CUBIC DIOPHANTINE INEQUALITIES FOR SPLIT FORMS

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ABSTRACT. Denote by $s_0^{(r)}$ the least integer such that if $s \geqslant s_0^{(r)}$, and F is a cubic form with real coefficients in s variables that splits into r parts, then F takes arbitrarily small values at nonzero integral points. We bound $s_0^{(r)}$ for $r \leqslant 6$.

1. INTRODUCTION

Let F be an indefinite homogeneous polynomial of degree $d \geqslant 2$ in s variables, with real coefficients. The form F *takes small values* if there exists $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ such that

$$|F(\mathbf{x})| < 1. \tag{1.1}$$

Schmidt [18] showed that if d is odd then there exists $s_0 = s_0(d)$ such that if $s \geqslant s_0$ then F takes small values.

Henceforth, let s_0 be the least integer with the above property, whenever such an integer exists. Naturally we seek upper bounds for s_0. For cubic forms, Freeman [8] holds the record $s_0(3) \leqslant 359\ 551\ 882$, improving significantly on previous work of Pitman [17]. Contrastingly, Baker, Brüdern and Wooley [2] have shown that any additive cubic form in seven variables takes small values (Baker [1] had already shown this for integral forms).

We consider an intermediate case where only some additive structure is present, a game recently played in a different context by Browning [3] (see also [4]) and continued by Dai and Xue [7]. Let r be a positive integer. A cubic form F in s variables *splits into r parts* if there exist positive integers $a_1, \ldots, a_r$ and nonzero cubic forms $C_1, \ldots, C_r$ such that $a_1 + \ldots + a_r = s$ and

$$F(x_1, \ldots, x_s) = \sum_{j=1}^{r} C_j(x_{A_{j-1}+1}, \ldots, x_{A_j}),$$

where $A_0 = 0$ and $A_j = a_1 + \ldots + a_j$ for $j = 1, 2, \ldots, r$. Denote by $s_0^{(r)}$ the least integer such that if F is a cubic form that splits into r parts and $s \geqslant s_0^{(r)}$ then F takes small values.

Theorem 1.1.

$$s_0^{(1)} \leqslant 358\ 817\ 445, \qquad s_0^{(2)} \leqslant 120\ 897\ 257, \qquad s_0^{(3)} \leqslant 35\ 042\ 291,$$
$$s_0^{(4)} \leqslant 8\ 324\ 100, \qquad s_0^{(5)} \leqslant 1\ 164\ 774, \qquad s_0^{(6)} \leqslant 77\ 027.$$

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Note that if $s \geqslant s_0^{(r)}$ and $\eta > 0$ then the inequality

$$|F(\mathbf{x})| < \eta$$

has a nontrivial solution for any cubic form F that splits into r parts, since the form $\eta^{-1}F$ takes small values. Note also that any cubic form that splits into seven parts takes small values, since any additive cubic form in seven variables takes small values. The improvement in $s_0^{(1)} = s_0(3)$ is obtained via two suggestions made in the introduction of [8]. Though this improvement is minor, we will justify it, since the ideas will also improve our other bounds and perhaps future related results. If one assumed that the parts were of roughly equal dimensions $a_1, \ldots, a_r$, then much better bounds could be obtained. For instance our methods easily show that if F splits into three parts of dimension at least 270 187 then F takes small values.

We briefly discuss the case that F has integer coefficients, wherein the inequality (1.1) reduces to $F(\mathbf{x}) = 0$. Heath-Brown [10] has shown that 14 variables suffice to ensure that a cubic form has a nontrivial zero, improving on Davenport's already spectacular previous record [6] of 16 variables. Under the additional premise that F is nonsingular, Heath-Brown [9] proved that only ten variables are necessary. This has been sharpened by Hooley [12, 13, 14, 15], who established the Hasse principle for nonary cubic forms defining hypersurfaces with at most ordinary double points.

For quadratic forms we have Margulis' celebrated proof of the Oppenheim conjecture (see [16]). This states that the values taken at integral points by an indefinite quadratic form in at least three variables, whose coefficients are not all in rational ratio, are dense on the real line. Margulis' result shows that $s_0(2) \leqslant 5$, since Meyer demonstrated that indefinite quadratic forms with integer coefficients in at least five variables represent zero nontrivially (see [19]). In fact $s_0(2) = 5$, for if $p \equiv 3 \mod 4$ is a prime then

$$a^2 + b^2 - p(c^2 + d^2) = 0$$

has no nontrivial integer solutions.

Now we summarise some of Freeman and Wooley's work on additive forms (see [20]). Let $\mathcal{F}(d)$ denote the least integer t such that any indefinite diagonal form of degree d with real coefficients in at least t variables takes small values. Then

$$\mathcal{F}(4) \leqslant 12, \qquad \mathcal{F}(5) \leqslant 18$$

and

$$\mathcal{F}(d) \leqslant d(\log d + \log\log d + 2 + o(1)).$$

Our discussion up to this point implies that $\mathcal{F}(2) = 5$ and $\mathcal{F}(3) \leqslant 7$.

Much less is known about diophantine inequalities for general forms. The only successful approach thus far has been to 'quasi-diagonalise' and then use results about additive forms. We shall also follow this pattern. If a cubic form F splits into r parts, we will use on each part a quasi-diagonalisation procedure due to Freeman [8], thereby approximating F by a diagonal form in some subspace. The resulting error term necessitates that we find a small solution to a diophantine inequality for additive cubic forms, and for this we use the work of Brüdern [5].

This paper is organised as follows. In §2, we slightly modify some results of Freeman [8], thereby obtaining the bound $s_0^{(1)} \leqslant 358\ 817\ 445$. We then recall two results of Brüdern [5]. In §3, we elaborate on our strategy for deducing the remaining bounds stated in Theorem 1.1. We quasi-diagonalise using different exponents for different parts. In §4 we implement our strategy.

Bold face will be used for vectors, for instance we shall abbreviate $(u_1, \ldots, u_n)$ to $\mathbf{u}$ and write $|\mathbf{u}| = \max |u_i|$. For a form F, denote by $|F|$ the maximum of the absolute values of its coefficients.

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2. Preliminary manoeuvres

In [8], "a trick used by Pitman [17]" is asserted to give $s_0(3) \leqslant 359\ 547\ 172$. Our treatment of this more closely resembles [11, p. 10]. Note that a finite number of integral vectors are linearly dependent over $\mathbb{Z}$ if and only if they are linearly dependent over $\mathbb{R}$.

Lemma 2.1. *Let F be a cubic form with real coefficients in s variables, let E be a positive real number, and let $n \geqslant 2$ be an integer. Let $N > 0$ be sufficiently large in terms of E and F. Suppose there exist linearly dependent nonzero vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{Z}^s$ such that for any $\mathbf{u} \in \mathbb{R}^n$ we have*

$$F(u_1\mathbf{x}_1 + \ldots + u_n\mathbf{x}_n) = \sum_{i \leqslant n} F(\mathbf{x}_i)u_i^3 + O(N^{-E}|\mathbf{u}|^3), \tag{2.1}$$

where the implicit constant may depend on F. Then F takes small values.

Proof. There exists $\mathbf{c} \in \mathbb{Z}^n - \{\mathbf{0}\}$ such that

$$c_1\mathbf{x}_1 = c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n,$$

and without loss of generality $c_2 \geqslant |c_i|$ for $i = 1, 2, \ldots, n$. Specialising $\mathbf{u} = (c_1, c_2, 0, \ldots, 0)$ in (2.1) yields

$$F(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) = c_1^3 F(\mathbf{x}_1) + c_2^3 F(\mathbf{x}_2) + O(N^{-E}c_2^3),$$

and we also have

$$\begin{aligned} F(c_1\mathbf{x}_1 + c_2\mathbf{x}_2) &= F(2c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + \ldots + c_n\mathbf{x}_n) \\ &= 8c_2^3 F(\mathbf{x}_2) + \sum_{i=3}^{n} c_i^3 F(\mathbf{x}_i) + O(N^{-E}c_2^3), \end{aligned}$$

so

$$c_1^3 F(\mathbf{x}_1) = 7c_2^3 F(\mathbf{x}_2) + \sum_{i=3}^{n} c_i^3 F(\mathbf{x}_i) + O(N^{-E}c_2^3). \tag{2.2}$$

Moreover,

$$\begin{aligned} c_1^3 F(\mathbf{x}_1) &= F(c_1\mathbf{x}_1) = F(c_2\mathbf{x}_2 + \ldots + c_n\mathbf{x}_n) \\ &= \sum_{i=2}^{n} c_i^3 F(\mathbf{x}_i) + O(N^{-E} c_2^3). \end{aligned} \tag{2.3}$$

Equations (2.2) and (2.3) yield

$$F(\mathbf{x}_2) \ll N^{-E}.$$

Since N is large we conclude that $|F(\mathbf{x}_2)| < 1$, so F takes small values. □

The following is the same as [8, Definition 2], except we do not insist that the quasi-diagonalising vectors be linearly independent.

Definition 2.2. Let n be a positive integer and E a positive real number. Let $\hat{w}_3^{(n)}(E)$ be the least nonnegative integer t such that if F is a form in more than t variables and N is sufficiently large in terms of s and E, then there exist $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ with $|\mathbf{x}_j| \leqslant N$ for $j = 1, 2, \ldots, n$ such that if $\mathbf{u} \in \mathbb{R}^n$ then

$$F(u_1\mathbf{x}_1 + \ldots + u_n\mathbf{x}_n) = \sum_{i \leqslant n} F(\mathbf{x}_i) u_i^3 + O(N^{-E}|F| \cdot |\mathbf{u}|^3).$$

Lemma 2.3. *Let $0 < \delta < 1$, let E_1, E_2 and E_3 be positive real numbers, and let n be a positive integer. Put*

$$E = \min(E_1\delta + \delta - 3, E_2 - E_2\delta - 3\delta, E_3 - 2), \qquad M = \hat{w}_3^{(n)}(E_2)$$

and

$$s = 1 + w_1^{(0)}(n(n+1)/2, E_3),$$

where $w_1^{(\cdot)}(\cdot,\cdot)$ is the positive integer defined in [8, Definition 1]. *Assume that $E > 0$. Then*

$$\hat{w}_3^{(n+1)}(E) \leqslant \max(s - 1, w_1^{(M)}(s(s+1)/2, E_1)).$$

Proof. We follow, *mutatis mutandis*, the proof of [8, Lemma 3]. Let

$$B > \max(s - 1, w_1^{(M)}(s(s+1)/2, E_1)),$$

let F be a cubic form with real coefficients in B variables, and let $N > 0$ be large. Again T is a subspace of $\mathbb{R}^B$, but now $\mathbf{x}$ is free to be any vector in $\mathbb{R}^B$, and $\mathbf{d}_1, \ldots, \mathbf{d}_{M+1}$ are not restricted to lie in U either. The vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^{M+1} \setminus \{0\}$ are no longer necessarily linearly independent. The independence of the vectors $\mathbf{d}_1, \ldots, \mathbf{d}_{M+1}$ nonetheless ensures that the vectors $\mathbf{b}_i$ are nonzero. The rest of the proof is identical to that of [8, Lemma 3]. □

The next corollary now follows in the same way as [8, Corollary 2].

Corollary 2.4. *Let $0 < \delta < 1$, let E be a positive real number, let n be a positive integer, and put*

$$s = 1 + \lceil (E+3)n(n+1)/2 \rceil.$$

Then

$$\hat{w}_3^{(n+1)}(E) \leqslant \lceil s(s+1)(E+3)/(2\delta) \rceil + \hat{w}_3^{(n)}((E+3\delta)/(1-\delta)).$$

We can now follow the proof of [8, Theorem 1] on [8, p. 34], for if $\mathbf{x}_1, \ldots, \mathbf{x}_9$ were linearly dependent then F would take small values by Lemma 2.1. This leads to the following observation.

Lemma 2.5. *Let ε be a positive real number, and let F be a real cubic form in s variables. Put $n = 9$, write $E = 24 + \varepsilon$, and let $N > 0$ be sufficiently large in terms of E and F. Suppose there exist $\mathbf{x}_1, \ldots, \mathbf{x}_9 \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ with $|\mathbf{x}_i| \leqslant N$ for $i = 1, 2, \ldots, 9$ such that* (2.1) *holds for any $\mathbf{u} \in \mathbb{R}^9$. Then F takes small values.*

In particular $s_0^{(1)} = s_0(3) \leqslant 1 + \hat{w}_3^{(9)}(24+\varepsilon)$, so it remains to bound $\hat{w}_3^{(9)}(24+\varepsilon)$ for some positive real number ε. To do so, we apply Corollary 2.4 eight times in succession, with well-chosen parameters δ. Firstly, $\hat{w}_3^{(1)}(E_1) = 0$ for any positive real number E_1. For $n = 1, 2, \ldots, 8$, Corollary 2.4 recursively bounds $\hat{w}_3^{(n+1)}(E_{n+1})$, where

$$E_{n+1} = (1 - \delta_n)E_n - 3\delta_n.$$

Given $E_9 = 24 + \varepsilon$, note that $E_8, \ldots, E_1$ are determined by $\delta_8, \ldots, \delta_1$.

Using Freeman's choice of parameters $\delta_8, \ldots, \delta_1$ on [8, p. 35] and $\varepsilon = 0.00001$ yields $\hat{w}_3^{(9)}(24 + \varepsilon) \leqslant 359\ 547\ 171$, which recovers $s_0(3) \leqslant 359\ 547\ 172$, as claimed in [8]. With $\delta_1 = 0.99999999999$ and $\varepsilon = 10^{-13}$, the choice of parameters given in Table 1 yields $\hat{w}_3^{(9)}(24 + \varepsilon) \leqslant 358\ 817\ 444$, so

$$s_0^{(1)} = s_0(3) \leqslant 358\ 817\ 445.$$

We use this method to obtain upper bounds for general $\hat{w}_3^{(n)}(E)$. The bound obtained is a function of $\delta_{n-1}, \ldots, \delta_1$, so a numerical optimisation procedure is necessary. We choose $\delta_1 = 0.99999999999$, and optimise the remaining parameters using the *Microsoft Excel* 'Solver', choosing the 'Evolutionary' method.

We shall also need the following results of Brüdern.

Theorem 2.6. [5, p. 2] *Let $\theta > 0$, and let $\boldsymbol{\lambda} \in \mathbb{R}^9$ with $|\lambda_1|, \ldots, |\lambda_9| \geqslant 1$. Then there exists a solution $\mathbf{t} \in \mathbb{Z}^9$ to the system*

$$|\lambda_1 t_1^3 + \ldots + \lambda_9 t_9^3| < 1, \qquad 0 < \sum_{i \leqslant 9} |\lambda_i t_i^3| \ll |\lambda_1 \cdots \lambda_9|^{1+\theta}.$$

Theorem 2.7. [5, p. 1] *Let $\theta > 0$, and let $\boldsymbol{\lambda} \in \mathbb{R}^8$ with $|\lambda_1|, \ldots, |\lambda_8| \geqslant 1$. Then there exists a solution $\mathbf{t} \in \mathbb{Z}^8$ to the system*

$$|\lambda_1 t_1^3 + \ldots + \lambda_8 t_8^3| < 1, \qquad 0 < \sum_{i \leqslant 8} |\lambda_i t_i^3| \ll |\lambda_1 \cdots \lambda_8|^{15/8+\theta}.$$

3. Strategy

For the remainder of this paper, let F be a cubic form with real coefficients in s variables that splits into r parts of dimensions $a_1 \leqslant \ldots \leqslant a_r$, put $\varepsilon = 10^{-13}$, and let N denote a large positive real number. Implicit constants in Vinogradov and Landau notation may henceforth depend on F.

We expect good bounds if many of our parts are large, since we can then quasi-diagonalise each large part. For example, if $r \geqslant 3$ and $a_1 > \hat{w}_3^{(3)}(24+\varepsilon)$ then F takes small values, by Lemma 2.5. (By the procedure outlined after Lemma 2.5, with $\delta_2 = 0.480769$, we obtain $\hat{w}_3^{(3)}(24+\varepsilon) \leqslant 270\ 186$.) Consequently, in proving Theorem 1.1, we need a method that is effective in the case that $a_1, \ldots, a_{r-2}$ are small. We will either quasi-diagonalise the largest part or the largest two parts.

Case: $r = 2, 3, 4$**.** Here we use nine quasi-diagonalising vectors.

Lemma 3.1. *Suppose $a_{r-1} < (s - \hat{w}_3^{(10-r)}(27-3r+\varepsilon))/(r-1)$. Then F takes small values.*

Proof. Now $a_r \geqslant s-(r-1)a_{r-1} > \hat{w}_3^{(10-r)}(E)$, where $E = 27-3r+\varepsilon$. By letting $\mathbf{x}_i$ be the $(A_{i-1}+1)$st standard basis vector for $i = 1, 2, \ldots, r-1$ and choosing $\mathbf{x}_r, \ldots, \mathbf{x}_9 \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ using the definition of $\hat{w}_3^{(10-r)}(E)$, we deduce for all $\mathbf{u} \in \mathbb{R}^9$ that

$$F(u_1\mathbf{x}_1 + \ldots + u_9\mathbf{x}_9) = \sum_{i \leqslant 9} F(\mathbf{x}_i)u_i^3 + O(N^{-E}c), \tag{3.1}$$

where $c = \max(|u_r|, \ldots, |u_9|)^3$. For $j = 1, \ldots, r-1$ we have $F(\mathbf{x}_j) \ll 1$, while for $i = r, \ldots, 9$ we have $|\mathbf{x}_i| \leqslant N$, and so $F(\mathbf{x}_i) \ll N^3$.

We may assume that $|F(\mathbf{x}_1)|, \ldots, |F(\mathbf{x}_9)| \geqslant 1$, since otherwise F takes small values. Let θ be sufficiently small compared to $\varepsilon = 10^{-13}$. By Theorem 2.6 we may choose $\mathbf{u} \in \mathbb{Z}^9 \setminus \{\mathbf{0}\}$ such that

$$\left| \sum_{i \leqslant 9} 2F(\mathbf{x}_i)u_i^3 \right| < 1$$

and $c \ll (N^3)^{9-r+\theta}$, giving $|F(u_1\mathbf{x}_1 + \ldots + u_9\mathbf{x}_9)| < 1$. This implies that F takes small values, for if $\mathbf{x}_1, \ldots, \mathbf{x}_9$ were linearly dependent then by (3.1) we could apply Lemma 2.1. $\square$

We wish to show that F takes small values. By Lemma 3.1 we may assume in the sequel that

$$a_{r-1} \geqslant (s - \hat{w}_3^{(10-r)}(27-3r+\varepsilon))/(r-1). \tag{3.2}$$

In this case we might quasi-diagonalise the largest two parts.

Lemma 3.2. *Let $E_1 > 3$ and E_2 be real numbers satisfying*

$$(E_1 - 3)(E_2 - 3(8-r)) > 18(9-r).$$

Suppose $a_{r-1} > \hat{w}_3^{(2)}(E_1)$ and $a_r > \hat{w}_3^{(9-r)}(E_2)$. Then F takes small values.

Proof. Since $E_1 > 3 + 18(9-r)/(E_2 - 3(8-r))$ we may choose $\alpha > 6/(E_2 - 3(8-r))$ such that $E_1 > 3 + 3\alpha(9-r)$. Let $\mathbf{w}_i$ be the $(A_{i-1}+1)$st standard basis vector for $i = 1, 2, \ldots, r-2$. There exist $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \ldots, \mathbf{y}_{9-r} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ such that $|\mathbf{x}_j| \leqslant N$ $(j = 1, 2)$, $|\mathbf{y}_k| \leqslant N^\alpha$ $(k = 1, 2, \ldots, 9-r)$ and for any $\mathbf{t} \in \mathbb{R}^{r-2}, \mathbf{u} \in \mathbb{R}^2$ and $\mathbf{v} \in \mathbb{R}^{9-r}$ we have

$$F(\mathbf{z}) = \sum_{i \leqslant r-2} F(\mathbf{w}_i) t_i^3 + \sum_{j \leqslant 2} F(\mathbf{x}_j) u_j^3 + \sum_{k \leqslant 9-r} F(\mathbf{y}_k) v_k^3 + O(N^{-E_1}|\mathbf{u}|^3 + N^{-\alpha E_2}|\mathbf{v}|^3), \tag{3.3}$$

where $\mathbf{z} = t_1\mathbf{w}_1 + \ldots + t_{r-2}\mathbf{w}_{r-2} + u_1\mathbf{x}_1 + u_2\mathbf{x}_2 + v_1\mathbf{y}_1 + \ldots + v_{9-r}\mathbf{y}_{9-r}$.

Let $\theta > 0$ be small in terms of E_1, E_2 and α. We may assume that $|F(\mathbf{w}_i)|, |F(\mathbf{x}_j)|, |F(\mathbf{y}_k)| \geqslant 1$, since otherwise F takes small values. For $j = 1, 2$ and $k = 1, 2, \ldots, 9-r$ we have $F(\mathbf{x}_j) \ll N^3$ and $F(\mathbf{y}_k) \ll N^{3\alpha}$. By Theorem 2.6 we may choose nonzero vectors $\mathbf{t} \in \mathbb{Z}^{r-2}$, $\mathbf{u} \in \mathbb{Z}^2$ and $\mathbf{v} \in \mathbb{Z}^{9-r}$ satisfying

$$|\mathbf{u}|^3 \ll N^3 (N^{3\alpha})^{9-r} N^\theta, \qquad |\mathbf{v}|^3 \ll (N^3)^2 (N^{3\alpha})^{8-r} N^\theta \tag{3.4}$$

and

$$\Big| \sum_{i \leqslant r-2} 2F(\mathbf{w}_i) t_i^3 + \sum_{j \leqslant 2} 2F(\mathbf{x}_j) u_j^3 + \sum_{k \leqslant 9-r} 2F(\mathbf{y}_k) v_k^3 \Big| < 1. \tag{3.5}$$

Substituting (3.4) into (3.3) yields

$$F(\mathbf{z}) = \sum_{i \leqslant r-2} F(\mathbf{w}_i) t_i^3 + \sum_{j \leqslant 2} F(\mathbf{x}_j) u_j^3 + \sum_{k \leqslant 9-r} F(\mathbf{y}_k) v_k^3 + O(N^{\mathcal{E}+\theta}), \tag{3.6}$$

where $\mathcal{E} = \max(3 + 3\alpha(9-r) - E_1, 3(2 + \alpha(8-r)) - \alpha E_2) < 0$. Since θ is small we also have $\mathcal{E} + \theta < 0$. Combining (3.5) and (3.6) now yields $|F(\mathbf{z})| < 1$. This implies that F takes small values, for if $\mathbf{w}_1, \ldots, \mathbf{w}_{r-2}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \ldots, \mathbf{y}_{9-r}$ were linearly dependent then by (3.3) we could apply Lemma 2.1. □

Choose E_1 so that

$$(s - \hat{w}_3^{(10-r)}(27 - 3r + \varepsilon))/(r-1) > \hat{w}_3^{(2)}(E_1). \tag{3.7}$$

It is advantageous to choose E_1 only slightly smaller than is necessary for (3.7). By (3.2) we now have $a_{r-1} > \hat{w}_3^{(2)}(E_1)$. We need s to be large enough to ensure that we can choose $E_1 > 3$. Choose E_2 so that

$$(E_1 - 3)(E_2 - 3(8-r)) > 18(9-r). \tag{3.8}$$

It is advantageous to choose E_2 only slightly larger than is necessary for (3.8). In view of Lemma 3.2 we may assume that $a_r \leqslant \hat{w}_3^{(9-r)}(E_2)$, which implies that

$$a_{r-1} \geqslant (s - \hat{w}_3^{(9-r)}(E_2))/(r-1).$$

If s is large enough then this bound will be better than our initial bound (3.2), in which case we can repeat this procedure until a contradiction is reached. This would imply that F takes small values.

For each r, the upper bound that we obtain for $s_0^{(r)}$ is not much greater than the theoretical limit of our approach, that being our upper bound for

$\hat{w}_3^{(10-r)}(27-3r+\varepsilon)$. Little is lost, therefore, from the guesswork and computer optimisation involved here.

Case: $r = 5, 6$. Here we can achieve better bounds using eight quasi-diagonalising vectors. We use the following analogously-proven variants of Lemmas 3.1 and 3.2. These rely on Theorem 2.7 instead of Theorem 2.6 as a key ingredient.

Lemma 3.3. *Suppose* $(r-1)a_{r-1} < s - \hat{w}_3^{(9-r)}(\varepsilon - 3 + 45(9-r)/8)$. *Then* F *takes small values.*

Lemma 3.4. *Let* E_1 *and* E_2 *be real numbers such* $H_1 > 0$ *and* $H_1 H_2 > 8 - r$, *where* $H_1 = (4E_1 - 33)/45$ *and* $H_2 = r - 8 + 8(E_2 + 3)/45$. *Suppose* $a_{r-1} > \hat{w}_3^{(2)}(E_1)$ *and* $a_r > \hat{w}_3^{(8-r)}(E_2)$. *Then* F *takes small values.*

The strategy in this case is much the same. Again the upper bound that we obtain for $s_0^{(r)}$ is not much greater than the theoretical limit of our approach, which is now our upper bound for $\hat{w}_3^{(9-r)}(\varepsilon - 3 + 45(9-r)/8)$.

4. Implementation

Our bounds for $\hat{w}_3^{(n)}(E)$ are gotten via the procedure outlined after Lemma 2.5. When $n \geqslant 3$, we specify our choices of $\delta_{n-1}, \ldots, \delta_2$ in §5, so that the reader may easily verify these bounds.

Case: $r = 2$. Assume for the sake of contradiction that $s \geqslant 120\ 897\ 257$ and F does not take small values. From Table 1 we see that $\hat{w}_3^{(8)}(21+\varepsilon) \leqslant 120\ 893\ 893$, so by Lemma 3.1 we have

$$a_1 \geqslant 120\ 897\ 257 - 120\ 893\ 893 > \hat{w}_3^{(2)}(14.6992).$$

Lemma 3.2 now gives

$$a_2 \leqslant \hat{w}_3^{(7)}(28.77) \leqslant 120\ 847\ 458$$

(see Table 2). Now $a_1 \geqslant 49\ 799 > \hat{w}_3^{(2)}(42)$, so by Lemma 3.2 we have

$$a_2 \leqslant \hat{w}_3^{(7)}(21.2308) < 54\ 000\ 000 < s/2$$

(see Table 2), contradiction.

Case: $r = 3$. Assume for the sake of contradiction that $s \geqslant 35\ 042\ 291$ and F does not take small values. From Table 3 we see that $\hat{w}_3^{(7)}(18+\varepsilon) \leqslant 35\ 037\ 484$, so by Lemma 3.1 we have

$$a_2 \geqslant (35\ 042\ 291 - 35\ 037\ 484)/2 > \hat{w}_3^{(2)}(12.705).$$

Lemma 3.2 now gives

$$a_3 \leqslant \hat{w}_3^{(6)}(26.1283) \leqslant 34\ 956\ 075$$

(see Table 3). Now $a_2 \geqslant 43\ 108 > \hat{w}_3^{(2)}(40)$, so by Lemma 3.2 we have

$$a_3 \leqslant \hat{w}_3^{(6)}(17.919) < 13\ 000\ 000$$

(see Table 4). Now $a_2 > 10\ 000\ 000 > \hat{w}_3^{(2)}(267)$, so by Lemma 3.2 we have

$$a_3 \leqslant \hat{w}_3^{(6)}(15.41) < 9\ 000\ 000 < s/3$$

(see Table 4), contradiction.

Case: $r = 4$**.** Assume for the sake of contradiction that $s \geqslant 8\ 324\ 100$ and F does not take small values. From Table 5 we see that $\hat{w}_3^{(6)}(15+\varepsilon) \leqslant 8\ 319\ 167$, so by Lemma 3.1 we have

$$a_3 \geqslant (8\ 324\ 100 - 8\ 319\ 167)/3 > \hat{w}_3^{(2)}(10.6989).$$

Lemma 3.2 now gives

$$a_4 \leqslant \hat{w}_3^{(5)}(23.69) \leqslant 8\ 300\ 761$$

(see Table 5). Now $a_3 > 7\ 779 \geqslant \hat{w}_3^{(2)}(20.935)$, so by Lemma 3.2 we have

$$a_4 \leqslant \hat{w}_3^{(5)}(17.02) \leqslant 3\ 532\ 167$$

(see Table 6). Now $a_3 \geqslant 1\ 597\ 311 > \hat{w}_3^{(2)}(143)$, so by Lemma 3.2 we have

$$a_4 \leqslant \hat{w}_3^{(5)}(12.7) < 2\ 000\ 000 < s/4$$

(see Table 6), contradiction.

Case: $r = 5$**.** Assume for the sake of contradiction that $s \geqslant 1\ 164\ 774$ and F does not take small values. From Table 7 we see that $\hat{w}_3^{(4)}(19.5+\varepsilon) \leqslant 1\ 149\ 469$, so by Lemma 3.3 we have

$$a_4 \geqslant (1\ 164\ 774 - 1\ 149\ 469)/4 > \hat{w}_3^{(2)}(15.215),$$

so by Lemma 3.4 we have

$$a_5 \leqslant \hat{w}_3^{(3)}(41.132) \leqslant 1\ 148\ 061$$

(see Table 7). Now $a_4 > 4\ 178 > \hat{w}_3^{(2)}(16)$, so by Lemma 3.4 we have

$$a_5 \leqslant \hat{w}_3^{(3)}(38.371) \leqslant 950\ 897$$

(see Table 8). Now $a_4 > 53\ 469 > \hat{w}_3^{(2)}(43)$, so by Lemma 3.4 we have

$$a_5 \leqslant \hat{w}_3^{(3)}(19.34) < 160\ 000 < s/5$$

(see Table 8), contradiction.

Case: $r = 6$**.** Assume for the sake of contradiction that $s \geqslant 77\ 027$ and F does not take small values. From Table 8 we see that $\hat{w}_3^{(3)}(111/8+\varepsilon) \leqslant 67\ 151$. By Lemma 3.3 we have

$$a_5 \geqslant (77\ 027 - 67\ 151)/5 > \hat{w}_3^{(2)}(11.52),$$

so by Lemma 3.4 we have

$$a_6 \leqslant \hat{w}_3^{(2)}(47) \leqslant 66\ 301.$$

Now $a_5 > 2\ 145 > \hat{w}_3^{(2)}(12)$, so by Lemma 3.4 we have

$$a_6 \leqslant \hat{w}_3^{(2)}(42+\varepsilon) \leqslant 50\ 761.$$

Now $a_5 > 5\ 253 \geqslant \hat{w}_3^{(2)}(17.76)$, so by Lemma 3.4 we have

$$a_6 \leqslant \hat{w}_3^{(2)}(21.56) \leqslant 8\ 621 < s/6,$$

contradiction.

5. Appendix

Table 1. δ_t values that produce our $\hat{w}_3^{(n)}(E)$ bounds.

t	$\hat{w}_3^{(9)}(24+\varepsilon) \leqslant 358\ 817\ 444$	$\hat{w}_3^{(8)}(21+\varepsilon) \leqslant 120\ 893\ 893$
8	0.1219508888575	
7	0.1390000051495	0.1384533753495
6	0.1616363780159	0.1609356877609
5	0.1916509521206	0.1922156002865
4	0.2380722861783	0.2388890203917
3	0.3174342105461	0.3164554189125
2	0.47495682232	0.4768207640774

Table 2. More δ_t values.

t	$\hat{w}_3^{(7)}(28.77) \leqslant 120\ 847\ 458$	$\hat{w}_3^{(7)}(21.2308) < 54\ 000\ 000$
6	0.1609827599744	0.16
5	0.1926282407656	0.2
4	0.2394541535068	0.24
3	0.3173479381534	0.3
2	0.4778402527145	0.5

Table 3. More δ_t values.

t	$\hat{w}_3^{(7)}(18+\varepsilon) \leqslant 35\ 037\ 484$	$\hat{w}_3^{(6)}(26.1283) \leqslant 34\ 956\ 075$
6	0.1622320717848	
5	0.1913883878295	0.190875904156
4	0.237715626746	0.2394408722191
3	0.3184353150435	0.3173069790669
2	0.4766081301145	0.4747480051924

Table 4. More δ_t values.

t	$\hat{w}_3^{(6)}(17.919) < 13\ 000\ 000$	$\hat{w}_3^{(6)}(15.41) < 9\ 000\ 000$
5	0.1922987759852	0.2
4	0.2382521051977	0.24
3	0.3199994105886	0.3
2	0.4736846151953	0.5

TABLE 5. More δ_t values.

t	$\hat{w}_3^{(6)}(15+\varepsilon) \leqslant 8\ 319\ 167$	$\hat{w}_3^{(5)}(23.69) \leqslant 8\ 300\ 761$
5	0.192823726901	
4	0.2397740002555	0.2374187457642
3	0.3178294478615	0.31819054621
2	0.4756096604837	0.4761905057407

TABLE 6. More δ_t values.

t	$\hat{w}_3^{(5)}(17.02) \leqslant 3\ 532\ 167$	$\hat{w}_3^{(5)}(12.7) < 2\ 000\ 000$
4	0.2397461759082	0.2
3	0.3189607203519	0.3
2	0.4774815418743	0.5

TABLE 7. More δ_t values.

t	$\hat{w}_3^{(4)}(19.5+\varepsilon) \leqslant 1\ 149\ 469$	$\hat{w}_3^{(3)}(41.132) \leqslant 1\ 148\ 061$
3	0.318181063807	
2	0.4761907830763	0.4746189532255

TABLE 8. δ_2 values that produce our $\hat{w}_3^{(3)}(E)$ bounds.

$\hat{w}_3^{(3)}(38.371) \leqslant 950\ 897$	0.4763156722225
$\hat{w}_3^{(3)}(19.34) < 160\ 000$	0.5
$\hat{w}_3^{(3)}(111/8+\varepsilon) \leqslant 67\ 151$	0.4726543938831

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