Modified Cramér-Rao Bound for M-FSK Signal Parameter Estimation in Cauchy and Gaussian Noise

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***Abstract*—The Cramér-Rao bound (CRB) provides an efficient standard for evaluating the quality of standard parameter estimators. In this paper, a modified Cramér-Rao bounds (MCRB) for modulation parameter estimations of *M*-ary frequency-shift-keying (*M*-FSK) signals is proposed under the condition of the Gaussian and non-Gaussian additive interference. We extend the MCRB to the estimation of a vector of non-random parameters in the presence of nuisance parameters. Moreover, the MCRB is applied to the joint estimation of phase offset, frequency offsets, frequency deviation, and symbol period of *M*-FSK signal with two important special cases of alpha stable distributions, namely, the Cauchy and the Gaussian. The extensive simulation studies are conducted to contrast the MCRB for the modulation parameter vector in different noise environments.**

***Index Terms*—Modified Cramér-Rao bound, parameter estimation, frequency-shift-keying, impulsive noise.**

I. Introduction

Cognitive radio (CR) is a promising technology for performance enhancement in vehicular networks, which allows the combination of artificial intelligence and software-defined radios. A major challenge of CR is the accurate environmental awareness and blind parameter estimations at CR receivers, which is a key feature for promoting efficient and secure communications in the CR context [1]. The Cramér-Rao bound (CRB) is the fundamental lower bound on the variance of any unbiased estimator, which has been proven that the optimal performance of any parameter estimator can be achieved [2]. However, when the observed signal contains modulated data, carrying out the statistical expectations involved in evaluating the CRB is extremely challenging. To avoid the high computational complexity caused by the unwanted parameters, a modified CRB (MCRB) is proposed as a better alternative [3].

Analytical MCRB for parameter estimations has been widely studied in several literatures, such as active and passive localization [4], blind demodulations [5], and channel identification [6]. Unfortunately, the methods introduced in these literatures rely on a strong assumption of the additive white Gaussian noise (AWGN) channel. Many experimental studies have shown that radio channels may experience non-Gaussian noise, such as the symmetric alpha stable (SαS) distribution. For example, non-Gaussian noise occurs in low frequency communications [7] and shallow underwater acoustic communications [8]. In [9], G. Yang et al. derived the CRB for joint timing and carrier phase offsets estimations of minimum shift keying signals in alpha-stable noise. In [10], M. Liu et al. investigated the CRB on the accuracy of estimating the direction-of-arrival parameter in impulsive noise. In [11], Y. Chen et al. provided the CRB of frequency estimations for complex sinusoid signals in symmetric alpha stable noise. To the best of our knowledge, the MCRB enabled joint parameter estimations for $\boldsymbol{M}$-ary frequency-shift-keying ($\boldsymbol{M}$-FSK) signals in Cauchy and Gaussian noise have not been studied.

Therefore, the MCRB used for parameter estimations is derived for $\boldsymbol{M}$-FSK signals in Cauchy and Gaussian noise. To be specific, we propose an analytical method to derive expressions of the MCRB for joint estimations of phase offset, frequency offsets, frequency deviation, and symbol period of $\boldsymbol{M}$-FSK signal in Cauchy and Gaussian noise. The contributions of this paper can be summarized as two aspects. First, we extend the MCRB to the joint estimation of a vector of non-random parameters in the presence of random nuisance parameters under Cauchy and Gaussian noise. Second, we derive the MCRB for a parameter vector from $\boldsymbol{M}$-FSK signal over fading channels in Cauchy noise.

II. System Model

The continuous-time baseband equivalent of FSK signal with phase and frequency offsets can be given by

$$s\left(t\right) = Ae^{j\theta}e^{j2\pi f_c t}\sum\nolimits_l e^{j2\pi f_\triangle s_l t}h\left(t - lT_b\right), \qquad (1)$$

where A, θ and f_c are the amplitude, the phase offset and frequency offset, respectively. $f_\triangle$ is the frequency deviation, and T_b is the symbol period. $h\left(t\right)$ denotes the shaping function. s_l is the data symbol transmitted during the l-th

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period, which takes equally likely values from the alphabet $s_l \in \{\tilde{s}_m \,|\tilde{s}_m = 2m - 1 - M, m = 1, \cdots, M\}$.

Assume that the signal is corrupted by additive noise. The received signal can be expressed as

$$r(t) = s(t) + w(t) \ , \tag{2}$$

where $s(t)$ represents a complex baseband FSK signal, and $w(t)$ is additive noise. When the signal is affected by fading channel and is corrupted by additive noise, the received signal can be derived by

$$r_\eta(t) = s_\eta(t) + w(t), \tag{3}$$

where $s_\eta(t)$ represents the FSK signal over fading channel

$$s_\eta(t) = Ae^{j\theta}e^{j2\pi f_c t}\sum\nolimits_l \eta_l e^{j2\pi f_\Delta s_l t}h(t - lT_b). \tag{4}$$

where η_l is non-constant fading gain.

The noise $w(t)$ is a random variable following the symmetric alpha-stable (SαS) distribution. The SαS distribution is defined by its characteristic function as

$$\varphi(\omega) = \exp\left(j\delta\omega - \gamma|\omega|^\alpha\right), \tag{5}$$

where α is the characteristic exponent, δ is the location parameter, and γ is the dispersion of the distribution. A stable distribution is called standard if $\delta = 0$ and $\gamma = 1$. The alpha stable distribution has no closed-form expression for the probability density function (PDF) except for two important special cases, namely, the Cauchy ($\alpha = 1$), and the Gaussian ($\alpha = 2$). The Cauchy distribution is given by

$$f_1(\gamma, \delta; x) = \frac{1}{\pi}\frac{\gamma}{\gamma^2 + (x - \delta)^2}. \tag{6}$$

Besides, the Gaussian distribution is given by

$$f_2(\gamma, \delta; x) = \frac{1}{\sqrt{\pi\gamma}}\exp\left(-\frac{(x - \delta)^2}{\gamma}\right). \tag{7}$$

The mixed signal to noise ratio (MSNR) is employed to describe the signal and noise power ratio in this paper.

$$\mathrm{MSNR} = 10\log_{10}\left(\sigma_s^2/\gamma\right), \tag{8}$$

where σ_s^2 is the signal variance, and γ is dispersion coefficient of the alpha stable noise.

III. MCRB FOR M-FSK PARAMETER ESTIMATION

A. MCRB in Cauchy Noise

To jointly estimate a parameter vector of $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}}$, we assume that a vector $\boldsymbol{\lambda}$ with finite dimensional can be uesed to represent $r(t)$ with adequate accuracy in the observed interval. The PDF of the received signal $r(t)$, with the Cauchy distribution noise can be expressed as

$$\rho(\boldsymbol{r}|\,\boldsymbol{s}, \boldsymbol{\lambda}) = \prod\nolimits_{l=1}^{L}(2\pi)^{-1}\gamma\left(\gamma^2 + |r_l - s_l|^2\right)^{-3/2}, \tag{9}$$

where L is the observation time and $\boldsymbol{s} \triangleq \{s_l\}$ is the vector of transmitted symbols. The log-likelihood function is

$$\begin{aligned}\ln\rho(\boldsymbol{r}|\,\boldsymbol{s}, \boldsymbol{\lambda}) &= L\log(\gamma) - L\log(2\pi) \\ &\quad - 1.5\sum\nolimits_{l=1}^{L}\log\left(\gamma^2 + |r_l - s_l|^2\right),\end{aligned} \tag{10}$$

Letting $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}} = (\lambda_1, \ldots, \lambda_4)^{\mathrm{T}}$ denote a vector parameter to be estimated. For any parameter $\lambda_m\,(m = 1, \cdots, 4)$, the fundamental lower bound on the error variance is given as

$$E_{\boldsymbol{r}}\left[\left(\hat{\lambda}_m - \lambda_m\right)^2\right] \geq \mathrm{MCRB}_\lambda(\lambda_m) = \left[\boldsymbol{I}_c^{-1}(\boldsymbol{\lambda})\right]_{m,m}. \tag{11}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda})]_{m,n} = E_{\boldsymbol{r}}\left[-\frac{\partial^2\ln\rho(\boldsymbol{r}|\,\boldsymbol{s}, \boldsymbol{\lambda})}{\partial\lambda_m\partial\lambda_n}\right], \tag{12}$$

where $\boldsymbol{I}_c(\boldsymbol{\lambda})$ is Fisher Information Matrix (FIM), $[\cdot]_{m,n}$ represents the element of the matrix at row m and column n, and $E_{\boldsymbol{r}}[\cdot]$ denotes statistical expectation with respect to the subscripted variable $\boldsymbol{r}$. First, we calculate the derivatives of the log likelihood function given in (10) with respect to the components of $\boldsymbol{\lambda}$ as follows.

$$\begin{aligned}\frac{\partial\ln\rho(\mathbf{r}|\,\mathbf{s}, \lambda)}{\partial\lambda_m} &= -\frac{\partial\left(1.5\sum\limits_{t=1}^{L}\log\left(\gamma^2 + |x(t) - s(t)|^2\right)\right)}{\partial\lambda_m} \\ &= -3\sum_{t=1}^{L}\frac{|w(t)|}{\gamma^2 + |w(t)|^2}\frac{\partial s(t)}{\lambda_m}.\end{aligned} \tag{13}$$

Then, by using (13), allows us to compute the $[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{m,n}$ as follows.

$$\begin{aligned}&[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{m,n} = E\left\{\frac{\partial^2\ln\rho(\mathbf{r}|\,\mathbf{s}, \lambda)}{\partial\lambda_m\partial\lambda_n}\right\} \\ &= 9E\left\{\sum_{t=1}^{L}\frac{|w(t)|}{\gamma^2 + |w(t)|^2}\frac{\partial s^*(t)}{\partial\lambda_m}\sum_{t=1}^{L}\frac{|w(t)|}{\gamma^2 + |w(t)|^2}\frac{\partial s(t)}{\partial\lambda_n}\right\} \\ &= 9E\left\{\sum_{t=1}^{L}\frac{|w(t)|^2}{\left(\gamma^2 + |w(t)|^2\right)^2}\frac{\partial s^*(t)\partial s(t)}{\partial\lambda_m\partial\lambda_n}\right\} \\ &= \frac{3}{5\gamma^2}\int E_s\left[\mathrm{Re}\left(\frac{\partial s^*(t)\partial s(t)}{\partial\lambda_m\partial\lambda_n}\right)\right]dt,\end{aligned} \tag{14}$$

where $E\left\{\frac{|w(t)|^2}{\left(\gamma^2 + |w(t)|^2\right)}\right\} = \frac{2}{15\gamma^2}$. In order to determine the expression of $[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{m,n}$, we should go through $E_s\left[\mathrm{Re}\left(\frac{\partial s^*(t)\partial s(t)}{\partial\lambda_m\partial\lambda_n}\right)\right]\,(m = n)$ which can be written as follows

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s(t)}{\partial\theta}\right|^2\right] = A^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}h^2(t - lT_b)\right], \tag{15}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s(t)}{\partial f_c}\right|^2\right] = 4\pi^2A^2t^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}h^2(t - lT_b)\right], \tag{16}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s(t)}{\partial f_\Delta}\right|^2\right] = 4\pi^2A^2t^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}s_l^2h^2(t - lT_b)\right], \tag{17}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s(t)}{\partial T_b}\right|^2\right] = A^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}l^2\dot{h}^2(t - lT_b)\right], \tag{18}$$

where $\dot{h}(t - T_b)$ is the derivative of $h(t - T_b)$ with respect to T_b. Denote $\Lambda = 3A^2/\left(5\gamma^2\right)$, $C = E_s[s_l s_l^*]$, $\Im_1 = \int_{T_b}\sum_{l=1}^{L}h^2(t - lT_b)dt$, $\Im_2 = \int_{T_b}t\sum_{l=1}^{L}h^2(t - lT_b)dt$, $\Im_3 = \int_{T_b}t^2\sum_{l=1}^{L}h^2(t - lT_b)dt$, $\Omega = \int_{T_0}\sum_{l=1}^{L}l^2\dot{h}^2(t - lT_b)dt$.

According to (14)-(18), the elements of $\boldsymbol{I}(\boldsymbol{\lambda}, s)$ are expressed as

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{1,1} = \Lambda\Im_1, \tag{19}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{2,2} = 4\pi^2\Lambda\Im_3, \tag{20}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{3,3} = 4\pi^2\Lambda C\Im_3, \tag{21}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{4,4} = \Lambda\Omega. \tag{22}$$

Using the same procedure, we can derive the $E_s\left[\mathrm{Re}\left(\frac{\partial s^*(t)\partial s(t)}{\partial\lambda_m\partial\lambda_n}\right)\right]$ $(m \neq n)$ as follows

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial\theta}\frac{\partial s(t)}{\partial f_c}\right)\right]=2\pi tA^2E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}h^2(t-lT_b)\right], \tag{23}$$

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial\theta}\frac{\partial s(t)}{\partial f_\Delta}\right)\right]=2\pi tA^2E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}s_l h^2(t-lT_b)\right], \tag{24}$$

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial f_c}\frac{\partial s(t)}{\partial f_\Delta}\right)\right]=(2\pi tA)^2E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}s_l h^2(t-lT_b)\right], \tag{25}$$

$$\begin{aligned} &E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial\theta}\frac{\partial s(t)}{\partial T_b}\right)\right]=E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial f_c}\cdot\frac{\partial s(t)}{\partial T_b}\right)\right] \\ &=E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial f_\Delta}\cdot\frac{\partial s(t)}{\partial T_b}\right)\right]=0. \end{aligned} \tag{26}$$

Substituting (23)-(26) into (14), the elements of $\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})$ for $m \neq n$ are expressed as

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{1,2} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{2,1} = 2\pi\Lambda\Im_2, \tag{27}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{1,3} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{3,1} = 2\pi\Lambda B\Im_2, \tag{28}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{2,3} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{3,2} = 4\pi^2\Lambda B\Im_3, \tag{29}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{1,4} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{4,1} = 0, \tag{30}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{2,4} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{4,2} = 0, \tag{31}$$

$$[\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{3,4} = [\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})]_{4,3} = 0. \tag{32}$$

where $B = E_s[s_l]$. According to (19)-(22) and (27)-(32), $\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})$ can be written as

$$\boldsymbol{I}_c(\boldsymbol{\lambda}, \boldsymbol{s})=\begin{bmatrix} \Lambda\Im_1 & 2\pi\Lambda\Im_2 & 2\pi\Lambda B\Im_2 & \\ 2\pi\Lambda\Im_2 & 4\pi^2\Lambda\Im_3 & 4\pi^2\Lambda B\Im_3 & \\ 2\pi\Lambda B\Im_2 & 4\pi^2\Lambda B\Im_3 & 4\pi^2\Lambda C\Im_3 & \\ & & & \Lambda\Omega \end{bmatrix}. \tag{33}$$

Substituting (33) in (11), the MCRBs for $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}}$ can be derived as

$$\mathrm{MCRB}^c_{\boldsymbol{\lambda}}(\theta) = \Im_3/(\Lambda(\Im_1\Im_3 - \Im_2\Im_2)), \tag{34}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\lambda}}(f_c) = \frac{C\Im_1\Im_3 - B^2\Im_2\Im_2}{4\pi^2\Lambda\Im_3(\Im_1\Im_3 - \Im_2\Im_2)}, \tag{35}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\lambda}}(f_\Delta) = 1\big/\big(4\pi^2\Lambda\Im_3\left(C - B^2\right)\big), \tag{36}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\lambda}}(T_b) = 1/(\Lambda\Omega). \tag{37}$$

Using a similar approach, we derive the FIM $\boldsymbol{I}_c(\boldsymbol{\mu}, \boldsymbol{s})$ for parameter vector of $\boldsymbol{\mu}=(\theta, f_c, f_\Delta)^{\mathrm{T}}$ which can be given as

$$\boldsymbol{I}_c(\boldsymbol{\mu}, \boldsymbol{s})=\begin{bmatrix} \Lambda\Im_1 & \Lambda 2\pi\Im_2 & 2\pi\Lambda B\Im_2 \\ 2\pi\Lambda\Im_2 & 4\pi^2\Lambda\Im_3 & 4\pi^2\Lambda B\Im_3 \\ 2\pi\Lambda B\Im_2 & 4\pi^2\Lambda B\Im_3 & 4\pi^2\Lambda C\Im_3 \end{bmatrix}. \tag{38}$$

Substituting (38) in (11), the MCRBs for $\boldsymbol{\mu} = (\theta, f_c, f_\Delta)^{\mathrm{T}}$ can be expressed as

$$\mathrm{MCRB}^c_{\boldsymbol{\mu}}(\theta) = \Im_3/(\Lambda(\Im_1\Im_3 - \Im_2\Im_2)), \tag{39}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\mu}}(f_c) = \frac{\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)}{4\pi^2\Lambda\Im_3(\Im_1\Im_3 - \Im_2\Im_2)}, \tag{40}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\mu}}(f_\Delta) = 1\big/\big(4\pi^2\Lambda\Im_3\left(C - B^2\right)\big). \tag{41}$$

For the parameter vector of $\boldsymbol{\upsilon} = (\theta, f_c)^{\mathrm{T}}$, we can obtain $\boldsymbol{I}_c(\boldsymbol{\upsilon}, \boldsymbol{s})$ by using the same procedure.

$$\boldsymbol{I}_c(\boldsymbol{\upsilon}, \boldsymbol{s}) = \begin{bmatrix} \Lambda\Im_1 & 2\pi\Lambda\Im_2 \\ 2\pi\Lambda\Im_2 & 4\pi^2\Lambda\Im_3 \end{bmatrix}. \tag{42}$$

Substituting (42) in (11), the MCRBs for $\boldsymbol{\upsilon} = (\theta, f_c)^{\mathrm{T}}$ are expressed as

$$\mathrm{MCRB}^c_{\boldsymbol{\upsilon}}(\theta) = \Im_3/(\Lambda(\Im_1\Im_3 - \Im_2\Im_2)), \tag{43}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\upsilon}}(f_c) = \Im_1\big/\big(4\pi^2\Lambda(\Im_1\Im_3 - \Im_2\Im_2)\big). \tag{44}$$

For the parameter vector of $\boldsymbol{\sigma} = (\theta, f_\Delta)^{\mathrm{T}}$, $\boldsymbol{I}_c(\boldsymbol{\sigma}, \boldsymbol{s})$ can be expressed as

$$\boldsymbol{I}_c(\boldsymbol{\sigma}, \boldsymbol{s}) = \begin{bmatrix} \Lambda\Im_1 & 2\pi\Lambda B\Im_2 \\ 2\pi\Lambda B\Im_2 & 4\pi^2\Lambda C\Im_3 \end{bmatrix}. \tag{45}$$

Substituting (45) in (11), the MCRBs for the parameter set $\sigma = (\theta, f_\Delta)^{\mathrm{T}}$, are written as

$$\mathrm{MCRB}^c_{\boldsymbol{\sigma}}(\theta) = C\Im_3\big/\big(\Lambda\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)\big), \tag{46}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\sigma}}(f_\Delta) = \Im_1\big/\big(4\pi^2\Lambda\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)\big). \tag{47}$$

For the parameter vector of $\boldsymbol{\nu} = (f_c, f_\Delta)^{\mathrm{T}}$, we can express $\boldsymbol{I}_c(\boldsymbol{\nu}, \boldsymbol{s})$ as

$$\boldsymbol{I}_c(\boldsymbol{\nu}, \boldsymbol{s}) = \begin{bmatrix} 4\pi^2\Lambda\Im_3 & 4\pi^2\Lambda B\Im_3 \\ 4\pi^2\Lambda B\Im_3 & 4\pi^2\Lambda C\Im_3 \end{bmatrix}. \tag{48}$$

Substituting (48) in (11), the MCRBs for the parameter set $\boldsymbol{\nu} = (f_c, f_\Delta)^{\mathrm{T}}$ written as

$$\mathrm{MCRB}^c_{\boldsymbol{\nu}}(f_c) = C/4\pi^2\Lambda\Im_3\left(C - B^2\right), \tag{49}$$

$$\mathrm{MCRB}^c_{\boldsymbol{\nu}}(f_\Delta) = 1\big/\big(4\pi^2\Lambda\Im_3\left(C - B^2\right)\big). \tag{50}$$

B. MCRB in Gaussian Noise

Considering an observation vector $\boldsymbol{r} = (r_1, r_2, \cdots, r_L)^{\mathrm{T}}$ in Gaussian noise, the PDF of the received signal $r(t)$ can be expressed as

$$p(\boldsymbol{r}|\,\boldsymbol{s}, \boldsymbol{\lambda}) = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{1}{\gamma}\int|r_l - s_l|^2 dt\right), \tag{51}$$

where $\boldsymbol{s} \triangleq \{s_l\}$ is the vector of transmitted symbols. The log-likelihood function becomes

$$\ln p(\boldsymbol{r}|\,\boldsymbol{s}, \boldsymbol{\lambda}) = -\frac{1}{\gamma}\int|r_l - s_l|^2 dt. \tag{52}$$

For any parameter $\lambda_m\,(m = 1, \cdots, 4)$, the fundamental lower bound on the error variance given as

$$E_{\boldsymbol{r}}\left[\left(\hat{\lambda}_m - \lambda_m\right)^2\right] \geq \mathrm{MCRB}_{\boldsymbol{\lambda}}(\lambda_m) = \left[\boldsymbol{I}_g^{-1}(\boldsymbol{\lambda})\right]_{m,m}, \tag{53}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda})]_{m,n} = E_{\mathbf{r}}\left[-\frac{\partial^2 \ln p\,(\boldsymbol{r}|\,\boldsymbol{s},\boldsymbol{\lambda})}{\partial\lambda_m\partial\lambda_n}\right], \tag{54}$$

where, $\boldsymbol{I}_g(\cdot)$ is FIM, $[\cdot]_{m,n}$ represents the factor of the matrix with row m and column n, and $E_{\boldsymbol{r}}[\cdot]$ denotes statistical expectation with respect to the subscripted variable $\mathbf{r}$. This gives

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{m,n} = \frac{2}{\gamma}\int E_{\mathbf{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial\lambda_m}\cdot\frac{\partial s(t)}{\partial\lambda_n}\right)\right]dt. \tag{55}$$

Similar to Section A, we can obtain the following equations

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{1,1} = \Re\Im_1, \tag{56}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{2,2} = 4\pi^2\Re\Im_3, \tag{57}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{3,3} = 4\pi^2\Re C\Im_3 \tag{58}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{4,4} = \Re\Omega, \tag{59}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{1,2} = [J(\lambda,s)]_{2,1} = 2\pi\Re\Im_2, \tag{60}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{1,3} = [J(\lambda,s)]_{3,1} = 2\pi\Re B\Im_2, \tag{61}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{2,3} = [J(\lambda,s)]_{3,2} = 4\pi^2\Re B\Im_3, \tag{62}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{1,4} = [J(\lambda,s)]_{4,1} = 0, \tag{63}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{2,4} = [J(\lambda,s)]_{4,2} = 0, \tag{64}$$

$$[\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s})]_{3,4} = [J(\lambda,s)]_{4,3} = 0. \tag{65}$$

where $\Re = 2A^2/\gamma$. Using (56)-(65), the FIM $\boldsymbol{J}_g(\boldsymbol{\lambda},\boldsymbol{s})$ is given by

$$\boldsymbol{I}_g(\boldsymbol{\lambda},\boldsymbol{s}) = \begin{bmatrix} \Re\Im_1 & 2\pi\Re\Im_2 & 2\pi\Re B\Im_2 & \\ 2\pi\Re\Im_2 & 4\pi^2\Re\Im_3 & 4\pi^2\Re B\Im_3 & \\ 2\pi\Re B\Im_2 & 4\pi^2\Re B\Im_3 & 4\pi^2\Re C\Im_3 & \\ & & & \Re\Omega \end{bmatrix}. \tag{66}$$

Substituting (66) in (53), the MCRBs for $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}}$ written as

$$\mathrm{MCRB}^g_{\boldsymbol{\lambda}}(\theta) = \Im_3/(\Re(\Im_1\Im_3 - \Im_2\Im_2)), \tag{67}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\lambda}}(f_c) = \frac{\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)}{4\pi^2\Re\Im_3\left(\Im_1\Im_3 - \Im_2\Im_2\right)}, \tag{68}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\lambda}}(f_\Delta) = 1/\left(4\pi^2\Re\Im_3\left(C - B^2\right)\right), \tag{69}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\lambda}}(T_b) = 1/(\Re\Omega). \tag{70}$$

For the parameter vector of $\boldsymbol{\mu} = (\theta, f_c, f_\Delta)^{\mathrm{T}}$, we can also obtain the MCRBs

$$\mathrm{MCRB}^g_{\boldsymbol{\mu}}(\theta) = \Im_3/(\Re(\Im_1\Im_3 - \Im_2\Im_2)), \tag{71}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\mu}}(f_c) = \frac{\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)}{4\pi^2\Re\Im_3\left(\Im_1\Im_3 - \Im_2\Im_2\right)}, \tag{72}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\mu}}(f_\Delta) = 1/\left(4\pi^2\Re\Im_3\left(C - B^2\right)\right), \tag{73}$$

For the parameter vector of $\boldsymbol{\upsilon} = (\theta, f_c)^{\mathrm{T}}$, the MCRBs given by

$$\mathrm{MCRB}^g_{\boldsymbol{\upsilon}}(\theta) = \Im_3/(\Re(\Im_1\Im_3 - \Im_2\Im_2)), \tag{74}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\upsilon}}(f_c) = \Im_1/\left(4\pi^2\Re\left(\Im_1\Im_3 - \Im_2\Im_2\right)\right). \tag{75}$$

For the parameter vector of $\boldsymbol{\sigma} = (\theta, f_\Delta)^{\mathrm{T}}$, the MCRBs written as

$$\mathrm{MCRB}^g_{\boldsymbol{\sigma}}(\theta) = C\Im_3/\left(\Re\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)\right), \tag{76}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\sigma}}(f_\Delta) = \Im_1/\left(4\pi^2\Re\left(C\Im_1\Im_3 - B^2\Im_2\Im_2\right)\right), \tag{77}$$

For the parameter vector of $\boldsymbol{\nu} = (f_c, f_\Delta)^{\mathrm{T}}$, the MCRBs expressed as

$$\mathrm{MCRB}^g_{\boldsymbol{\nu}}(f_c) = C/\left(4\pi^2\Re\Im_3\left(C - B^2\right)\right), \tag{78}$$

$$\mathrm{MCRB}^g_{\boldsymbol{\nu}}(f_\Delta) = 1/\left(4\pi^2\Re\Im_3\left(C - B^2\right)\right). \tag{79}$$

C. MCRB for Fading Channel with Cauchy Noise

In this section we consider the MCRB for modulation parameter estimations of FSK signals over the fading channel with Cauchy noise. Since the noise $w(t)$ is Cauchy noise, the PDF of the received signal $r(t)$ can be expressed as

$$\rho\,(\boldsymbol{r}_\eta|\,\boldsymbol{s}_\eta,\boldsymbol{\lambda}) = \prod\nolimits_{l=1}^{L}(2\pi)^{-1}\gamma\left(\gamma^2 + \left|r^l_\eta - s^l_\eta\right|^2\right)^{-3/2}, \tag{80}$$

where $\boldsymbol{s}_\eta \triangleq \left\{s^l_\eta\right\}$ is the vector of transmitted symbols is affected by fading channel. The log-likelihood function is

$$\begin{aligned}\ln\rho\,(\boldsymbol{r}_\eta|\,\boldsymbol{s}_\eta,\boldsymbol{\lambda}) &= L\log(\gamma) - L\log(2\pi)\\ &\quad - 1.5\sum\nolimits_{l=1}^{L}\log\left(\gamma^2 + \left|r^l_\eta - s^l_\eta\right|^2\right),\end{aligned} \tag{81}$$

Let $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}} = (\lambda_1,\ldots,\lambda_4)^{\mathrm{T}}$ be the modulation parameter vector. Based on (11) and (12), this bound on the variance of any unbiased estimate is given by

$$E_{\boldsymbol{r}}\left[\left(\hat{\lambda}_m - \lambda_m\right)^2\right] \geq \mathrm{MCRB}_\lambda(\lambda_m) = \left[\boldsymbol{I}^{-1}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)\right]_{m,n}. \tag{82}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{m,n} = E_{\boldsymbol{r}}\left[-\frac{\partial^2\ln\rho\,(\boldsymbol{r}_\eta|\,\boldsymbol{s}_\eta,\boldsymbol{\lambda})}{\partial\lambda_m\partial\lambda_n}\right], \tag{83}$$

where $[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{m,n}$ is Fisher Information Matrix (FIM). Using the same procedure in section A, $[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{m,n}$ can be written as

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{m,n} = \frac{3}{5\gamma^2}\int E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*_\eta(t)}{\partial\rho_m}\cdot\frac{\partial s_\eta(t)}{\partial\rho_n}\right)\right]dt. \tag{84}$$

Similarly, $E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*_\eta(t)}{\partial\rho_m}\cdot\frac{\partial s_\eta(t)}{\partial\rho_n}\right)\right]$ can be written as follows

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s_\eta(t)}{\partial\theta}\right|^2\right] = A^2 E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}\eta^2_l h^2(t - lT_b)\right], \tag{85}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s_\eta(t)}{\partial f_c}\right|^2\right] = 4\pi^2A^2t^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}\eta^2_l h^2(t - lT_b)\right], \tag{86}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s_\eta(t)}{\partial f_\Delta}\right|^2\right] = 4\pi^2A^2t^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}s^2_l\eta^2_l h^2(t - lT_b)\right], \tag{87}$$

$$E_{\boldsymbol{s}}\left[\left|\frac{\partial s(t)}{\partial T_b}\right|^2\right] = A^2E_{\mathbf{s}}\left[\sum\nolimits_{l=1}^{L}\eta^2_l l^2\dot{h}^2(t - lT_b)\right]. \tag{88}$$

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*_\eta(t)}{\partial\theta}\frac{\partial s_\eta(t)}{\partial f_c}\right)\right] = \frac{A(t)}{2\pi t}E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}\eta^2_l h^2(t - lT_b)\right], \tag{89}$$

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s_\eta^*(t)}{\partial\theta}\frac{\partial s_\eta(t)}{\partial f_\Delta}\right)\right]=\frac{A(t)}{2\pi t}E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}s_l\eta_l^2h^2(t-lT_b)\right], \tag{90}$$

$$E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s^*(t)}{\partial f_c}\frac{\partial s(t)}{\partial f_\Delta}\right)\right]=A(t)E_{\boldsymbol{s}}\left[\sum\nolimits_{l=1}^{L}s_l\eta_l^2h^2(t-lT_b)\right], \tag{91}$$

$$\begin{aligned}&E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s_\eta^*(t)}{\partial\theta}\frac{\partial s_\eta(t)}{\partial T_b}\right)\right]=E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s_\eta^*(t)}{\partial f_c}\cdot\frac{\partial s_\eta(t)}{\partial T_b}\right)\right]\\&=E_{\boldsymbol{s}}\left[\mathrm{Re}\left(\frac{\partial s_\eta^*(t)}{\partial f_\Delta}\cdot\frac{\partial s_\eta(t)}{\partial T_b}\right)\right]=0.\end{aligned} \tag{92}$$

where $A(t) = (2\pi tA)^2$. We assumed that, $\Lambda = 6A^2/(5\gamma^2)$, $\Upsilon_1 = \int_{T_b}\sum_{l=1}^{L}\eta_l^2h^2(t-lT_b)dt$, $\Upsilon_2 = \int_{T_b}t\sum_{l=1}^{L}\eta_l^2h^2(t-lT_b)dt$, $\Upsilon_3 = \int_{T_b}t^2\sum_{l=1}^{L}\eta_l^2h^2(t-lT_b)dt$, $\Psi=\int_{T_0}\sum_{l=1}^{L}l^2\eta_l^2\dot{h}^2(t-lT_b)dt$. The elements of $\boldsymbol{I}_\eta(\boldsymbol{\lambda},s)$ are expressed as

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{1,1}=\Lambda\Upsilon_1, \tag{93}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{2,2}=4\pi^2\Lambda\Upsilon_3, \tag{94}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{3,3}=4\pi^2\Lambda C\Upsilon_3, \tag{95}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{4,4}=\Lambda\Psi. \tag{96}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{1,2}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{2,1}=2\pi\Lambda\Upsilon_2, \tag{97}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{1,3}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{3,1}=2\pi\Lambda B\Upsilon_2, \tag{98}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{2,3}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{3,2}=4\pi^2\Lambda B\Upsilon_3, \tag{99}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{1,4}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{4,1}=0, \tag{100}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{2,4}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{4,2}=0, \tag{101}$$

$$[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)]_{3,4}=[\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s})]_{4,3}=0. \tag{102}$$

According to (93)-(102), $\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)$ can be written as

$$\boldsymbol{I}_\eta(\boldsymbol{\lambda},\boldsymbol{s}_\eta)=\begin{bmatrix}\Lambda\Upsilon_1 & 2\pi\Lambda\Upsilon_2 & 2\pi\Lambda B\Upsilon_2 & \\ 2\pi\Lambda\Upsilon_2 & 4\pi^2\Lambda\Upsilon_3 & 4\pi^2\Lambda B\Upsilon_3 & \\ 2\pi\Lambda B\Upsilon_2 & 4\pi^2\Lambda B\Upsilon_3 & 4\pi^2\Lambda C\Upsilon_3 & \\ & & & \Lambda\Psi\end{bmatrix}. \tag{103}$$

Application of (103) to (82) yields,

$$\mathrm{MCRB}_{\boldsymbol{\lambda}}^{\eta}(\theta)=\Upsilon_3/(\Lambda(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)), \tag{104}$$

$$\mathrm{MCRB}_{\boldsymbol{\lambda}}^{\eta}(f_c)=\frac{C\Upsilon_1\Upsilon_3-B^2\Upsilon_2\Upsilon_2}{4\pi^2\Lambda\Upsilon_3(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)}, \tag{105}$$

$$\mathrm{MCRB}_{\boldsymbol{\lambda}}^{\eta}(f_\Delta)=1/\left(4\pi^2\Lambda\Upsilon_3\left(C-B^2\right)\right), \tag{106}$$

$$\mathrm{MCRB}_{\boldsymbol{\lambda}}^{\eta}(T_b)=1/(\Lambda\Psi). \tag{107}$$

Similarly, for the parameter vector of $\boldsymbol{\mu}=(\theta,f_c,f_\Delta)^{\mathrm{T}}$, the MCRBs can be expressed as

$$\mathrm{MCRB}_{\boldsymbol{\mu}}^{\eta}(\theta)=\Upsilon_3/(\Lambda(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)), \tag{108}$$

$$\mathrm{MCRB}_{\boldsymbol{\mu}}^{\eta}(f_c)=\frac{\left(C\Upsilon_1\Upsilon_3-B^2\Upsilon_2\Upsilon_2\right)}{4\pi^2\Lambda\Upsilon_3(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)}, \tag{109}$$

$$\mathrm{MCRB}_{\boldsymbol{\mu}}^{\eta}(f_\Delta)=1/\left(4\pi^2\Lambda\Upsilon_3\left(C-B^2\right)\right). \tag{110}$$

For the parameter vector of $\boldsymbol{\upsilon}=(\theta,f_c)^{\mathrm{T}}$, we can also obtain the MCRBs as follow,

$$\mathrm{MCRB}_{\boldsymbol{\upsilon}}^{\eta}(\theta)=\Upsilon_3/(\Lambda(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)), \tag{111}$$

$$\mathrm{MCRB}_{\boldsymbol{\upsilon}}^{\eta}(f_c)=\Upsilon_1/\left(4\pi^2\Lambda(\Upsilon_1\Upsilon_3-\Upsilon_2\Upsilon_2)\right). \tag{112}$$

For the parameter vector of $\boldsymbol{\sigma}=(\theta,f_\Delta)^{\mathrm{T}}$, the MCRBs are written as

$$\mathrm{MCRB}_{\boldsymbol{\sigma}}^{\eta}(\theta)=C\Upsilon_3/\left(\Lambda\left(C\Upsilon_1\Upsilon_3-B^2\Upsilon_2\Upsilon_2\right)\right), \tag{113}$$

$$\mathrm{MCRB}_{\boldsymbol{\sigma}}^{\eta}(f_\Delta)=\Upsilon_1/\left(4\pi^2\Lambda\left(C\Upsilon_1\Upsilon_3-B^2\Upsilon_2\Upsilon_2\right)\right). \tag{114}$$

For the parameter vector of $\boldsymbol{\nu}=(f_c,f_\Delta)^{\mathrm{T}}$, we can express the MCRBs as

$$\mathrm{MCRB}_{\boldsymbol{\nu}}^{\eta}(f_c)=C/4\pi^2\Lambda\Upsilon_3\left(C-B^2\right), \tag{115}$$

$$\mathrm{MCRB}_{\boldsymbol{\nu}}^{\eta}(f_\Delta)=1/\left(4\pi^2\Lambda\Upsilon_3\left(C-B^2\right)\right). \tag{116}$$

We can make the following remarks concerning the described derivation.

Remark 1: The MCRBs of the parameter vector $\boldsymbol{\lambda}$ are dependent on the amplitude A and dispersion coefficient γ in Cauchy and Gaussian noise. When MSNR is given, the MCRBs with Gaussian noise is less than MCRBs with Cauchy noise.

Remark 2: For the parameter vector of $\boldsymbol{\lambda}$, MCRBs for modulation parameter estimation of $\boldsymbol{M}$-FSK signal are dependent on the amplitude A, dispersion coefficient γ, shaping function $h(t)$ and observation time L. In addition, MCRBs for frequency offsets f_c and frequency deviation f_Δ are affected by data symbols $\{s_l\}$.

Remark 3: For the case of $\boldsymbol{M}$-FSK modulated sequence transmitted over fading channel, the fading gain is one of important factor affecting MCRBs. In other words, the worse the fading channel, the higher the MCRBs.

IV. Numerical Results

In this section, the numerical simulation results of the derived MCRBs are presented followed with the corresponding analysis. In simulations, the signal amplitude A is unity, the symbol time T_b is normalized to unity, the carrier frequency f_c is set to $10/T_b$, the carrier phase θ is set arbitrarily between 0 and 2π, the frequency deviation f_Δ is set to $2/T_b$. The observation time is 100 so that there are 100 symbols available for estimations. The parameters of the Cauchy noise ($\alpha = 1$) and Gaussian noise ($\alpha = 2$) are selected as a location parameter $\delta = 0$ and a dispersion coefficient $\gamma = 1$. The consider a fading channel is modeled as an independent Rayleigh flat fading channel.

Fig.1 and Fig.2 demonstrate the MCRBs of the joint estimation parameter vector $\boldsymbol{\lambda} = (\theta, f_c, f_\Delta, T_b)^{\mathrm{T}}$ and the MCRBs of the individual estimation parameter as a function of MSNR in Cauchy and Gaussian noise. It can be observed that jointly estimating MCRBs and individually estimating MCRBs are almost identical for frequency deviation f_Δ and symbol duration T_b. For carrier phases θ and carrier frequency f_c, jointly estimations of the MCRBs are higher than individual estimations of that when MSNR is given. This can be explained that $\mathrm{MCRB}_{\boldsymbol{\lambda}}(\theta)$ and $\mathrm{MCRB}_{\boldsymbol{\lambda}}(f_c)$ are dependent on the frequency deviation f_Δ, and the carrier phases θ and carrier frequency f_c interact with each other. Additionally, comparing Fig.1 and Fig.2, it is observed that the MCRBs

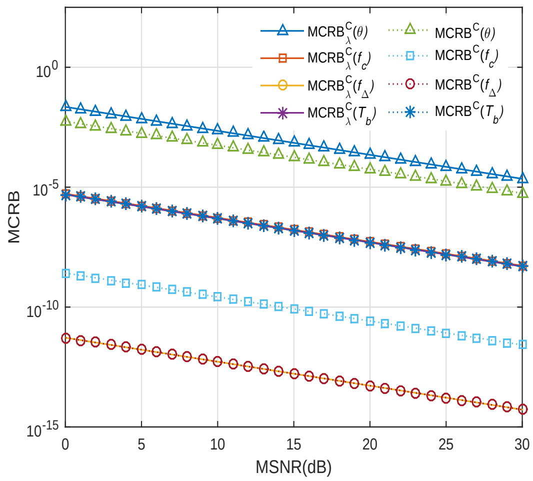

Fig. 1. MCRB of joint estimation and individual estimation for ***M***-FSK signals in Cauchy noise

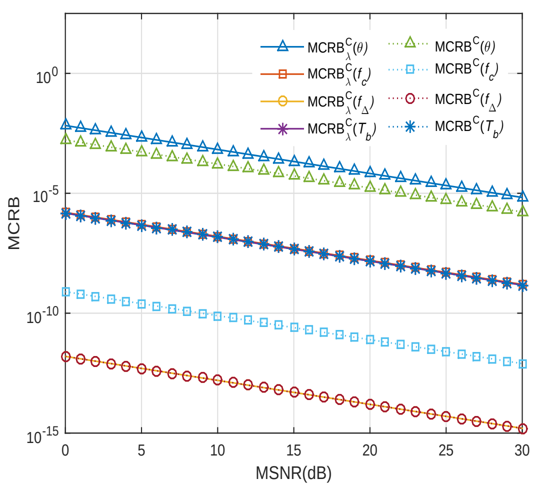

Fig. 2. MCRB of joint estimation and individual estimation for ***M***-FSK signals in Gaussian noise

with Gaussian noise is less than MCRBs with Cauchy noise for given amplitude A.

In Fig. 3, we evaluate the MCRBs of the joint estimation parameter vector $\boldsymbol{\lambda} = (\theta, f_c, f_{\Delta}, T_b)^{\mathrm{T}}$ and the MCRBs of the individual estimation parameter when FSK modulated signal is affected by fading channel and is corrupted by additive Cauchy noise. From the figure, we note that the MCRBs with joint estimations are also higher than MCRBs with individual estimations for the carrier phases θ and carrier frequency f_c. Moreover, comparing Fig.3 and Fig.1, we can find out that the MCRBs for modulation parameter estimations from FSK signals over fading channel are all less than MCRBs with Cauchy noise for given MSNR, indicating that the worse the fading channel, the higher the MCRBs.

V. Conclusion

This paper has derived the MCRB to estimate the modulation parameters of ***M***-FSK signals in Cauchy noise and

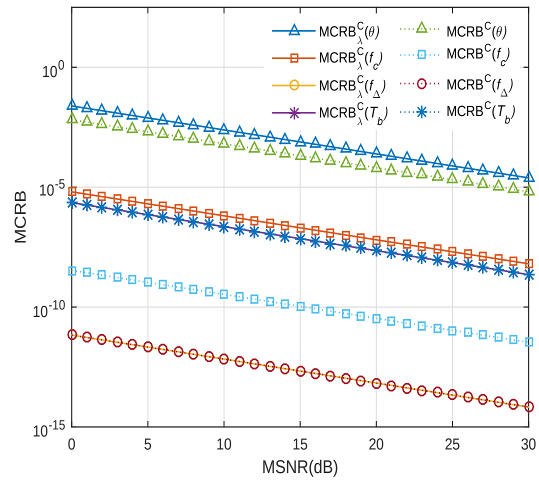

Fig. 3. MCRB of joint estimation and individual estimation for ***M***-FSK signals over Fading channel

Gaussian noise. The modulation parameters considered are the frequency offsets, carrier phases, frequency deviation and symbol duration. In addition, we investigated the MCRB for a parameter vector $\boldsymbol{\lambda}$ from ***M***-FSK signal over fading channels in Cauchy noise. Theoretical analysis and simulations demonstrated that the smaller the characteristic exponent α of the noise, the higher the MCRBs for given MSNR. And the worse the fading channel, the higher the MCRBs for given MSNR. The MCRB approach seems to be applicable in many other modulation types, such as PSK and QAM .

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