# Evaluating the Holevo Cramér-Rao Bound for Multiparameter Quantum Metrology

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Only with the simultaneous estimation of multiple parameters are the quantum aspects of metrology fully revealed. This is due to the incompatibility of observables. The fundamental bound for multiparameter quantum estimation is the Holevo Cramér-Rao bound (HCRB) whose evaluation has so far remained elusive. For finite-dimensional systems we recast its evaluation as a semidefinite program, with reduced size for rank-deficient states. We show that it also satisfies strong duality. We use this result to study phase and loss estimation in optical interferometry and three-dimensional magnetometry with noisy multiqubit systems. For the former, we show that, in some regimes, it is possible to attain the HCRB with the optimal (single-copy) measurement for phase estimation. For the latter, we show a nontrivial interplay between the HCRB and incompatibility and provide numerical evidence that projective single-copy measurements attain the HCRB in the noiseless 2-qubit case.

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*Introduction.*—Measuring physical quantities with ever increasing precision underlies both technological and scientific progress. Quantum mechanics plays a central role in this challenge. On the one hand, the unavoidable statistical uncertainty due to quantum fluctuations is a fundamental limitation to high precision metrology. On the other hand, quantum-enhanced metrological schemes that take advantage of nonclassical features, such as entanglement, coherence, or squeezing, have been proposed and implemented experimentally [1–7]. Myriad metrological applications are intrinsically multiparameter [8], e.g., sensing electric, magnetic, or gravitational fields [9], force sensing [10,11], imaging [12,13], and superresolution [14–18]. As a consequence, the field of multiparameter quantum metrology has been growing rapidly, both theoretically [19–39] and experimentally [40–44].

The mathematical framework behind quantum metrology is quantum estimation theory [45], pioneered by Helstrom [46–48] and Holevo [49–51]. In particular, multiparameter quantum estimation highlights a defining trait of quantum theory, absent in single-parameter estimation: incompatibility of observables [52,53]. Because of this, multiparameter quantum estimation is much more challenging, but also serves as a test bed for understanding quantum measurements.

Precision bounds for multiparameter estimation are given in terms of *matrix* inequalities for the mean square error matrix (MSEM) $\mathbf{\Sigma}$, see Eq. (1). However, matrix bounds are, in general, not tight for multiparameter quantum estimation. Instead, the Holevo Cramér-Rao bound (HCRB) [51,54] is the most fundamental *scalar* lower bound imposed by quantum mechanics on the weighted mean square error (WMSE) $\mathrm{Tr}[W\mathbf{\Sigma}]$ (for a positive definite $W$). The HCRB represents the best precision attainable with global measurements on an asymptotically large number of identical copies of a quantum state [55–59]. Implementing such collective measurements is exceptionally challenging [41,60], but in some cases the HCRB is attained by single-copy measurements: for pure states [61] and for displacement estimation with Gaussian states [51].

Despite its importance, the HCRB has been used more as a mathematical object in asymptotic quantum statistics [62] than applied to concrete metrological problems. Indeed, the HCRB is considered hard to evaluate, even numerically, being defined through a constrained minimization over a set of operators. Closed-form results for nontrivial cases are known only for qubits [63], two-parameter estimation with pure states [64], and two-parameter displacement estimation with two-mode Gaussian states [65,66], while a numerical investigation has been attempted for pure states and Hamiltonian parameters [67]. The evaluation of the HCRB thus remains a major roadblock in the development of multiparameter quantum metrology.

This Letter removes this roadblock by providing a recipe for evaluating the HCRB numerically for finite-dimensional systems. Our main result recasts the optimization required for evaluating the HCRB as a semidefinite program (SDP). This was shown only for displacement estimation with Gaussian states [66]. We present an SDP whose complexity grows with the rank of the state instead of a naive dependence on the Hilbert space dimension. The application of our recipe to evaluate the HCRB for two well-known metrological problems provides new insights. In particular, we provide numerical evidence that single-copy attainability of the HCRB with projective measurements is possible in nontrivial cases.

*Multiparameter quantum estimation.*—We consider a generic finite-dimensional quantum system with Hilbert space $\mathcal{H} \cong \mathbb{C}^d$, denote the space of linear operators ($d \times d$ matrices) on $\mathcal{H}$ as $\mathcal{L}(\mathcal{H}) \cong \mathbb{C}^{d\times d}$, and the space of observables (Hermitian matrices) as $\mathcal{L}_{\mathsf{h}}(\mathcal{H})$.

The state of the system $\rho_{\boldsymbol{\theta}} \in \mathcal{L}_{\mathsf{h}}(\mathcal{H})$ is parametrized by a real vector $(\theta_1, \ldots, \theta_n)^{\mathsf{T}} = \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^n$ [68], the collection $\{\rho_{\boldsymbol{\theta}}\}$ for all the values of $\boldsymbol{\theta}$ is called the quantum statistical model. The goal is to simultaneously estimate all $n$ parameters by measuring possibly multiple copies of $\rho_{\boldsymbol{\theta}}$. After measurement, classical data are processed with an estimator $\tilde{\boldsymbol{\theta}}$, a function from the space of measurement outcomes $\Omega$ to the space of parameters $\Theta$. The MSEM of the estimator

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\boldsymbol{\Pi}, \tilde{\boldsymbol{\theta}}) = \sum_{\omega\in\Omega} p(\omega|\boldsymbol{\theta})[\tilde{\boldsymbol{\theta}}(\omega) - \boldsymbol{\theta}][\tilde{\boldsymbol{\theta}}(\omega) - \boldsymbol{\theta}]^{\mathsf{T}}, \quad (1)$$

quantifies the precision of the estimation. The probability of observing the outcome $\omega$ is given by the Born rule $p(\omega|\boldsymbol{\theta}) = \mathrm{Tr}(\rho_{\boldsymbol{\theta}}\Pi_\omega)$; the measurement is described by a positive operator valued measure (POVM): $\boldsymbol{\Pi} = \{\Pi_\omega \succeq 0, \omega \in \Omega | \sum_{\omega\in\Omega} \Pi_\omega = \mathbb{1}_d\}$, without loss of generality, we consider $\Omega$ to be a finite set [69].

We consider locally unbiased estimators that satisfy

$$\sum_{\omega\in\Omega} [\tilde{\theta}_i(\omega) - \theta_i] p(\omega|\boldsymbol{\theta}) = 0, \qquad \sum_{\omega\in\Omega} \tilde{\theta}_i(\omega) \frac{\partial p(\omega|\boldsymbol{\theta})}{\partial \theta_j} = \delta_{ij}. \quad (2)$$

For this class of estimators, the matrix Cramér-Rao bound (CRB) on the MSEM is [70]

$$\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\boldsymbol{\Pi}, \tilde{\boldsymbol{\theta}}) \succeq F(\rho_{\boldsymbol{\theta}}, \boldsymbol{\Pi})^{-1} \quad (3)$$

($A \succeq 0$ if and only if $A$ is positive semidefinite); the classical Fisher information matrix (FIM) $F(\rho_{\boldsymbol{\theta}}, \boldsymbol{\Pi})$ is defined as

$$F(\rho_{\boldsymbol{\theta}}, \boldsymbol{\Pi}) = \sum_{\omega\in\Omega} p(\omega|\boldsymbol{\theta}) \left(\frac{\partial \log p(\omega|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right) \left(\frac{\partial \log p(\omega|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right)^{\mathsf{T}}, \quad (4)$$

where $\partial f(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ is the gradient of the function $f$. For locally unbiased estimators, the MSEM is the covariance matrix (CM) and the bound is attainable: there is always an estimator in this class with a CM equal to the inverse FIM [62,71]. To meaningfully compare the precision of different multiparameter estimators, it is customary to consider a scalar cost function, the WMSE $\mathrm{Tr}[W\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\boldsymbol{\Pi}, \tilde{\boldsymbol{\theta}})]$, with $0 \prec W \in \mathbb{S}^n$ ($\mathbb{S}^n$ is the set of real symmetric $n$-dimensional matrices).

The most widely known lower bound for the MSEM in quantum estimation relies on the (real symmetric) quantum Fisher information matrix (QFIM), defined as $J^S_{ij} = \mathrm{Re}(\mathrm{Tr}[\rho_{\boldsymbol{\theta}} L_i L_j])$, where $L_i \in \mathcal{L}_{\mathsf{h}}(\mathcal{H})$ are the symmetric logarithmic derivatives (SLDs) satisfying $2\partial\rho_{\boldsymbol{\theta}}/\partial\theta_i = L_i\rho_{\boldsymbol{\theta}} + \rho_{\boldsymbol{\theta}} L_i$ [45–47]. For single-parameter estimation, the SLD bound is always attainable by measuring single copies of the state; however for multiple parameters it is in general not attainable. Moreover, as a consequence of the noncommutativity of operators, for multiparameter estimation the QFIM is not the unique quantum generalization of the classical FIM [72]. Another important one is the (complex Hermitian) matrix $J^R_{ij} = \mathrm{Tr}[\rho_{\boldsymbol{\theta}} \tilde{L}_i \tilde{L}_j^\dagger]$, where the right logarithmic derivatives (RLDs) $\tilde{L}_i \in \mathcal{L}(\mathcal{H})$ satisfy $\partial\rho_{\boldsymbol{\theta}}/\partial\theta_i = \rho_{\boldsymbol{\theta}}\tilde{L}_i$ [73,74]. Both matrices give valid matrix bounds $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\boldsymbol{\Pi}, \tilde{\boldsymbol{\theta}}) \succeq (J^{\{S,R\}})^{-1}$. The corresponding scalar bounds for the WMSE are $C^S_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) = \mathrm{Tr}[W(J^S)^{-1}]$ and $C^R_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) = \mathrm{Tr}[W\mathrm{Re}(J^R)^{-1}] + ||\sqrt{W}\mathrm{Im}(J^R)^{-1}\sqrt{W}||_1$, where $||A||_1 = \mathrm{Tr}[\sqrt{A^\dagger A}]$ is the trace norm [51,71,75]. Whether $C^S_{\boldsymbol{\theta}}$ is larger than $C^R_{\boldsymbol{\theta}}$ depends on the model.

Holevo introduced a tighter bound, the HCRB $C^H_{\boldsymbol{\theta}}$ [50,51],

$$\mathrm{Tr}[W\boldsymbol{\Sigma}_{\boldsymbol{\theta}}(\boldsymbol{\Pi}, \tilde{\boldsymbol{\theta}})] \geq C^H_{\boldsymbol{\theta}} \geq \max\{C^S_{\boldsymbol{\theta}}, C^R_{\boldsymbol{\theta}}\}. \quad (5)$$

Both inequalities can be tight. In particular [76],

$$C^H_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) = C^S_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) \Leftrightarrow D_{\boldsymbol{\theta}} = \mathbb{0}_n, \quad (6)$$

where $(D_{\boldsymbol{\theta}})_{ij} \equiv \mathrm{Im}(\mathrm{Tr}[L_j L_i \rho_{\boldsymbol{\theta}}])$ is a skew-symmetric matrix [71]. Condition (6) is called weak commutativity and quantum statistical models satisfying it are asymptotically classical [77].

*Computing the bound with an SDP.*—The HCRB is obtained as the result of the following minimization [51,75]:

$$C^H_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) = \min_{V\in\mathbb{S}^n, \boldsymbol{X}\in\mathcal{X}_{\boldsymbol{\theta}}} (\mathrm{Tr}[WV] | V \succeq Z[\boldsymbol{X}]), \quad (7)$$

with the Hermitian $n \times n$ matrix $Z[\boldsymbol{X}]_{ij} = \mathrm{Tr}[X_i X_j \rho_{\boldsymbol{\theta}}]$ and the collection $\boldsymbol{X}$ of operators $X_i \in \mathcal{L}_{\mathsf{h}}(\mathcal{H})$ in the set

$$\mathcal{X}_{\boldsymbol{\theta}} = \{\boldsymbol{X} = (X_1, \ldots, X_n) | \mathrm{Tr}[X_i \partial_j \rho_{\boldsymbol{\theta}}] = \delta_{ij}\}. \quad (8)$$

For a density matrix with rank $r < d$, we can restrict the operators $X_i$ to the quotient space $\mathcal{L}^r_{\mathsf{h}}(\mathcal{H}) = \mathcal{L}_{\mathsf{h}}(\mathcal{H})/\mathcal{L}_{\mathsf{h}}[\ker(\rho_{\boldsymbol{\theta}})]$, with dimension $\tilde{d} = 2dr - r^2$. For any $X \in \mathcal{L}_{\mathsf{h}}(\mathcal{H})$, any scalar quantity evaluated in the eigenbasis of $\rho_{\boldsymbol{\theta}}$ is independent of the diagonal block of $X$ corresponding to the kernel of $\rho_{\boldsymbol{\theta}}$ [51,78] (see Sec. I of the Supplemental Material [79] for details).

We introduce a basis $\lambda_i$ of Hermitian operators for $\mathcal{L}^r_{\mathsf{h}}(\mathcal{H})$, orthonormal with respect to the Hilbert-Schmidt inner product $\mathrm{Tr}[\lambda_i \lambda_j] = \delta_{ij}$. Using such a basis, each operator $X_i \in \mathcal{L}^r_{\mathsf{h}}(\mathcal{H})$ corresponds to a real valued vector

$\boldsymbol{x}_i \in \mathbb{R}^{\tilde{d}}$. With some abuse of notation, we use $\boldsymbol{X}$ to denote also the collection of these real vectors, i.e., the $\tilde{d} \times n$ real matrix with $\boldsymbol{x}_i$ as columns. The quantum state also belongs to $\mathcal{L}^r_{\mathsf{h}}(\mathcal{H})$ and therefore corresponds to a vector $\boldsymbol{s_\theta}$ in the chosen basis. This corresponds to the generalized Bloch vector [89,90] when working in the full space $\mathcal{L}_{\mathsf{h}}(\mathcal{H})$.

A quantum state induces an inner product on $\mathcal{L}^r_{\mathsf{h}}(\mathcal{H})$ via

$$Z[\boldsymbol{X}]_{ij} = \mathrm{Tr}[X_i X_j \rho_{\boldsymbol{\theta}}] = \boldsymbol{x}_i^{\mathsf{T}} S_{\boldsymbol{\theta}} \boldsymbol{x}_j, \tag{9}$$

where $S_{\boldsymbol{\theta}} \succeq 0$ is the Hermitian matrix representing the inner product in the chosen basis. With this choice, we can write $Z[\boldsymbol{X}] = \boldsymbol{X}^{\mathsf{T}} S_{\boldsymbol{\theta}} \boldsymbol{X}$ so that the matrix inequality on the rhs of Eq. (7) reads $V \succeq \boldsymbol{X}^{\mathsf{T}} S_{\boldsymbol{\theta}} \boldsymbol{X}$. Crucially, this last matrix inequality can be converted to a linear matrix inequality (LMI) by using the Schur complement condition for positive semidefiniteness [91],

$$V - B^\dagger B \succeq 0 \Leftrightarrow \begin{pmatrix} V & B^\dagger \\ B & \mathbb{1} \end{pmatrix} \succeq 0, \tag{10}$$

for any matrix $B$ and identity matrix 1 of appropriate size. Thus, we can rewrite the minimization problem in Eq. (7) as

$$\begin{aligned} &\underset{V\in\mathbb{S}^n, \boldsymbol{X}\in\mathbb{R}^{\tilde{d}\times n}}{\text{minimize}} && \mathrm{Tr}[WV] \\ &\text{subject to} && \begin{pmatrix} V & \boldsymbol{X}^{\mathsf{T}} R_{\boldsymbol{\theta}}^\dagger \\ R_{\boldsymbol{\theta}} \boldsymbol{X} & \mathbb{1}_{\tilde{r}} \end{pmatrix} \succeq 0 \\ &&& \boldsymbol{X}^{\mathsf{T}} \frac{\partial \boldsymbol{s_\theta}}{\partial \boldsymbol{\theta}} = \mathbb{1}_n, \end{aligned} \tag{11}$$

where the matrix $R_{\boldsymbol{\theta}}$ can be *any* $\tilde{r} \times \tilde{d}$ matrix [with $rd = \mathrm{rank}(S_{\boldsymbol{\theta}}) \le \tilde{r} \le \tilde{d}$] satisfying $S_{\boldsymbol{\theta}} = R_{\boldsymbol{\theta}}^\dagger R_{\boldsymbol{\theta}}$, e.g., a Cholesky-like decomposition. Here $\partial \boldsymbol{s_\theta}/\partial \boldsymbol{\theta}$ is a matrix with the vector components of the operators $\partial \rho_{\boldsymbol{\theta}}/\partial \theta_i = \partial_i \rho_{\boldsymbol{\theta}}$ as columns; this is the Jacobian matrix of $s_{\boldsymbol{\theta}}$ only if the basis $\{\lambda_i\}$ is parameter independent. The program (11) can be readily recognized as a convex minimization problem [92], the solutions of an LMI form a convex set and the objective function is linear. It can be converted to an SDP (see Sec. II of the Supplemental Material [79] for details), which can be solved numerically using efficient and readily available algorithms with a guarantee of global optimality. In practice, the program (11) can be fed directly to a numerical modeling framework, such as CVX [93] or YALMIP [94].

For every convex minimization, called the primal problem, there exists a maximization, the dual problem, that yields a lower bound to the solution of the former. This property is known as weak duality [92]. Strong duality means that the solution to the primal and the dual problems coincide. Not every SDP satisfies it, but it is a desirable property that certifies an unambiguous solution. A sufficient condition for strong duality is Slater's condition. Qualitatively this means that there must be optimization variables satisfying the inequality constraints strictly.

We now show that our convex optimization problem (11) satisfies Slater's condition as long as $J^S \succ 0$, i.e., a nonsingular quantum statistical model. We denote by $\boldsymbol{L}$ the matrix with the real vectors representing the SLDs as columns. Upon noticing that $(\boldsymbol{L}^{\mathsf{T}} \partial \boldsymbol{s_\theta}/\partial \boldsymbol{\theta})_{ij} = \mathrm{Tr}[L_i \partial_j \rho_{\boldsymbol{\theta}}] = (J^S)_{ij}$, it is easy to show that the matrices $\boldsymbol{X} = \boldsymbol{L}(J^S)^{-1}$ and $V = (J^S)^{-1} + V'$, with an arbitrary $V' \succ 0$, satisfy both constraints in (11). For this choice of $V$ and $\boldsymbol{X}$, the matrix inequalities in (7) and (11) are strict.

An analytical optimization over $V$ in (7) leads to [71]

$$h_{\boldsymbol{\theta}}(\boldsymbol{X}) = \min_{V\in\mathbb{S}^n} (\mathrm{Tr}[WV] | V \succeq Z[\boldsymbol{X}]) \tag{12}$$

$$= \mathrm{Tr}[W \mathrm{Re} Z(\boldsymbol{X})] + ||\sqrt{W} \mathrm{Im} Z[\boldsymbol{X}] \sqrt{W}||_1, \tag{13}$$

so that $C^H_{\boldsymbol{\theta}}(\rho_{\boldsymbol{\theta}}; W) = \min_{X\in\mathcal{X}_{\boldsymbol{\theta}}} h_{\boldsymbol{\theta}}(\boldsymbol{X})$; no general closed-form solution for this last optimization is known. From our previous convexity argument, we also infer that $h_{\boldsymbol{\theta}}(\boldsymbol{X})$ is a convex function of $\boldsymbol{X}$, being a partial minimization of an affine function over a convex set [92]. This may not be apparent from (13) since the second term is not convex; the sum of the two terms is convex as long as the matrix $Z[\boldsymbol{X}]$ is positive semidefinite and the identity (10) can be used.

*Optical interferometry with loss.*—Optical interferometry, where the goal is to measure a phase difference between two optical paths, is a prime example of quantum metrology [2]. In some instances, one may wish to estimate both the phase and the loss induced by a sample in one arm of a Mach-Zehnder interferometer [95].

We consider initial states with a fixed photon number $N$ across two modes $|\psi_{\mathrm{in}}\rangle = \sum_{k=0}^N c_k |k, N-k\rangle$. These include, for example, N00N states and Holland-Burnett states [96]. The evolved state after the lossy interferometer, with one arm characterized by a transmissivity $\eta$ and a phase shift $\phi$, has a direct sum form $\rho_{\phi,\eta} = \bigoplus_{k=0}^N p_l |\psi_l\rangle\langle\psi_l|$, where each $|\psi_l\rangle$ corresponds to $l$ lost photons [97] (see Sec. III of the Supplemental Material [79] for details). For this problem, it is possible to obtain the SLDs $L_\phi$ and $L_\eta$ analytically, as well as the QFIM $J^S = \mathrm{diag}(J^S_{\phi\phi}, J^S_{\eta\eta})$. Crucially, this multiparameter estimation problem is *never* asymptotically classical, since [95]

$$\mathrm{Im}(\mathrm{Tr}[L_\phi L_\eta \rho_{\phi,\eta}]) = -\frac{J^S_{\phi\phi}}{2\eta}. \tag{14}$$

Hence, the weak commutativity condition (6) never holds if the model is nonsingular; thus, we get $C^H_{\boldsymbol{\theta}} > C^S_{\boldsymbol{\theta}} > C^R_{\boldsymbol{\theta}} = 0$ (the RLD bound is completely uninformative [95]). Equation (14) also means that phase and loss cannot be jointly estimated with the same precision obtainable by estimating each parameter individually and there exists a trade-off between precisions. Following Crowley *et al.* [95]

we focus on a strategy to estimate $\phi$ with the best possible precision and still get an estimate of $\eta$, by considering the projective POVM $\mathbf{\Pi}_\phi$ obtained from the spectral decomposition of the SLD $L_\phi$.

More concretely, we study Holland-Burnett states, a family of states particularly resilient to imperfections [98]; we also fix $W = \mathbb{1}_2$. Figure 1 shows the classical CRB $C_\phi^C = \mathrm{Tr}[F(\rho_{\phi,\eta}, \mathbf{\Pi}_\phi)^{-1}]$, along with the HCRB (computed by solving the SDP numerically) and the SLD-CRB $C_{\phi,\eta}^S$, as a function of $\eta$ for $N$ up to 14. Figure 1(a) shows that the HCRB is over 30% tighter than the SLD bound, especially for intermediate transmissivities. Figure 1(b) shows that the measurement we consider attains the HCRB for certain values of $N$ and $\eta$; i.e., the relative difference is zero up to numerical noise. Even when the bound is not attained, the relative difference remains small at around 4% for $N = 14$.

For generic one-photon states ($N = 1$), we have found the analytical conditions for the HCRB to be attained by $\mathbf{\Pi}_\phi$. For $|\psi_{\mathrm{in}}\rangle = c_0|0,1\rangle + c_1|1,0\rangle$ (with $|c_0|^2 + |c_1|^2 = 1$), we have $C_\phi^C = C^H(\rho_{\phi,\eta}; \mathbb{1}_2)$ as long as $|c_1|^2 \geq 1/2$ or $(1 - |c_1|^2/|c_0|^2)/2 \leq \eta \leq 1$. The relative difference $1 - C^H/C_\phi^C$ is at most 4.9% and always zero for $\eta \geq 1/2$ (see Sec. III. A of the Supplemental Material [79] for details). A numerical analysis on random states for higher values of $N$ suggests that there is indeed a threshold value of $\eta$, increasing with $N$, above which $\mathbf{\Pi}_\phi$ attains the HCRB.

Finally, we remark that working in the space $\mathcal{L}_{\mathrm{h}}^r(\mathcal{H})$ provides a distinct advantage for the numerics, since the Hilbert space dimension is $(N^2 + 3N + 2)/2$, while $\rho_{\phi,\eta}$ has rank $r = N + 1$,whereby $\tilde{d} = (N+1)^3 < (N+1)^4$.

*3D magnetometry.*—Noiseless 3D magnetometry, another illustrative example of multiparameter quantum metrology, has been studied terms of the QFIM [9]. Here, we highlight the necessity of using the HCRB for this problem and present results on 3D magnetometry using $M$ qubits in the presence of dephasing noise. The parameters to be estimated $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ appear via the single-qubit Hamiltonian $H^{(j)}(\boldsymbol{\varphi}) = \boldsymbol{\varphi} \cdot \boldsymbol{\sigma}^{(j)}$, where $\boldsymbol{\sigma}^{(j)}$ is a vector of Pauli operators acting on the $j$th qubit. The parameters are imprinted on the probe state via the unitary $U_{\boldsymbol{\varphi}} = \otimes_j^M \exp[-\mathrm{i}H^{(j)}(\boldsymbol{\varphi})]$. This is followed by local dephasing along the $z$ axis described by the single-qubit map $2\mathcal{E}_\gamma[\rho] = (1 + \sqrt{1-\gamma})\rho + (1 - \sqrt{1-\gamma})\sigma_z\rho\sigma_z$, with $\gamma \in [0, 1]$; an approximation valid when the sensing time is short.

We use as probe states the family of 3D–Greenberger-Horne-Zeilinger (GHZ) states

$$|\psi_M^{\text{3D-GHZ}}\rangle = \frac{1}{\mathcal{N}} \sum_{k=1}^{3} |\phi_k^+\rangle^{\otimes M} + |\phi_k^-\rangle^{\otimes M}, \qquad (15)$$

which was shown to present Heisenberg scaling in the noiseless case [9]; $|\phi_k^\pm\rangle$ are the eigenvectors corresponding to the $\pm 1$ eigenvalues of the $k$th Pauli matrix and $\mathcal{N}$ is the normalization. The final state for which we compute the bound is $\rho_{\boldsymbol{\varphi}} = \mathcal{E}_\gamma^{\otimes M}[U_{\boldsymbol{\varphi}}|\psi_M^{\text{3D-GHZ}}\rangle\langle\psi_M^{\text{3D-GHZ}}|U_{\boldsymbol{\varphi}}^\dagger]$; for the numerical results, we choose equal parameter values $\varphi_i = 1 \ \forall \ i$ and $W = \mathbb{1}_3$.

In Fig. 2 we show the nontrivial relationship between the HCRB, the SLD-CRB, and incompatibility for this quantum statistical model, as a function of the dephasing strength $\gamma$. We quantify the incompatibility of the model with the magnitude of the matrix $D_{\boldsymbol{\theta}}$, capturing the violation of the weak commutativity condition (6); in particular, we use the Frobenius norm $||D_{\boldsymbol{\varphi}}||_F^2 = \sum_{ij} |(D_{\boldsymbol{\varphi}})_{ij}|^2$. Figure 2(a) shows that the relative difference $1 - C^S/C^H$ is monotonically decreasing for 2 and 3 qubits, while it has a non-monotonic behavior for 5 or more qubits. Figure 2(b) shows that this behavior is not always reflected at the level of incompatibility; this is remarkably different from the simple monotonic relationship found for two-parameter pure state models [61,64]. Furthermore, while the matrices $D_{\boldsymbol{\theta}}$ have a comparable magnitude for different number of qubits, the relative differences do not, e.g., being around 0.2% for 5 qubits and around 30% for 2 qubits.

Our SDP formulation grants us a previously inaccessible ease in the evaluation of the HCRB. In turn, this enables us to get these glimpses into the noncommutative information geometry of three-parameter mixed state problems of nontrivial dimension. From this example, we see that the pair of matrices $J^S$ and $D_{\boldsymbol{\theta}}$ are not sufficient for a complete description of a given state's performance for multiparameter estimation.

Finally, we concentrate on the noiseless ($\gamma = 0$) case with 2 qubits, noting that a single qubit does not allow us to estimate all the components of $\boldsymbol{\varphi}$ [9]. In Fig. 2 we see that

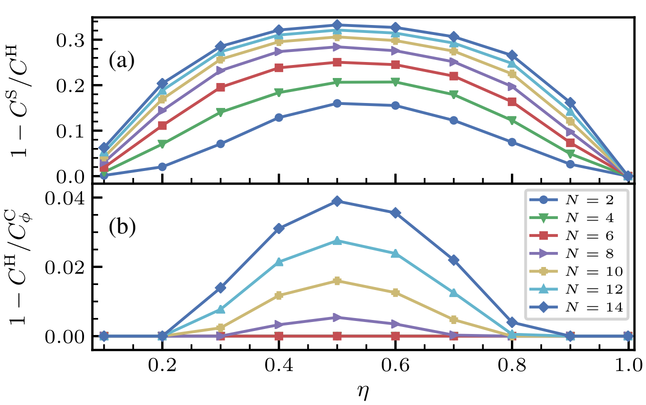


FIG. 1. Relative difference between different CRBs for simultaneous estimation of phase and loss, as a function of the transmissivity $\eta$, for $N$-photons Holland-Burnett probe states and $W = \mathbb{1}_2$. (a) Relative difference between the SLD-CRB and the HCRB. (b) Relative difference between the HCRB and the classical CRB for the optimal phase measurement $C_\phi^C = \mathrm{Tr}[F(\rho_{\phi,\eta}, \mathbf{\Pi}_\phi)^{-1}]$; this quantity is zero (up to numerical noise) for $N \leq 6$.

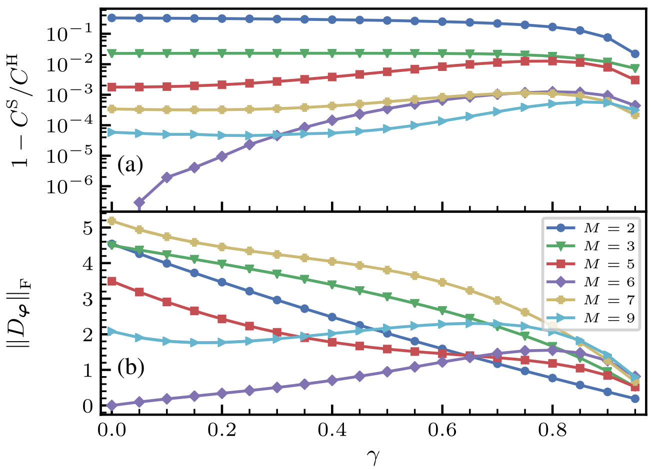


FIG. 2. Comparison of the SLD-CRB, the HCRB (both with $W = \mathbb{1}_3$), and incompatibility for a $M$-qubit 3D-GHZ probe state undergoing a 3D phase rotation with equal values of parameters $\varphi_i = 1$, followed by local dephasing of strength $\gamma$ in the $z$ direction. (a) Relative difference $1 - C^S/C^H$ between the SLD-CRB and HCRB. (b) Incompatibility of the quantum statistical model, quantified by the Frobenius norm of the matrix $(D_{\boldsymbol{\varphi}})_{ij} = \mathrm{Im}(\mathrm{Tr}[L_j L_i \rho_{\boldsymbol{\varphi}}])$, showing incompatibility for all $\gamma$ (except for $M = 6$ and $\gamma = 0$). Data for $M = 4, 8$ is not shown on the plot, since such models appear to be asymptotically classical, with $\|D_{\boldsymbol{\varphi}}\|_F \lesssim 10^{-7}$ (consistently, we find $1 - C^S/C^H \lesssim 10^{-6}$).

for $M = 2$ the SLD bound is considerably looser than the HCRB, with a relative difference around 30% for $\gamma = 0$. On the contrary, we conjecture that the HCRB is attainable with single-copy projective measurements. We base this on the numerical equality between the HCRB and a numerical minimization of the classical scalar CRB over all 2-qubit projective measurements. For 5000 random initial states with parameter values taken from five sets, the relative difference between the two quantities was always found to be smaller than $10^{-4}$ (see Sec. IV.A of the Supplemental Material [79] for details). While for pure states the HCRB is always attainable with single-copy measurements, the optimal POVM needs not be projective [61], making it harder to implement experimentally. This finding shows that optimal protocols for 3D magnetometry with 2 qubits may be not too far from experimental reach.

*Conclusions.*—We have shown how to evaluate the HCRB by solving an SDP, making it more easily accessible than previously believed. This enabled us to study two examples—optical interferometry and 3D magnetometry—and gather numerical evidence that the HCRB is attainable by single-copy projective measurements, whereas the SLD bound is not. These findings suggest that there may be further unstudied cases where the HCRB is easier to attain than naively expected. They also illustrate the potential of our formulation to enable new discoveries in multiparameter quantum estimation, which should aid a deeper quantitative understanding of quantum measurements more generally.

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