Multimodality, Uncertainty and Aggregation

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Contents

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Finally, I would like to thank Paul Dunne and my wife Agnieszka without whose help this work would never have been printed.

to Agnieszka

Summary

The prime purpose of this thesis is to devise a method for aggregating beliefs in decision situations involving conflict. In the process of conducting this investigation it has been found that a completely fresh approach to interpreting and modelling uncertainty is required.

The major mathematical tool employed throughout this work is Catastrophe Theory. The relevant aspects of this subject are presented in the first chapter and are repeatedly used in the three main sections of the thesis.

A considerable part of the work is concerned with the new way of eliciting statements about beliefs. A number of illustrations is included in order to provide an intuitive feel for this interpretation of probability. The proposed method gives a basis for an aggregation scheme. Catastrophe Theory provides the framework for constructing aggregation rules sensitive to aspects like conflict, grouping and precision of information. Some particular models are described in detail.

In another section the geometry of a certain type of mixtures is analysed. Mixtures can be used for modelling aggregation problems and their main properties are discussed.

0. Preface

Catastrophe Theory

The early seventies witnessed the emergence of a new mathematical theory. Introduced by Thom (11) Catastrophe Theory quickly established itself as a branch of Singularity Theory and became a recognised part of Pure Mathematics. However, Thom had created the subject with an intention to model various phenomena in natural sciences. He believed he was making a contribution to philosophy. Thom's "disciples" hoped to extend his ideas to other fields such as the social sciences. The general enthusiasm inevitably carried over into the realms of Statistics. The time seemed to be ripe for a wide range of applications. The early work was done, among others, by Zeeman (28,29) and Harrison (29).

After the initial avalanche of models had died down a little, it suddenly became very fashionable, in the mid seventies and thereafter, to criticise and discredit all the work in which Catastrophe Theory was being used. Admittedly, the catastrophists had contributed to their own downfall by an often indiscriminate use of Thom's famous models. Sussmann (27) lists a number of cases where, in his view, Catastrophe Theory models had been applied inappropriately. Zeeman and Harrison do not escape his axe either, despite the fact that Zeeman has been acknowledged as Thom's "first officer". Clearly Sussmann is questioning Zeeman's credentials as a social scientist and not as a mathematician.

Nevertheless, in a brief spell all the early excitement vanished and the number of applications fell considerably. The subject had built up such a bad reputation that most social scientists took it as a point of honour to both criticise it and avoid any connections with it. Nowadays a layman may almost have an impression that Catastrophe Theory has been refuted as a mathematical theory.

No doubt the rise and fall of Catastrophe Theory is not unique in social science. Nevertheless it is slightly unusual that a sound mathematical method had been put to so much misuse and abuse. After all some, more theoretical, applications have been

successful. Smith (24,26) and Cobb (17-21) have managed to "slip through" a number of results without too much hostility. Obviously, once things settle down a bit, a more serious approach should allow Catastrophe Theory to make a significant contribution to mathematical modelling in Social Science.

Probability Theory

Statistics has inherited a burden of interpreting probability measures, one of the oldest tasks of modern philosophy. Measure Theory provides an easy calculus, but fails to answer questions concerning interpretation, updating and aggregation of probability measures. This century a number of new approaches have emerged challenging the most basic concepts of sharpness and additivity. Kolmogorov's axioms are under scrutiny in a way reminiscent of Euclidean axioms of geometry.

Philosophy

Twentieth century philosophy of science has inevitably affected trends in Statistics. Carnap (3) regards the quantitative theories as the ultimate objectives of all sciences. He believes that more and more fundamental concepts can be quantified. This belief is in line with the general tendency to discretise most of the basic concepts such as time, length and mass. Terms like "chronon" and "hodon" are to represent the basic units. In a sense this is not saying anything new: the Greeks have postulated the existence of elementary particles and called them "atoms". Just because Dalton has called a much larger composite by the same name does not mean that the Greeks have been contradicted.

The "digital approach" with irreducible units has infected the approach to measurability of beliefs. Astonishingly, it is also here that the resistance to discrete concepts has risen. Walley and Fine (39) and others have seriously questioned the modern approach to probability theory and have replaced it with a system based on non-additivity and non-sharpness of beliefs. In general, the insistence of precise pictures of reality has been criticised by the development of Fuzzy Subsets (see, for instance, Kauffman (4)).

Thus there appears to be a new trend towards a continuous and smooth reformulation of some scientific concepts. Contrary to the popular belief, Catastrophe Theory also propagates a continuous frame of reference. After all, although Thom's theory appears to be concerned with discontinuous change, it deals with sudden raptures by analysing a continuous underlying structure. Therefore, however discrete, every model is embedded into a continuous framework.

There appear to be two diverse trends in the modern philosophy of science. They clash in many areas and, in particular, in the structure of beliefs dispute.

Outline of the Dissertation

The prime interest of this work centres around the interpretation of the basic concept of probability. The idea to redefine this concept has been instigated by the work on the aggregation problem. The main difficulty within the aggregation dispute seems to be the inherent structure of the Kolmogorov system in which there is a place for exactly one measure. Using this single measure it has proved very difficult to construct a structure where several measures could be credibly combined into an aggregate representation. Encouraged by the recent attempts at reformulation of probability concepts we embarked on erecting a brand new model.

The starting point is Catastrophe Theory. The necessary concepts are outlined in Chapter 1. We make a special effort to introduce the geometry of the Butterfly Catastrophe which is central to most of our later analysis. The Butterfly, and its properties, are less known than the famous Cusp Catastrophe model. Most of the authors bypass the four-dimensional control space of the Butterfly, but we examine it carefully. We believe that the Butterfly will be of a much greater use in modelling conflict in Social Sciences than the Cusp.

What has Catastrophe Theory to do with the model of probability? Once again the inspiration comes from the aggregation problem. While it appeared reasonably natural to use catastrophe models to model conflict associated with amalgamation of different beliefs

we have also found that it may be advantageous to use a similar structure when defining a single measure. After all a unified theory is more appealing.

In Chapter 2 we describe our approach. The fundamental component is an "energy" function defined on a suitable space **W** . It is a smooth potential function and Catastrophe Theory is used to analyse its properties. Sample spaces and events are subsets of **W** determined by this energy function. A probability measure can be defined using the same method. We give an illustration of the method by considering Bernoulli Trials. The important aspect of this formulation is the inherent use of Catastrophe Theory. Alternative events are viewed as competing regimes and dynamics decide the likelihood of their occurrence. For convenience, we adhere to Kolmogorov's axioms, but other methods can be formulated in our language: for instance in order to set up an "upper" and "lower" probability model it is sufficient to superimpose two or more energy functions over the same space **W** .

The aggregation problem is tackled in Chapter 4. We operate within the Decision Theoretic framework and we consider only simple systems where at most three conflicting decisions are in competition. We use the energy approach to construct the Decision Space. Energy functions now become the expected loss functions.

The energy approach is designed to give a more general structure than either Probability Theory or Decision Theory. In fact in Chapter 2 we use terms like "spaces of alternatives" to denote any space with an associated measurable function. Energy functions create a dynamic structure and set up an "energy field" over each space. Attractors of those systems are termed "observables" and Catastrophe Theory is used to analyse their multimodal construction.

We take a small detour in Chapter 3, where we discuss the mixture model introduced by Smith (24). The model is generalised to the case when the scale parameter becomes an extra control factor. We also discuss benefits of using j-components mixtures vis a vis a j-modal Cobb (21) type density.

The model for probability presented here should be treated as an illustration and an experiment. It is quite clear that fresh formulations are possible. What was once viewed as a "natural" representation of beliefs has been shown to be fallible. A parallel can be drawn with the Euclidean geometry: Apparently there is nothing natural about a space of curvature zero, and humans can perceive positive or negative curvatures just as easily.

Notation

Unless otherwise stated R denotes the real line, Z is the set of integers and t is used to label the time axis.

Statements of the form

$$X \sim G(a,b)$$

mean that G is a distribution function of the random variable X and (a,b) is the parameter space.

1. Catastrophe Theory

1.1 Introduction

Throughout this thesis we shall use simple models from Catastrophe Theory. It is therefore appropriate to introduce this subject. We shall content ourselves to a very shallow treatment concentrating on aspects directly relevant to the rest of the work. For a complete description the reader should consult Thom (11), Poston and Stewart (9) or Zeeman (13-16).

Catastrophe Theory is concerned with the study of the qualitative development of form. In particular, sudden changes in this development are of interest. Any given process can be modelled by a parametrised equation, referred to as the potential function. Even when this model is perfectly continuous and smooth in all its variables, the resulting process may exhibit sudden changes in behaviour. Classification of all types of such phenomena, known as catastrophes, is the object of Catastrophe Theory.

Mathematically the problem reduces to the analysis of parametrised polynomial equations of various degrees. Catastrophes correspond to appearances and disappearances of critical points of these curves or surfaces. A complete classification of qualitative types is available for curves with parameter spaces of dimension not greater than 5.

1.2 Basic Definitions and Results

Loosely speaking any smooth curve can be locally approximated by its Taylor series expansion. Catastrophe Theory concerns itself mainly with the qualitative properties of curves near their critical points.

Definition

Let $f:R \to R$ be C^∞ . x_0 is a singularity of order k , i.e. of type x^k , if

$$\frac{\partial^i f}{\partial x^i}(x_0) = 0 \quad \text{for } i = 1,\ldots,k+1.$$

and

$$\frac{\partial^{k+2} f}{\partial x^{k+2}}(x_0) \neq 0$$

Denote a singularity of order k by A_k (N.B. refer to A_1, simply as a "singularity").

We shall work with potential functions of the following kind:

$$V: X \times C \to R, \qquad V \text{ is } C^\infty$$

where

$$X \subset R^r$$
$$C \subset R^s$$

are open subsets.

X = "Behaviour Space" - in our applications r is usually 1.

C = "Control Space" or parameter space. In our applications s will never be higher than 4.

Write $V_c(x)$ or $V(x,c)$ for $V : X \times C \to R$ and $x \in X$, $c \in C$.

For an example of a potential function and illustrations of the definitions below see section 1.3.1.

Definition

r = corank of the potential function

s = codimension of the potential function

Definition

Let $V(x,c) = 0$ be a potential function.

Define

$$M = \left\{(x,c) \in X \times C: \frac{\partial V}{\partial x}(x,c) = 0\right\}$$

as the Catastrophe Manifold of V, i.e. the set of critical points of V.

The geometry of M is our prime interest.

Definition

Let $\chi : M \to C$ be the canonical projection defined by

$$\chi(x,c) = c$$

known as the catastrophe map.

Definition

Let

$$\Sigma = \left\{ (x,c) \in X \times C : \frac{\partial V}{\partial x}(x,c) = 0,\ \frac{\partial^2 V}{\partial x^2} = 0 \right\}$$

be known as the singularity set of V .

Definition

Let

$$(X|M) = \left\{ x \in X : \textit{there exists } c \in C \textit{ s.t. } \frac{\partial V}{\partial x}(x,c) = 0 \right\}$$

Definition

A point $(x_0,c_0) \in M$ is called a *bifurcation point* if for any neighbourhood N_{c_0} of c_0 in C the projection

$$\Pi_{c_0} : N_{c_0} \to (X|M)$$

defined by

$$\Pi_{c_0}(c) = x = \left\{ \frac{\partial V}{\partial x}(c) \right\}^{-1} (0)$$

is discontinuous at c_0 .

Denote by B_V the set of bifurcation points of V .

Intuitively $(x_0,c_0) \in M$ is a bifurcation point if the corresponding potential function $V(x_0,c_0)$ changes topological type at c_0 , i.e. gains or loses a stationary point. It will not come as a great surprise that

Lemma 1.1

M is a smooth submanifold of $X \times C$.

Lemma 1.2

$$B_v = \chi(\Sigma)$$

Thus B_V is a set of inflection points of V .

Definition

A *catastrophe* is a singularity of χ .

The main result from Catastrophe Theory we need is the following.

Theorem 1

Any singularity of χ is locally equivalent to one of type A_k with $k \leq s$.

It is important to note that the topological complexity of critical points is only dependent on the dimension of the control space. From a practical point of view we can draw two conclusions:

(i) any potential function is equivalent to some polynomial of a finite degree;

(ii) complexity of the critical points is independent of the corank, and therefore, we should aim to reduce the dimension of the behaviour space to 1 whenever possible.

1.3 Two Catastrophes

At this stage it is common to go through the classification theorem and list all the existing catastrophes in each codimension. That is completely superfluous for our purpose and we shall only describe two types of singularities. At the same time the analysis presented will be reasonably thorough. We shall not attempt to present the full mathematical context, but simply treat the reader as a practical statistician interested in applying the method.

1.3.1 Cusp Catastrophe

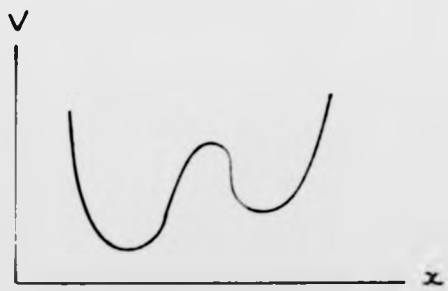
Consider the following potential function .

$$V(x;a,b) = \frac{1}{4}x^4 - \frac{1}{2}bx^2 - ax \tag{c1}$$

where $x \in X$, the behavioural variable, is of dimension 1, and $(a,b) \in C = R^2$ is the control space.

This family of parametrised curves contains basically two qualitatively different types as is illustrated below:

$$\left(\frac{a}{2}\right)^2 > \left(\frac{b}{3}\right)^3$$



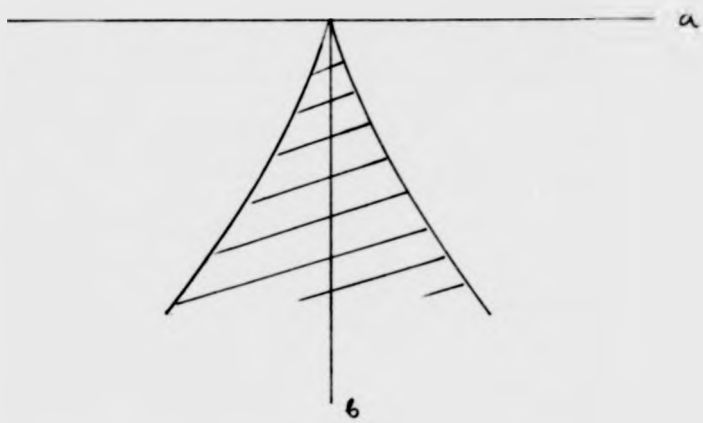
$$\left(\frac{a}{2}\right)^2 < \left(\frac{b}{3}\right)^3$$

diag 1·1

There exists a continuous boundary between the two types given by

$$\left\{\frac{a}{2}\right\}^2 = \left\{\frac{b}{3}\right\}^3 \tag{c2}$$

Let us examine the control space of V :



diag 1.2

The potential $V(x;a,b)$, is bimodal over the shaded region and unimodal outside it. On the boundary V has an inflection point either to the right or to the left of the single minima.

What about the origin of the control space? $(0, 0) \in C$ appears to have some special properties:

(i) it is the only non-smooth point on the boundary;

(ii) Any neighbourhood of $(0, 0)$ is homeomorphic to the whole control space.

Property (ii) says that any neighbourhood of the origin contains all possible types of functions in the family. Note that the origin is the only point with that property.

Let us re-examine the situation using the notation and results of the previous sec-

tion.

Then

$$M = \left\{ (x,a,b) : x^3 - bx - a = 0 \right\}$$

is the catastrophe manifold of V .

$$\Sigma = \left\{ (x,a,b) \in M : 3x^2 - b = 0 \right\}$$

is the singularity set.

$$B_V = \left\{ (a,b) : \left(\frac{a}{2}\right)^2 \leq \left(\frac{b}{3}\right)^3 \right\}$$

Finally, we can see that

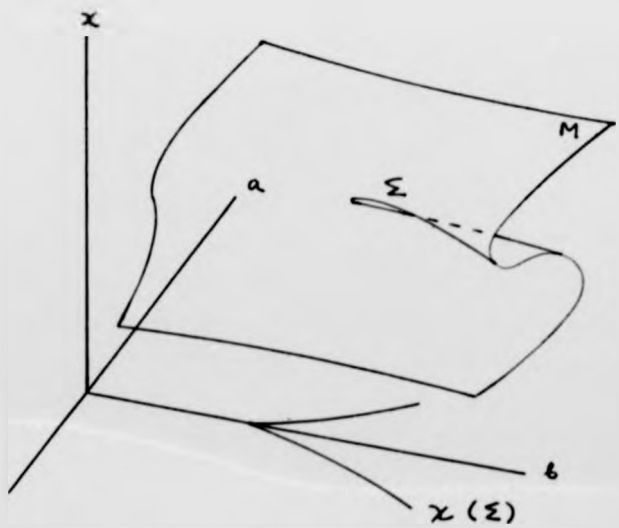
$$(X|M) = R$$

$$\chi(\Sigma) = \left\{ (a,b) : b = 3x^2,\ a = -2x^3 \right\}$$

and the only singular point of $\chi(\Sigma)$ is the origin (0, 0, 0)

We refer to this singularity as the *Cusp Catastrophe* . It is easily seen to be of the type A_2 .

The following geometric illustration of the canonical cusp catastrophe is quoted by all authors:



diag 1·3

The curved surface in R^3 is the catastrophe manifold M . It is smooth at all points. The singularity set Σ is the red curve in R^3 . Planes parallel to the x - axis touch M along points of Σ only. The natural projection of Σ onto the control space gives the wish-bone shaped curve - the boundary of the bifurcation set B_V .

It is worth stressing the importance of this boundary. Write

$$\delta = \left(\frac{b}{3}\right)^3 - \left(\frac{a}{2}\right)^2 \tag{c3}$$

δ is known as Cardano discriminant of the cubic equation. In our context $\delta > 0$ corresponds to two local minima of V and $\delta < 0$ to just one local minimum.

1.3.2 Butterfly Catastrophe

The next member of the cuspoid family has the potential function

$$V(x) = \frac{1}{6}x^6 - \frac{1}{4}dx^4 - \frac{1}{3}cx^3 - \frac{1}{2}bx^2 - ax \tag{b}$$

with $C = (a,b,c,d)$ as the control space.

We have to examine the shape of V and V' for various $c \in C$. First consider

$$\frac{dV}{dx} = V' = x^5 - dx^3 - cx^2 - bx - a = 0 \tag{1.1}$$

$$V'' = 5x^4 - 3dx^2 - 2cx - b = 0 \tag{1.2}$$

$$V''' = 20x^3 - 6dx - 2c \tag{1.3}$$

Simultaneously, these equations give the conditions for cusp points in the topology of V . In particular the number of real roots of (1.3) determines the number of cusp occurring.

Its discriminant is

$$\tau = \frac{d^3}{5} - \frac{c^2}{2} \tag{1.4}$$

If $\tau < 0$, (1.3) has only one real root, and consequently V exhibits only one cusp. Otherwise, (1.3) has three real roots and V has three corresponding cusp points. We shall now proceed to examine the behaviour of V over various regions of C .

(1) Case $d < 0$, fixed

It is enough to examine the potential function to see that the problem is practically reduced to a cusp potential:

(a) terms x^6 and x^4 are both positive

(b) term x^3 can be eliminated by a change of coordinates

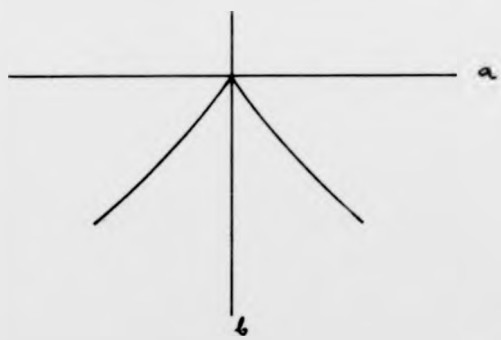
Thus,

(i) $c = 0$ gives a single cusp at

$$x = 0$$

$$a = 0$$

$$b = 0$$



diag 1.4

(ii) $c < 0$

From (1.3):

$$c = x(10x^2 - 3d)$$

gives the x - coordinate of the cusp.

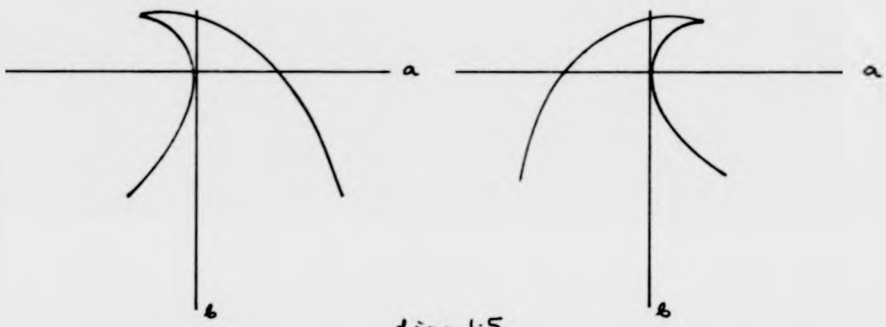
$$\Rightarrow x < 0$$

Say

$$c = c_0 < 0$$
$$x = x_0 < 0$$

and the coordinates of the cusp become

$$a = x_0^3(6x_0^2 - d) < 0$$
$$b = -15x_0^4 + 3dx_0^2 < 0$$
$$c = c_0 < 0$$



diag 1·5

The (a,b) - sections of the bifurcation set for $c < 0$ and $c > 0$

(iii) $c > 0$

Say $c = c_0 > 0$. Bifurcation set is the mirror image of case (ii) with the cusp point at

$$x = x_0 > 0$$
$$a = x_0^3(6x_0^2 - d) > 0$$
$$b = -15x_0^4 + 3dx_0^2 < 0$$
$$c = c_0 > 0$$

Clearly, the case $d < 0$ can only be of interest as a "passing state" of the butterfly potential.

(2) Case $d > 0$, fixed and $c = 0$

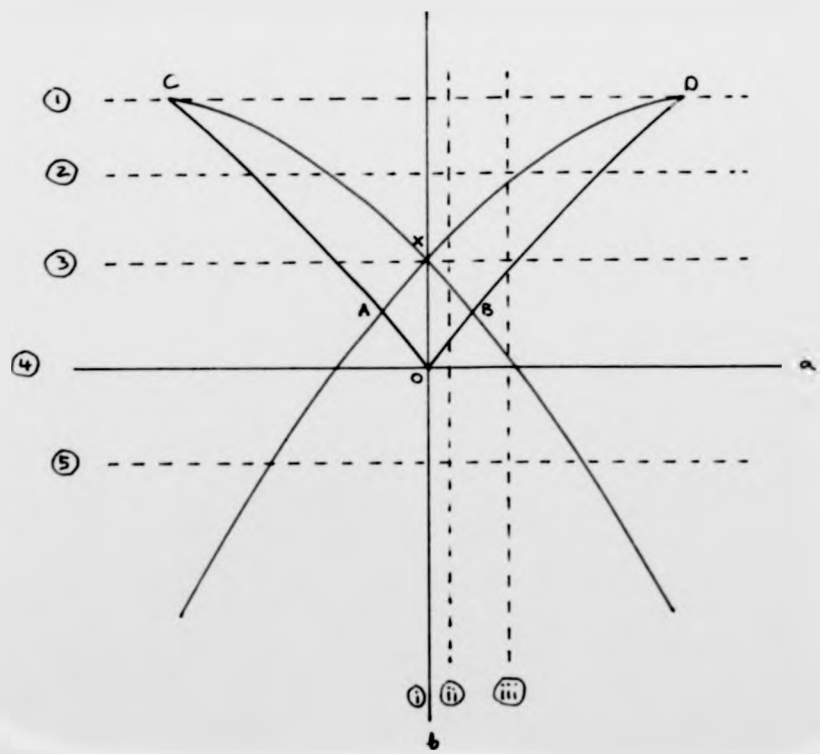
We shall examine the shape of the (a,b) - section of the bifurcation set and look at the corresponding sections of the catastrophe manifold (i.e. the surface $V' = 0$) as well as the potential functions' shapes at these control points.

The catastrophe manifold is given by the equation (1.1):

$$V'(x) = x^5 - dx^3 - bx - a = 0 \tag{1.5}$$

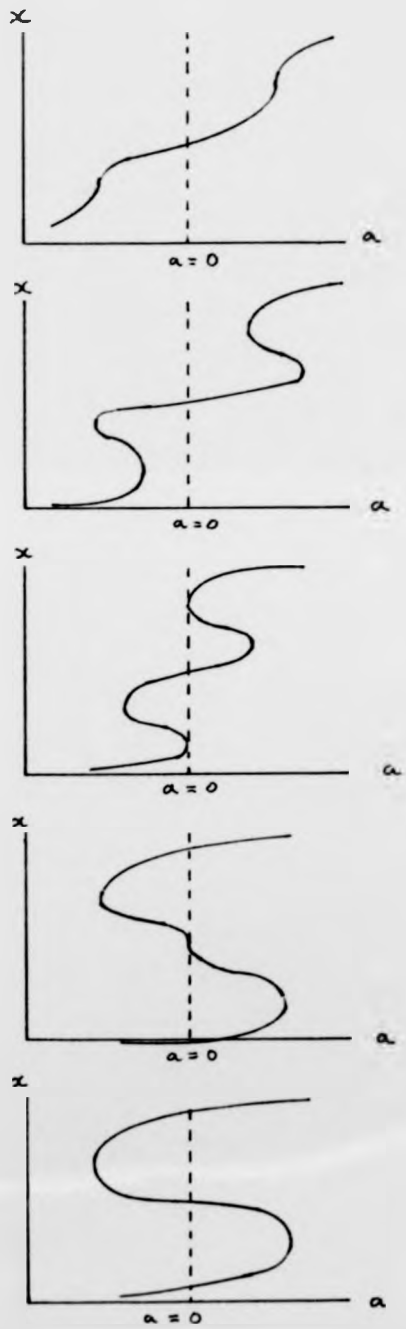
with $c = 0$ and $d > 0$, a constant.

Equations (1.1) and (1.2) together with $c = 0$ constraint give rise to the following shape of the (a,b) - sections of the bifurcation set:



diag 1.6

Corresponding (a,x) - sections of the catastrophe manifold, $\frac{dV}{dx} = 0$:



section ①

section ②

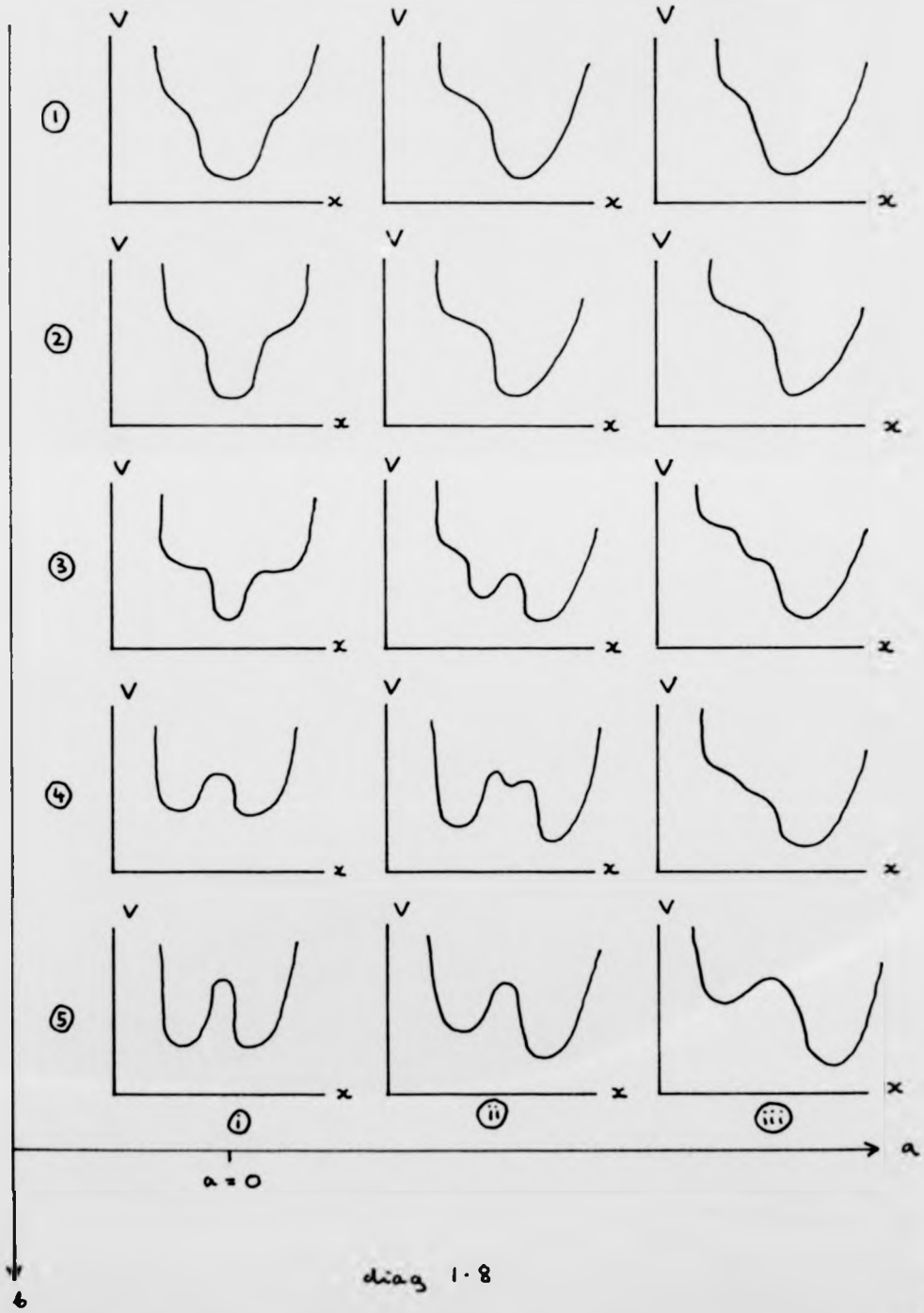
section ③

section ④

section ⑤

diag 1.7

Corresponding potential functions for points in (a,b) - plane lying on the intersections of broken lines in diagram 1.6:



diag 1·8

The diagram can be reflected in the b - axis to obtain a perfectly symmetric picture for $a < 0$.

Consider again diagram 1.6. The black figures indicate the number of local minima exhibited by the potential function. The red and green lines dividing those regions correspond to inflection points of the potential function.

The equations of the "butterfly" shape are:

$$V(x) = x^5 - dx^3 - bx - a = 0 \tag{1.6}$$

$$V'(x) = 5x^4 - 3dx^2 - b = 0 \tag{1.7}$$

Eliminating b from (1.6) we obtain

$$a = -4x^5 + 2dx^3 \tag{1.8}$$

$$b = 5x^4 - 3dx^3 \tag{1.9}$$

The three cusp points are given by

$$\frac{\partial b}{\partial x} = 0$$

$$20x^3 - 6dx = 0$$

i.e

$$x = 0 \quad or \quad x^2 = \frac{3d}{10}$$

But, from (1.9) :

$$x^2 = \frac{3d \pm \sqrt{9d^2 + 20b}}{10}$$

Therefore,

No real solutions for $20b < -9d^2$

Four real solutions for $\frac{-9d^2}{20} < b < 0$

Three real solutions for $b = 0$

Two real solutions for $b > 0$

(all clear from the diagram 1.6)

Thus, b - coordinates of the cusps are

$$b = -\frac{9d^2}{20} \quad , \quad b = 0$$

From (1.8) we get

$$\frac{\partial a}{\partial z} = 0 = -20z^4 + 6dz^2$$

$$Therefore \quad z^2 = 0 \quad or \quad z^2 = \frac{3d}{10}$$

So, the a - coordinates of the cusps are

$$a = 0 \quad or \quad a^2 = \frac{6^2 \times 3}{25^2 \times 10} d^5$$

Hence cusps occur at points

O, *coordinates* $(0, 0, 0)$

D, *coordinates* $\left(\frac{3d}{10}, \frac{6\sqrt{3}}{25\sqrt{10}} d^{5/2}, -\frac{9d^2}{20} \right)$

C, *coordinates* $\left(-\frac{3d}{10}, -\frac{6\sqrt{3}}{25\sqrt{10}} d^{5/2}, -\frac{9d^2}{20} \right)$

We now proceed to find the coordinates of the quadrant $OAXB$, the region of most interest as its interior defines a family of potentials with three local minima

Starting with X :

$$z^5 - dz^3 - bz = 0 \quad with \quad a = 0$$

Thus $z = 0$ or

$$z^2 = \frac{d \pm \sqrt{d^2 + 4b}}{2}$$

But at X (1.5) has a double root, i.e.

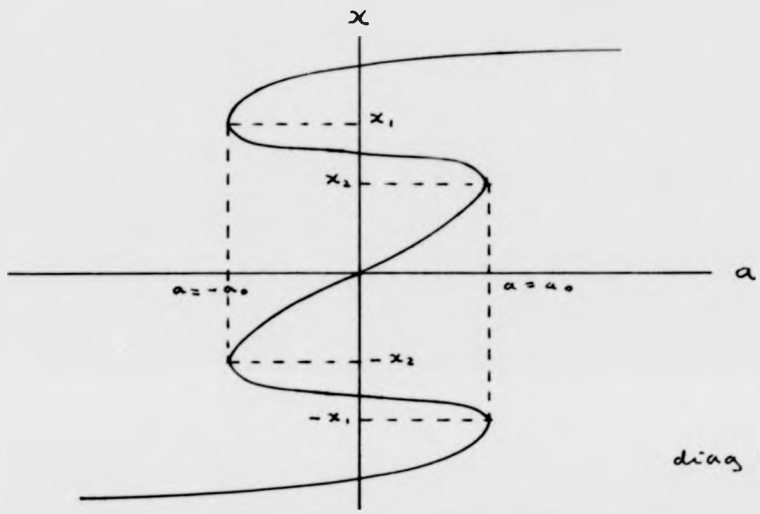
$$d^2 + 4b = 0 \quad \text{and} \quad z^2 = \frac{d}{2}$$

Hence the coordinates of X are

$$\left(\pm \sqrt{\frac{d}{2}}, 0, -\frac{d^2}{4} \right)$$

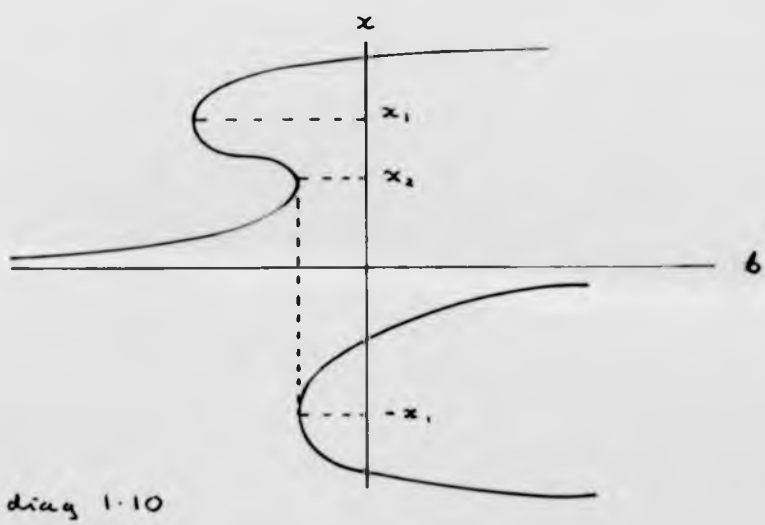
Let (a,b) - coordinates of B be (a_0,b_0) . Then A has coordinates $(-a_0,b_0)$. Before we find the values of a_0 and b_0 let us examine the geometry of the potential function and the catastrophe manifold at those points.

The (a,x) - section of the catastrophe manifold at $b = b_0$ looks as follows:



diag 1·9

The (b,x) - section of the same manifold at $a = a_0$ has the following shape:



diag 1·10

Let us analyse the curves in diagrams 1.9 and 1.10.

The equation of the curve in the diagram 1.9 is given by (1.6) with $b = b_0$. Writing it as a function

$$a(x) = x^5 - dx^3 - b_0 x \tag{1.10}$$

we see that $a(x)$ has four turning points at $x_1, x_2, -x_2, -x_1$ s.t.

$$a(x_2) = a(-x_1) = -a(x_1) = -a(-x_2) \tag{1.11}$$

This implies that

$$\frac{\partial a}{\partial x} = 5x^4 - 3dx^2 - b_0 = 0 \tag{1.12}$$

has four distinct real roots of the form

$$x_1 = +\left\{\frac{3d + \sqrt{9d^2 + 20b_0}}{10}\right\}^{\frac{1}{2}} \tag{1.13}$$

$$x_2 = +\left\{\frac{3d - \sqrt{9d^2 + 20b_0}}{10}\right\}^{\frac{1}{2}} \tag{1.14}$$

and $-x_1$ $-x_2$.

Note that the condition $\frac{9d^2}{20} \leq b_0 \leq 0$ for all roots to be real is satisfied at b_0.

Similarly, consider the curve in diagram 1.10 as a function

$$b(x) = x^4 - dx^2 - \frac{a}{x}, \quad x \neq 0 \tag{1.15}$$

This curve has three turning points s.t.

$$b(x_2) = b(-x_1)$$

$$\frac{dV}{dx} = x^5 - dx^3 - bx - a_0 = 0 \tag{1.16}$$

the original quintic form of the equation (1.15) has five real roots in x, namely x_1, x_2 (repeated root), $-x_1$ (repeated root).

Thus (1.16) can be factorised as

$$(x - x_1)(x - x_2)^2(x + x_1)^2 = 0$$

and

$$x_1 + 2x_2 - 2x_1 = \textit{coefficient of the } x^4 \textit{ term} = 0$$

giving

$$x_2 = \tfrac{1}{2}x_1 \tag{1.17}$$

Putting (1.13) and (1.14) into (1.17) we obtain

$$2\left|\frac{3d - \sqrt{9d^2 + 20b_0}}{10}\right|^{\frac{1}{2}} = \left|\frac{3d + \sqrt{9d^2 + 20b_0}}{10}\right|^{\frac{1}{2}}$$

Hence

$$\frac{9d}{5} = \sqrt{9d^2 + 20b_0}$$

$$b_0 = -\frac{179}{500}d^2$$

Putting this back into (1.13) and (1.14) we get

$$x_1 = 2\sqrt{\frac{3}{25}d} \tag{1.19}$$

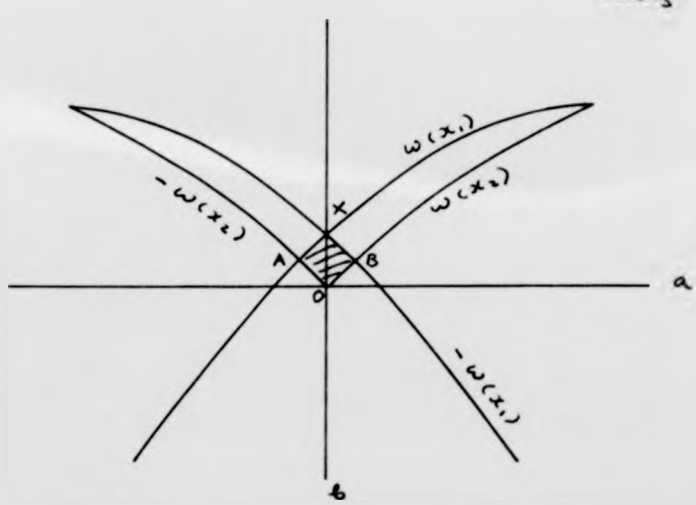
$$x_2 = \sqrt{\frac{3}{25}d} \tag{1.19}$$

Finally, from (1.16) we can quickly find that

$$a_0 = \frac{631\sqrt{3}}{12500}d^{5/2} \tag{1.20}$$

Thus, we now have the coordinates of all corner points of the quadrangle containing the 3 - minima region of the bifurcation set:

diag 1.11



We are now in a position to find the conditions for the existence of the third minimum of $V(x)$.

Denote by $w(x_1)$ the branch of the bifurcation set corresponding to the root x_1 of the "generalised" equation (1.12), i.e.

$$5x^4 - 3dx^2 - b = 0$$

namely

$$x_1 = + \left\{ \frac{3d + \sqrt{9d^2 + 20b_0}}{10} \right\}^{\frac{1}{2}}$$

Similarly, the remaining branches are $-w(x_1)$, $w(x_2)$ and $-w(x_2)$ where x_2 is the other positive root of (1.12).

Note that branches $\pm\ w(x_1)$ exist for

$$b \geq -\frac{9}{20}d^2$$

and branches $\pm\ w(x_2)$ exist only for

$$-\frac{9}{20}d^2 \leq b \leq 0$$

The region of three minima is bounded by

$$w(x_1) \text{ and } -w(x_1) \quad \text{for} \quad \frac{-9}{20}d_2 \leq b \leq b_0$$

and

$$w(x_2) \text{ and } -w(x_2) \quad \text{for} \quad b_0 \leq b \leq 0$$

Recall, that the (a,b) - section of the bifurcation set is given equations (1.6) and (1.7). Simplifying we get the required region as

$$a^2 = 4x^6(\ d^2 - 4dx^4 + 4x^2)$$

where

$$x^2 = \frac{3d + \sqrt{9d^2 + 20b}}{10} \quad \text{if } -\frac{d^2}{4} \leq b \leq b_0$$

$$= \frac{3d - \sqrt{9d^2 + 20b}}{10} \quad \text{if } b_0 \leq b \leq 0$$

Hence the 3 minima region is given by the following inequalities:

$$d > 0 \qquad (*)$$

$$-\frac{d^2}{4} < b < 0 \qquad (**)$$

$$a^2 < \frac{4}{10^3}\left[108d^3 \pm (36d^2 + 20b)\sqrt{9d^2 + 20b} + 180db\right] \times \qquad (***)$$

$$\left\{d^2 - \frac{4d}{10^2}\left[18d^2 \pm 6d\sqrt{9d^2 + 20b} + 20b\right] + \frac{6}{5}d \pm \frac{2\sqrt{9d^2 + 20b}}{5}\right\}$$

with $+$ if $-\frac{d^2}{4} \leq b \leq b_0$

$-$ if $b_0 \leq b \leq 0$

We can now summarise these conditions by defining

$$\eta(a,b,c=0,d) > 0$$

if and only if $(*)$, $(**)$, $(***)$ all hold, otherwise $\eta < 0$.

Thus η is positive on the three minima region, and negative everywhere else.

(3) Case d > 0, fixed and c > 0, fixed

We aim to generalise conditions η for the case $c \neq 0$. Since the manifold is perfectly symmetric around $c = 0$ it is enough to look at the case $c > 0$.

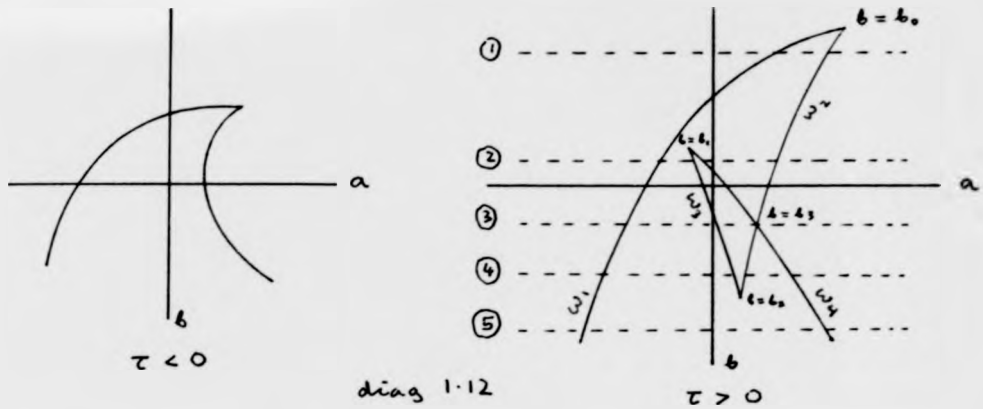
First let us look at the geometry of the (a,b) - section of the bifurcation set for $c > 0$ and $d > 0$.

Recall, the discriminant given by (1.4):

$$\tau = \frac{d^3}{5} - \frac{c^2}{2}$$

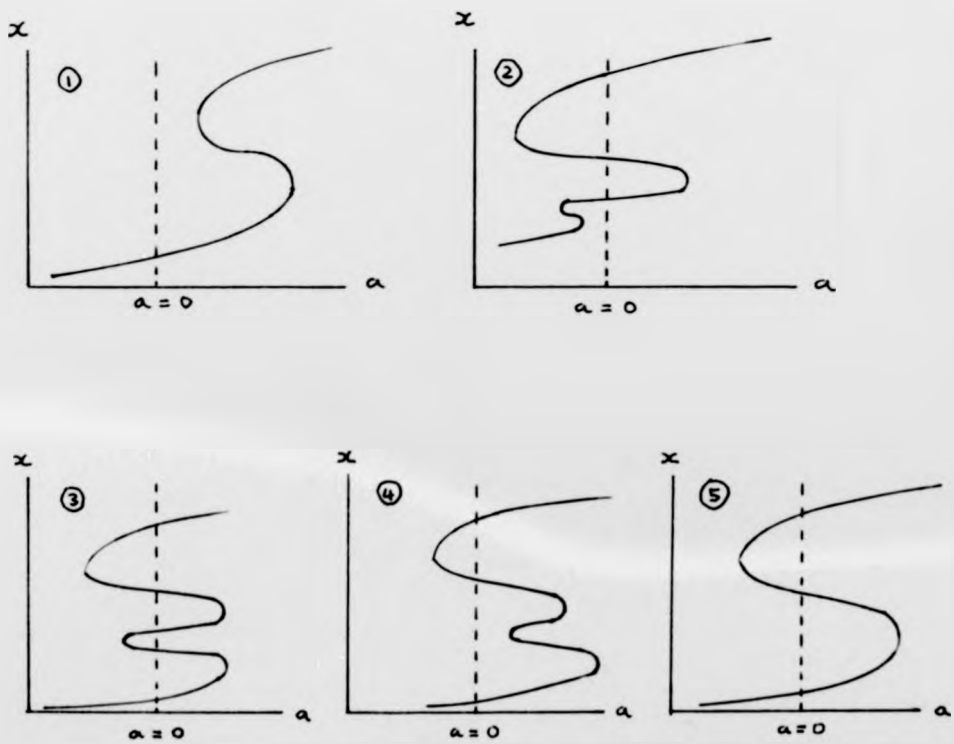
It will replace $(*)$ in the system of inequalities η:

$\tau > 0$ is a necessary condition for $V(x)$ to have three local minima whenever $c \neq 0$.



diag 1·12

For the case $\tau > 0$, consider the following (a,x) - section of the catastrophe manifold.



diag 1·13

Notice that if $c < 0$ all the pictures have to be reflected in the $a = 0$ axis.

In order to complete the generalisation of η it is necessary to solve the following equations:

(i)

$$\frac{d^2V}{dx^2} = 5x^4 - 3dx^2 - 2cx - b = 0 \tag{1.2}$$

subject to $\tau > 0$. This equation will have 0, 1, 2, 3, 4, 3, 2 real roots accordingly as b increases (refer to diagram 1.12 for $\tau > 0$). The branches of the bifurcation set (a,b) - section will be given by those roots, say

$$x_i = \omega_i(b,c,d) \qquad i = 1,2,3,4.$$

(ii) Putting (1.1) and (1.2) together we obtain

$$a = x^2(-4x^3 + 2dx + c) \tag{1.21}$$

(iii) Again referring to diagram 1.12 we have to determine the coordinates of the cusp points in order to define regions of b over which particular real roots ω_i exist, i.e. we must find end points of the branches of the bifurcation set. It is clear from the diagram that

$$\begin{aligned} \omega_1 &\ \textit{exists for}\ b \geq b_0 \\ \omega_2 &\ \textit{exists for}\ b_0 \leq b \leq b_2 \\ \omega_3 &\ \textit{exists for}\ b_1 \leq b \leq b_2 \\ \omega_4 &\ \textit{exists for}\ b_1 \leq b \end{aligned} \tag{1.22}$$

Thus exact computation of all the condition can cause some problems as it involves solving a quartic (1.2). However it is not necessary for us to have the exact solution. Clearly the method is analogous to the case $c = 0$. Only all the equations and inequalities become functions of c .

Therefore we can state

Theorem 2

For fixed $d > 0$ and fixed $c > 0$, with $\tau > 0$

$$V(x) = \frac{1}{6}x^6 - \frac{1}{4}dx^4 - \frac{1}{3}cx^3 - \frac{1}{2}bx^2 - ax \qquad \text{(b)}$$

exhibits three local minima over the following region of the sub-control space (a,b) :

$$x_j(-4x_j^3 + 2dx_j + c) < a < x_i(-4x_i^3 + 2dx + c)$$

where x_i, x_j are roots of $\dfrac{d^2V}{dx^2} = 0$, and take the following particular values:

$$x_j = \omega_3 \text{ and } x_i = \omega_4 \text{ if } b_1 \le b \le b_3$$
$$x_j = \omega_3 \text{ and } x_i = \omega_2 \text{ if } b_3 \le b \le b_2$$

[Note: diagram 1.12 is drawn for the case when the "pocket" does not cross ω_1 . This, however, occurs when τ is large enough. The above condition has to be suitably adjusted. This complicates explicit calculations even further, but the intuitive idea is as simple as the case described here.]

Summary - Effects of control parameters.

The above analysis is not very helpful for getting an intuitive feel for the properties of the Butterfly Catastrophe. Neither is it particularly easy to appreciate the sensitivity of the shape of the potential function to changes in control variables.

This section will concentrate on these general aspects, and an effort will be made to minimise tedious calculations.

The control variables of $V(x)$ can be crudely divided into two pairs:

(i) a and c control the symmetry of the system: a affects the position of the unique minimum and the relative heights of two/three minima when these occur; c affects the position of the cusp and the shape of the bifurcation set.

(ii) b and d are the "bifurcation factors": they control the number of stationary points of V ; b causes bimodality while d creates a "split within a split" and causes trimodality.

It must be remembered that groups (i) and (ii) interact at all times in the sense that "critical values" of b and d depend on particular values of a and c respectively, etc.

This brings us to the two discriminants which effectively link symmetry factors to bifurcation factors.

$$\delta = \left(\frac{b}{3}\right)^3 - \left(\frac{a}{2}\right)^2 \tag{c3}$$

is the Cardano discriminant of the cubic and it determines completely the qualitative behaviour of Cusp Catastrophe. Here, however, it is no longer independent, and a new discriminant emerges

$$\tau = \frac{d^2}{5} - \frac{c^2}{2} \tag{1.4}$$

τ can be thought of as the "discriminant within a discriminant", since it is constructed as the discriminant of $\frac{d^3 V}{dx^3} = 0$, which is a cubic.

The pair (δ, τ) are a good ,practical way of summarising the qualitative behaviour of V . It is a much simpler approach than using the η equations. It must be remembered that τ is the independent discriminant, whilst δ is sensitive to values of τ. τ gives an immediate answer to the question "Can V be trimodal?", but δ, designed to answer "Is V bimodal?" gives only a qualified reply.

We are nevertheless able to give a string of weak results and conclusions to describe properties of τ and δ and their relationship.

Notation

Write the control space $C = \{a,b,c,d\}$ as a Cartesian product

$$C = \mathbf{A} \times \mathbf{D}$$

where $\mathbf{A} = (a,b)$ and $\mathbf{D} = (c,d)$ are two-dimensional subspaces of C .

Definition

Let $A \subseteq \mathbf{A}$. Define

$$A^{+} = \left\{ (a,b) \in A \ \ s.t. \ \ \delta(a,b) > 0 \right\}$$

Similarly, for $D \in D$ define

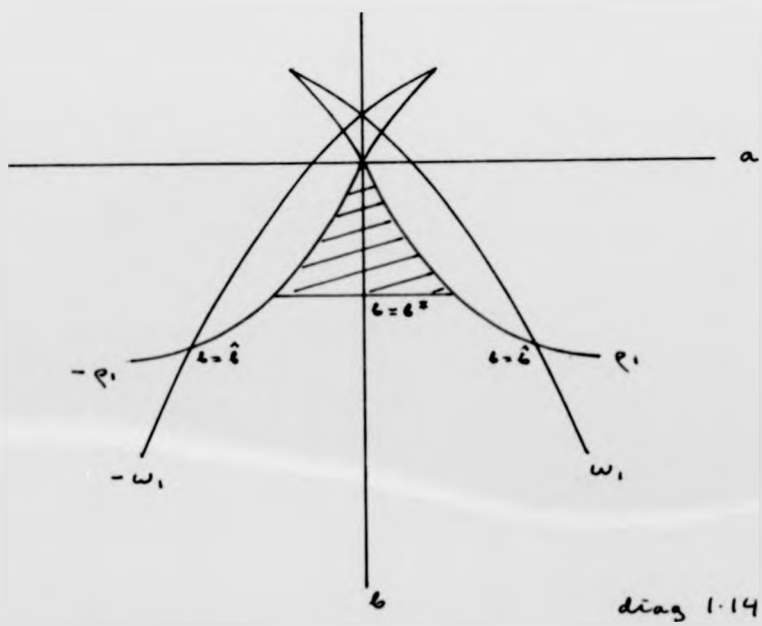
$$D^{+} = \left\{ (c,d) \in D \ \ s.t. \ \ \tau(c,d) > 0 \right\}$$

Lemma 1

Let A be a bounded subset of $\mathbf{A}^{+}$. Then there exists a $D \in \mathbf{D}$ s.t. $V(x)$ is bimodal on $A \times D$.

Proof

Consider the following (a,b) - section of the bifurcation set, together with the wish-bone shaped $\delta = 0$ curve:



diag 1·14

WLOG take A to be the set bounded by ρ_1, $-\rho_1$ and line $b = b^*$.

It is now enough to prove that A is contained in the bimodal region of an (a,b) - section of the bifurcation set for some $D \in \mathbf{D}$ whenever b^* is finite.

Choose $D \in \mathbf{D}$ with the following properties:

(i) $c = 0$

(ii) $d > 0$

We will show that for any fixed b^*, we can choose $d(b^*) > 0$ s.t.

$$D = \left\{ (c,d) : c = 0 \, , \, d > d(b^*) \right\}$$

will be as required.

Refer to the diagram: it is enough to show that for any fixed b^* it is always possible to choose d s.t.

$$b \geq b^*$$

(In fact equality holds when $d = d(b^*)$).

Recall that the size of the pocket is an increasing function of d. So is the intercept of ω_1 and the a - axis. This can be seen by combining equations (1.13) (with $b = 0$) and (1.8) to get the value of the intercept as

$$a = \frac{2}{5}\left(\frac{3}{5}\right)^{3/2} d^{5/2}$$

Similar calculation will yield intersection of ω_1 and any b - coordinate. In each case this intercept will be an increasing function of d. Therefore $d(b^*)$ can be chosen as required.

Note that only in the most general case we will require $D \subset \mathbf{D}'$.

Lemma 2

$V(x)$ trimodal on $A \times D \subset \mathbf{A} \times \mathbf{D}$ implies $D \subseteq \mathbf{D}'$, but *not* conversely.

Proof

Refer to diagram 1.11. If $V(z)$ is trimodal then this diagram is the relevant section of D and obviously $D \subset \mathbf{D}'$.

But conversely we can easily choose a region A s.t. V is not trimodal on $A \times V$ for any $D \subset \mathbf{D}'$.

Lemma 3

$V(z)$ trimodal on $A \times D \subset \mathbf{A} \times \mathbf{D}$ does *not* imply $A \subset \mathbf{A}'$.

Proof

Refer again to diagram 1.11. Region AOBX meets $\mathbf{A}'$ only at the origin.

Corollary

If $c = 0$, $d > 0$, then

$$\left\{ V: A \in \mathbf{A}' \right\} \cap \left\{ V: V \textit{ trimodal} \right\} = \phi$$

Lemma 4

If $V(z)$ is trimodal on $A \times D \subset \mathbf{A} \times \mathbf{D}$ and $A \subset \mathbf{A}'$, then $c \neq 0$

Proof

Follows from corollary above as the intersection is now non-empty.

The above results are intuitively more obvious than analytically. They can be useful for quick tests on trimodality, as well as tests on "availability of trimodality", i.e. they can provide an indication that there is a possibility of a third mode occurring should some of the parameters evolve in a particular manner.

The main reason for introducing (δ, τ) in place of η equations is that the former can be more easily handled in statistical inference and estimation.

1.4 Remarks

(1) The Cusp and Butterfly Catastrophes are essentially sufficient for our purposes. Nevertheless more complicated models exist and may have to be used. In particular straightforward generalisations of the cusp and the butterfly will occasionally be referred to in later chapters. Unfoldings of an A_{k+2} singularity can be written in the form

$$V_{k+2}(x) = \frac{1}{k+2}x^{k+2} + \frac{1}{k}a_k x^k + \frac{1}{k-1}a_{k-1}x^{k-1} + \cdots + \frac{1}{2}a_2 x^2 + a_1 x \qquad \text{(g)}$$

for an even integer $k \geq 2$. The control space $C = (a_k, \ldots, a_1)$ has codimension k. V_{k+2} exhibits at most $k - 1$ local minima.

The family of potentials $\left\{ V_{k+2}: k \geq 2, \textit{ even integer} \right\}$ is called the "cuspoid family" of catastrophes and is defined over a 1-dimensional behaviour space. It can be further generalised to multidimensional behaviour spaces if necessary.

(2) The V_{k+2} potential is usually referred to as the "canonical model". A function $F : U \to R$ is said to be equivalent to V_{k+2} if it is of the same topological type. Stewart (10) defines such "topological equivalence" of two smooth functions

$$f: U \subseteq R^n \to R$$
$$g: V \subseteq R^n \to R$$

as follows.

Suppose WLOG $f(u) = 0 = g(v)$. Then f and g are equivalent near u and v if there exist neighbourhoods U_1 of u and V_1 of v, in U and V respectively, and a diffeomorphism

$$\phi: U_1 \to V_1$$

s.t. the diagram

$$\begin{tikzcd} U_1 \arrow[dr, "f"] \arrow[d, "\phi"'] & \\ V_1 \arrow[r, "g"'] & \mathbb{R} \end{tikzcd}$$

commutes.

(3) Potentials like $V_{k,2}$ will be used as expected loss functions, densities and, more generally, energy functions. Their basic characteristic, multimodality, will prove crucial in our approach to statistical modelling.

2. General Framework

2.1 Introduction

When viewed as a branch of Measure Theory Probability Theory is a closed book. However, both the quantitative development as well as the interpretation of probability range far beyond the realms of abstract mathematics. We wish to examine the latter aspect: the motivation and the link with the real world that probability measures claim to possess.

Constructing probability spaces consists of two parts:

1. Identifying a sample space, say Ω, with a suitable algebra of events, say A.
2. Defining and interpreting a probability measure on (Ω, A).

Part 1 has generally attracted little attention. In Part 2 the "interpretation" aspect has been extremely controversial. Opinions have been so diverse that some even claim that classes of probability measures have to be defined on every Ω, and an existence of a single probability measure is just a restricted, special case (see, for instance, Walley and Fine (39)). Clearly the problem is a philosophical one and not mathematical.

It appears that there exist numerous difficulties associated with Part 1 as well. Indeed, this and other issues which we are planning to discuss are all interwoven. Decision Theory has always lain on the border of Probability Theory. Its internal structure is mathematically equivalent to that of Probability Theory. We intend to consider a more general framework in which decision spaces and sample spaces are going to be examples of "spaces of alternatives" which we define later.

The aim is to construct some kind of a new general structure and then attack problems like "updating" and "aggregation".

2.2 Some Philosophy

Barnett (30) identifies four basic approaches to probability interpretation:

(i) *classical* - A uniform measure is set on to a chosen partition of Ω . This leads to a circular definition of probability based on a concept of symmetrical "equally likely" events. Although this fact alone is not usually regarded as a major objection, the approach fails to explain how individuals are supposed to recognise those mysterious types of events. Principles of "cogent reason" and "insufficient reason" only provide an intuitive picture. Borel claims that everyone has his own "primitive notion" of the concept. All these rather vague arguments have meant that the classical approach has been largely abandoned.

(ii) *frequentist* - This assumes that the relative frequency of occurrence of an event converges. This is an empirical approach and it aims to create an "objective" model of the world. The early protagonists of this method were Laplace and Venn, but the mathematical basis was properly set up by Von Mises (37). Fundamental concepts of this approach are "repeatable experiments", mutual exclusion, independence and conditional probability. The main criticism of the frequentist view concerns the crucial notion of "repeatable experiments". It requires countably many copies of the sample space at any time in order to calculate the probability of occurrence of any event. In many situations we may wish to assign probabilities to outcomes which are clearly "one - off". Frequentists would like to be able to do this in all circumstances, but the "repeatable experiments" framework does not always provide a valid interpretation.

(iii) *logical* - Probability is a measure of implication. In this approach the concept of probability becomes a part of logic. The treatment is mainly axiomatic, and numerical values are not thought of as essential com-

ponents. The logical method was developed by Keynes, Jeffreys and Carnap (32). The "Principle of Insufficient Reason" and frequentist methods are often used in a practical context. Critics object to the inflexibility of the abstract mathematical structure of the logical approach.

(iv) *subjective*- Probabilities are measured by individuals' disposition towards bets. The governing law is "coherence". What is coherence? Basically it is the aim of an individual to conform to Kolmogorov's axioms and thus to avoid the ignominy of a "sure loss" from his bets. This approach rejects the necessity of a universal probability structure and relies on each individual to construct his own probability model of the world. The entire philosophy is a stark contrast to the frequentist view. Opponents criticise the lack of objectivism and the inherent dependence on personalist viewpoints.

There are two other modern approaches not mentioned by Barnett.

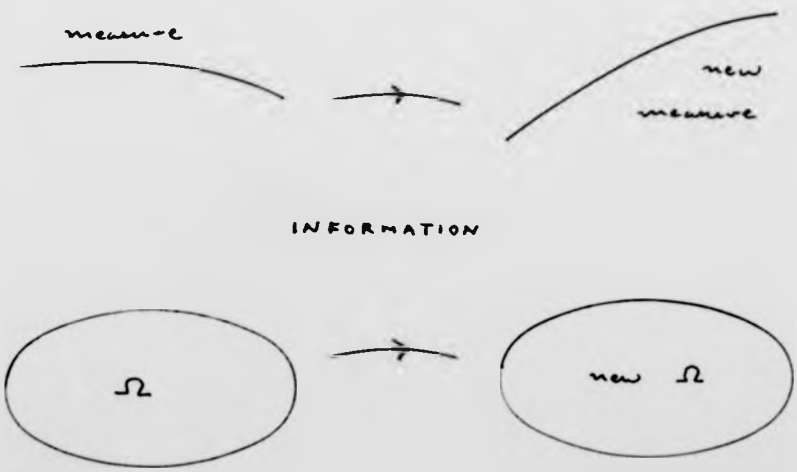
(v) *entropy approach* - Probabilities are calculated by maximising entropy, i.e. minimising information, subject to the given constraints. It is a physical approach and is described by Williams (56).

(vi) *fuzzy approach* - Intervals are used to represent uncertainty. Instead of a single valued probability of an event a pair, "upper" and "lower" probabilities, are assigned to each event. Usually a subjectivist view is used as a basis for this construction. Thus, in terms of gambles, the lower probability of an event A is the largest price an individual is willing to pay for the gamble on A when he stands to receive 1 unit if A occurs. The upper probability of an event A is the lowest price an individual is prepared to accept in return for a bet on A . In general, it is claimed that this leads to a non-additive probability model. Effectively this approach questions the existence of a unique measure on any sample

space. It is a strikingly unorthodox view, and it is in direct conflict with the frequentist ideal. See, for instance, Walley and Fine (39).

Apart from the basic notion of probability several other issues have led to disagreement. The most famous problem is the one of change in beliefs. Measure Theory lends little help in this matter, and each school of statistics prescribes its own method. The mathematical structures are reasonably similar, but once again interpretation varies. All schools agree that a change in belief corresponds to a change in information. New beliefs are conditional on the information received. Subjectivists employ Bayes Theorem. Samplists do not object to the mathematics of that theorem, but they disagree about the way in which Bayesians apply it. On the whole they do not accept the suitability of some subjective information. The entropists repeatedly use the minimisation of information principle, and new information is entered in the form of constraints on the probabilities in the model. Williams (56) claims that Bayes Theorem and Jeffreys rule are special cases of that principle.

It is possible to summarise the philosophy behind each approach with the following diagram:



and an equation:

$$\textit{Change in Belief} = \textit{function of (Change in Information)}$$

Several comments spring to mind:

(i) Most approaches do not recognise any changes in the structure of sample spaces. After all if Ω is chosen to be large enough any information can only restrict the support set of the measure. In no situation can this sample space be actually enlarged, c.f. Williams (56): events of prior probability zero cannot have positive posterior probability.

(ii) The "Information Space" remains a mystery. What exactly is information and how can it be measured? Is it a vector or scalar quantity? Can we ever lose information or do we always gain some? Each approach in its own right tries to answer these questions indirectly. After all, concepts like significance levels, support sets, likelihoods, the principle of minimum information and Fisher's information all in some way attempt to evaluate the state or the increase in information. Yet none of the above agree to the meaning of "information". Indeed each method interprets this concept in a totally different way.

(iii) Our equation, $\Delta\, belief = f(\Delta\, information)$, suggests the existence of some dynamic structure here. Especially if we can define information in such a way that it is measurable. However, thus far, generally the increments of information have been presented as discrete and often "large". None of the methods above are sensitive to small changes in information. Consequently calculus procedures are not likely to be helpful.

(iv) The relation between information and time, and thus, indirectly, between probability and time has never really been examined. Naturally, we assume that any new information comes to us in the future, but still many problems remain, e.g.

(a) Is the rate of flow of information relevant?

(b) Can beliefs be altered in periods of time when no information is received?

Time Series models are updated at points of time. These are generally discrete and are introduced as reference points for collecting information. In quantum mechanics more effort is made to relate probabilities to calculus.

In short, we believe that sample spaces and their associated measures can be successfully viewed as functions of time. Thus the triple

$$\left\{ \Omega, A, P \right\}$$

should be written as

$$\left\{ \Omega(t), A(t), P(t) \right\}$$

This extra parametrisation creates no new problems. Whenever undesirable we can postulate, in particular cases, the constant case

$$\left\{ \Omega(t), A(t), P(t) \right\} = \left\{ \Omega, A, P \right\}, \qquad \text{for } all\ t.$$

We intend to show that, in many situations modelling can be simplified and clarified by reference to the t-axis.

Note, incidentally, that Decision Theory suffers from exactly the same problems. Sample spaces are replaced by decision spaces and probability density functions are replaced by various utility, risk and expected loss functions all of which have the same mathematical structure. Incidentally, no method has ever been proposed to update utilities.

In general our approach is to introduce a dynamic structure on any space using measurable maps defined on it.

It is our intention to propose a completely new way for modelling uncertainty. In a traditional set up events and their probabilities form the primitive structure. We look one

stage further back and begin our modelling by first constructing an underlying structure for events and probabilities.

The basic element of our representation is an energy function. All the energy functions we will look at are potential functions of the type described in Chapter 1. The concepts of events and probabilities will be generated by the energy function. In this way events are viewed as secondary concepts appearing on the surface of the model. The entire structure is evolved from the underlying dynamic provided by the energy function.

We begin with an example.

Introductory Example

N players compete in a golf tournament over 4 rounds. After 2 rounds there is to be a cut reducing the field by a half. An observer is given the list of all competitors and is asked to construct a model representing his beliefs about the prospects of each participant. The model is supposed to assign the probability of winning the competition to each player. Our observer is requested to produce two distributions: one prior to the commencement of play on the first day and one after the cut has been made at the end of the second round. Scores of all players will be available to him.

How should the model be constructed and how might it be constructed in practice ?

Ideally, a Bayesian observer would devise a prior distribution based on his knowledge about each competitor. He would then update all probabilities by some suitable function of scores in the first two rounds. If a particular player is eliminated at the cut his posterior probability is reduced to 0 and all remaining probabilities are normalised. A non-Bayesian would confine his assessment to those players who made the cut and would assign the probabilities according to the scores. He would, no doubt, refuse to commit himself before the first tee-shot.

In practice it seems doubtful that any observer would go through the pains of the above procedures. Consider the following simplified scheme.

Before the start of the competition our observer (we shall call him **O**) chooses a subset of size $n \leq N$, say A_n , of players he considers as main contenders. He then assigns probabilities $\left\{ p_1, \ldots, p_n \right\}$ to each one of them with $\sum_{i=1}^{n} p_i < 1$, and sets $P(A_n{}^C) = 1 - \sum p_i = p_0$. This gives his prior distribution.

After the second round he picks a new set of size k , say B_k , of all those players he still believes to be in contention. He then proceeds to assign new probabilities $\left\{ q_1, \ldots, q_k \right\}$ to each member of B_k using scores and his prior estimates as information. Again he ensures $\sum_{i=1}^{k} q_i < 1$ and $P(B_k{}^C) = 1 - \sum q_i = q_0$.

Thus the prior distribution is concentrated on ($n + 1$) points while the posterior is concentrated on ($k + 1$) points. However, if $n < N$, **O** cannot ensure that B_k is a subset of A_n or even that k is smaller than n . Therefore **O** is faced with a possibility that his posterior will be concentrated not only on points from A_n , but also on several points of $A_n{}^C$. Since **O** has treated $A_n{}^C$ as a "single point" he could be forced to add new points to his initial sample space.

When modelling beliefs of some individual we must use a theoretical structure flexible enough to cope with many complex situations. It would be nice to be able to adopt a Bayesian model in all circumstances, but in practice we may find its scope restrictive. An individual may be capable of expressing statements about uncertainty without adhering to any particular models or obeying any sets of axioms. If we tried to "stretch" his views to fit into some rigid framework we could easily distort his picture of reality.

For instance, in the above example, **O** may, quite possibly, turn out to be far less worried about coherence than we have previously assumed . He may, say, ignore all players outside A_n and effectively assign probability 0 to $A_n{}^C$ (i.e. take $p_0 = 0$). Nevertheless, when constructing B_k he may need to include some players from $A_n{}^C$, and he will have to assign positive probabilities to those players. In other words, events with

prior probability zero could end up with positive posterior probability. Bayesians and Entropists would definitely object to that!

The above example could be made more complicated if we removed the information about the original entry into the tournament. Suppose our observer **O** does not know either the size of the entry (N) or names of all competitors. His information may be partial: he knows a set of N_0 players definitely competing; he may speculate about some other entrants; but there is a subset of players he has never heard of. Under such conditions he cannot specify his sample space, but that need not stop him from expressing his opinion about chances of various players. He may well proceed using the earlier described analysis. After the half way cut his sample space will crystallise, but he may be forced to consider events which he never even listed in his prior model.

In our view modelling human beliefs using sample spaces and coherent probability measures as fundamental concepts can run into difficulties. The above observer, **O** , could often fail to conform to a whole set of axioms and still remain a successful predictor or gambler. And even should he turn out to be a disaster we may still wish to be able to model his beliefs.

An important question to consider is what precisely is it that an individual examines when faced with a problem like the one described above? Does he treat the victory of each competitor as an event and tries to estimate the plausibility of its occurrence? Or does he try to assess the potential of each competitor to become a future winner?

In our opinion a typical observer considers the latter problem. Thus his tendency would be to weigh the relative evidence pointing towards various players, and he would be less interested in quoting standard probabilities. We shall attempt to construct a new method for representing beliefs, which is more adaptive to a less rigid type of analysis. In our model we shall use a different primitive concept to describe uncertainty.

The *energy function* is the fundamental concept we shall employ. It will determine the structure of every model involving uncertainty. In particular it will generate the event

space. Thus no longer will it be a prerequisite to specify the sample space of a model. Events, which we will term "alternatives", will become secondary concepts as indeed will probability measures.

The definitions and basic properties of our method are described in the next section. Let us introduce this approach in loose terms by applying it to the above problem.

In order to help the reader to construct an intuitive picture of our philosophy we present just one more illustration.

Consider a smooth elastic surface in R^2 curved to create a number of "hills" and "valleys". A silver ball rolled accross this surface will move along various geodesics until it looses all its kinetic energy, gets caught inside the rim of one of the valleys, and is brought to rest by the gravity at the bottom.

We interpret the above physical picture in the following way. The curved surface is the energy function, denoted by E, which generates the observer's "uncertainty field". Gravity adds the natural gradient dynamic given by

$$\frac{dx}{dt} = -\frac{dE}{dx}$$

where x provides the local Euclidean measure and t refers to the time axis.

The valleys correspond to events or possible outcomes. The silver ball is interpreted as a dynamic random variable whose realisation is the particular "event-valley" in which it finally comes to a halt. The elasticity of the surface is viewed as dependence of the energy function E on "elasticity parameters" θ: Thus when we alter the shape of the surface by changing the parametrisation we affect the underlying structure of the model by moving, removing and adding "valleys" and "hills".

In our philosophy concepts like a sample space or an event are nothing sacred. The reader should realise that the surface described above comprises much more than an ordinary algebra of events. Only a subset of points on our surface can be identified with standard events. These points correspond to the "hearts of the valleys" where the silver ball

may come to rest. Other points can never be observed in the usual sense. But our formulation is dynamic, and therefore parameter induced earthquakes can destroy some valleys as well as create new ones. In such a context a traditional concept of a probability measure becomes almost irrelevant. Instead we consider the "attraction region" of each valley inside which a silver ball is trapped. A standard probability measure can be deduced from a more precise definition of an "attraction region" and will be discussed later.

We first summarise these ideas.

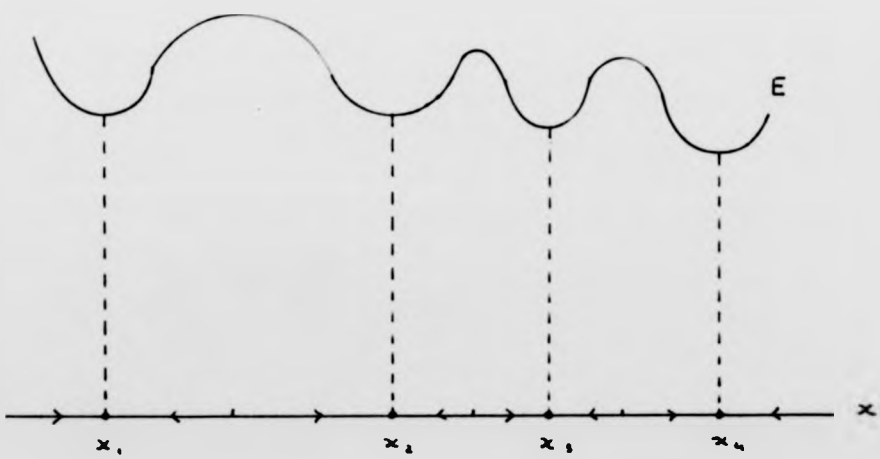
Consider a potential function defined on a real line as follows:

$$E: R \times V \to R$$

is a potential function on R parametrised by some $V \subset R^r$. We shall refer to E as an energy function and we shall use it to describe beliefs of an individual about any problem involving uncertainty. The event space and the probability measure are determined by E . "Possible outcomes" are defined to be points $x \in R$ corresponding to the minima of E . The associated probability measure is induced from the dynamic on R generated by E :

$$\frac{dx}{dt} = -\frac{dE}{dx} \qquad x \in R$$

Thus events are the stable equilibria of the dynamic. The probability of an event is proportional to the size of its basin of attraction. The shape of E is controlled by the parameter space V .



"Possible outcomes" = $\{x_1, x_2, x_3, x_4\}$

diag 2 · 1

In our example, the Bayesian observer is using an N - modal energy function to specify his beliefs. His posterior distribution is more concentrated: information has reduced the number of modes. O is unable to classify such vast amounts of information and his beliefs can be modelled by an energy function with only ($n + 1$) local minima. This energy function determines the prior event set Ω_1 containing ($n + 1$) points. The information provided by the first two rounds alters the shape of E , and, in particular, affects the location of the local minima. This gives rise to a new event set , Ω_2 , with ($k + 1$) points in it.

The energy function contains a complete picture of O 's beliefs. It can list the set of possible outcomes he considers at any point in time and evaluate the associated probabilities. It can cope with incoherence and changes in the event structure.

2.3 Basic Definitions

Throughout the rest of the chapter we will never go beyond the scope of the Euclidean spaces. The following concepts will be used repeatedly:

W = "World " Space. The largest domain we shall use. It can be thought of as a continuum which contains any sample or decision space mentioned. It is a smooth manifold, and, in one - dimensional cases, it will inevitably be represented by a subset of the real line. In general, the most complicated version of **W** will be of the form $W = \bigcup_{\alpha \in I} R_\alpha$ where $R_\alpha \subset R^{n_\alpha}$, for $n_\alpha \in Z^+$, $\alpha \in I$, some index set, is a differentiable manifold.
See section 2.4.2 for an example of a World Space.

Θ = Parameter (Control) Space, R^n , n as large as necessary.

T = "Time" Space, R^s , with s usually equal to 1, provides extra parameterisation.

Each of the above spaces will be equipped with a local Borel measure, which will be denoted in various ways as convenient.

Usual Euclidean topology can be defined locally for any subspace R_n of W. Thus $B_\epsilon(c)$ will denote a ball of radius ϵ around $c \in R_n$ and, more generally, a neighbourhood N_c will denote any connected subset of R_n containing c .

E = Space of Energy functions of the form:

$$E: W \times \Theta_E \times T \to R \tag{2.1}$$

Each $E \in$ **E** will be C^∞ at all points θ of Θ_E , subspace of $\boldsymbol{\Theta}$, and all $w \in$ **W** , $t \in T$. Thus E is a potential function in the sense defined in Chapter 1.

Define codimension of E to be the dimension of Θ_E .

$\frac{\partial^r E}{\partial x^r}(x,\theta)$ will denote the value of the rth derivative of E w.r.t. the local measure on **W** evaluated at the point $(x,\theta) \in \mathbf{W} \times \boldsymbol{\Theta}$.

We are going to describe situations involving uncertainty using models of the following structure.

Definition

A *model* is defined as a triple (**W**, E, T) where $E \in$ **E** . The dimension of the model is the codimension of E .

Thus, for instance, we will replace a standard probability model

$$(\Omega , A , P)$$

by

$$(\mathbf{W} , E , T)$$

E will determine Ω as a subset of **W** and will generate a probability measure on the algebra of events, A . T will give extra parametrisation to handle development of Ω and the change in the structure of P .

The dimension of the model introduces an equivalence relation on $\mathbf{E}$, but it is a weak concept in our context.

Definition

Define $\mathbf{A}_E$, the Space of Alternatives of the model $(\mathbf{W}, E, T)$, to be the set

$$A_E = \bigcup_{\theta \in \Theta_E} \bigcup_{t \in T} \left\{ x \in \mathbf{W} : \frac{\partial E}{\partial x}(x, \theta, t) = 0 \right\} \tag{2.2}$$

Thus $\mathbf{A}_E$ is the set of all fixed points of E w.r.t. the measure on $\mathbf{W}$ under any parametrisation (θ, t) of E . Trivially, $\mathbf{A}_E \subset \mathbf{W}$ for all $E \in \mathbf{E}$.

Definition

$x \in \mathbf{W}$ is a *stable alternative* if

$$\frac{\partial E}{\partial x}(x, \theta, t) = 0 \quad \text{for } all \quad \theta \in \Theta, \text{ for } all \ t \in \mathbf{T}$$

Definition

$x \in \mathbf{A}_E$ is *observable* if it is a stable equilibrium of the dynamic induced on $\mathbf{W}$ by E in some parametrisation $(\theta, t) \in \Theta_E \times T$.

In the same vein we can introduce two complementary concepts:

Definition

$x \in A_E$ is *unobservable* if it is an unstable equilibrium of the dynamic.

Definition

$x \in A_E$ is *transient* if it is not a fixed point of the dynamic in any parametrisation.

See section 2.4.2 for an intuitive illustration of these concepts.

Thus every $E \in \mathbf{E}$ determines a space of alternatives. However, this mapping is not injective and many E can lead to the same $\mathbf{A}$. It is worth our while to classify various spaces of alternatives without any reference to energy functions.

Definition

N_a is a proper neighbourhood of $a \in \mathbf{A}$ in $\mathbf{W}$ if N_a is a subset of $\mathbf{A}$.

Definition

$\mathbf{A}$ is discrete if no $a \in \mathbf{A}$ has a proper neighbourhood in $\mathbf{W}$.

Definition

$\mathbf{A}$ is locally continuous if there exists some $a \in A$ which has a proper neighbourhood in $\mathbf{W}$.

Definition

$\mathbf{A}$ is a piece-wise continuous if $\mathbf{A}$ is locally continuous at every $a \in A$.

Definition

$\mathbf{A}$ is continuous if it is a proper neighbourhood of every $a \in \mathbf{A}$.

Example: Gaussian Energy Function

Consider a model (R, $n(\theta,1)$, T) where the energy function

$$n: R \times \Theta \times T \to [0,1] \times T$$

has the Gaussian form

$$n_t(z_t,\theta_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_t - \theta_t)^2}, \qquad t \in T.$$

The space of alternatives turns out to be

$$A = \bigcup_{\theta \in \Theta = R} \left\{ z \in R : z = \theta \right\} = R, \qquad \text{for } all\ t \in T.$$

Thus T provides an extra dimension to the parameter space.

Analogous models can be defined for any unimodal density on R . Note that $x \in R$ is observable if it is a mode in some parametrisation.

Discrete densities are not differentiable, therefore their energy functions do not correspond to pdf's. However, they can always be defined by assigning values to unobservable points and demanding:

(i)

$$\frac{\partial E}{\partial \mu}(x;\theta,t) = 0 \qquad \text{for } x \in A_E$$

(ii)

$$E(x,\theta,t) = P_t(X = x;\, \theta)$$

An example and an alternative formulation is provided in the next section.

An analogous argument holds for decision spaces. Now energy functions take the shape of expected losses, utilities, etc. Spaces of alternatives correspond to decision spaces.

2.4 An Illustration: Energy Models for Bernoulli Trials

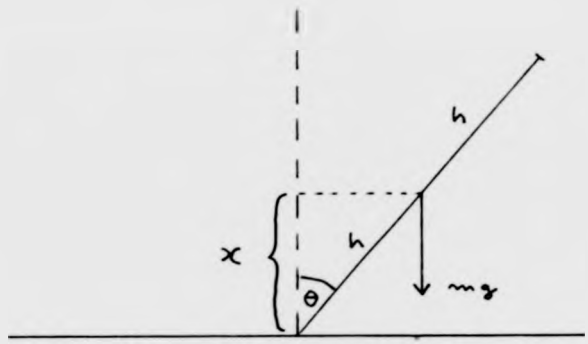
2.4.1 Introduction

In an attempt to construct models for discrete distributions we shall begin by considering a simple "coin-tossing" experiment. It will then be quite straightforward to extend the method to Bernoulli trials.

The motivation behind our approach has a direct physical basis in this case. Therefore, it is perhaps appropriate to treat discrete distributions as a starting point for our "energy interpretation" of probability.

2.4.2 A Model for a "Fair Coin"

Let us examine the physical shape of the coin. Like any object under gravity it will come to rest in a position minimising its potential energy.



diag 2·2

Clearly, the energy E depends only on the height of the centre of gravity, x, above the horizontal.

Assume, for convenience, $mg = 1$, where m = mass of the coin. Then the potential energy of the coin is given by

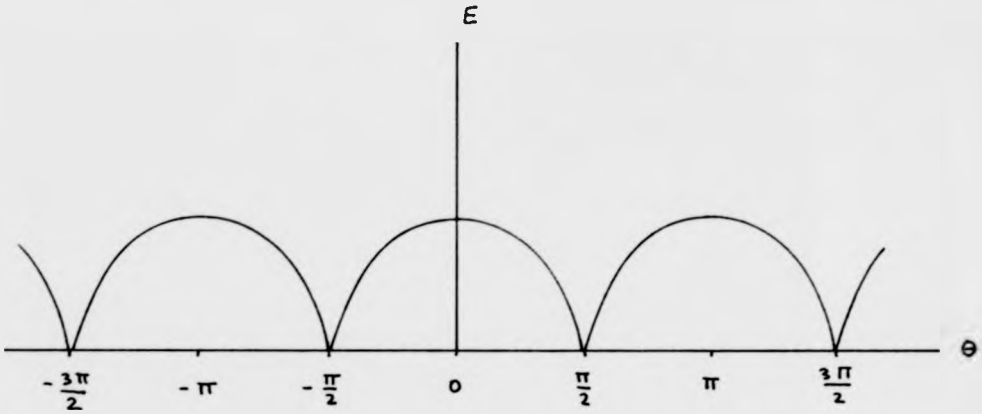
$$E = x$$

$$= h \cos\theta , \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

We can extend the definition of the angle θ to the whole real line by adding π for each half turn of the coin. This produces a more general form of the energy function

$$E = | h \cos\theta | , \quad \theta \in R$$

which looks as follows:



diag 2·3

Define

$$\frac{dE}{d\theta}\left(\frac{2n+1}{2}\pi\right) = 0\,, \quad n \in Z$$

Then the natural dynamic on θ induced by E and given by

$$\dot{\theta} = -\frac{dE}{d\theta} \tag{2.3}$$

gives

$$\theta = \frac{2n+1}{2}\pi\,, \quad n \in Z$$

as the minima of potential, and

$$\theta = n\pi\,, \quad n \in Z$$

as the maxima of potential.

The dynamic (2.3) can be described qualitatively by the phase portrait:

diag 2.4

We may call the equilibria states in the usual way:

$$\theta = -\frac{2n-1}{2}\pi \qquad \textit{Heads}$$

$$\theta = n\pi \qquad \textit{Edge}$$

$$\theta = \frac{2n+1}{2}\pi \qquad \textit{Tails}$$

Next we postulate that the space of alternatives of the coin consists of two observable states, "Heads" and "Tails", and one unobservable state, "Edge". Using the standard metric on the real line we can induce a probability measure on the space of alternatives determined by E .

Definition

Define the probability of occurrence of an equilibrium state to be proportional to the size of the basin of attraction of this state under the given dynamic.

Consequently, by enforcing Kolmogorov's axioms, we arrive at

$$P\left(\theta = -\frac{2n-1}{2}\pi\right) = P\left(\theta = \frac{2n+1}{2}\pi\right) = \frac{1}{2}$$

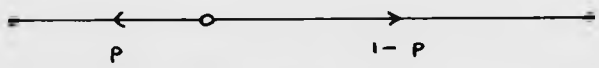
$$P(\theta = n\pi) = 0$$

Note that it is sufficient to use the interval $\left| -\frac{\pi}{2}, \frac{\pi}{2} \right|$ as the World Space W of the coin. In this case $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$ are the observable states, $\theta = 0$ is unobservable and other states are transient.

2.4.3 Generalisation to Bernoulli Trials

Consider a Bernoulli Trial with probability p of success. It would be natural to extend the idea of the energy function corresponding to such an experiment from the energy function of the coin. However, we no longer have the analogy of a physical object.

Suppose we adopt the opposite approach and start with the phase portrait. By a direct analogy it must look as follows:



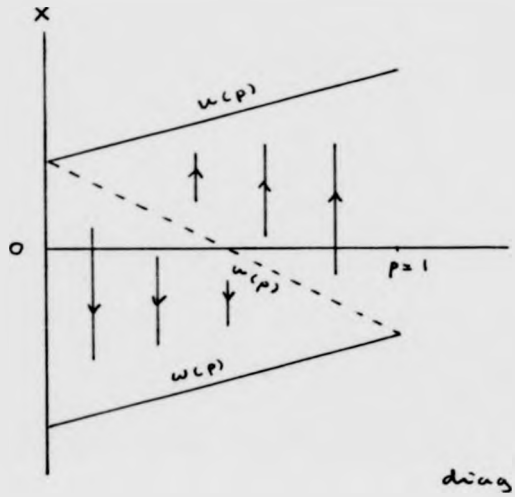
diag 2·5

It has two attractors, with respective basins of attraction of size p and $1 - p$ and one repellor dividing the two basins.

Let the World Space of the model be X. Let the equilibrium states (the alternatives) be given by

$$x = u(p) \quad , \quad minimum$$
$$x = v(p) \quad , \quad maximum$$
$$x = w(p) \quad , \quad minimum$$

Graphically, we could express them in the following way:



$$u(p) = \frac{p+1}{3}$$

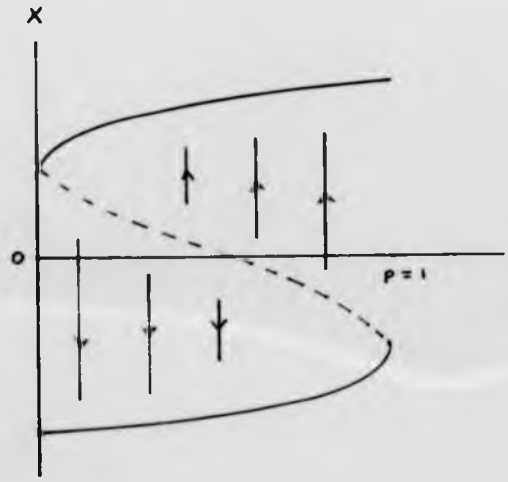
$$v(p) = \frac{1-2p}{3}$$

$$w(p) = \frac{p-2}{3}$$

diag 2.6

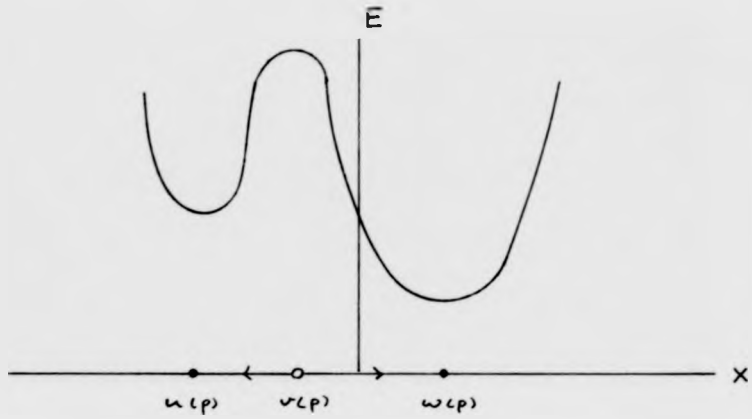
Note that $v(p)$ acts as a separatrix between observable (stable) states given by $u(p)$ and $w(p)$.

The graph can be smoothed and approximated by a cubic with the same properties.



diag 2.7

Since this is a graph of the stationary values of an energy function, the actual energy function will be equivalent to a quartic with two minima $u(p)$ and $w(p)$ separated by a maximum $v(p)$.



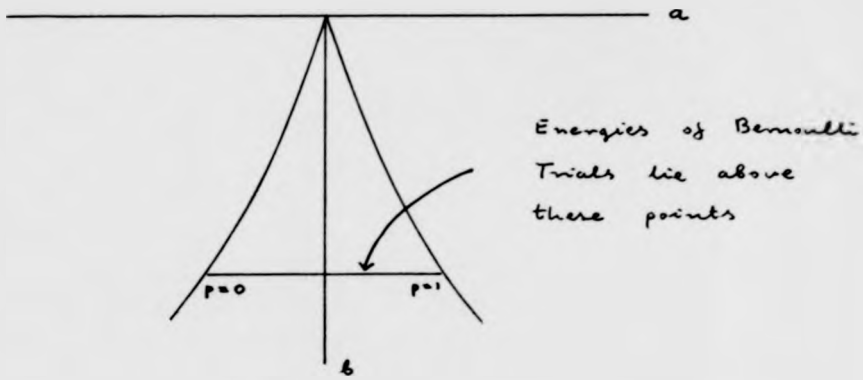
diag 2·8

An example of such a potential function, with attractors at 0 and 1, is presented in the next section.

2.4.4 Link with Canonical Cusp Catastrophe

As can be seen from earlier diagrams Bernoulli Trials have models behaving very much like a cusp catastrophe. In fact for a fixed positive value of the splitting factor, Bernoulli Trials can be described by a path joining the boundaries of the bifurcation set in the control space.

This can be seen on the familiar diagram.



diag 2.9

Thus p can be viewed as the normal factor with boundary conditions $P \in \{0,1\}$ ensuring at least two stationary values of E .

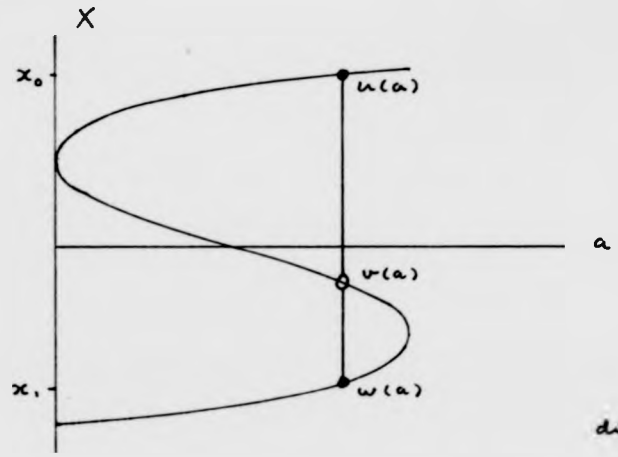
Let us leave the Bernoulli Trials for a moment and have a look at a number of applications of bimodal energy functions to decision problems.

First suppose we are modelling a decision problem using a cusp catastrophe potential with a fixed splitting factor $b = b_0 > 0$:

$$V(x) = \frac{1}{4}x^4 - \frac{1}{2}b_0x^2 - ax$$

Suppose the competing decisions lie in the neighbourhoods of $x = x_1$ and $x = x_0$. We are interested in the likelihood of a switch between the decisions as a enters the bifurcation set.

For each $a \in$ *Bifurcation set* define $u(a)$, $w(a)$ to be the values of x corresponding to the minima of V near x_0 and x_1 respectively, and $v(a)$ be the value of x corresponding to the maximum of V .



diag 2.10

Then define

$$p_a = \text{probability of a switch from } x_1 \text{ to } x_0 \text{ at } a$$

$$= \frac{u(a)-v(a)}{u(a)-w(a)}$$

Thus p_a increases with a as it traverses the bifurcation set left to right. We use the position of the separatrix to define a measure on the bifurcation set. In this way we generalise other switching rules used in similar situations:

1) Maxwell Rule, defined by

$$p_a = \begin{cases} 1 & \text{if } a > m \\ 0 & \text{if } a < m \end{cases}$$

where m satisfies $V(u(m)) = V(w(m))$.

2) Delay Rule, defined by

$$p_a = \begin{cases} 1 & \text{if } a \geq d \\ 0 & \text{if } a < d \end{cases}$$

where d lies on the right boundary of the bifurcation set.

Suppose we wish to model a decision situation when the two conflicting alternatives are stable and fixed at $x = 0$ and $x = 1$. One interesting model can be constructed as follows.

Let E be an energy function satisfying the differential equation

$$\frac{dE}{dx} = x(x - p)(x - 1), \qquad p \in R$$

Then

$$E(x) = \frac{1}{4}x^4 - \frac{1}{3}(p + 1)x^3 + \frac{1}{2}px^2 + constant \qquad (*)$$

It turns out that this potential has a number of interesting properties. The energy function can be mapped on to a canonical cusp catastrophe by substituting

$$y = x - \frac{1 + p}{3}$$

to give

$$E(y) = \frac{1}{4}y^4 - \frac{1}{2}by^2 - ay + constant$$

with

$$a = \frac{1}{27}(p + 1)(2p - 1)(p - 2) \qquad \textit{Normal factor}$$

$$b = \frac{1}{3}(1 - p + p^2) \qquad \textit{Splitting factor}$$

The energy function (*) has three stationary values at $x = 0, p, 1$. The fact that E pivots on $x = 0$ (since $E(0) = 0$ for all p) gives the equation an unbalanced look. We can restore the symmetry by adding a constant term $\frac{(1 - 2p)}{24}$.

The final form of the potential is

$$E(x) = \frac{1}{4}x^4 - \frac{1}{3}(p + 1)x^3 + \frac{1}{2}px^2 + \frac{1 - 2p}{24} \qquad (**)$$

Let us examine some properties of E viewed as a function of p .

Lemma 1.1

$$E(0) + E(1) = 0, \quad \text{for } all\ p$$

Thus the alternatives obey a certain kind of "conservation of energy" law.

Lemma 1.2

$$E(0) > E(1) \iff p < \tfrac{1}{2}$$

The global minimum of E has the larger basin of attraction. Using the Maxwell Rule the switch occurs at $p = \tfrac{1}{2}$.

Lemma 1.3

For $p \in [0, 1]$, $z = 0$ and $z = 1$ are the only alternatives.

At a first glance this result implies that we need only consider the family $\left\{ E_p(z) : 0 \leq p \leq 1 \right\}$ to model our decision problem.

Lemma 1.4

When $p = 0$, $z = 0$ becomes an inflection point and $z = 1$ remains the only observable. When $p = 1$ the roles are reversed.

The Delay Rule commands us to switch only when the current preference is no longer available. Other rules are also possible. We discuss some of them in 2.5.

Let us examine the behaviour of E when p lies outside the $[0, 1]$ interval.

Lemma 1.5

When $p < 0$, $z = 0$ becomes a maximum. A new minimum emerges at $z = p$. Similarly, when $p > 1$, $z = 1$ becomes a maximum.

Lemma 1.6

E is symmetrical for three values of p : $p = -1$, $p = \frac{1}{2}$ and $p = 2$. The respective axes of symmetry go through $z = 0$, $z = p$ and $z = 1$.

Proof

E is symmetric $\Leftrightarrow$ $a = 0$ $\Leftrightarrow$ $p = -1, \frac{1}{2}, 2$. Axes of symmetry follow by Lemma 1.5.

So the roles of the stationary points can be interchanged. The Maxwell switching points correspond to the axes of symmetry.

In a practical context the decision maker would need to relate the value of p to his information. His actions would be determined by his choice of the switching rule and the relationship

$$p = I(t)$$

where, in general, $I : T \to R^r$ (r an integer) is a bijective map which we will refer to as the *information function* . In the case with two alternatives the range of I is one-dimensional.

Definition

Information function is said to be *bounded* if its range is homeomorphic to $[0, 1]^r$.

Theorem 1

Let E be given by (**), $p = I(t)$, I information function.

Then E determines a set of stable alternatives , $A_E = \left\{ 0, 1 \right\}$ $\Leftrightarrow$ I is bounded.

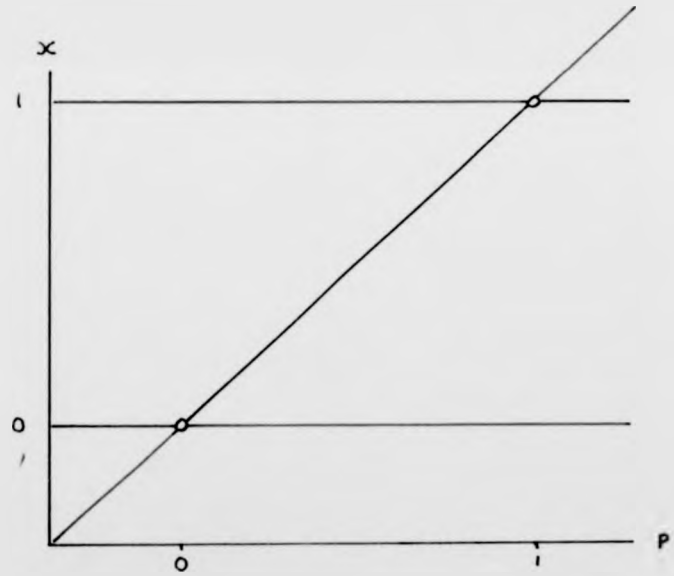
Proof

Follows directly from Lemma 1.3.

Mathematically the result is not a great revelation, but its implications for modeling decision problems are quite exciting. The main negative inference from the theorem is that unbounded information inevitably leads to unstable alternatives. Intuitively this means that a decision maker who cannot ensure deterministic information is in no position to list his options.

Potential functions can be used in situations when sample or action spaces are difficult to specify in advance. This method should also be applicable in predictive models to deal with outliers.

The behaviour pattern of the stationary points of E in the bimodal case is pictured below:



local minimum

local maximum

o inflection point

diag 2·11

The above diagram illustrates a common phenomenon which has thus far been largely ignored. An innocuous binary decision problem is determined by the behaviour of a single control p . Whenever the value of p lies inside the $[0,1]$ region the problem is trivial. Our model offers a facility to deal with the situation when the information fails to conform and falls outside the $[0,1]$ interval. A new option $x = p$ emerges whenever $p \notin [0,1]$. Lemma 1.6 predicts when this new alternative becomes optimal under the Maxwell Rule. The most important aspect of our model is that the Decision Maker is able to alter his action space according to the information received and is not constrained by an erroneous "a priori" choice.

To complete the mapping of E on to the canonical cusp catastrophe we have to increase the dimension of the parameter space.

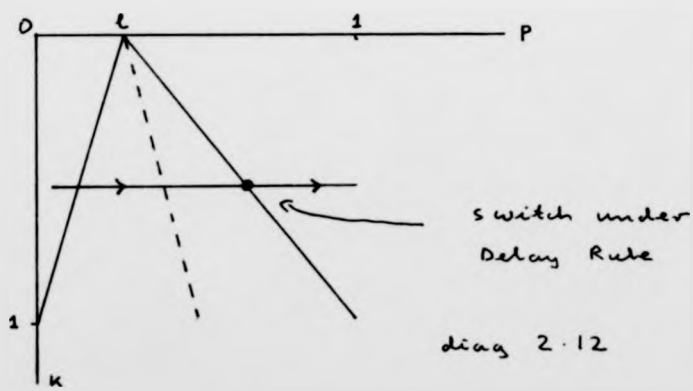
To do this consider the first derivative of the extended energy function given by

$$F'_l(x) = x[(x - l) - \frac{p-d}{k}](x - 1), \qquad l \in [0,1]$$

where $F_l: X \times C \to R$ with $C = (p, k)$. $k \in (0,\infty)$ is the scale parameter and $l \in [0,1]$ is the location parameter. For each $l \in [0,1]$, F_l is homeomorhic to the canonical cusp catastrophe.

Intuitively k affects the switching rule between the alternatives $x = 0$ and $x = 1$ and l induces a bias towards either option.

The control surface C is presented below. Its main characteristic is the straight boundary of the bimodal region.



diag 2·12

The bifurcation set is enclosed by the lines $p = l(1 - k)$ and $p = l(1 - k) + k$ with $(p = l, k = \lim_{\epsilon \to 0} \epsilon)$ as the cusp point. Traversing across the bifurcation set on a path parallel to th p - axis a decision maker using a Delay Rule switches sooner for small values of k . As k approaches 0 the switch is almost instantaneous.

The presence of the location parameter l implies that, strictly speaking, our model is a section of the butterfly catastrophe with l as the bias factor and no access to the third mode.

The "dual" energy function $-F_l$ can be applied in testing. For instance, consider a quality control model given by

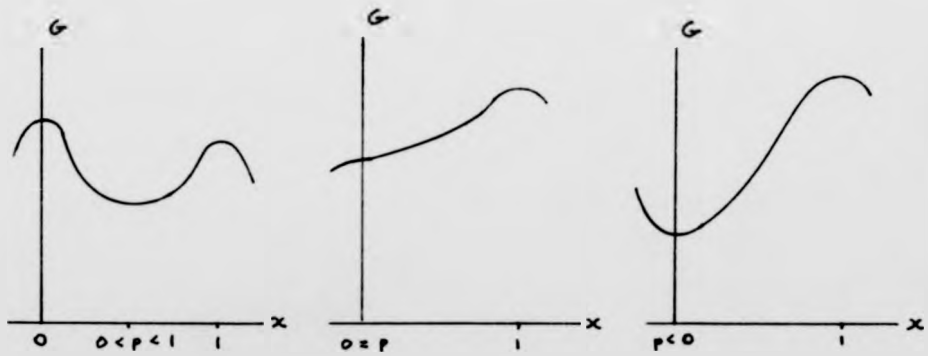
$$G'_l = -F'_l = -x[(x-l) - \frac{p-l}{k}](x-1)$$

where the two maxima at $x = 0$ and $x = 1$ correspond to accepting and rejecting a tested batch. The unique minimum at $x = p$ can be interpreted as a Likelihood Ratio statistic in a sequential test with

l = prior belief about batch quality;

k = risk factor.

Whenever the test ratio hits the boundary of $[0,1]$ the decision to accept/reject is taken.



Case $l = 0$, $k = 1$: as p decreases past 0, the decision $x = 0$ is taken.

diag 2.13

The potential function E in (**) can evolve in yet another direction. Suppose we replace E by

$$H'(x) = x(x - p)^3(x - 1)$$

Then H retains the topological characteristics of E . But if we perturb the middle term of the above equation to end up with

$$B'(x) = x[x - (p - \epsilon)][(x - p)][x - (p + \epsilon)](x - 1)$$

where $0 \leq \epsilon \leq p \leq 1$,

then the energy function B exhibits three local minima whenever $\epsilon > 0$. A full version of B unfolds like the butterfly catastrophe.

The above model can be applied to decision situations with imprecise information. For instance, using our earlier interpretation of p as an information function, in certain situations

$$p = I(t)$$

could turn out to have a stochastic structure and only a region $[p - \epsilon, p + \epsilon]$ could be specified as a range of I . Under such conditions a decision maker choosing between options $x = 0$ and $x = 1$ may defer his decision if ϵ is large enough, effectively "sitting on the fence" at $x = p$.

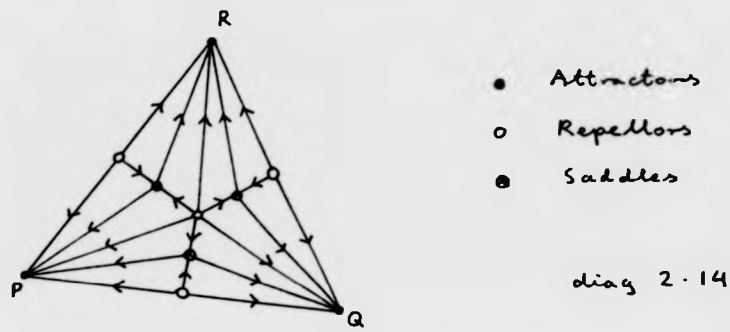
The same energy function B provides an interpretation for a betting scheme with "upper" and "lower" probabilities. A gambler is indifferent between bets on $x = 0$ and $x = 1$ when the odds fall in the region $[p - \epsilon, p + \epsilon]$. Thus ϵ measures the *indeterminacy* of the gambler's beliefs.

In the next section we state the natural extension of Theorem 1 to the multimodal case.

2.4.5 General Discrete Sample Space

Phase portraits can be used to construct models for a general discrete distribution. We restrict ourselves to a countable number of observables. If, in fact, this number is finite, say n, then we represent the phase portrait, generated by the energy function, by a $(n - 1)$ - simplex.

Take the case $n = 3$, for instance. One possible phase portrait looks as follows:



diag 2 · 14

Analogously, the probability of an event is proportional to the area of its basin of attraction. Thus only attractors are observable.

In higher dimensions the phase portraits become more difficult to draw. The principles still remain the same.

It is always possible to construct an energy function for a given $(n - 1)$ - simplex. This is proved in general in the next section. Let us first look at the restricted case when the space of alternatives is a constant function of time. We can devise an energy function by extending the potential (**).

Consider the differential equation

$$\frac{\partial V}{\partial x}(x) = x(x - p_1)(x - 1)(x - p_2)\ldots(x - p_{n-1})(x - (n-1))$$

where $n \geq 3$ is an integer. Then $\left\{ 0, 1, \ldots, n-1 \right\}$ is the set of stable alternatives.

Define

$$I: T \to R^{n-1}$$

by

$$I(t) = (p_1 = p_1(t), \ldots, p_{n-1} = p_{n-1}(t))$$

Then the statement of Theorem 1 can be extended to

Theorem 2

V generates a set of stable alternatives, $A_V = \left\{ 0, 1, \ldots, n-1 \right\} \iff I$ is bounded.

Thus the information determines the complexity of the space of alternatives.

2.4.6 Formal Conclusions

It has proved more natural to use the basin of attraction in defining measures on discrete spaces rather than the values of energy function at observable points. Since the basic components of the model $(\mathbf{W}, E, T)$ do not include a measure we are free to define it as we like. It is important to remember that it is the energy functions that determine measures and not vice versa.

Let us go back for a moment to a standard probability model for a discrete sample space and try to induce its associated energy function. Let Ω be a sample space with a finite number of atoms $\omega_1, \ldots, \omega_n$. Suppose there exists a probability measure P on Ω satisfying Kolmogorov's axioms. Assume that the support set of P contains exactly m atoms of Ω .

Then we can construct an energy function for this model using the following result:

Theorem

Let Δ_m be a (m-1)-simplex with vertices in Ω defined by:

$$(x_1, \cdots, x_m) \in [0,1]^m$$
$$\sum_{i=1}^{m} x_i = 1$$

Put $\omega_i = (0,\ldots,1,\ldots,0) \in \Omega$. Then there exists a function

$$E: \Delta_m \to R$$

with the following properties:

(i) E is C^∞ on the interior of Δ_m and right continuous and infinitely right differentiable on the boundary of Δ_m

(ii) There exists an injection $b: \Omega \to S$, where

S = set of stationary values of E, i.e. the natural dynamic:

$$\dot{x} = -\frac{dE}{dx}$$

has all elements of Ω as its fixed points.

(iii) Each vertex of Δ_m is in the support set of P . Thus $\omega \in \Omega$ is observable $<=> P(\omega) > 0$.

(iv) For each $\omega \in \Omega$,

$$p(\omega) \propto A(\omega)$$

where $A(\omega)$ = volume of the basin of attraction of ω .

Proof

It is sufficient to construct one potential function possessing the required properties.

Consider

$$V: \Delta_m \to R$$

defined by

$$V(x_1, \ldots, x_m) = 1 - \left(\sum_{i=1}^{n} c_i x_i^2\right)$$

Then V is minimised only on the vertices ω_i of Δ_m . Therefore V induces the required type of dynamic on Δ_m . The sizes of the basins of attraction can be adjusted by the choice of constants c_i .

2.4.7 Comments

(i) Energy models can easily be constructed for discrete spaces of alternatives. In some simple cases there is a nice physical interpretation not only for observables and unobservables, but also the World Space continuum containing them (c.f. angle θ a coin makes with the vertical).

(ii) "Dynamics" and "basins of attraction" are concepts previously never associated with probabilities. Perhaps they can be of value.

(iii) Catastrophe Theory is once again employed to define an underlying continuous structure beneath the surface of a discrete situation.

2.5 Another Illustration: Perception and Uncertainty

Subjectivists often use gambles to elicit statements about probability. The energy approach provides another method of elicitation.

Consider an individual asked to choose between two outcomes $X = 0$ and $X = 1$ modelled by a Bernoulli Trial

$$X = \begin{cases} 0 & w.p.\ p \\ 1 & w.p.\ 1 - p \end{cases}$$

The value p is to be viewed as a summary of the individual's beliefs. It is an information function and its value is supposed to help the individual to commit himself to one of the two alternatives. The structure of this information function is analogous to the one described on page 60. The ivdividual must assess the value of p in order to decide which outcome is more likely.

In our formulation let X denote the real line and $X = 0$, $X = 1$ are the two alternatives.

Let $\mathbf{E}$ be a family of potential functions generating all Bernoulli Trials. Thus any $E \in \mathbf{E}$ is of the form

$$E: X \times [0,1] \to R$$

with the following properties:

(i) For each $0 < p < 1$

$$E_p : X \to R$$

has exactly 3 stationary values on $[0,1] \subseteq X$: local minima at $X = 0$ and $X = 1$, local maximum at $X = p$;

(ii) $E_0 : X \to R$ has exactly one local minimum at $X = 1$;

(iii) $E_1 : X \to R$ has exactly one local minimum at $X = 0$.

The interval $[0,1]$ forms the World Space of any Bernoulli Trial with $X = 0$ and $X = 1$ as the observable outcomes and $X = p$ as the unobservable event.

Definition

Let $E \in \mathbf{E}$. A point $a \in [0,1]$ is called a *Maxwell point* if

(i) $E_a(0) = E_a(1)$

(ii) $E_z(0) > E_z(1)$ if $z < a$

and $E_z(0) < E_z(1)$ if $z > a$

Suppose we model individual's beliefs by a potential function $E \in \mathbf{E}$. We ask two basic questions:

(a) For what values of p does the individual back the event $X = 1$?

(b) Given an initial value of p , say p_0 , how does the individual react to changes in p ?

The answer to (b) is of a greater interest. At this stage we are not going to look at the issues involved in the updating of p . We shall assume that p is tractable (at least to the individual himself) and we can model individual's perception of p by a dynamic of the form

$$\dot{p} = q(t) \tag{2.4}$$

Each separate dynamic may lead to different behaviour patterns. At any time t the energy function of the individual is

$$E_{p(t)} : X \to R$$

where $p(t)$ is a solution of (2.4).

The only other thing we need to know is the decision mechanism resulting in a choice between $X = 0$ and $X = 1$.

Definition

An *action function* associated with a dynamic $p(t)$ is defined by

$$A(t;\, p(t)) = \begin{cases} 0 & \text{if } X = 0 \textit{ is chosen at time } t \\ 1 & \text{if } X = 1 \textit{ is chosen at time } t \end{cases}$$

Definition

A *switching point* is a discontinuity of A.

Any individual's behaviour can be completely described by a triple

$$(\, E,\, p(t),\, A(t;\, p(t))\,)$$

Let us look at several possible behaviour patterns. We assume that a given individual has reacted in a certain fashion to a situation involving uncertainty, and we have been able to model his response using the following energy function:

$$E_p(x) = \frac{1}{4}x^4 - \frac{1}{3}(p+1)x^2 + \frac{1}{2}px^2 + \textit{constant}$$

with $0 \le p \le 1$.

Note that E_p has a Maxwell point at $p = \frac{1}{2}$.

(a) Let

$$A(t) = \begin{cases} 0 & \text{if } t > \frac{1}{2} \\ 1 & \text{if } t \le \frac{1}{2} \end{cases}$$

and let the dynamic be of the form

$$p = t \qquad 0 \le t \le 1$$

Maxwell point coincides with the switching point here. This model represents what may be termed the "rational action". The individual minimises energy by switching to the more plausible outcome at the

earliest opportunity.

(b) More general action function is of the form

$$A(t) = \begin{cases} 0 & \text{if } t > p_0 \\ 1 & \text{if } t \leq p_0 \end{cases}$$

The dynamic is as in (a). This time the individual switches at an arbitrary time. So certain bias is introduced on his information. The same effect can be achieved with the action function in (a) and by either

(i) choosing E_p with a Maxwell point at p_0 ;

(ii) changing the dynamic to $p(t) = 2p_0 t$.

In other words either the energy function has a bias, as in case (i), or the rate of flow of information is different to the outsider.

(c) Delayed action.

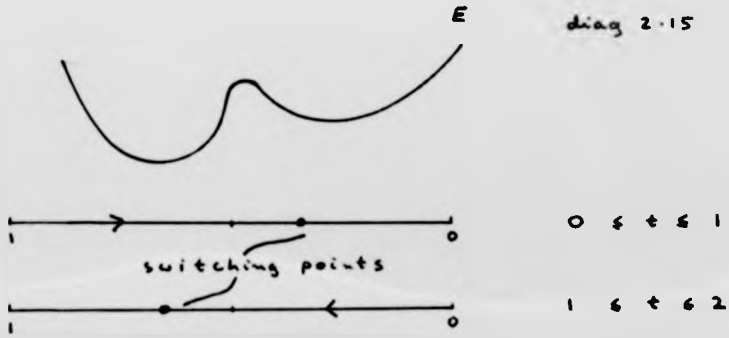
Consider a dynamic given by

$$p(t) = \begin{cases} t & \text{for } 0 \leq t \leq 1 \\ 2-t & \text{for } 1 \leq t \leq 2 \end{cases}$$

and the action function defined by

$$A(t) = \begin{cases} 1 & \text{if } t < \tfrac{1}{2}+\epsilon \\ 0 & \text{if } \tfrac{1}{2}+\epsilon \leq t < \dfrac{3}{2}+\epsilon \\ 1 & \text{if } t \geq \dfrac{3}{2}+\epsilon \end{cases}$$

The switch occurs well past the Maxwell point as p progresses in either direction.



Our individual is slow to acknowledge fresh information. He continues to back his current choice for a longer period than his information suggests. The extreme case ($\epsilon = \frac{1}{2}$) is to stick with one local minimum for as long as it exists.

Energy functions can be used to model various types of human responses when faced with uncertainty. It is possible to represent almost any form of behaviour. The question remains whether it would be possible to determine what particular energy function is suitable in a given situation.

It is worth stressing that the energy approach does not presuppose any particular probability structure. For instance, case (a) above naturally leads to a simple additive method. But in case (c) it seems more appropriate to formulate an upper and lower probability model to explain the delayed adaption pattern. Here the basic notion upheld by Walley and Fine (39) of a difference between the "buying price" and "selling price" is well exemplified (see also section 2.2 paragraph (vi)) .

Elicitation remains a poorly examined part of Probability Theory. This analysis is an attempt to direct attention to the possibility of a dynamic foundation of the basic concept of probability. A successful interpretation is quite likely to come from that direction.

2.6 Updating Problems

Thus far the "time" axis T has not played any significant role in our analysis. Now we shall take $T = R$ and write energy equations in terms of $t \in T$.

Consider a model $(\mathbf{W}, E, T)$ written as

$$E(\ x(t),\ \theta(t)\ ) = 0$$

and

$$A_t = \bigcup_{\theta \in \Theta_a} \left\{ x \in W\colon \frac{\partial E}{\partial x}(x(t), \theta(t)) = 0 \right\}$$

Derivative w.r.t. the measure on $\mathbf{W}$ gives the "horizontal" structure on $\mathbf{W} \times \Theta \times T$.

Writing x and θ as functions of t we can obtain the general form of the "vertical" structure by considering

$$\frac{\partial E}{\partial x}\,\dot{x}(t) + \frac{\partial E}{\partial \theta}\,\dot{\theta}(t) = 0 \tag{2.5}$$

Suppose, in general, this equation can be solved by

$$\dot{\theta}(t) = u(x,\theta,t) \tag{2.6}$$

$$\dot{x} = v(x,\theta,t) \tag{2.7}$$

Let us examine several desirable properties we would like these solutions to have.

(i) It seems dangerous to assume $\dot{x}(t) = 0$, i.e. $A(t)$ independent of t , yet whenever A is a sample space or a decision space this very assumption is inevitably made. In decision theory in particular such an attitude can be a gross oversimplification. Surely the options open to a decision maker often change and it is not always possible to define a decision space large enough to contain all the choices. Conversely, some options may vanish and it is not always clear that the utility structure can be "smoothly" altered to accommodate this fact.

(ii) Orthogonally to the plane

$$\dot{x} = v(x,\theta,t)$$

or under assumption $\dot{x}(t) = 0$, the change in θ ,

$$\dot{\theta}(t) = u(x,\theta,t)$$

is considered as the main aspect of modelling. We are going to refer to it as "updating". The form of $u(x,\theta,t)$ is not clear in general. Intuitively, it seems desirable that

$$\dot{\theta}(t) \;\alpha\; \Delta(\ information\ ) \tag{2.8}$$

and therefore, it is vital to define the concept "information" precisely.

(iii) Our meaning of "information" we use is slightly different to the one currently found in the literature. Fisher's information is a measure of sharpness of a distribution. Williams' entropy approach identifies new

information with extra constraints on his existing distribution. We think of information as an impulse altering the energy structure of a model. In particular this corresponds to a vector in a Θ -space. Thus "motion" is our equivalent of information. Clearly, inherently the updating function must be a function of information.

$$u(x,\theta,t)$$

simply translates the information into a definite movement in the parameter space. Concepts like "discounting" can be defined within this formulation.

Example: One-Parameter Exponential Family

Consider a distribution taken from the exponential family.

$$f(\omega \mid \theta_1,\theta_2) \propto \exp\left\{\theta_1\omega + \theta_2 a(\omega)\right\}$$

$\omega \in R$ is the parameter of interest, $\theta = (\theta_1,\theta_2)$ is the hyper-parameter space. f is a conjugate prior for a random variable Y with a distribution

$$f(y \mid \omega) \propto \exp\left\{\omega y + b(y)a(\omega)\right\}, \quad \text{for } y \in R$$

Then given the outcome $Y = y$ the updating function for $\theta \in R^2$ is given by

$$u_y(\theta_1,\theta_2) = (\theta_1 + y, \theta_2 + b(y))$$

An analogous updating procedure can be employed using our formulation. The interpretation of all the maps used is different from the usual one, even though the energy function E happens to be algebraically equivalent to f. The updating equation given above can be viewed as a description of the way in which the shape of the energy function alters our time. Of course, in distinction to the probabilistic equation, we are not constrained only to use this updating rule since our structure is looser than the traditional one. In this example it must be remembered that the energy function which has the mathematical form of the exponential density is in fact defined on the $R \times \Theta$ space and

must not be confused with a density. We put

$$E_1(\omega,\theta,t) = f(\omega;\theta), \quad \text{for } all\ t \quad with\ \theta = (\theta_1,\theta_2)$$

Thus $A_t = R$ for all t and $\dot{\omega}(t) = 0$. The vertical structure is given by

$$(\dot{\theta}_1(t), \dot{\theta}_2(t)) = \left(1, \frac{db(t)}{dt} \right)$$

with initial conditions

$$\theta(t_0) = (\theta_{10},\theta_{20}), \quad b(t_0) = 0$$

Hence

$$\theta_1(t) = t + \theta_{10}$$
$$\theta_2(t) = b(t) + \theta_{20}$$

where $t \in T$ is a realisation of another system with a distribution

$$E_2(t|\omega) = f(t;\omega) \ \alpha \ \exp\left\{ \omega t + b(t)a(\omega) \right\}$$

Here the structure of E_1 is sensitive to realisations of E_2. Note that the space of alternatives of E_1 becomes a parameter space of E_2 whenever Bayesian type updating is considered.

Any "law of motion" along the $\mathbf{W} \times \mathbf{\Theta}$ space is a function of information. But not all information has to be of the form in the above example. It need not be created by an interaction between two energy systems. For instance, vague prior information falls into the latter category.

2.7 Aggregation

2.7.1 Introduction

In recent years a lot of attention has been given to the so called "aggregation problem". In both probability theory and decision theory issues of amalgamation of beliefs or group decision making have at last been tackled.

In probability theory the basic question can be phrased as follows:

(P) Let P and Q be two separate measures on some sample space Ω yielding values P(A) and Q(A) for some $A \in \mathbf{A}$, algebra of events on Ω . Does there exist an "aggregate measure" R = function of P and Q on Ω , and if so, what is the precise functional relationship and in particular the value R(A)?

An analogous question in decision theory could sound like this:

(D) If individuals D_1 and D_2 choose d_1 and d_2 as their respective optimal decisions for the same problem (B_i, L_i) , i=1,2, on a decision space D, where (B,L) represent their belief and loss structure, then what decision

$$d^* = \textit{function of } d_1 \text{ and } d_2$$

will they make together?

2.7.2 An Overview of Recent Approaches

French (45) has recently published a paper outlining most modern methods in aggregation. He has omitted to mention the last one in the following list:

(1) Bayesian Approach - an "investigator" treats expert opinions as data. Usually log odds are assumed to have Normal distributions. For details see French (42,43,45), Lindley (47), Morris (49,50), Winkler (53,54).

(2) Linear Opinion Pool Method - the traditional weighted averaging of experts probabilities or log odds. Propagated by McConway (48).

(3) Stochastic Approach - Each expert updates his beliefs by other expert opinions. Matrix of all their beliefs converges under certain assumptions. See De Groot (41).

(4) "Non-Additive" Approach - Based on interval type elicitation aggregation measures retain many of the basic properties of fuzzy probabilities. They additionally obey certain desirable criteria. Developed by Walley (55).

In addition to listing and describing the main approaches French (45) has finally attempted to specify the actual problem faced in aggregation. He came out with these

basic types:

(a) Expert Problem - an external aggregator does the assessing;

(b) Group Decision Problem - the full set of experts is responsible for the final outcome;

(c) Textbook Problem - a group is asked to produce a joint probability assessment for an unspecified purpose.

From a mathematical point of view (a), (b) and (c) appear structurally equivalent. The differences must then be philosophical. Unfortunately none of the authors seem to be aware of French's classification. Their main weakness seems to be the failure to specify the actual problem. Since, in the end, each one of them is working on a different set of assumptions, it is hardly surprising when they come up with different solutions. For example, consider the disagreement about the Marginalisation Property: Lindley (47) and McConway (48) working under different assumptions come up with opposite conclusions.

Once again it is felt that certain amount of rigour and specification of fundamental concepts would not come amiss in the aggregation dispute.

We wish to present a more general approach. As usual we shall try to erect a dynamic structure flexible enough to include any of the older approaches as a special case. In many ways the methods listed above have been too specific. They may well have produced adequate results for specific situation, but have never provided a general answer to the aggregation question.

2.7.3 The Energy Approach

Let us begin by specifying the elements of the problem in our language.

Let $E_i(x,\theta_i,t) = 0$ be the energy equations of n separate models

$$(W, E_i, \Theta_i \subseteq \Theta) \quad i = 1,\ldots,n.$$

Note that

(i) E_i specifies a space of alternatives A_i , for each i.

(ii) $\theta_i \in \Theta_i$ are control spaces of each E_i , not necessarily of the same codimension in R^k.

(iii) Each E_i is additionally parametrised by the same space T.

The $\left\{E_i\right\}$ are said to be *aggregatable* if there exists a continuous 1-1 map

$$a: \Theta_1 \times \cdots \times \Theta_n \to \Theta^*$$

and some energy function $E \in \mathbf{E}$

$$E: W \times \Theta^* \times T \to R \tag{2.9}$$

s.t. A_E , the space of alternatives determined by E, contains $\bigcup_{i=1}^{n} A_i$ as a subset.

That is the most general scheme for aggregating within the framework of spaces of alternatives. Several points are worth noting:

(i) any previously used method is a special case of above;

(ii) the problem reduces to determining the map a and in particular the dimension of Θ^* ;

(iii) although we provide a general guideline we are no nearer finding the map a , if indeed such a map exists uniquely.

In Chapter 4 we are going to be more specific and attempt, in some degree, to solve the problem in the decision theoretic framework (D).

2.8 Conclusions

In this chapter we have tried to describe problems involving measurable spaces from a slightly different perspective. The energy approach is not designed to provide quick mathematical techniques for various branches of statistics - although one particular energy function described above has proved to be very useful in certain practical applications. In general our aim is to reformulate all the fundamental statistical concepts. This

does not imply that our approach has to be used in all circumstances. The purpose of presenting it here is two-fold:

1. To introduce a common denominator and framework for interpreting probability and decision problems;

2. To provide a starting point for an investigation of updating and aggregation.

The last few sections were concerned with a brief outline of this approach. At present it is not claimed that purpose 2 has been achieved. However, we believe that some initial ground work has been done. There are many directions for improvement: the dynamic set up presented above is deterministic; extra parametrisation can provide basis for stochastic extensions.

It is vital that problems of complexity in the aggregation dispute are properly formulated and embedded within some comprehensible framework. All past attempts seem to be disconnected and suspended in the air. The main object of this discussion is to put problems of updating and aggregation on a firm ground.

3. Asymmetric Mixture and Catastrophes

3.1 Introduction

In his Ph.D. theses J. Smith (24) has proved that a mixture of two identically shaped but differently located expected loss functions has the topology of the back-to-back cusp catastrophe whenever the expected loss functions are of a certain type. This class includes the Normal expected loss constructed from Normal beliefs and a conjugate normal loss. We shall refer to Smith's model as the "symmetric mixture" to underline the basic characteristic of his set up.

It seems only natural to attempt to generalise this result to the case when the expected losses are not identical. We shall only look at a very mild extension: both components will still be of the same type, but they will have different scale parameters. Such a problem seems to be of a greater practical interest as we are more likely to encounter decision situations with each participant having either a different variance or a different tolerance for losses. As we shall show even this very gentle perturbation of Smith's assumptions leads to many complications. The problem evolves dramatically. In some special cases the equations of the cusp point are at least one degree higher than in the original problem. We shall present the geometric view of the situation, and solve the Smith problem using this method.

In the main section of this chapter we shall prove that the existence and uniqueness of the cusp singularity depends on exactly one condition. To arrive at that result we will first examine the properties of the derivatives of the expected loss function. Later we will show that the Normal expected loss always satisfies the condition in question, and therefore the cusp singularity occurs. In another example, using a polynomial function, we will show that the condition, although satisfied, can lead to other solutions than those intuitively expected. It has been impossible to prove that this condition is an inherent property of T-type functions.

The fact that a unique cusp point exists does not necessarily help in finding its exact location. In the general case there is not enough data to find the cusp point, while in the Normal case the equations to solve are extremely complicated. No doubt they can be solved using numerical methods. Luckily, in the polynomial example, the coordinates of the cusp point can be found explicitly.

Finally, we look at the relation between mixtures and the cuspoid family of catastrophes. In particular, we point out the natural embedding of a 2-component mixture within the Butterfly catastrophe.

3.2 Type T functions and their properties

3.2.1 Definitions

The type T function $E(\theta,p)$, where p is the scale parameter, is defined as follows:

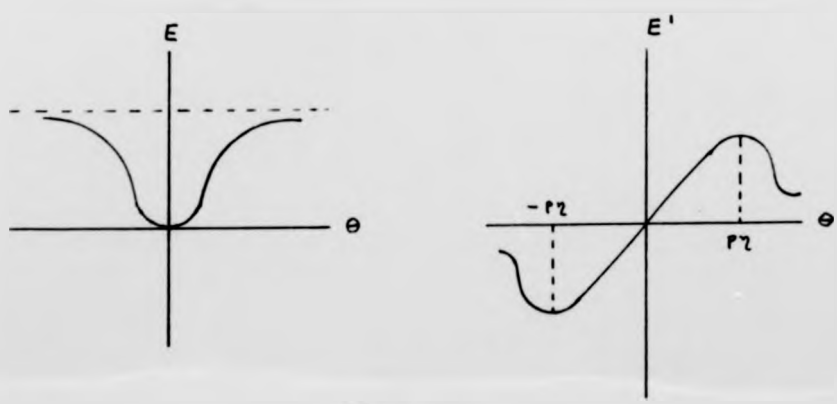
For all $p > 0$, $E(\theta,p)$ is C^{∞} , generic, symmetric about $\theta = 0$, strictly increasing with $|\theta|$, $\lim_{\theta\to\infty} E(\theta,p) = 1$, and

(i) E'' has a unique zero in $(0,\infty)$ at $p\eta$;

(ii) E''' has a unique zero in $(0,\infty)$ at $p\lambda$;

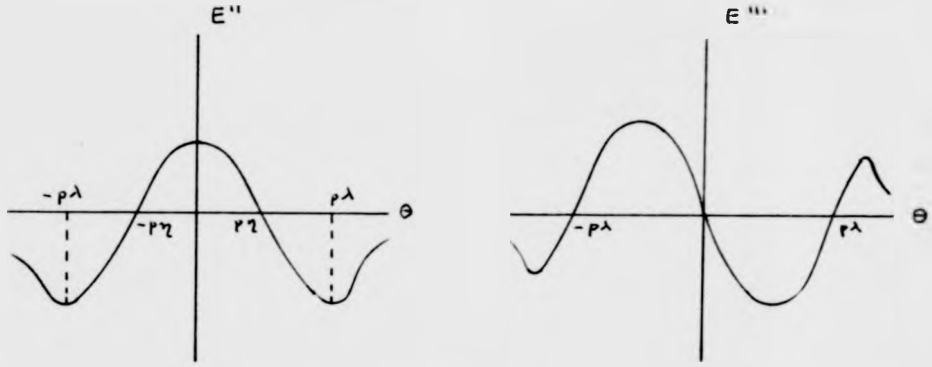
(iii) $(E'''/E')((0,p\eta)) \cap (E'''/E')((p\eta,p\lambda)) = \phi$

Consequently, E and its derivatives must look as follows:



diag 3·1 a

The second and third derivatives of E can be seen below:



diag 3.1 e

Example

The inverted Normal

$$E(\theta,p) = 1 - \frac{k^2}{p^2} \exp\left\{ -\frac{\theta^2}{2p} \right\}$$

where k is a constant, is of type T with $\eta = \frac{1}{\sqrt{p}}$ and $\lambda = \sqrt{\frac{3}{p}}$. See section 3.4.2 for a polynomial example.

We shall require to use two other functions of derivatives of E, which are defined by

$$R(\theta,p) = \frac{E'''(\theta,p)}{E'(\theta,p)} \tag{3.1}$$

$$S(\theta,p) = \frac{E''(\theta,p)}{E'(\theta,p)} \tag{3.2}$$

Let us examine the properties of each of these functions. Their behaviour in the vicinity of $p\eta$ and $p\lambda$ is crucial for our analysis.

3.2.2 Properties of $R(\theta,p)$

R1 R is symmetric about $\theta = 0$ (since both E' and E''' are antisymmetric about the origin).

R2

$$R(\theta,p) < 0 \quad \text{for} \quad 0 = \theta < p\lambda$$
$$R(p\lambda,p) = 0$$
$$R(\theta,p) > 0 \quad \text{for} \quad \theta > p\lambda$$

R3

$$\lim_{\theta \to 0} R(\theta,p) = c(p) < 0$$

and

$c(p)$ is an increasing function of p with

$$\lim_{p \to \infty} c(p) = -0$$

Proof Follows from continuity of R . Since $E(\theta,p)$ is increasingly "flat" as p increases by definition, $c(p)$ is also strictly increasing and approaches 0 from below.

R4 $\theta = 0$ is a local minimum of $R(\theta,p)$, for all p .

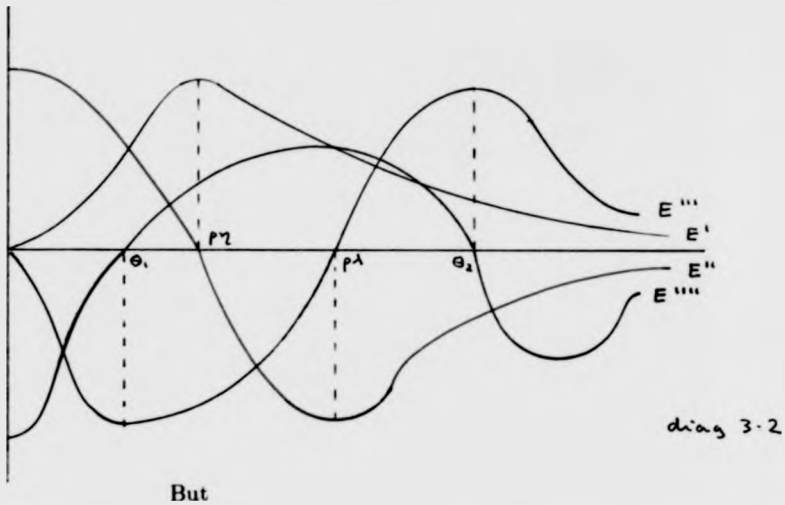
Proof Trivial by R1 and R2.

R5 R is strictly increasing on $(0,\theta_2]$, where θ_1 and θ_2 are positive roots of $E''''(\theta,p) = 0$ with $\theta_1 < \theta_2$.

Proof *Step* 1 . Consider the region $[min(\theta_1,p\eta), \theta_2]$

$$R'(\theta,p) = \frac{E''''(\theta,p) - R(\theta,p)E''(\theta,p)}{E'(\theta,p)}$$
$$< 0 \iff E''''(\theta,p) - R(\theta,p)E''(\theta,p) > 0$$

The first four derivatives of E can be seen below.



diag 3·2

But

(i) $\theta_1 < p\lambda < \theta_2$ *as* $E'''(p\lambda,p) > 0$:

(ii) $R'(\theta,p) > 0$ *as* $E'''(\theta_1,p) = 0$ and R and E'' have opposite signs at θ_1 ;

Case $\theta_1 < p\eta$:

(iii) $R'(\theta,p) > 0$ on $[p\eta,\theta_2]$ since E''' is increasing and E' is decreasing there;

(iv) $\theta_1 < \theta < p\eta \Rightarrow E'''(\theta,p) > 0$

and $R(\theta,p)\,E''(\theta,p) < 0 \Rightarrow R'(\theta,p) > 0$.

Case $\theta_1 > p\eta$:

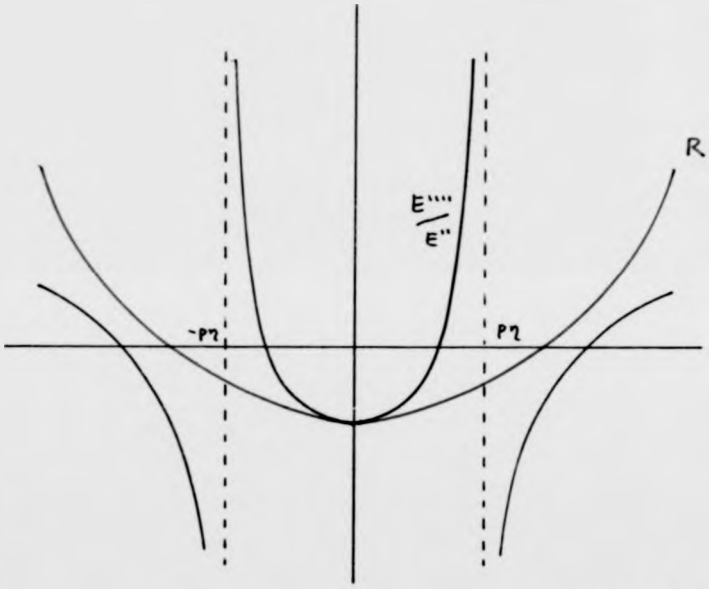
(v) For $p\eta < \theta < \theta_1$, E''' is decreasing slower than E' since E'''' is increasing and E'' is decreasing. Thus $R'(\theta,p) > 0$.

Step 2 Consider region $(0,\theta_1)$.

$$R''(\theta,p) = 0 \iff \frac{E''''(\theta,p)}{E''(\theta,p)} = R(\theta,p)$$

Therefore any turning point of R for $\theta > 0$ must lie on

$(E''''/E'')(\theta,p)$. Consider the shape of E''''/E'' . It is strictly increasing on $(0,\theta_1)$. Thus R would have an even number of turning points to make sure $\theta = 0$ is a local minimum of R . But then not all turning points of R could lie on E''''/E'' . Thus R touches E''''/E'' at $\theta = 0$ and never meets it again in $(0,\theta_2)$.



diag 3·3

Hence $R''(\theta,p) > 0$ on $(0,\theta_2)$.

R6 Let $q < p$.

Then $R(\theta,q)$ and $R(\theta,p)$ meet at a unique $\theta = \mu_R(p/q)$ with

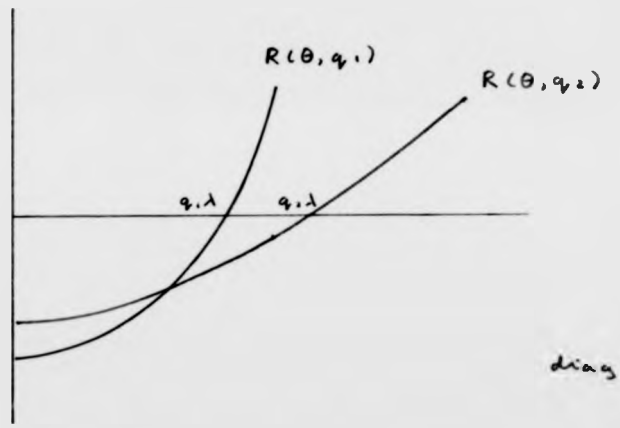
$$0 < \mu_R(p/q) < q\lambda$$

and

$$\lim_{p \to \infty} \mu_R(p/q) = q\lambda$$

Proof Follows by R2 and R3.

We are now in position to sketch the pair of functions $R(\theta,q)$ in the region $[0,\theta_2]$:



diag 3.4

The behaviour of R for $\theta > \theta_2$ is of minor interest to us. In fact, if E is of exponential type R will be always increasing, but if E is a polynomial R may have another turning point.

3.2.3 Properties of $S(\theta,p)$

S1 S is antisymmetric about $\theta = 0$.

Proof E'' is symmetric and E' is antisymmetric.

S2

$$\lim_{\theta \to 0} S(\theta,p) = +\infty$$

S3

$$S(\theta,p) > 0 \quad \text{for } \theta < p\eta$$
$$S(p\eta,p) = 0$$
$$S(\theta,p) < 0 \quad \text{for } \theta > p\eta$$

S4 S is strictly decreasing on $(0,p\lambda)$

Proof Consider the region $(0,p\eta)$. E'' is decreasing and E' is increasing $\Rightarrow$ S is decreasing. Consider region $(p\eta,p\lambda)$.

$$S'(\theta,p) = R(\theta,p) - [S(\theta,p)]^2 < 0$$

since $R(\theta,p) < 0$ on $(p\eta,\lambda p)$ by R2.

S5 $S'(\theta,p)$ is an increasing function of θ, for all p.

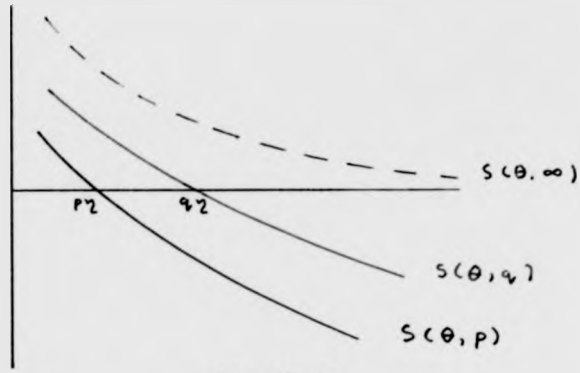
S6 Let $p < q$. Then $S(\theta,q)$ and $S(\theta,p)$ never meet in $(0,p\lambda)$, and

$$S(\theta,q) > S(\theta,p), \text{ for } all\ \theta \in (0,p\lambda)$$

S7 The family of curves $\left\{ S(\theta,p) : p \geq 1 \right\}$ is bounded from above by $S(\theta,\infty)$ defined by

$$S(\theta,\infty) = \lim_{p \to \infty} S(\theta,p)$$

See the diagram below.



diag 3·5

S8 Let p < q. Then $S(\theta,p) = -S(\theta,q)$ has a unique root in $(0,q\lambda)$ at

$$\theta = \mu_S(p/q)$$

with

$$q\eta \le \mu_S(p/q) \le p\eta, \quad \text{for } all\ p \ge q$$

In the same way as R , the behaviour of S for $\theta > p\lambda$ depends on whether E is exponential or polynomial. In the former case S has no further turning points, but in the latter case S will have another minimum and approach 0 thereafter.

3.3 The Main Problem

3.3.1 The Model

Our attention will be focused on the following model.

Let $E(\delta,p)$ be of the type T. Consider the mixture

$$E^* = \alpha E(\delta+\mu, p_1) + (1-\alpha)E(\delta-\mu, p_2) \tag{3.3}$$

with $0 \le \alpha \le 1$, $\mu > 0$ as control parameters and p_i as the coefficients of spread.

Since only the relative sizes of p_1 and p_2 are relevant we shall concentrate on the case $p_1 = 1$ and $p_2 = p$.

Of course, by putting $p_1 = p_2 = p$ we reduce the problem to Smith's Theorem. We shall refer to the general case as the "asymmetric mixture".

Our main strategy will be to search for cusps in the topology of the mixture. A mixture is an example of a potential function and the importance of energy functions in statistical modelling cannot be overstated. Consider, for instance, the problem involving randomised and mixture decision rules. Smith (25, pp.20-1) discusses the shape of the Pareto boundary in the multiperson decision-making context. Having pointed out that this boundary need not, in general, be convex Smith lists conditions under which a set of mixture rules posesses a concave Pareto boundary. Under the same conditions Smith shows that randomised decision rules can occur if and only if a mixture rule exhibits a catastrophe.

Another application of energy function is described by Zeeman, Harrison *et al* (29). In order to gain a fuller understanding and be in a position to interpret his model Zeeman searches for a cusp in the behaviour surface of his potential function. Both the qualitative and quantitative properties of his model are dependent on the existence and the location of the cusp point. The topology of mixtures is of vital importance in many statistical applications. Our aim is to describe a more general model than Smith's, which we believe has more practical importance.

3.3.2 Review of the Smith Method

We now state the result and proof of the simple case by adapting Smith's Theorem contained in Smith (24) to our notation.

Smith's Theorem

Let $E(\delta,1)$ be of the type T and be written simply $E(\delta)$. Then

$$E^{*} = \alpha E(\delta+\mu) + (1-\alpha)E(\delta-\mu) \tag{3.4}$$

has a unique cusp at $(\delta, \alpha, \mu) = (0, \frac{1}{2}, \eta)$.

Proof

If a cusp occurs at $\delta = D$, then the first three derivatives of E^* vanish at that point giving

$$AE'(\mu+\delta) + E'(D-\mu) = 0 \tag{3.5.1}$$

$$AE''(\mu+\delta) + E''(D-\mu) = 0 \tag{3.5.2}$$

$$AE'''(\mu+\delta) + E'''(D-\mu) = 0 \tag{3.5.3}$$

$$\textit{where} \quad A = \frac{\alpha}{1-\alpha}, \ \mu > 0.$$

But E is symmetric about 0, and from diagram 1 it follows that

$$AE'(\mu+\delta) = E'(\mu-D) \tag{3.5.4}$$

$$AE''(\mu+\delta) = -E''(\mu-D) \tag{3.5.5}$$

$$AE'''(\mu+\delta) = E'''(\mu-D) \tag{3.5.6}$$

$E(\delta)$ has no stationary points in the region $\left\{D: |D| > \mu\right\}$.

Properties (i) and (ii) of type T functions imply $\eta < \lambda$ (refer to diagram 1).

Property (i) and equation (3.5.5) imply

$$\begin{aligned} \mu - D &\leq \eta \\ \mu + D &\geq \eta \end{aligned} \tag{3.5.7}$$

Property (ii) and equation (3.5.6) imply

$$\mu + D < \lambda \tag{3.5.8}$$

Thus

$$\mu - D \in (0,\eta) \tag{3.5.9}$$
$$\mu + D \in (\eta,\lambda)$$

Divide (3.5.6) by (3.5.4)

$$R(\mu - D) = R(\mu + D) \tag{3.5.10}$$

Therefore property (iii) and equations (3.5.9) imply

$$\mu - D = \mu + D = \eta$$

Therefore $D = 0$ is the necessary condition for a cusp in $E^{*}(\delta)$, and

$$\mu = \eta$$

at the cusp point.

Now (3.5.4), (3.5.5), (3.5.6) become

$$(A - 1)E'(\mu) = 0 \tag{3.5.11}$$
$$(A + 1)E''(\mu) = 0 \tag{3.5.12}$$
$$(A - 1)E'''(\mu) = 0 \tag{3.5.13}$$

But $E'(\mu) > 0 => A = 1 => \alpha = \frac{1}{2}$ But $A + 1 > 0$ therefore $E''(\mu) = 0 => \mu = \eta$ as required.

Therefore the cusp occurs at

$$(\delta, \alpha, \mu) = (0, \tfrac{1}{2}, \eta)$$

By the above method it is easy to first of all restrict the possible location of the cusp, and then pin point it using the third property of type T functions.

3.3.3 The Asymmetric Mixture using the Smith Method

There is no reason to suspect that the asymmetric mixture given by equation (3.3) should be of a different topological type than Smith's special case. We shall therefore search for a cusp point in it. Initially we apply the Smith method.

Consider

$$E^{*}(\delta) = \alpha E(\delta+\mu,1) + (1-\alpha)E(\delta-\mu,p) \tag{3.6}$$

where $E(\theta,q)$ is of type T.

If $E^{*}(\delta)$ has a cusp at $\delta = D$, then

$$AE'(\mu+D,1) + E'(D-\mu,p) = 0 \tag{3.7.1}$$

$$AE''(\mu+D,1) + E''(D-\mu,p) = 0 \tag{3.7.2}$$

$$AE'''(\mu+D,1) + E'''(D-\mu,p) = 0 \tag{3.7.3}$$

where $A = \dfrac{\alpha}{1-\alpha}$, $\mu > 0$. But $E(\theta,p)$ is symmetric at 0, for all p. Thus

$$AE'(\mu+D,1) = E'(\mu-D,p) \tag{3.7.4}$$

$$AE''(\mu+D,1) = -E''(\mu-D,p) \tag{3.7.5}$$

$$AE'''(\mu+D,1) = E'''(\mu-D,p) \tag{3.7.6}$$

Search for cusps only in $\left\{ D: |D| \leq \mu \right\}$.

Property (i) of type T functions and equation (3.7.5) imply

$$\begin{aligned} \mu + D &> \eta \\ \mu - D &< p\eta \end{aligned} \tag{3.7.7}$$

Property (ii) and equation (3.7.6) imply

$$\mu + D < \lambda \tag{3.7.8}$$

We must distinguish between two cases:

(i) $\lambda > p\eta$

Then one of the following holds:

(a)

$$\eta < \mu - D < \mu + D < p\eta \tag{3.7.9}$$

(b)

$$\mu - D < \eta < \mu + D < \lambda$$

(c)

$$\eta < \mu - D < p\eta < \mu + D < \lambda$$

(ii) $p\eta > \lambda$

Then one of the following holds:

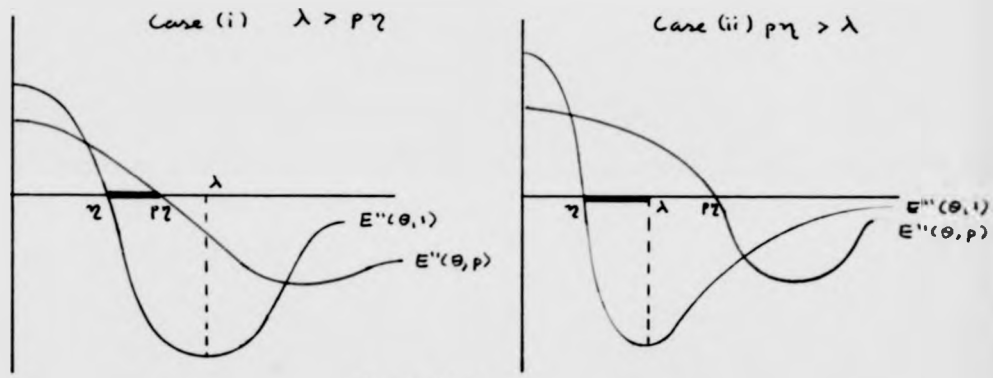
(a)

$$\eta < \mu - D < \mu + D < \lambda \tag{3.7.10}$$

(b)

$$\mu - D < \eta < \mu + D < \lambda$$

This difference can be seen on the following diagrams.



diag 3·6

The green region of the θ - axis signifies the possible placement of the $\mu - D$ and $\mu + D$ values.

So, summarising

$$\eta < \mu + D < \lambda \tag{3.7.11}$$
$$\mu - D < min(\lambda, p\eta)$$

These inequalities are insufficient to pin point the μ and δ coordinates of the cusp. Also, using (3.7.6) divided by (3.7.4), i.e.

$$R(\mu + D, 1) = R(\mu - D, p) \tag{3.7.12}$$

does not help, since for $p \neq 1$ we cannot apply property (iii) and this equation has many solutions.

Therefore Smith's approach fails to find the exact placement of the cusp. Neither does it disprove the existence of one.

3.3.4 The Geometric View of the Asymmetric Mixture

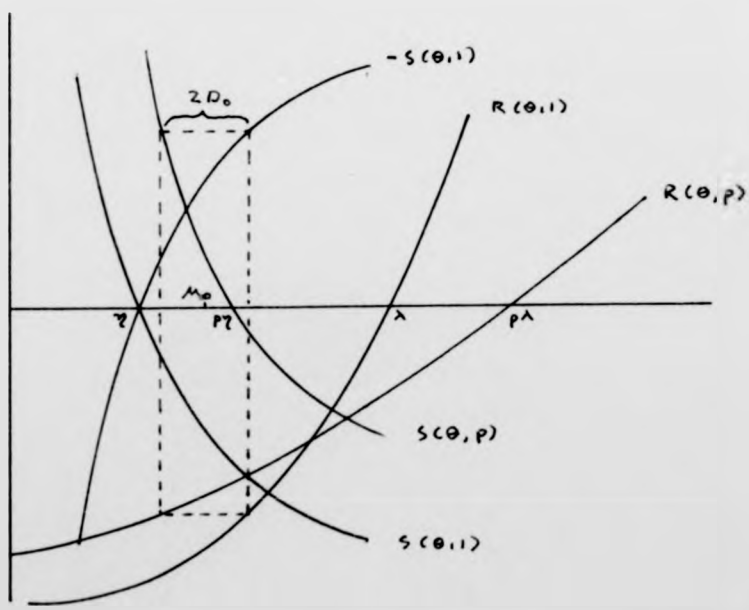
If the expected loss function given by (3.6) has a unique cusp point it must satisfy the set of equations (3.7.4), (3.7.5), (3.7.6) whether or not Smith's method works. Let us examine these equations again.

Dividing (3.7.6) by (3.7.4) and (3.7.5) by (3.7.4) we obtain

$$R(\mu+D,1) = R(\mu-D,p) \tag{3.8r}$$

$$S(\mu+D,1) = -\ S(\mu-D,p) \tag{3.8s}$$

If the cusp point exists at some (μ_0,D_0) it must satisfy both of these equations simultaneously. From the properties of R and S discussed earlier we can produce a geometric view of the situation. The diagram 3.7 is the central part of our argument. Together with the analogous diagram of Smith's special case it underlines the relative complexity of the general problem and the restricted case.



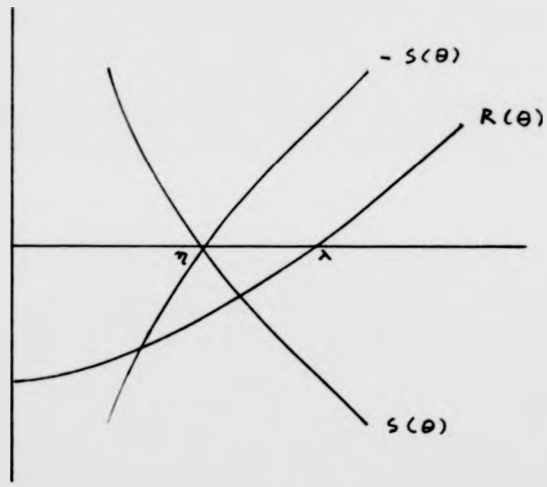
diag 3·7 : Case $\lambda > p\eta$

The system of equations (3.8) has a solution (μ_0, D_0) if it is possible to fit the corners of the rectangle width $2D_0$ to lie on the four relevant curves as pictured above.

We can use this method to solve Smith's special case. System (3.8) reduces for $p = 1$ to

$$\begin{aligned} R(\mu+D) &= R(\mu - D) \\ S(\mu+D) &= -S(\mu-D) \end{aligned} \tag{3.9}$$

This gives the following diagram.



diag 3.8

$$\begin{aligned} R(\mu+D) &= R(\mu - D) \\ S(\mu) &= -S(\mu) \end{aligned}$$

which gives a unique cusp at $(\delta = 0, \mu = \eta)$ for $\theta \in [0,\infty)$. Similarly for $\theta < 0$ we get another cusp at $\mu = -\eta$.

The geometric approach gives a quick solution to Smith's problem. We are now ready to tackle the general case.

3.3.5 The Existence and Uniqueness Theorem

Using the geometry of R and S functions we can determine the necessary and sufficient conditions for the existence of a unique cusp point in the topology of the asymmetric mixture.

Lemma 1.1

$\mu_S(p)$ is a continuous and increasing function of p, for all $p \geq 1$.

Proof

Follows directly from S8: $q\eta \leq \mu_S(p/q) \leq p\eta$, for *all* $p \leq q$.

Lemma 1.2

$\mu_R(p)$ is a continuous and increasing function of p, for all $p \geq 1$.

Proof

The minimum of R, $c(p)$, is an increasing function of p by R3. $R(\theta,p)$ is an increasing function of p for each $\theta \in [0,\lambda]$. Since $R(\theta,1)$ is also increasing on that region, the intercept $\mu_R(p)$ is increasing. The continuity of $\mu_R(p)$ follows by R6:

$$0 < \mu_R(p) < \lambda, \quad \lim_{p \to \infty} \mu_R(p) = \lambda$$

Theorem 1

Consider the system

$$E^*(\delta,p) = \alpha E(\delta+\mu,1) + (1-\alpha)E(\delta-\mu,p)$$

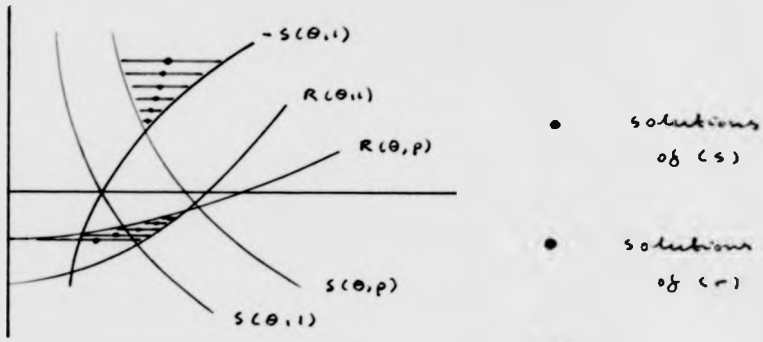
for a fixed $p > 1$, E of type T, $\mu > 0$, $0 \leq \alpha \leq 1$.

Then E^* exhibits a cusp catastrophe with a unique cusp point over $(\delta, \alpha, \mu) \Longleftrightarrow$

$$\mu_R(p) \geq \mu_S(p) \qquad \text{(M)}$$

Proof

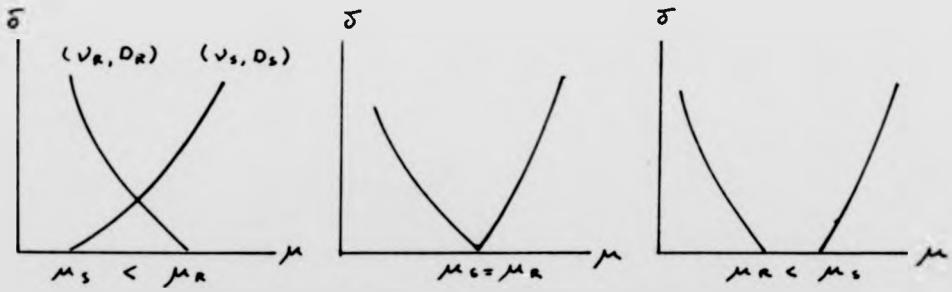
Consider solutions of equations (3.8r) and (3.8s) separately in the plane (μ,δ).



diag 3-9

The solutions of (s) are $(\nu_S(p)\,,\,D_S(p))$. They exist only for $\nu_S(p) > \mu_S(p)$ and $D_{S(p)} > 0$. $\nu_S(p)$ is an increasing function of $D_S(p)$.

Similarly, the solutions of (r) are $(\nu_R(p)\,,\,D_R(p))$. They exist only if $\nu_R(p) < \mu_R(p)$ and $\nu_R(p)$ is a decreasing function of $D_R(p)$.



diag 3·10

Therefore a common solution exists $<=>$ $\mu_R \geq \mu_S$, and it is unique.

Corollary 1.1

The system (3.6) exhibits a cusp at $D = 0$,

$$\alpha = \frac{E'(\mu_0,p)}{E'(\mu_0,p) + E'(\mu_0,1)}$$
$$<=> \mu_R(p) = \mu_S(p) \quad with \quad \mu_0 = \mu_R(p)$$

Proof

Put $\delta = 0$, $\mu = \mu_0 = \mu_R(p)$ into (3.7.4) for the α -coordinate of the cusp point.

The condition (M) acts as a discriminant on the class of type T functions. It is not possible to prove from the given specifications whether or not (M) is an intrinsic property of this class. Neither is it obvious that (M) is independent of p . We may well have three subclasses among type T functions:

(i) T_M - If $E \in T_M$ then (M) holds for all $p \geq 1$;

(ii) $T_{M,P}$ - If $E \in T_{M,P}$ then (M) holds for all $p \in P \subset [1,\infty)$;

(iii) $T_{M,I}$ - If $E \in T_{M,I}$ then (M) holds for $<=> p = 1$.

Clearly, $T_M \subseteq T_{M,P} \subseteq T_{M,I} \subset T$, but nothing more strict can be induced in general.

We know that T_M is not empty as it includes the Normal mixture.

Another issue to be resolved is the behaviour of the asymmetric mixture over the full (μ , α , p) control space.

Corollary 1.2

The cusp point coordinates ($\delta = D$,$\mu = m$, $\alpha = a$) are continuous in p .

Proof

It is sufficient to prove continuity at $p = 1$. The result follows from continuity of solutions of (3.8r) and (3.8s) and Lemmas 1.1 and 1.2.

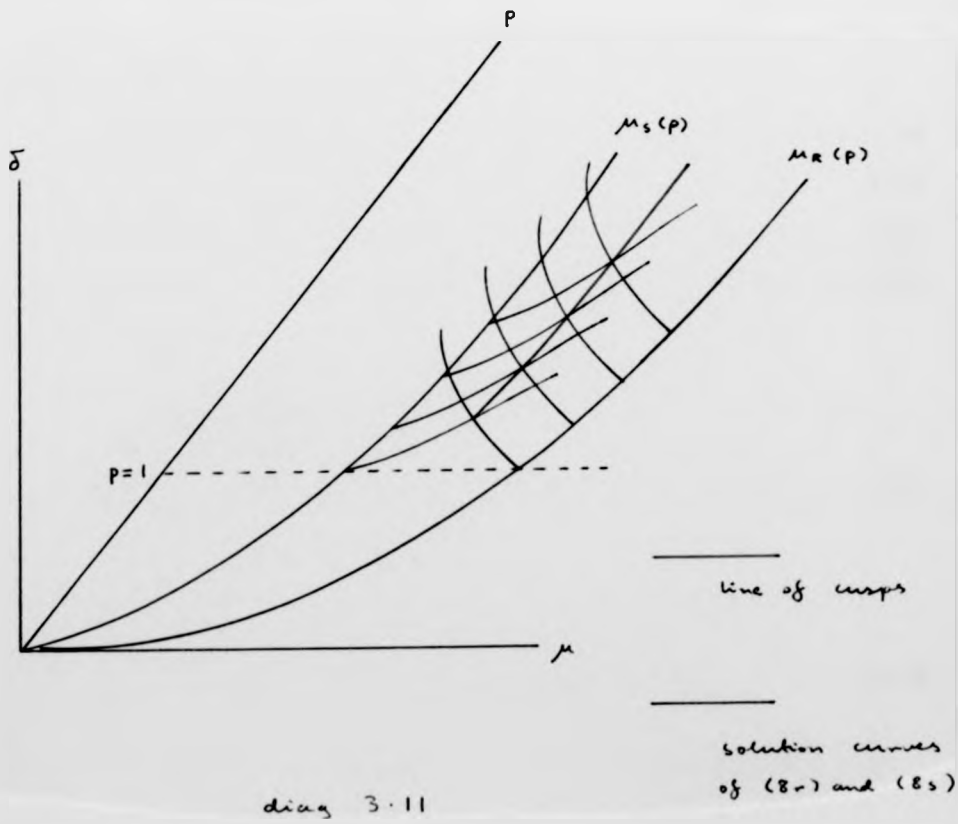
Corollary 1.3

The system (3.6) exhibits no higher order catastrophes than cusps over (μ , α , p) control space.

Proof

Let first $E \in T_M$. By Theorem 1 $E^*(\delta,p)$ exhibits a unique cusp in each p - section of (μ , α , p) . Corollary 1.2 states that the progress of the cusp points in the p-direction is continuous.

In order to display higher order catastrophes some section of the control space would have to have at least two isolated cusps in it. If $E \notin T_M$, then E^* can behave no worse.



diag 3·11

Above we picture the control space section (μ, p) and the behaviour axis δ for the system (3.6) and $E \in T_M$. The line of cusps is continuous and can never bifurcate. No other cusps can emerge at any other point. Thus there is no possibility of higher order catastrophes. Yet with three dimensions we would expect them. This leads us to

Theorem 2

The system (3.6) can be embedded in a control space of a Butterfly catastrophe by a projection

$$(\mu, \alpha, p) \to (\mu, \alpha, c = f(p^{1}, \alpha, \mu), d = d_0 < 0)$$

where f is some continuous function increasing in p^{1}.

The last result says that an asymmetric mixture is basically a section of the Butterfly catastrophe. The constraint $d = d_0 < 0$ ensures that trimodality does not occur.

3.3.6 Digression: Who Needs Mixtures?

L. Cobb (21) analyses topological complexities of mixture densities vis a vis multimodal densities in the extended Pearson family of distributions. In particular he notes that a mixture of j components (M_j , say) requires a parameter space with $3j - 1$ dimensions, whereas the corresponding multimodal density with j modes (K_j) has codimension equal to only $2j$.

In the language of Chapter 2 this simply says that if

$$E \in \mathbf{E}$$

determines a space of alternatives A and E is a mixture of j components $E_k \in \mathbf{E}$, $k = 1, \ldots, j$ with each E_k of type T, and

$$E(x) = \sum_{k=1}^{j} \alpha_k E_k(x), \quad \sum_{k=1}^{j} \alpha_k = 1, \quad x \in W$$

then (i) E has codimension $3j - 1$ whenever codimension $(E_k) \leq 2$, for all k .

(ii) E exhibits at most j modes.

Cobb suggests that E can be replaced by another energy function, namely K_j, which is topologically equivalent to an unfolding of a $A_{2(j-1)}$ singularity.* K_j requires only a $2(j-1)$ - dimensional control space and still displays up to j modes.

It is questionable if this reduction in dimensionality is desirable. Obviously, from the point of view of estimation, a considerable amount of work and time can be saved by using smaller parameter spaces. However, methodologically it is far more important to include all aspects to increase the model's sensitivity.

Theorem 2 provides some clues. It is far wiser to treat M_j as a special case of K_{j+1} rather than as an extension of K_j. An arbitrary decision to use M_j creates an artificial restriction on the number of available modes in any modelling system.

Example 1

Consider a Normal mixture model for beliefs (R, P, T) with

$$P_t(X = x \mid m, v) = \alpha n(m, v) + (1 - \alpha) n(-m, 1)$$

where $X = R$, for *all* $t \in T$.

We have reduced the parameter space to just (m , α , v) by eliminating the overall scale and location parameters.

In 3.4.1 we shall show that $P \in T_M$ and so results from previous sections apply. Thus P displays a cusp singularity over (m, α) for each $v = 1$.

The corresponding Cobb density is the "Bimodal Normal" N_3 defined by

$$f_3(x \mid a, b) = c \exp\left\{ -\frac{1}{4}x^4 + \frac{1}{2}bx^2 + ax \right\}$$

and the effective codimension is 2. The "next up" Cobb density is N_5,

$$f_5(x \mid a, b, c, d) = c \exp\left\{ -\frac{1}{6}x^6 + \frac{1}{4}dx_4 + \frac{1}{3}cx_3 + \frac{1}{2}bx^2 + ax \right\}$$

(*) An unfolding of $A_{2(j-1)}$ has codimension $2j - 2$ and displays up to j modes.

Theorem 2 enables us to embed the control space of P within the control space of f_5. This may be advantageous for several reasons:

(i) The f_5 model is more general than P in the same way as P is more general than f_3;

(ii) f_5 can be used as a test for trimodality and the appropriateness of a bimodal model;

(iii) Extended Pearson family densities are computationally easier to handle (See Cobb (20)).

Example 2 Anorexia Nervosa

It has been interesting to monitor the progress made by catastrophists in their attempt to model the mental disorder known as Anorexia Nervosa. Mathematically the modelling went through three stages:

1. *Cusp Catastrophe Model* with a two-dimensional control space;

2. *Butterfly Catastrophe Model* with the control space expanded now to four dimensions. This development led to finding a cure for the illness, which could not be predicted in the bimodal structure offered by 1.

3 E_6 *Catastrophe* with five-dimensional control space and two-dimensional behaviour space. In this way further aspects of the illness could be observed and dealt with.

The details of the work are described by Zeeman (16) and Calahan (1).

The two examples carry one important message. It often proves profitable to increase dimensionality of any model in order to include more aspects of the problem in question. Strangely enough a more general model may well provide a quicker answer.

Define a relation $\leq$ on elements of E by

$$E_1 \leq E_2 \iff codim\ E_1 \leq codim\ E_2$$

Then the relationship between a mixture and its "neighbouring" canonical models is

$$K_j \leq M_j \leq K_{j+1}$$

It can never be a mistake to run K_{j+1} in parallel to M_j whenever a j component mixture seems appropriate. In fact, since M_j is equipped to predict at most j modes, any K_k, with $k > j$ can be tried if only in order to confirm validity of the M_j hypothesis.

The choice lies with the experimenter. Whenever speed is required K_j will do better than M_j. For an accurate analysis K_{j+1} and higher order polynomials may often have to be employed.

Regardless of mathematical considerations mixtures possess many desirable properties useful in modelling aggregation and uncertainty problems. One particular advantage of an M_j model is the direct interpretation of parameters. For instance, the two - component mixture is generated by the natural parameters of location (μ), scale (p) and relative importance (α) . In a potential function of the cuspoid family these essential characteristics are sometimes difficult to isolate. The most interesting feature of the asymmetric mixture, the scale ratio p , is not an independent control factor of the Butterfly model and has to be expressed as a function of other parameters. Yet, in a practical context, the behaviour of p may be of major interest. Indeed p may turn out to be much more tractable than the control parameters of the Butterfly.

The embedding in Theorem 2 is non - linear, consequently the parameter spaces of mixtures and multimodal densities not only differ in dimension but also in structure. Practical considerations will decide what type of model is chosen. In the next chapter we look at an aggregation model in an "industrial relations" setting (see

page 129). One of the component models used is a mixture. It would be difficult and unwise to replace this mixture with a polynomial without losing the natural interpretation of the parameters.

3.4 Examples of Type T Functions

3.4.1 The Exponential Case: Normal Expected Loss

Let us look at Smith's fundamental example and generalise it.

$$E(\theta,p) = 1 - \frac{k^{1/2}}{p^{3/2}} \exp\left\{ - \frac{\theta^2}{2p} \right\} \tag{3.10}$$

where $p = k + V$. (k,V) are measures of spread of the loss and the belief functions respectively.

E is, of course, of type T and we can look at the mixture

$$E^*(\delta) = \alpha E(\delta+\mu,p_1) + (1-\alpha)E(\delta-\mu,p_2) \tag{3.3}$$

The bifurcation set and the cusp point of E^* are given by

$$(E^*)'(\delta) = (E^*)''(\delta) = (E^*)'''(\delta) = 0$$

First put

$$G(\theta,p) = \frac{k^{1/2}}{p^{3/2}} \exp\left\{ - \frac{\theta^2}{2p} \right\}$$

Thus $E(\theta,p) = 1 - pG(\theta,p)$

Note that

$$G'(\theta,p) = -\frac{1}{p}\theta G(\theta,p)$$

$$G''(\theta,p) = \frac{1}{p}\left\{\frac{\theta^2}{p} - 1\right\}G(\theta,p)$$

$$G'''(\theta,p) = \frac{1}{p^2}\left\{3\theta - \frac{\theta^3}{p}\right\}G(\theta,p)$$

And so

$$E'(\theta,p) = \theta G(\theta,p)$$

$$E''(\theta,p) = \left\{1 - \frac{1}{p}\theta^2\right\} G(\theta,p)$$

$$E'''(\theta,p) = \frac{\theta}{p}\left\{\frac{\theta^2}{p} - 3\right\} G(\theta,p)$$

Thus $E''(\theta,p)$ has a unique a unique positive zero at $\theta = \sqrt{p}$. $E'''(\theta,p)$ has a unique positive zero at $\theta = \sqrt{3p}$.

We can now calculate explicitly the functions R and S :

$$R(\theta,p) = \frac{E'''(\theta,p)}{E'(\theta,p)} = \left(\frac{\theta}{p}\right)^2 - \frac{3}{p} \tag{3.11}$$

$$S(\theta,p) = \frac{E''(\theta,p)}{E'(\theta,p)} = \frac{1}{\theta} - \frac{\theta}{p} \tag{3.12}$$

$R(\theta,p)$ is a simple parabola and it satisfies all the properties described in section 3.1. In fact it turns out that R has exactly one minimum at $\theta = 0$, and no other turning points at all.

Similarly, S behaves well in the region $\theta > p\eta$, and

$$\lim_{\theta \to \infty} S(\theta,p) = -\frac{1}{p}$$

So S is a hyperbola with asymptotes $\theta = 0$ and $\theta = -\frac{1}{p}$.

In order to check whether or not the Normal mixture exhibits a cusp catastrophe we must examine the behaviour of $\mu_R(p)$ and $\mu_S(p)$.

Lemma 3.1

If E is the Normal expected loss, then the condition (M) holds for all $p > 1$.

Proof

$\mu_R(p)$ is the solution of

$$R(\theta,p) = R(\theta,1)$$

and by (3.11) we get

$$\left(\frac{\theta}{p}\right)^2 - \frac{3}{p} = \theta^2 - 3$$

Hence

$$\theta^2 = \frac{3p}{1+p}$$

So

$$\mu_R(p) = \sqrt{\frac{3p}{p+1}}$$

$\mu_S(p)$ is the solution of

$$S(\theta,p) = -S(\theta,1)$$

and so (3.12) implies

$$\frac{1}{\theta} - \frac{\theta}{p} = -\frac{1}{\theta} + \theta$$

$$\theta^2 = \frac{2p}{p+1}$$

giving

$$\mu_S(p) = \sqrt{\frac{2p}{1+p}}$$

Thus

$$\mu_R(p) > \mu_S(p), \quad \text{for } all\ p > 1$$

Theorem 3

If E is the Normal expected loss, then E^*, given by (3.6), exhibits a unique cusp in the half-plane $\mu > 0$ for each $p > 1$.

Moreover, the δ-coordinate of the cusp point, $D_0 \neq 1$.

Proof

The result follows immediately from Lemma 3.1 and Theorem 1.

$D_0 \neq 0$ follows from Corollary 1.1.

We achieve an intuitively appealing result in the Normal case: the cusp point moves away from the origin, but exists for all p. But we are still a long way from finding the explicit coordinates of the cusp point. To do this we must still solve

$$R(\mu+\delta,1) = R(\mu-\delta,p) \tag{3.8}$$
$$S(\mu=\delta,1) = -S(\mu-\delta,p)$$

Using (3.11,3.12) and rearranging this reduces to

$$(\delta^2 - \mu^2)[(1 - p)\delta - (1 + p)\mu] = 2\mu p \tag{3.13}$$
$$(\delta - \mu)^2 - 3p = p^2[(\delta + \mu)^2 - 3]$$

The top one of these equations is a cubic in δ and μ and the two equations are very difficult to solve simultaneously. Once again let us compare the above set of equations with the reduced case $p = 1$:

$$\mu^2 = \delta^2 + 1 \tag{3.14}$$
$$(\delta - \mu)^2 = (\delta + \mu)^2$$

The top equation is now only a quadratic, and we quickly arrive at solutions

$$\delta = 0, \ \mu = \pm 1$$

The general problem involves solving an equation one degree higher than in Smith's case.

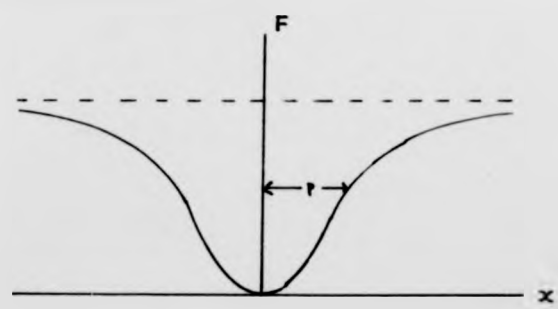
3.4.2 The Polynomial Case

Let us now consider another expected loss function.

Define

$$F(x,p) = \frac{x^2}{p^2 + x^2}, \quad p \geq 1 \tag{3.15}$$

F has certainly got the right shape and p plays the role of the coefficient of spread. This can be seen below.



diag 3.12

Lemma 4.1

$F(x,p)$ is of type T, for all $p \neq 0$.

Proof

(a) F is clearly C^{∞}, symmetric, increasing in $|x|$ and bounded by 1.

(b)

$$F'(x,p) = \frac{2p^2 x}{(p^2 + x^2)^2}$$

$$F''(x,p) = \frac{2p^2(p^2 - 3x^2)}{(p + x^2)^3}$$

Therefore $F''(x,p) = 0$ has a unique positive root at $x = \frac{p}{\sqrt{3}} = p\eta_F$

(c)

$$F'''(x,p) = \frac{24p^2 x(x^2 - p^2)}{(p^2 + x^2)^4}$$

Therefore $F'''(x,p) = 0$ has a unique positive root at $x = p = p\lambda_F$

(d)

$$R_F(x,p) = \frac{F''(x,p)}{F'(x,p)} = \frac{12(x^2 - p^2)}{(p^2 + x^2)^2}$$

Note that

(i) $R_F(x,p)$ is increasing on $[0, p\lambda_F]$

(ii) $R_F(x,p) < 0$ on $[0, p\lambda_F]$ and $R_F(p\lambda_F, p) = 0$

(iii) $R_F(x,p) > 0$ for $x > p\lambda_F$.

$$(i),(ii),(iii) \Rightarrow R_F((0, p\eta_F)) \cap R_F((p\eta_F, p\lambda_F)) = \phi$$

We have to examine the properties of R_F and $S_F(x,p) = F''(x,p)$ to determine the existence of a cusp point in

$$F^*(x) = \alpha F(x+\mu,1) + (1-\alpha)F(x-\mu,p) \tag{3.16}$$

We know that

$$R_F(x,p) = \frac{12(x^2 - p^2)}{(p^2 + x^2)^2} \tag{3.17}$$

$$S_F(x,p) = \frac{p^2 - 3x^2}{x(p^2 + x^2)} \tag{3.18}$$

Lemma 4.2

$\mu_{R_F}(p) = \mu_{S_F}(p)$, for all p .

Proof

To find $\mu_{R_F}(p)$ we must solve

$$R_F(x,p) = R_F(x,1)$$

which gives

$$\frac{12(x^2 - p^2)}{(p^2 + x^2)^2} = \frac{12(x^2 - 1)}{(1 + x^2)^2}$$

Hence

$$3x^4 + x^2(1 + p^2) - p^2 = 0 \tag{3.19}$$

with solutions

$$x^2 = -\frac{1}{6}\left\{(1 + p^2) \pm \sqrt{(1+p^2)^2 + 12p^2}\right\}$$

This gives two real solutions and the positive one of those is the required

$$\mu_{R_F}(p) = \frac{1}{\sqrt{6}}\left[\sqrt{(1+p^2)^2 + 12p^2} - (1+p^2)\right]^{\frac{1}{2}} \tag{3.20}$$

To find $\mu_{S_F}(p)$ consider

$$S_F(x,p) = -S_F(x,p)$$

i.e.

$$\frac{p^2 - 3x^2}{p^2 + x^2} = -\frac{1 - 3x^2}{1 + x^2}$$

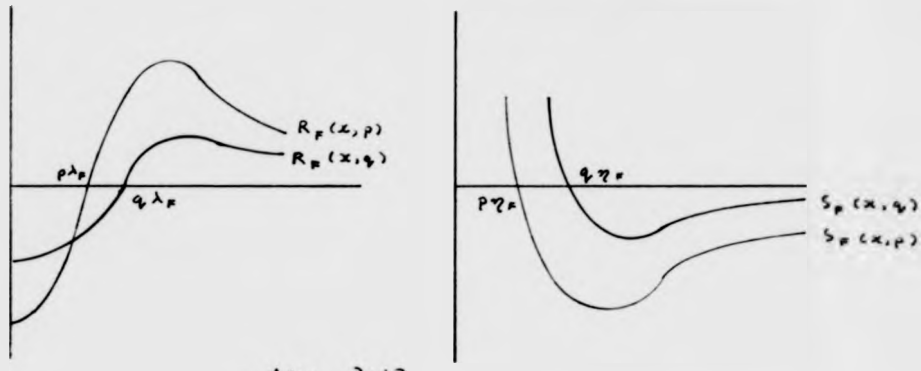
Hence

$$3x^4 + x^2(1 + p^2) - p^2 = 0$$

which is the same equation as (3.19).

$$\mu_{R_F}(p) = \mu_{S_F}, \quad \text{for } all\ p$$

The functions R_F and S_F look as follows: Take $q > p$.



diag 3.13

It can be shown that $S_F\ (x,p)$ never meets $S_F\ (x,q)$ if $p \neq q$:

Suppose the contrary. Then

$$\frac{p^2 - 3x^2}{p^2 + x^2} = \frac{q^2 - 3x^2}{q^2 + x^2} \quad \text{for } some\ x > 0$$

Hence

$$4p^2x^2 = 4q^2x^2 \iff x = 0 \text{ or } p = q, \ \textit{contradiction.}$$

Notice that R_F and S_F behave differently for large x than do their exponential counterparts. Because of the tail behaviour of the expected loss functions the polynomial functions are bounded, while the exponential ones are not.

We can now draw the obvious conclusion.

Theorem 4

F^* defined by (3.16) exhibits a unique cusp in the half plane $\mu > 0$ for every $p > 1$. The coordinates of the cusp point are

$$(x,\alpha,\mu) = \left(0, \frac{p^2}{p^2 + \dfrac{(p^2+\mu_0)^2}{(1+\mu_0)^2}}, \mu_0 \right)$$

where

$$\mu_0 = \frac{1}{\sqrt{6}}\left[\sqrt{(1+p^2)^2 + 12p^2} - (1+p^2)\right]^{1/2}$$

Proof

Follows immediately from Theorem 1 which can be applied since $F(x,p)$ is of type T by Lemma 4.1.

The coordinates of the cusp point follow from Corollary 1.1 and Lemma 4.2:

$$x = 0$$

$$\mu = \mu_{R_F}(p) = \mu_0$$

$$\alpha = \frac{F'(\mu_0,p)}{F'(\mu_0,p) + F'(\mu_0,1)} = \frac{p^2}{p^2 + \dfrac{(p^2+\mu_0^2)^2}{(1+\mu_0^2)^2}}$$

where

$$\mu_0 = \frac{1}{\sqrt{6}}\left[\sqrt{(1+p^2)^2 + 12p^2} - (1+p^2)\right]^{1/2}$$

by equation (3.20).

Consider the behaviour of the cusp point as p increases:

$$\alpha = \frac{p^2}{p^2 + \dfrac{(p^2+\mu_0^2)^2}{(1+\mu_0^2)^2}}$$

increases as p increases to give more weight to the component with smaller spread.

Meanwhile, $\mu_0(p)$ increases as p increases to the upper bound of $\lambda_F = 1$. This can be seen from the diagram or from the general properties of R functions. It can also be shown directly:

$$\begin{aligned} g(p) &= 1 - [\mu_0(p)]^2 \\ &= 1 - \frac{1}{6}\left[\sqrt{(1+p^2)^2 + 12p^2} - (1 + p^2)\right] \\ &= \frac{1}{6}\left[(7 + p^2) - \sqrt{(7 + p^2)^2 - 48}\right] \end{aligned}$$

> 0, for *all* p

and

$$g(p) \to 0 \quad as \quad p \to \infty$$

i.e.

$$\mu_0(p) \to 1 \quad as \quad p \to \infty$$

We finally have an example where the exact coordinates of the cusp point can be found explicitly, for all p .

3.5 Conclusions

The attempt to analyse the properties of the asymmetric mixture has been partially successful. The problem is far more complicated than it appears at a first glance. The relative complexity of the general situation compared to Smith's special case is best summarised by diagrams 3.7 and 3.8. While in Smith's case the solution is simply forced upon us the general picture yields little or no clues of how to proceed.

But the geometry of this system affords us at least the possibility of finding a relatively simple necessary and sufficient condition for existence and uniqueness of cusp singularity. The proof is based on a purely geometric argument and does not give us the explicit coordinates of the cusp point. It may well be that the most general case will not yield these coordinates.

Looking at the special cases the situation improves only slightly. The extension of the fundamental example used by Smith behaves as badly as the general case. We can obtain the equations of the bifurcation set and the cusp point, but they are very awkward to solve. We do know that a solution exists, if that is any consolation.

Perhaps the most important benefit of looking at the extended problem is focused in Theorem 2. The increase in the dimension of the control space enables us to embed a 2-component mixture into the Butterfly potential function, and, in

general, to relate an M_j model to the cuspoid family.

Whenever a mixture is fitted to a problem it is wise to be aware of the fact that M_j restricts us to just j modes and may sometimes suppress a few others.

4. Aggregate Decision Making and Conflict

4.1 Motivation

The recipe suggested in 2.6.3 can readily be applied to Decision Theory. There appears to be a greater need for new methodology here than in the abstract Probability Theory. Computationally we face fewer difficulties since normalising constants are of no importance. No major inroads have been made into classifying utility and expected utility functions, therefore Decision Theory foundations seem more recipient to new approaches. Pragmatists will no doubt be more interested in getting some quick decisions out of their models, and our methods set out to achieve this task. At first glance the mathematical techniques may appear cumbersome, but the routines are easy to apply. And, of course, we believe they provide accurate modelling facilities.

The models used in this chapter consist of the usual triple

$$(\mathbf{W}, E, T)$$

where E is invariably an expected loss function, parametrised by a subset of R^* , yielding a decision space D as its space of alternatives. Most often we have

$$D \subseteq \mathbf{W} = R$$

The aggregation function is introduced in the decision theoretic context and is a restricted case of the map defined in 2.6.3.

4.2 The General Scheme

4.2.1 Introduction

The method presented here is designed to provide a practical tool for aggregating decisions in conflict situations.

By "aggregation" is meant both the problem of amalgamating separate decisions processes into one resultant action as well as combining several attributes of a single decision. "Conflict" is handled using Catastrophe Theory in its simplest form.

4.2.2 The Scheme

Consider a class of real-valued, C^2 expected loss functions written in the form

$$E_i : A_i \times V_i \to R$$

WLOG assume $i \in \{1,\ldots,n\}$ throughout. A_i are the decision spaces and V_i are the environment spaces.

We proceed using the ideas from 2.6.3, and consequently the aggregation procedure is based on the assumption that the above loss functions can be represented by a single expected loss of the same structure:

$$E : X \times W \to R$$

where X is the decision space and W, the environment space, is constructed from all the V_i's.

In general, no restriction is made to the dimension of any of the decision spaces or the environment spaces. However, for simplicity, the rest of the scheme will be presented under the assumption that the decision spaces are all one dimensional.

The process then consists of three stages.

(1) Local Optimisation

Define a dynamic associated with each expected loss function E_i by

$$\frac{da_i}{dt} = -\frac{\partial E_i(v_i, a_i)}{\partial a_i} \qquad (4.0)$$

$$a_i \in A_i, \quad v_i \in V_i$$

Then each E_i is optimised using a map

$$opt(E_i) : V_i \to A_i$$

defined by

$$opt(E_i) = a_i^*(v_i)$$

where a_i^* is an attractor of (4.0). Write

$$\mathbf{opt} = (\, opt(E_1), \cdots, opt(E_n)\,)$$

$$\mathbf{opt} : V_1 \times \cdots \times V_n \to A_1 \times \cdots \times A_n$$

(2) Aggregation

To construct W we begin by combining the environment spaces $V_1, \ldots, V_n$:

$$\epsilon : V_1 \times \cdots \times V_n \to V_\epsilon$$

Then put $V = V_\epsilon \bigcup V_0$ where V_0 is an environment space disjoint from $V_1, \ldots, V_n$.

The aggregating function σ maps local decision spaces onto the control space of E :

$$\sigma : A_1 \times \cdots \times A_n \times V \to W$$

The aggregation is completed by combining σ with the local optimisation operator **opt** :

$$\sigma^* = \sigma.(\mathbf{opt}, id) : V_1 \times \cdots \times V_n \times V \to W$$

(3) Final Optimisation

Lastly, by defining a dynamic on X , analogous to (4.0), we optimise E to obtain the final decision x^* :

$$opt(E) : W \to X$$

$$opt(E) = x^*(\omega), \quad \omega \in W$$

Example

A decision maker, D , intends to use advice of two subordinates A and B to reach a decision $x \in X = R$. He constructs an expected loss function of the form

$$E: X \times V \to R$$

where V is the environment space. V consists of:

(i) D 's own preferences and beliefs about the problem;

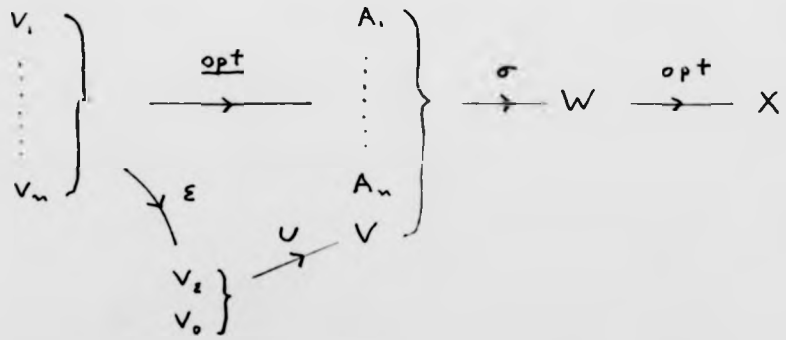
(ii) Information about A 's and B 's preferences and beliefs;

(iii) (a^*, b^*) , the actions advised by A and B .

Notice that it is not necessary for D to have the complete knowledge of (i), (ii) and (iii) in order to construct V and hence E . If, however , all information is available D proceeds according to the outlined scheme.

4.2.3 Summary and Comments

The scheme described here can be summarised by a diagram.



Thus the whole process can be written as

$$opt.\sigma.(\text{opt}, id) : V_1 \times \cdots V_n \times (Im(\epsilon) \cup V_0) \to X$$

where Im denotes the image of a map.

While σ has been called the aggregating function, ϵ can be regarded as the influence map. If we view the aggregator as a person distinct from the n decision makers then when viewed as follows the construction of W has some intuitive appeal. V is the the control space of the aggregation function. It consists of two components: V_e = "aggregator's perception of decision maker's environment", and V_0 = "aggregator's independent environment".

In most aggregation schemes V and $A_1 \times \cdots \times A_n$ would now be sufficient to produce the final decision. We introduce one extra stage and use σ to construct a control space of the final decision process.

If the development of the process is viewed over time, clearly we will require

$$(V_0 \times V_1 \times \cdots \times V_n)_{t+1}$$

to be dependent on z_t^*. In this way the cycle is completed. Notice that environmental spaces and decision spaces interact in both directions.

The ultimate aim must be the classification of all processes of the above type. The method is determined by properties of σ and ϵ. In particular the part played by σ is of major importance. In this work we shall only look at very simple cases, and distinguish two major types according to the dimension of Im (σ).

Evolution over time is another important aspect.

The T element of the triple is designed to take care of that. Two cases are to be considered:

(i) the development of a model over time,

and

(ii) the sequential aggregation using the same model.

Within the energy approach such analysis becomes possible.

Finally, a word must be said about the practical context of our scheme. The general description does not state who is doing the actual aggregating. French (45) has listed four types of aggregation problems (see page 76), and the reader will no doubt wonder which case we are covering. In short the answer is "All of them". Our aggregating function σ is designed to construct a control space W for any of the problems listed by French. But, of course, the exact form of σ and the structure of the enviroment spaces will reflect the practiacl context. For instance, if we are dealing with the "Expert Problem" the energy function

$$E: X \times W \to R$$

will represent the expected loss function of the external investigator. W will be constructed from the decisions of the contributing experts as well as from other information available to the aggregator and contained in V_0. The particular form of

$$\sigma: A_1 \times \cdots \times A_n \times (V_e \bigcup V_0) \to W$$

will be chosen by the aggregator.

In other situations, such as the "Textbook Problem", the choice of σ and the complexity of W are the responsibility of the whole group.

So far we have presented a general outline of our scheme. We now turn to look at some more specific situations.

4.3 Cusp Aggregation Rules

4.3.1 Definition of a Catastrophic Aggregation Rule

Following the notation of the previous section let $E_1, \ldots, E_n$ be a set of C^∞ expected loss functions

$$E_i : A_i \times V_i \to R \quad i = 1, \ldots, n$$

Let

$$E : X \times W \to R$$

be their aggregate expected loss, and let W be constructed according to the same scheme with

$$\sigma : A_1 \times \cdots \times A_n \times V \to W$$

as the aggregating function. Viewing E as a potential function we can classify σ - functions according to the topological type of E .

Definition

σ is called a Catastrophic Aggregation Rule if

$$dim\ X \geq 1 \quad \text{and} \quad dim\ W \geq 2$$

Thus in particular, σ is a

(i) Cusp Aggregation Rule (CAR) if

$$dim\ X = 1, \quad dim\ W = 2$$

(ii) Butterfly Aggregation Rule (BAR) if

$$dim\ X = 1, \quad dim\ W = 4$$

etc.

At present no attempt will be made to examine any more complicated rules such as the Umbilics or the Double Cusp.

4.3.2 Standard Aggregation Rule

Let $F^{*}(\boldsymbol{\mu}, \mathbf{R})$ be any continuous multivariate belief function of a parameter θ with mean $\boldsymbol{\mu}$ an $n \times 1$ column vector and covariance matrix $\mathbf{R}$.

Let $L^{*}(\mathbf{a} - \boldsymbol{\theta}, \mathbf{V})$ be an associated loss function, $\mathbf{a} \in \mathbf{A}$ being the action space.

Suppose, WLOG, that the n marginal distributions combine with n respective marginal loss functions to produce n expected loss functions, which are all C^{*}. Call these $E_1, \ldots, E_n$.

Suppose additionally that the optimisation on $E_1, \ldots, E_n$ gives

$$\mathbf{opt} = \boldsymbol{\mu} \tag{4.1}$$

Meanwhile, the multivariate expected loss function is of the form

$$E(\mathbf{a}) = 1 - \exp\left\{ - f(\mathbf{a} - \boldsymbol{\mu}, \mathbf{P}) \right\} \tag{4.2}$$

where f is a polynomial in $(\mathbf{a} - \boldsymbol{\mu})$ and $\mathbf{P}$ is its spread matrix constructed from $\mathbf{R}$ and $\mathbf{V}$. Call $\mathbf{P}^{-1}$ the *interaction matrix* of the process.

We wish to aggregate the actions of the n decision makers. (4.1) gives us the local optimisation. (4.2) will help us to construct V. Then we shall choose a σ to complete the process.

Let

$$E: X \times W \to R$$

be the aggregate expected loss function.

Define

$$V_e = (\mathbf{P}, \mathbf{R}), \qquad V_0 = (a_0, b_0)$$

Now let $\sigma : \mathbf{A} \times V \to W$ be a CAR

Thus $W = (a, b)$, say

Definition

Call σ a *standard* CAR if

$$a = 1^T \mathbf{R}^{-1}(\mathbf{a} - \bar{\mathbf{a}}) + a_0 \qquad (4.3)$$
$$b = (\mathbf{a} - \bar{\mathbf{a}})^T \mathbf{P}^{-1}(\mathbf{a} - \bar{\mathbf{a}}) + b_0$$

a is going to play the role of the normal factor, and b is going to be the splitting factor.

Properties of the Normal Case

Using the same notation, put

F = Multivariate Normal

L = Multivariate Conjugate Normal Loss

It can be shown that

$$\mathbf{P} = \mathbf{R} + \mathbf{V}$$

so that (4.3) becomes

$$a = 1^T \mathbf{R}^{-1}(\mathbf{a} - \bar{\mathbf{a}}) + a_0 \qquad (4.4)$$
$$b = (\mathbf{a} - \bar{\mathbf{a}})^T (\mathbf{R} + \mathbf{V})^{-1}(\mathbf{a} - \bar{\mathbf{a}}) + b_0$$

Let us examine a couple of simple situations. Some of the examples presented below display a number of desirable properties. For instance, in case (1) we end up with an expanded version of the linear opinion pool (c.f. Mc Conway (48)). Not only do we obtain a consensus distribution but we can also keep track of the amount of dissent among the contributors. Thus our model analyses the problem in a higher dimension. A similar approach in case (2) gives rise to a more general version of the Smith model. The solution we arrive at is sensitive to interactions between the contributors due to an introduction of non-zero correlations into their joint belief and loss structures.

(1) Beliefs and Losses uncorrelated, i.e. **R** and **V** diagonal. Then (4.4) becomes

$$a = \sum_{i=1}^{n} \frac{1}{r_i}(a_i - \bar{a}) + a_0$$
$$b = \sum_{i=1}^{n} \frac{(a_i - \bar{a})^2}{r_i + v_i} + b_0$$

$$\mathbf{R} = diag(r_i), \quad \mathbf{V} = diag(v_i)$$

We further simplify by taking opt $= (\mu_1, \ldots, \mu_n)$ with $\bar{\mu} = 0$. Then

$$a = \sum_{i=1}^{n} \frac{\mu_i}{r_i} + a_0$$

$$b = \sum_{i=1}^{n} \frac{\mu_i^{\,2}}{r_i + v_i} + b_0$$

is the final form of σ^{*} .

This model has the following properties:

1. (a) If r_i and v_i are large then in the limit

$$(a,b) = (a_0,b_0)$$

In other words, the aggregator takes little notice of the contributors whenever their beliefs lack precision;

(b) But, if instead, r_i and v_i are small (a,b) may end up very far from (a_0,b_0) . In this case the aggregator disregards his own biases.

So if the contributors base their beliefs on observations from, say, some DLM with precisions increasing in time a situation of type (a) may evolve into one of type (b).

2. Suppose additionally $r_i = r$ for all i .

Then

$$a = a_0$$

The aggregator is not going to lean towards any particular section of contributors under this restriction.

The amount of conflict in the model will depend directly on

$$\sum_{i=1}^{n} \mu_i^2$$

So we are really considering the distribution of the vector μ . Aggregator treats the inputs as data. Removing the restriction $\bar{\mu} = 0$, the pair

$$(\bar{\mu}, \sum \mu_i^2)$$

is treated as a summary of the group's intentions and inner conflict. There is nothing particularly new about looking at such a summary. What is novel is the idea of treating the components of the summary as control factors of the cusp potential function. Precisions act as weighing coefficients. Note that the method can be used when nothing at all is known about the manner in which the contributors arrived at their individual decisions. For instance, if the aggregator is given

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$$

as his only data, he can still construct a model of the form

$$a = c_1\bar{\mu} + a_0$$

$$b = c_2 \sum_{i=1}^{n} \mu_i^2 + b_0$$

The constants c_1 and c_2 calibrate the potential function and, in some respect, represent the aggregator's dependence on his contributors.

3. If the tolerance to losses, v_i, is large the effect is similar to the earlier case when $(v_i + r_i)$ was large. Essentially b will lie close to b_0. Clearly if the contributors have large margins for error it is unlikely that much conflict between them can develop.

(2) $n = 2$, Beliefs and Losses correlated.

Say

$$\mathbf{R} = \begin{vmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{vmatrix} \quad \textit{with } -1 < \rho < 1$$

$$\mathbf{R}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{vmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{vmatrix}$$

$$\mathbf{V} = \begin{vmatrix} k_1^2 & \delta k_1 k_2 \\ \delta k_1 k_2 & k_2^2 \end{vmatrix}$$

$$\mathbf{P} = \mathbf{R} + \mathbf{V} = \begin{vmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{vmatrix}$$

$$\mathbf{P}^{-1} = \frac{1}{p_1 p_2 - p_{12}^2} \begin{vmatrix} p_2 & -p_{12} \\ -p_{12} & p_1 \end{vmatrix}$$

Hence σ gives (4.4) as

$$a = \frac{(a_1 - \bar{a})}{\sigma_1^2(1-\rho^2)} + \frac{(a_2 - \bar{a})}{\sigma_2^2(1-\rho^2)} + a_0$$

$$b = \Big\{(\sigma_2^2 + k_2^2)(a_1 - \bar{a})^2 - 2(\rho\sigma_1\sigma_2 + \delta k_1 k_2)(a_1 - \bar{a})(a_2 - \bar{a})$$

$$+ (\sigma_1^2 + k_1^2)(a_2 - \bar{a})^2\Big\} / \Big\{(1-\rho^2)\sigma_1^2\sigma_2^2$$

$$+ (1 - \delta^2)k_1^2k_2^2 + k_1^2\sigma_2^2 + k_2^2\sigma_1^2 - 2\rho\delta k_1k_2\sigma_1\sigma_2\Big\} + b_0$$

The last result looks more interesting if we further simplify by putting

$$\text{opt} = (\mu, -\mu)$$
$$\sigma = \sigma_1 = \sigma_2, \quad k = k_1 = k_2$$

Then

$$\sigma^* = \text{opt}.\sigma$$

gives

$$a = a_0 \tag{4.5}$$
$$b = 2\mu^2 / [(1 - \rho)\sigma^2 + (1 - \delta)k^2] + b_0$$

Note that equation (4.5) generalises Smith's Theorem to the case of correlated beliefs and losses. If we put $\rho = \delta = 0$, $a_0 = 2\log\left(\frac{\alpha}{1 - \alpha}\right)$, $b_0 = -1$ we obtain his result

$$a = 2\log\left(\frac{\alpha}{1 - \alpha}\right)$$
$$b = \frac{2\mu^2}{\sigma^2 + k^2} - 1$$

So, the rule proposed here produces a recognised result as a special case. We can isolate the following main properties of the above model:

1. a_0 represents the aggregator's initial bias towards either contributor. This bias is quite independent from the usual weighting factors since we have assumed $\sigma_1 = \sigma_2 = \sigma$.

2. The splitting factor is clearly a function of μ, ρ, δ, σ^2, k^2 with the following properties:

   (a) If σ^2 and k^2 are very large b will lie close to b_0 as in the previous example;

(b) If the denominator in the equation for b is constant, then b will clearly be an increasing function of μ as in Smith's original model and all our earlier cases;

(c) We must examine carefully the sensitivity of the model to changes in δ and ρ .

(i) $\delta = constant < 1$. Then

$$b = \frac{2\mu^2}{(1-\rho)\sigma^2 + c}$$

Thus

$$\frac{2\mu^2}{2\sigma^2 + c} \leq b \leq \frac{2\mu^2}{c}$$

Therefore b is increasing with ρ . So for a constant value of μ the splitting factor increases with correlation. This is what we would expect: a difference in opinion (μ) is more likely in the absence of correlation, therefore the conflict is greater if differences occur among correlated contributors;

(ii) $\delta = 1$. Now

$$\frac{\mu^2}{\sigma^2} \leq b \leq \infty$$

If the contributors have perfectly correlated loss structures and still produce different conclusions we can face an enormous conflict with

$$\lim_{\rho \to 1} b(\rho) = \infty$$

(iii) Analogous analysis holds when ρ is held constant;

(iv) When $\delta = \rho = 0$ the model is reduced to Smith's s mixture.

(3) $n = 3$, two correlated contributors.

Suppose the beliefs and losses have the following spread matrices:

$$\mathbf{R} = \begin{bmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma^2 & \rho\sigma^2 \\ 0 & \rho\sigma^2 & \sigma^2 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} v^2 & 0 & 0 \\ 0 & v^2 & \delta v^2 \\ 0 & \delta v^2 & v^2 \end{bmatrix}$$

Then the interaction matrix is the inverse of

$$\mathbf{P} = \mathbf{R} + \mathbf{V} = \begin{bmatrix} \sigma^2 + v^2 & 0 & 0 \\ 0 & \sigma^2 + v^2 & x(\sigma^2 + v^2) \\ 0 & x(\sigma^2 + v^2) & \sigma^2 + v^2 \end{bmatrix}$$

where

$$x = \frac{\rho\sigma^2 + \delta v^2}{\sigma^2 + v^2}$$

Thus $-1 \leq x \leq 1$ as a function of ρ and δ .

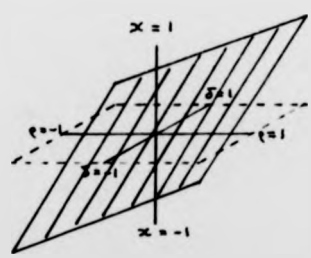
Hence the Standard CAR model for the situation takes form

$$a = \frac{\mu_1}{\sigma^2} + \frac{\mu_2 + \mu_3}{\sigma^2(1 + \rho)}$$

$$b = \frac{\mu_1^2}{\sigma^2 + v^2} + \frac{\mu_2^2 + \mu_3^2 - 2x\mu_2\mu_3}{(\sigma^2 + v^2)(1 - x^2)}$$

where opt = (μ_1, μ_2, μ_3) and $\bar{\mu} = 0$.

The dependence of x on ρ and δ is portrayed below:



diag 4.1

Let us consider the properties of this particular model. Denote by A_i the indivi-

dual contributors taking decisions a_i, $i = 1,2,3$, respectively.

1. a is a decreasing function of ρ with

$$\lim_{\rho \to 1} a(\rho) = \frac{\mu_1}{\sigma^2} + \frac{\mu_2 + \mu_3}{2\sigma^2}$$

and

$$\lim_{\rho \to 1} a(\rho) = -\infty$$

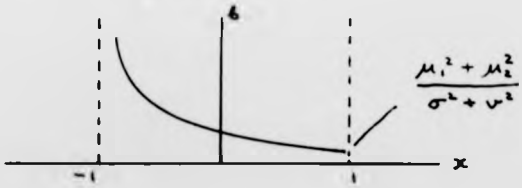
The normal factor was not affected by correlation in our previous case, but this time ρ affects the weights attached to the opinion of each contributor. The aggregator intends to give progressively less weight to the opinions of A_2 and A_3 as ρ increases. When $\rho = 1$ he treats their inputs as one by averaging them. However if $\rho < 0$ their opinions gain extra strength. Ultimately, there is a heavy bias towards A_2 and A_3 if they are of the same sign. Note that when $\rho = 0$ we are back at the general uncorrelated situation when the usual weighted average of all three inputs is taken.

2. The correlation coefficient x affects only the second component of b.

Case $\mu_2 = \mu_3$.

Now

$$b = \frac{\mu_1^2}{\sigma^2 + v^2} + \frac{2\mu_2^2}{(\sigma^2 + v^2)(1 + x)}$$



diag 4.2

Thus when

(i) $x = -1$: we face imminent conflict;

(ii) $x = 0$: usual uncorrelated situation;

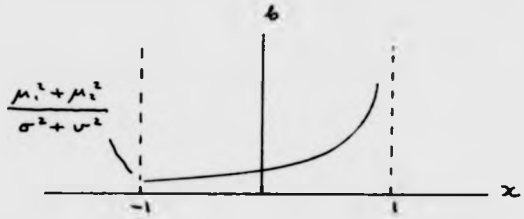
(iii) $x = 1$: we treat A_2 and A_3 as a single contributor.

Case $\mu_2 = -\mu_3$.

Then

$$b = \frac{\mu_1^{\,2}}{\sigma^2 + v^2} + \frac{2\mu_2^{\,2}}{(\sigma^2 + v^2)(1 - x)}$$

This produces a reflection of the last picture:



diag 4·3

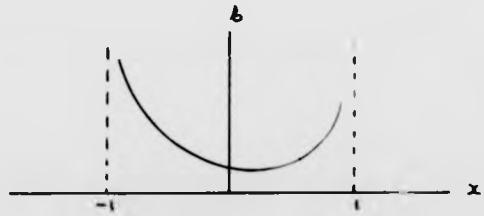
In particular

(i) $x = -1$: A_2 and A_3 are treated as one;

(ii) $x = 0$: standard uncorrelated case;

(iii) $x = 1$: conflict is imminent.

General case: WLOG take $\mu_2 > \mu_3 > 0$.



diag 4·4

The conflict is minimal at $x = \dfrac{\mu_3}{\mu_2}$.

If $x = 1$ or -1 we face an imminent conflict as the situation is explosive irrespective of the current choices of μ_2 and μ_3 .

4.3.3. Simple Projection Rule

With identical notation as in previous chapters consider again the set of n expected loss functions

$$E_i : A_i \times V_i \to R \quad i = 1,\ldots,n$$

Similarly let

$$\sigma : A_1 \times \cdots \times A_n \times V \to W$$

be the aggregating function, and let V be constructed as before.

Definition

σ is called a *projection rule* if

(i) $\dim W = n$

(ii) $W = (\omega_1,\ldots,\omega_n)$ and $\omega_i = \sigma_i\ (a_i, p_i)$ where

$$a_i \in A_i$$

$$\mathbf{P} = \begin{vmatrix} p_1 & p_{ij} \\ p_{ji} & p_n \end{vmatrix} \quad n \times n$$

is the interaction matrix and $\sigma = (\sigma_1, \ldots, \sigma_n)$.

Thus the projection rule is used when there is a one-to-one correspondence between each component decision and one control factor. In such cases we can loosely speak of "independent" contributions of n decision makers.

Definition

If σ is a projection rule then it is called *simple* if additionally each σ_i is a linear function of a_i,

$$\omega_i = p_i a_i + a_{i0} \quad i = 1,\ldots,n$$

Clearly, the topology of the projection rules is only dependent on the number n of decision makers.

In this chapter we only look at the cusp rules, so we consider the case $n = 2$.

Let $E_i : A_i \times V_i \to R \quad i = 1,2$, and WLOG put

$$P = \begin{vmatrix} p_1 & 0 \\ 0 & p_2 \end{vmatrix}$$

as the interaction matrix. Define $\sigma : A_1 \times A_2 \times V \to W$ by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = A_\theta \begin{pmatrix} a_1 p_1 \\ a_2 p_2 \end{pmatrix}$$

where $W = (\alpha, \beta)$

$$A_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$V = (P, \theta), \quad \theta \in V_0$$

The rotation matrix is introduced in order to allow the aggregator the choice in the angle of projection of the two contributing decisions onto his control space.

Example Simulation : Demand v. Industrial Unrest

Introduction

Background : Suppose we wish to construct a model of an industrial conflict. Typically we consider a factory with a sizable work force. We are interested in the dependence of *output* of the plant on the *demand* and the state of industrial relations.

The object of the exercise is to enable any participant in an industrial dispute to monitor the situation. Thus the management, the unions and the government should be able to use the model presented below. The conclusions each might draw from it could, of course, be quite different.

For the sake of consistency the reader may assume that this model has been constructed by the management as a means to anticipate and control strike situations.

The demand is the easier of the two factors to monitor. We are going to measure it in terms of orders received by the company. To quantify industrial relations we introduce the concept of *industrial unrest* (IU). This factor is much harder to measure. Intuitively, IU represents the level of dissatisfaction with the management and general conditions felt

by the work force.

Initially we intend to look at two other factors which influence the output and describe the effect of demand and attitudes of the workers. One such factor we will refer to as "pressure". It measures the amount of power, influence and desire for change felt by the work force. The other aspect is "intensity", and it describes the strength of feeling about any issue faced by the workers.

Our "empirical" control factors of demand and industrial unrest can be related to "pressure" and "intensity". We first make the following observations:

(i) When demand is constant an increase in IU corresponds to a drop in output. Initially the decrease in production is smooth and hardly noticeable. But when IU reaches a sufficiently high level the response of the output is often discontinuous.

(ii) With a constant level of IU a rise in demand leads to an increase in the power of the work force and hence to a corresponding increase in "pressure".

This type of behaviour - control interdependence has often been modelled by a cusp catastrophe potential. Sussmann (27) has heavily criticised this approach, but we will persist with it because the Cusp provides a simple and effective geometric interpretation of the problem. Our model must be seen as no more than a "first approximation". An interpretation with a more developed control space will undoubtedly paint a more accurate picture. But it will still retain many of the basic features of the cusp model. The restricted case we present here is primarily designed to illustrate the potential of our aggregation technique.

The model we are proposing is empirically testable. The "demand" and *IU* factors can be quantified along the lines indicated in models A_1 and A_2 below. The aggregation is then achieved by a straightforward application of the projection rule. A strike can then be predicted by examining the evolution of the parameters of the aggregated potential.

The Model : The Cusp Catastrophe has a 2 - dimensional control space. In order to model output as a function of only two factors we must relate the four concepts defined above to each other.

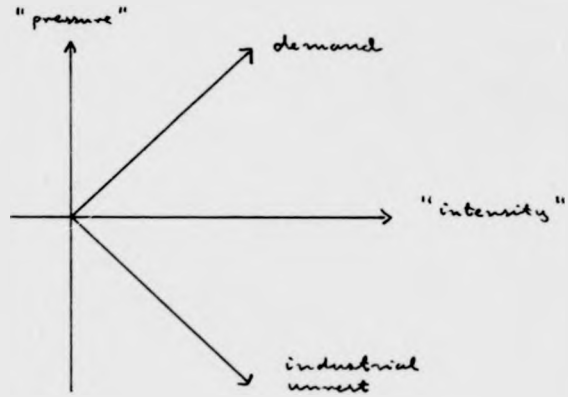
We postulate

(1) The effect of "pressure" and "intensity" is orthogonal;

(2) "Pressure" and "intensity" are both increasing functions of demand;

(3) IU is an increasing function of "intensity" but a decreasing function of "pressure".

Consequently the reduced control space looks as follows:



The energy function we are going to use is equivalent to the Canonical Cusp Catastrophe and is given by

$$E(x) = 1 - k(a,b)\exp\left\{ - [\frac{1}{4}x^4 - \frac{1}{2}bx^2 - ax] \right\}$$

with

$x =$ output meeting quality standards

and $C = (a,b)$, is the control space, where

$a =$ "pressure" - Normal factor;

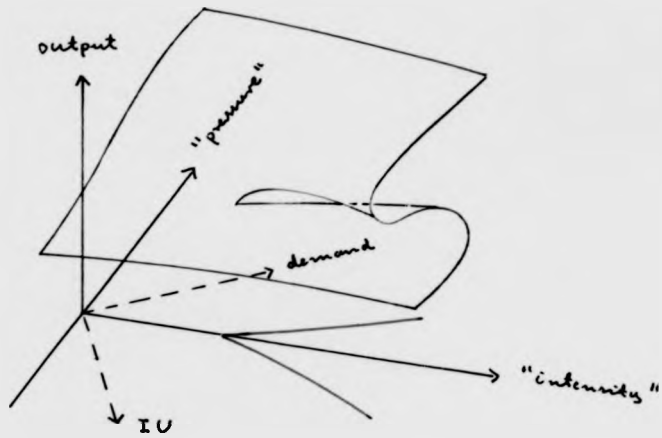
$b =$ "intensity" - Splitting factor.

The postulates (2) and (3) then imply

$\omega =$ $a + b$ = demand

$u =$ $b - a$ = industrial unrest

The proposed model is shown below.



The bifurcation set of the model defines the *conflict region* of the dispute. A discontinuity of output corresponds to either a strike or a return to work.

We investigate u and ω separately and then aggregate using a simple projection rule.

Model A_1 - Demand

Assuming a "business cycle" of, say, 4 years we can model the demand using a Seasonal DLM. Let

$y_t =$ orders (or log orders)

$\theta_{1t} =$ underlying market demand

Then put

$$y_t = (1,0)\boldsymbol{\theta}_t + v_t, \quad v_t \sim N(0;V)$$

$$\boldsymbol{\theta}_t = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \boldsymbol{\theta}_{t-1} + \delta\boldsymbol{\theta}_t$$

$$\delta\boldsymbol{\theta}_t \sim N\left[\, 0 \,;\, V_\theta \,\right]$$

where

$$\boldsymbol{\theta}_t = \begin{pmatrix} \theta_{1t} \\ \theta_{2t} \end{pmatrix}$$

and $2\pi\phi = T = 4$ *years* .

Updating

$$\left(\boldsymbol{\theta}_t \mid D_t \right) \sim N\left[\begin{matrix} n_t \\ b_t \end{matrix} ; \; C_t \right]$$

where

$$n_t = n_{t-1}\cos\phi + b_{t-1}\sin\phi + A_{1,t}e_t$$
$$b_t = -n_{t-1}\sin\phi + b_{t-1}\cos\phi + A_{2,t}e_t$$
$$C_t = R - A_t Y_t A_t^T$$
$$R = Var\left[\begin{matrix} \theta_t \\ \beta_t \end{matrix} \mid D_{t-1} \right] = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$Y_t = Var(y_t \mid D_{t-1}) = r_{11} + V$$

$$e_t = y_t - \hat{y}_t = y_t - n_{t-1}\cos\phi - b_{t-1}\sin\phi$$

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_t = \begin{pmatrix} r_{11} \\ r_{12} \end{pmatrix} / Y_t$$

Hence

$$n_t = n_{t-1}\cos\phi + b_{t-1}\sin\phi + \frac{r_{11}}{r_{11} + V}(y_t - n_{t-1}\cos\phi - b_{t-1}\sin\phi)$$

Also

$$C_t = R - A_t Y_t A_t^T = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

with

$$C_{11} = r_{11} - \frac{r_{11}^2}{r_{11} + V} = \frac{r_{11}V}{r_{11} + V}$$

Therefore observer's beliefs about the level of demand, θ_{1t} , at time t , are

$$\theta_{1t} \sim N[n_t; C_{11}]$$

Using the conjugate normal loss function of the form

$$L(a_1,\theta) = 1 - \exp\left\{ - \frac{1}{2k_1}(\theta - a_1)^2 \right\} \qquad (*)$$

where $\theta = \theta_1$, $a_1 \in A_1$ is the observer's decision about the demand level and k_1 is a constant which in practice depends on profit margins.

We obtain the expected loss function

$$E_1(a_1) = 1 - \left(\frac{k_1}{k_1 + C_{11}}\right) \exp\left\{ - \frac{1}{k_1 + C_{11}}(a_1 - n_t)^2 \right\}$$

where

$$v_1 = (k_1, C_{11}, n_t) \in V_1$$

the environment space.

If no component of V_1 is dependent on a_1, the optimisation gives

$$opt(E_1) = a_{1,t}^* = n_t$$

Model A_2 : Industrial Unrest

Industrial disputes arise when conflicting interests of the management and the work force attain a sufficiently high level. To model the development of industrial unrest let

$\phi_t =$ level of industrial conflict at time t .

We postulate that ϕ_t is a bimodal function, and the two modes correspond to the interests of each competing group. The separation of the two modes represents the split between the two sides and the height of the modes illustrates their relative power.

Therefore we can use a mixture to model ϕ_t :

$$\phi_t = \alpha_t N(\mu_t; C_t) + (1 - \alpha_t) N(-\mu_t; C_t)$$

where

$\mu_t =$ "alienation or polarisation" between the management and the work force;

$\alpha_t =$ "relative influence" or support for each side;

$C_t =$ sharpness of views or determination of each group.

Note that more generally the scale parameters of each mixture component could be different. In that case an asymmetric mixture would have to be used.

The bimodal structure of ϕ_t allows us to monitor sudden changes of moods and attitudes of either side.

The estimation of all the parameters can be done by either a survey or a study of various data such as absenteeism (see, for instance, Zeeman (29)).

Using a loss function analogous to (*) we obtain a weighted conjugate normal expected loss

$$E_2(a_2) = \alpha_t \left[1 - \left(\frac{k_2}{k_2 + C_t}\right)^{1/2} \exp\left\{ - \frac{(a - \mu_t)^2}{k_2 + C_t} \right\} \right] + (1 - \alpha_t) \left[1 - \left(\frac{k_2}{k_2 + C_t}\right)^{1/2} \exp\left\{ - \frac{(a + \mu_t)^2}{(k_2 + C_t)} \right\} \right]$$

where $a_2 \in A_2$ is the observer's decision about the level of industrial conflict, and

$$v_2 = (k_2, C_t, \alpha_t, \mu_t) \in V_2$$

is the environment. If all components of V_2 are independent of a_2, then the optimisation gives

$$opt(E_2) = a_{2,t}^{*}(v_2)$$

and a^{*} need not be single-valued nor continuous function of v_2 .

Aggregation

In order to combine models A_1 and A_2 we must first construct V.

Take $V = V_e \bigcup V_0$ with $V_e = \mathbf{P}$, $V_0 = (a_{10}, a_{20})$ the interaction matrix. Since we have assumed independence of models A_1 and A_2 , $\mathbf{P}$ will be diagonal:

$$\mathbf{P} = \begin{vmatrix} \dfrac{k_1}{k_1 + C_{11}} & 0 \\ 0 & \dfrac{k_2}{k_2 + C_t} \end{vmatrix} = \begin{vmatrix} p_1 & 0 \\ 0 & p_2 \end{vmatrix}$$

Now our model for the aggregate expected loss function is

$$E: X \times W \to R$$

where

$$\sigma : A_1 \times A_2 \times V \to W$$

is a simple projection rule, thus requiring

$$W = (a, b)$$

with

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \omega \\ u \end{pmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{P} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

where

a = "pressure",

b = "intensity",

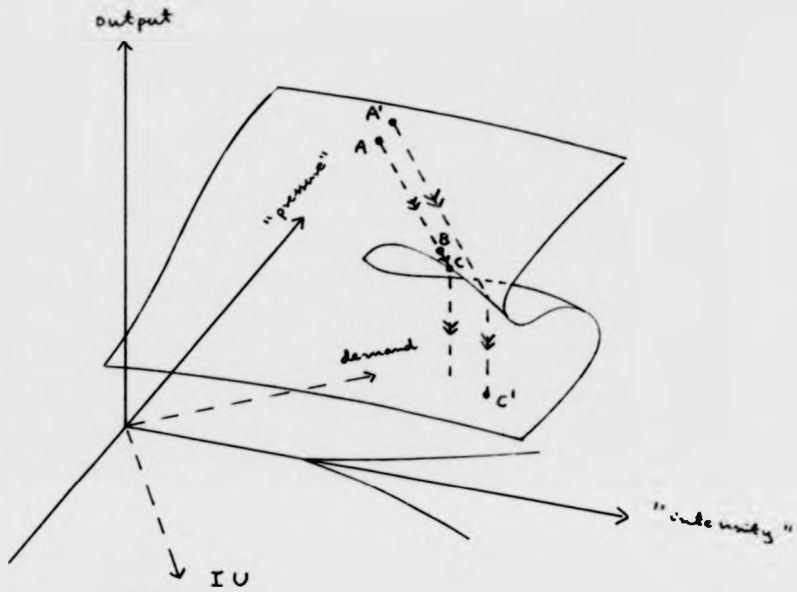
ω = demand,

u = industrial unrest.

In this way we have constructed a model of an industrial situation by first examining each control variable separately. The controls (a, b) have been introduced because of their natural interpretation as normal and splitting factors of the canonical cusp catastrophe. The components (ω, u) , on the other hand, are easier to estimate in practice.

We have assumed a smooth development of the market leading to a continuous demand curve. This assumption can be relaxed without altering the global structure of the model. Yet we have allowed a discontinuous contribution from the industrial relations aspect. The "double jump" effect is known as a *cascading catastrophe* and is almost impossible to track in any other way. In industrial relations literature such a phenomena are called "wild - cat strikes" (see Lane (5)).

A possible dynamic associated with this type of dispute is shown below.



Even though the demand remains steady, a sudden deterioration of industrial relations displaces the system from A deeply into the conflict zone at B . A failure of initial negotiations is now sufficient to spark off a strike at C . A "wild cat" dispute is characterised by a direct jump from A' over the threshold to C' .

An afterthought : An "inverted", or Dual, Cusp Model we have introduced in 2.4.4 can be used to devise an alternative model of industrial conflict.

Consider the differential equation

$$\frac{dG}{dx} = -x(x - p_t)(x - d_t) = 0$$

G exhibits a unique minimum at $x = p_t$. By interpreting x as the output, d_t as the demand and p_t as the *level of industrial cooperation* (effectively $-IU$), then G can be

used as an energy function for our problem.

The factory produces at the minimum of G . Thus if $p_t \leq 0$ there is a standstill of production. The output reaches the full capacity when $p_t \geq d_t$. The development of p_t and d_t need not be continuous.

4.3.4 Double Conflict

Let E_1 and E_2 be two expected loss functions, both with topological structure equivalent to that of a canonical cusp catastrophe.

Aggregating such expected losses may be of great interest in many practical contexts where each contributor faces internal conflict of his own even before confronting his adversary.

Clearly in this situation

$$\mathbf{opt} : V_1 \times V_2 \to A_1 \times A_2$$

may be a discontinuous function on some regions of $V_1 \times V_2$.

We shall model the aggregation process using a CAR:

$$\sigma : A_1 \times A_2 \times V \to W = (a,b)$$

where V is constructed from V_1 and V_2 . Let **P** be the interaction matrix for E_1 , E_2. The methods discussed earlier yield two possible candidates for σ : the standard and the projection rule.

Let us however look at another possible rule.

Internal and External Conflict

WLOG suppose that $E_i : V_i \times A_i \to R$ is given by

$$E_i(a_i) = 1 - c(\alpha_i, \beta_i)\exp\left\{ -\left[\frac{1}{4}a_i^4 - \frac{1}{2}\beta_i a_i^2 - \alpha_i a_i\right]\right\} \tag{4.6}$$

where

$$V_i = (\alpha_i, \beta_i) \quad i = 1,2$$

The expected loss (4.6) is essentially a cusp catastrophe potential function.

Definition

Internal conflict (IC) for E_i is defined by

$$\delta_i = \left(\frac{\beta_i}{3}\right)^3 - \left(\frac{\alpha_i}{2}\right)^2$$

Clearly if $\delta_i < 0$, E_i is unimodal, etc.

Definition

External conflict (EC) between E_1 and E_2 is defined by

$$\Delta = (\mathbf{a} - \bar{\mathbf{a}})^T \mathbf{P}^{-1}(\mathbf{a} - \bar{\mathbf{a}})$$

Thus $\Delta : V_1 \times V_2 \to (R \geq 0)$, and, due to properties of E_1 and E_2, may well be a discontinuous function on some regions of $V_1 \times V_2$ (where $\delta_i > 0$).

Total Conflict

As Im σ is two dimensional we require two control factors. One of them will probably be the average level of decision, the other will have to be related to the conflicts in the system.

By *Total Conflict* (TC) we will mean the splitting factor of the aggregate expected loss. This total conflict will obviously spring out from the two types of conflict defined above.

The table below indicates an intuitive relationship between the three types of conflict.

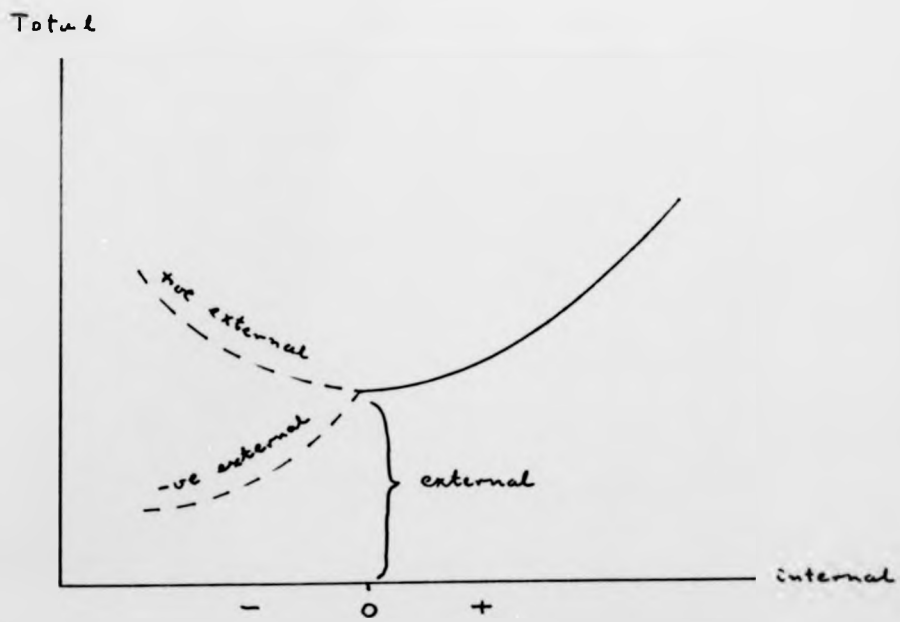
External	+	0	+	0	+	0
Internal	+	+	0	0	−	−
Total	++	+	−	0	−+	0

where + = *high positive conflict*

− = *high negative conflict*

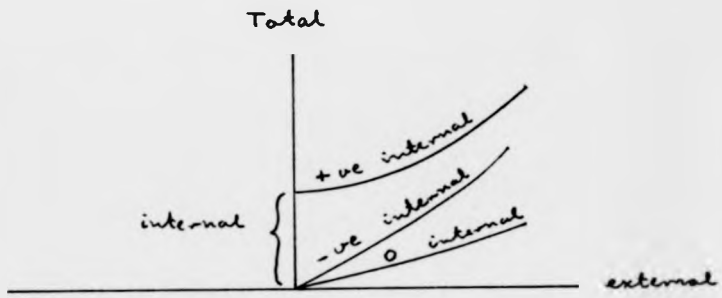
0 = *no conflict*

Graphically this relationship should look something like this:



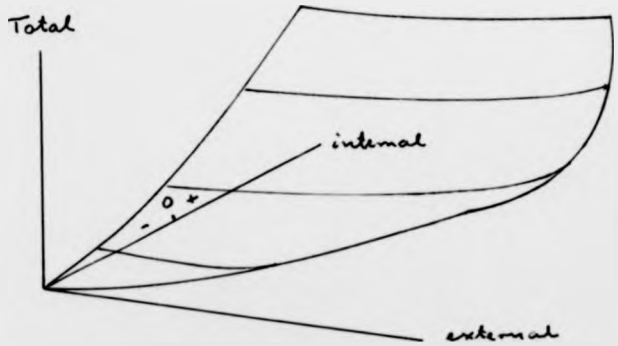
diag 4·5

Thus when internal conflict is negative there are two possibilities for total conflict depending on the sign of the external conflict.



diag 4·6

Combining these two graphs we get



diag 4·7

Consider the following function:

$$z(x,y) = e^{x} + e^{xy} + e^{y}, \quad y \geq 0 \tag{4.7}$$

It has roughly the required shape if we put

x = internal conflict

y = external conflict

z = total conflict

Also there seems to exist a natural interpretation for each element of (4.7). Clearly e^x and e^y measure the contribution of each type of conflict, while e^{xy} measures the contribution of the interacted conflicts. The latter term is only significant when large y "meets" negative x, which seems quite natural.

Aggregation

We propose the following σ to be used in double conflict situations.

$$\sigma : A_1 \times A_2 \times V \to W = (a,b)$$

with

$$a = 1^T \mathbf{R}^{-1}(\mathbf{a} - \bar{\mathbf{a}}) + a_0$$

$$b = \exp\left\{1^T \boldsymbol{\delta}\right\} + \exp\left\{-\Delta 1^T \boldsymbol{\delta}\right\} + \exp\left\{\Delta\right\} + b_0$$

where

$$\mathbf{P}^{-1} = \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} \subseteq \mathbf{V}$$

and $\mathbf{R}$ is the covariance matrix of beliefs.

Let us look at the simplest case when

$$\mathbf{P} = \begin{vmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{vmatrix}$$

where ρ and σ^2 have been obtained from the α's and β's in the equation (4.6).

The behaviour of this model differs from the Standard CAR through its splitting factor. We therefore only need to concentrate on the properties of the Total Conflict.

General remarks:

1. TC is a non-negative function of IC and EC and various precisions. Therefore

b_0 is usually a negative constant representing the tolerance of the aggregator to the conflict generated by the contributors.

2. Double Conflict is designed to be used when both contributors face bimodal expected losses. It is an extension of the Standard CAR in the sense that when $IC = 0$, the TC is essentially an increasing function of Δ , which plays the role of the splitting factor in the Standard CAR.

3. The inherent availability of bimodality in E_1 and E_2 enables us to introduce the notion of "negative internal conflict " when bimodality does not occur. "Negative IC" must be distinguished from a structural unimodality of the earlier models. It represents some kind of inner confidence of the contributor and is linked with high precision of beliefs and intolerance to losses.

Properties of TC:

1. External Conflict.

$$\mathbf{P}^{-1} = \frac{1}{\sigma^2(1-\rho^2)}\begin{vmatrix} 1 & -\rho \\ -\rho & 1 \end{vmatrix}$$

Hence

$$\Delta(\mathbf{a};\sigma^2,\rho) = \frac{(a_1 - a_2)^2}{2\sigma^2(1-\rho)}$$

(i) $\sigma^2 \to \infty \Rightarrow \Delta \to 0$.

No precision means the aggregator cannot attach any significance to disagreement among A_1 and A_2 .

(ii) $\rho \to 1$ then $\Delta \to \infty$ unless $a_1 = a_2$.

Conflict explodes if perfectly correlated contributors clash.

(iii) As ρ decreases Δ decreases since disagreement is less surprising when A_1 and A_2 are less correlated.

(iv) $\rho = 0$. Then

$$\Delta = \frac{1}{2\sigma^2}(a_1 - a_2)^2$$

(v) In general Δ is an increasing function of the difference in decisions taken by A_1 and A_2 .

2. Internal Conflict.

The IC component of TC is given by

$$e^{\delta} = e^{\delta_1 + \delta_2}$$

Thus TC is an increasing function of IC.

3. The overall dependence of TC on IC and EC is summerised by the diagram 4.7.

Normal Case of Double Conflict (Approximation Method)

In order to arrive at the Normal Beliefs situation we replace the expected loss function given by (4.6) with a mixture used by Smith.

$$E_i(x_i) = \alpha_i E(x_i - \mu_i) + (1 - \alpha_i)E(x_i + \mu_i)$$

where

$$E(y - \nu) = 1 - \left(\frac{k}{k + V}\right)^{1/2} \exp\left\{ - \frac{(y - \nu)^2}{2(k + V)} \right\}$$

Measuring x_i in units of $(k_i + V_i)$ we can map E_i onto a canonical cusp catastrophe and obtain a surface of stationary values of E_i satisfying

$$x_i^3 - b_i x_i - a_i = 0 \tag{4.8}$$

where

$$b_i = 3(\mu_i^2 - 1)$$
$$a_i = 2\log(\alpha_i / 1 - \alpha_i)$$

The internal conflict, σ_i, is then the discriminant of the cubic (4.8):

$$\delta_i = \left(\frac{b_i}{3}\right)^3 - \left(\frac{a_i}{2}\right)^2$$

If $\sigma_i > 0$ then E_i will have three stationary values:

$$x_{i1}^{*} = 2(r_i)^{1/3}\cos\left(\frac{\theta_i}{3}\right) \qquad (\text{ min })$$

$$x_{i2}^{*} = 2(r_i)^{1/3}\cos\left(\frac{\theta_i + 2\pi}{3}\right) \qquad (\min)$$

$$x_{i3}^{*} = 2(r_i)^{1/3}\cos\left(\frac{\theta_i + 4\pi}{3}\right) \qquad (\max)$$

where

$$r_i = \left(\frac{b_i^3}{27}\right)^{1/2}$$

$$\theta_i = \cos^{-1}\left(\frac{a_i/2}{(b_i^3/27)^{1/2}}\right)$$

Since $0 < \theta_i < \pi$, we have

$$x_{i1}^{*} > x_{i3}^{*} > x_{i2}^{*}$$

The interaction matrix for E_1 and E_2 is

$$\mathbf{P}^{-1} = \begin{bmatrix} k_1 + V_1 & k_{12} + V_{12} \\ k_{12} + V_{12} & k_2 + V_2 \end{bmatrix}^{-1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

The covariance matrix is

$$\mathbf{R} = \begin{bmatrix} V_1 & V_{12} \\ V_{12} & V_2 \end{bmatrix}$$

So the Double Conflict aggregation procedure yields

$$a = \mathbf{1}^T\mathbf{R}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + x_0$$

$$b = \exp\left\{\mathbf{1}^T\boldsymbol{\delta}\right\} + \exp\left\{-\Delta\mathbf{1}^T\boldsymbol{\delta}\right\} + \exp\left\{\Delta\right\} + b_0$$

to form $W = (a, b)$ as the control space of

$$E: X \times W \to R$$

The optimisation gives, say,

$$opt = (x_1^{*}, x_2^{*})$$

Then

$$\Delta = (\mathbf{x} - \bar{\mathbf{x}})^T\mathbf{P}^{-1}(\mathbf{x} - \bar{\mathbf{x}})$$

$$\boldsymbol{\delta} = \begin{vmatrix} \left(\frac{b_1}{3}\right)^3 - \left(\frac{a_1}{2}\right)^2 \\ \left(\frac{b_2}{3}\right)^3 - \left(\frac{a_2}{2}\right)^2 \end{vmatrix}$$

Notice that using Maxwell Rule, the sign of a_i determines which root of (4.8) we will choose:

$$x_i^* = \begin{cases} x_{i1}^* & \text{if} \quad a_i > 0 \\ x_{i2}^* & \text{if} \quad a_i < 0 \end{cases}$$

If $a_i = 0$ the local decision is ambiguous.

Normal Case of Double Conflict (Exact Method)

Consider again the mixture

$$E_i(x_i) = \alpha E(x_i - \mu_i) + (1 - \alpha)E(x_i + \mu_i)$$

where

$$E(y) = 1 - \left\{\frac{k}{k + V}\right\}^{\frac{1}{2}} \exp\left\{-\frac{y^2}{2(k + V)}\right\}$$

Instead of using an approximation to canonical cusp catastrophe we can find the exact shape of the bifurcation set of E_i as follows: Define

$$E(y) = 1 - \frac{1}{p}G(y)$$

where

$$G(y) = \frac{k}{(k + V)^{3/2}} \exp\left\{-\frac{y^2}{2(k + V)}\right\}$$

and

$$p = (k + V)^{-1}$$

Then clearly, (see Chapter 3)

$$E'(y) = yG(y)$$
$$E''(y) = -(y^2p - 1)G(y)$$
$$E'''(y) = y[(yp)^2 - 3p]G(y)$$

Bifurcation set is given by equations

$$E_i(x_i) = E_i' = 0$$

i.e.

$$\alpha_i(x_i - \mu_i)G(x_i - \mu_i) + (1 - \alpha_i)(x_i + \mu_i)G(x_i + \mu_i) = 0 \tag{4.9}$$

$$\alpha_i[1 - p_i(x_i - \mu_i)^2]G(x_i - \mu_i) + (1 - \alpha_i)[1 - p_i(x_i + \mu_i)^2]G(x_i + \mu_i) = 0 \tag{4.10}$$

Dividing (4.9) by (4.10) we get:

$$\frac{(x_i - \mu_i)}{1 - p_i(x_i - \mu_i)^2} = \frac{(x_i + \mu_i)}{1 - p_i(x_i + \mu_i)^2}$$

Hence

$$\mu_i^2 = x_i + \frac{1}{p_i} \tag{4.11}$$

Putting (4.11) back into (4.9) we obtain the equation of the bifurcation set as

$$\alpha_i = \left[1 + \frac{1}{p_i c_i}\exp\left\{\tfrac{1}{2} p_i c_i - \frac{1}{p_i c_i}\right\}\right]^{-1} = B(\mu_i) \tag{4.12}$$

where

$$c_i = \left[\mu_i + (\mu_i^2 - \frac{1}{p_i})^{\frac{1}{2}}\right]^2$$

So the internal conflict, corresponding to the bimodal region of E_i, is given by

$$\delta_i = |B(\mu_i) - \tfrac{1}{2}| - |\alpha_i - \tfrac{1}{2}|, \quad \text{for } 0 \le \alpha_i \le 1$$

We can now use the exact

$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ in place of δ.

4.4 Butterfly Aggregation Rule

4.4.1 Introduction

Using notation analogous to that in the previous sections consider

$$E: X \times W \to R$$

the aggregate expected loss function constructed for the set $E_1, \cdots, E_n$ of expected loss functions

$$E_i : A_i \times V_i \to R \quad i = 1,\ldots,n$$

In this chapter we will be solely concerned with the case

$$dim\ W = 4$$

i.e. the aggregation function

$$\sigma : A_1 \times \cdots \times A_n \times V \to W$$

has image of dimension four.

Thus E is qualitatively equivalent to the Butterfly potential.

The discussion that follows is a natural extension of cusp models to cases involving three local minima of potential. We will be looking both at new models and extend some of the cusp models presented earlier. It is felt, in general, that butterfly models are of much greater importance, and it is intended, eventually, to treat cusp models as their special cases. This will be true, in particular, of the " Double Conflict " model described in the last section.

4.4.2 Butterfly Aggregation Rules

The geometry of the canonical Butterfly Catastrophe is described in 1.3.2. Using that analysis we can now develop various constructions of the 4 - dimensional control space W of

$$E: X \times W \to R$$

discussed in the introduction.

(1) Simple Butterfly Aggregation Rule

The most trivial construction is an exact analogue of the corresponding CAR case.

Let $n = 4$ and consider

$$\sigma: A_1 \times \cdots \times A_4 \times V \to W = (\omega_1, \cdots, \omega_4)$$

an aggregation function which can be split into components

$$\sigma = (\sigma_1, \cdots, \sigma_4)$$

with

$$\sigma_i: A_i \times V_i \to \omega_i$$

Such models can only be applicable if we can identify the independence of all four components and project them on to the appropriate axes of the control space.

There is also a possibility of some rotation of the axes, say

$$\begin{bmatrix} \omega_1 \\ \\ \omega_4 \end{bmatrix} = A_\theta \begin{bmatrix} a_1 \\ \\ a_4 \end{bmatrix} + \omega_0$$

where $a_i \in A_i$ and A_θ is the rotation matrix with θ a function of $V_1 \times \cdots \times V_4$ and ω_0 a translation in the W space.

Essentially such models require

(i) clear independence of the four inputs;

(ii) identifiability of each input with either one exact control axis in W or with some rotation and displacement of the four orthogonal axes in W .

In practice these conditions will rarely be satisfied.

(2) Extended Double Conflict

Let

$$E_i(z_i) = 1 - k(\alpha_i, \beta_i)\exp\left\{ - \left[\frac{1}{4}z_i^4 - \frac{1}{2}\beta_i z_i^2 - \alpha_i z_i\right]\right\}$$

where $z_i \in A_i$, $i = 1,2$ $V_i = (\alpha_i, \beta_i)$.

Recall the Double Conflict aggregation method discussed earlier.

Define

$$\delta_i = \left(\frac{\beta_i}{3}\right)^3 - \left(\frac{\alpha_i}{2}\right)^2 \tag{4.13}$$

and then put

$$\delta = \delta_1 + \delta_2$$

as the *internal conflict* of the system.

Next define the *external conflict* by

$$\Delta = (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{P}^{-1}(\mathbf{x} - \bar{\mathbf{x}})$$

where $\mathbf{P}^{-1}$ is the interaction matrix.

The aggregation method proposed in 4.2.4 defines

$$\sigma : A_1 \times A_2 \times V \to W = (a,b)$$

by

$$a = \mathbf{1}^T \mathbf{R}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + a_0$$
$$b = e^{\delta} + e^{\Delta} + e^{-\Delta\delta} + b_0$$

where $\mathbf{R}$ is the covariance matrix.

This method can be extended to a Butterfly rule by treating Δ and δ as separate factors. The final result is then far more sensitive and should lead to decisions more acceptable to both sides.

Thus define

$$\sigma: A_1 \times A_2 \times V \to W = (a,b,c,d)$$

with

$$\begin{aligned} a &= \mathbf{1}^T \mathbf{R}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + a_0 \\ b &= \Delta + b_0 \\ c &= c_0 + l(p_{11} - p_{22}) \\ d &= \delta + d_0 \end{aligned} \tag{4.14}$$

where l is a linear function of the difference between the " precisions " of the two sides. However, it is not an essential term and may be left out.

Notice that the splitting factor in the original method has now been divided into the splitting and the butterfly factors. We have explained, in chapter 1, the roles which these two factors play. This can now be appreciated in a practical context:

(i) b causes the split of the minima as an external evidence of the difference of opinions;

(ii) d measures the internal uncertainty of each proponent and causes yet another split, perhaps leading to a compromise solution;

(iii) c is related to the precision of the information available to each side, and therefore it will sway the position of the cusp(s) accordingly.

(iv) In the special case when

$$\mathbf{P} = \begin{vmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{vmatrix}$$

we can use the earlier analysis of the Double Conflict aggregation to describe the behaviour of this model:

(a) The normal factor is the same as in all our previous examples;

(b)

$$\Delta = \frac{(x_1 - x_2)^2}{2\sigma^2(1 - \rho)}$$

and so the splitting factor is represented by the EC whose properties we have examined in 4.3.4;

(c) Similarly,

$$\delta = \delta_1 + \delta_2$$

gives the butterfly factor;

(d) The constant terms (a_0, b_0, c_0, d_0) represent the aggregator's bias towards either contributor and his resistance to conflict and compromise. The latter, d_0 , may be positive or negative depending on whether or not the aggregator is conducive to a compromise solution;

(e) To determine the qualitative type of the expected loss we can use the methods described in 1.3.2.

(3) General BAR

Let us now consider a more general situation with

$$E_i : A_i \times V_i \to R \quad i = 1, \ldots, n$$

and $dim V_i \leq 2$ ensuring that the E_i are at most bimodal.

Then we can naturally extend the above scheme by putting

$$\delta = \sum_{i=1}^{n} \delta_i$$

where δ_i is given by (4.13), and then use the equations (4.14) to define the aggregation map. The only problem comes with c , and l will have to be replaced by some map which will polarise all the opinions and then move the cusp towards the most " precise " group. Alternatively, l may be left out altogether.

Note that if, for some i , E_i turns out to be unimodal it will have no positive contribution to the internal conflict. In the extreme case, when all the E_i are unimodal, the butterfly factor will probably be negative (depending on the size of d_0), and the compromise opinion will not emerge. But surely we would expect this to be the case when each individual is confident about his own views and has no internal conflict.

(4) Double Butterfly Conflict

We have not yet looked at the case when one or more of the component expected losses are themselves trimodal. Let us look at this in the case $n = 2$ and refer to the situation as the *Double Butterfly* aggregation problem.

WLOG let

$$E_i(x_i) = 1 - k(\alpha_i,\beta_i,\gamma_i,\xi_i)\exp\left\{ - \left(\frac{1}{6}x_i^6 - \frac{1}{4}\xi_i x_i^4 - \frac{1}{3}\gamma_i x_i^3 - \frac{1}{2}\beta_i x_i^2 - \alpha_i x_i\right)\right\}$$

$i = 1,2 \quad x_i \in A_i$

Thus E_i are equivalent to the canonical Butterfly. The two relevant discriminants are

$$\delta_i = \left(\frac{\beta_i}{3}\right)^3 - \left(\frac{\alpha_i}{2}\right)^2$$

the internal conflict, and

$$\tau_i = \frac{\xi_i^3}{5} - \frac{\gamma_i^2}{2}$$

the *internal compromise* .

The results of the section 1.3.2 are useful in determining the shape of E_i according to the values of (δ_i,τ_i) . Also the name of the τ_i discriminant becomes clear in this context:

the more positive value of τ_i , the more likely is the compromise region to emerge.

When aggregating two Butterflies it may be more useful perhaps to employ a higher order catastrophe. On the other hand not much more can be gained by further increasing the number of minima of the expected loss function. In fact, in many cases people will be more interested in the reduction of the minima. The problem presented here can be perfectly satisfactorily handled by CAR models. When more sensitivity is required we propose the following BAR model to aggregate the two above:

Let

$$\sigma: A_1 \times A_2 \times V \to W = (a,b,c,d)$$
$$V = \left\{\delta_1,\delta_2,\tau_1,\tau_2\right\}$$

be defined by

$$a = 1^T \mathbf{R}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + a_0$$
$$b = (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{P}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + b_0$$
$$c = \tau_1 - \tau_2 + c_0$$
$$d = \delta_1 + \delta_2 + d_0$$

Only the bias factor requires some explanation. Basically, the aggregated model will show some bias towards that contributor who shows more flexibility and willingness to compromise. But this " orientation " of the bias is purely arbitrary. It can be argued that the bias should, in fact , be directed towards the more uncompromising and confident contributor. Therefore the sign reversal on the bias factor is acceptable if preferred.

4.4.3 Comments and Conclusions

Butterfly Aggregation Rules have been presented here as the natural extension of the Cusp Aggregation Rules. But it is perhaps more appropriate to look at the latter as a special simplified case of the former. Butterfly models are obviously more sensitive and accurate. If enough information is available it is clearly an advantage to consider more aspects in order to produce efficient decisions. These models will be particularly useful when dealing with highly conflicted groups and trying somehow to bring them together. In such

cases CAR models would only amplify all the existing disagreements, and provide few clues on the possible cures, whilst the BAR models might be able to detect any areas where some compromise could be reached.

In many cases, however, the use of BAR models would not be justified. Sometimes there are not enough independent inputs to merit the use of a four-dimensional control space. Also, in some cases, the computations involved in identifying all four factors are too heavy to warrant the use of a BAR model. And, of course, in most practical situations the issue of trimodality does not arise, and CAR (or simpler) models are sufficient to illustrate all the complexities of the situation.

Perhaps the best practical advice that can be offered at present is to perform the original aggregation using a CAR model. If this does not take account of all the aspects and does not help in finding acceptable decisions, then a Butterfly model must be the one to be tried next.

4.4.4 Normal Cases of Some BAR

Throughout this section the contributor expected loss functions will be of the form

$$E_i(x_i) = \alpha_i E(x_i - \mu_i) + (1 - \alpha_i)E(x_i + \mu_i)$$

where E is the Normal expected loss function given by

$$E(y) = 1 - \left(\frac{k}{k+V}\right)^{\frac{1}{2}} \exp\left\{ - \frac{y^2}{2(k+V)} \right\}$$

Following the results of section 4.2.4, the bifurcation set of E_i can be obtained either by approximation or exactly, and the respective values of the internal conflict are as follows:

$$\delta_i = \left(\frac{b_i}{3}\right)^3 - \left(\frac{a_i}{2}\right)^2$$

where

$$b_i = 3(\mu_i^2 - 1)$$
$$a_i = 2\log(\alpha_i / 1 - \alpha_i)$$

$$\delta_i = |B(\mu_i) - \tfrac{1}{2}| - |\alpha_i - \tfrac{1}{2}|, \quad \text{for } 0 \le \alpha_i \le 1$$

where $B(\mu_i)$ is given by (4.12) in 4.3.4.

(1) Extended Double Conflict

The aggregation function $\sigma: A_1 \times A_2 \times V \to W = (a,b,c,d)$ is given by (4.14) in the section 4.3.3(2).

In the Normal case (using the same notation as before) we have

$$\mathbf{R} = \begin{vmatrix} V_1 & V_{12} \\ V_{12} & V_2 \end{vmatrix}$$

$$\mathbf{P}^{-1} = \begin{vmatrix} k_1 + V_1 & k_{12} + V_{12} \\ k_{12} + V_{12} & k_2 + V_2 \end{vmatrix}^{-1} = \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix}$$

Using the approximate δ_i first, σ becomes (assuming $x_1 + x_2 = 0$)

$$a = 1^T \begin{vmatrix} V_1 & V_{12} \\ V_{12} & V_2 \end{vmatrix}^{-1} \mathbf{x} + a_0$$

$$b = \mathbf{x}^T \begin{vmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{vmatrix} \mathbf{x} + b_0$$

$$c = p_{11} - p_{22} + c_0$$

$$d = \sum_{i=1}^{2} \left\{ (\mu_i^2 - 1)^3 - (\log \frac{\alpha_i}{1 - \alpha_i})^2 \right\} + d_0$$

Consider two particular cases:

(i) $V_1 = V_2 = V, \quad V_{12} = 0$

$k_1 = k_2 = k, \quad k_{12} = 0$

Then

$$a = \frac{1}{V}(x_1 + x_2) + a_0 = a_0$$

$$b = \frac{1}{k + V}(x_1^2 + x_2^2) + b_0$$

(i) $V_1 = V_2 = V, \quad V_{12} = \rho V$

$$k_1 = k_2 = k, \quad k_{12} = 0$$

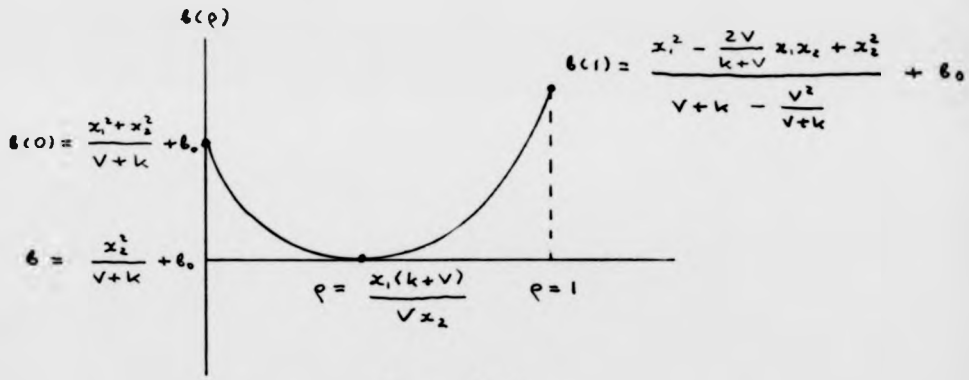
$$\mathbf{P}^{-1} = \frac{1}{(k + V)^2 - V^2\rho^2} \begin{bmatrix} V+k & -V\rho \\ -V\rho & V+k \end{bmatrix}$$

and hence

$$a = \frac{x_1 + x_2}{V(1 + \rho)} + a_0 = a_0$$

$$b = \frac{\left[x_1 - \frac{V\rho}{V+k} x^2\right]^2}{(V + k)\left[1 - \frac{V^2\rho^2}{(V+k)^2}\right]} + \frac{x_2^2}{V+K} + b_0$$

It is worthwhile to examine the dependence of b on ρ for constant x_1, x_2, b_0, k, V :



diag 4.8

Note also that

$$b(1) - b(0) = \frac{V^2}{(V + k)[(V + k)^2 - V^2]} \left\{(x_1 - x_2)^2 - 2\frac{k}{V} x_1 x_2\right\}$$

$$b(1) \geq b(0) \iff \frac{k}{V} \leq \frac{(x_1 - x_2)^2}{2x_1x_2}$$

The increase in the correlation of the two beliefs does not relate to the splitting factor in a linear manner. In fact, if the last inequality holds (which may, for instance, happen if x_1^* and x_2^* are far apart) the conflict between parties with perfect correlation may be greater than that of independent parties.

If we use the exact δ_i this will only affect the butterfly factor, which is independent of the decision space. In order to create the third ("compromise") minimum we require

(i) $d > 0$, i.e. at least one contributor has a bimodal expected loss;

(ii) $b < 0$

(iii) (b,d) in the trimodality region (see section 1.3.2).

(2) Double Butterfly

Consider

$$E_i(x_i) = \alpha_{1i}E(x_i - \mu_{1i}) + \alpha_{2i}E(x_i - \mu_{2i}) + \alpha_{3i}E(x_i - \mu_{3i}) , \quad with \sum_{j=1}^{3} = 1$$

where once again

$$E(y) = 1 - (3k)^{\frac{1}{2}}\exp\left\{ - \frac{3y^2}{2} \right\}$$

by putting $k + V = \frac{1}{3}$, for convenience.

Assume additionally $\mu_{2i} = 0$. Then following Smith and Harrison (22) , $E_i(\delta_i)$ exhibits a unique Butterfly point at

$$(\delta_i,\alpha_{1i},\alpha_{3i},\mu_{1i},\mu_{3i}) = (0,r,r,1,-1)$$

where

$$r = [2(1 + 2e^{-3/2})]^{-1}$$

Taylor series expansion around the Butterfly point gives the following approximation of E_i to the canonical butterfly:

$$x_i = \phi_i + \frac{1}{6}[3(\alpha_{3i} - \alpha_{1i}) + 2(\mu_{3i} + \mu_{1i})]$$

$$d_i = 10[(\alpha_{2i} - 0.31) + \frac{2}{3}(\frac{\mu_{1i} - \mu_{3i}}{2} - 1)]$$

$$c_i = - \frac{10}{3}(\mu_{3i} + \mu_{1i})$$

$$d_i = - 7(\alpha_{2i} - 0.31)$$

$$a_i = \frac{10}{27}\left[2(\mu_{3i} + \mu_{1i}) - 3(\alpha_{3i} - \alpha_{1i})\right]$$

Thus for each E_i we can calculate the internal conflict

$$\delta_i = \left(\frac{b_i}{3}\right)^3 - \left(\frac{a_i}{2}\right)^2$$

$$= -\left[\frac{7}{3}(\alpha_{2i} - 0.31)\right]^3 - \left(\frac{5}{27}\right)^2\left[2(\mu_{3i} + \mu_{1i}) - 3(\alpha_{3i} - \alpha_{1i})\right]^2$$

and the internal compromise

$$\tau_i = \frac{d_i^3}{3} - \frac{c_i^2}{2}$$

$$= \frac{10^3}{2}(\alpha_{2i} - 0.31) + \frac{1}{3}(\mu_{1i} - \mu_{3i}) - \frac{2}{3}\Big]^3 - \frac{100}{6}(\mu_{3i} + \mu_{1i})^2$$

The aggregation function

$$\sigma: A_1 \times A_2 \times V \to W = (a,b,c,d)$$

is given by

$$a = 1^T \mathbf{R}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + a_0$$
$$b = (\mathbf{x} - \bar{\mathbf{x}})^T \mathbf{P}^{-1}(\mathbf{x} - \bar{\mathbf{x}}) + b_0$$
$$c = \tau_1 - \tau_2 + c_0$$
$$d = \delta_1 + \delta_2 + d_0$$

where

$$\mathbf{R} = \begin{vmatrix} V_1 & V_{12} & V_{13} \\ V_{12} & V_2 & V_{23} \\ V_{13} & V_{23} & V_3 \end{vmatrix}$$

$$\mathbf{V} = \begin{vmatrix} k_1 & k_{12} & k_{13} \\ k_{12} & k_2 & k_{23} \\ k_{13} & k_{23} & k_3 \end{vmatrix}$$

$$\mathbf{P} = \mathbf{R} + \mathbf{V}$$

Consider two particular cases:

(i) $V_1 = V_2 = V_3 = V, \; V_{ij} = 0$ if $i \neq j$

$k_1 = k_2 = k_3 = k, \; k_{ij} = 0$ if $i \neq j$

Then, assuming $\bar{x} = 0$

$$a = \frac{1}{V}(x_1 + x_2 + x_3) + a_0 = a_0$$

$$b = \frac{1}{k + V}(x_1^2 + x_2^2 + x_3^2) + b_0$$

(ii) $V_1 = V_2 = V_3 = V,\ V_{12} = V_{23} = V_{13} = \rho V$

$k_1 = k_2 = k_3 = k,\ K_{ij} = 0$ if $i \neq j$

Then

$$a = \frac{1}{V}\left(x_1 + \frac{x_2 + x_3}{1 + \rho}\right) + a_0$$

$$b = \frac{x_1^2 + x_2^2}{V + k} + \frac{\left(x_3 - \frac{\rho V}{V + k}x_2\right)^2}{(V + k)\left(1 - \frac{V^2\rho^2}{(V + k)^2}\right)} + b_0$$

Once again b is a quadratic in ρ , and the splitting factor is not a decreasing function of ρ .

A Double Butterfly is a natural extension of a Double Conflict at the level of the (a,b) - section of the bifurcation set. The butterfly factor is constructed from the internal conflict in each case. Note that bimodal expected losses may have much larger internal conflicts than trimodal ones. The main structural difference comes in the construction of the bias factor. One can treat the bimodal case as having a zero compromise, and hence the bias in the Double Conflict is a either constant or a linear function of the precision difference.

4.5 A Remark

Intuitively aggregation must sometimes lead to multimodal expected loss functions. The energy approach provides a natural framework for modelling such phenomena.

Above we have only looked at simple cases where only two or three regimes appear in competition. The object is to illustrate the potential of the method. In the end the complexity of any model, within the aggregation dispute or anywhere else, is arbitrary. The investigator decides how much information and how many aspects are going to be

included. The models described here possess the variable amount of sensitivity and accuracy to suit circumstances.

5. Conclusions

The original aim of this work was to construct models for aggregation of beliefs. Special emphasis was to be placed on decision making in the face of conflict. From the study of recent literature it soon became clear that the aggregation debate had no specific direction and various researches were only concerned with isolated issues. No general framework existed and even the most basic elements of the problem have not been clearly defined. Thus the first task was to set up some foundations and proceed from there.

It soon became apparent that the traditional probability theory could not provide the axiomatic set up to tackle the aggregation issue. As was mentioned in the preface Measure Theory had not been equipped with any means to amalgamate separate measures. Obviously we had to look for methodology elsewhere. Non-additive methods seemed very attractive as they could cope with problems such as incoherence and inconsistency of any of the group members whose beliefs we were combining. However these methods added a lot of other complications especially when elicitation was concerned.

Finally a new formulative model was devised. This differs very little from the traditional set up in the sense that Kolmogorov axioms are being obeyed. The emphasis is moved away from the probability measure and placed upon a certain "invisible" energy function. This creates a kind of a "gravitational field" and both observable and unobservable events are subjected to its force. The basic assumption is that every model of uncertainty had an associated energy function which generated such a field.

Philosophically our approach is closest perhaps to the propensity interpretation of probability. The "Fair Coin" model, presented in Chapter 2, best illustrates this resemblance.

Once the concept of the measure has been removed from the focal point of the theory the aggregation issues can be reviewed in a fresh light. In Chapter 4 we looked at one particular method. An aggregator is placed in a position where he can choose the geometric structure of his decision problem. We only looked at cases where two or three

conflicting sets of options are available, but that was felt to be sufficient to introduce the method. In any case, in practice it is rarely cost-efficient to consider a more polarised situation and still hope to achieve a working consensus. The basic advantage in amalgamating energy functions lies in the fact that they do not have to obey any laws of probability. The difficulties associated with using measures in aggregation stems from the fact that no one quite knows what laws they are supposed to obey. Inevitably *ad hoc* methods are being disguised as "laws of nature". This criticism is levelled in particular at the Bayesian models of Lindley (47) and French (12), who appear to be especially dogmatic.

In our view, *ad hoc* methods are unavoidable. We believe it is futile to try to establish exact laws governing disputes. The only problem lies in finding efficient *ad hoc* rules. The models suggested in this work should be treated as empirical. We first create the framework in which it seems easier to manouver. Then we define a set of rules. These rules are neither to difficult to use nor too insensitive to capture conflict. The reader should view the aggregation models as dependent on the asserted structure of uncertainty. But the converse is not true. Should the models prove to be unacceptable the suggested interpretation of probability here can still survive on its own merits.

The basic tool used throughout this work is Catastrophe Theory. At first it appeared to be the most natural way of modelling conflict. Later we found that the description of any model of uncertainty can incorporate potential functions. In this way Catastrophe Theory models ended up in almost all corners of this dissertation.

It can be said that the philosophy behind all our methodology is based on the belief that , no matter how polarised, all situations have an underlying smooth, and perhaps multimodal, structure.

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