that exceeds \( v_s \) and allows \( i \) to win against \( t_{-i} \), where \( \phi_i^{-1}(\cdot) \) denotes a bounded inverse of \( \phi_i(\cdot) \):

\[
\phi_i^{-1}(x) = \begin{cases} 
\bar{r}_i & \text{if } x < \phi_i(\bar{r}_i) \\
y & \text{s.t. } \phi_i(y) = x & \text{if } x \in [\phi_i(\bar{r}_i), \phi_i(\underline{r}_i)]
\end{cases}.
\]

The mechanism in Corollary 2 is simply the second-price auction with an optimal reserve when bidders are ex-ante identical. Thus, the optimal design is exactly the same as in cash auctions (Myerson 1981) or equity auctions where bidders’ opportunity costs are common knowledge (Liu 2016). This correspondence reflects properties that hold generally for both cash and equity auctions: (1) the regularity condition requires virtual valuations to strictly increase in the direction where incentive compatibility requires the winning probability to weakly increase, and (2) standard formats select in this “reinforced” direction of incentive compatibility. To see what these two properties mean, consider equity auctions: (1) incentive compatibility requires the winning probability to weakly decrease in \( r_i \), whereas regularity requires the virtual valuation to strictly decrease in \( r_i \), and (2) the winning probability in standard formats strictly decreases in \( r_i \). Thus, when the regularity condition holds, this
“reinforced” direction is in line with revenue maximization. It follows that with ex-ante identical bidders, standard formats are optimal when the regularity condition holds.

We now solve for the optimal mechanism in a general setting in which Assumption 1 need not hold. We follow the general approach of Myerson (1981), making minor adjustments for the fact that incentive compatibility requires that the winning probability $G_i$ decrease in $r_i$, whereas in cash auctions, the winning probability must increase in bidder type. We then show that in equity auctions the features of the optimal mechanism differ dramatically according to whether the regularity condition is or is not violated. These differences are more striking than the analogous differences that emerge in cash auctions.

Let $\tilde{F}_i(r_i)$ be the cdf of $\tilde{f}_i(r_i)$. For any $y \in [0, 1]$, define
\begin{equation}
    b_i(y) = \phi_i\left(\tilde{F}_i^{-1}(y)\right) \quad \text{and} \quad B_i(y) = \int_0^y b_i(z) \, dz.
\end{equation}

Define $C_i(\cdot)$ to be the concave hull of $B_i(\cdot)$. Define
\begin{equation}
    \bar{\phi}_i(r_i) = C_i'(\tilde{F}_i(r_i))
\end{equation}
for $r_i \in [r_i, \bar{r}_i]$. For any vector of bidder types $t$, let $N(t)$ be the set of bidders for whom $\bar{\phi}_i(r_i)$ is both maximal and higher than $v_s$

\begin{equation}
    N(t) = \left\{ i | \bar{\phi}_i(r_i) \geq \max \left\{ \max_{j \neq i} \bar{\phi}_j(r_j), v_s \right\} \right\}.
\end{equation}

Proposition 3 The following constitutes an optimal equity mechanism:

(i) the winning rule is
\begin{equation}
    \bar{W}_i(t) = \begin{cases} 1/|N(t)| & \text{if } i \in N(t) \\ 0 & \text{if } i \notin N(t) \end{cases},
\end{equation}
for all $i$ and $t$, and

(ii) the equity-retention rule for the winning bidder is
\begin{equation}
    \bar{Q}_i(t) = \bar{\phi}_i^{-1}\left(\max\left\{ \max_{j \neq i} \bar{\phi}_j(r_j), v_s \right\}\right),
\end{equation}
which is the maximum value of the break-even $r_i$ that corresponds to a virtual valuation exceeding $v_s$ that allows $i$ to win against $t_{-i}$, and

\begin{equation}
    \bar{\phi}_i^{-1}(x) \equiv \begin{cases} \max\{y | y \in [r_i, \bar{r}_i] \text{ and } \phi_i(y) = x\} & \text{if } x < \bar{\phi}_i(\bar{r}_i) \\ \bar{r}_i & \text{if } x \in [\bar{\phi}_i(\bar{r}_i), \bar{\phi}_i(\bar{r}_i)] \end{cases}.
\end{equation}
When the design problem is not regular, virtual valuations increase in \( r_i \) on some range. Because incentive compatibility requires the allocation probability to be weakly decreasing in \( r_i \), always allocating the asset to a bidder with the highest virtual valuation is not incentive compatible. Proposition 3 concavifies with an adjusted virtual valuation that pools types on the increasing range and treats them equally. The adjusted virtual valuation is weakly decreasing in \( r_i \). Allocating the asset to a bidder with the highest adjusted virtual valuation maximizes a seller’s revenue subject to the incentive compatibility requirement.

Regardless of whether the regularity condition holds for a bidder, in the optimal design, the type of that bidder with the highest break-even equity stake extracts no rents. This implication is less direct than it may seem: in a non-optimal, but incentive-compatible mechanism, the type with the lowest rent of a given bidder varies across mechanisms, and need not be the one with the highest break-even equity stake for that bidder.\(^9\) Nonetheless, a key insight is that while the bidder’s rent \( h_i(t_i, t_i) \) can be non-monotone, \( M_i(r_i) \) (which equals \( h_i/v_i \)) must be non-increasing by (17). By (9) and individual rationality, it follows that if \( h_i(t_i, t_i) \) is zero for any type \( t_i \), then \( h_i \) for a type with the highest break-even equity stake must also be zero. Thus, in the optimal design, for all bidders, any type with the highest break-even equity stake earns zero rent.

### 3.1 Illustrations of virtual valuations in a one-dimensional setting

We illustrate virtual valuations in the setting of Che and Kim (2010) where \( x_i \) is a deterministic function of \( v_i \) with \( dx_i/dv_i < 1 \) so that both gross valuation and net valuation increase in \( v_i \). Abusing notation slightly, we use \( f_i(v_i) \) to denote the pdf of \( v_i \), and let \( F_i(v_i) \) denote the cdf. \[ r_i = \frac{x_i(v_i)}{v_i} \] strictly monotone in \( v_i \). Then

\[
\tilde{f}_i(r_i) = f_i(v_i) \left| \frac{dv_i}{dr_i} \right|, \tag{36}
\]

and the expected value of \( v_i \) given \( r_i \) in (19) becomes \( a_i(r_i) = v_i \). The form of \( \int_{a_i(r_i)}^0 a_i(r) \tilde{f}_i(r) \, dr \) in (20), however, will depend on whether \( x_i(v_i)/v_i \) is decreasing or increasing in \( v_i \). If \( x_i(v_i)/v_i \) is

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\(^9\)As an extreme illustration, suppose the seller faces a single bidder \( i \) and (suboptimally) lets the bidder win for free. Then the lowest-rent bidder type has the lowest net valuation \( v_i - x_i \), which, as we show in section 3.1, need not correspond to the bidder with the highest break-even equity stake.
strictly increasing in \( v_i \), then \( r_i \) corresponds to \( v_i \), and \( \tilde{f}_i (r) \, dr = f_i (v) \, dv \). Therefore,

\[
\int_{\Sigma_i} a_i (r) \, \tilde{f}_i (r) \, dr = \int_{v_i}^r v \, f_i (v) \, dv. \tag{37}
\]

If, instead, \( \frac{x_i (v_i)}{v_i} \) strictly declines in \( v_i \), then \( \Sigma_i \) corresponds to \( \tilde{v}_i \) and \( \tilde{f}_i (r) \, dr = -f_i (v) \, dv \), so

\[
\int_{\Sigma_i} a_i (r) \, \tilde{f}_i (r) \, dr = -\int_{v_i}^{\tilde{v}_i} v \, f_i (v) \, dv = \int_{v_i}^{\tilde{v}_i} v \, f_i (v) \, dv. \tag{38}
\]

The corresponding virtual valuations follow directly from (20). Importantly, the virtual valuation takes a different form depending on whether \( \frac{x_i (v_i)}{v_i} \) increases or decreases in \( v_i \), reflecting the differences between (37) and (38). Next, we examine a more complicated setting where \( \frac{x_i (v_i)}{v_i} \) is not monotone in \( v_i \), and show that virtual valuations take yet further different forms.

**Example 2 (hybrid case):** \( r_i = \frac{x_i (v_i)}{v_i} \) nonmonotone in \( v_i \) (\( \cap \)-shaped in \( v_i \)).

Let \( v_i \) be uniformly distributed on \([5, 15]\) with \( x_i = 0.6v_i - 1 \) for \( v_i \in [5, 10] \) and \( x_i = 0.2v_i + 3 \) for \( v_i \in [10, 15] \), and let \( v_s = 0 \). Then \( x_i = 2 \) at \( v_i = 5 \), \( x_i = 5 \) at \( v_i = 10 \), and \( x_i = 6 \) at \( v_i = 15 \), and \( \frac{x_i}{v_i} = 0.6 - \frac{1}{v_i} \) for \( v_i \in [5, 10] \), and \( \frac{x_i}{v_i} = 0.2 + \frac{3}{v_i} \) for \( v_i \in [10, 15] \).

Even though \( x_i \) is deterministic and strictly increasing in \( v_i \), because multiple points correspond to the same \( r_i \), aggregation is still needed. We have \( r_i = 0.4 \) and \( \tilde{r}_i = 0.5 \). Each \( r_i \in (0.4, 0.5) \) corresponds to two points:

\[
0.6 - \frac{1}{v_i} = r_i \Rightarrow v_i = \frac{1}{0.6 - r_i} \quad \text{and} \quad 0.2 + \frac{3}{v_i} = r_i \Rightarrow v_i = \frac{3}{r_i - 0.2}. \tag{39}
\]

To calculate the associated density \( \tilde{f}_i (r) \) and to find the proper weighting in calculating \( a_i (r_i) \), observe that the absolute value of the derivative of \( v_i \) over \( r_i \) at these two points are

\[
\left| \frac{dv_i}{dr_i} \right| = \frac{1}{(0.6 - r_i)^2} \quad \text{and} \quad \left| \frac{dv_i}{dr_i} \right| = \frac{3}{(r_i - 0.2)^2}.
\]

Because \( v_i \) is uniformly distributed over an interval of length 10, the density over \( v_i \) is 0.1. Hence, the density over \( r_i \) is 0.1 \( \left| \frac{dv_i}{dr_i} \right| \) summed over the two points, yielding\(^{10}\)

\[
\tilde{f}_i (r_i) = \frac{0.1}{(0.6 - r_i)^2} + \frac{0.3}{(r_i - 0.2)^2}.
\]

\(^{10}\) In this example the derivative of \( r_i \) is not continuous at \( \tilde{r}_i \). Were \( r_i \) continuous and differentiable in \( v_i \), then, from the first-order condition, \( \tilde{f} (r_i) \) would be infinite at \( \tilde{r}_i \). This would not violate the premise of no atom at any \( x_i/v_i \), because the integral of \( f_i (r_i) \) goes to zero as the width of the interval vanishes.
To calculate \( a_i(r_i) \), refer to equation (19). Noting that the weight of mass at each of these two points is 0.1\(|\frac{d\nu_i}{dr_i}|\tilde{f}_i(r_i)\), and that \( \nu_i \) relates to \( r_i \) via (39), we have

\[
a_i(r_i) = \frac{0.1}{\tilde{f}_i(r_i)} \left( \frac{1}{0.6 - r_i} \right) + \frac{0.3}{\tilde{f}_i(r_i)} \left( \frac{3}{r - 0.2} \right).
\]

Substituting for these components into (20) yields the virtual valuation.

**Discussion.** These examples share the salient feature that \( \nu_i - x_i \) increases in \( v_i \) so that a high type is unambiguously defined. In cash auctions, the relevant bidder type is the net valuation \( \nu_i - x_i \), and in the optimal design the zero-rent type has the lowest net valuation. In contrast, in equity auctions, the relevant bidder type is \( r_i \), and in the optimal design the zero-rent type has the highest \( r_i \). Thus, when \( \frac{x_i(v_i)}{\nu_i} \) first increases and then decreases in \( v_i \), the zero-rent-type has an intermediate net valuation. Moreover, when \( r_i = \frac{x_i(v_i)}{\nu_i} \) increases in \( v_i \), the zero-rent-type in the optimal design has the highest net valuation, which is the opposite of what arises in cash auctions.

Our one-dimensional representation result does more than reduce the dimensionality of analysis. In a one-dimensional setting like Example 2, although the relationship between \( x_i \) and \( v_i \) is monotone, it is not the final form of our representation result because \( r_i \) evolves non-monotonically. In such settings, the representation result helps to derive the virtual valuation and identify the lowest rent type. Further, the representation result shows \( r_i \) is the relevant bidder type: as a function of \( r_i \), the virtual valuation (20) takes a simple and unique form. In contrast, even in a one-dimensional setting, if one uses \( v_i \) or \( v_i - x_i \) as the bidder type, the functional form of the virtual valuation will differ depending on the monotonicity of \( r_i \).

### 4 Global Violation of Regularity Condition

Assumption 1 requires virtual valuations to decrease in \( r_i \), i.e., to move in line with incentive compatibility. The virtual valuation measures available rents and thus tends to increase with a bidder’s net valuation \( \nu_i - x_i \). In cash auctions, the net valuation is the relevant bidder type, and hence, the regularity condition holds under mild assumptions—a monotone hazard condition on the distribution suffices. In contrast, in equity auctions the relevant bidder type is given by the break-even equity stake \( r_i \), and a lower \( r_i \) need not correspond to a higher
net valuation. This is true even when \( x_i \) depends deterministically on \( v_i \), and more so, due to aggregation, when \( x_i \) and \( v_i \) are distributed on a two-dimensional space.

**Definition 2** The regularity condition is **globally violated** for bidder \( i \) if for any types \( t_1 \) and \( t_2 \) such that \( \phi_i(t_1) > \phi_i(t_2) \), incentive compatibility requires \( G_i(t_1) \leq G_i(t_2) \).

Global violations imply that virtual valuations go in the opposite direction of the regularity requirement over the entire domain: From Theorem 1 part \( (iv) \), the regularity condition is globally violated for bidder \( i \) if \( \phi_i(r_i) \) increases in \( r_i \) over \( [r_i, \bar{r}_i] \).

**Proposition 4** Suppose that in the one-dimensional representation, \( x_i \) is deterministically related to \( v_i \). Then, for any \( f_i(v_i) \) strictly positive over \( [v_i, \bar{v}_i] \) and \( x_i(v_i) < v_i \), there exist \( \epsilon_1, \epsilon_2 > 0 \) such that the regularity condition is globally violated for bidder \( i \) if \( \frac{d\ln x_i}{d\ln v_i} \in (1, 1 + \epsilon_1) \) and \( | \frac{d}{dv_i} \left( \frac{d\ln x_i}{d\ln v_i} \right) | < \epsilon_2 \) at all \( v_i \in [v_i, \bar{v}_i] \).

Proposition 4 shows that global violations occur when \( x_i(v_i) \) is slightly more than unit elastic in the reduced dimensionality space of Corollary 1. This condition imposes no restrictions on the scale of opportunity costs relative to \( v_i \). This means that global violations can arise in “vanilla” economies in which opportunity costs are tiny, making the “efficiency cost” of not allocating the asset to a bidder with the highest valuation high.

This feature that virtual valuations go against the regularity requirement over the entire domain can never arise in cash auctions or in equity auctions with constant opportunity costs. In cash auctions, the virtual valuation is

\[
\psi_i(y_i) = y_i - \frac{1 - D_i(y_i)}{d_i(y_i)},
\]

where \( y_i \equiv v_i - x_i \) is bidder \( i \)'s net valuation and \( D(\cdot) \) and \( d(\cdot) \) are the cdf and pdf of \( y_i \). Thus, \( \psi_i(y_i) < y_i < \bar{y}_i = \psi_i(\bar{y}_i) \) for \( y_i < \bar{y}_i \). Similarly, \( \phi_i(y_i) < y_i < \bar{y}_i = \phi_i(\bar{y}_i) \) for \( y_i < \bar{y}_i \) for virtual valuations in equity auctions with constant opportunity costs. Thus, even though the virtual valuation may decline over some range of \( v_i \) in violation of the regularity condition, it cannot decrease over the entire range.

The global violation of the regularity condition represents the greatest possible mismatch between revenue maximization and incentive compatibility. Revenue maximization requires
a seller to allocate the asset to bidders with higher virtual valuations as much as possible, but incentive compatibility demands the opposite. This mismatch has novel implications:

**Lemma 3** If the regularity condition is globally violated for bidder $i$, then $\bar{\phi}_i$ in (32) is the same for all types of bidder $i$.

**Corollary 3** If the regularity condition is globally violated for bidder $i$, then in the optimal design, bidder $i$’s expected winning probability is independent of his type.

**Corollary 4** If the regularity condition is globally violated for all bidders, then it is optimal for the seller to disregard all bids and sell to any bidder $i \in \arg\max_j \bar{\phi}_j$ (if $\bar{\phi}_i > v_s$), asking for fraction $1 - \bar{r}_i$ of equity. The bidder always accepts the offer.

Boone and Mulherin (2007) find that in corporate takeovers that use equities as payments, selling firms typically only ask a subset of potential bidders, often a single bidder, to participate. Consistent with this, we show that if the regularity condition is globally violated for all bidders, then it is optimal for a seller first to identify those bidders whose adjusted virtual valuation is maximal, and to exclude bidders with lower adjusted virtual valuations. The seller selects one non-excluded bidder and sells to that bidder when the maximal adjusted virtual valuation exceeds $v_s$. The seller demands the highest equity share that this selected bidder $i$ would be willing to cede regardless of his type, hence demanding share $1 - \bar{r}_i$. Note that when bidders are ex-ante heterogeneous, multiple bidders can have the highest adjusted virtual valuation but their values of $\bar{r}_i$ can differ. When this is so, the seller can choose any of these bidders, but the seller must tailor its equity demand to the selected bidder.

**Corollary 5** Suppose bidders are ex-ante identical and the regularity condition is globally violated. Then expected seller revenue in the optimal equity auction does not vary with the number of bidders $n$. In contrast, in a standard first- or second-price format, revenue falls with $n$.

When the regularity condition is globally violated, not only are standard (first- or second-price) equity auction designs not optimal, even with optimal reserve prices, but expected seller revenue falls as competition rises. The greater is $n$, the lower is the winner’s break-even equity stake $r_i$, and this corresponds to a lower virtual valuation $\phi_i$ when the regularity
condition is globally-violated. By (23), seller revenue falls. Indeed, as \( n \) goes to infinity, the winner’s break-even stake approaches the \( r \) associated with the lowest virtual valuation. This contrasts with cash auctions (or equity auctions with constant opportunity costs), in which with ex-ante identical bidders, revenues in standard auction formats strictly rise with \( n \) (when the highest net valuation exceeds \( v_s \)), and a seller extracts full rents as \( n \) goes to infinity.

The result that competition reduces seller revenues when the regularity condition is globally violated reflects the intuition highlighted earlier that standard auction formats select in a “reinforced” direction of incentive compatibility. Thus, with global violations of regularity, the lowest virtual-valuation bidder is selected; and increased competition reduces this “lowest” virtual valuation. In contrast, in cash auctions (or equity auctions with constant opportunity costs), the regularity condition can be locally violated, but it cannot be violated at sufficiently high net-valuations, and as \( n \) grows large so does the probability of a high valuation.

If opportunity costs are common knowledge, optimal equity auctions generate higher revenues than cash auctions, even with ex-ante heterogeneous bidders (Hansen 1985; Liu 2016). Corollary 5 shows that if adverse selection is so severe that the regularity condition is globally violated, optimal equity auctions generate less revenues than cash auctions when \( n \) is large, consistent with Che and Kim (2010). In fact, this holds as long as virtual valuations for equity auctions are distorted at high types (i.e., if \( \phi (r) \) is less than the maximum net valuation).\(^{11}\)

We now give a simple way to implement the optimal mechanism if the regularity condition is globally violated for some bidders, showing how to distinguish between bidder populations.

**Corollary 6** Suppose the regularity condition is globally violated for bidders 1 through \( \hat{n} \), but not bidders \( \hat{n} + 1 \ldots n \), where \( \hat{n} < n \). Then it is optimal to sell sequentially in two stages: the seller first sells to bidders \( \{\hat{n} + 1 \ldots n\} \) via the modified mechanism in Proposition 3 that replaces \( v_s \) with \( \max \{v_s, \max_{j=1 \ldots \hat{n}} \bar{\phi}_j\} \). If the asset is not sold in the first stage, the seller sells to the first \( \hat{n} \) bidders, using the mechanism in Corollary 4.

Of note, Corollaries 3 through 6 hold as long as regularity is largely, even if not globally, violated: the corollaries follow as long as \( \bar{\phi}_i \) (see Lemma 3) is the same for all types of bidder

\(^{11}\)A sufficient condition for such distortion is that for a positive measure of types that correspond to the lowest break-even stake, the associated net valuation is strictly less than the maximum net valuation.
for which global violation of regularity is a sufficient, but not necessary, condition.

**Revenue Comparisons.** Our analysis provides insights into the forces that affect revenue comparisons of optimal equity and cash auctions. If uncertainty solely concerns valuations, then optimal equity auctions generate higher revenues, reflecting that equity bids tie payments to bidder type (Hansen 1985). When bidders have two-dimensional private information, additional forces come into play. First, if the regularity condition is largely violated, it favors cash over equity, a difference that is enhanced by more bidders because the adverse selection causes equity revenues to rise slowly at best with the number of bidders. This effect tends to rise with increased two-dimensional uncertainty that magnifies the adverse selection.

Second, as Proposition 1 shows, the distribution of net valuations in the one-dimensional representation for equity mechanisms second-order stochastically dominates the distribution of net valuations for the original two-dimensional distribution (and hence for cash auctions). The effects of this stochastic dominance are subtle. With few bidders, second-order stochastic dominance lets a seller set the reserve more efficiently, reducing the probability of no sale, and raising revenues. In contrast, with many bidders, the probability of no sale is small and what matters primarily is the upper-tail of the distribution, so that second-order stochastic dominance reduces revenues. Thus, this feature favors equity when there are few bidders, but it favors cash when \( n \) is large. Both of these effects are stronger when the extent of two-dimensional uncertainty increases, which increases the differences in the distributions.

Reflecting the combined forces, optimal equity auctions yield more revenues than cash auctions if there are few bidders or moderate two-dimensional uncertainty, but cash auctions do better with many bidders and high two-dimensional uncertainty. Example 3 illustrates.

**Example 3.** \( n \) ex-ante identical bidders bid with equity or cash. Bidder \( i \)'s expected valuation is \( v_i = 2x_i + y_i \), where \( y_i \) is independently and uniformly distributed on \([1, 3]\). Bidder \( i \)'s opportunity cost \( x_i \) is uniformly distributed over \([4 - d, 4 + d]\), where \( d \in [0, 4] \).

Implicitly, \( d \) measures the extent of two-dimensional uncertainty. In cash auctions, the regularity condition holds for all \( d \). In contrast, in equity auctions, the regularity condition is violated unless \( d \) is very close to zero. For a given \( d \), we use the fraction of all type pairs \((v_i, x_i)\) for which \(\frac{d}{dr} \phi (r) > 0\) at their corresponding \( r \) to measure the extent of violations.
Figure 1 shows how violations of the regularity condition in equity auctions rise with $d$, leading the extent of pooling in the optimal mechanism to rise.

Figure 1: **Violations.** $d$ is on the horizontal axis. The vertical axis is the fraction of probability mass for which the regularity condition is violated.

Figure 2 plots how the expected revenue differences between optimal equity and cash auctions varies with $d$ for different numbers of bidders. The non-monotone curvature ("ripples") in the evolution of the expected revenue difference with $d$ illustrates the possible complicated interactions of the different forces. With one or two bidders, equity auctions always generate higher revenues, even with extensive two-dimensional uncertainty. With more than two bidders, cash auctions generate higher expected revenues as long as there is sufficient two-dimensional uncertainty, a difference that rises with $n$. Once $d$ is sufficiently large, further increases in $d$ amplify revenue differences between the two auction designs for all $n$. Thus, with substantial two-dimensional uncertainty, optimal equity auctions greatly outperform optimal cash auctions with one bidder, but the opposite holds once there are enough bidders.

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$^{12}$The regularity condition for cash auctions is satisfied for all $d$ in this example. Thus, a second-price auction with optimal reserve is always optimal. In contrast, with equity auctions, the mechanism in Proposition 3 becomes the optimal mechanism when the regularity condition is violated.
5 Surplus-maximizing mechanisms

Lemma 1 holds for any incentive-compatible mechanism. As a result, our one-dimensional representation can be used to analyze mechanisms that maximize expected social surplus:

\[ \eta = \int \left( \sum_{i=1}^{n} W_i(t)(v_i - x_i) + W_0(t)v_s \right) f(t) \, dt. \] (42)

To proceed, we identify a ‘surplus valuation’, which is the analogue of a virtual valuation:

**Definition 3** Bidder \( i \)'s surplus valuation at \( r_i \in [\underline{r}_i, \bar{r}_i] \) in equity auctions is

\[ \kappa_i(r_i) \equiv (1 - r_i) a_i(r_i). \] (43)

**Theorem 3** In any incentive-compatible mechanism of equity auctions, expected surplus is:

\[ \eta = \int_{S_1 \times \ldots \times S_n} \left( \sum_{i=1}^{n} W_i(t) \kappa_i(r_i) + W_0(t)v_s \right) f(t) \, dt_1 \ldots dt_n. \] (44)

By (24), the surplus valuation \( \kappa_i(r_i) \) is the expected net valuation at \( r_i \). Comparing (44) with (23) suggests that the role \( \kappa_i(r_i) \) plays in surplus maximizing is analogous to that played
by the virtual valuation in revenue maximizing. Indeed, analysis of surplus-maximizing mechanisms follows that of revenue-maximizing mechanisms, simply by replacing ‘virtual valuation’ with ‘surplus valuation’. Thus, the regularity condition for surplus maximization requires \( \kappa_i(r_i) \) rather than \( \phi_i(r_i) \) to strictly decrease over \([r_i, \bar{r}_i]\). When this regularity condition holds for all \( i \), then in the surplus-maximizing mechanism: (1) a bidder with the highest surplus valuation wins whenever this surplus valuation exceeds the seller’s valuation, and (2) the equity share \( r \) retained by the winning bidder \( i \) solves \( \kappa_i(r) = \max_{j \neq i} \{ \kappa_j(r_j), v_s \} \).

When the regularity condition is violated for one or more bidders, a surplus-maximizing mechanism can be constructed along the lines for revenue-maximization (Proposition 3) via concavification. We sketch this structure out in the appendix.

The properties of cash and equity auctions for surplus-maximizing mechanisms differ. With cash, a second-price auction (with optimal reserve) always maximizes surplus. With equity, however, a second-price auction typically fails to maximize surplus when bidders are ex-ante heterogeneous, even if the design is regular. This difference reflects that in cash auctions, bidders have the same “surplus valuation” function—the surplus valuation is given by a bidder’s net valuation, even when bidders are ex-ante heterogeneous—whereas in equity auctions, the surplus valuation function \( \kappa_i(\cdot) \) differs across heterogeneous bidders. As a result, the surplus-maximizing equity mechanism does not have a second-price format.\(^{13}\)

Moreover, the regularity condition for surplus maximization in cash auctions always holds: the net valuation is trivially an increasing function of itself. In contrast, in equity auctions, the analogous regularity condition can be violated everywhere.\(^{14}\) When such global violations arise, disregarding all bids and selling to any bidder with the highest adjusted surplus-valuation maximizes expected surplus (when the adjusted surplus valuation exceeds \( v_s \)). Thus, in equity auctions, pooling can be both revenue-maximizing and surplus-maximizing. This contrast reinforces how with severe adverse selection, incentive compatibility requires “inferior” types (in terms of either virtual or surplus valuations) to win with a weakly higher probability. As a result, a mechanism designer can do no better than to make the winning

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\(^{13}\)Note that if bidders have publicly-known but heterogeneous opportunity costs, then regularity holds and the surplus-maximizing equity mechanism reduces to the efficient mechanism identified by Hansen (1985).

\(^{14}\)Lemma 4 in the appendix provides a condition for the regularity condition to be locally violated. This condition implies that global violations occur when the extreme adverse selection (decreasing bidding strategy) condition identified by Che and Kim (2010) holds.
probability independent of bidder type.

Finally, in equity auctions, when the regularity condition is sometimes but not always violated, the support over which adjusted surplus valuations pool typically differs from that over which adjusted virtual valuations pool, implying that revenue-maximizing mechanisms are not efficient. Figure 3 reveals that in the canonical uniform-uniform setting of Example 3, violations are greater for surplus maximization than revenue maximization. In contrast, in cash auctions, the opposite holds because the regularity condition for surplus maximization can never be violated, while it can be for revenue maximization.

![Figure 3: Violations: Surplus-maximizing vs. Revenue-maximizing mechanisms. The horizontal axis measures $d$. The vertical axis is fraction of probability mass with violations of regularity condition. Surplus-maximizing: green line. Revenue-maximizing: red line.](image)

To see the intuition for the difference, observe that the virtual valuation (20) differs from the surplus valuation (43) by the second term in (20). This term is zero at $r_i = r_i^*$, and it is negative (it is subtracted) for all $r_i > r_i^*$. For the most part, this term decreases in $r_i$, causing regularity to be violated to a lesser extent for revenue maximization. So, too, in cash auctions, the virtual valuation (41) differs from the surplus valuation by the second term. This second term is zero for the highest net-valuation and it is strictly negative elsewhere, generally (but not always) trending upward in net valuation—in the direction mandated by regularity. The difference is that, without this second term, the regularity condition always holds for
surplus maximization: adding this second term can only lead to violations of regularity, but not reduce them. Thus, from an efficiency perspective, revenue-maximizing mechanisms can only over-pool in cash auctions, whereas they tend to under-pool in equity auctions.

6 Conclusion

In equity auctions when bidders’ values and opportunity costs are private information, adverse selection can arise that results in conflict between revenue maximization, which requires a seller to allocate the asset to bidders with higher virtual valuations; and incentive compatibility, which demands that bidder types with lower break-even equity requirements be weakly more likely to win. Unlike in cash auctions, virtual valuations and incentive compatibility may go globally in opposite directions. We derive the implications for the optimal design. For example, if adverse selection is extreme for all bidders, a seller does best to identify bidders with the highest adjusted virtual valuation, exclude all others, select one non-excluded bidder, and demand the highest equity share that this bidder would cede regardless of his type. Thus, our work provides guidance on standard formats, identifying when they cease to be an appropriate selling mechanism; and when they still perform well given simple modifications. These insights also hold for equity mechanisms that maximize expected social surplus.

The general principles identified should extend to auctions in which bidders offer ordered securities other than equities. For example, the optimality of simple ways to sell when adverse selection is severe for some or all bidders should hold for a given class of ordered securities. Liu and Bernhardt (2018) investigate settings in which a seller can combine different classes of ordered securities, showing that a seller can then do better when adverse selection obtains.
7 Bibliography


Skrzypacz, A. 2013. “Auctions with contingent payments: An overview”, *International*
8 Appendix: Proofs

Proof of Lemma 2: Consider a plane where $x_i$ is on the x-axis and $v_i$ is on the y-axis. Then, $R$ is the distance from point $(x_i = rR, v_i = R \sqrt{(1+r^2)})$ to the origin. Now consider the fan-shaped sector in the first quadrant defined by lines $x_i/v_i = r$ and $x_i/v_i = r + \epsilon$. Denote the opening angle between the two lines by $d\theta$. From the property of a fan-shaped sector that the length of the arc at radius $R$ is $Rd\theta$, the total mass between $R$ and $dR$ is proportional to $R$. □

Proof of Theorem 1: We first prove the “only if”. Conditions (i) – (iii) follow from the text. To prove condition (iv), note that if it is not profitable for $t_i$ to mimic $t'_i$, and vice versa, then

$$(q_i(t'_i) - r_i) G_i(t'_i) \leq (q_i(t_i) - r_i) G_i(t_i)$$

and

$$(q_i(t_i) - r'_i) G_i(t_i) \leq (q'_i(t_i) - r'_i) G'_i(t_i).$$

Adding these two equations yields $(r'_i - r_i) (G_i(t'_i) - G_i(t_i)) \leq 0$, which yields condition (iv). By (9) and individual rationality, $M_i(r_i) \geq 0$ for all $r_i$, establishing condition (v).

To prove the “if” part, assume conditions (i) through (v) hold. We show it is not profitable for a type $t_i$ to mimic a type $t'_i$, i.e., $m_i(t_i, t'_i) - m_i(t_i, t_i) \leq 0$. By conditions (i) and (ii),

$$m_i(t_i, t'_i) - m_i(t_i, t_i) = m_i(t_i, t'_i) - m_i(t'_i, t'_i) + M_i(r'_i) - M_i(r_i)$$

$$= (r'_i - r_i) G_i(t'_i) + M_i(r'_i) - M_i(r_i). \quad (45)$$

If $r'_i = r_i$, then $m_i(t_i, t'_i) - m_i(t_i, t_i) = 0$, hence deviation is not profitable. Now consider a report $t'_i \neq t_i$ for which $r'_i \neq r_i$. By conditions (i) – (iii) and the fact that an absolutely continuous function is the definite integral of its derivative, (18) holds and (45) yields

$$m_i(t_i, t'_i) - m_i(t_i, t_i) = (r'_i - r_i) G_i(t'_i) + \int_{r'_i}^{r_i} G_i(r) \, dr. \quad (46)$$

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By condition (iv), the right-hand side of (46) is non-positive regardless of whether \( r'_i > r_i \) or \( r'_i < r_i \). Thus, \( m_i (t_i, t'_i) - m_i (t_i, t_i) \leq 0 \), i.e., deviation is not profitable. This establishes incentive compatibility. By condition (v) of the theorem, (18), and \( G_i \geq 0 \), we have \( M_i (r_i) \geq 0 \) for all \( r_i \). Therefore, by (9), individual rationality holds.

**Proof of Theorem 2.** Let \( \pi_s,i \) be bidder \( i \)'s contribution to the seller’s expected profit:

\[
\pi_{s,i} = \int_{S_i} G_i (t_i) (v_i - x_i - v_s) f_i (t_i) dt_i - \int_{S_i} h_i (t_i, t_i) f_i (t_i) dt_i.
\]  

(47)

Adding \( v_s \), the seller’s expected revenue becomes:

\[
\pi_s = \sum_{i=1}^{n} \pi_{s,i} + v_s.
\]  

(48)

To aid analysis, define

\[
A_i (r_i) = \int_{\bar{r}_i}^{r_i} a_i (r) \tilde{f}_i (r) dr
\]  

(49)

and

\[
l_i (r) = \lim_{\epsilon \to 0} E [v_i - x_i - v_s | r \leq x_i/v_i \leq r + \epsilon],
\]  

(50)

where the expectation is taken over \( S_i \). We next rewrite \( \pi_i \) in terms of \( r_i \). Integrate the first term on the RHS of (47) using (15):

\[
\int_{S_i} G_i (t_i) (v_i - x_i - v_s) f_i (t_i) dt_i = \int_{\bar{r}_i}^{r_i} G_i (r_i) l_i (r_i) \tilde{f}_i (r_i) dr_i.
\]  

(51)

Similarly, by (9), the second term on the RHS of (47) becomes

\[
\int_{S_i} h_i (t_i, t_i) f_i (t_i) dt_i = \int_{\bar{r}_i}^{r_i} M_i (r_i) v_i f_i (t_i) dt_i = \int_{\bar{r}_i}^{r_i} M_i (r_i) a_i (r_i) \tilde{f}_i (r_i) dr_i.
\]  

(52)

Substituting (51) and (52) into (47) yields

\[
\pi_{s,i} = \int_{\bar{r}_i}^{r_i} G_i (r_i) l_i (r_i) \tilde{f}_i (r_i) dr_i - \int_{\bar{r}_i}^{r_i} M_i (r_i) a_i (r_i) \tilde{f}_i (r_i) dr_i.
\]  

(53)

Substituting (18) for \( M_i (r_i) \) and using (49), we rewrite the second integral in (53) as

\[
\int_{\bar{r}_i}^{r_i} \left( M_i (\bar{r}_i) + \int_{\bar{r}_i}^{r_i} G_i (r) dr \right) dA_i (r_i) = M_i (\bar{r}_i) A_i (\bar{r}_i) + \int_{\bar{r}_i}^{r_i} A_i (r_i) G_i (r_i) dr_i.
\]  

(54)

Next, substitute (54) into (53) to obtain

\[
\pi_{s,i} = \int_{\bar{r}_i}^{r_i} G_i (r_i) \left( l_i (r_i) - \frac{A_i (r_i)}{\tilde{f}_i (r_i)} \right) \tilde{f}_i (r_i) dr_i - M_i (\bar{r}_i) A_i (\bar{r}_i).
\]  

(55)
We now return to the original two-dimensional framework and write \( \pi_{s,i} \) as

\[
\pi_{s,i} = \int_{S_i} G_i(t_i) \left( l_i(r_i) - \frac{A_i(r_i)}{f_i(r_i)} \right) f_i(t_i) \, dt_i - M_i(\bar{r}_i) A_i(\bar{r}_i)
\]

\[
= \int_{S_i \times \ldots \times S_n} W_i(t) \left( l_i(r_i) - \frac{A_i(r_i)}{f_i(r_i)} \right) f(t) \, dt_1 \ldots dt_n - M_i(\bar{r}_i) A_i(\bar{r}_i).
\]

Noting \( l_i(r) = (1 - r) a_i(r) - v_s \) by \( x_i = (x_i/v_i) v_i \) and (50), plugging in (48) and substituting for the virtual valuation, the theorem follows.

**Derivations for Example 1.** We use the uniform-uniform distribution of \((x_i, y_i)\) rather than that of \((x_i, v_i)\). For \( \frac{x_i}{v_i} = r_i \), we have \( \frac{x_i}{x_i + y_i} = r_i \), yielding

\[
y_i = \left( \frac{1}{r_i} - 1 \right) x_i.
\]

For a given \( r_i \in (0, 1) \), (56) is the straight line through \((0,0)\) with slope \( k(r_i) \equiv \frac{1}{r_i} - 1 = \frac{1-r_i}{r_i} \).

If \( k(r_i) \leq 1 \), then this line forms a right-angle triangle with the \( x_i \) axis and line \( x_i = 1 \). The length of the segment inside the unit square is therefore \((1 + k(r_i)^2)^{0.5}\). Similarly, if \( k(r_i) > 1 \), the length of the line segment inside the square is \( \left(1 + \frac{1}{k(r_i)^2}\right)^{0.5} \). Combining, for all \( r_i \), we write the length of the line segment inside the square as

\[
\left\{ \left(1 + \min \left\{ k(r_i), \frac{1}{k(r_i)} \right\} \right)^2 \right\}^{0.5} = \left\{ \left(1 + \min \left\{ \frac{1-r_i}{r_i}, \frac{r_i}{1-r_i} \right\} \right)^2 \right\}^{0.5}.
\]

The angle corresponding to this line is \( \theta(r_i) \equiv \arctan \left( \frac{1}{r_i} - 1 \right) \), which has derivative \( \frac{d\theta}{dr_i} = \frac{-1}{r_i^2 + (1-r_i)^2} \). Thus, the mass in the differential fan-shaped sector with slopes between \( k(r_i) \) and \( k(r_i + dr_i) \) is \( \frac{1}{2} \left( 1 + \min \left\{ \frac{1-r_i}{r_i}, \frac{r_i}{1-r_i} \right\} \right)^2 / (r_i^2 + (1-r_i)^2) \, dr_i \), reflecting that the area of a fan-shaped sector is \(1/2\) times the radius squared times the opening angle. This yields (26).

We now calculate \( a_i(r_i) \) in (19). If \( r_i \geq 0.5 \), then \( k(r_i) \leq 1 \) and line (56) leaves the unit square at point \((x_i = 1, y_i = 1/r_i - 1)\) where \( v_i = x_i + y_i = 1/r_i \). Consider the fan-shaped sector inside the square bounded by lines with angles \( \theta \) and \( \theta + d\theta \). Taking the mass-weighted expectation of \( v_i \) in this sector yields \( a_i(r_i) = \frac{2}{3 r_i} \), where the factor \( 2/3 \) comes from the property of a fan-shaped sector that the length of the arc at radius \( R \) is \( R d\theta \)—and hence the total mass between \( R \) and \( dR \) is proportional to \( R \), and that the value of \( v_i \) at radius \( R \) is proportional to \( R \). Including the other scenario in which \( r_i < 0.5 \), we have

\[
a_i(r_i) = \frac{2}{3} \left\{ 1 + \frac{1}{k(r_i)} \right\} = 1/ (1 - r_i) \quad \text{if} \quad r_i \geq 0.5
\]

\[
\text{if} \quad r_i < 0.5.
\]

(57)
That is, \( a_i(r_i) = \frac{2}{3\max\{r_i,1-r_i\}} \). By Corollary 1, (25) follows.

**Proof of Proposition 1:** In the one-dimensional distribution prescribed by Corollary 1, for any \( r \), conditional on \( r_i \in [r,r+\epsilon] \) where \( \epsilon \) is infinitesimal, \( i \)'s net valuation is \( a_i(r)(1-r) \). By (19),

\[
a_i(r)(1-r) \equiv \lim_{\epsilon \to 0} E\left[v_i(1-r) \mid r \leq x_i/v_i \leq r + \epsilon\right] \quad (58)
\]

\[
= \lim_{\epsilon \to 0} E\left[v_i - x_i \mid r \leq x_i/v_i \leq r + \epsilon\right]. \quad (59)
\]

Thus, the net valuation distribution for the original two-dimensional distribution is a mean-preserving spread of that for the one-dimensional representation of Corollary 1. \( \square \)

**Proofs of Proposition 2 and Corollary 2:**

**Step 1:** To prove Corollary 2, we first show that truth-telling is an equilibrium. Compare the profits of bidder \( i \) with type \( r_i \) from truthful reporting and under-reporting (i.e., reporting a type pair that corresponds to a lower break-even equity stake than \( r_i \)). (1) If truthful and under-reporting both result in winning, then by (29), profits are the same. (2) If truthful and under-reporting both result in losing, profits are also the same (zero). (3) If truthful reporting results in losing, but under-reporting results in winning, under-reporting is strictly unprofitable because \( \max_{j \neq i} \{ \phi_j(r_j) \}, v_s > \phi_i(r_i) \) and \( Q_i < r_i \) by (29). Summarizing, truthful reporting weakly dominates under-reporting. A similar argument shows that truthful reporting weakly dominates over-reporting. Further, individual rationality is satisfied because a bidder can always ensure a zero profit by reporting a sufficiently high \( r_i \).

Next, we show \( M_i(\tilde{r}_i) = 0 \) for all \( i \). By (10), it suffices to consider \( G_i(\tilde{r}_i) > 0 \). By (27) and (29), when bidder \( \tilde{r}_i \) wins in equilibrium, \( Q_i = \tilde{r}_i \) and hence \( M_i(\tilde{r}_i, \tilde{r}_i) = 0 \). Thus, \( \pi_{s,a} \) (equation (22)) is zero, which is its maximum possible value. Further, because a bidder with the maximum virtual valuation \( \phi_i(x_i) \) wins when \( \phi_i(x_i) > v_s \), \( \pi_{s,b} \) obtains its maximum possible value. Because this mechanism maximizes \( \pi_{s,a} \) and \( \pi_{s,b} \) simultaneously, it is optimal.

**Step 2:** We show the “if” part of Proposition 2 by assuming (27) and (28). By (10), (27), and (28), it is without loss to denote \( m_i(t_i, t'_i) \) by \( m_i(r_i, r'_i) \). Equation (10) yields

\[
m_i(r_i, r'_i) - m_i(r_i, r'_i) = (q_i(r_i) - r_i)G_i(r_i) - (q_i(r'_i) - r_i)G_i(r'_i)
\]

\[
= (q_i(r_i) - r_i)G_i(r_i) - (q_i(r'_i) - r'_i)G_i(r'_i) + (r_i - r'_i)G_i(r'_i),
\]

33
which, by (28), yields

\[ m_i(r_i, r_i) - m_i(r_i, r'_i) = \int_{r_i}^{r'_i} G_i(r) \, dr - (r'_i - r_i) G_i(r'_i) \geq 0, \]

where the inequality follows because \( W_i(r_i, t_{-i}) \) is weakly decreasing in \( r_i \) and hence so is \( G_i(r_i) \). This establishes that truth-telling is an equilibrium. Further, by (28) and (10), \( m_i(\bar{r}_i, r_i) = 0 \). By arguments similar to those in Step 1, the mechanism is optimal.

\textbf{Step 3:} We prove the “only if” part of Proposition 2. As Corollary 2 shows, a mechanism exists that simultaneously maximizes \( \pi_{s,a} \) and \( \pi_{s,b} \). An optimal mechanism cannot do worse, so it must also maximize both \( \pi_{s,a} \) and \( \pi_{s,b} \). Thus, we have (27). Similarly, we have \( m_i(\bar{r}_i, r_i) = 0 \) for all \( i \). By (4) and (27), \( G_i \) is a function of \( r_i \) only and is continuous in \( r_i \), and \( M_i(r_i) \ ((18)) \) is differentiable everywhere. By Theorem 1, \( q_i \) is a function of \( r_i \) only. Thus (28) follows upon equating the right-hand sides of (10) and (18).

\textbf{Proof of Proposition 3:} As \( C_i(\cdot) \) is the concave hull of \( B_i(\cdot) \), we have

\[ C_i(y) = \max_{\omega, z_1, z_2} \left\{ \omega B_i(z_1) + (1 - \omega) B_i(z_2) \right\} \]

s.t. \( \{\omega, z_1, z_2\} \in [0,1]^3 \) and \( \omega z_1 + (1 - \omega) z_2 = y \).

Thus, \( C_i(\cdot) \) is the lowest concave function on \([0,1]\) such that \( C_i(y) \geq B_i(y) \) for all \( y \).

It is straightforward to show that truth telling is an equilibrium. We next establish optimality. Given any allocation rule \( W_i(\cdot) \) and associated \( G_i(\cdot) \) as defined in (4), we have

\[ \int_{\xi_i}^{r_i} G_i(r_i) \left( C'_i \left( \tilde{F}_i(r_i) \right) - b_i \left( \tilde{F}_i(r_i) \right) \right) \tilde{f}_i(r_i) \, dr_i \]

\[ = G_i(r_i) \left( C_i \left( \tilde{F}_i(r_i) \right) - B_i \left( \tilde{F}_i(r_i) \right) \right) \left| \int_{\xi_i}^{r_i} \left( B_i \left( \tilde{F}_i(r_i) \right) - C_i \left( \tilde{F}_i(r_i) \right) \right) dG_i(r_i) \right| + \int_{\xi_i}^{r_i} \left( B_i \left( \tilde{F}_i(r_i) \right) - C_i \left( \tilde{F}_i(r_i) \right) \right) dG_i(r_i) \]

\[ = \int_{\xi_i}^{r_i} \left( B_i \left( \tilde{F}_i(r_i) \right) - C_i \left( \tilde{F}_i(r_i) \right) \right) dG_i(r_i) \geq 0, \]

(60)

where the inequality in (60) follows because \( B_i(\tilde{F}_i(r_i)) \leq C_i(\tilde{F}_i(r_i)) \) and \( G_i(\cdot) \) is non-increasing in \( r_i \) by incentive compatibility.

Let \( \tilde{G}_i(r_i) \) be the distribution corresponding to allocation rule \( \tilde{W}(\cdot) \) in (33). Then (60) holds as an equality,

\[ \int_{\xi_i}^{r_i} \tilde{G}_i(r_i) \left( C'_i \left( \tilde{F}_i(r_i) \right) - b_i \left( \tilde{F}_i(r_i) \right) \right) \tilde{f}_i(r_i) \, dr_i = 0, \]

(61)
because when \( B_i(\bar{F}_i(r_i)) < C_i(\bar{F}_i(r_i)) \), the concave hull \( C_i(\cdot) \) is a straight line locally, implying the derivative \( C'_i(\cdot) = \bar{\phi}_i(\cdot) \) is constant. Thus \( G_i(\cdot) \) is constant and \( dG_i(\cdot) = 0 \). Further,

\[
\int \left( \sum_{i=1}^{n} W_i(t)(\bar{\phi}_i(t_i) - v_s) \right) f(t) dt_1...dt_n \geq \int \left( \sum_{i=1}^{n} W_i(t)(\bar{\phi}_i(t_i) - v_s) \right) f(t) dt_1...dt_n
\]

where the second inequality follows from (60). Moreover, from (61), we have

\[
\int \left( \sum_{i=1}^{n} W_i(t)(\phi_i(t_i) - v_s) \right) f(t) dt_1...dt_n = \int \left( \sum_{i=1}^{n} W_i(t)(\phi_i(t_i) - v_s) \right) f(t) dt_1...dt_n.
\]

Substituting (63) into the left-hand-side of (62) yields

\[
\int \left( \sum_{i=1}^{n} W_i(t)(\phi_i(t_i) - v_s) \right) f(t) dt_1...dt_n \geq \int \left( \sum_{i=1}^{n} W_i(t)(\phi_i(t_i) - v_s) \right) f(t) dt_1...dt_n.
\]

Thus, \( \bar{W}_i(\cdot) \) maximizes the term \( \pi_{s,a} \) in (21). The mechanism in the proposition also maximizes \( \pi_{s,b} \) by having \( M_i(\bar{r}_i) = 0 \) for all \( i \). Optimality of the mechanism follows.

**Proof of Proposition 4:** Observe that if, for some \( d_1 > 0 \),

\[
\frac{d \ln x_i}{d \ln v_i} \in (1, 1 + d_1)
\]

at all \( v_i \in [v_i, \bar{v}_i] \) then

\[
\frac{d_1}{d_1} v_i < x_i < \frac{d}{d_1} \left( \frac{v_i}{\bar{v}_i} \right)^{1+d_1}.
\]

This observation means that for any \( r^* \in \left( \frac{d}{d_1}, 1 \right) \) (the value of \( r^* \) is unimportant), there exists a \( d_1^* > 0 \), such that for any \( d_1 \in (0, d_1^*) \), we have \( \frac{d_1}{d_1^*} < r^* \) for \( \forall v_i \in [v_i, \bar{v}_i] \). By (64),

\[
\frac{d \ln x_i}{d \ln v_i} = \frac{d x_i}{d v_i} \frac{v_i}{x_i} \leq 1 + d_1 \iff \frac{d x_i}{d v_i} \leq \frac{x_i}{v_i} (1 + d_1).
\]

Define \( d_1^{*} \equiv \min \{ d_1^*, \frac{1}{r^*} - 1 \} \). Then, for all \( d_1 \in (0, d_1^{*}) \), \( \frac{d_1}{d_1^*} < r^* \) and (66) yield \( \frac{d x_i}{d v_i} < 1 \).
To proceed further, in (64) let \( d_1 \in (0, d_1^{**}) \). We have

\[
\frac{d}{dv_i} \phi_i(v_i) = 1 - \frac{dx_i}{dv_i} - \left( \frac{d}{dv_i} \int_{v_i}^{v_i} vF_i(v) \frac{x_i(v_i)}{v_i^2} \right) \frac{d}{dv_i} \left( \frac{d\ln x_i(v_i)}{d\ln v_i} - 1 \right) - \int_{v_i}^{v_i} vF_i(v) \frac{x_i(v_i)}{v_i^2} \frac{d}{dv_i} \left( \frac{d\ln x_i(v_i)}{d\ln v_i} - 1 \right) - \int_{v_i}^{v_i} vF_i(v) \frac{x_i(v_i)}{v_i^2} \left( \frac{d}{dv_i} \frac{d\ln x_i(v_i)}{d\ln v_i} \right).
\]

We first show that the absolute values of \( \int_{v_i}^{v_i} vF_i(v) \) and its derivative are both bounded:

\[
\left| \frac{\int_{v_i}^{v_i} vF_i(v)}{f_i(v_i)} \right| \leq \frac{\bar{v}_i}{\min_{v_i} \{f_i(v_i)\}} \equiv B_1
\]

and

\[
\left| \frac{d}{dv_i} \int_{v_i}^{v_i} vF_i(v) \right| = |v_i - \int_{v_i}^{v_i} vF_i(v) f_i'(v_i) | \leq \bar{v}_i \left( 1 + \max_{v_i} \left\{ \left| f_i'(v_i) / f_i^2(v_i) \right| \right\} \right) \equiv B_2.
\]

Further, the absolute values of \( \frac{x_i}{v_i^2} \) and its derivative are both bounded:

\[
\left| \frac{x_i}{v_i^2} \right| < \bar{v}_i \equiv B_3 \quad \text{and} \quad \frac{d}{dv_i} \frac{x_i}{v_i^2} = \frac{1}{v_i^2} \frac{dx_i}{dv_i} - \frac{2x_i}{v_i^3}
\]

and thus

\[
\left| \frac{d}{dv_i} \frac{x_i}{v_i^2} \right| < \frac{1}{v_i^2} + \frac{2x_i}{v_i^3} \equiv B_4.
\]

Thus,

\[
\frac{d}{dv_i} \phi_i(v_i) > 1 - \frac{dx_i}{dv_i} - (B_2B_3 + B_1B_4) \left( \frac{d\ln x_i(v_i)}{d\ln v_i} - 1 \right) - B_1B_3 \left( \frac{d}{dv_i} \frac{d\ln x_i(v_i)}{d\ln v_i} \right). \quad (67)
\]

Let \( \epsilon_1 \in (0, d_1^{**}) \). When \( \frac{d\ln x_i}{d\ln v_i} \in (1, 1 + \epsilon_1) \), \( \frac{dx_i}{dv_i} < 1 \). In addition, by (67), there exists \( \epsilon_2 > 0 \) such that when \( \left| \frac{d}{dv} \left( \frac{d\ln x_i}{d\ln v_i} \right) \right| < \epsilon_2 \), \( \frac{d}{dv} \phi_i(v_i) > 0 \) at all \( v_i \in [\bar{v}_i, \bar{v}] \). Because \( r_i \) strictly increases in \( v_i \) by \( \frac{d\ln x_i}{d\ln v_i} > 1 \), \( \phi_i \) strictly increases in \( r_i \), completing the proof.

**Proof of Lemma 3:** If the regularity condition is globally violated for bidder \( i \), then \( B_i(\cdot) \) in (31) is convex. Therefore, the concave hull \( C_i(\cdot) \) is a straight line on \([0, 1]\).

**Sketch of Analysis of Surplus-maximizing mechanisms.** The analysis directly follows that for revenue-maximizing mechanism. Here we define the key notation; substituting into the analogous arguments for revenue-maximization yields the results.

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Let \( \eta_i \) be bidder \( i \)'s contribution to expected social surplus (over the value when the seller retains the asset):

\[
\eta_i = \int_{S_i} G_i(t_i) (v_i - x_i - v_s)f_i(t_i)\,dt_i.
\]

Adding the surplus of \( v_s \) when the seller retains the asset, total expected surplus is

\[
\eta = \sum_{i=1}^{n} \eta_i + v_s. \quad (68)
\]

We next rewrite \( \eta_i \) in terms of \( r_i \) using (15) and (19):

\[
\eta_i = \int_{\mathbb{R}_i} G_i(r_i) (\kappa_i(r_i) - v_s) \mathcal{f}_i(r_i)\,dr_i. \quad (69)
\]

Returning to the original two-dimensional framework, we have

\[
\eta_i = \int_{S_1 \times \ldots \times S_n} \left( W_i(t) (\kappa_i(r_i) - v_s) \right) f(t)\,dt_1 \ldots dt_n.
\]

Substituting yields the formulation in (44).

**Definition 4** The design problem for surplus maximization is regular if \( \kappa_i(r_i) \) strictly decreases over \([r_i, \bar{r}_i]\) for all \( i \).

**Proposition 5** Suppose that the design problem for surplus maximization is regular. Then the following constitutes a surplus-maximizing equity mechanism:

(i) the winning rule is

\[
W_i(t) = \begin{cases} 
1 & \text{if } \kappa_i(r_i) > \max \{ \max_{j \neq i} \{ \kappa_j(r_j) \}, v_s \} \\
0 & \text{if } \kappa_i(r_i) < \max \{ \max_{j \neq i} \{ \kappa_j(r_j) \}, v_s \}
\end{cases},
\]

for all \( i \) and \( t \).

(ii) the equity-retention rule for the winning bidder is

\[
Q_i(t) = \kappa_{i}^{-1} \left( \max \{ \max_{j \neq i} \{ \kappa_j(r_j) \}, v_s \} \right),
\]

where \( \kappa_{i}^{-1} (\cdot) \) denotes a bounded inverse of \( \kappa_i (\cdot) \):

\[
\kappa_{i}^{-1} (x) \equiv \begin{cases} 
y \in [r_i, \bar{r}_i] & \text{s.t. } \kappa_i(y) = x \\
0 & \text{if } x < \kappa_i(\bar{r}_i) \\
1 & \text{if } x \in [\kappa_i(\bar{r}_i), \kappa_i(r_i)]
\end{cases}.
\]

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In cash auctions, the regularity condition—which mandates that the surplus valuation be increasing in net valuation—always holds because the surplus valuation is the net valuation. By contrast, the regularity condition in equity auctions that \( \kappa_i(r_i) \) decrease in \( r_i \), need not always hold. Indeed, in one-dimensional deterministic setting of Che and Kim (2010), we have

**Lemma 4** If \( x_i \) is deterministic of \( v_i \) and \( \frac{dx}{dv_i} \in (0,1) \), then the regularity condition holds for bidder \( i \) if and only if \( r_i \) is strictly decreasing in \( v_i \).

**Proof:** By \( \frac{dx}{dv_i} \in (0,1) \), bidder \( i \)'s net valuation strictly increases with \( v_i \). Thus net valuation strictly decreases in \( r_i \) if and only if \( r_i \) strictly decreases in \( v_i \). The lemma follows.

When the regularity condition is violated for one or more bidders, a surplus-maximizing mechanism can again be constructed along the lines for the revenue-maximizing mechanism, defining an adjusted surplus valuation \( \bar{\kappa}_i(r_i) \) that is constructed via \( \kappa_i(r_i) \) in a similar way as how \( \bar{\phi}_i(r_i) \) is constructed via \( \phi_i(r_i) \). For any vector of bidder types \( t \), let \( N(t) \) be the set of bidders for whom \( \bar{\kappa}_i(r_i) \) is both maximal and higher than \( v_s \).

**Proposition 6** The following constitutes a surplus-maximizing equity mechanism:

(i) the winning rule is

\[
\bar{W}_i(t) = \begin{cases} 
1/|N(t)| & \text{if } i \in N(t) \\
0 & \text{if } i \notin N(t)
\end{cases}
\] (73)

for all \( i \) and \( t \), and

(ii) the equity-retention rule for the winning bidder is the same as that in part (ii) of Proposition 5 with \( \kappa_i(r_i) \) replaced by \( \bar{\kappa}_i(r_i) \).

If the regularity condition for surplus maximization is globally violated for all bidders, then as with revenue-maximizing mechanisms, pooling can be optimal. That is, the best a mechanism designer can do is to randomize:

**Corollary 7** If the regularity condition for surplus maximization is globally violated for all bidders, then it maximizes expected surplus to disregard all bids and sell to any bidder \( i \in \arg \max_j \bar{\kappa}_j \) (if \( \bar{\kappa}_i > v_s \)), asking for equity share \( 1 - \bar{r}_i \). The bidder always accepts the offer.