THE PROBABILISTIC POINT OF VIEW ON THE GENERALIZED FRACTIONAL PDES

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Abstract

This paper aims at unifying and clarifying the recent advances in the analysis of the fractional and generalized fractional PDEs of Caputo and Riemann-Liouville type arising essentially from the probabilistic point of view. This point of view leads to the path integral representation for the solutions of these equations, which is seen to be stable with respect to the initial data and key parameters and is directly amenable to numeric calculations (Monte-Carlo simulation). In many cases these solutions can be compactly presented via the wide class of operator-valued analytic functions of the Mittag-Leffler type, which are proved to be expressed as the Laplace transforms of the exit times of monotone Markov processes.

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1. Introduction

The simplest Cauchy problem for the linear equation of Caputo or Caputo-Dzherbashyan type

$$D^{\beta}_{a+\star}f(x) = -\lambda f(x), \quad f(a) = Y,$$

with $\beta > 0$ is known to have the unique solution

$$f(x) = E_{\beta}(-\lambda x^{\beta})Y,$$

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where E_β is the Mittag-Leffler function. Lots of research in the modern theory of fractional PDEs is devoted to various extensions of this equation, when λ is replaced by an unbounded operator in some Banach space (for instance, a diffusion operator) and $D^\beta_{a+\star}$ by various versions of generalized derivatives including mixtures of fractional derivatives.

Generalized fractional calculus was initially developed by extending fractional integrals to the integral operators with more general integral kernels and then defining the fractional derivatives as the compositions of these integrals with usual derivatives.

Our alternative approach was suggested in [39] and was motivated by probabilistic interpretation of fractional derivatives. It starts with the definition of the one-sided generalized fractional derivatives as the generators of monotone Markov or sub-Markov processes. The generalized fractional integrals is then defined as the corresponding right inverse operators. The objective of this paper is to overview the development of the probabilistic approach and to explain in detail how the most fundamental examples of generalized fractional operators and related fractional equations fit to the general probabilistic framework. To make the text accessible to the readers with a mild background in probability we stress everywhere when possible the analytic counterparts of the formulas and their interpretations.

The paper is organised as follows. In Section 2 we show how our generalized mixed fractional operators can be equivalently introduced from the three points of views, bringing together the languages and methods of probability theory, operator semigroups and generalized functions (and related pseudo-differential equations) and giving the meaning to the generalized fractional integrals in terms of Dynkin's martingale, potential operators and fundamental solutions, respectively.

In Section 3 we recall the main generalized fractional operators discussed in the literature and derive their probabilistic representations as the generators or potential operators of appropriate Markov processes or semigroups. For completeness, we explain briefly how the introduction of these operators can be motivated via two analytic approaches: interpolation from integer-valued iterations and the dressing by the operators of multiplication and the change of variables. Thus we show how various different operators scattered through the literature arise in a unified way from simple general ideas.

Section 4 deals with the simplest linear fractional PDEs. Once the generalized fractional derivatives are expressed as the generators of certain Markov processes, probability theory provides a universal tool for obtaining solutions to various boundary-value processes via the so-called Dynkin martingale. This approach leads directly to the uniqueness of the solutions

and their integral representations, but often fails to provide the existence of (sufficiently regular) solutions. However, in the case of monotone Markov processes that correspond to the equations with one-sided fractional derivatives the existence problem can be settled in a unified way leading to the introduction of the new class of generalized operator-valued Mittag-Leffler type functions. These functions, as their classical counterpart, can be represented as the Laplace transforms of positive measures, expressed in terms of the transition probabilities of the corresponding stochastic processes, or, in analytic language, the Green functions of the Cauchy problems for the corresponding generalized mixed-fractional derivative operators. These measures turn out to be the distributions of the exit times of the underlying processes. This representation implies both the well-posedness and natural regularity properties of the solutions in various classes of classical and generalized solutions. It is also well suited for numeric schemes, because of its explicit integral form.

Section 5 we touch upon more general classes of fractional linear problems including two-sided problems and the equations with higher order and partial fractional derivatives.

Section 6 is devoted to additional bibliographic comments and developments.

The following notations for function spaces will be used:

For a closed or open subset S of $\mathbf{R}^d$, $B(S)$ and $C(S)$ are the Banach spaces of bounded measurable and continuous functions on S respectively, equipped with the sup-norm, $C_\infty(S)$ is the closed subspace of $C(S)$ consisting of functions vanishing at infinity, $C_{uc}(S)$ is the closed subspace of $C(S)$ consisting of uniformly continuous functions.

$C^k(S)$ is a Banach space of k times continuously differentiable functions with bounded derivatives on S with the norm being the sum of the sup norms of the function itself and all its partial derivatives up to and including order k. For a closed S the derivatives on the boundary are understood as the limits of the derivatives defined in their interiors.

Apart from these standard notations, we introduce some more specific ones. For a subset $A \subset S$ let

$$C_{kill\,A}(S) = \{f \in C(S) : f|_A = 0\}, \quad C_{const\,A}(S) = \{f \in C(S) : f|_A \text{ is a constant}\}.$$

Moreover, for $A \subset S$ we shall consider the space $C(A)$ to be the subset of $B(S)$ obtained by setting the values of the functions to zero outside A.

By $\mathbf{1}_A$ we shall denote the indicator function of a set A. Specifically, $\mathbf{1}_{\geq a}$ is the indicator of the half-line $\{y \geq a\}$.

The letters $\mathbf{E}$ and $\mathbf{P}$ will be used to denote the expectation and probability with respect to various Markov processes.

2. Generalized fractional operators and Markov processes

2.1. **Preliminaries: standard fractional derivatives.** Let $I_a f$ be the integration operator defined on the set of continuous curves $f \in C([a,b])$ as $I_a f(x) = \int_a^x f(t)\,dt$. Integration by parts yields

$$I_a^2 f(x) = \int_a^x (I_a f)(y)\,dy = \int_a^x (x-y) f(y)\,dy.$$

Similarly by induction one gets the following formula for the *iterated Riemann integral*

$$I_a^n f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)dt. \tag{2.1}$$

This formula motivates the definition of the (left) *fractional* or *Riemann-Liouville* (RL) integral of order $\beta > 0$:

$$I_a^\beta f(x) = I_{a+}^\beta f(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} f(t)dt. \tag{2.2}$$

Noting that the derivation is the inverse operation to usual integration, the definition (2.2) of the fractional integral suggests two notions of fractional derivative, the so-called *RL (left) derivatives* of order $\beta \in (n, n+1)$, n a nonnegative integer (where $x > a$):

$$D_{a+}^\beta f(x) = \frac{d^{n+1}}{dx^{n+1}} I_{a+}^{n+1-\beta} f(x) = \frac{1}{\Gamma(n+1-\beta)}\frac{d^{n+1}}{dx^{n+1}}\int_a^x (x-t)^{n-\beta} f(t)dt, \tag{2.3}$$

and the (left) *Caputo-Dzherbashyan (CD) derivatives* of order $\beta \in (n, n+1)$:

$$D_{a+*}^\beta f(x) = I_a^{n+1-\beta}\frac{d^{n+1}}{dx^{n+1}} f(x) = \frac{1}{\Gamma(n+1-\beta)}\int_a^x (x-t)^{n-\beta}\left[\frac{d^{n+1}}{dt^{n+1}} f\right](t)dt. \tag{2.4}$$

Remark 2.1. The CD derivative was initially introduced by Liouville (see [23]) and actively studied by Dzherbashyan, see e.g. [14], and Caputo.

In this paper we shall be mostly concern with the case $\beta \in (0,1)$ and its extensions, and occasionally look at the derivatives of higher order as the compositions of the derivatives of order $\beta \in (0,1)$. For $\beta \in (0,1)$ the definitions of the RL and CD derivatives turn to

$$D_{a+}^\beta f(x) = \frac{d}{dx} I_{a+}^{1-\beta} f(x) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x (x-t)^{-\beta} f(t)dt, \quad x > a, \tag{2.5}$$

and respectively

$$D^{\beta}_{a+\star}f(x) = I^{1-\beta}_{a}\left[\frac{d}{dx}f\right](x) = \frac{1}{\Gamma(1-\beta)}\int_a^x (x-t)^{-\beta}\left[\frac{d}{dt}f\right](t)dt, \quad x > a. \tag{2.6}$$

As is seen by direct calculations (see e.g. Appendix in [39]), for smooth enough f,

$$D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{f(x-z)-f(x)}{z^{1+\beta}}dz + \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{2.7}$$

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{x-a}\frac{f(x-z)-f(x)}{z^{1+\beta}}dz + \frac{f(x)-f(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{2.8}$$

implying

$$D^{\beta}_{a+\star}f(x) = D^{\beta}_{a+}[f-f(a)](x) = D^{\beta}_{a+}f(x) - \frac{f(a)}{\Gamma(1-\beta)|x-a|^{\beta}}. \tag{2.9}$$

In particular it follows that for smooth bounded integrable functions, the RL and CD derivatives coincide for $a = -\infty$, and one defines the *fractional derivative in generator form* as their common value:

$$\begin{aligned}\frac{d^{\beta}}{dx^{\beta}}f(x) &= D^{\beta}_{+}f(x) = D^{\beta}_{-\infty+}f(x) = D^{\beta}_{-\infty+\star}f(x)\\ &= \frac{1}{\Gamma(-\beta)}\int_0^{\infty}\frac{f(x-z)-f(x)}{z^{1+\beta}}dz.\end{aligned} \tag{2.10}$$

Another useful rewriting of (2.7) and (2.8) (used in Section 3) is obtained by the change of the variable of integration:

$$D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(-\beta)}\int_a^{x}\frac{f(y)-f(x)}{(x-y)^{1+\beta}}dz + \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \tag{2.11}$$

$$D^{\beta}_{a+\star}f(x) = \frac{1}{\Gamma(-\beta)}\int_a^{x}\frac{f(y)-f(x)}{(x-y)^{1+\beta}}dz + \frac{f(x)-f(a)}{\Gamma(1-\beta)(x-a)^{\beta}}. \tag{2.12}$$

When $\beta \in (0,1)$ and $x < a$, the corresponding *right derivatives* can be introduced by the formulas

$$D^{\beta}_{a-}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{a-x}\frac{f(x+z)-f(x)}{z^{1+\beta}}dz + \frac{f(x)}{\Gamma(1-\beta)(a-x)^{\beta}}, \tag{2.13}$$

$$D^{\beta}_{a-\star}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{a-x}\frac{f(x+z)-f(x)}{z^{1+\beta}}dz + \frac{f(x)-f(a)}{\Gamma(1-\beta)(a-x)^{\beta}}, \tag{2.14}$$

implying

$$D^{\beta}_{a-\star}f(x) = D^{\beta}_{a-}[f-f(a)](x) = D^{\beta}_{a-}f(x) - \frac{f(a)}{\Gamma(1-\beta)(a-x)^{\beta}}. \tag{2.15}$$

The right RL and CD derivatives coincide for $a = \infty$, and one defines the *fractional derivative in generator form* as their common value:

$$\frac{d^\beta}{d(-x)^\beta} f(x) = D^\beta_- f(x) = D^\beta_{\infty-} f(x) = D^\beta_{\infty-\star} f(x)$$
$$= \frac{1}{\Gamma(-\beta)} \int_0^\infty \frac{f(x+z) - f(x)}{z^{1+\beta}} dz. \tag{2.16}$$

2.2. **Generalized fractional differential operators.** The fractional derivative $d^\beta f/dx^\beta$, $\beta \in (0,1)$, from (2.10) was suggested as a substitute to the usual derivative df/dx, which can model some kind of memory by taking into account the past values of f. An obvious extension widely used in the literature (see e.g. [18], [57], [74], [25] and references therein) represent various mixtures of such derivatives, both discrete and continuous,

$$\sum_{j=1}^N a_j \frac{d^{\beta_j} f}{dx^{\beta_j}}, \quad \int_0^1 \frac{d^\beta f}{dx^\beta} \mu(d\beta). \tag{2.17}$$

To take this idea further, one can observe that $d^\beta f/dx^\beta$ represents a weighted sum of the increments of f, $f(x-y) - f(x)$, from various past values of f to the 'present value' at x. From this point of view, the natural class of *generalized mixed fractional derivative* represent the *causal integral operators*

$$D^{(\nu)}_+ = -L'_\nu, \quad L'_\nu f(x) = \int_0^\infty (f(x-y) - f(x))\nu(dy), \tag{2.18}$$

with some positive measure ν on $\{y : y > 0\}$ satisfying the *one-sided Lévy condition*:

$$\int_0^\infty \min(1, y)\nu(dy) < \infty, \tag{2.19}$$

which is just the condition ensuring that L_ν is well-defined at least on the set of bounded infinitely smooth functions on $\{y : y \geq 0\}$. The sign $-$ is introduced to comply with the standard notation of the fractional derivatives, so that, for instance,

$$\frac{d^\beta}{dx^\beta} f(x) = D^\beta_{-\infty+} = D^{(\nu)}_+$$

with $\nu(y) = -1/[\Gamma(-\beta) y^{1+\beta}]$ (note that $\Gamma(-\beta) < 0$).

The dual operators to L_ν are given by the *anticipating integral operators* (weighted sums of the increments from the 'present' to any point 'in future'):

$$D^{(\nu)}_- = -L_\nu, \quad L_\nu f(x) = \int_0^\infty (f(x+y) - f(x))\nu(dy). \tag{2.20}$$

As one of the most studied examples of $D_+^{(\nu)}$ let us mention the so called *tempered fractional derivatives* given by the measure

$$\nu(dy) = Ce^{-\lambda y}y^{-1-\beta} \tag{2.21}$$

with constants $C, \lambda > 0$ (see [6] and [53]). Section 3.4 yields another point of view on this derivative.

Looking at (2.10) with 'probabilistic eyes' one recognises in the operators $-d^\beta f/dx^\beta$, $\beta \in (0,1)$ the generators of stable Lévy subordinators with the inverted direction (see e. g. [53], Ch. 3 and [38], Ch. 1 and 8), the operators $-d^\beta f/d(-x)^\beta$ representing the generators of these subordinatots with the standard direction. Thus the probabilistic point of view suggests to take as fully mixed extensions of these operators the general Lévy subordinators, which leads again to the same operators (2.18) and (2.20), the condition (2.19) being the well known condition of the theory of Lévy processes defining the *one-sided Lévy measures*.

Remark 2.2. Standard Lévy subordinators are defined as increasing processes, and thus operators (2.18) (which are more natural for fractional derivatives) are marked by primes.

One can weight differently the points in past or future depending on the present position, and one can also add a local part to complete the picture, leading to the operators

$$D_+^{(\nu,\gamma)} = -L^l_{\nu,\gamma}, \quad L^l_{\nu,\gamma}f(x) = \int_0^\infty (f(x-y) - f(x))\nu(x,dy) - \gamma(x)\frac{df}{dx}, \tag{2.22}$$

with a non-negative $b(x)$ and transition kernel $\nu(x,.)$, $\int \min(1,y)\nu(x,dy) < \infty$, which capture in full the idea of 'weighting the past' and which can be called the *one-sided*, namely *left* or *causal*, *weighted mixed fractional derivatives of order at most one*. Symmetrically, one can define the *right* or *anticipating weighted mixed fractional derivatives of order at most one* as

$$D_-^{(\nu,\gamma)} = -L^r_{\nu,\gamma}, \quad L^r_{\nu,\gamma}f(x) = \int_0^\infty (f(x+y) - f(x))\nu(x,dy) + \gamma(x)\frac{df}{dx}. \tag{2.23}$$

Remark 2.3. Notice that L^l and L^r are dual only if ν and b do not depend on x.

From the probabilistic point of view the extension to (2.22) and (2.23) represents the transition from time homogeneous monotonic processes to arbitrary decreasing or increasing Feller processes; operators (2.22) and (2.23) are known to represent the general form of the generators of such processes (Courrège theorem, see e.g. [38]).

To link with the theory of pseudo-differential operators, it is worth noting that the operators $D_+^{(\nu)}$ and $D_-^{(\nu)}$ are ΨDOs with the symbols $-\psi_\nu(-p)$ and $-\psi_\nu(p)$, where

$$\psi_\nu(p) = \int_0^\infty (e^{ipy} - 1)\nu(dy)$$

is the symbol of the operator L_ν.

If ν is finite, the operators $D_+^{(\nu)}$ are bounded, which is not the case for usual derivatives. Thus the proper extensions of the derivatives represent only the operator $D_+^{(\nu)}$ arising from infinite measures ν satisfying (2.19). The operators arising from finite ν can be better described as the mixtures of finite differences approximating the derivatives.

The operators $D_\pm^{(\nu)}$ represent the extensions of the fractional derivatives $D^\beta_{-\infty+}$ and $D^\beta_{\infty-}$ in generator form. Looking for the corresponding extensions of the operators $D^\beta_{a\pm}$ and $D^\beta_{a\pm*}$ with a finite a we note that, by (2.8) and (2.14), D^β_{a+*} (resp. D^β_{a-*}) is obtained from $D^\beta_{-\infty+}$ (resp. $D^\beta_{\infty-}$) by the restriction of its action to the space $C_{const\,(-\infty,a]}(\mathbf{R})$ (resp. $C_{const\,[a,\infty)}(\mathbf{R})$. Therefore, the *analogs of the CD derivatives* should be defined as

$$\begin{aligned} D^{(\nu)}_{a+*}f(x) &= -\int_0^{x-a}(f(x-y)-f(x))\nu(dy) - \int_{x-a}^\infty (f(a)-f(x))\nu(dy),\\ D^{(\nu)}_{a-*}f(x) &= -\int_0^{a-x}(f(x+y)-f(x))\nu(dy) - \int_{x-a}^\infty (f(a)-f(x))\nu(dy). \end{aligned} \tag{2.24}$$

By (2.9) and(2.13), the operators D^β_{a+} or D^β_{a-}, the *analogs of the Riemann-Liouville derivatives*, are obtained by further restricting the actions of D^β_+ and D^β_- to the spaces $C_{kill\,(-\infty,a]}(\mathbf{R})$ and $C_{kill\,[a,\infty)}(\mathbf{R})$:

$$\begin{aligned} D^{(\nu)}_{a+}f(x) &= -\int_0^{x-a}(f(x-y)-f(x))\nu(dy) + \int_{x-a}^\infty f(x)\nu(dy),\\ D^{(\nu)}_{a-}f(x) &= -\int_0^{a-x}(f(x+y)-f(x))\nu(dy) + \int_{x-a}^\infty f(x))\nu(dy). \end{aligned} \tag{2.25}$$

Similarly, the derivatives with a finite a based on the general one-sided operators (2.23) are obtained by the same reduction of these operators,

that is, say for the left derivatives, they are given by

$$D^{(\nu,\gamma)}_{a+*}f(x) = -\int_0^{x-a}(f(x-y)-f(x))\nu(x,dy)$$
$$-(f(a)-f(x))\int_{x-a}^{\infty}\nu(x,dy)+\gamma(x)\frac{df}{dx},$$
$$D^{(\nu,\gamma)}_{a+}f(x) = -\int_0^{x-a}(f(x-y)-f(x))\nu(x,dy)+f(x)\int_{x-a}^{\infty}\nu(x,dy)+\gamma(x)\frac{df}{dx}. \tag{2.26}$$

From the probabilistic point of view it is then seen that the operators (2.8) are the generators of the modifications of the stable subordinators obtained by forbidding them (interrupting on an attempt) to cross the boundary $x = a$ with an $a \in \mathbf{R}$, that is, all jumps aimed to jump over the chosen barrier-point a are forced to land exactly at a. On the other hand, the operators (2.7) are the generators of the modifications of the stable subordinators obtained by killing them on an attempt to cross the boundary $x = a$. Applying the same procedure to operators (2.23) and (2.22) lead to the operators (2.24) and (2.26), which represent thus the generators of Markov processes interrupted or killed on an attempt to cross the boundary point a.

Finally one can further extend these mixed derivatives by including additional killing mechanisms or by mixing killing and stopping (say, by working with the linear combinations of of CD and RL derivatives). Such extension leads to the following mixed derivative operators:

$$D^{(\nu,\gamma,S,R)}_{a+}f(x) = -\int_0^{x-a}(f(x-y)-f(x))\nu(x,dy)$$
$$-(f(a)-f(x))S(a,x)+f(x)K(a,x)+\gamma(x)\frac{df}{dx}, \tag{2.27}$$

where $S(a,x)$ and $K(a,x)$ are the rates of stopping and killing respectively. Some formulas simplifies (this is especially relevant for the extensions to manifolds) if one counts jumps by their finite points (rather than relative to x) leading to the following modified version of (2.27):

$$D^{(\nu,\gamma,S,R)}_{a+}f(x) = -\int_a^{x}(f(s)-f(x))\tilde{\nu}(x,ds)$$
$$-(f(a)-f(x))S(a,x)+f(x)K(a,x)+\gamma(x)\frac{df}{dx}. \tag{2.28}$$

Remark 2.4. One can show that these operators $-D^{(\nu,\gamma,S,R)}_{a+}$ represent the most general generators (under some mild technical condition) of decreasing sub-Markov Feller processes on $[a,\infty)$.

2.3. **Generalized fractional integrals: shift invariant case.** Let us see what should be the proper analog of the fractional integral exploiting approaches from probability, semigroup theory and the generalized functions/ΨDEs theory. Let us consider first the case of operators (2.24) arising from the Lévy subordinators. We shall talk about the left derivatives for definiteness.

The operator $d^\beta/dx^\beta \circ I^\beta_{-\infty}$ acts as the identical operator on functions with a compact support. Hence, in the language of the semigroups of operators, the operator $I^\beta_{-\infty}$ is the potential operator of the strongly continuous semigroup of linear operators in $C_\infty(\mathbf{R})$ generated by $-d^\beta/dx^\beta$, that is the limit of the resolvent operator $R_\lambda = (\lambda + d^\beta/dx^\beta)^{-1}$, as $\lambda \to 0$. Notice that I^β_a is just the reduction of $I^\beta_{-\infty}$ to the space $C_{kill\,(-\infty,a]}(\mathbf{R})$. As is worth noting $I^\beta_{-\infty}$ is an unbounded operator in $C_\infty(\mathbf{R})$, but is bounded when reduced to $C_{kill\,(-\infty,a]}((-\infty,b])$ for any $b > a$. Thus we can define the *generalized fractional integral* I^ν_a as the potential operator of the semigroup generated by L'_ν reduced to the space $C_{kill\,(-\infty,a]}(\mathbf{R})$.

On the other hand, the usual fractional integral (2.2) solves the boundary value problem $D^\beta g = f$ with $g(x) = 0$ for $x \le a$. Recall that for a Feller process $X_t(x)$ and a function f from the domain of its generator L the process $f(X_t(x)) - \int_0^t Lf(X_s(x))\,ds$ is a martingale, called Dynkin's martingale (see e.g. [38]). Applying to this martingale Doob's optional sampling theorem shows that

$$f(x) = \mathbf{E} f(X_\tau(x)) - \mathbf{E}\int_0^\tau g(X_s(x))\,ds, \tag{2.29}$$

where $g = Lf$ and τ is the exit time of $X_t(x)$ from a domain (at least if $\mathbf{E}\tau < \infty$). This implies that

$$I^\beta_a f(x) = \mathbf{E}\int_0^{\tau_x} f(x - X^\beta_t)\,dx, \tag{2.30}$$

where X^β_t is the stable subordinator and τ_x is the time when X^β_t reaches the point $x - a$ (and thus $x - X^\beta_t$ reaches a). Therefore, from the probabilistic point of view, the *generalized fractional integral*, representing the analog of I^β_a for the case of the generalized derivative $D^{(\nu)}_{a+}$, is the path integral

$$I^\nu_a f(x) = \mathbf{E}\int_0^{\tau_x} f(x - X^\nu_t)\,dt, \tag{2.31}$$

where X^ν_t is the Lévy subordinator generated by operator (2.20) and τ_x the time for this process to reach a.

Finally, as is known (see e.g. [40]), the fundamental solution (vanishing on the negative half-line) to the fractional derivative operator d^β/dx^β is

$U^\beta(x) = x_+^{\beta-1}/\Gamma(\beta)$, so that the usual fractional integral $I_a^\beta f(x)$ represents the integral operator with the kernel being the fundamental solution of d^β/dx^β or, in other words, the convolution with this fundamental solution, restricted to the space $C_{kill\,(-\infty,a]}(\mathbf{R})$. Thus, from the point of view of the theory of PDEs and ΨDEs, the *generalized fractional integral*, representing the analog of I_a^β for the case of the generalized derivative $D_{a+}^{(\nu)}$, is the integral operator

$$I_a^{(\nu)} f(x) = \int_0^{x-a} f(x-z) U^{(\nu)}(dz), \tag{2.32}$$

where $U^{(\nu)}(dz)$ is the fundamental solution to the operator L'_ν (we denoted it $U^{(\nu)}(dz)$, as it turns out to be a measure, see below).

These three facets of the generalized fractional integral were given by analogy. Let us see now that they are all well defined and in fact represent the same objects. It is well known that the operators L_ν and L'_ν generate Feller processes on $\mathbf{R}$ (called Lévy subordinators) and the corresponding strongly continuous semigroups in spaces $C_\infty(R)$ and $C_{uc}(\mathbf{R})$. The latter space is much less used as the former, but is handy for us as it includes the spaces of functions that are constants on the halflines. Recall that the potential measure is defined as the integral kernel of the potential operator. The potential measure for any Lévy subordinator is known to exist and to equal the vague limit

$$U^{(\nu)}(M) = \int_0^\infty G_{(\nu)}(t, M)\, dt$$

of the measures $\int_0^K G_{(\nu)}(t,.)\, dt$, $K \to \infty$, such that $U^{(\nu)}(M)$ is finite for any compact M whenever ν does not vanish (see e.g. [65] or [42]). More precisely, for any $\lambda > 0$,

$$U^{(\nu)}([0,z]) \le \frac{e^{\lambda z}}{\phi_\nu(\lambda)}, \quad \phi_\nu(\lambda) = -\psi_\nu(i\lambda) = \int_0^\infty (1 - e^{-\lambda y})\nu(dy). \tag{2.33}$$

Here $G_{(\nu)}(t, dy)$ is the Green function of the Cauchy problem for the operator L'_ν. Thus the definition of the $I_a^{(\nu)}$ from the point of view of the semigroup theory is correct.

Turning to the probabilistic definition we note that the process obtained from a subordinator by killing on the attempt to cross the boundary is a well defined Feller sub-Markov process, so that formula (2.52) arising from a Dynkin's martingale is well defined and represents a unique solution to the corresponding boundary problem. From this uniqueness it follows that the definitions of the generalized integral from the semigroup and probabilistic points of view coincide.

Finally, from the definitions of the potential measure it follows that the potential measure represents a fundamental solution vanishing on the negative half-line. Therefore to fix the definition arising from the PDEs theory it is only needed to show the uniqueness of such fundamental solution. This is the content of the following assertion from [41] (see also [42]), which also includes the λ-potential measures defined as

$$U_\lambda^{(\nu)}(A) = \int_0^\infty e^{-\lambda t} G_{(\nu)}(t, A)\, dt. \tag{2.34}$$

These measures are known to be bounded (see e.g. [65] or [42]):

$$\|U_\lambda^{(\nu)}\| = \int_0^\infty U_\lambda^{(\nu)}(dx) \le \frac{1}{\phi_\nu(\lambda)}. \tag{2.35}$$

Proposition 2.1. Let the measure ν on $\{y : y > 0\}$ satisfy (2.19).

(i) For any $\lambda > 0$, the λ-potential measure $U_\lambda^{(\nu)}$ represents the unique fundamental solution of the operator $\lambda - L'_\nu$.

(ii) If the support of ν is not contained in a lattice $\{\alpha n, n \in \mathbf{Z}\}$, with some $\alpha > 0$, the measure $U^{(\nu)}(dy)$ represents the unique fundamental solution to the operator $-L'_\nu$, up to an additive constant.

(iii) Let $\{\alpha n, n \in \mathbf{Z}\}$ be the minimal lattice (that cannot be further rarified) containing the support of ν, so that for any $k \in \mathbf{Z}$, $k > 1$, there exists $n \in \mathbf{Z}$ such that αn belongs to the support of ν and $n/k \notin \mathbf{Z}$. Then any two fundamental solutions to the operator $-L'_\nu$ differ by a linear combination of the type

$$G(x) = \sum_{n \in \mathbf{Z}} a_n \exp\{2\pi n i x/\alpha\} \tag{2.36}$$

with some numbers a_n. In particular, $U^{(\nu)}(dy)$ is again the unique fundamental solution vanishing on the negative half-line.

The following result summarizes the properties of generalized fractional integrals.

Proposition 2.2. (i) Let a measure ν on $\{y : y > 0\}$ satisfy (2.19). For any generalized function $g \in D'(\mathbf{R})$ supported on the half-line $[a, \infty)$ with any $a \in \mathbf{R}$, and any $\lambda \ge 0$, the convolution $U_\lambda^{(\nu)} \star g$ with the λ-potential measure (2.34) is a well-defined element of $D'(\mathbf{R})$, which is also supported on $[a, \infty)$. This convolution represents the unique solution (in the sense of generalized function) of the equation $(\lambda - L'_\nu) f = g$, or equivalently

$$D_+^{(\nu)} f = -\lambda f + g,$$

supported on $[a,\infty)$.

(ii) If $\lambda>0$ and $g\in C_\infty(\mathbf{R})\cap C_{kill\,(-\infty,a]}(\mathbf{R})$, then

$$f(x)=(U_\lambda^{(\nu)}\star g)(x)=R'_\lambda g(x)=\int_{-\infty}^{\infty}g(x-y)U_\lambda^{(\nu)}(dy)$$

$$=\int_0^{x-a}g(x-y)\int_0^\infty e^{-\lambda t}G_{(\nu)}(t,dy)\,dt \tag{2.37}$$

belongs to the domain of the operator L'_ν (considered as the generstor of the Feller semigroup on $C_\infty(\mathbf{R})$) and thus represents the classical solution to the equation $(\lambda-L'_\nu)f=g$, or equivalently

$$D_+^{(\nu)}f=D_{a+}^{(\nu)}f=D_{a+*}^{(\nu)}f=-\lambda f+g. \tag{2.38}$$

(iii) If reduced to the space $C_{kill\,(-\infty,a]}((-\infty,b])$ with $b>a$ (this space is invariant under T'_t and hence under all R'_λ), the potential operator R'_0 with the kernel $U^{(\nu)}$ becomes bounded and hence

$$(U^{(\nu)}\star g)(x)=R'_0g(x)=I_a^{(\nu)}g(x) \tag{2.39}$$

belongs to the domain of L'_ν and thus represents the classical solution to the equation

$$-L'_\nu f=D_+^{(\nu)}f=D_{a+}^{(\nu)}f=D_{a+*}^{(\nu)}f=g \tag{2.40}$$

on $C_{kill\{a\}}([a,b])$.

P r o o f. (i) Since the generalized functions $U_\lambda^{(\nu)}$ and g have support bounded from below, their convolution $U_\lambda^{(\nu)}\star g$ is well-defined and solves the equation $(\lambda-L'_\nu)f=g$. Uniqueness follows from the uniqueness of the fundamental solution.

(ii) Since L'_ν generates a semigroup, which preserves the space $C_{kill\,(-\infty,a]}(\mathbf{R})$, this space is also invariant under the resolvent $R'_\lambda=(\lambda-L'_\nu)^{-1}$. Since the image of the resolvent always coincides with the domain of the generator (see e.g. [16]), $R'_\lambda g$ belongs to the intersection of the space $C_{kill\,(-\infty,a]}(\mathbf{R})$ and the domain of $D_+^{(\nu)}$.

(iii) As was mentioned above, the potential operator

$$R'_0g(x)=(U^{(\nu)}\star g)(x)=\int_0^{x-a}g(x-y)U^{(\nu)}(dy)$$

is bounded on $C_{kill\,(-\infty,a]}(\mathbf{R})\cap C((-\infty,b])$. Hence the required equation can be obtained from (ii) by passing to the limit $\lambda\to 0$. □

In particular, applying (2.37) to $L'_\nu = -d^\beta/dx^\beta$ and comparing with the classical solution of the fractional linear equation via the Mittag-Leffler function yields the integral representation

$$\beta z^{\beta-1} E'_\beta(-\lambda z^\beta) = \int_0^\infty e^{-\lambda t} G_\beta(t,z)\,dt = U^\beta_\lambda(z), \tag{2.41}$$

which is equivalent to Zolotarev's formula

$$E_\beta(s) = \frac{1}{\beta}\int_0^\infty e^{sx} x^{-1-1/\beta} G_\beta(1, x^{-1/\beta})\,dx, \tag{2.42}$$

thus yielding a proof of this formula from the semigroup theory by-passing subtle analytic manipulations of Zolotarev's initial derivation (see [70] and [71]).

For general ν the classical interpretation of the solution $R'_\lambda g(x)$ is subtle for $g \in C([a,\infty))$ not vanishing at a. However, $R'_\lambda g(x)$ may well belong to the domain locally, outside the boundary point a. And in fact, the requirement for the solution to belong to the domain outside a boundary point is common for the classical problems of PDEs. The following assertion illustrates this point concretely.

Proposition 2.3. Under the assumptions of Proposition 2.2 let the potential measure $U^{(\nu)}(dy)$ have a continuous density, $U^{(\nu)}(y)$, with respect to Lebesgue measure. Let $g \in C^1[a,b]$ and is continued as zero to the left of a. Then the function $f(x) = R'_0 g(x)$ is continuously differentiable in $(a,b]$. Hence it satisfies the equation $D^{(\nu)}_{a+*} f = g$ locally, at all points from $(a,b]$.

P r o o f. From the formula for $R'_0 g(x)$ it follows that

$$(d/dx)R'_0 g(x) = \int_0^{x-a} \frac{d}{dx} g(x-y) U^{(\nu)}(y)\,dy + g(a)U^{(\nu)}(x-a),$$

which is well-defined and continuous for $x \ge a$. The limit from the right of $(d/dx)R'_0 g(x)$ as $x \to a$ is $g(a)U^{(\nu)}(0)$, which may cause a jump when this function crosses the value $x = a$. □

Apart from generalized solutions arising from duality as considered above, one uses also the notions of generalized solution by approximation. Namely, for a measurable bounded function $g(x)$ on $[a,\infty)$, a continuous curve $f(x)$, $t \ge a$, is the *generalized solution via approximation* to the problem $D^{(\nu)}_+ f = -\lambda f + g$ on $C([a,b])$, if there exists a sequence of curves $g^n(.) \in C_{kill\,(-\infty,a]}(\mathbf{R})$ such that $g^n \to g$ a.s., as $n \to \infty$, and the corresponding classical (i.e. belonging to the domain) solutions $f^n(x)$, given by (2.37) with $g^n(x)$ instead of $g(x)$, converge point-wise to $f(t)$, as $n \to \infty$.

The following assertion is a consequence of Proposition 2.2.

PROPOSITION 2.4. For any measurable bounded function $g(x)$ on $[a,\infty)$, formula (2.37) (resp. (2.39)) supplies the unique generalized solution by approximation to problem (2.38) (resp. (2.40)) on $[a,b]$ for any $b>a$.

2.4. **Generalized fractional integrals: weighted mixed derivatives.** Let us turn to operators (2.22). It is known (see e.g. [38]) that the operator $L^l_{\nu,\gamma}$ generates a conservative Feller semigroup T_t in $C_\infty(\mathbf{R})$ with invariant core $C^1_\infty(\mathbf{R})$ whenever the following conditions hold:

(i) $b\in C^1(\mathbf{R}^d)$ and $b\ge 0$;

(ii) $\nabla\nu(x,dy)$, the gradient of the Lévy kernel with respect to x, exists in the weak sense as a signed measure that depends weakly continuously on x, in the sense that $\int f(y)\nabla\nu(x,dy)$ is a continuous function for any $f\in C(\mathbf{R}^d)$ with a support separated from zero;

(iii)

$$\sup_x\int \min(1,y)\nu(x,dy)<\infty,\quad \sup_x\int \min(1,y)|\nabla\nu(x,dy)|<\infty, \tag{2.43}$$

and for any $\epsilon>0$ there exists a $K>0$ such that

$$\sup_x\int_K^\infty \nu(x,dy)<\epsilon,\quad \sup_x\int_K^\infty |\nabla\nu(x,dy)|<\epsilon,\quad \sup_x\int_0^{1/K} y\nu(x,dy)<\epsilon. \tag{2.44}$$

Next result shows the existence of the potential measures describing the integral kernel of the resolvent and potential operators of this Feller semigroup:

$$R_\lambda g(x)=\int_0^\infty e^{-\lambda t}(T_tg)(x)\,dt=\int_0^\infty dte^{-\lambda t}\int_{-\infty}^x P_{(\nu,\gamma)}(t,x,dy)g(y)$$
$$=\int_{-\infty}^x \Pi^\lambda_{(\nu,\gamma)}(x,dy)g(y), \tag{2.45}$$

with $\lambda\ge 0$, where $P_{(\nu,\gamma)}$ denote the transition probabilities of the semigroup T_t and $\Pi^\lambda_{(\nu,\gamma)}(x,dy)$ the λ-potential measure.

PROPOSITION 2.5. Let a kernel $\nu(x,dy)$ and a function b satisfy assumptions (i)-(iii) above. Let the inequality

$$\nu(x,dy)\ge\tilde\nu(dy) \tag{2.46}$$

hold with some non-vanishing measure $\tilde\nu$ satisfying (2.19). Then the following holds.

(i) For any nonincreasing function f we have the comparison principle for semigroups:

$$T_t f \geq \tilde{T}'_t f, \tag{2.47}$$

where $\tilde{T}'_t$ is the semigroup generated by the operator $L'_{\tilde{\nu}}$.

(ii) The potential operator $\Pi_{(\nu,\gamma)} = \Pi^0_{(\nu,\gamma)}$ of the semigroup T_t is well defined and satisfies the comparison principle for potential operators:

$$(\Pi_{(\nu,\gamma)}\mathbf{1}_{\geq a})(x) = \Pi_{(\nu,\gamma)}(x,[a,x]) \leq U^{(\tilde{\nu})}([a,x]) \leq \frac{e^{\lambda z}}{\phi_\nu(\lambda)}, \quad x > a, \tag{2.48}$$

where by $\Pi_{(\nu,\gamma)}$ we denoted both the integral operator and the measure representing its integral kernel (see (2.33) for the last estimate).

(iii) The same holds for the λ-potential operators $\Pi^\lambda_{(\nu,\gamma)}$, $\lambda > 0$:

$$(R_\lambda \mathbf{1}_{\geq a})(x) = (\Pi^\lambda_{(\nu,\gamma)}\mathbf{1}_{\geq a})(x) = \Pi^\lambda_{(\nu,\gamma)}(x,[a,x]) \leq U^{(\tilde{\nu})}_\lambda([a,x]) \leq \frac{1}{\phi_\nu(\lambda)}, \quad x > a \tag{2.49}$$

(see (2.35) for the last estimate).

(iv) If additionally $\nu(x,dy) \leq \bar{\nu}(dy)$ with some non-vanishing $\bar{\nu}(dy)$ satisfying (2.19), then $T_t f \leq \bar{T}'_t f$ for a non-increasing function f.

P r o o f. Let us consider the case of $b = 0$ (by approximating the derivative with finite differences we can reduce the general case to this one).

(i) Notice that

$$(L - L'_{\tilde{\nu}})f(x) = \int_0^\infty (f(x-y) - f(x))(\nu(x,dy) - \tilde{\nu}(dy)),$$

which is positive for any nonincreasing function f. Let us write the difference between the actions of T_t and $\tilde{T}'_t$ in the standard form via the difference of the generators (see e.g. [37]):

$$T_t f - \tilde{T}'_t f = \int_0^t T_{t-s} f (L - L'_{\tilde{\nu}})\tilde{T}'_s f.$$

Since $\tilde{T}'_s$ preserves the set of nonincreasing functions, $\tilde{T}'_s f$ is nonincreasing. Hence, $(L - L'_{\tilde{\nu}})\tilde{T}'_s f \geq 0$ and hence $T_t f - \tilde{T}'_t f \geq 0$, yielding (2.47).

(ii) By changing f to $-f$ it follows from (2.47) that

$$T_t f \leq \tilde{T}'_t f \tag{2.50}$$

for any nondecreasing function f.

Applying (2.50) to the indicator functions of half-lines shows that

$$\mathbf{P}(X_t(x) > c) \leq \mathbf{P}(x - X_t > c),$$

where X_t is the decreasing process generated by L and X_t is the subordinator defined by the measure $\tilde{\nu}$. By the definition of potential operator of the process X_t it equals (whenever it exists)

$$\Pi_{(\nu,\gamma)} f(x) = \int_0^\infty T_t f(x)\, dt.$$

In particular,

$$(\Pi_{(\nu,\gamma)} \mathbf{1}_{\geq a})(x) \leq \int_0^\infty (\tilde{T}'_\nu \mathbf{1}_{\geq a})(x)\, dx = U^{(\tilde{\nu})}([a,x]),$$

yielding (2.48).

(iii), (iv) are fully analogous to (ii). □

Consequently, under the assumptions of Proposition 2.5, the potential operator $\Pi_{(\nu,\gamma)}$ is an integral operator, which is well defined for functions with a support bounded below. It is a bounded operator when reduced to the spaces of functions with a fixed compact support. The integral kernel of the potential operator, $\Pi_{(\nu,\gamma)}(x, dy)$, is such that, for any x, the measure $\Pi_{(\nu,\gamma)}(x, \cdot.)$ is supported on the set $(-\infty, x]$. Consequently, from the point of view of the semigroup theory, we can define the *generalized mixed fractional (weighted) integral* $I_a^{(\nu,\gamma)}$ as the potential operator $\Pi_{(\nu,\gamma)}$ reduced to the space $C_{kill\,(-\infty,a]}(\mathbf{R})$:

$$I_a^{(\nu,\gamma)} g(x) = \int_a^x g(y) \Pi_{(\nu,\gamma)}(x, dy) \tag{2.51}$$

On the space $C_{kill\,(-\infty,a]}(\mathbf{R})$ the composition $D_{a+}^{(\nu,\gamma)} \circ I_a^{(\nu,\gamma)}$ acts as identity, as it should be.

Since the process generated by $L^l_{(\nu,\gamma)}$ and stopped at the attempt to cross the given boundary-point a is a well defined Feller process with a regular boundary point, we can apply the theory of Dynkin's martingale that implies that if a function g solves the boundary-value problem $D^\beta g = f$ with $g(x) = 0$ for $x \leq a$, then it can be expressed as the path integral yielding the *probabilistic definition of the generalized fractional integral*:

$$I_a^{(\nu,\gamma)} g(x) = \mathbf{E} \int_0^{\tau_x} g(X_t^\nu(x))\, dt, \tag{2.52}$$

where $X_t^\nu(x)$ is the decreasing process generated by operator (2.23) started at x and τ_x the time for this process to reach a.

As the fundamental solutions are less in use for the operators that are not shift invariant, we shall not give detail on this interpretation here.

The results of the previous section extends now automatically to the case of weighted integrals. For instance, the following holds.

Proposition 2.6. Let the assumptions of Proposition 2.5 hold. (i) If $\lambda > 0$ and $g \in C_\infty(\mathbf{R}) \cap C_{kill\,(-\infty,a]}(\mathbf{R})$, then

$$f(x) = R_\lambda g(x) = \int_{-\infty}^{x} \Pi^\lambda_{(\nu,\gamma)}(x, dy) g(y))$$

belongs to the domain of the operator $L^l_{\nu,\gamma}$ and thus represents the classical solution to the equation $(\lambda - L^l_{\nu,\gamma})f = g$, or equivalently, to the equation

$$D^{(\nu,\gamma)}_+ f = D^{(\nu,\gamma)}_{a+} f = D^{(\nu)}_{a+*} f = -\lambda f + g. \tag{2.53}$$

(ii) If reduced to the space $C_{kill\,(-\infty,a]}((-\infty, b])$ with $b > a$ (this space is invariant under T_t and hence under all R_λ), the potential operator R_0 becomes bounded and hence the integral (2.51) belongs to the domain of $L^l_{\nu,\gamma}$ and thus represents the classical solution to the equation (2.53) with $\lambda = 0$.

Similarly, Proposition 2.4 on the generalized solutions has the straightforward extension

The mixed fractional integrals can be also defined when arising from more general fractional derivatives (2.28). Namely, they represent the potential operators of the semigroups and processes generated by (2.28) and killed at the boundary. Their probabilistic or path integral representation can be written as in (2.52):

$$I^{(\nu,\gamma,S,R)}_{a+} f(x) = \mathbf{E} \int_0^{\tau_x} f(X_t(x))\, dt, \tag{2.54}$$

where X_t is the decreasing sub-Markov process generated by the operator

$$D^{(\nu,\gamma,S,R)}_{a+} f(x) = -\int_a^x (f(s) - f(x))\tilde{\nu}(x, ds) + f(x)[K(a,x) + S(a,x)] + b(x)\frac{df}{dx}.$$

Alternatively, using the Feynman-Kac theory, it can be rewritten as

$$I^{(\nu,\gamma,S,R)}_{a+} f(x) = \mathbf{E} \int_0^\infty f(Y_t(x)) \exp\{-\int_0^t [K(a, Y_s(x)) + S(a, Y_s(x))] ds\}\, dt, \tag{2.55}$$

where Y_t is the decreasing Markov process on $[a, \infty)$ generated by the operator

$$D^{(\nu,\gamma,S,R)}_{a+} f(x) = -\int_a^x (f(s) - f(x))\tilde{\nu}(x, ds) + b(x)\frac{df}{dx}.$$

Remark 2.5. It is natural to ask (especially in order to link our mixed fractional integrals with generalized integrals from [4]), which integral operators can be represented as our mixed fractional integrals, that is, as

potential operators to the semigroups generated by decreasing processes. This question is essentially answered in Chapter 2 Section 4 of [16]. Namely, in this book the criterion is given for operators to represent potential operators of killed Markov processes (even though this criterion is not very easy to check). In our case just the additional requirement arises that the kernel $\Pi(x, dy)$ of such operator should have support on the half-line $(-\infty, x]$ for any x.

2.5. **Further extensions of mixed fractional operators.** Extensions of the fractional operators and the fractional differential equations introduced above corresponds to monotone stochastic processes. They do not exhaust all interesting problems. Apart from the extensions with $\beta \in (1, 2)$ (which we shall not touch here, see [39]) important additional situations include two-sided problems (very natural in the fractional calculus of variations) and multidimensional problems. Unlike the problems considered above, working with potential operators for these cases is usually much more complicated, because we cannot in general use the potential operator of the free problem (without a boundary) and just reduce it to problems with a one-sided support. On the other hand, the representation via Dynkin's martingale works usually equally well in all cases. However, it gives only uniqueness of the solution and the integral representation for it, that is, a generalized solution in probabilistic sense. The difficulty lies in the identification of the analytic characteristics of thus obtained generalized solutions and of the conditions when they are classical.

The most general extension of mixed fractional derivatives within the probability theory arises from an arbitrary Markov process by interrupting it on the attempt to cross a boundary. We refer to [39] for detail and only mention here briefly three examples.

(i) *Two-sided problem mixed with a diffusion* (see detail in [21]). The typical case is the problem

$$(D^{\beta_1}_{a+*} + D^{\beta_2}_{b-*} + A)f = g, \quad f(a) = f_a, \quad f(b) = f_b,$$

with a given function g and a diffusion operator A. Extending the derivatives $D^{\beta_1}_{a+*}$ and $D^{\beta_2}_{b-*}$ to fully mixed derivatives $-L'_\nu$ and L_ν leads to the two-sided problem

$$(L + A)f = g, \quad f(a) = f_a, \quad f(b) = f_b, ,$$

with

$$Lf(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))\nu(x, dy) + ga(x)\frac{df}{dx}$$
$$+ (f(b) - f(x)) \int_{b-x}^{\infty} \nu(x, dy) + (f(a) - f(x)) \int_{-\infty}^{a-x} \nu(x, dy). \qquad (2.56)$$

The generalized solution to this problem (the two-sided analog of the fractional integral) can be given by Dynkin's martingale (2.29) with τ the exit time from (a,b) of the process $X_t(x)$ generated by $L+A$, or simply the process generated by

$$\int_{a-x}^{b-x}(f(x+y)-f(x))\nu(x,dy)+ga(x)\frac{df}{dx}+Af(x),$$

as these two processes coincide before exiting (a,b). Spectral problems for two-sided derivatives will be analyzed in Section 5.1.

(ii) *One-sided multidimensional fractional equations.* These are the equations in the orthant

$$O=\{x=(x_1,\cdots,x_k):x_j\geq a_j\}$$

of the type

$$(D^{\beta_1}_{a_1+*}+\cdots+D^{\beta_k}_{a_k+*})f(x_1,\cdots,x_k)=g(x_1,\cdots,x_k),$$

where D_j acts on the variable x_j. When f is given on the boundary of the orthant O, the generalized solution is well defined by Dynkin's martingale (2.29) with τ the exit time from the interior of O.

(iii) *Fully multidimensional case.* The analog of RL derivative arising from a process in $\mathbf{R}^d$ and a domain $D\subset\mathbf{R}^d$ is the generator of the process killed on leaving D. For CD version this is more subtle, as we have to specify a point where a jump crosses the boundary. The most natural model assumes that a trajectory of a jump follows the shortest path. Namely, assume that A is a generator of a Feller process $X_t(x)$ in $\mathbf{R}^d$ with the generator of type

$$Af(x)=(\gamma(x),\nabla)f(x)+\int_{\mathbf{R}^d}(f(x+y)-f(x))\nu(x,dy) \tag{2.57}$$

with a kernel $\nu(x,.)$ on $\mathbf{R}^d\setminus\{0\}$ such that

$$\sup_x\int_{\mathbf{R}^d}\min(1,|y|)\nu(x,dy)<\infty, \tag{2.58}$$

that is, in the terminology of [37], [38], a generator or order at most one.

Let D be an open convex subset of $\mathbf{R}^d$ with boundary ∂D and closure $\bar{D}$. For $x\in\mathbf{R}^d$ and a unit vector e let $L_{x,e}=\{x+\lambda e,\lambda\geq 0\}$ be the ray drawn from x in the direction e. For $x\in D$, let

$$\lambda(x,y/|y|)=\max\{R>0:x+Ry/|y|\in\bar{D}\}$$

$$R_D(x,y)=\begin{cases}x+y, & \text{if}\,|y|\leq\lambda(x,y/|y|)\\ x+\lambda(x,y/|y|)y/|y|, & \text{if}\,|y|\geq\lambda(x,y/|y|)\end{cases} \tag{2.59}$$

The process $X_t^s(x)$ in D interrupted and stopped on an attempt to cross ∂D can be defined by the condition of stopping at ∂D and the generator

$$A_{D\star}f(x) = (\gamma(x), \nabla)f(x) + \int_{D(x)} [f(R_D(x,y)) - f(x)]\nu(x,dy), \quad (2.60)$$

which represents a multidimensional extension of the CD boundary operator. The unique generalized solution for a boundary value problem $A_{D\star}f = g$ with a given function g and given values of f on ∂D can be again given via Dynkin's martingale.

Finally, in the spirit of the most general extension of one-sided mixed derivatives (2.28), the most general multidimensional mixed derivative of order at most one in the open set $S \subset \mathbf{R}^n$ can be defined as the pseudo-differential operator of order at most one generating (with negative sign) a sub-Markov process in the closure $\bar{S}$:

$$D_{a+}^{(\nu,\gamma,S,R)}f(x) = -\int_{\bar{S}} (f(s) - f(x))\tilde{\nu}(x,ds) + f(x)K(x) + (b(x)\nabla f(x))) \quad (2.61)$$

with a measure ν such that $\sup_x \int |s-x|\nu(x,ds) < \infty$.

In all cases above the corresponding mixed fractional integral can be defined as the potential operators of the corresponding sub-Markov processes, with their path integral or probabilistic interpretation given in terms of Dynkin's martingale.

By the probabilistic representation for various generalized derivatives discussed below we shall understand their representation in forms (2.26), (2.27), (2.28), (2.61), which explicitly reveal the structure of the Markov process they generate.

3. Probabilistic representations for the fundamental fractional operators

Apart from the basic RL fractional operators, the next popular fractional operators are possibly the Hadamard and Erdélyi-Kober operators. We shall now remind their definitions together with their popular extensions and then derive their probabilistic representations. Next we shall describe two general analytic approaches, which can be used to motivate the introduction of these operators and their further extensions. Finally we shall briefly discuss the multiplicative interpolations of derivatives introduced by Hilfer and possible generalizations leading to the random fractional operators.

3.1. Hadamard-Kilbas fractional operators.

The *Hadamard fractional integral* of fractional order $q > 0$ and a boundary point $a > 0$ is defined by

the formula (where $x > a$)

$$ {}^H I_a^q g(x) = \frac{1}{\Gamma(q)} \int_a^x \left(\ln \frac{x}{s}\right)^{q-1} \frac{g(s)}{s} ds = \frac{1}{\Gamma(q)} \int_1^{x/a} (\ln y)^{q-1} g(x/y) \frac{dy}{y}, \tag{3.62} $$

the second version being obtained by the change $x/s = y$, $s = x/y$, $ds = -xdy/y^2$. The corresponding *Hadamard fractional derivative of RL type* of order $q \in (0,1)$ is defined as

$$ {}^H D_{a+}^q f(x) = x \frac{d}{dx} \circ {}^H I_a^{1-q} f(x) = x \frac{d}{dx} \frac{1}{\Gamma(1-q)} \int_1^{x/a} (\ln y)^{-q} f(x/y) \frac{dy}{y}, \tag{3.63} $$

and the *Hadamard fractional derivative of CD type* of order $q \in (0,1)$ as

$$ {}^H D_{a+*}^q f(x) = {}^H I_a^{1-q} \circ x \frac{d}{dx} f(x) = \frac{1}{\Gamma(1-q)} \int_a^x \left(\ln \frac{x}{s}\right)^{-q} f'(s)\, ds, \tag{3.64} $$

both again for $x > a$.

Proposition 3.1. If f is continuously differentiable, then

$$ {}^H D_{a+}^q f(x) = \frac{1}{\Gamma(1-q)} (\ln \frac{x}{a})^{-q} f(x) $$

$$ -\frac{q}{\Gamma(1-q)} \int_0^{x-a} \left(\ln \frac{x}{x-y}\right)^{-q-1} (f(x-y) - f(x)) \frac{ds}{x-y} $$

$$ = \frac{1}{\Gamma(1-q)} (\ln \frac{x}{a})^{-q} f(x) - \frac{q}{\Gamma(1-q)} \int_a^x \left(\ln \frac{x}{s}\right)^{-q-1} (f(s) - f(x)) \frac{ds}{s} \tag{3.65} $$

$$ {}^H D_{a+*}^q f(x) = [{}^H D_{a+}^q (f(.) - f(a))](x) $$

$$ = \frac{1}{\Gamma(1-q)} \left(\ln \frac{x}{a}\right)^{-q} (f(x) - f(a)) - \frac{q}{\Gamma(1-q)} \int_a^x \left(\ln \frac{x}{s}\right)^{-q-1} (f(s) - f(x)) \frac{ds}{s}. \tag{3.66} $$

P r o o f. We get from (3.63) that

$$ {}^H D_{a+}^q f(x) = \frac{x}{a} \frac{1}{\Gamma(1-q)} \left(\ln \frac{x}{a}\right)^{-q} f(a) \frac{a}{x} + \frac{x}{\Gamma(1-q)} \int_1^{x/a} (\ln y)^{-q} f'(x/y) \frac{dy}{y^2} $$

$$ = \frac{1}{\Gamma(1-q)} \left(\ln \frac{x}{a}\right)^{-q} f(a) + \frac{1}{\Gamma(1-q)} \int_a^x \left(\ln \frac{x}{s}\right)^{-q} f'(s) ds. \tag{3.67} $$

Using $f'(s) = (f(s) - f(x))'$ and integrating by parts and taking into account that

$$ \lim_{s \to x} (f(s) - f(x))(\ln(x/s))^{-q} = 0, $$

yields

$$
\begin{aligned}
{}^{H}D^{q}_{a+}f(x) &= \frac{1}{\Gamma(1-q)}\left(\ln\frac{x}{a}\right)^{-q} f(a) - \frac{1}{\Gamma(1-q)}(\ln\frac{x}{a})^{-q}(f(a)-f(x)) \\
&\quad -\frac{q}{\Gamma(1-q)}\int_a^x \left(\ln\frac{x}{s}\right)^{-q-1}(f(s)-f(x))\frac{ds}{s} \\
&= \frac{1}{\Gamma(1-q)}\left(\ln\frac{x}{a}\right)^{-q} f(x) - \frac{q}{\Gamma(1-q)}\int_0^{x-a}(\ln\frac{x}{x-y})^{-q-1}(f(x-y)-f(x))\frac{dy}{x-y}.
\end{aligned}
$$

On the other hand, the r.h.s. of (3.64) equals the second term in (3.67) implying (3.102). □

Since

$$
\int_{x-a}^{x}\left(\ln\frac{x}{x-y}\right)^{-q-1}\frac{dy}{x-y} = \int_0^a\left(\ln\frac{x}{s}\right)^{-q-1}\frac{ds}{s} = \frac{1}{q}\left(\ln\frac{x}{a}\right)^{-q},
$$

one can write equivalently that

$$
{}^{H}D^{q}_{a+}f(x) = f(x)\int_{x-a}^{x}\nu(x,y)\,dy - \int_0^{x-a}(f(x-y)-f(x))\nu(x,y)\,dy,
$$

$$
\nu(x,y) = \frac{q}{\Gamma(1-q)(x-y)}(\ln\frac{x}{x-y})^{-q-1}. \tag{3.68}
$$

This representation shows that the Hadamard derivatives represent particular case of the general operators (2.26), revealing the probabilistic meaning of these operators: $-{}^{H}D^{q}_{a+}$ is seen to generate the process on $[0,\infty)$ obtained from the process generated by the operator

$$
-{}^{H}D^{q}_{0+}f(x) = \int_0^x (f(x-y)-f(x))\nu(x,y)\,dy, \tag{3.69}
$$

by killing the latter process when attempting to cross the boundary $x=a$.

Notice also that for $a=0$ both derivatives are well defined and coincide with operator (3.69) (with inverse sign), which generates a well defined process on $[0,\infty)$ and represents a Hadamard analog of the fractional derivative in generator form (2.10).

Kilbas in [26] introduced the generalized Hadamard operators:

$$
{}^{HK}I^{q}_{a,\mu}g(x) = \frac{1}{\Gamma(q)}\int_a^x (s/x)^{\mu}\left(\ln\frac{x}{s}\right)^{q-1}\frac{g(s)}{s}ds, \quad x>a>0, \tag{3.70}
$$

$$
{}^{HK}D^{q}_{a+,\mu}f(x) = x^{-\mu}(x\frac{d}{dx})x^{\mu}\circ{}^{H}I^{1-q}_{a,\mu}f(x), \tag{3.71}
$$

$$
{}^{HK}D^{q}_{a+*,\mu}f(x) = {}^{H}I^{1-q}_{a,\mu}\circ\left(x^{-\mu}(x\frac{d}{dx})x^{\mu}\right)f(x). \tag{3.72}
$$

Extending the calculations above yields the following result (we omit similar calculations, more general case is considered below):

Proposition 3.2. If f is continuously differentiable, then

$$
{}^{HK}D^q_{a+}f(x) = \frac{1}{\Gamma(1-q)}\left(\ln\frac{x}{a}\right)^{-q}(x/a)^{-\mu}f(x)
$$

$$
-\int_a^x (f(s)-f(x))\nu_{HK}(x,s)\,ds + f(x)\frac{\mu}{\Gamma(1-q)}\int_a^x\left(\ln\frac{x}{s}\right)^{-q}(x/s)^{-\mu}\frac{ds}{s}. \tag{3.73}
$$

$$
{}^{HK}D^q_{a+*}f(x) = \frac{1}{\Gamma(1-q)}\left(\ln\frac{x}{a}\right)^{-q}(x/a)^{-\mu}(f(x)-f(a))
$$

$$
-\int_a^x (f(s)-f(x))\nu_{HK}(x,s)\,ds + f(x)\frac{\mu}{\Gamma(1-q)}\int_a^x\left(\ln\frac{x}{s}\right)^{-q}(x/s)^{-\mu}\frac{ds}{s} \tag{3.74}
$$

with

$$
\nu_{HK}(x,s) = \frac{q}{s\Gamma(1-q)}\left(\ln\frac{x}{s}\right)^{-q-1}\left(\frac{x}{s}\right)^{-\mu}.
$$

The distinguishing property of this case as compared with the standard Hadamard case is the presence of the killing term also in the CD version of the derivative.

For $a=0$ both derivatives coincide and equal

$$
{}^{HK}D^q_{0+*}f(x) = {}^{H}D^q_{0+}f(x) = -\int_0^x (f(s)-f(x))\nu_{HK}(x,s)\,ds
$$

$$
+f(x)\frac{\mu}{\Gamma(1-q)}\int_0^x\left(\ln\frac{x}{s}\right)^{-q}(x/s)^{-\mu}\frac{ds}{s}
$$

$$
= -\int_0^x (f(s)-f(x))\nu_{HK}(x,s)\,ds + \mu^q f(x). \tag{3.75}
$$

This operator (with the negative sign) generates a decreasing process on $[0,\infty)$ with the uniform killing rates μ^q.

This process and its stopped versions arising from $a>0$ can be well called the *Hadamard-Kilbas processes*. Of course, they belong to the general class of processes generated by (2.28).

3.2. **Erdélyi-Kober-Kiryakova-Luchko fractional operators.** Turning to Erdélyi-Kober operators. Let us deal directly with the generalized Erdélyi-Kober integrals. The corresponding fractional calculus was developed by V. Kiryakova and Yu. Luchko (see e.g. [29], [30], [32], [69]) and we shall refer to these operators as the Erdélyi-Kober-Kiryakova-Luchko (EKKL) operators. The corresponding integrals are defined by the formulas

$$
{}^{EKKL}I^{\gamma,q}_{a,\beta}g(x) = \frac{\beta x^{-\beta(\gamma+q)}}{\Gamma(q)}\int_0^x (x^\beta - s^\beta)^{q-1}s^{\beta\gamma+\beta-1}g(s)\,ds
$$

$$= \Omega \circ x^{-(\gamma+q)} I_0^q x^\gamma \circ \Omega^{-1} g(x), \quad x \geq 0, \tag{3.76}$$

where I_0^q is the standard RL fractional integral and Ω_β is the operator changing the variable: $\Omega_\beta f(x) = f(x^\beta)$. The choice of the lower bound $a = 0$ makes these operators quite specific. Namely, by the change of the variable the integral (3.76) rewrites in another useful form:

$$^{EKKL}I_{a,\beta}^{\gamma,q} g(x) = \frac{\beta}{\Gamma(q)} \int_0^1 (1-u^\beta)^{q-1} u^{\beta\gamma+\beta-1} g(ux)\, du. \tag{3.77}$$

This expression shows that all these integral operators with various β, γ, q commute.

The classical Erdélyi-Kober operators are given by this formula with $\gamma = 0$. For $q \in (0,1)$, the corresponding RL type derivatives can be defined (see [29]) via the three equivalent expressions (simple check shows that these expressions coincide):

$$^{EKKL}D_{0+,\beta}^{\gamma,q} f(x) = \frac{x^{1-\beta-\beta\gamma}}{\Gamma(1-q)} \frac{d}{dx} \int_0^x (x^\beta - u^\beta)^{-q} u^{\beta(\gamma+q+1)-1} f(u)\, du, \tag{3.78}$$

$$^{EKKL}D_{0+,\beta}^{\gamma,q} f(x)$$

$$= \left(\frac{x}{\beta} \frac{d}{dx} + \gamma + 1 \right) \left[\frac{\beta x^{-\beta(1+\gamma)}}{\Gamma(1-q)} \int_0^x (x^\beta - u^\beta)^{-q} u^{\beta(\gamma+q+1)-1} f(u)\, du \right], \tag{3.79}$$

and

$$^{EKKL}D_{0+,\beta}^{\gamma,q} = \Omega_\beta \circ (x^{-\gamma} \frac{d}{dx} I_{0+}^{1-q} x^{\gamma+q}) \circ \Omega_\beta^{-1}. \tag{3.80}$$

The probabilistic representation of these derivatives is as follows.

Proposition 3.3. If f is continuously differentiable, then

$$^{EKKL}D_{0+,\beta}^{\gamma,q} = -\int_0^x (f(s)-f(x)) \nu_{EKK}(x,s)\, ds + f(x) \frac{\Gamma(\gamma+q+1)}{\Gamma(1+\gamma)}, \tag{3.81}$$

with

$$\nu_{EKKL}(x,s) = -\frac{\beta x^{-\beta\gamma}}{\Gamma(-q)} (x^\beta - s^\beta)^{-q-1} s^{\beta(\gamma+q+1)-1}. \tag{3.82}$$

P r o o f. Changing the variable of integration in (3.78) to $u = xs$ and then differentiating yields

$$^{EKKL}D_{0+,\beta}^{\gamma,q} = \frac{x^{1-\beta-\beta\gamma}}{\Gamma(1-q)} \frac{d}{dx} x^{\beta(\gamma+1)} \int_0^1 (1-s^\beta)^{-q} s^{\beta(\gamma+q+1)-1} f(xs)\, ds$$

$$= \frac{\beta(\gamma+1)}{\Gamma(1-q)} \int_0^1 (1-s^\beta)^{-q} s^{\beta(\gamma+q+1)-1} f(xs)\, ds$$

$$+\frac{x}{\Gamma(1-q)}\int_0^1 (1-s^\beta)^{-q} s^{\beta(\gamma+q+1)-1} s f'(xs)\,ds.$$

Returning back to the variables $s = u/x$ yields

$$^{EKKL}D^{\gamma,q}_{0+,\beta} = \frac{x^{-\beta(\gamma+1)}}{\Gamma(1-q)}\int_0^x (x^\beta - u^\beta)^{-q} u^{\beta(\gamma+\delta+1)-1}[\beta(\gamma+1)f(u)+uf'(u)]\,du.$$

Integrating by parts via $f'(u) = (f(u)-f(x))'$ yields

$$^{EKKL}D^{\gamma,q}_{0+,\beta} = -q\beta\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x (f(u)-f(x))(x^\beta-u^\beta)^{-q-1}u^{\beta(\gamma+q+1)}u^{\beta-1}\,du$$

$$-\beta\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x (f(u)-f(x))(x^\beta-u^\beta)^{-q}u^{\beta(\gamma+q+1)-1}(\gamma+q+1)\,du$$

$$+\beta\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x f(u)(x^\beta-u^\beta)^{-q}u^{\beta(\gamma+q+1)-1}u^{\beta-1}(\gamma+1)\,du$$

$$= -q\beta\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x (f(u)-f(x))(x^\beta-u^\beta)^{-q-1}u^{\beta(\gamma+q+1)}u^{\beta-1}\,du$$

$$-q\beta\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x (f(u)-f(x))(x^\beta-u^\beta)^{-q}u^{\beta(\gamma+q+1)-1}\,du$$

$$+\beta(\gamma+1)f(x)\frac{x^{-\beta(1+\gamma)}}{\Gamma(1-q)}\int_0^x (x^\beta-u^\beta)^{-q}u^{\beta(\gamma+q+1)-1}\,du.$$

Performing cancellations and using the integral

$$\int_0^x (x^\beta-u^\beta)^{-q}u^{\beta(\gamma+q+1)-1}\,du = \frac{1}{\beta}x^{\beta(\mathbf{1}+\gamma)}B(1-q,\gamma+q+1)$$

$$= \frac{\Gamma(1-q)\Gamma(1+q+\gamma)}{\beta\Gamma(2+\gamma)}x^{\beta(\mathbf{1}+\gamma)},$$

yields (3.81). □

The processes generated by (3.81) (with the negative sign) can be called the *Erdélyi-Kober-Kiryakova-Luchko process*. This is a decreasing process on $[0,\infty)$ with the constant killing rate $\Gamma(\gamma+q+1)/\Gamma(\gamma)$.

Of course one can obtain formula for the interrupted version of these process generated by the fractional derivatives $^{EKKL}D^{\gamma,q}_{a+,\beta}$ and $^{EKKL}D^{\gamma,q}_{a+8,\beta}$ with $a > 0$. We shall do it below in a more general setting. The lower bound $a = 0$ is specific both for yielding the commuting class of operators and by the fact that the underlying processes have uniform killing rates.

3.3. **Elementary analytic approaches to generalized fractional operators.** The development of the generalized fractional calculus was initially led and motivated by the theory of special functions, namely the class of the so-called G-functions and more general H-functions. This development is well presented in the literature (see e.g. [29], [30], [32], [52], [58]).

We introduce here two elementary approaches both leading rapidly to the wide class of operators that cover the majority of operators discussed in the literature. Then we derive the probabilistic representation for these operators revealing the stochastic processes that govern the solutions of the corresponding PDEs.

1. *Method of iterations.* Extending the approach leading to the definition (2.1) of the RL integral let us look at a more general integral operator of the Riemann-Stieltjes type

$$I_{a,G}f(x) = \int_a^x f(u)dG(u), \tag{3.83}$$

where $G(u)$ is a non decreasing function. Assuming that G is differentiable, $G'(u) = g(u)$, it simplifies to $I_{a,G}f(x) = \int_a^x g(u)f(u)\,du$. By direct induction and integration by parts one gets the formula for the iterations:

$$I^n_{a,G}f(x) = \frac{1}{(n-1)!}\int_a^x (G(x)-G(t))^{n-1}g(t)f(t)dt, \tag{3.84}$$

for differentiable G and

$$I^n_{a,G}f(x) = \frac{1}{(n-1)!}\int_a^x (G(x)-G(t))^{n-1}f(t)dG(t), \tag{3.85}$$

in general case. In full analogy with the definition (2.1) formula suggests the definition of the *fractional RL-Stieltjes integral* by the formula

$$I^q_{a,G}f(x) = \frac{1}{\Gamma(q)}\int_a^x (G(x)-G(t))^{q-1}f(t)dG(t) \tag{3.86}$$

for any $q > 0$.

If we take $g(u) = 1/u$ and $a > 0$, formula (3.86) turns to formula (3.62) defining the Hadamard integral.

In [24] U. N. Katugampola aimed at obtaining some interpolation between the standard RL integral and the Hadamard integral and for this purpose applied the above scheme of reasoning with the power function $g(u) = s^\rho$, $\rho > -1$, which yields the integral

$$I^q_{a,\rho}f(x) = \frac{(\rho+1)^{1-q}}{\Gamma(q)}\int_a^x (x^{\rho+1}-t^{\rho+1})^{q-1}t^\rho f(t)\,dt. \tag{3.87}$$

U. N. Katugampola was seemingly unaware of the Erdélyi-Kober theory and did not recognise the link with the classical Erdélyi-Kober integral (3.76) with $\gamma = 0$:

$$^{EKK}I^{0,q}_{a,\beta}g(x) = x^{-\beta q}\beta^q I^q_{a,\beta+1}f(x). \tag{3.88}$$

The difference is therefore just in a multiplier that does not seem to be very essential from the first sight. However, as was noted above, with precisely this multiplier the Erdélyi-Kober-Kiryakova-Luchko integrals with various indices become commuting operators, which creates the beautiful link with the theory of H-functions and leads to the multi-indices Mittag-Leffler functions and the multi-indices Erdélyi-Kober integrals (see e.g. [32]).

To extend this commuting class of operators, it was suggested by Kalla (see references in [32]) to analyse the operators of the type

$$I^{\Phi}_0 g(x) = \int_0^1 \Phi(u)g(ux)\,du, \tag{3.89}$$

parametrized by functions Φ. However, by the direct change of the variables $u = e^{-\xi}, x = e^y$, these operators turn just to the general shift invariant integral operators

$$f \mapsto \int_0^\infty f(y-\xi)K(\xi)d\xi, \quad K(\xi) = \Phi(e^{-\xi})e^{-\xi}.$$

Thus by the universal change of variables the whole class of Erdélyi-Kober-Kiryakova-Lucko operators is inserted in the class of such shift invariant operators. However, these concerns only the operators with the lower bound $a = 0$.

Operators (3.83) were applied in [62], [63] for solving certain ordinary ODEs via the 'fractional tools'.

2. *Substitution and dressing.* Assuming G is strictly monotone and continuous and rewriting integral (3.83) via the method of substitution,

$$I_{a,G}f(x) = \int_{G(a)}^{G(x)} f(u(G))dG, \tag{3.90}$$

with $u(G)$ the inverse function of G, suggests that the fractional RL-Stieltjes integral (3.86) can be obtained directly from the corresponding standard RL integral just by the change of variable. Namely, denoting by Ω the operator acting as $\Omega_G f(x) = f(G(x))$ and fixing the function G by the condition $G(a) = a$, one sees that $I_{a,G} = \Omega_G \circ I_a \circ \Omega_G^{-1}$ and

$$I^q_{a,G} = \Omega_G \circ I^q_a \circ \Omega_G^{-1}, \tag{3.91}$$

that is, $I_{a,G}$ and $I^q_{a,G}$ (defined by (3.86)) are obtained from the corresponding standard integrals via 'dressing' with the operator Ω of the change of variable.

Once the idea of dressing is introduced, it is natural to exploit its further by 'dressing' with respect to other linear transformations of functions, the simplest one being the operators of multiplication $M_w f(x) = w(x)f(x)$ with a function w. Using such dressing in combination with (3.91) leads to the following generalized *dressed fractional integral operator*

$$I^q_{a,G,w} = M_w^{-1} \circ \Omega_G \circ I^q_a \circ \Omega_G^{-1} \circ M_w, \tag{3.92}$$

or concretely

$$I^q_{a,G,w} f(x) = \frac{1}{\Gamma(q)w(x)} \int_a^x (G(x) - G(t))^{q-1} f(t)w(t)dG(t). \tag{3.93}$$

Assuming G to be differentiable, it becomes

$$I^q_{a,G,w} f(x) = \frac{1}{\Gamma(q)w(x)} \int_a^x (G(x) - G(t))^{q-1} f(t)w(t)G'(t)\, dt. \tag{3.94}$$

Formula (3.92) suggests the definition of the corresponding derivatives as the dressed versions of the standard fractional derivatives. Namely, working for simplicity with differentiable G and noting that

$$\Omega \circ \frac{d}{dx} \circ \Omega^{-1} f(x) = \frac{1}{G'(x)} f'(x),$$

we obtain, for the case for $q \in (0,1)$, the following *RL and CD dressed derivatives*:

$$\begin{aligned} {}^A D^q_{a+,G,w} f(x) &= M_w^{-1} \circ \Omega_G \circ D^q_{a+} \circ \Omega_G^{-1} \circ M_w f(x) \\ &= \frac{1}{w(x)G'(x)\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_a^x \frac{w(t)G'(t)f(t)}{[G(x)-G(t)]^\alpha} dt \right), \end{aligned} \tag{3.95}$$

$$\begin{aligned} {}^A D^q_{a+*,G,w} f(x) &= M_w^{-1} \circ \Omega_G \circ D^q_{a+*} \circ \Omega_G^{-1} \circ M_w f(x) \\ &= \frac{1}{w(x)\Gamma(1-\alpha)} \int_a^x \frac{[w(t)f(t)]'}{[G(x)-G(t)]^\alpha}\, dt. \end{aligned} \tag{3.96}$$

Notation ${}^A D$ reflects the fact that operators (3.94), (3.95), (3.96) were seemingly first introduced and studied by O. Agrawal in [3].

Remark 3.1. The authors of [27] introduced and studied the modified Hadamard integrals, which are seen to be obtained from the usual Hadamard integrals via dressing with the operator M_w with $w(x) = x$.

As an alternative to (3.92), one can use the dressing by multiplications and change of variables in a different order, and also use different functions in the input and output yielding the operators

$$I^q_{a,G,w,v} = \Omega_G \circ M_v \circ I^q_a \circ M_w \circ \Omega_G^{-1}, \tag{3.97}$$

the corresponding derivatives being

$$D^{q;G,w,v}_{a+} = \Omega_G \circ M_v \circ D^q_{a+} \circ M_w \circ \Omega_G^{-1}, \tag{3.98}$$

$$D^{q;G,w,v}_{a+*} = \Omega_G \circ M_v \circ D^q_{a+} \circ M_w \circ \Omega_G^{-1}. \tag{3.99}$$

The Erdélyi-Kober-Kiryakova-Luchko fractional operators are obtained via these formulas with all functions G, w, v being power functions.

3.4. **Probabilistic representations.** Of course, differential equations involving dressed operators can be reduced to the usual fractional equations via dressing and redressing. This method was developed in detail in [7] (where it was called the transmutation method) for the case of Erdélyi-Kober operators. However, if one is interested in mixtures of the derivatives of different kind (say, dressed by different operators), one needs to have explicit representation for each mixed component.

Proposition 3.4. If G is differentiable increasing function with $G(a) = a$ and f is continuously differentiable, then

$$\begin{aligned} {}^A D^q_{a+,G,w} f(x) &= -\int_a^x (f(s) - f(x))\nu_A(x,s)\,ds \\ &+ \frac{f(x)}{\Gamma(1-q)}\left[\frac{1}{(G(x)-G(a))^q} - q\int_a^x \frac{w(u)-w(x)}{w(x)(G(x)-G(u))^{1+q}} G'(u)\,du\right], \end{aligned} \tag{3.100}$$

$$\begin{aligned} {}^A D^q_{a+*,G,w} f(x) &= -\int_a^x (f(s)-f(x))\nu_A(x,s)\,ds + \frac{w(a)}{w(x)}\frac{f(x)-f(a)}{\Gamma(1-q)(G(x)-a)^q} \\ &+ \frac{f(x)}{\Gamma(1-q)}\left[\frac{w(x)-w(a)}{w(x)(G(x)-G(a))^q} - q\int_a^x \frac{w(u)-w(x)}{w(x)(G(x)-G(u))^{1+q}} G'(u)\,du\right], \end{aligned} \tag{3.101}$$

with

$$\nu_A(x,s) = -\frac{1}{\Gamma(-q)}\frac{w(s)}{w(x)(G(x)-G(s))^{1+q}} G'(s). \tag{3.102}$$

P r o o f. It is most handy to use the first formula in (3.95) and the formula for the standard derivatives (2.11) implying that

$${}^A D^q_{a+,G,w} f(x)$$

$$= \frac{1}{\Gamma(-q)w(x)} \int_a^{G(x)} \frac{(wf)(G^{-1}(y)) - (wf)(x)}{(G(x)-y)^{1+q}} dy + \frac{f(x)}{\Gamma(1-q)(G(x)-a)^q}.$$

Changing variable of integration yields

$$^A D^q_{a+,G,w} f(x)$$

$$= \frac{1}{\Gamma(-q)} \int_a^x \frac{(w(u)/w(x))f(u) - f(x)}{(G(x)-G(u))^{1+q}} G'(u)\, du + \frac{f(x)}{\Gamma(1-q)(G(x)-a)^q},$$

which can be rewritten as

$$^A D^q_{a+,G,w} f(x) = \frac{1}{\Gamma(-q)} \int_a^x \frac{w(u)(f(u)-f(x))}{w(x)(G(x)-G(u))^{1+q}} G'(u)\, du$$

$$+ f(x) \left[\frac{1}{\Gamma(1-q)(G(x)-G(a))^q} + \frac{1}{\Gamma(-q)} \int_a^x \frac{w(u)-w(x)}{w(x)(G(x)-G(u))^{1+q}} G'(u)\, du \right],$$

as yielding (3.100).

Similarly,

$$\Omega_G \circ D^q_{a+*} \circ \Omega_G^{-1} f(x) = \frac{1}{\Gamma(-q)} \int_a^x \frac{(f(u)-f(x))G'(u)\, du}{(G(x)-G(u))^{1+q}} + \frac{f(x)-f(a)}{\Gamma(1-q)(G(x)-a)^q}.$$

Dressing by M_w and rearranging yields (3.101). □

If the function $w(x)$ has a constant sign everywhere and the coefficient at $f(x)$ is positive (for instance, if $w(x)$ is an increasing positive function), the operator $-^A D^q_{a+,G,w}$ generates a decreasing Markov process with killing that can be called *Agrawal's process*.

As a hallmark of the dressing with the multiplication operators, the CD derivative also gets a killing term (with the rate given by the last square bracket in (3.101)) and the equation $^A D^q_{a+*} f = {}^A D^q_{a+}(f - f(a))$ does not hold anymore. However, the RL and CD derivatives still coincide if $f(a) = 0$.

Needless to stress that these operators also fit the general framework of the operators (2.28).

Let us turn to the derivatives (3.98), (3.99) representing the extension of the Erdélyi-Kober-Kiryakova-Luchko operators.

Proposition 3.5. If G is differentiable increasing function with $G(a) = a$ and f is continuously differentiable, then

$$D^{q;G,w,v}_{a+} = -\int_a^x (f(s)-f(x))\nu_{G,w,v}(x,s)\, ds + f(x)(\Omega_G \circ M_v \circ D^q_{a+} w)(x), \tag{3.103}$$

$$D^{q;G,w,v}_{a+*} = -\int_a^x (f(s)-f(x))\nu_{G,w,v}(x,s)\, ds + f(x)(\Omega_G \circ M_v \circ D^q_{a+*} w)(x)$$

$$+\frac{(f(x)-f(a))w(x)v(G(x))}{\Gamma(1-q)(G(x)-a)^q}, \tag{3.104}$$

with

$$\nu_{G,w,v}(x,s)=-\frac{v(G(x))w(G(s))G'(s)}{\Gamma(-q)(G(x)-G(s))^{1+q}} \tag{3.105}$$

P r o o f. By (2.11),

$$M_v\circ D^q_{a+}\circ M_w\circ\Omega_G^{-1}f(x)$$
$$=\frac{v(x)}{\Gamma(-q)}\int_a^x\frac{w(y)f(G^{-1}(y))-w(x)f(G^{-1}(x))}{(x-y)^{1+q}}+\frac{f(G^{-1}(x))w(x)v(x)}{\Gamma(1-q)(x-a)^q}.$$

Therefore,

$$D^{q;G,w,v}_{a+}=\frac{v(G(x))}{\Gamma(-q)}\int_a^{G(x)}\frac{w(y)f(G^{-1}(y))-w(G(x))f(x)}{(G(x)-y)^{1+q}}dy+\frac{f(x)w(G(x))v(G(x))}{\Gamma(1-q)(G(x)-a)^q}$$
$$=\frac{v(G(x))}{\Gamma(-q)}\int_a^x\frac{w(G(u))f(u)-w(G(x))f(x)}{(G(x)-G(u))^{1+q}}G'(u)du+\frac{f(x)w(G(x))v(G(x))}{\Gamma(1-q)(G(x)-a)^q},$$

and consequently

$$D^{q;G,w,v}_{a+}=\frac{v(G(x))}{\Gamma(-q)}\int_a^x\frac{w(G(u))(f(u)-f(x))}{(G(x)-G(u))^{1+q}}G'(u)du$$
$$+f(x)\left[\frac{v(G(x))}{\Gamma(-q)}\int_a^x\frac{w(G(u))-w(G(x))}{(G(x)-G(u))^{1+q}}G'(u)du+\frac{f(x)w(G(x))v(G(x))}{\Gamma(1-q)(G(x)-a)^q}\right].$$

The expression in the square bracket rewrites as

$$\frac{v(G(x))}{\Gamma(-q)}\int_a^{G(x)}\frac{w(y)-w(G(x))}{(G(x)-y)^{1+q}}dy+\frac{f(x)w(G(x))v(G(x))}{\Gamma(1-q)(G(x)-a)^q}$$

yielding (3.103).

Similarly,

$$D^{q;G,w,v}_{a+*}=\frac{v(G(x))}{\Gamma(-q)}\int_a^{G(x)}\frac{w(y)f(G^{-1}(y))-w(G(x))f(x)}{(G(x)-y)^{1+q}}dy$$
$$+\frac{(f(x)w(G(x))-f(a)w(a))v(G(x))}{\Gamma(1-q)(G(x)-a)^q},$$

yielding (3.104). □

As was noted above, dressing fractional derivatives by the multiplication operators leads to killing even in the CD version. Therefore it is natural to modify the obtained CD-type derivatives by subtracting the killing term. In fact, performing thus procedure with the dressing by the exponential function $e^{-\lambda x}$ leads to the *tempered fractional derivatives* defined by the Lévy kernel (2.21) (see [6] and [53]).

3.5. **Multiplicative interpolations of Hilfer.** Let us touch upon the multiplicative interpolations between (or multiplicative mixtures of) the CD and RL derivatives introduced by Hilfer (see e.g. [23]), which are defined by the formulas

$$^{Hi}D^{\alpha,\beta}_{a+} = I^{(1-\alpha)\beta} D I^{(1-\alpha)(1-\beta)}_{a+}, \quad \beta \in (0,1), \tag{3.106}$$

so that for $\beta = 0$ this operator turns to the RL derivative and for $\beta = 1$ to the CD one.

From the point of view of probability, these derivatives do not lead to any new processes, as the following statement shows.

Proposition 3.6. If f is continuously differentiable and $\beta < 1$, then

$$^{Hi}D^{\alpha,\beta}_{a+} f(x) = D^{\alpha}_{a+} f(x), \tag{3.107}$$

that is, for smooth f, the interpolated derivatives turn to the usual RL derivatives.

P r o o f. Using the definition and integration by parts we get

$$I^{(1-\alpha)(1-\beta)}_{a+} f(x) = \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} \int_a^x (x-t)^{(1-\alpha)(1-\beta)-1} f(t) dt$$

$$= -\frac{1}{\Gamma[(1-\alpha)(1-\beta)]} \int_a^x f(t) \frac{d}{dt} \frac{(x-t)^{(1-\alpha)(1-\beta)}}{(1-\alpha)(1-\beta)} dt$$

$$= \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} f(a) \frac{(x-a)^{(1-\alpha)(1-\beta)}}{(1-\alpha)(1-\beta)} + (I^{(1-\alpha)(1-\beta)+1}_{a+} f')(x).$$

Therefore

$$D I^{(1-\alpha)(1-\beta)}_{a+} f(x) = \frac{(x-a)^{(1-\alpha)(1-\beta)-1}}{\Gamma[(1-\alpha)(1-\beta)]} f(a) + (I^{(1-\alpha)(1-\beta)}_{a+} f')(x),$$

and

$$I^{(1-\alpha)\beta} D I^{(1-\alpha)(1-\beta)}_{a+} f(x) = (I^{(1-\alpha)}_{a+} f')(x)$$

$$+ \frac{f(a)}{\Gamma[(1-\alpha)\beta]} \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} \int_a^x (x-s)^{(1-\alpha)\beta-1} (s-a)^{(1-\alpha)(1-\beta)-1} ds$$

$$= D^{\alpha}_{a+*} f(x) + \frac{f(a)}{\Gamma[(1-\alpha)\beta]} \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} (x-a)^{-\alpha} B[(1-\alpha)\beta, (1-\alpha)(1-\beta)]$$

$$= D^{\alpha}_{a+*} f(x) + \frac{f(a)}{\Gamma(1-\alpha)(x-a)^{\alpha}} = D^{\alpha}_{a+} f(x).$$

□

For completeness, let us note that the working with derivatives (3.106) allows one to distinguish succinctly the scales of the regularity classes of the solutions to the spectral problems:

$$^{Hi}D^{\alpha,\beta}_{a+}f = I^{(1-\alpha)\beta}_{a+}DI^{(1-\alpha)(1-\beta)}_{a+}f = \lambda f. \tag{3.108}$$

In fact, applying the operator $I^{(1-\alpha)(1-\beta)}_{a}$ to the both sides of this equation leads to the spectral problem for the CD operator:

$$D^{\alpha}_{a+*}g = \lambda g, \quad g = I^{(1-\alpha)(1-\beta)}_{a+}f, \tag{3.109}$$

and applying further the usual derivative D leads to to the spectral problem for the RL operator:

$$D^{\alpha}_{a+}h = \lambda h, \quad h = g' = D^{1-(1-\alpha)(1-\beta)}_{a+}f. \tag{3.110}$$

The natural boundary value condition $g(a) = Y$ for the CD problem (3.109) leads to the integral boundary condition $I^{(1-\alpha)(1-\beta)}_{a+}f(a) = Y$ for the Hilfer spectral problem for f. In particular, this condition turns to

$$I^{(1-\alpha)}_{a+}f(a) = Y \tag{3.111}$$

for the RL spectral problem.

3.6. **Random fractional integrals.** Iterated integrals (3.83) and the corresponding fractional integrals were considered so far only for the case of differentiable $G(x)$. The general approach presented here suggests further extensions. Firstly one can look at non-smooth increasing G, for instance, choosing as G the famous Cantor staircase. Aiming to assess the general features of such integrals, one can look at random fractional integrals arising from choosing $G(x)$ as an increasing process, for instance, a Lévy subordinator. Even more exotic calculus can be introduced attempting to build fractional versions of more general indefinite stochastic integrals. For instance, one can start with the Wiener or Ito integral

$$I_W f(x) = \int_0^x f(y)dB(y),$$

where $dB(y)$ is the Ito stochastic differential of the standard Brownian motion. Simple (but lengthy) calculations show that if f is a smooth function (or even a smooth adapted processes on the Wiener space) vanishing at zero, the iterated integral can be expressed as the following usual (non-stochastic) integral:

$$(I_W)^k f(x) = \frac{1}{k!}\int_0^x f'(y)(x-y)^{k/2}H_k\left(\frac{B(x)-B(y)}{\sqrt{x-y}}\right)dy, \tag{3.112}$$

where

$$H_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2} = \left(x - \frac{d}{dx}\right)^k \mathbf{1}, \quad k \in \mathbf{N},$$

are the Hermite polynomials. Extending $H_k(x)$ to fractional values of k (by using either fractional derivative in the first equality or the fractional power in the second equality of the last formula) would naturally define the *fractional Wiener integral*. We shall not explore this topic further here.

Remark 3.2. In the theory of probability by fractional Wiener integral one understands another object, namely the integral $\int_0^x f(y)\, dB^H(y)$ with B^H the fractional Brownian motion.

4. Simplest linear equations and generalized Mittag-Leffler functions

4.1. **Equations with shift invariant mixed derivatives.** If a non-negative measure ν on $\{y : y > 0\}$ satisfy (2.19) and $g \in C_{const\,(-\infty,a]}(\mathbf{R}) \cap C_{uc}(\mathbf{R})$, the resolvent operator R'_λ from (2.37) yields the unique solution $R'_\lambda g$ of the equation

$$D^{(\nu)}_{a+*} f(x) = -\lambda f(x) + g(x) \tag{4.113}$$

in the domain of the generator of the semigroup T_t on $C_{uc}(\mathbf{R})$. This function equals $g(a)/\lambda$ to the left of a.

However, $R'_\lambda g$ is not the solution we are mostly interested in, as it prescribes the boundary value at a, rather than solves the boundary value problem. By prescribing the boundary value one necessarily takes the solution out of the domain of the generator.

The natural way to state properly the boundary value problem

$$D^{(\nu)}_{a+*} f(x) = -\lambda f(x) + g(x), \quad f(a) = Y, \quad x \geq a, \tag{4.114}$$

is by turning it to the problem with the vanishing boundary value, which is a usual trick in the theory of PDEs. Namely, introducing the new unknown function $u = f - Y$ we see that u must solve the problem

$$D^{(\nu)}_{a+} u(x) = -\lambda u(x) - \lambda Y + g(x), \quad u(a) = 0, \quad x \geq a, \tag{4.115}$$

just with $g - \lambda Y$ instead of g. We can thus *define the solution to* (4.114) to be the function $f = u + Y$, where u solves (4.115). Let us stress for clarity that in (4.115) the r.h.s. $g(x) - \lambda Y$ is considered to be continued as zero to the left of a.

Remark 4.1. It is also possible to work directly with the resolvents of the CD derivatives (see [20], [21]), but the approach via equation (4.115) seems to be simpler.

Taking first $g = 0$ we find the solution to (4.114) to be

$$f(x) = Y + u(x) = Y - \lambda Y \int_0^{x-a} \int_0^\infty e^{-\lambda t} G_{(\nu)}(t, dy)\, dt$$
$$= \lambda Y \int_0^\infty e^{-\lambda t} \left(\int_{x-a}^\infty G_{(\nu)}(t, dy) \right) dt, \tag{4.116}$$

since

$$\lambda \int_0^\infty \int_0^\infty e^{-\lambda t} G_{(\nu)}(t, dy)\, dt = 1.$$

Integrating by parts and taking into account that $\int_{x-a}^\infty G_{(\nu)}(0, dy) = 0$ for $x > a$, we get in this case the alternative expression:

$$f(x) = Y \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} \left(\int_{x-a}^\infty G_{(\nu)}(t, dy) \right) dt.$$

Restoring g we arrive at the following.

Proposition 4.1. For any g supported on $[a, \infty)$ the unique solution to problem (4.114) in the sense defined above is given by the formula

$$f(x) = Y \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} \left(\int_{x-a}^\infty G_{(\nu)}(t, dy) \right) dt$$
$$+ \int_0^{x-a} g(x-y) \int_0^\infty e^{-\lambda t} G_{(\nu)}(t, dy)\, dt \tag{4.117}$$

This solution can be classified as classical (from the domain of the generator) or generalized (in the sense of the generalized functions or by approximation) according to Proposition 2.2 applied to problem (4.115).

In analogy with the derivative d^β/dx^β, one can define the family of *generalized Mittag-Leffler functions* depending on the positive parameter z as

$$E_{(\nu),z}(-\lambda) = \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} \left(\int_z^\infty G_{(\nu)}(t, dy) \right) dt$$
$$= 1 - \lambda \int_0^\infty e^{-\lambda t} \left(\int_0^z G_{(\nu)}(t, dy) \right) dt = 1 - \lambda \int_0^z U_\lambda^{(\nu)}(dy). \tag{4.118}$$

Since the function $\int_{x-a}^\infty G_{(\nu)}(t, dy)$ increases with t, its derivative (in the sense of generalized functions) is well-defined as a positive measure (and as a function almost everywhere), and therefore the function $E_{(\nu),z}(-\lambda)$

are well defined and continuous for $\lambda \geq 0$. They are completely monotone function of these λ and are bounded by 1:

$$|E_{(\nu),z}(-\lambda)| \leq \int_0^\infty \frac{\partial}{\partial t}\left(\int_z^\infty G_{(\nu)}(t,dy)\right) dt = \left(\int_z^\infty G_{(\nu)}(t,dy)\right)\Bigg|_0^\infty = 1. \tag{4.119}$$

Moreover, $E_{(\nu),z}(0) = 1$ and the solution (4.116) to problem (4.115) is expressed as

$$f(x) = YE_{(\nu),x-a}(-\lambda) + \int_0^{x-a} g(x-y)U_\lambda^{(\nu)}(dy), \tag{4.120}$$

where the λ-potential measure is expressed in terms of $E_{(\nu),z}$ by the equation

$$\int_0^z U_\lambda^{(\nu)}(dy) = (1 - E_{(\nu),z}(-\lambda))/\lambda. \tag{4.121}$$

If the measures $G_{(\nu)}(t,dy)$ have densities with respect to Lebesgue measure, $G_{(\nu)}(t,y)$, then the λ-potential measure also has a density, $U_\lambda^{(\nu)}(y)$, and (4.121) rewrites as

$$U_\lambda^{(\nu)}(y) = -\frac{1}{\lambda}\frac{\partial E_{(\nu),y}(-\lambda)}{\partial y}. \tag{4.122}$$

However, only for the case of the derivative d^β/dx^β, due to the particular scaling property of G_β, one has the additional relation $E_{(\nu),z}(-\lambda) = E_\beta(-\lambda z^\beta)$.

In case of $L_\nu = -d^\beta/dx^\beta$,

$$\int_1^\infty G_\beta(t,y)dy = \int_1^\infty t^{-1/\beta}G_\beta(1,t^{-1/\beta}y)dy = \int_{t^{-1/\beta}}^\infty G_\beta(1,x)dx.$$

so that

$$\frac{\partial}{\partial x}\int_1^\infty G_\beta(t,y)dy = \frac{1}{\beta}t^{-1-1/\beta}G_\beta(1,t^{-1/\beta}),$$

and therefore we again arrive at formula (2.42).

Let us now turn to the extension of the linear equations to the Banach-space-valued setting, that is, to the equations

$$D_{a+*}^{(\nu)}\mu(x) = A\mu(x) + g(x), \quad \mu(a) = Y, \quad x \geq a. \tag{4.123}$$

If $\mu(a) = Y = 0$, this turns to the RL type equation

$$D_{a+}^{(\nu)}\mu(x) = A\mu(x) + g(x), \quad \mu(a) = 0, \quad x \geq a. \tag{4.124}$$

As above, we shall define the solution to (4.123) as the function $\mu(x) = Y + u(x)$, where $u(x)$ solves the problem

$$D_{a+}^{(\nu)}u(x) = Au(x) + AY + g(x), \quad u(a) = 0, \quad x \geq a. \tag{4.125}$$

Notice also that the assumption of e^{tA} to be a contraction naturally extends the case $A = -\lambda$ with $\lambda > 0$, as $e^{-\lambda t} \leq 1$, and allows one to define the operator-valued generalized Mittag-Leffler functions by the operator-valued integral

$$E_{(\nu),z}(A) = \int_0^\infty e^{tA} \frac{\partial}{\partial t} \left(\int_z^\infty G_{(\nu)}(t, dy) \right) dt$$
$$= 1 + A \int_0^\infty e^{tA} \left(\int_0^z G_{(\nu)}(t, dy) \right) dt. \tag{4.126}$$

THEOREM 4.1. *(i) Let the measure ν on $\{y : y > 0\}$ satisfy* (2.19) *and let A be the generator of the strongly continuous semigroup e^{tA} of contractions in the Banach space B, with the domain of the generator $D \subset B$. Then the $\mathcal{L}(B, B)$-valued potential measure,*

$$U_{-A}^{(\nu)}(M) = \int_0^\infty e^{tA} G_{(\nu)}(t, M)\, dt, \tag{4.127}$$

of the semigroup $T_t' e^{tA}$ on the subspace $C_{kill(a)}([a, b], B)$ of $C_{uc}((-\infty, b], B)$ is well-defined as a σ-finite measure on $\{y : y \geq 0\}$ such that for any $z > 0$, $\lambda > 0$,

$$\|U_{-A}^{(\nu)}([0, z])\| \leq e^{\lambda z}/\phi_\nu(\lambda).$$

Therefore, the potential operator (given by convolution with $U_{-A}^{(\nu)}$) of the semigroup $T_t' e^{tA}$ on $C_{kill(a)}^k([a, b], B)$ is bounded for any $b > a$.

(ii) For any $g \in C_{kill(a)}([a, b], B)$, the B-valued function

$$f(x) = \int_0^{x-a} U_{-A}^{(\nu)}(dy) g(x - y) = \int_0^{x-a} \int_0^\infty e^{tA} G_{(\nu)}(t, dy)\, dt\, g(x - y) \tag{4.128}$$

belongs to the domain of the generator of the semigroup $T_t' e^{tA}$ and represents the unique solution to problem (4.124) *from the domain. For any $g \in C([a, b], B)$ (continued as zero to the left of a), this function represents the unique generalized solution to* (4.124)*, both by approximation and in the sense of generalized functions.*

(iii) For any $g \in C([a, b], B)$ (continued as zero to the left of a) and $Y \in B$, the function

$$f(x) = Y + \int_0^{x-a} U_{-A}^{(\nu)}(dy)(AY + g(x - y))$$
$$= E_{(\nu),x-a}(A)Y + \int_0^{x-a} U_{-A}^{(\nu)}(dy) g(x - y) \tag{4.129}$$

represents the unique generalized solution to problem (4.123) *or* (4.125).

P r o o f. (i) The norms of the operator-valued measure $U_A^{(\nu)}$ are estimated by the measure $U^{(\nu)}$, because e^{tA} are contractions. (ii) and (iii) follow in the same way as for real-valued A above. □

4.2. **Equations with weighted mixed derivatives.** Let us turn to equations arising from the operator $L^l_{(\nu,\gamma)}$ from (2.22) and the corresponding derivative (2.26), assuming everywhere that the assumptions of Proposition 2.5 hold.

The linear problem

$$D^{(\nu,\gamma)}_{a+*} f(x) = -\lambda f(x) + g(x), \quad f(a) = Y, \quad x \geq a, \tag{4.130}$$

will be dealt with in the same way as (4.114). Namely, its solution will be defined as the function $f = u + Y$, where u solves the equation

$$D^{(\nu,\gamma)}_{a+} u(x) = -\lambda u(x) - \lambda Y + g(x), \quad u(a) = 0, \quad x \geq a. \tag{4.131}$$

Taking first $g = 0$ we find (by (2.51) for $\lambda = 0$ and by the definition of resolvent (2.45) for $\lambda > 0$) that the solution to (4.114) equals

$$f(x) = Y + u(x) = Y - \lambda Y \int_a^x \int_0^\infty e^{-\lambda t} P_{(\nu,\gamma)}(t, x, dy)\, dt, \tag{4.132}$$

where $P_{(\nu,\gamma)}(t, x, dy)$ are the transition probabilities for the process generated by $L^l_{(\nu,\gamma)}$. Consequently, we get for $x > a$ that

$$\begin{aligned} f(x) &= \lambda Y \int_0^\infty e^{-\lambda t} \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right) dt \\ &= Y \int_0^\infty e^{-\lambda t} \frac{\partial}{\partial t} \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right) dt. \end{aligned} \tag{4.133}$$

As in the case of shift invariant mixtures, the expression under the bracket is monotonic in t and hence the derivative is well defined almost everywhere and is positive. Of course, strictly speaking, the last formula holds only in case of absolutely continuous (in t) function $\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy)$. In the general case, the last formula should be written more precisely as the Stiltjes integral

$$f(x) = Y \int_0^\infty e^{-\lambda t} d_t \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right), \tag{4.134}$$

where d_t is the differential with respect to the variable t.

Restoring g we arrive at the following.

Proposition 4.2. Let the assumptions of Proposition 2.5 hold. Then for any $\lambda \geq 0$ and any g supported on $[a, \infty)$ the unique solution to problem (4.130) in the sense defined above is given by the formula

$$f(x) = Y \int_0^\infty e^{-\lambda t} d_t \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right) + \int_a^x g(y) \Pi^\lambda_{(\nu,\gamma)}(x, dy). \tag{4.135}$$

We can now define the family of *generalized Mittag-Leffler functions* as

$$E^{(\nu,\gamma)}_{x,a}(-\lambda) = \int_0^\infty e^{-\lambda t} d_t \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right) = 1 - \lambda \Pi^\lambda_{(\nu,\gamma)}(x, [a, x]). \tag{4.136}$$

In explicitly probabilistic form it can be written also as

$$E^{(\nu,\gamma)}_{x,a}(-\lambda) = \lambda Y \mathbf{E} \int_0^\infty e^{-\lambda t} \mathbf{1}_{y \leq a}(X^t_x)\, dt, \tag{4.137}$$

where X^t_x is the Feller process generated by $L^l_{(\nu,\gamma)}$ (a simpler expression will be given below in (4.150)). These functions are completely monotone. Moreover, it follows that the differential of E with respect to a exists as a measure so that

$$\Pi^\lambda_{(\nu,\gamma)}(x, dy) = d_y E^{(\nu,\gamma)}_{x,y}(-\lambda)$$

and the solution (4.135) rewrites as

$$f(x) = Y E^{(\nu,\gamma)}_{x,a}(-\lambda) + \int_a^x g(y) d_y E^{(\nu,\gamma)}_{x,y}(-\lambda). \tag{4.138}$$

Similarly the operator-valued equation

$$D^{(\nu)}_{a+*} \mu(x) = A\mu(x) + g(x), \quad \mu(a) = Y, \quad x \geq a \tag{4.139}$$

is analyzed for the case of A generating a semigroup of contractions, e^{tA}, in a Banach space B.

The operator-valued generalized Mittag-Leffler functions are defined by the operator-valued integral

$$E^{(\nu,\gamma)}_{x,a}(A) = \int_0^\infty e^{At} d_t \left(\int_{-\infty}^a P_{(\nu,\gamma)}(t, x, dy) \right) = 1 + A \Pi^{-A}_{(\nu,\gamma)}(x, [a, x]), \tag{4.140}$$

where $\Pi^{-A}_{(\nu,\gamma)}$ is the operator-valued potential measure,

$$\Pi^{-A}_{(\nu,\gamma)}(x, dy) = \int_0^\infty e^{At}\, dt\, P_{(\nu,\gamma)}(t, x, dy), \tag{4.141}$$

and the following direct extension of Theorem 4.1 is obtained.

THEOREM 4.2. *Let the assumptions of Proposition 2.5 hold and let A be the generator of the strongly continuous semigroup e^{tA} of contractions in the Banach space B. Then the $\mathcal{L}(B,B)$-valued potential measure* (4.141) *is well defined. Moreover, for any $g \in C([a,b],B)$ (continued as zero to the left of a) and $Y \in B$, the function*

$$f(x) = E^{(\nu,\gamma)}_{x,a}(-\lambda)Y + \int_a^x [d_y E^{(\nu,\gamma)}_{x,y}(-\lambda)]g(y), \tag{4.142}$$

where in the square bracket sits the operator-valued Stiltjes measure, represents the unique generalized solution to problem (4.139).

4.3. **Power series expansion for generalized Mittag-Leffler functions.** We have constructed the solutions to the linear problems (4.124) and (4.123) only for the case of A generating a contraction semigroup (with a direct extension to the case of a uniformly bounded semigroup e^{tA}). This restriction was ultimately linked with formula (4.136) for the generalized Mittag-Leffler function, from which it is not seen directly that it can be extended to negative λ. We shall see that this is always possible providing by-passing the power series representation for these functions.

From the definition (4.136) it follows that

$$E^{(\nu,\gamma)}_{x,a}(-\lambda) = 1 - \lambda(R_\lambda \mathbf{1}_{\geq a})(x)$$

with the resolvent operator (2.45). But in the space $C_{kill\,(-\infty,a]}((-\infty,b])$ for any $b > a$, the potential operator $I^{(\nu,\gamma)}_{a+}$ is the bounded right inverse to $D^{(\nu,\gamma)}_{+}$, so that the resolvent equals

$$R_\lambda = (\lambda + (D^{(\nu,\gamma)}_{+})^{-1} = I^{(\nu,\gamma)}_{a+}(1 + \lambda I^{(\nu,\gamma)}_{a+})^{-1} = \sum_{k=0}^{\infty}(-\lambda)^k (I^{(\nu,\gamma)}_{a+})^{k+1},$$

the series being convergent at least for $\lambda \|I^{(\nu,\gamma)}_{a+}\|_b < 1$, where $\|I^{(\nu,\gamma)}_{a+}\|_b$ is the norm of the operator $I^{(\nu,\gamma)}_{a+}$ on the space $C_{kill(-\infty,a]}((-\infty,b])$. Therefore

$$E^{(\nu,\gamma)}_{x,a}(-\lambda) = \left[\sum_{k=0}^{\infty}(-\lambda I^{(\nu,\gamma)}_{a+})^k \mathbf{1}_{\geq a}\right](x). \tag{4.143}$$

This yields the series representation for the generalized Mittag-leffler functions. Since the series has a non-vanishing convergence radius (for any finite x), it defines the continuation of the Mittag-leffler functions $E^{(\nu,\gamma)}_{x,a}(-\lambda)$ to a neighborhood of zero.

In order for this series to be convergent for all $\lambda \in \mathbf{C}$ one has to impose some additional assumptions on the kernel ν, the simplest one being a lower bound of a stable type.

Proposition 4.3. Under the assumptions of Proposition 2.5 let $\nu(x, dy)$ have the lower bound of the β-fractional type

$$\nu(x, dy) \geq (-1/\Gamma(-\beta))C_\nu y^{-1-\beta}\, dy \tag{4.144}$$

with some $\beta \in (0, 1)$ and $C_\nu > 0$. Then, for any $x > a$,

$$\Pi_{(\nu,\gamma)}(x, [a, x]) \leq C_\nu U^\beta[0, x-a] = C_\nu(I_0^\beta \mathbf{1})(x-a) = C_\nu(x-a)^\beta/\Gamma(\beta), \tag{4.145}$$

$$\Pi^\lambda_{(\nu,\gamma)}(x, [a, x]) \leq C_\nu U^\beta_\lambda[0, x-a] \leq \frac{C_\nu}{|\lambda|}[E_\beta(|\lambda|(x-a)^\beta) - 1], \tag{4.146}$$

$$E^{\nu,b}_{x,a}(\lambda) \leq \max(1, C_\nu)E_\beta(|\lambda|(x-a)^\beta). \tag{4.147}$$

P r o o f. All inequalities follow from the comparison principle of Proposition 2.5 and equation (4.136). □

Proposition 4.3 implies that the function $E^{\nu,b}_{x,a}(\lambda)$ is well defined as a whole analytic function of λ and series (4.143) converges for all λ. This allows one to get a more or less direct extension of Theorem 4.1 to arbitrary strongly continuous semigroups e^{tA} (not necessarily contractions), see detail in [42].

4.4. **Probabilistic solutions of linear equations and probabilistic representation for generalized Mittag-Leffler functions.** Solutions to linear fractional equations constructed above are expressed in terms of the transition probabilities of underlying processes. The derivation was performed analytically, via resolvents. It is instructive to give an alternative, pure probabilistic derivation of these results. this approach leads also to the new representations for the solutions.

The main tool here is the Dynkin martingale. Namely, as is well known in probability theory, if X_t is a Feller process X^x_t in $\mathbf{R}^d$ generated by the operator L and f is a function from the domain of L, then the process

$$M^{f,\lambda}_t = f(X^x_t)e^{-\lambda t} + \int_0^t e^{-\lambda s}(\lambda - L)f(X_s)\, ds$$

is a martingale for any $\lambda \geq 0$, called the *Dynkin martingale*. Let τ be a stopping time with a finite expectation $\mathbf{E}\tau$. Then one can use Doob' optional sampling theorem to deduce the following *Dynkin formula*:

$$f(x) = \mathbf{E}[f(X^x_\tau)e^{-\lambda\tau} + \int_0^\tau e^{-\lambda s}(\lambda - L)f(X^x_s)\, ds] \tag{4.148}$$

Suppose now that f is a solution for the problem (4.130) with $Y = 0$ (chosen for simplicity, as the general case is reduced to this anyway). Recalling that the fractional derivative is the negation of the generator of

the corresponding Markov process and choosing the stopping time $\tau_{x,a}$ to be the time of exit of the process from the interval (a,∞) we find that

$$f(x) = \mathbf{E}\int_0^{\tau_{x,a}} e^{-\lambda s} g(X_s^x)\, ds. \tag{4.149}$$

It is of course not surprising that this formula coincides with the expression (4.135) with $Y = 0$. In fact, recalling that g is considered to be continued as zero to the left of a, we have

$$\mathbf{E}\int_0^{\tau_{x,a}} e^{-\lambda t} g(X_t^x)\, dt = \int_0^\infty e^{-\lambda t} g(X_t^x)\, dt = \int_0^\infty e^{-\lambda t}\int_a^x P_{(\nu,\gamma)}(t,x,dy) g(y)\, ds.$$

Thus (4.149) yields the probabilistic representation for the solution (4.135) with $Y = 0$.

Next, we get from (4.136) (or from (4.132)) that

$$E_{x,a}^{(\nu,\gamma)}(-\lambda) = 1 - \lambda\int_0^\infty e^{-\lambda t}\int_a^x P_{(\nu,\gamma)}(t,x,dy)\, ds$$

$$= 1 - \lambda\mathbf{E}\int_0^{\tau_{x,a}} e^{-\lambda t}\, dt = 1 - \lambda\mathbf{E}\frac{1 - e^{-\lambda\tau_{x,a}}}{\lambda} = \mathbf{E}e^{-\lambda\tau_{x,a}},$$

yielding the fundamental probabilistic representation for the generalized Mittag-Leffler function:

$$E_{x,a}^{(\nu,\gamma)}(-\lambda) = \mathbf{E}e^{-\lambda\tau_{x,a}}. \tag{4.150}$$

Thus this function is just the Laplace transform of the exit time of the underlying decreasing Markov process. For instance, for the shift invariant case, the function (4.118) equals

$$E_{(\nu),x}(-\lambda) = \mathbf{E}\exp\{-\lambda\tau_x^{(\nu)}\}, \tag{4.151}$$

where $\tau_x^{(\nu)}$ is the exit time from the interval $[0,x)$ for the Lévy subordinator generated by the operator L_ν from (2.20). For the classical Mittag-Leffler function formula (4.151) reduces to the known formula

$$E_\beta(-\lambda x^\beta) = \mathbf{E}\exp\{-\lambda\tau_x^\beta\} \tag{4.152}$$

(seemingly first derived in [11], see also [55]), where τ_x^β is the corresponding exit time for the β-stable subordinator.

Remark 4.2. The remarkable formula (4.153) and its direct application to the Banach space valued problems seem to be poorly appreciated by the 'fractional community'.

The solution to the spectral problem (4.123) with $g(x) = 0$ can now be written in the following remarkably simple form

$$\mu(x) = \mathbf{E} \exp\{A\tau_x^{(\nu)}\}Y, \tag{4.153}$$

that explicitly unifies the solutions for all mixed fractional derivatives of order up to and including 1.

As another example let us consider the Cauchy problem

$$^H D^q_{a+*} f(x) = Af(x), \quad f(a) = Y, \tag{4.154}$$

in the Banach space B, with the Hadamard derivative (3.102) and A as above. Its solution writes down as

$$\mu(x) = \mathbf{E} \exp\{A\tau_{x,a}\}Y, \tag{4.155}$$

where $\tau_{x,a}$ is the exit time from the interval $(a, x]$ for the Hadamard process generated by (3.69) and started at x.

4.5. **Some examples.** Let us present some examples.

(i) Generalized *fractional Schrödinger equation*:

$$D^{(\nu,\gamma)}_{a+*}\psi = -iH\psi, \tag{4.156}$$

where H is a self-adjoint operator in a Hilbert space $\mathcal{H}$. Since H generates a unitary group, and hence a contraction semigroup, Theorem 4.2 applies. Similarly one can deal with *fractional Schrödinger equation* with the complex parameter

$$D^{(\nu,\gamma)}_{a+*}\psi = \sigma H\psi, \tag{4.157}$$

if H is a negative self-adjoint operator. Specific examples of these equations were analyzed recently in [17].

(ii) Generalized *fractional Feller evolution*, where the operator A generates a Feller semigroup in $C_\infty(\mathbf{R}^d)$ and a Feller process, for instance, a diffusion or a stable-like process.

(iii) Generalized fractional evolutions generated by ΨDOs with spatially homogeneous symbols (or with constant coefficients):

$$D^{(\nu,\gamma)}_{a+*} f = -\psi(-i\nabla)f + g, \quad f|_{t=a} = f_a, \tag{4.158}$$

under various assumptions on symbols $\psi(p)$ ensuring that $-\psi(-i\nabla)$ generates a semigroup. In this case the solution given by Theorem 4.2 can be constructed explicitly via the Fourier transform, see detail in [42].

5. **Further linear equations**

5.1. **Two-sided problems.** We shall now touch upon the theory of two-sided problems that includes the problem

$$D^{(\nu)}_{a+*}f + D^{(\nu)}_{b-*}f = -\lambda f + g, \quad f|_{t=a} = f_a, \quad f|_{t=b} = f_b. \tag{5.159}$$

In its general form it is the problem of the type

$$D^{(\nu)}_{[a,b]*}f = -Lf = -\lambda f + g, \quad f|_{t=a} = f_a, \quad f|_{t=b} = f_b, \tag{5.160}$$

with L from (2.56). It turns out that essential simplification for the two-sided problems occurs in case of the Lévy measures with densities. We shall consider only this case (also omitting for simplicity the usual drift) thus choosing

$$Lf(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))\nu(x,y)dy$$

$$+ (f(b) - f(x)) \int_{b-x}^{\infty} \nu(x,y)dy + (f(a) - f(x)) \int_{-\infty}^{a-x} \nu(x,y)dy. \tag{5.161}$$

As follows from the discussions of the previous section, by shifting the unknown function, the problem can be reduced to the problem with the vanishing boundary conditions. To solve the latter problem one has to construct the Feller process and the resolvent generated by the operator

$$-D^{(\nu)}_{[a,b]}f(x) = L_{kill}f(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))\nu(x,y)dy - k(x)f(x), \tag{5.162}$$

with

$$k(x) = k_a(x) + k_b(x) = \left(\int_{-\infty}^{a-x} + \int_{b-x}^{\infty}\right) \nu(x,y)dy,$$

describing the process killed on the boundary. As was mentioned already, these semigroup and the resolvent cannot be obtained from the corresponding objects for the operator without boundary just by reducing it to the space of functions vanishing outside (a,b).

We shall denote by primes the derivatives with respect to the variable x. For instance, $\nu'(x,y) = (\partial/\partial x)\nu(x,y)$.

The following conditions on ν will be assumed throughout this section:

(A) $\nu(x,y)$ is a continuous function of two variables on the set $(x \in [a,b], y \neq 0)$ having a continuous derivative $\nu'(x,y)$ such that

$$z\nu(x,z) \leq \varkappa \int_z^{\infty} \nu(x,y)dy \tag{5.163}$$

with a constant $\varkappa < 1$ and sufficiently small z;

(B) $\nu(x,y) \le \tilde{\nu}(y)$, $\nu'(x,y) \le \tilde{\nu}(y)$ with a function $\tilde{\nu}$: $\int (1 \wedge y)\tilde{\nu}(y)dy < \infty$;

(C) $k(x) \to \infty$ as $x \to a$ or $x \to b$.

Note that (C) is just the assumption that the Lévy kernel $\nu(x,y)dy$ is unbounded. Assumption (A) is not too restrictive, at least it holds for all standard examples. Say, for α-stable processes (classical fractional derivatives) it holds with $\omega = \alpha$.

Let us introduce the special notations for our main Banach spaces: $C_0 = C_{kill\{a,b\}}([a,b])$ equipped with the sup-norm $\|.\|$, $C_{00} = \{f \in C_0 : f' \in C_0\}$ equipped with the norm $\|f\|_{00} = \|f\| + \|f'\|$.

LEMMA 5.1. *Under conditions (A)-(C) the operator $-K$ of multiplication by $-k(x)$ from* (5.173) *generates (i) a semigroup group of contractions in $C([a,b])$, (ii) a strongly continuous semigroup of contractions in C_0 with the domain*

$$D_K = \{f \in C_0 : \lim_{x\to a} k(x)f(x) = 0, \lim_{x\to b} k(x)f(x) = 0\},$$

(iii) a uniformly bounded semigroup in $C_0 \cap C^1([a,b])$, (iv) a uniformly bounded strongly continuous semigroup in C_{00}.

P r o o f. It is mostly straightforward. Let us prove only (iii) and (iv). We have

$$[e^{-tk(x)}f(x)]' = e^{-tk(x)}f'(x) - tk'(x)e^{-tk(x)}f(x).$$

The first term is bounded: $\|e^{-tk(.)}f'(.)\| \le \|f'\|$. The second term is bounded away from the boundaries. Let us estimate it in a neighborhood of $x = b$ (neighborhoods of a are analogous). Near $x = b$ the main (unbounded) part of $k(x)$ is $k_b(x) = \int_{b-x}^{\infty} \nu(x,y)dy$ and thus

$$tk'(x)e^{-tk(x)}f(x) \sim t\nu(x,b-x)(b-x)f'(b)e^{-tk(x)}$$

$$+t(b-x)f'(b)e^{-tk(x)}\int_{b-x}^{\infty} \nu'(x,y)dy.$$

For $b - x \le 1$ we estimate the two terms by (A) and (B) respectively:

$$t\nu(x,b-x)(b-x)|f'(b)|e^{-tk(x)} \le \varkappa|f'(b)|tk_b(x)e^{-tk_b(x)} \le \varkappa|f'(b)|,$$

$$t(b-x)|f'(b)|e^{-tk(x)}\int_{b-x}^{\infty} \nu'(x,y)dy$$

$$\le t|f'(b)|\int_0^{\infty}(1\wedge y)\nu'(x,y)dy \le t|f'(b)|\int_0^{\infty}(1\wedge y)\tilde{\nu}(y)dy.$$

Thus both terms are uniformly bounded. Moreover, if $f'(b) = 0$, then $\lim_{x\to b}[e{-}tk(x)f(x)]' = 0$, so that $e^{-tK} : C_0 \cap C^1([a,b]) \to C_{00}$ for $t > 0$. This implies the strong continuity of e^{-tK} in C_{00}. □

THEOREM 5.1. *Assume that (A)-(C). Then the operator L_{kill} of* (5.173) *generates a Feller semigroup in C_0 and a bounded semigroup in C_{00}. The domain of this semigroup in C_0 contains the space*

$$D_{kill} = \{f \in C_{00} : L_{kill}f(x) \to 0, \text{ as } x \to a, x \to b\}.$$

P r o o f. *Step 1.* Let us introduce the approximation L_h of L_{kill}, $h \in (0,1)$, by the formula

$$L_h f(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))(1 - \chi_h(y))\nu(x,y)dy - k(x)f(x), \quad (5.164)$$

where χ_h a continuous even function $\mathbf{R} \to [0,1]$ such that $\chi_h(z) = 1$ for $z \in [-h,h]$ and $\chi_h(z) = 0$ for $|z| \geq 2h$. Since L_h differs from the operator $-K$ by a bounded operator (in both $C([a,b])$ and $C^1([a,b])$), we can conclude from Lemma 5.1 and the perturbation theory that L_h generates bounded semigroups T_t^h in the spaces $C([a,b])$ and $C^1([a,b])$. Moreover, the perturbation series representation for the semigroup T_t^h has the form (see e.g. [38])

$$T_t^h = e^{-tK} + \sum_{m=1}^{\infty} \int_{0\leq s_1 \leq \cdots \leq s_m \leq t} ds_1 \cdots ds_m$$
$$\times e^{-(t-s_m)K}(L_h + K)e^{-(s_m - s_{m-1})K} \cdots (L_h + K)e^{-s_1 K}.$$

It follows from this formula and Lemma 5.1 that the spaces C_0 and C_{00} are invariant under T_t^h. It is also straightforward to see that T_t^h is strongly continuous in both C_0 and C_{00}. Since the operator L_h is conditionally positive, it follows that T_t^h is a contraction in C_0. The domain of the semigroup T_t^h in C_0 is given by those $f \in C_0$ such that $L_h f(x) \to 0$, as $x \to a$ and $x \to b$. Moreover, $(T_t^h f(x) - f(x))/t \to L_h f(x)$ uniformly on any closed interval $[a', b'] \subset (a,b)$ and for any $f \in C_0 \cap C^1([a,b])$. The domain of the semigroup T_t^h in C_{00} is given by those $f \in C_{00}$ such that $L_h f(x) \to 0$ and $(L_h f)'(x) \to 0$, as $x \to a$ and $x \to b$.

Step 2. The next key step is to show that the semigroups T_t^h are bounded in C_{00} uniformly in h. To this end let us differentiate the equation $\dot{f}_t = L_h f_t$ satisfied by $f_t = T_t^h f$ with f from the C_{00}. Thus for $g_t = \partial f_t/\partial x$ we get the equation

$$\dot{g}_t(x) = \Omega_h g_t(x) = L_h g_t(x)$$

$$+L'_h f_t(x) - f_t(x)\chi_h(b-x)\nu(x,b-x) + f_t(x)\chi_h(a-x)\nu(x,a-x), \quad (5.165)$$

with

$$L'_h f(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))(1-\chi_h(y))\nu'(x,y)dy$$
$$-f_t(x)\left(\int_{-\infty}^{a-x} \nu'(x,y)dy + \int_{b-x}^{\infty} \nu'(x,y)dy\right).$$

Writing

$$-f(x)\chi_h(b-x)\nu(x,b-x) + f(x)\chi_h(a-x)\nu(x,a-x)$$
$$= \int_x^b g(y)\,dy\chi_h(b-x)\nu(x,b-x) + \int_a^x g(y)\,dy\chi_h(a-x)\nu(x,a-x)$$
$$= \int_x^b (g(y)-g(x))\,dy\chi_h(b-x)\nu(x,b-x) + \int_a^x (g(y)-g(x))\,dy\chi_h(a-x)\nu(x,a-x)$$
$$+g(x)[\chi_h(b-x)(b-x)\nu(x,b-x) + \chi_h(x-a)(x-a)\nu(x,a-x)],$$

we get

$$\Omega_h g(x) = \tilde{L}_h g + L'_h f,$$

with

$$\tilde{L}_h g(x) = \int_{a-x}^{b-x} (g(x+y) - g(x))(1-\chi_h(y))\nu(x,y)dy - \tilde{k}(x)g(x), \quad (5.166)$$

where

$$\tilde{k}(x) = k_a(x) + k_b(x) - \chi_h(b-x)(b-x)\nu(x,b-x) - \chi_h(x-a)(x-a)\nu(x,a-x).$$

The key point is now the observation that, by (5.163),

$$\tilde{k}(x) \geq (1-\varkappa)k(x).$$

Hence we can show by the same perturbation argument as used above for L_h that $\tilde{L}_h$ generates a strongly continuous semigroup in C_0. Moreover, since all terms in the expression for $\tilde{L}_h$ are conditionally positive operators this semigroup is a group of contractions and thus is bounded uniformly in h. Expressing f via g as above in $L'_h f$, we observe that this operator becomes a uniformly bounded operator in $C([a,b])$ and hence, using again perturbation argument, we conclude that the semigroups T_h^t in C_{00} are uniformly bounded in h.

Step 3. Let us show now that the operators T_t^h converge strongly in C_0, as $h \to 0$. To compare these operators for different h we shall use the following standard (and easy to prove) formula (see e.g. [38])

$$(T_t^{h_1} - T_t^{h_2})f = \int_0^t T_{t-s}^{h_2}(L_{h_1} - L_{h_2})T_s^{h_1}\,ds \quad (5.167)$$

expressing the difference of the semigroups in terms of the difference of their generators. For arbitrary $h_1 > h_2$ and $f \in C_{00}$, we have

$$(L_{h_1} - L_{h_2})\phi(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))(\chi_{h_2} - \chi_{h_1}(y))\nu(x,y)dy$$

and thus

$$|(L_{h_1} - L_{h_2})\phi(x)| \le \int_{|y|\le h_1} \|f'\||y|\nu(x,y)dy \le \|f'\| \int_{|y|\le h_1} |y|\tilde{\nu}(y)\, dy$$

Since $\|(T_t^h f)'\|$ is uniformly bounded by Step 2, we have

$$\|(T_t^{h_1} - T_t^{h_2})f\| = o(1)t\|f'\|_{C_\infty^1}, \quad h_1 \to 0. \tag{5.168}$$

Therefore the family $T_t^h f$ converges in C_0 to a family $T_t f$, as $h \to 0$, for any $f \in C_{00}$. By the standard density argument this convergence holds also for any $f \in C_0$ and the limiting family T_t specifies a strongly continuous semigroup in $C_\infty(\mathbf{R}^d)$.

Step 4. Writing

$$\frac{T_t - f}{t} = \frac{T_t - T_t^h f}{t} + \frac{T_t^h - f}{t}$$

and noting that by (5.168) the first term is of order $o(1)\|f'\|$, as $h \to 0$, allows one to conclude that, for any $f \in C_0 \cap C^1([a,b])$, $(T_t f(x) - f(x))/t \to L_{kill} f(x)$ uniformly on any closed interval $[a', b'] \subset (a,b)$. It follows that D_{kill} belongs to the domain of L_{kill}. □

Having proved Theorem 5.1, we can safely apply the resolvent of the operator L_{kill} to obtain the solutions for the equations

$$D_{[a,b]}^{(\nu)} f = -\lambda f + g$$

with vanishing boundary conditions, which are classical (lie in the domain of the generator of L_{kill}) for $g \in C_0$ and generalized otherwise.

By shifting the solutions to (5.160) can be reduced to the boundary problem with vanishing boundary conditions. For instance, writing $f = u + f^0$ with $f_0 = f_a + (x-a)(f_b - f_a)/(b-a)$, the spectral problem

$$D_{[a,b]*}^{(\nu)} f = -Lf = -\lambda f, \quad f|_{t=a} = f_a, \quad f|_{t=b} = f_b, \tag{5.169}$$

is reduced to the problem

$$D_{[a,b]}^{(\nu)} u = -Lf = -\lambda u - \lambda f^0 - D_{[a,b]*}^{(\nu)} f^0, \quad u(a) = u(b) = 0, \tag{5.170}$$

which is solved by the equation

$$f(x) = f^0(x) + u(x) = f^0 + R_{kill}^\lambda(\lambda f^0 + D_{[a,b]*}^{(\nu)} f^0). \tag{5.171}$$

Using the technique of Dynkin's martingale (see Section 4.4), this can be expressed in probabilistic (or path integral) terms as

$$f(x) = f^0(x) + \mathbf{E}\int_0^{\tau_x} e^{-\lambda s}(\lambda f^0 + D^{(\nu)}_{[a,b]*} f^0)(X^x_s)\, ds, \tag{5.172}$$

where X^x_s is the process generated by L_{kill} and τ_x is its killing time.

Alternatively, to solve the equations with CD derivatives one can work directly with the process generated by (5.173) by stopping it when reaching the boundary. To justify this approach one needs the following result.

THEOREM 5.2. *Assume that (A)-(C). Then the operator L of* (5.173) *generates a Feller semigroup in $C([a,b])$ and also a strongly continuous semigroup in the space $C'_0 = \{f \in C^1([a,b]) : f'(a) = f'(b) = 0\}$.*

P r o o f. It is a consequence of Theorem 5.1. Let us consider the semigroup generated by L in the space C'_0. Differentiating the equation $\dot f = Lf$ we get (after some handy cancellations) for the derivative $g = f'$ the equation

$$\dot g = (Lf)' = \int_{a-x}^{b-x} (f(x+y) - f(x))\nu(x,y)dy - g(x))$$

$$\times \left(\int_{b-x}^{\infty} \nu(x,y)dy + \int_{-\infty}^{a-x} \nu(x,y)dy\right) + \int_{a-x}^{b-x} (f(x+y) - f(x))\nu'(x,y)dy$$

$$+ (f(b) - f(x)) \int_{b-x}^{\infty} \nu'(x,y)dy + (f(a) - f(x)) \int_{-\infty}^{a-x} \nu'(x,y)dy. \tag{5.173}$$

By Theorem 5.1, the first two terms on the r.h.s generates a well-defined Feller semigroup in C_0. All other terms (the terms with ν') are bounded when expressed in terms of g, and hence application of the perturbation theory yields the claimed result. □

5.2. **Mixed RL-CD-Hilfer boundary-value problems.** As was noted (see (3.111)), the usual boundary-value problem for RL derivative imposes a bit artificial integral boundary condition on the unknown function. However, under certain assumptions on the source function g the usual Cauchy problem,

$$D^{(\nu)}_{a+} f(x) = -\lambda f(x) + g(x), \quad f(a) = Y, \tag{5.174}$$

can be solved. To see how it works let us analyse a bit more general case of mixed RL and CD derivatives:

$$(\gamma D^{(\nu)}_{0+*} + \delta D^{(\nu)}_{0+}) f(x) = -\lambda f(x) + g(x), \quad f(0) = Y, \tag{5.175}$$

where γ, δ are nonnegative constants, g is a locally integrable function vanishing to the left of 0, and the derivative operators are of form (2.24) and (2.25). For $\gamma = 0$, problem (5.175) turns to the RL problem (5.174).

Acting as earlier, we make a substitution of the unknown function $f(x) = u(x) + Y$, which turns (5.175) to the problem

$$(\gamma+\delta)D^{(\nu)}_{0+*}u(x) = (\gamma+\delta)D^{(\nu)}_{0+}u(x) = -\lambda u(x) - \lambda Y + g(x) - \delta Y \int_x^\infty \nu(dy), \tag{5.176}$$

with $u(0) = 0$, because RL and CD derivatives coincide for functions vanishing at the boundary and $D^{(\nu)}_{0+}\mathbf{1}_{\geq 0}(x) = \int_x^\infty \nu(dy)$. Since g is locally integrable and thus an element of the space of generalized functions $D'(\mathbf{R})$, we can apply Proposition 2.2 to get the unique generalized solution to this problem

$$u(x) = \int_0^\infty [g(x-y) - \delta Y \int_{x-y}^\infty \nu(dz) - \lambda Y] U^{(\nu)}_\lambda(dy).$$

In order to satisfy the boundary requirement classically, that is, to have $\lim_{x\to 0} u(x) = 0$, it is sufficient that

$$\sup_y \left| g(y) - \delta Y \int_y^\infty \nu(dz) \right| < \infty \tag{5.177}$$

at least for y from some neighborhood of the origin. If g is continuous for $x > 0$ and satisfies this condition we obtain the classical solution to problem (5.176) and thus to problem (5.175).

Similarly one can solve the boundary-value problem with mixed Hilfer and CD derivatives of the type

$$(\gamma D^{(\nu)}_{0+*} + \delta^{Hi} D^{\alpha,\beta}_{a+})f(x) = -\lambda f(x) + g(x), \quad f(a) = Y, \tag{5.178}$$

The sufficient condition (5.177) for the classical solvability turns to the condition

$$\sup_y \left| g(y) - \delta Y \frac{y^{-\alpha}}{\Gamma(1-\alpha)} \right| < \infty, \tag{5.179}$$

at least for y from some neighborhood of the origin, because ${}^{Hi}D^{\alpha,\beta}_{a+}\mathbf{1}_{\geq 0}(x) = x^{-\alpha}/\Gamma(1-\alpha)$.

5.3. **Higher order and partial derivatives and related equations.** Let us mention briefly how the equations with higher order and partial derivatives can be dealt with.

1. Let us consider, for instance, the equation

$$(D^{(\nu)}_{a+*})^k f(x) = -\lambda f(x), \quad f(a) = Y_0, \quad D^{(\nu)}_{a+*})^l f(a) = Y_l, \quad l = 1, \cdots k-1, \tag{5.180}$$

where $D^{(\nu)}_{a+*}$ is given by (2.24). Introducing the vector-valued unknown function $F = (f_0, \cdots, f_{k-1})$ with $f_0 = f$, $f_l = (D^{(\nu)}_{a+*})^l f$, $l = 1, \cdots, k-1$, problem (5.180) rewrites in the vector form as

$$(D^{(\nu)}_{a+*})^k F(x) = AF(x) = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ & \cdots & & \\ 0 & 0 & \cdots 0 & 1 \\ -\lambda & 0 \cdots & & \end{pmatrix} F(x), \quad F(a) = Y = \begin{pmatrix} Y_0 \\ Y_1 \\ \cdots \\ Y_{k-1} \end{pmatrix}. \tag{5.181}$$

By (4.153) the solution to this problem can be written as $F(x) = \mathbf{E} \exp\{A\tau^{(\nu)}_x\} Y$.

2. Let us consider the partial fractional differential equation

$$D^{(\nu_2),x_2}_{a_2+*} D^{(\nu_1),x_1}_{a_1+*} f(x_1, x_2) = -\lambda f(x_1, x_2) + g(x_1, x_2), \quad x_1 \geq a_1,\, x_2 \geq a_2, \tag{5.182}$$

with the simplest vanishing boundary conditions $f(a_1, x_2) = f(x_1, a_2) = 0$, where $D^{(\nu_j),x_j}_{a_j+*}$ acts on the variable x_j, $j = 1, 2$. Introducing the vector-valued function $F = (f_1, f_2)$, $f = f_1, f_2 = D^{(\nu_1),x_1}_{a_1+*} f$, we rewrite equation (5.182) in the matrix form:

$$DF(x) = \begin{pmatrix} D^{(\nu_1),x_1}_{a_1+*} f_1 \\ D^{(\nu_1),x_1}_{a_1+*} f_2 \end{pmatrix} = AF(x) = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix} F = \begin{pmatrix} f_2 \\ -\lambda f_1 \end{pmatrix}. \tag{5.183}$$

The operator D generates a Markov process on the triples (j, x_1, x_2), or, in other words on the two copies of the orthant $\{x_1 \geq a_1, x_2 \geq x_2\}$ such that x_1 is decreasing according to the generator $-D^{(\nu_1),x_1}_{a_1+*}$ on one of the copies and x_2 is decreasing independently according to the generator $-D^{(\nu_2),x_2}_{a_2+*}$ on the other copy. The solution can be again expressed either via the resolvent of D or via Dynkin's martingale.

6. **Additional bibliographical comments**

The literature on fractional calculus is enormous. We shall mention only the sources most closely related to the probabilistic point of view of the present paper. Some historical reviews can be found in e.g. [23], [30], the wealth of physical and economics application e.g. in [33], [66], [67], [68], numerical methods are dealt with in monographs [9], [48]. One of the main impetus for the physics interest in fractional equations was in fact probabilistic in nature. It was inspired by their appearance as the scaled limits of continuous time random walks, see e. g. [53], [36], [47] and extensive bibliography therein.

Fractional Schrödinger equation is getting popularity in physics community, see e. g. [46], [56], [10], [17] and references therein. For the fractional versions of the wave equations we refer to [60]. Of importance for physics are also fractional kinetic equations with application to statistical mechanics and fractional stochastic PDEs. For these developments we refer to [35], [43] and [49], [50], [72], [73] and references therein.

An insightful collection of references on nonlinear fractional equations and their applications can be found in [59]. Mild forms are studied in detail in [44], [45] and [42], where they are applied to the theory of fractional Hamilton-Jacobi-Bellman equations.

For various approaches to equations in bounded domains we refer to [2], [13], [19], [54] Two-sided and multidimensional problems appear naturally in the fractional calculus of variations, see [51], [5]. The optimization problems of this theory are formulated in terms of the certain class of fractional equations on bounded domains, the so-called fractional Euler-Lagrange equations. Their analysis was initiated seemingly in [12]. Problems with partial fractional derivatives are studied in detail in [1].

A handy tool for dealing with fractional equations is based on the method of duality, which we did not discussed here, see [40].

The method of generalized Mittag-Leffler functions developed above cannot be directly extended to deal with non-autonomous equations of the type $D^{(\nu)}_{a+*}f(x) = A(x)f(x) + g(x)$ with a family of operators $A(x)$ depending on x. The relevant modification of the theory is developed in [41], [42] and is based on the method of chronological operator-valued Feynman-Kac formulae.

The present paper is mostly based on the ideas suggested by the author in [39] and further developed in [20], [21], [22], [44] and [41].

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