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ISOMETRIC COPIES OF DIRECTED TREES IN ORIENTATIONS OF GRAPHS

TARAS BANAKH, ADAM IDZIK, OLEG PIKHURKO, IGOR PROTASOV, KRZYSZTOF PŚCZOLKA

Abstract. The isometric Ramsey number \( R(\overrightarrow{H}) \) of a family \( \overrightarrow{H} \) of digraphs is the smallest number of vertices in a graph \( G \) such that any orientation of the edges of \( G \) contains every member of \( \overrightarrow{H} \) in the distance-preserving way. We observe that for any finite family \( \overrightarrow{H} \) of finite acyclic graphs the isometric Ramsey number \( R(\overrightarrow{H}) \) is finite, and present upper bounds for \( R(\overrightarrow{H}) \) in some special cases. For example, we show that the isometric Ramsey number of the family of all oriented trees with \( n \) vertices is at most \( n^{2n+o(n)} \).

1. Introduction

In this paper we consider the “isometric” version of the result of Cochet and Duchet [6] who proved (generalizing a result of Rödl [11]) that for every acyclic digraph \( \overrightarrow{H} \) there exists a finite graph \( G \) such that every orientation of \( G \) contains an isomorphic copy of \( \overrightarrow{H} \).

First we recall the necessary definitions from Graph Theory. A graph is a pair \( G = (V_G, E_G) \) consisting of a set \( V_G \) of vertices and a set \( E_G \) of two-element subsets of \( V_G \), called the edges of \( G \). By a digraph we will mean a pair \( \overrightarrow{G} = (V_G, E_G) \) consisting of a set \( V_G \) of vertices and a set \( E_G \subset V_G \times V_G \) of directed edges, where neither loops \((x, x)\), nor pairs of opposite arcs \((x, y)\) and \((y, x)\) are allowed. By an orientation of a graph \( \overrightarrow{G} = (V_G, E_G) \) we understand a function \( \overrightarrow{G} \to V_G^2 \) assigning to each edge \( e \in E_G \) an ordered pair \( e = (a, b) \in V_G^2 \) such that \( e = \{a, b\} \). In this case the pair \( \overrightarrow{G} = (V_G, \{\overrightarrow{e} \in E_G\}) \) is a digraph called an orientation of \( G \).

A sequence \((v_0, \ldots, v_n)\) of distinct vertices of a graph \( G \) is called a path in \( G \) if for every positive \( i \leq n \) the unordered pair \( \{v_{i-1}, v_i\} \) is an edge of \( G \). The length of the path \((v_0, \ldots, v_n)\) is \( n \), that is, the number of edges. The distance \( d_G(x, y) \) between two vertices \( u, v \) of a graph \( G \) is the smallest length of a path in \( G \) connecting the vertices \( u \) and \( v \). If \( u \) and \( v \) cannot be connected by a path, then we write \( d_G(u, v) = \infty \) and assume that \( \infty > n \) for all \( n \in \omega \). A graph \( G \) is called connected if any two vertices \( u, v \) can be connected by a path in \( G \).

The distance in a digraph is taken with respect to the underlying undirected graph.

A sequence \((v_0, \ldots, v_n)\) of distinct vertices of a digraph \( \overrightarrow{G} \) is called a directed path in \( \overrightarrow{G} \) if for every positive \( i \leq n \) the ordered pair \((v_{i-1}, v_i)\) is an edge of \( G \). A directed cycle is a sequence \((v_0, \ldots, v_n)\) of distinct vertices with \((x_i, x_{i+1})\) being a directed edge for each residue \( i \) modulo \( n + 1 \). A digraph \( \overrightarrow{G} \) is acyclic if it contains no directed cycles. It is well-known that each graph \( G \) admits an acyclic orientation \( \overrightarrow{G} \); take any linear order \( \leq \) on the set \( V_G \) of vertices and for any edge \((u, v) \in E_G\) put \((u, v) \in E_G\) if and only if \( u < v \).

Following Rado’s arrow notations, for a graph \( G \) and a digraph \( \overrightarrow{H} \) we write \( G \to \overrightarrow{H} \) if for every orientation \( \overrightarrow{G} \) of \( G \) there exists an injective function \( f : V_G \to V_H \) such that an oriented pair \((u, v) \) of vertices of \( \overrightarrow{H} \) is a directed edge in \( \overrightarrow{H} \) if and only if \((f(u), f(v))\) is a directed edge in \( \overrightarrow{G} \). (Thus we require that \( f \) induces an isomorphism of undirected graphs and preserves all edge orientations.) If, moreover, \( d_{\overrightarrow{G}}(u, v) = d_G(f(u), f(v)) \) for every pair of vertices \( u, v \in V_H \), then we write \( G \Rightarrow \overrightarrow{H} \) and say that \( f \) is an isometric embedding of \( H \) in \( \overrightarrow{G} \). Since each graph \( G \) admits an acyclic orientation, the arrow \( G \to \overrightarrow{H} \) implies that the digraph \( \overrightarrow{H} \) is acyclic.

Given a graph \( G \) and a class \( \overrightarrow{H} \) of digraphs, we write \( G \to \overrightarrow{H} \) (resp. \( G \Rightarrow \overrightarrow{H} \)) if for every oriented graph \( \overrightarrow{H} \in \overrightarrow{H} \) we have \( G \to \overrightarrow{H} \) (resp. \( G \Rightarrow \overrightarrow{H} \)). In this case the family \( \overrightarrow{H} \) necessarily consists of acyclic digraphs. For a natural number \( n \in \mathbb{N} \) by \( \overrightarrow{T}_n \) we denote the class of oriented trees on \( n \) vertices. By a tree we understand a connected graph without cycles. For \( n \in \mathbb{N} \), the directed path \( I_n \) is the digraph with \( V_{I_n} = \{0, \ldots, n - 1\} \) and \( E_{I_n} = \{(i - 1, i) : 0 < i < n\} \).

For a class \( \overrightarrow{H} \) of digraphs let \( R(\overrightarrow{H}) \) (resp. \( \overrightarrow{R}(\overrightarrow{H}) \)) be the smallest number of vertices of a graph \( G \) such that \( G \to \overrightarrow{H} \) (resp. \( G \Rightarrow \overrightarrow{H} \)). If no graph \( G \) with \( G \to \overrightarrow{H} \) (resp. \( G \Rightarrow \overrightarrow{H} \)) exists, then we put \( R(\overrightarrow{H}) = \infty \) (resp.
\[ \mathbb{I}(\vec{H}) = \infty. \]

The number \( \mathbb{R}(\vec{H}) \) (resp. \( \mathbb{R}(\vec{H}) \)) is called the (isometric) Ramsey number of the family \( \vec{H} \). If the family \( \vec{H} \) consists of a unique digraph \( \vec{H} \), then we write \( \mathbb{R}(\vec{H}) \) and \( \mathbb{I}(\vec{H}) \) instead of \( \mathbb{R}(\{\vec{H}\}) \) and \( \mathbb{I}(\{\vec{H}\}) \), respectively.

By Theorem B of Cochand and Duchet [8], for every finite acyclic digraph \( \vec{H} \), the Ramsey number \( \mathbb{R}(\vec{H}) \) is finite. This implies that for every finite family \( \vec{H} \) of finite acyclic digraphs the Ramsey number \( \mathbb{R}(\vec{H}) \leq \sum_{\vec{H} \in \mathbb{H}} \mathbb{R}(\vec{H}) \) is finite, too. In Section 2 we shall apply a deep Ramsey result of Dellamonica and Rödl [7] to prove that the isometric Ramsey number \( \mathbb{I}(\vec{H}) \) is finite, too.

For the family \( \vec{T}_n \) of oriented trees on \( n \) vertices Kohayakawa, Luczak and Rödl [9] proved that \( \mathbb{R}(\vec{T}_n) = O(n^3 \log n) \). In this paper for every \( n \in \mathbb{N} \) we construct a graph \( G_n \) with \( < 2^{2n-1} \) vertices such that \( G_n \Rightarrow \vec{T}_n \), showing that \( \mathbb{I}(\vec{T}_n) < 2^{2n-1} \). Using Bollobás’ [3] bounds on the order of graphs of large girth and large chromatic number, we shall improve the upper bounds \( \mathbb{I}(\vec{T}_n) \leq \mathbb{I}(\vec{T}_n) < 2^{2n-1} \) to \( \mathbb{I}(\vec{T}_n) = o(n^{2n}) \) and \( \mathbb{I}(\vec{T}_n) = o(n^{4n}) \). In Theorem 4.2 using random graphs we improve the latter upper bound to \( \mathbb{I}(\vec{T}_n) \leq (4e + o(1))n^2 \ln n \) obtained by Kohayakawa, Luczak and Rödl [9] to the upper bound \( (K + o(1))n^2 \ln n \) for \( K = \min_{x \geq 1} \frac{16x^2}{x^2 + 1} \approx 98.8249 \ldots \). In Section 5 we search for long directed paths in arbitrary orientations of graphs. In the final Section 6 we prove that every infinite graph \( G \) admits an orientation containing no directed path of infinite diameter in \( G \). Some other results and problems related to coloring and orientations in graphs can be found in [10].

2. The Isometric Ramsey Number for a Finite Acyclic Digraph

In this section we prove that each finite acyclic digraph \( \vec{H} \) has finite isometric Ramsey number \( \mathbb{I}(\vec{H}) \). The idea of the proof of this result was suggested to the authors by Yoshiharu Kohayakawa.

**Theorem 2.1.** For any finite acyclic digraph \( \vec{H} = (V, \vec{E}) \), the isometric Ramsey number \( \mathbb{I}(\vec{H}) \) is finite.

**Proof.** Clearly, it is enough to prove the theorem when the graph \( \vec{H} \) is connected. Fix any vertex \( h \) of \( H \) and consider the digraph \( \vec{H} \) with

\[ V_{\vec{H}} := V_{\vec{H}} \times \{0, 1\} \text{ and } E_{\vec{H}} := \{(h, 0), (h, 1)\} \cup \{(u, 0), (v, 1) : (u, v) \in E_{\vec{H}}\}. \]

Observe that the digraph \( \vec{H} \) is acyclic, connected and contains isometric copies of \( H \) and the graph \( \vec{H} \) with the opposite orientation. Being acyclic, the graph \( \vec{H} \) admits a linear ordering \(<\) of vertices such that \( u < v \) for any directed edge \( (u, v) \in E_{\vec{H}} \).

By Theorem 1.8 of [7], there exists a finite graph \( G \) with a linear ordering of vertices such that for any 2-coloring of its edges there exists a monotone isometric embedding \( f : V_{\vec{H}} \rightarrow V_G \) such that the set \( \{f(u), f(v)\} : (u, v) \in E_{\vec{H}}\) is monochrome. In this case we shall say that the embedding \( f \) is monochrome. The monotonicity of \( f \) means that \( f \) preserves the order of vertices.

We claim that \( G \Rightarrow \vec{H} \). Given any orientation \( \vec{G} \) of the graph \( G \), color an edge \( \{u, v\} \in E_G \) with \( u < v \) in green if \( (u, v) \in E_{\vec{G}} \) and in red if \( (u, v) \in \vec{E}_{\vec{G}} \). By the Ramsey property of \( G \), there exists a monochrome monotone isometric embedding \( f : V_{\vec{H}} \rightarrow V_G \). If the color of the monochromatic set \( C = \{f(u), f(v)\} : (u, v) \in E_{\vec{H}}\) is green, then the map \( g_0 : V_{\vec{H}} \rightarrow V_G, g_0 : v \mapsto f(v, 0) \), is a required isometric isomorphic embedding of \( \vec{H} \) into \( \vec{G} \). If the color of \( C \) is red, then the map \( g_1 : V_{\vec{H}} \rightarrow V_G, g_1 : v \mapsto f(v, 1) \), is an isometric isomorphic embedding of \( \vec{H} \) into \( \vec{G} \). In both cases we get \( G \Rightarrow \vec{H} \). \( \Box \)

**Corollary 2.2.** Any finite family \( \vec{H} \) of finite acyclic digraphs has finite isometric Ramsey number \( \mathbb{I}(\vec{H}) \). \( \Box \)

**Corollary 2.3.** For every \( n \in \mathbb{N} \) the family \( \vec{T}_n \) of directed trees on \( n \) vertices has finite isometric Ramsey number \( \mathbb{I}(\vec{T}_n) \). \( \Box \)

**Remark 2.4.** The proof of [7] Theorem 1.8 proceeds by a more general induction involving amalgamation and hypergraphs, and seems to give very bad bounds on the isometric Ramsey number \( \mathbb{I}(\vec{A}_n) \) for the family \( \vec{A}_n \) of all acyclic digraphs on \( n \) vertices. It would be interesting to get some reasonable upper bound on this function.
3. Simple bounds for the isometric Ramsey numbers $\mathbb{IR}(\vec{T}_n)$

In this section we prove some simple upper bounds on the isometric Ramsey numbers $\mathbb{IR}(\vec{T}_n)$ and $\mathbb{IR}(\vec{I}_n)$. First we present a simple example of a graph witnessing that $\mathbb{IR}(\vec{T}_n) < 2^{2n-1}$. The construction of this graph exploits rectangular products of graphs. By definition, the rectangular product $G \times H$ of two graphs $G, H$ is a graph such that $V_{G \times H} = V_G \times V_H$ and an unordered pair $\{(g, h), (g', h')\} \subset G \times H$ is an edge of $G \times H$ if and only if either $\{g, g'\} \in E_G$ and $h = h'$ or $g = g'$ and $\{h, h'\} \in E_H$. It can be shown that for any vertices $(g, h), (g', h')$ of $G \times H$ we get

$$d_{G \times H}((g, h), (g', h')) = d_G(g, g') + d_H(h, h').$$

For an (oriented) graph $G$ by $|G|$ we denote the cardinality of the set $V_G$ of vertices of $G$. For a cardinal number $m$ by $K_m$ we denote the complete graph on $m$ vertices.

**Lemma 3.1.** Let $\vec{T}, \vec{T}'$ be two families of finite oriented trees such that for every oriented tree $\vec{T} \in \vec{T}'$ there is an oriented subtree $\vec{T} \in \vec{T}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. For any graph $G$ with $G \Rightarrow \vec{T}$ we get $G \times K_{|G|+1} \Rightarrow \vec{T}'$.

**Proof.** Let $G' = G \times K_{|G|+1}$. To prove that $G' \Rightarrow \vec{T}'$, take any oriented tree $\vec{T}' \in \vec{T}'$ and any orientation $\vec{G}'$ of the graph $G'$. By our assumption, for the tree $\vec{T}'$ there exists an oriented subtree $\vec{T} \in \vec{T}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. Let $t'$ be the unique element of the set $V_{\vec{T}'} \setminus V_{\vec{T}}$ and $t \in V_{\vec{T}}$ be the unique vertex of $\vec{T}$ such that $(t', t)$ or $(t, t')$ is an edge of $\vec{T}$.

For every vertex $u$ of the complete graph $K_{|G|+1}$, consider the subgraph $G'_u = G' \times \{u\}$ of $G'$ and its orientation $\vec{G}'_u$, inherited from the orientation $\vec{G}'$. Since $G \Rightarrow \vec{T}$, there is an isometric embedding $f_u : \vec{T} \rightarrow \vec{G}'$. By the Pigeonhole Principle, there are two distinct vertices $u, w$ in $K_{|G|+1}$ such that $f_u(t) = (g, u)$ and $f_w(t) = (g, w)$ for some vertex $g$ of the graph $G$. Now look at the orientation of the edges $\{t, t'\}$ and $\{(g, u), (g, w)\}$ in the digraphs $\vec{T}_T$ and $\vec{G}'$.

If either $(t, t') \in E_T$, and $((g, u), (g, w)) \in E_{\vec{G}'}$, or $(t', t) \in E_T$, and $((g, w), (g, u)) \in E_{\vec{G}'}$, then we define a map $f : \vec{T} \rightarrow G'$ by $f(t') = (g, w)$ and $f(\vec{T}) = f_u$ and observe that $f$ is an isometric embedding of $\vec{T}$ into $\vec{G}'$.

If either $(t, t') \in E_T$, and $((g, u), (g, w)) \in E_{\vec{G}'}$, or $(t', t) \in E_T$, and $((g, w), (g, u)) \in E_{\vec{G}'}$, then we define a map $f : \vec{T} \rightarrow G'$ by $f(t') = (g, u)$ and $f(\vec{T}) = f_w$ and observe that $f$ is an isometric embedding of $\vec{T}$ into $\vec{G}'$.

**Corollary 3.2.** If for some $n \in \mathbb{N}$ a graph $G$ satisfies the isometric Ramsey relation $G \Rightarrow \vec{T}_n$, then $G \times K_{|G|+1} \Rightarrow \vec{T}_n+1$.

**Theorem 3.3.** For every $n \in \mathbb{N}$ $\mathbb{IR}(\vec{T}_{n+1}) \leq \mathbb{IR}(\vec{T}_n)(\mathbb{IR}(\vec{T}_n) + 1)$ and $\mathbb{IR}(\vec{T}_n) < 2^{2n-1}$.

**Proof.** The inequality $\mathbb{IR}(\vec{T}_{n+1}) \leq \mathbb{IR}(\vec{T}_n)(\mathbb{IR}(\vec{T}_n) + 1)$ follows from Corollary 3.2. Indeed, for every $n \in \omega$ we can choose a graph $G$ with $|G| = \mathbb{IR}(\vec{T}_n)$ vertices and $G \Rightarrow \vec{T}_n$. By Corollary 3.2 the graph $G' = G \times K_{|G|+1}$ satisfies the relation $G' \Rightarrow \vec{T}_n+1$ and hence

$$\mathbb{IR}(\vec{T}_{n+1}) \leq |G'| = |G|(|G| + 1) = \mathbb{IR}(\vec{T}_n)(\mathbb{IR}(\vec{T}_n) + 1).$$

It remains to prove that $\mathbb{IR}(\vec{T}_n) + 1 \leq 2^{2n-1}$ for $n \in \mathbb{N}$. For $n = 1$ we get the equality $\mathbb{IR}(\vec{T}_1) + 1 = 1 + 1 = 2^0$. Assume that for some $n \in \mathbb{N}$ we have proved that $\mathbb{IR}(\vec{T}_n) + 1 \leq 2^{2n-1}$. Then

$$\mathbb{IR}(\vec{T}_{n+1}) + 1 \leq \mathbb{IR}(\vec{T}_n)(\mathbb{IR}(\vec{T}_n) + 1) + 1 \leq (2^{2n-1} - 1)2^{2n-1} + 1 = 2^{2n} - 2^{2n-1} + 1 \leq 2^{2n}.$$
(3) There exists a constant $C$ such that for any numbers $k, g \geq 3$ and $m = \text{Erdős}(k, g)$ we have the inequality \[ s \geq m \cdot \ln m \leq C k, \] which implies that $\text{Erdős}(k, g) = o(k^{g-2})$ as $\max\{k, g\} \to \infty$. \hfill $\square$

Write $G \to \vec{H}$ if for every orientation $\vec{G}$ of $G$ and every $\vec{H} \in \vec{H}$ there is an injective map $f : V_{\vec{G}} \to V_{\vec{H}}$ such that for every directed edge $(x, y)$ of $H$ the pair $(f(x), f(y))$ is a directed edge of $\vec{G}$. (Note that we do not require that $f$ induces isomorphism, that is, $G$ can have extra edges inside the set $f(V_{\vec{H}})$.) Another function related to $\mathcal{IR}(\vec{H})$ is Burr’s function $\text{Burr}(\vec{H})$ assigning to every family $\vec{H}$ of oriented trees the smallest number $k$ such that $G \to \vec{H}$ for every graph $G$ with chromatic number $\chi(G) \geq k$. If such number $k$ does not exist, then we put $\text{Burr}(\vec{H}) = \infty$. By the Gallai-Hasse-Roy-Vitaver Theorem $\cite[Theorem 3.13]{13}$, the chromatic number $\chi(G)$ of a finite graph $G$ is equal to $\max\{n \in \mathbb{N} : G \to \vec{I}_n\}$. This equality implies that $\text{Burr}(\vec{I}_n) = n$ for every $n \in \mathbb{N}$. In $\cite{10}$ Burr considered the numbers $\text{Burr}(\vec{T}_n)$ and proved that $\text{Burr}(\vec{T}_n) \leq (n - 1)^2$. This upper bound was improved to the upper bound $\text{Burr}(\vec{T}_n) \leq \frac{1}{2} n^2 - \frac{1}{2} n + 1$ in $\cite{2}$. According to (still unproved) Conjecture of Burr $\cite{4}$, the equality $\text{Burr}(\vec{T}_n) = 2n - 2$ holds for all $n \geq 2$.

**Proposition 3.5.** For any $n \in \mathbb{N}$ and a subclass $\vec{H} \subset \vec{T}_n$ we get the upper bound $\mathcal{IR}(\vec{H}) \leq \text{Erdős}(\text{Burr}(\vec{H}), 2n - 2)$.

**Proof.** Fix a graph $G$ of cardinality $|G| = \text{Erdős}(\text{Burr}(\vec{H}), 2n - 2)$ with chromatic number $\chi(G) \geq \text{Burr}(\vec{H})$ and girth $g(G) \geq 2n - 2$. Let us prove that $G \to \vec{H}$. Take any orientation $\vec{G}$ of $G$ and $\vec{H} \in \vec{H}$. Since $G \to \vec{H}$, there is an orientation-preserving injection $f : \vec{H} \to \vec{G}$. Since $\vec{H}$ is a connected graph with at most $n$ vertices and $g(G) \geq 2n - 2$, the map $f$ is an isometric embedding. So, $G \Rightarrow H$. \hfill $\square$

Combining Proposition 3.5 with known upper bounds $\text{Burr}(\vec{I}_n) = n$ and $\text{Burr}(\vec{T}_n) \leq \frac{1}{2} n^2 - \frac{1}{2} n + 1$ we get the following upper bounds for the isometric Ramsey numbers $\mathcal{IR}(\vec{I}_n)$ and $\mathcal{IR}(\vec{T}_n)$.

**Corollary 3.6.** For every $n \in \mathbb{N}$ we get the upper bounds $\mathcal{IR}(\vec{I}_n) \leq \text{Erdős}(n, 2n - 2) = o(n^{2n-4}) = o(n^{2n})$ and $\mathcal{IR}(\vec{T}_n) \leq \text{Erdős}(\frac{1}{2} n^2 - \frac{1}{2} n + 1, 2n - 2) = o((\frac{1}{2} n^2 - \frac{1}{2} n + 1)^{2n-4}) = o(n^{4n})$.

In Theorem 3.6 we shall improve the upper bound $o(n^{4n})$ for $\mathcal{IR}(\vec{T}_n)$ to the upper bound $n^{2n + o(n)}$.

**Remark 3.7.** By Theorem 3 in $\cite{10}$, $\mathcal{R}(\vec{I}_n) \geq n^2/2$ for all $n \in \mathbb{N}$. This yields the lower bound $\frac{1}{2} n^2 \leq R(\vec{I}_n) \leq \mathcal{IR}(\vec{I}_n) \leq \mathcal{IR}(\vec{T}_n)$ for the isometric Ramsey numbers $\mathcal{IR}(\vec{I}_n)$ and $\mathcal{IR}(\vec{T}_n)$.

**Remark 3.8.** It can be shown that $\mathcal{IR}(\vec{I}_1) = \mathcal{IR}(\vec{T}_1) = 1 = |K_1|$, $\mathcal{IR}(\vec{I}_2) = \mathcal{IR}(\vec{T}_2) = 2 = |K_2|$, $\mathcal{IR}(\vec{I}_3) = 5 = |C_5|$, $\mathcal{IR}(\vec{T}_3) = 6 = |K_2 \times K_3|$, $\mathcal{IR}(\vec{I}_4) \leq 30 = |C_5 \times K_6|$, $\mathcal{IR}(\vec{T}_4) \leq 42 = |K_2 \times K_3 \times K_7|$.

**Question 3.9.** What is the exact value of the isometric Ramsey numbers $\mathcal{IR}(\vec{I}_4)$ and $\mathcal{IR}(\vec{T}_4)$? Are they distinct?

## 4. Isometric copies of directed trees in orientations of random graphs

In this section we shall apply the technique of random graphs and shall improve the upper bound $\mathcal{IR}(\vec{T}_n) = o(n^{4n})$ established in Corollary 3.6 to the upper bound $\mathcal{IR}(\vec{T}_n) \leq (4e + o(1))(n^2 \ln n)^n = n^{2n + o(n)}$.

First we prove some technical lemmas. The first of them uses the idea of the proof of Theorem 1 in $\cite{3}$.

**Lemma 4.1.** A graph $G = (V_G, E_G)$ satisfies $G \Rightarrow \vec{T}_n$ for some $n \in \mathbb{N}$ if there exist sequences $(w_k)_{k=1}^{n-1}$ and $(d_k)_{k=1}^{n-1}$ of positive real numbers such that for every $2 \leq k < n$ the following conditions hold:

1. For every set $S = \{s_1, \ldots, s_{k-1}\} \subset V_G$ of cardinality $k - 1$ and every $v \in V_G \setminus S$, we have that $|Y| \leq d_k$, where $Y$ consists of $y \in V_G \setminus (S \cup \{v\})$ such that $(y, v) \in E_G$ and $\text{dist}_{G \setminus v}(y, s_i) \leq i$ for some $1 \leq i < k$. 

We will denote the number of non-isomorphic oriented trees with $n$ arcs by $\text{OCT}_n$. In $\cite{3}$ it was proved that $\mathcal{OCT}_n \leq 2^n n^{3n}$. This implies that $\text{Burr}(\vec{T}_n) \leq (2n)^n$. In $\cite{10}$ it was shown that $\text{Burr}(\vec{T}_n) \leq (n - 1)^2$.
Lemma 4.3

Proof. For a subset $U \subset V_G$ by $G[U]$ we denote the induced subgraph $G[U] = (U, E[U])$ of $G$, where $E[U] = \{(u, v) \in E_G : u, v \in U\}$. Also, let us write $(G, U) \Rightarrow \vec{T}_k$, meaning that, for every $\vec{T} \in \vec{T}_k$, every orientation $\vec{G}$ of $G$ contains a copy of $\vec{T}$ which lies inside $U$ and is an isometric subgraph of $G$.

We shall inductively prove that for every $1 \leq k \leq n$ and every set $U \subset V_G$ of size $|U| > \sum_{i=1}^{k-1} w_i$, we have $(G, U) \Rightarrow \vec{T}_k$. The base case $k = 1$ is trivial. Suppose that this holds for some $k$. Take any $U \subset V_G$ with $|U| > \sum_{i=1}^{k-1} w_i$. Take any orientation $\vec{G}(G[U])$ of $E(G[U])$ and any directed tree $\vec{T} \in \vec{T}_{k+1}$. Let $u$ be a pendant vertex of $\vec{T}$. By symmetry, assume that $(v, u)$ is an arc in $\vec{T}$, that is, the arc in $\vec{T}$ goes from the unique neighbor $v$ of $u$ to $u$.

Let $W$ be the set of vertices in $U$ whose out-degree in $G[U]$ is at most $d_k + k - 1$. We claim that $|W| \leq w_k$. Suppose not. Then $|W| > w_k$ and Item 2 guarantees that $W$ spans more than $(d_k + k - 1)w_k$ edges in $G$, each edge contributing to out-degree of some vertex in $W$. Thus $(d_k + k - 1)|W| \geq (d_k + k - 1)w_k$, which is a desired contradiction showing that $|W| \leq w_k$.

Thus $U' = U \setminus W$ has size $|U'| = |U| - |W| > \left(\sum_{i=1}^{k-1} w_i\right) - w_k = \sum_{i=1}^{k-1} w_i$. By inductive assumption, $U'$ has a $G$-isometric copy $\vec{T}'$ of the oriented tree $\vec{T} - u$. Let $\{s_1, \ldots, s_{k-1}\}$ be an enumeration of the set $S := V_{\vec{T}'} \setminus \{v\} \subset U'$ such that $\text{dist}(s_i, v) \leq i$ for every $i < k$. Let $Y$ be defined as in Item 1 with respect to $v$ and $\{s_1, \ldots, s_{k-1}\}$. By Item 1, $|Y| \leq d_k$. On the other hand, the neighbor $v \in V_{\vec{T}'} \subset U \setminus W$ of $u$ must have out-degree in $U \setminus S$ greater than $d_k + k - 1 - |S| = d_k$. Thus there is an out-neighbor of $v$ which is in $U \setminus (W \cup Y)$. Let $u$ be mapped to this vertex. Then $(v, u) \in \vec{G}(G[U])$ is oriented from $v$ to $u$, as desired. Since $d_{G - v}(u, s_i) > i$ for each $i < k$, the addition of $u$ cannot violate the $G$-isometry property (since all vertices of $\vec{T} - u$ are embedded into $S \cup \{v\}$). This gives the required embedding of $\vec{T}'$ and finishes the proof.

Our next elementary lemma yields an upper bound on the sum of a geometric progression.

Lemma 4.2. For positive real numbers $a, c$ with $a > 1 + \frac{1}{c}$ we get $\frac{a^n - 1}{a - 1} < (1 + c)a^{n-1}$ for every $n \in \mathbb{N}$.

Proof. The inequality is equivalent to $a^n - 1 < (1 + c)a^{n-1}(a - 1) = a^n - a^{n-1} + ca^{n-1}(a - 1)$ and to $a^{n-1} - 1 < ca^{n-1}(a - 1)$. The latter inequality follows from $a^{n-1} < ca^{n-1}(a - 1)$, which is equivalent to $1 < c(a - 1)$.

In the proof of Lemma 4.4 we shall use the following Chernoff-type bounds; for a proof see e.g. [1] §A.1.

Lemma 4.3 (Chernoff bounds). Let $X_1, \ldots, X_n$ be independent random variables taking values in $\{0, 1\}$ and let $EX$ be the expected value of their sum $X = \sum_{i=1}^{n} X_i$. Then

$$\mathbb{P}\{X \geq C \cdot EX\} \leq \left(\frac{eC}{2C - 1}\right)^{EX} \quad \mathbb{P}\{X \geq (1 + c)EX\} \leq e^{-\frac{c^2}{2}EX} \quad \mathbb{P}\{X \leq (1 - c)EX\} \leq e^{-\frac{c^2}{2}EX}$$

for every $C > 1$ and $0 < c < 1$.

Lemma 4.4. For positive integers $n, N$ the inequality $|\mathbb{R}(\vec{T}_n)| \leq N$ holds if there exist real numbers $c, p \in (0, 1)$, $C \in (1, \infty)$ satisfying the following inequalities:

1. $c^2 p N > 3 \ln(3N)$;
2. $(1 - C + C \ln C)p(1 + c)^n(pN)^{n-2} > (n - 1) \ln N + \ln(1 + c) + \ln(3)$;
3. $c^2 C S(1 + c)^2 n(pN)^{2n-4} > N \ln 2 + \ln(3n)$;
4. $(n - 1)(n - 2) + \frac{2C}{(1-c)p} + 2C C^2 (n - 1)(1 + c)^n(pN)^{n-2} < N$.

Proof. Assume that the numbers $n, N, p, c, C$ satisfy the assumptions of the lemma. Let $G = G(N, p)$ be a random graph on $N$ vertices in which an edge $\{u, v\} \subset V_G$ appears with probability $p$. We shall prove that with non-zero probability the random graph $G$ has $G \Rightarrow \vec{T}_n$.

Let $h := (1 + c)^n(pN)^{n-2}$.

For every positive integer $k < n$ let

$$d_k = Cp\ h \quad \text{and} \quad w_k = \frac{2(d_k + k - 1)}{(1 - c)p}.$$
Chernoff bound implies that any fixed vertex of $G$ has degree $\geq (1+c)p(N-1)$ with probability $< e^{-\frac{1}{2}p(N-1)}$. Consequently, with probability $P_1 > 1 - Ne^{-\frac{1}{2}p(N-1)}$ all vertices of $G$ have degree $< (1+c)pN$. The condition (1) implies that $-\frac{1}{2}p(N-1) < -\ln(3N)$ and hence

$$P_1 > 1 - Ne^{-\frac{1}{2}p(N-1)} > 1 - Ne^{-\ln(3N)} = \frac{2}{3}.$$  

For every $k < n$, take any pairwise distinct points $v, s_1, \ldots, s_{k-1} \in V_G$. If the maximum degree of $G$ is at most $(1+c)pN$, then for every $i < k$ the ball $B(s_i, i) = \{x \in V_G : \text{dist}_G(x, s_i) \leq i\}$ has cardinality

$$|B(s_i, i)| \leq \sum_{j=0}^{i} (1+c)pN^j = \frac{(1+c)pN^{i+1} - 1}{(1+c)pN - 1} < (1+c)((1+c)pN)^i.$$  

The latter strict inequality can be derived from Lemma 4.2 and the inequality $cpN \geq c^2pN > 3\ln(3N) \geq 3$.

By above, the set $X$ of vertices of $G - v$ at distance at most $i < k$ in $G - v$ from some $s_i$ has size at most $(1+c) \sum_{i=1}^{k-1} ((1+c)pN)^i = (1+c)(1+c)pN^{k-1} < (1+c)^{k+1}(pN)^{k-1} \leq h$. Consider the set $\bar{Y}$ of neighbors of $v$ that fall into the set $X$. The definition of $X$ does not depend on the edges incident to $v$, so conditioned on $X$ (of size at most $h$) the size of $Y$ is dominated by $Y' \sim Bin(h, p)$. Chernoff bound shows that the probability that $Y'$ is at least $Cp\ln = C\bar{E}Y'$ is at most $(\frac{e^{-1}}{e})^h$. Since the number of possible choices of $v, s_1, \ldots, s_{k-1}$ is equal to $\binom{N}{(N-k)!} \leq N^k$, with probability

$$P_2 \geq 1 - \sum_{k=1}^{N-1} N^k(\frac{e^{-1}}{e})^h = 1 - \left(\frac{e^{-1}}{e}\right)^h \frac{N^{n-1} - 1}{N - 1} > 1 - (1+c)N^{n-1}(\frac{e^{-1}}{e})^h$$  

the condition (1) of Lemma 4.1 is satisfied or we have a vertex of degree $\geq (1+c)pN$. We claim that $P_2 > \frac{2}{3}$. It suffices to prove that

$$(1+c) + (n-1)\ln N + ph(C - 1 - C \ln C) < -\ln(3).$$

But this follows from condition (2).

Next, we prove that with probability $> \frac{2}{3}$ the condition (2) of Lemma 4.1 holds. Take any positive $k < n$ and put $\bar{w}_k = \min\{m \in \mathbb{N} : w_k < m\}$. For any fixed set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$, the number of edges it spans is $Bin(\binom{\bar{w}_k}{2}, p)$. By Chernoff bound, the probability that it is less than $(1-c)p(\bar{w}_k^2)$ is less that $e^{-\frac{1}{2}c^2p(\bar{w}_k^2)}$. The probability $P_{3, k}$ that some set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$ spans less than $(1-c)p(\bar{w}_k^2)$ edges is $P_{3, k} < (\frac{N}{\bar{w}_k})e^{-\frac{1}{2}c^2p(\bar{w}_k^2)} < 2Ne^{-\frac{1}{2}c^2p\bar{w}_k(\bar{w}_k+1)}$. We claim that $P_{3, k} < \frac{1}{3m}$ which will follow as soon as we show that $N\ln 2 - \frac{1}{2}c^2p\bar{w}_k(\bar{w}_k+1) < -\ln(3n)$. For this it suffices to check that $\frac{1}{2}c^2p\bar{w}_k(\bar{w}_k+1) > N\ln 2 + \ln(3n)$.

This follows from the chain of the inequalities

$$\frac{1}{2}c^2\bar{w}_k(\bar{w}_k+1) > \frac{1}{2}c^2w_k^2 \geq c^2C^2R^2 = c^2C^2(1+c)^{2n}pN^{2n-4} > N\ln 2 + \ln(3n),$$

the last inequality postulated in (3). Therefore, $P_{3, k} < \frac{1}{3m}$ and the probability $P_3$ that for every $k < n$ every set $W \subset V[G]$ of cardinality $|W| > w_k$ spans at least

$$(1-c)p\left(\frac{\bar{w}_k}{2}\right) > (1-c)p(w_k + 1)/2 = (d_k + k - 1)(w_k + 1) > (d_k + k - 1)w_k$$

edges is $> 1 - \sum_{k=1}^{n-1} P_{3, k} > 1 - \frac{n-1}{3m} > \frac{2}{3}$. So, with probability $> \frac{2}{3}$ the condition (2) of Lemma 4.1 holds.

Since $(1 - P_1) + (1 - P_2) + (1 - P_3) < 1$, there is a non-zero probability that the random graph $G = G(N, p)$ satisfies the conditions (1) and (2) of Lemma 4.1.

It remains to show that the condition (3) of Lemma 4.1 holds, too. For this observe that

$$\sum_{k=1}^{n-1} w_k = \sum_{k=1}^{n-1} \frac{2(Cph + k - 1)}{(1-c)p} = \frac{2}{(1-c)p} \sum_{k=1}^{n-1} (k - 1) + \frac{2C}{1-c} (n-1)h =$$

$$= \frac{(n-1)(n-2)}{(1-c)p} + \frac{2C}{1-c} (n-1)(1+c)pN^{n-2} < N.$$  

The last inequality follows from the condition (4) of the Lemma.

Now it is legal to apply Lemma 4.1 and conclude that $G \Rightarrow \mathcal{T}_n$ and hence $\mathcal{IR}(\mathcal{T}_n) \leq |G| = N$.  

Now we are able to prove the promised upper bound $\mathcal{IR}(\mathcal{T}_n) \leq (4e + o(1))(n^2 \ln n)^n = n^{2n + o(n)}$.  

\[\square\]
Theorem 4.5. For every \( \varepsilon \in (0, 1) \) there is \( n_\varepsilon \in \mathbb{N} \) such that \( \Re(\tilde{T}_n) \leq (4e(1+\varepsilon)n^2\ln n)^n \) for all \( n \geq n_\varepsilon \).

Proof. Choose any positive \( \delta, c \in (0, 1) \) such that
\[
(1+\delta)(1+c) < 1 + \varepsilon \quad \text{and} \quad 4(1+\delta) \frac{1-c}{2+c} > 2 + \delta.
\]
For every \( n \in \mathbb{N} \) let \( N \) be the smallest integer number, which is greater than
\[
\frac{(2+c)c^n}{1-c}(n-1)(1+c)^n(4(1+\delta)n^2\ln n)^{n-2}
\]
and let
\[
p := \frac{4(1+\delta)n^2\ln n}{N}.
\]
So, \( N > \frac{(2+c)c^n}{1-c}(n-1)(1+c)^n(pN)^{n-2} \geq N - 1 \). It is easy to see that
\[
N = o\left((4e(1+\varepsilon)n^2\ln n)^n\right)
\]
and for \( C = e^n \) the conditions (1),(3),(4) of Lemma 4.4 hold for all sufficiently large \( n \). To verify the condition (2), observe that
\[
1 - C + C \ln C)p(1+c)^n(pN)^{n-2} \geq (1 - e^n + e^n \ln e^n)p\left(\frac{(N-1)(1-c)}{(2+c)e^n(n-1)}\right) =
\]
\[
\frac{1 + e^n(n-1) - 1 - c - N - 1}{e^n(n-1) - 1} pN = \left(1 + \frac{1}{e^n(n-1)}\right)\frac{N - 1 - c}{N} 4(1+\delta)n^2\ln n >
\]
\[
> \left(1 + \frac{1}{e^n(n-1)}\right)\frac{N - 1}{N}(2 + \delta) n^2\ln n = (2 + \delta + o(1)) n^2\ln n.
\]
On the other hand, \( (n-1)\ln N + \ln(1+c) + \ln 3 = (2 + o(1)) n^2\ln n \). So, the condition (2) holds for large \( n \). Applying Lemma 4.4, we conclude that
\[
\Re(\tilde{T}_n) \leq N \leq (4e(1+\varepsilon)n^2\ln n)^n
\]
for all sufficiently large \( n \).

By Corollary 4.6 and Theorem 4.5 \( \Re(\tilde{T}_n) = o(n^{2n}) \) and \( \Re(\tilde{T}_n) \leq n^{2n+o(n)} \).

Question 4.6. What is the growth rate of the sequence \( \Re(\tilde{T}_n) \)? Is \( \Re(\tilde{T}_n) = n^{o(n)} \)?

The technique developed for the proof of Theorem 4.5 allows us to improve the upper bound
\[
\Re(\tilde{T}_n) \leq (4(5e)^2 + o(1)) n^4\ln n,
\]
obtained by Kohayakawa, Łuczak and Rödl in (the proof of) Theorem 1 of [9], and replace the constant
\( 4(5e)^2 = 2500e^8 \approx 7452395.96 \) by a much smaller constant \( K \approx 98.82 \ldots \).

Theorem 4.7. Let \( K := \min_{x>1} \frac{16x^2}{1 - x + x\ln x} \approx 98.8249 \ldots \) For any positive \( \varepsilon > 0 \) there exists \( n_\varepsilon \in \mathbb{N} \) such that \( \Re(\tilde{T}_n) < (K + \varepsilon)n^4\ln n \) for all \( n \geq n_\varepsilon \). Consequently, \( \Re(\tilde{T}_n) < 99n^4\ln n \) for all sufficiently large \( n \).

Proof. We indicate which changes should be made in the proof of Theorem 4.5 to obtain Theorem 4.7.

In the condition (1) of Lemma 4.4, the inequality \( d_{G-\varepsilon}(y, s_i) \leq i \) should be replaced by \( d_{G-\varepsilon}(y, s_i) \leq 1 \).

In the proof of Lemma 4.4, the constant \( h \) should be redefined as \( h := (1+c)(n-2)pN \) and the conditions (1–4) of Lemma 4.4 should be changed to the conditions:

1. \( c^2pN > 3\ln(3N) \);
2. \( (1 - C + C\ln C)(1+c)(n-2)p^2N > (n-1)\ln N + \ln(1+c) + \ln(3) \);
3. \( (cC(1+c)(n-2)pN)^2 > N \ln 2 + \ln(3n) \);
4. \( \frac{n(n-1)}{1-c}\varepsilon + \frac{2C(1+c)}{(1-c)p}(n-1)(n-2)pN < N \).

Now we are able to prove Theorem 4.7. Let \( C \approx 4.92155 \ldots \) be the unique real number in \((1, \infty)\) such that
\[
\frac{16C^2}{1 - C + C\ln C} = K := \min_{x>1} \frac{16x^2}{1 - x + x\ln x} \approx 98.8249 \ldots \)

\(^{1}\)The approximate values of \( C \) and \( K \) were found by the online WolframAlpha computational knowledge engine at [www.wolframalpha.com](http://www.wolframalpha.com)
Given any \( \varepsilon > 0 \), choose real numbers \( \delta, c \in (0, 1) \) such that \( K\delta < \varepsilon \) and
\[
4(1 + \delta)\frac{(1-c)^2}{(1+c)^3} > 4 + \delta.
\]

For every \( n \in \mathbb{N} \) let \( p := \frac{1 - \varepsilon}{2c(1-c)n^2} \) and let \( N \) be the smallest integer, which is greater than \( K(1 + \delta)n^4 \ln n \).

It is easy to see that \( N = o((K + \varepsilon)n^4 \ln n) \) and the conditions (1'), (3') and (4') are satisfied for all sufficiently large \( n \). To see that (2') holds, observe that
\[
(1 - C + C \ln C)(1 + c)(n - 2)p^2N \geq \frac{(1 - C + C \ln C)(1 + c)(1 - c)^2}{(2C(1+c)^2n^2)^2}(n - 2)K(1 + \delta)n^4 \ln n = \frac{1 - C + C \ln C}{C^2}(1 - c)^2(1 + \delta)n^4 \ln n = (1 + \delta)K\frac{16}{4(1+c)^3}(1 - c)^2(1 + \delta)n^4 \ln n > (4 + \delta)(n - 2) \ln n = (4 + \delta + o(1))n \ln n.
\]

On the other hand,
\[
(n - 1) \ln N + \ln(1 + c) + \ln 3 \leq (n - 1) \ln(1 + K(1 + \delta)n^4 \ln n) + \ln(1 + c) + \ln 3 = (4 + o(1))n \ln n,
\]
so for large \( n \) the condition (2') is satisfied, too.

Applying the modified version of Lemma 4.4 we get
\[
\mathcal{R}(\overline{T}_n) \leq N \leq (K + \varepsilon)n^4 \ln n
\]
for all sufficiently large numbers \( n \). \( \square \)

5. Long directed paths in orientations of a graph

By the Gallai-Hasse-Roy-Vitaver Theorem [13, Theorem 3.13], each finite graph \( G \) has chromatic number
\[
\chi(G) = \max\{n \in \mathbb{N} : G \twoheadrightarrow \overline{I}_n\},
\]
where the symbol \( G \twoheadrightarrow \overline{I}_n \) means that each orientation of \( G \) contains a simple directed path of length \( n \). Having in mind this characterization, for every graph \( G \) consider the numbers
\[
\bar{\chi}_I(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \overline{I}_n\}, \quad \bar{\chi}_T(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \overline{T}_n\},
\]
and observe that \( \bar{\chi}_I(G) \leq \bar{\chi}_I(G) \leq \chi(G) \) and
\[
\bar{\chi}_T(G) \leq \bar{\chi}_I(G) \leq \chi(G) \text{ and}
\]
\[
\bar{\chi}_I(G) \leq \sup\{\text{diam}(G') + 1 : G' \text{ is a connected component of } G\}.
\]

Observe that \( \mathcal{R}(\overline{I}_n) \) (resp. \( \mathcal{R}(\overline{T}_n) \)) is equal to the smallest cardinality \( |G| \) of a graph \( G \) with \( \bar{\chi}_I(G) \geq n \) (resp. \( \bar{\chi}_T(G) \geq n \)). So, the characteristics \( \bar{\chi}_I \) and \( \bar{\chi}_T \) determine the isometric Ramsey numbers \( \mathcal{R}(\overline{I}_n) \) and \( \mathcal{R}(\overline{T}_n) \).

We shall show that a graph \( G \) has \( \bar{\chi}_I(G) \leq 2 \) if and only if \( G \) is a comparability graph. We recall that a graph \( G \) is called a comparability graph if \( G \) admits a transitive orientation \( \bar{G} \) (that is, for any directed edges \( (x, y) \) and \( (y, z) \) of \( \bar{G} \) the pair \( (x, z) \) is a directed edge of \( \bar{G} \)); equivalently, the set \( V_G \) of vertices of \( G \) admits a partial order such that a pair \( \{u, v\} \) of distinct vertices of \( G \) is an edge of \( G \) if and only if \( u \) and \( v \) are comparable in the partial order. By the results of Ghouila-Houri and of Gilmore and Hoffman (see [4, Theorem 6.1.1]), comparability graphs can be characterized as graphs \( G \) whose every cycle of odd length has a triangular chord (more precisely, for every \( (2n + 3) \)-cycle on \( v_0, \ldots, v_{2n+2} \) with \( n \geq 1 \), there is a residue \( i \) modulo \( 2n + 3 \) such that \( \{v_i, v_{i+2}\} \in E_G \)). More information on comparability graphs can be found in Chapter 6 of the survey [4].

Proposition 5.1. A graph \( G \) has \( \bar{\chi}_I(G) \leq 2 \) if and only if \( G \) is a comparability graph.

Proof. If \( G \) is comparability graph, then \( G \) has a transitive orientation \( \bar{G} \). It follows that for any directed path \( (v_0, v_1, v_2) \) in \( \bar{G} \) the pair \( (v_0, v_2) \) is an edge of \( \bar{G} \) and hence \( d_{\bar{G}}(v_0, v_2) \leq 1 \). This means that \( G \not\twoheadrightarrow \overline{I}_3 \) and hence \( \bar{\chi}_I(G) \leq 2 \).

If \( G \) is not a comparability graph, then \( G \) contains an odd cycle without a triangular chord. It is easy to see that any orientation \( \bar{G} \) of the cycle \( C \) contains a directed path \( (v_0, v_1, v_2) \). Since \( C \) has no triangular chords, \( d_{\bar{G}}(v_0, v_2) = 2 \), which means that \( \{v_0, v_1, v_2\} \) is an isometric copy of \( \overline{I}_3 \) in \( \bar{G} \) and in \( G \). Therefore, \( \bar{\chi}_I(G) \geq 3 \). \( \square \)

Problem 5.2. Characterize graphs \( G \) with \( \bar{\chi}_I(G) \leq 3 \) (\( \bar{\chi}_I(G) \leq n \) for \( n \geq 4 \)).

Problem 5.3. Characterize graphs \( G \) with \( \bar{\chi}_T(G) \leq 2 \) (\( \bar{\chi}_T(G) \leq n \) for \( n \geq 3 \)).
Remark 5.4. Any cycle $C$ of odd length $n \geq 5$ satisfies $\bar{\chi}_I(C) = 3$ and $\bar{\chi}_T(C) = 2$.

Now we prove a weak 3-space property for the number $\bar{\chi}_I(G)$. By a weak homomorphism $f : G \to H$ of graphs $G, H$ we understand a function $f : V_G \to V_H$ such that for every edge $\{u, v\}$ of $G$ we have either $f(u) = f(v)$ or $\{f(u), f(v)\}$ is an edge of $H$. For a weak homomorphism $f : G \to H$ and vertex $y$ of $H$ the preimage $f^{-1}(y)$ is a graph with the set of edges $\{\{u, v\} \in E_G : f(u) = y = f(v)\}$.

**Proposition 5.5.** If $f : G \to H$ is a weak homomorphism of finite graphs, then

$$\bar{\chi}_I(G) \leq \max \left\{ \sum_{y \in F} \bar{\chi}_I(f^{-1}(y)) : F \subseteq V_H, |F| \leq \chi(H) \right\}.$$

Proof. By definition of the chromatic number $\chi(H)$, there exists a coloring $c : V_H \to \{1, \ldots, \chi(H)\}$ of the graph $H$ such that for every edge $\{u, v\}$ of $G$ the colors $c(u)$ and $c(v)$ are distinct. For every $y \in H$ choose an orientation $\tilde{G}_y$ of the graph $G_y = f^{-1}(y)$ such that $\tilde{G}_y \not\cong \tilde{I}_k$ for $k = \tilde{\chi}_I(G_y) + 1$. Let $\tilde{G}$ be the orientation of the graph $G$ such that for an edge $\{u, v\}$ of $G$ the ordered pair $(u, v)$ is an edge of $\tilde{G}$ if and only if either $c(f(u)) < c(f(v))$ or $(u, v)$ is an edge of $\tilde{G}_y$ for some $y \in H$.

We claim that the digraph $\tilde{G}$ contains no isometric copy of the graph $\tilde{I}_{m+1}$, where

$$m = \max \left\{ \sum_{y \in F} \bar{\chi}_I(G_y) : F \subseteq V_H, |F| \leq \chi(H) \right\}.$$

Suppose on the contrary that $\tilde{G}$ contains a directed path $(v_0, \ldots, v_m)$ such that $d_G(v_0, v_m) = m$. It follows that $(c(f(v_0)), \ldots, c(f(v_m)))$ is a non-decreasing sequence of numbers in the interval $\{1, \ldots, \chi(H)\}$. Consequently, for every number $i$ in the set $C = \{c(f(v_0)), \ldots, c(f(v_m))\}$ the set $J_i = \{j \in \{0, \ldots, n\} : c(f(v_j)) = i\}$ coincides with some subinterval $[a_i, b_i]$ of $\{0, \ldots, n\}$ and the set $\{f(v_j) : j \in [a_i, b_i]\}$ is a singleton $\{y_i\}$ for some vertex $y_i \in H$. It follows that $(v_{a_i}, \ldots, v_{b_i})$ is a directed path isometric to $\tilde{I}_{[a_i, b_i]}$ in the graph $G_{y_i}$ and hence $|[a_i, b_i]| \leq \tilde{\chi}_I(G_{y_i})$. The choice of the orientation $\tilde{G}$ guarantees that the set $F = \{y_i : i \in C\}$ has cardinality $|F| = |C| \leq \chi(H)$. Then

$$m + 1 = |[0, m]| = \sum_{i \in C} |[a_i, b_i]| \leq \sum_{i \in C} \tilde{\chi}_I(G_{y_i}) = \sum_{y \in F} \tilde{\chi}_I(G_y) \leq m,$$

which is a desired contradiction.

\[\Box\]

6. InFinite directed paths in orientations of graphs

Now we discuss the problem of existence of infinite directed paths in orientations of graphs. Consider the infinite digraphs $\tilde{I}_\omega$ and $\tilde{I}_{-\omega}$ with $V_{\tilde{I}_\omega} = \omega = V_{\tilde{I}_{-\omega}}$, $E_{\tilde{I}_\omega} = \{(i, i+1) : i \in \omega\}$, and $E_{\tilde{I}_{-\omega}} = \{(i+1, i) : i \in \omega\}$.

First, observe that Theorem 3.3 implies the following:

**Corollary 6.1.** There exists a countable graph $G$ such that $G \nRightarrow \tilde{I}_n$ for every $n \in \mathbb{N}$.

On the other hand, we shall prove that each graph $G$ admits an orientation containing no isometric copy of the digraphs $\tilde{I}_\omega$ or $\tilde{I}_{-\omega}$ and, more generally, no directed paths of infinite diameter in $G$. (For a subset $A \subseteq V_G$ of a graph $G$ its diameter is defined as $\text{diam}(A) = \sup\{d_G(u, v) : u, v \in A\} \in \omega \cup \{\infty\}$.)

A sequence $(v_n)_{n \in \omega} \in V_G^\omega$ of distinct vertices of a graph $G$ is called an $\omega$-path in $G$ if for every $n \in \omega$ the pair $\{v_n, v_{n+1}\}$ is an edge of $G$. An $\omega$-path $\omega$-path in $G$ is called a directed edge $\tilde{G}$ of $G$ if for every $n \in \omega$ the pair $(v_n, v_{n+1})$ (resp. $(v_{n+1}, v_n)$) is an isometric copy of $\tilde{G}$ in $G$.

The Ramsey Theorem implies that every orientation of the complete countable graph $K_\omega$ contains $\tilde{I}_\omega$ or $\tilde{I}_{-\omega}$. On the other hand, we have the following result:

**Theorem 6.2.** Every graph $G$ has an orientation $\tilde{G}$ containing no directed $\omega$-paths of infinite diameter in $G$. This implies that $G \nRightarrow \tilde{I}_\omega$ and $G \nRightarrow \tilde{I}_{-\omega}$.

Proof. Without loss of generality, the graph $G$ is connected. Fix any vertex $o$ in $G$ and for every vertex $v$ of $G$ let $d_G(o, v)$ be the smallest length of a path linking the vertices $v$ and $o$. Choose an orientation $\tilde{G}$ of $G$ such that for any edge $\{u, v\}$ in $G$ with $d_G(o, u) = d_G(o, v) + 1$ the pair $(u, v)$ is an edge of $\tilde{G}$ if $d_G(o, u)$ is even and $(v, u)$ is an edge of $\tilde{G}$ if $d_G(o, u)$ is odd.
We claim that the orientation $\vec{G}$ contains no directed $\omega$-paths of infinite diameter. To derive a contradiction, assume that $(v_n)_{n \in \omega}$ is a directed $\omega$-path of infinite diameter. Fix any even number $n \in \omega$ such that $\|v_0\| < n$. Since the $\omega$-path $(v_n)_{n \in \omega}$ has infinite diameter, there exists a number $k \in \omega$ such that $\|v_k\| \geq n$. We can assume that $k$ is the smallest number with this property. Taking into account that $\|v_n\| - \|v_{n+1}\| \leq 1$ for all $n \in \omega$, we conclude that $\|v_k\| = n > \|v_0\|$ and $\|v_{k-1}\| = n - 1$, and hence $(v_{k-1}, v_k)$ is an edge of $\vec{G}$. Let also $m$ be the smallest number such that $\|v_m\| \geq n + 1$. For this number we get $\|v_{m+1}\| = n + 1$, $\|v_{m-1}\| = n$ and hence $(v_m, v_{m-1})$ is a directed edge $\vec{G}$. Since both pairs $(v_{k-1}, v_k)$ and $(v_m, v_{m-1})$ are directed edges of the oriented graph $\vec{G}$, the $\omega$-path $(v_n)_{n \in \omega}$ is not directed in $\vec{G}$. Since the graphs $\vec{I}_\omega$ and $\vec{I}_{-\omega}$ have infinite diameters, the digraph $\vec{G}$ does not contain isometric copies of $\vec{I}_\omega$ or $\vec{I}_{-\omega}$.

\textbf{Remark 6.3.} Theorem 6.2 implies that every locally finite graph $G$ admits an orientation containing no directed $\omega$-paths.

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\textbf{References}


T.Banakh: Institute of Mathematics, Jan Kochanowski University in Kielce (Poland) and Ivan Franko University of Lviv (Ukraine)
E-mail address: t.o.banakh@gmail.com

A.Idzik: Institute of Mathematics, Jan Kochanowski University in Kielce (Poland) and Institute of Computer Science, Polish Academy of Sciences, Warsaw (Poland)
E-mail address: adidzik@gmail.com

O.Pikhurko: Mathematics Institute and DIMAP, University of Warwick, Coventry, UK
E-mail address: O.Pikhurko@warwick.ac.uk

I.Protasov: Faculty of Cybernetics, Taras Shevchenko National University in Kyiv, Ukraine
E-mail address: i.v.protasov@gmail.com

K.Pszczola: Instytut Matematyki i Kryptologii, Wojskowa Akademia Techniczna, Warsaw, Poland
E-mail address: pszczola@fr.pl