Interpreting Observations of Ion Cyclotron Emission from Energetic Ion Populations in Large Helical Device plasmas

by

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Go forth on your path, as it exists only through your walking.
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3.5 Panels reprinted with permission from A. Bustos et al. 2011 *Nucl. Fusion* **51** 083040 [14]. Left and middle panels: Initial fast ion distribution computed with HFREYA [15] represented as a function of the parameters $\lambda$, $E$, $\rho$ and $\phi$ corresponding to pitch, energy, plasma radius (expressed in terms of normalized toroidal flux $(\Psi/\Psi_{LFS})^{1/2}$) and toroidal angle respectively. Right panels: Time evolution of the average energy (top) and persistence (bottom) of perpendicular NBI hydrogen in LHD plasmas computed with the code ISDEP [16]. The persistence is defined as the probability that fast ions have remained in the plasma at a given time. Three phases are identified: prompt losses ($t \sim 0.5 \times 10^{-4}$ to $10^{-4}$s), slowing-down phase ($t \sim 10^{-4}$ to $10^{-2}$) and loss of confinement $t > 10^{-2}$, computed with ISDEP.

3.6 Left: Panel reprinted with permission from T. Ozaki; E. Veshchev; T. Ido; A. Shimizu; P. Goncharov; S. Sudo; *Review of Scientific Instruments* **2012**, 83, DOI: 10.1063/1.4742925, Copyright ©2012 American Institute of Physics [17]. Angularly-resolved velocity distribution of fast neutral particles measured (trapped) with ARMS-Neutral Particle Analyser showing particle loss characteristics at different magnetic field values. The graph is expressed in polar coordinates. Positive angle is measured from magnetic axis to a given sightline. The minimum measurable energy is 15keV, which appears as the lowest white half-circle, followed by 30 and 45keV. The energy resolution is a few keV. Colouring is on a log scale. Right: Panel reprinted with permission from A. Bustos et al. 2011 *Nucl. Fusion* **51** 083040 doi:10.1088/0029-5515/51/8/083040 [14]. Velocity space distribution computed with the code ISDEP [16] and representing the escape of perpendicular NBI hydrogen in LHD when they hit the vacuum vessel. The different graphs correspond to different times, early times at the top and later times at the bottom. A wedge-shaped, strongly anisotropic distribution is visible, which diffuses in velocity space at later times.

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**xi**
3.7 Time evolution (horizontal axis, in units of proton gyroperiods $\tau_H$) of the energy density of different field components and particle populations in four PIC-hybrid simulations. Left panels are for LHD hydrogen plasma 79126 with sub-Alfvénic 40 keV NBI protons, for which $v_{NBI}/V_A = 0.872$. Right panels are for LHD hydrogen plasma 79003 with super-Alfvénic 36.5 keV NBI protons for which $v_{NBI}/V_A = 1.125$. Top panels are for propagation angle of $k$ with respect to $B_0$ of 89.5°. Bottom panels are for propagation angle 85°. Concentration $\xi = n_{NBI}/n_e$ is chosen at a level that gives rise to saturation of the MCI within the simulation run time: $\xi = 5 \times 10^{-4}$ at top left and bottom right; $\xi = 7.5 \times 10^{-4}$ at top right; and $\xi = 5 \times 10^{-3}$ at bottom left. Red trace: bulk protons. Cyan: NBI protons. Green $z$-component and magenta $y$-component of the fluctuating part of the magnetic field. Dark blue: $x$-component of the electric field. The excitation of the $y$-component of the magnetic field is due to the NBI protons that have a velocity component along the $z$ direction, while $k$ is along the 1D simulation domain ($\hat{x}$ direction). The NBI protons typically release 10-15% of their energy to the background protons and to the electric and magnetic fields. Saturation of the instability occurs after $\approx 1 \mu$s to 100$\mu$s. This time is similar to the time window used to compute the LHD power spectra (see Fig 2. of [7]).
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Bright spots at sequential proton cyclotron harmonics along the fast-Alfvén branch result from the MCI, driven by NBI protons, for waves propagating in the $\hat{x}$ direction, almost perpendicular to the background magnetic field. The dark trace corresponds to $\omega = kv_{NBI}$ and lies either above or below $\omega = kV_A$, which defines the boundary between regions of $(\omega,k)$ space that can in principle resonate with sub-Alfvénic or super-Alfvénic NBI protons. The excitations occur along the fast Alfvén branch and preferentially close to modes satisfying $\omega = kv_{NBI}$.

3.9 Power spectra of the fluctuating $z$-component of the magnetic field in the four PIC-hybrid simulations whose energy evolution and spatio-temporal Fourier transforms are shown in the corresponding panels of Fig. 3.7 and Fig 3.8, respectively. Power spectra are obtained using different orientations of the background magnetic field with respect to the spatial domain of the 1D3V simulation, which defines the direction of $k$. Power spectra are obtained by taking the spatio-temporal Fourier transform $\delta B_z(t) = B_z(t) - B_{0,z}$, averaged over the simulation duration and summed between $k = 0$ and $k = 25\Omega_H/V_A$.

Peaks at multiple successive proton cyclotron harmonics are captured.

3.10 Comparison of measured and simulated ICE spectra, plotted on dB scales. Top panels: measured LHD ICE power spectra during sub-Alfvénic (left) and super-Alfvénic (right) perpendicular proton NBI, reproducing Fig. 3.2. Bottom panels: power spectra of $\delta B_z^2/B_0^2$ obtained from our PIC-hybrid simulations for parameters corresponding to LHD plasmas 79126 (left) and 79003 (right), see Table 3.1, reproducing the bottom two panels of Fig. 3.9.
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3.12 Linear fit of early phase growth rate $\gamma_\ell$ inferred from the simulations, normalized to $\Omega_H$, against the square root of $\xi = n_{NBI}/n_e$, subject to the constraint that for a zero density beam, the line should intersect the origin. It is evident that the ICE growth rate $\gamma_\ell/\Omega_H$ scales as $\sqrt{\xi}$. Left panel is for cyclotron harmonic mode $\ell = 11$ of the excited wave in the ICE emitting region of LHD plasma 79126 with locally sub-Alfvénic 40keV NBI protons, and each point is for a value of $\xi$ between $0.5 \times 10^{-4}$ and $7.5 \times 10^{-4}$. Right panel is for cyclotron harmonic mode $\ell = 12$ of the excited wave in the ICE emitting region of LHD plasma 79003 with locally super-Alfvénic 36.5keV NBI protons, and the values of $\xi$ are between $1 \times 10^{-4}$ and $2 \times 10^{-3}$. 
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3.14 Enhanced sub-Alfvénic NBI proton relaxation via the magnetoacoustic cyclotron instability under LHD plasma 79126 edge conditions. The NBI protons have an initial energy of 25keV corresponding to $v_{NBI}/V_A = 0.7$. Top left: Energy density time evolution of the different components of the electric and magnetic fields and of the background and NBI beam protons, see Fig. 3.7. Top right: Power spectrum of the $z$-component of the fluctuating part of the magnetic field in our PIC-hybrid simulations, on a dB scale, obtained as in Fig. 3.11. Bottom left: spatio-temporal fast Fourier transform of $\delta B_z$, the fluctuating part of the $z$-component of the magnetic field represented on a log$_{10}$ scale. The dark straight line shows $\omega/k = v_{NBI}$, as in Fig. 3.8. The grey curves are the growth rates $\gamma$ computed from the simulations with associated dark error bars. All fits of the growth rates are computed between $0.20\tau_H$ and $3.3\tau_H$. Bottom right: time evolution of the spatial fast Fourier transform of $\delta B_z$, on a log$_{10}$ scale, whose original LHD 79126 40keV NBI proton wavenumber time evolution appears in Fig. 3.11. The beam density $\xi = 0.0005$. The angle between the background magnetic field and the $k$ vector is $89.5^\circ$. . .
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3.16 Absolute kinetic energy density time evolution of the bulk and NBI protons in the enhanced sub-Alfvénic simulation under LHD 79126 edge plasma parameters (left) and the enhanced super-Alfvénic simulation under LHD 79003 edge plasma parameters (right). The fast ions release $\approx 10\%$ of their initial energy to the electric and magnetic fields and to the bulk protons which experience a relative kinetic energy increase of $\approx 0.1\%$ to $\approx 15\%$. 

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3.17 Dependence of early phase growth rates in the simulations on NBI ion concentration $\xi$, obtained across multiple cyclotron harmonics $\ell$. Top and bottom panels correspond to LHD plasmas 79003 and 79126 respectively. Left: collapsed plot of $(\gamma_\ell/\Omega_H)/\alpha_\ell$, versus $\xi$. Here $\gamma_\ell = \gamma_\ell (\xi)$ is the growth rate inferred from the simulations for mode number $\ell$, and depends on the relative NBI density $\xi$ at the ICE location, and $\alpha_\ell$ is the calculated slope from the best linear fit of $\gamma_\ell$ as a function of $\sqrt{\xi}$. Right: translated compensated plot of the quantity $(\gamma_\ell/\Omega_H)/\sqrt{\xi}$ versus $\xi$, with $\gamma_\ell$ inferred from simulations. From the linear theory of the MCI, $\gamma_\ell/\Omega_H = \alpha_\ell\sqrt{\xi}$, which means that $(\gamma_\ell/\Omega_H)/\alpha_\ell = \sqrt{\xi}$ such that the plot $(\gamma_\ell/\Omega_H)/\alpha_\ell$ with respect to $\sqrt{\xi}$ should bring the identity. This is the collapsed plot. Similarly, $(\gamma_\ell/\Omega_H)/\sqrt{\xi} = \alpha_\ell$, a quantity that does not depend on $\xi$, but on the mode value $\ell$ only, corresponds to the compensated plot. The values of $\xi$ typically span one order of magnitude, between $10^{-4}$ and $10^{-3}$. These graphs show that within error bars, $\gamma_\ell \propto \sqrt{\xi}$.

3.18 (Top) Time evolution of the change in the electric and magnetic field energy density and in the beam and background proton kinetic energy in 2 massless electron PIC-hybrid simulations. (Bottom) Spectral power of the fast Fourier transform of $\delta B_z$ taken over the whole simulations domain over $20 \tau_H$. The graphs are obtained from simulations initialized with LHD plasma parameters 79126 with sub-Alfvénic 40 keV NBI proton distributed in velocity-space as $f_{NBI} = \delta (v_\parallel) \frac{1}{2\pi v_{r\perp}} \exp \left[-(v_\perp - v_{NBI})^2/v_{r\perp}^2\right]$, with $v_{r\perp}^2/v_{NBI}^2 = 0.1$ (left panels) and $f_{NBI} = \frac{1}{2\pi \delta (v_\parallel) \delta (v_\perp - v_{NBI})}$ (right panels), both with $\xi = 0.0005$. The saturation energy of the NBI is decreased with the inclusion of a finite spread. However, examination of the power spectra in Fig. 3.19 averaged over the whole simulation show an increase in intensity at lower harmonics up to approximately the lower hybrid frequency. The power is distributed more evenly in the finite spread case than in the delta peaked distribution scenario.
3.19 Top: Power spectra of the \( z \)-component of the fluctuating part of the magnetic field generated by simulations initialized with LHD plasma parameters 79126 with sub-Alfvénic 40 keV NBI proton. The left (right) panel corresponds to a simulation which has 10% (0%) perpendicular thermal spread \( v_{r,\perp}^2/v_{NBI}^2 \) in the proton beam distribution as described in Fig. 3.18. The relative beam density is \( \xi = 0.0005 \). The power spectra are obtained by taking the spatio-temporal Fourier transform of \( \delta B_z \), averaged over 20 proton gyrations and summed between \( k = 0 \) and \( k = 25\Omega_H/V_A \). Middle: \((k,t)\) plots showing the time evolution of the spectral power of wavenumbers on a \( \log_{10} \) scale. They are obtained by taking the square modulus of the spatial Fourier transform of \( \delta B_z (k,t) \). Bottom: Time evolution of the power of several wavenumbers \( kV_A/\Omega_H \) on a \( \log_2 \) scale corresponding to a specific proton cyclotron harmonic through the dispersion relation, as shown on the bottom panels of Fig. 3.18.

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4.1 Description of the filter bank used for the acquisition of the ICE data in LHD. Reproduced from T. Akiyama, 34th ITPA conference meeting.
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4.4 Description of ICE in LHD (left) and approximate location of the ICE emission during perpendicular deuterium NBI (right).

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6.4 Inertial hybrid kinetic dispersion relation for LHD 79003 plasma parameter. The relative beam densities $\xi$ are (from to left to right), $\xi = 7.5 \times 10^{-5}$, $\xi = 7.5 \times 10^{-4}$ and $\xi = 7.5 \times 10^{-3}$.

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Declarations

The present work is submitted to the University of Warwick in support of the author application for the Degree of Doctor of Philosophy. It is been carried out by the author unless otherwise stated and has not been submitted in any previous application for any other degree or any other university. This thesis has been undertaken between October 2014 up until October 2018 under the supervision of Prof. Richard O. Dendy and Prof. Sandra C. Chapman.

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Chapter 3


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Abstract

Ion cyclotron emission (ICE) has been reported from most magnetic confinement fusion devices (MCF), both large tokamaks and stellarators. The ICE phenomenology is rich as it results from fusion products and from neutral beam injection (NBI) energetic ions. These can drive ICE on confined trajectories or during MHD instabilities as they are expelled from the plasma. In either scenario, an inversion in velocity space perpendicular to the local magnetic field is thought to take place. In particular ICE was observed during DT experiments in both JET and TFTR, driven by the 3.5MeV $\alpha$ particle. Self-consistent hybrid [19] and full [20] particle-in-cell (PIC) simulations of the magnetoacoustic cyclotron instability (MCI) [21] captured ICE features related to JET ICE. The linear scaling of the ICE intensity with the $\alpha$ particle density, used as a proxy for the measured neutron flux, was reported in Ref. [22]. Ion cyclotron emission has been proposed as a noninvasive diagnostic in ITER [23].

In this thesis, we study ICE detected in the Large Helical Device (LHD) heliotron stellarator with hybrid PIC simulations. Unlike full PIC, hybrid PIC treats the electron as a neutralising fluid while the ion species are still fully kinetic. We focus on NBI-driven ICE in hydrogen plasmas and explore the sub- and super-Alfvénic regime where NBI proton speeds $v_{NBI}$ are either lower or higher than the Alfvén speed which is determined at the emission location. Our simulations show that relaxation of these protons drive the MCI, at perpendicular and oblique propagation angles, and spectral peaks at proton cyclotron harmonics are obtained.

Deuterium was introduced for the first time during the 2017 LHD campaign [24] and brought ICE measurements with it. The spectral features of NBI-driven ICE are reported to qualitatively and quantitatively vary across LHD plasma discharges. We interpret the observations to be the result of the different edge plasma density which controls the ratio $v_{NBI}/V_A$ and affects the ICE power spectra as suggested by hybrid PIC simulations. Doppler-shifted ICE is detected during MHD instabilities [25]. Fusion-born protons could be driving the ICE and account for the large Doppler shifts as explored via hybrid simulations. We perform cepstra analysis on the measured spectra to identify which cyclotron harmonics are present.

The hybrid kinetic dispersion relation with inertial electrons is derived, generalizing the massless case [26]. Solutions are obtained in the case of background Maxwellians and energetic ring beams and compared against the full- and neutral-kinetic dispersion relations.
Chapter 1

Introduction

1.1 Fusion energy

Fusion is the mechanism whereby stars generate energy by combining light elements under the strong pressure forces resulting from their gravity. For more than half a century, many experiments have been conducted in an attempt to generate fusion on the ground. At a temperature of 1keV in the sun, the hydrogen atoms fuse at a low rate. To achieve fusion on earth, the reaction of deuterium (D) with tritium (T)

\[ ^2D + ^3T \rightarrow ^4He + n + 17.6\text{MeV} \] (1.1)

into a 3.5MeV α particle and a 14.1 MeV neutron is considered the most promising way due to its high cross section, at higher temperatures. Magnetic Confinement Fusion (MCF) is one approach to generate such fusion power by means of strong magnetic fields to confine a plasma while it is being heated. Different designs are being tested including the tokamak and the stellarator. The goal of those devices is to go beyond breakeven to achieve ignition: to overcome the losses to create a self-burning plasma which maintains fusion reactions through the heat it generates (temperature maintained by the trapped-α particles). This is reflected in the product of the density with the temperature and confinement time \( nT\tau_E \), also called triple product which sets a lower limit on the temperature of approximately 10keV to ignite the International Thermonuclear Experimental Reactor (ITER). ITER results from the collaboration between 35 countries and is considered to be the largest such one on earth. It has entered its construction phase in 2007 and the first plasma is expected in 2025. ITER should deliver 500MW of power for 50MW of input power (referred as to \( Q = 10 \)). For comparison, the Joint European Torus (JET) at Culham generated 16MW of fusion power for 24MW of input power in 1997.
Controlled thermonuclear fusion is driving a wide variety of fields (heating, cryogeny, material science, MHD, supercomputing) and ITER should demonstrate the operation of these integrated technologies. The major goal is to generate a self-sustained DT plasma. On the longer term, the design and operation of a fusion power plant should be realised with the Demonstration Power Plant (DEMO), before commercial fusion reactors arise. The DT reactions generate neutrons together with an extreme heat load, therefore a beryllium blanket will directly face the plasma to protect the inner walls and the magnets in ITER. In advanced scenarios, it should be replaced to breed tritium with $^6\text{Li}$ and $^7\text{Li}$. A neutron reaction such as $p + ^{11}\text{B} \rightarrow 3\alpha + 8.7\text{MeV}$ whose elements are plentiful is considered as a mean of power generation in Ref. [27]. A Moore’s-like law for the progress of the fusion triple product is shown in Fig. 1.1: it has doubled every 1.8 years while Moore’s original law says that the number of transistors in a microprocessor has doubled every two years. In addition to the fundamental technology progresses it brings, thermonuclear fusion could also be a part to the solution of the energy demand which is currently globally met by oil [28] and this while IPCC has recently issued a report warning of the risks related to a $2^\circ$C global warming increase, recommending policy makers to act to contain it to $1.5^\circ$C [29].
1.2 Tokamaks and stellarators

The interest in tokamaks, invented by Soviet physicists Igor Tamm and Andrei Sakharov, attracted much interest after the announcement that the Soviet T-3 tokamak had achieved electron temperatures of 1keV, a major result. Scepticism prevailed and Soviets invited a UK delegation at the Kurchatov Institute during Cold War in 1968 to let them make their own temperature measurements. The Soviet observations was confirmed and the tokamak concept started to be explored worldwide. A tokamak is a device used to develop controlled thermonuclear fusion, illustrated in Fig. 1.2. It consists of an arrangement of magnetic field lines that are wound around a torus generating helix shapes. The development of the safety factor permitted to reduce detrimental instabilities including the cancellation of the pinch effect. The safety factor corresponds to the ratio of the number of times a magnetic field line goes around the toroidal direction as it goes along the poloidal direction and needs to be greater than one. Instabilities were strongly suppressed provided the safety factor is above 1 implying the wavelength of the (potentially) unstable mode is larger than the device. A solenoid creates an axial magnetic field by means of currents looping around it. Therefore a plasma created at its center would be confined but free to move along its axis. This can be circumvented by bending the solenoid to join its ends and close it in a donut shape. The magnets around it are closer on the inner side of the donut than on the outer side. The charged particles undergo a magnetic drift (vertical) that depends on their charge as a result of this magnetic field gradient. This loss of particles is avoided by joining top and bottom with a poloidal magnetic field. In a tokamak a time varying magnetic flux in the plasma column creates the toroidal current. This current also heats the plasma through the Joule effect which becomes less effective as the temperature increases. The stellarator concept was invented in 1951 by Spitzer with the shape-eight idea. A schematic representation is shown in Fig. 1.3. The aforementioned drift is always upward (or downward) in a torus whereas if the torus is twisted in a shape of eight, a magnetic field line on the low field side (outer part of the torus) would later be on the high field side and the drift would change sign. Instead of generating a current in the plasma, the magnetic field is externally generated by magnetic fields specifically arranged. While a tokamak is 2d mostly, the magnetic field arrangement is 3d which made their design complicated. They have been the object of renew interest with the advent of supercomputing which has allowed more precise design. The stellarator, contrary to the tokamak which operates in pulsed mode, works in steady state conditions. Tokamaks and stellarators are heated with
Figure 1.2: Reproduced from Ref. [2]. Representation of a tokamak displaying the magnetic field coils which generate the toroidal magnetic field. An increasing current in the inner poloidal field coils create a rising magnetic flux. In turn, a toroidal electric field is set up and results in a current which induces a poloidal magnetic field. The addition of the poloidal and toroidal magnetic fields establishes magnetic field lines that wind around the torus in a helical shape. A drawback comes from the fact that the current cannot be increased indefinitely and as such the tokamak does not operate in continuous mode.

Figure 1.3: Reproduced from Ref. [3]. The complex 3D layout of the magnetic field coils generate the helical field lines winding around the stellarator. Contrary to the tokamak, no current flowing through the plasma is necessary to sustain the poloidal magnetic field. The stellarator has the advantage to operate in steady state mode.
neutral-beam injection (NBI), electron cyclotron resonance heating (ECRH) as well as with ion cyclotron resonance heating (ICRH).

1.3 Ion cyclotron emission

Suprathermal ion cyclotron emission [30, 31] (ICE) is detected from all large toroidal magnetic confinement fusion (MCF) plasmas including TFR [32], PDX [33], JET [34], TFTR [5], JT-60U [35], ASDEX-U [36], KSTAR [37], DIII-D [38] and is observed in stellarators, LHD [7, 39], W7-AS [40]. ICE is notable as the first collective radiative instability driven by confined fusion-born ions that was observed in deuterium-tritium (D-T) plasmas in JET and TFTR [41–44]. The frequency spectrum of ICE typically exhibits narrow peaks at values which can be identified with sequential local cyclotron harmonics of the energetic ions, as shown in Fig. 1.4. The numerical value of the ion cyclotron frequency \( \Omega_c = Z_i eB/m_i \), where \( Z_i \) is the ion charge and \( m_i \) its mass is determined by the local value of the magnetic field strength which can be identified in the emitting region, from which the radial location is inferred. Typically, but not invariably, this is at the outer mid-plane edge of the plasma; ICE from the core plasma has been reported recently from DIII-D [45] and from ASDEX-Upgrade [46], a development which suggests great potential for the exploitation of ICE as a diagnostic for energetic particles in ITER [23]. The measured ICE power scaled with the measured neutron activity in JET DT plasmas, see Fig. 1.5. The measurements could be taken by wave reflectometry [47, 48] or with a B-dot probe as was designed and successfully implemented in ASDEX-U for wavenumbers, polarisation and amplitude measurements including ICE [49]. In a nutshell, the different types of ICE identified so far are [50]:

- mICE or minority ions ICRH-accelerated driven ICE
- Beam-driven ICE
- Fusion Products (FP-ICE)
- Central-ICE
Figure 1.4: (Left) Reproduced from Ref. [4]. JET ion cyclotron spectrum peaking at sequential alpha particle cyclotron harmonics for D and DT. (Right) Reproduced from Ref. [5] TFTR ion cyclotron spectrum in DT beam injection.

Figure 1.5: Reproduced from Ref. [4]. (Left) Large orbit excursions of \( \alpha \)-particles during JET DT. (Right): Correlation between ICE power of the second \( \alpha \) cyclotron harmonic with function of neutron activity over six order of signal intensity.
1.4 The Large Helical Device

The Large Helical Device (LHD), shown in Fig. 1.6, is a stellarator [51] which is located in Toki in the prefecture of Gifu in the central region of Japan. It is owned by the National Institute for Fusion Science (NIFS) and is the second largest superconducting stellarator worldwide after Wendelstein 7-X. The Large Helical Device, which started operating in 1998, is characterised by a major radius of 3.6m and a minor radius of 0.65m and has a background magnetic field of up to 3T. The methods for heating the plasma include Ion Cyclotron Heating (ICH) and Electron Cyclotron Heating (ECH) along with Neutral Beam Injection (NBI). The achieved plasma parameters and the new goals for LHD are shown in Table 1.7. The first deuterium LHD campaign started in March 2017, from which ICE data are studied in this work in Chapter 4, and should extend over a period of 9 years. To that purpose, LHD is equipped with three tangential (tNBI, 180keV for hydrogen) and two radial (also called perpendicular, rNBI, 40keV for hydrogen) neutral beam injection systems. The former are negative-ion sources based, while the latter are positive-ion based sources. The rNBIs have been upgraded to increase their power from 6 to 9MW. The divertor has been closed to stop particle recycling (plasma ions hitting the divertor and returning in the plasma as neutrals [52]) from the divertor plates. In addition, neutron diagnostics were installed [6]. Among important topics, the isotope effect will be studied. This translates into the empirical scaling, resting on experiments, that the energy confinement time \( \tau_e \) scales with the ion atomic number (A) as \( \tau_e \propto A^{0.5} \) although gyro-Bohm scaling predicts that the confinement time scales as \( \tau_e \propto A^{-0.2} \) [6]. The confinement of Energetic Particles (EP) will also be further studied during the LHD deuterium experiments using Charge eXchange Neutral Particle Analyzer (CX-NPA) as described in Chapter 3 when discussing the estimation of the NBI fast protons velocity distribution function. This measurement technique is complemented with Scintillator Lost Ion Probes (SLIP) measurements. The interactions between EP and EP-driven instabilities are also considered [6], such as with Alfvén Eigenmodes (AE’s) [53] and with the Energetic Interchange Mode (EIC) [25]. Studies of confinement properties will gain from neutron diagnostics because the neutron emission rate is governed by the reactions between the energetic deuterium ions and the bulk ions and provide data which are integrated in velocity space [6].

The detection of ICE, which is directly related to this work, has been reported in LHD hydrogen plasmas during NBI heating, for which a typical ICE power spectrum is shown in Fig. 1.8 and has peaks at the fundamental cyclotron frequency.
of the fast proton and at its harmonics (evaluated in the outer edge). The interpretation of these measurements are the object of Chapter 3. ICE has also been observed during Toroidal Alfvén eigenmodes in LHD as shown in Fig. 1.9. Recent ICE measurements from the first deuterium campaign are studied in Chapter 4.

<table>
<thead>
<tr>
<th>Plasma Parameters</th>
<th>Achieved</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ion temperature</td>
<td>(8.1,\text{keV at } 1 \times 10^{19},\text{m}^{-3})</td>
<td>(10,\text{keV at } 2 \times 10^{19},\text{m}^{-3})</td>
</tr>
<tr>
<td>Electron temperature</td>
<td>(20,\text{keV at } 2 \times 10^{19},\text{m}^{-3})</td>
<td>(10,\text{keV at } 2 \times 10^{19},\text{m}^{-3})</td>
</tr>
<tr>
<td>Density</td>
<td>(1.2 \times 10^{20},\text{m}^{-3}) at (T_e) of 0.25 keV</td>
<td>(4 \times 10^{19},\text{m}^{-3}) at (T_e) of 1.3 keV</td>
</tr>
<tr>
<td>Beta</td>
<td>5.1% at 0.425 T</td>
<td>5% at 1-2 T</td>
</tr>
<tr>
<td>Steady-state operation</td>
<td>54 min. 28 sec (0.5 MW)</td>
<td>1 hour</td>
</tr>
<tr>
<td></td>
<td>(1 keV, (4 \times 10^{19},\text{m}^{-3}))</td>
<td>(3 MW)</td>
</tr>
<tr>
<td></td>
<td>47 min. 39 sec (1.2 MW)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2 keV, (1 \times 10^{20},\text{m}^{-3}))</td>
<td></td>
</tr>
</tbody>
</table>

Figure 1.6: Top view of the Large Helical Device, Google Image 2018.

Figure 1.7: Table reproduced from Ref. [6]. Achieved and targeted plasma parameters in LHD.
Figure 1.8: Figure reproduced from Ref. [7]. ICE power spectrum driven by perpendicular NBI in LHD plasma 73331.

Figure 1.9: Figure reproduced from Ref. [8]. Correlations is shown between the time evolution of the power of the ICE fundamental cyclotron frequency measured with a high-frequency magnetic probe (top panel) and the low frequency magnetic field fluctuations identified as the toroidal Alfvén eigenmode (TAE) together with the measured lost fast ion flux measured with the Scintillator Lost Ion Probe (SLIP)
1.5 Context

Most of the richness in plasma physics stems from the extreme variety in the different time and length scales at play, due to the large difference in masses between the electrons and the ions. As a corollary, plasma modelling can prove to be challenging such that an important aspect of the research is to find a trade-off between models that are tractable to solve (on a modern computer for example) while retaining enough accuracy by allowing the coupling between the different scales present in the plasma. We illustrate this wide range of lengths and times scales with the cold plasma dispersion relations shown in Fig. 1.10 representing the relation between frequencies $\omega$ and wavenumbers $k$ existing in a magnetised plasma. The frequencies span the ion cyclotron frequency to the electron cyclotron frequency, and well beyond (plasma frequencies if bigger), through to intermediate frequencies such as the lower and upper hybrid frequencies involving both electrons and ions physical parameters. At large scales, waves well below the ion cyclotron frequency exist and can be predominant. The plots represent two extreme cases corresponding to waves propagating parallel (left) or perpendicular (right) to the background magnetic field. The physics of ICE is related to the latter and more specifically involves the compressional Alfvén wave at multiple harmonics of the cyclotron frequency of energetic ions, such as neutral-beam injected (NBI) ions or fusion products (FP) in fusion plasmas. More specifically, it is thought that ICE results from the resonance between the fast-Alfvén wave, supported by the majority thermal plasma, and ion-cyclotron waves sustained by an energetic ion population that has undergone an inversion in velocity-space. The instability leading to this resonance is called the magnetoacoustic cyclotron instability [21, 54] and its origin will be further developed in the next sections. We illustrate typical numerical dispersion relations obtained from PIC-hybrid simulations in Fig. 1.11. The left panel shows the numerical dispersion relation for a pure thermal plasma while the right panel includes a thermal plasma along with a minority fast ion population which exhibits an inversion in velocity space. The resulting dispersion relation displays strong excitations at multiple cyclotron harmonics along the fast Alfvén wave. The scales intensities broadly differ between the two plots. The lower hybrid frequency in these simulations is around $40\Omega_H$ which helps to compare this range of frequencies on Fig. 1.10. These waves are electromagnetic, while Bernstein modes, for example, which are warm plasma waves, are electrostatic.

The present thesis continues and extends the work presented in Refs. [20] and [19] which were dedicated to the simulations of ICE resulting from the 3.5 MeV fusion-
born alpha particles in JET DT plasmas. The simulations were carried out using a full-PIC code and a hybrid-PIC code respectively. Both methods treat the ions fully kinetically, while the electrons are fully kinetic or modelled as a fluid. The latter is an example of multiscale model which highly resolves ion physics while alleviating full resolution of the electrons by using a generalised Ohm’s law [13]. In this work, we interpret novel ICE measurements from the Large Helical Device (LHD) heliotron-stellarator, which complements the work of [20] and [19] that was dedicated to tokamaks. We focus on NBI fast ions, which have typically much lower energies than FP ions. We perform simulations for both pure perpendicular propagations and inclined a few degrees from it in Chapter 3, which was not previously explored in the context of PIC simulations of the MCI in fusion plasmas. The first LHD deuterium plasma campaign took place in 2017 from which ICE measurements were taken with the particular features that the cyclotron harmonics spectral peaks were shifted. By carefully analysing which harmonics are the most intense and how they are offset, we hypothesised that these are the signatures of fusion-born protons and support it with hybrid-PIC calculations in Chapter 4. We investigate the ICE linear physics in various kinetic models in Chapter 6 after deriving the hybrid kinetic dispersion relation retaining electron inertia in Chapter 5. The long-term goal is to obtain both qualitative and quantitative information on the energetic ion population velocity distribution based on ICE measurements such as power spectra. We present a review of ICE measurements in tokamak plasmas and their interpretation over the past 40 years before presenting the hybrid approximation to then move to
Figure 1.11: Numerical dispersion relations resulting from hybrid kinetic simulations. The graphs are obtained by computing the spatiotemporal Fourier transform of $\delta B_z$ output by the simulations. The frequency is normalised to the proton cyclotron frequency $\Omega_H$ and the wavenumber accordingly to the proton skin depth. The left panel shows the dispersion relation of a thermal plasma and the right panel includes an energetic ion beam in addition to the thermal plasma. The parameters are for LHD plasma 79003 as presented in Chapter 3.

present simulation results.
1.6 ICE observations in large tokamaks

1.6.1 Early measurements

In JET

ICE was first reported from the TFR tokamak [32]. Measurements followed from JET, where ICE was detected with an ICRF heating antenna in receiver mode from Ohmic and NBI-heated plasmas. Strongly suprathermal spectral peaks at cyclotron harmonics and half harmonics of protons in deuterium plasmas were described in [34]. Large radial drift excursions of fusion-born ions were identified as the underlying driver for JET ICE, because these give rise to a velocity-space inversion at the emission location; collective radiation through the MCI then follows. Inverted sawtooth oscillations in the ICE signals from sawtoothing JET Ohmic plasmas was also reported early [55]. Its intensity was proportional to the measured deuterium fusion reactivity $R_{DD}$ over three orders of magnitude. From observations of time delays relative to the soft X-ray (SXR) sawtooth crash, see Fig. 3 of [55], the ICE sawtoothing was interpreted in terms of an enhancement of fusion activity in the edge plasma following the arrival of a sawtooth heat pulse.

With the introduction of tritium into JET in 1991 [56], ICE was detected at successive cyclotron harmonics of $\alpha$ particles. The intensity of this ICE extended the linear correlation with the measured neutron flux [4] to over six decades of signal intensity across all classes of JET plasma.

No clear correlation of ICE with fishbone bursts was obtained from JET. During H-mode discharges, two types of ELMs, identified as small and large, were reported. The antenna-plasma coupling resistance $R_c$ is always observed to increase during ELM events, so that the correlation of ICE with small ELMs was attributed to a change in $R_c$ rather than to intrinsic changes in the ICE power. In contrast, ICE disappeared for about 20ms at the time of large ELM events which were believed to expel fast ions from the plasma edge, thereby extinguishing the ICE source. The frequency matching to local magnetic field strength that follows from the observation of ICE spectral peaks at successive cyclotron harmonics; the disappearance of ICE with large ELMs which penetrate 10-30cm into the edge plasma; and correlation of the time-evolution of the ICE signal with edge $\alpha$ particle density, all point to emission that originates from the outer midplane edge (Fig. 10 [4]) and is driven by fusion-born ions.

A population inversion [57, 58] in the shape of a hollow shell could exist in the plasma core, and potentially drive ICE there if the fusion reactivity was growing fast enough. This was not the case in JET DT plasmas, but may underly recent
observations of transient core ICE from DIII-D [45] and ASDEX-Upgrade [46]. ICE driven by fusion-born $^3$He ions was not observed from JET, consistent with their maximum radial drift excursions that are too small to reach the edge; unlike in TFTR, where $^3$He ICE was detected [5].

In TFTR

Demonstrations of fusion power generation soon followed in TFTR deuterium and tritium plasmas and DT-supershots [59], during which ICE was also reported [5]. Dedicated radio-frequency probes were installed close to the top and the bottom of the vacuum vessel, situated in a vertical plane intersecting the magnetic axis, at poloidal angles of ±90° with respect to the outer midplane. Each array of seven probes could distinguish between electrostatic and electromagnetic signals. ICE was observed at cyclotron harmonics of both fusion products ($^3$He in DD supershots and both $^4$He and $^3$He in T and in DT supershots) and of NBI-injected D and T. As with ICE in JET, the emission originated from the outer midplane edge plasma (slightly beyond the edge, neglecting Doppler shift for α’s and inside for the NBI driven ICE), but there were significant differences with respect to JET, particularly in the time evolution of the TFTR ICE signals. Typically, during the first 50 to 200 ms following the NBI trigger, cyclotron frequencies of the fusion products dominated the measured ICE spectrum, but then died out. They were replaced by ICE spectral peaks at multiple cyclotron harmonics of the injected D and T, until the NBI injectors were turned off. This contrasts sharply with ICE from JET, where ICE due to fusion-born ions persisted for the entire duration of the discharge. In subsequent TFTR experiments, He was puffed [60], forcing the plasmas to transit from typical supershots to pure L-modes. This changed the electron critical density and hence the local value of the ratio $v_\alpha/V_A$ ($\ll 1$), where $v_\alpha$ is the α-particle birth velocity and $V_A$ is the local Alfvén speed in the ICE-emitting region at the plasma edge. In TFTR supershots, the α-particle fusion products are sub-Alfvénic in the edge plasma. As discussed in section 1.7.1, this may account for the transient nature of the ICE, which is restricted to the early stage of the discharge. A plasma shifting from supershot to L-mode leads to early α-driven ICE generation lasting for ≈200 ms which then soon extinguishes, typical of the supershot phase, see Fig. 7 of [5]. At the transition, the α-driven ICE is then regenerated at the time of the He puff as this leads to an increase of the edge density which in turn increases $v_\alpha/V_A$, such that 3.5 MeV α-particles become super-Alfvénic. The α-driven ICE signal persists during that phase until the NBI is turned off, see Fig. 8 of [5]. In the pure L-mode TFTR plasmas, ICE prevailed for the entire duration of the discharge,
until the beams were switched-off, see Fig. 9 of [5]. The peak intensity of ICE due to fusion-born ions, evaluated at the fundamental frequency which corresponds to the \( \alpha \)-particle cyclotron frequency \( \Omega_\alpha \) at the edge, correlated with the measured neutron flux. The spectrum of NBI Triton-ICE from TFTR displays relatively lower intensity at harmonic numbers 3 and 6, which may be due to cyclotron resonant absorption by the bulk deuterium, see Fig. 2 of [61].

In JT-60

In early ICE measurements from the JT-60 tokamak, an electrostatic perturbation was observed at the second cyclotron harmonic of protons during perpendicular H NBI into hydrogen plasmas [62]. Correlation with charge-exchange analysers placed the origin of the ICE signal at the edge of the torus where a fast ion tail was generated. The ICE was observed only if the beam protons had energies no higher than 40 keV and the intensity of the ICE appeared to be proportional to tail temperature. It was considered that resonance with ICE waves was further energising fast ions such that the tail did not undergo significant thermalization. The intensity of the ICE signal was greater at higher plasma density. A possible interpretation includes an increase in the ratio of proton injection speed to local Alfvén speed, see Fig. 2 [63], although the instability considered was electrostatic and not observed at energies above 40keV. ICE from JT-60 plasmas was observed in limiter magnetic geometry. In this configuration, NBI protons can undergo large banana orbit excursions which, it was noted, could give rise to velocity-space anisotropy at the edge and perhaps drive the Harris instability [63], which is predominantly electrostatic. Concern was raised about enhanced losses generated during perpendicular NBI in addition to ripple loss and fishbone instabilities.

In JT-60U

Subsequent to Ref. [63], JT-60 was upgraded with a divertor to enable H-mode operation, and become JT-60 U, from which new ICE measurements were reported in [35]. These included the very interesting detection of ICE from fusion-born tritons in the core of JT-60U deuterium plasmas; no corresponding phenomenon was reported from JET or TFTR. The parallel and perpendicular electric fields were measured by means of three RF probes installed below an ICRF antenna. The first three harmonics of tritons were observed at 21MHz, corresponding to a major radius of 3.1-3.2m. The emission from the core (due to T) was positively correlated with ELMs and peaked at its onset, as can be seen from the \( D_\alpha \) signal in Fig 16 of Ref.
Conversely, ELMs anti-correlated with ICE signals that originated from the edge. The third D harmonic (82MHz corresponding to a major radius of 3.87m) was at its minimal amplitude at the maximum of the ELM. ICE from the outer edge at proton cyclotron harmonics was also identified. The underlying instability was identified with the MCI due to the relatively much smaller parallel component of the electric field that was observed. This implies excitation of predominantly electromagnetic waves, specifically the fast Alfvén wave which is generated by the MCI. As in JET and in TFTR [60, 64], no ICE occurred in conjunction with toroidal Alfvén eigenmodes (TAEs).

1.6.2 Recent measurements

In JET

Simultaneous use of different antennas enabled the observation of ICE from JET plasmas during ICRH scenarios [65] designed to simulate the behaviour of fusion product ions, using the ICRH-heated minority proton population. The ICE originates from the outer midplane edge of the plasma, as for fusion product-driven ICE (FP-ICE). A distinctive feature of minority ion-driven ICE (mICE) is the observed time delay between the ICRH antenna turn-on and the initiation of ICE. This is accounted for by the time needed for initially thermal ICRH-resonant protons to be accelerated and form a tail of fast ions in velocity space that is capable of destabilising waves on the fast-Alfvén ion-cyclotron branch. The threshold for detecting mICE is primarily governed by the radio frequency power $P_{rf}$ delivered to the plasma and by the stored energy in the fast ions $W_{fast}$.

In JET hydrogen plasmas, ICE has been detected with the Sub-Harmonic Arc Detection (SHAD) system from ICRF-wave accelerated minority fast $^3$He ions [66, 67]. The frequency range of the SHAD system is limited such that only the fundamental cyclotron frequency can be detected. The ICE spectra featured sub-structures spaced by a few kHz thought to correspond to the excitations of Cyclotron Alfvénic Eigenmodes (CAEs) [68]. The SHAD system detected ICE quickly after the radio frequency power $P_{rf}$ ramp-up for most plasmas, suggestive that the drive is supported by the minority injected $^3$He. Another instance of ICE was also detected with modulated ICRH power when it reached its flat-top, shortly after the step-down of the NBI pulse, and following the disappearance of a magnetic fluctuation identified as a $3/2$ neoclassical tearing mode. The intensity of this mode correlated with fast ion losses when the ICRH was reaching its peak. It is thought that classical fast ion confinement was recovered when the tearing-mode died out. This can account for
the subsequent ICE emission in the edge as a result of fast $^3$He large orbit excursions. This is supported by energy profiles calculations with the PION code [67] and related computed fast ion trajectories. Full 1D3V PIC and hybrid-PIC simulations were run [67] using the local plasma and fast ion parameters, including both parallel and perpendicular velocities of the fast ions at the last closed flux surface. These were obtained using magnetic moment conservation for 2MeV $^3$He ions, moving perpendicularly to the magnetic field at the resonance location. These velocities then specify the initial ring-beam and drifting-Maxwellian distribution functions that are used to characterise the $^3$He ion population at the start of the PIC and PIC-hybrid simulations. The resulting simulated power spectra display peaks at successive $^3$He cyclotron harmonics, including the fundamental as detected with the SHAD system. The measurements illustrated the potential use of ICE as a complementary passive diagnostics of fast ions physics.

**In JT-60U**

ICE driven by NBI and fusion-born energetic ions was identified in JT-60U [62, 69], and the spatial structures of the wave components in the poloidal and toroidal directions were studied [70]. Positive ion based NB (P-NB’s) was used for perpendicular and tangential beam injection, along with negative-ion based NB (N-NB) in the tangential direction. Two sets of ICRF antennae, used as pickup loops and separated by 1.67m, were arrayed in the toroidal direction. Each antenna is made of two straps separated by 44 cm, see Fig. 1 of [70]; this enables measurement of toroidal mode numbers by computing phase differences obtained from adjacent toroidal straps. The poloidal straps of the antenna are used to distinguish between electrostatic and electromagnetic perturbations [71]. A short description of the system is provided in Ref. [71]. ICE during perpendicular P-NBI of D ions was detected at multiple cyclotron harmonics of $\Omega_D$, at a value implying a magnetic field strength corresponding to the outermost flux surface at the outer midplane. These observed frequencies and subsequent harmonics lay just below the cyclotron frequency and its related harmonics at the outermost flux surface at the outer midplane. Rich structure was present in the dynamical evolution of the ICE: for example, odd harmonics were excited at the beginning of the NBI whereas even modes were strongly driven later. Fluctuations were also reported in proton NBI heated hydrogen JT-60 U plasmas [70]. FP-ICE, distinctive for its sharp spectral peaks, was only observed during tangential NBI and the cyclotron frequencies correspond to a locus slightly beyond the outermost magnetic surface. Two intense peaks were detected at the onset of the 80keV tangential P-NBI followed later by perpendicular P-NBI. These were identi-
fied as the two first cyclotron harmonics of FP \(^3\)He in the plasma midplane edge. Its intensity weakened or disappeared with increasing density, and the \(^3\)He fundamental was especially sensitive \cite{71}. As the perpendicular P-NBI power increased from 4 to 8MW, it was accompanied by the generation of broad D cyclotron harmonics peaks. This feature is very similar to the transient observation of FP-ICE in TFTR \cite{5}, mentioned in section 1.6.1. A single peak identified as FP T ICE arose later, after the start of the tangential N-NBI (450keV, 3MW) at higher density and neutron emission rate compared to conditions under which the FP \(^3\)He ICE fundamental and its harmonics were excited. The FP-T peak frequency was slightly below the value of \(\Omega_T\) at the outermost magnetic surface. The intensity scaled linearly with the measured neutron flux unlike FP \(^3\)He ICE. As mentioned earlier, wavenumbers were measured by means of two ICRF antennae, each consisting of an array of four loops set up in two rows and two columns: the straps (vertical arrays) are separated by 44 cm. It was concluded that some ICE excitations could exhibit structure with very low \(m\). ICE due to FPs displayed toroidal mode structures, whereas ICE due D-NBI did not. The magnitude of \(k_\parallel\) was higher for T than for \(^3\)He, estimated to be 10m\(^{-1}\) and 3m\(^{-1}\) at their fundamental cyclotron frequency respectively. The finite toroidal wavenumber of FP-ICE was further exploited to distinguish the fundamental of FP \(H\) from the second harmonic of D-NBI in ICE data \cite{69} owing to the zero toroidal wavenumber of NBI-driven ICE in JT-60 U. The signal is sensitive to the density and was observed during weak magnetic shear operation \cite{69, 72}, which could be a necessary condition underlying FP \(H\) ICE. Harmonic splitting, reported in JET FP-ICE \cite{73, 74}, was also observed in JT-60 U and corresponded to FP \(^3\)He ICE propagating in both toroidal directions.

1.6.3 Contemporary measurements of ion cyclotron emission

In ASDEX-Upgrade

ICE measurements from ASDEX-Upgrade, reported in \cite{36, 50}, were measured with cross dipole antennas and a voltage probe inside an ICRH antenna. Four types of ICE were identified: FP, beam and ICRH-driven ICE, all occurring in the edge; and core ICE. In NBI-heated discharges with power above 5MW, ICE bursts at the fundamental of \(^3\)He were followed by diffuse D NBI ICE, similar to JT-60U and TFTR. When the power was above 10MW, a spectral peak just below the second D harmonic was identified to be FP \(H\) driven-ICE as the signal strongly correlated with the neutron rate, reminiscent of FP \(H\) driven-ICE in JT-60 U, see Section 1.6.2. Distinction between local or global instability was pointed out: the local case
corresponding to growth rate $\gamma$ being stronger than the period of the fast ion drifts and the global scenario associated with the many ion passes to excite the waves, the Compressional Alfvén Eigenmode.

New sets measurements have been taken using [75] as diagnostics consists of a system of pairs of B-dot probes [46] (inductors) installed on both the low- and high-field sides (LFS and HFS) of the torus. Each pair of probes is oriented perpendicular to each other, except for one whose probes are parallel to one another to obtain wavenumber measurements. Slow digitizers collect the signals which rectify RF waves into DC voltages by mean of log detectors [75]. These detectors are able to take in two RF inputs to capture the phase between the signals. They strongly attenuate frequencies below 10MHz and above 50MHz to minimize the effect of waves on the outputs which leads to information loss. This has been circumvented by splitting the signals from the pair of probes oriented parallel to each other: half goes to the slow digitizer and half is directed to a newly fast digitization system with characteristics. This consists in 2-channels, a bandwidth of 125MHz and 500MB of internal memory allowing to sample two RF signals during 1ms every 10ms for 8 sec. The system removes the frequencies launched by ICRF antennae and triggers on demand for memory constraint. A hydrogen plasma in H-mode demonstrated the capabilities of the diagnostics [75]: classical ICE was observed to originate from the edge, driven by H-NBI (52 and 72keV), at a frequency of 30MHz. The signal was modulated, as a result of correlation with ELM crashes.

Importantly, ICE emitted from the core is also reported from D plasmas in ASDEX-Upgrade, heated with 60keV D-NBI in both H-mode and L-mode regimes, following work in [36, 50]. The fundamental of H/second D harmonic is observed, along with the second H harmonic/fourth D harmonic in plasmas at lower field magnetic field. This core ICE takes two forms. It appears as a transient at the beginning of the NBI phase ($\approx$ first 100ms), during which an inversion or anisotropy in velocity space could exist [58]. The ICE signal is sensitive to the magnetic field value on axis which evolves with the pressure profile, as reported in [36]. In addition, core ICE is observed during steady-state operation and dies out after as long as one second when the NBI changes orientation from tangential to more perpendicular. Splitting is observed within the ICE spectral peaks with separations of 100-200kHz and there is additional fine structure with separations in the low tens of kHz, characteristic of the emission. These structures are also seen in ICE associated with ICRH minority H heating (mICE) in ASDEX [36]. Fast ion full-orbit tracking codes indicate that pitch angles corresponding to $v_\parallel/v$ above 0.45 for co-current and and below -0.72 for counter-current beam operations lead to confined FP H trajectories. These ions
are super-Alfvénic in the core of AUG, and thus more likely to drive ICE by the magnetoacoustic cyclotron instability (MCI) [76] than to the D-NBI ions which are sub-Alfvénic at that location.

In DIII-D

Recent measurements of ion cyclotron emission originating from the plasma core were recently reported from the DIII-D tokamak and correlate with beam ion losses [77]. The ICE signals were obtained using an antenna strap in D NBI deuterium plasmas, heated by eight deuterium NBI beams. These are oriented either mainly parallel or mainly perpendicular with respect to the toroidal magnetic field. The injection direction can be co-current (6 beams; parallel) or counter-current (2 beams; anti-parallel). The measured magnetic fluctuations at the second harmonic of deuterium (∼30MHz) were substantial during tangential injection, particularly in the co-current direction. A signal probably corresponding to the 4th harmonic of deuterium at the edge was also reported. ICE during perpendicular NBI injection was extremely weak in comparison, despite losses being larger in this configuration. These differences were investigated by focusing on the spatial extent of the beam and on the loss strike structures along the wall; during quiescent H-mode plasmas characterized by reversed current directions. Six beams were oriented in the counter-current direction, half of them perpendicular and the other half parallel. Prompt losses and ionization profiles were computed with a Monte Carlo code that takes the scrape-off layer into account. The profiles between the different beams were similar (density of ionizations and wall strikes) but the locations were quite different, indicating the ICE measurements dependence on the beam geometries. Utility of Helium-driven ICE was further emphasized from measurements in Helium H-mode plasmas. The signal was suggested to be the fourth harmonic of He in the plasma core and broadly followed the edge density time evolution. The results were consistent, as for D, with the beam ion losses and wall strikes. The Helium cyclotron frequency being degenerate with D causes difficulties in distinguishing between those two species solely from ICE measurements. However, as described in Section 1.6.2, toroidal mode number measurements distinguish between FP H-driven ICE and the second harmonic of D, which could be relevant to He-driven ICE.

In 2017, following the aforementioned study, two antenna loops introduced in the outer carbon wall were restored along with an antenna probe for low and high bandpass measurements [45], completing the initial system of two antennae straps. These two systems are separated by 50° toroidally. The acquisition system can resolve signals frequencies up to 200MHz, digitized at 200MSamples/s for about 5s
worth 32GB of data per discharge. A module in OMFIT (One Modeling Framework for Integrated Tasks) was dedicated to the processing of the data, by slices of 80ms and computed with Windowed fast Fourier transform (WFFT) by slices of 80µs and then joined together to give the final spectrogram. Three types of ICE signals were identified: classical ICE whose frequencies \( f \) typically correspond to harmonics of \( f_{ci} \), the ion cyclotron frequency, with \( 5 \leq f \leq 100\text{MHz} \). The second type of magnetic fluctuations, sometimes observed, is attributed to compressional Alfvén eigenmodes (CAE’s) with \( f < f_{ci} \) (2-10MHz). Novel emission corresponding to whistler waves with \( f \gg f_{ci} \) (100-200MHz), destabilized by MeV runaway electrons, is identified [78] with the same ICE diagnostic. The ICE originates both from the plasma centre (during L-mode phase plasma) at \( R = 1.8\text{m} \), as inferred from the EFIT equilibrium reconstruction code [45] and from the outboard edge. ICE driven by H, \(^3\text{He} \) or T has not yet been identified. A very rich phenomenology is emerging from the DIII-D ICE measurements [79], including the detection of core ICE in both early NBI plasmas and during L-mode phases. The more common edge ICE is typically driven during multi-beam NBI heated H-mode plasmas and exhibits spectral peaks which are broader than those associated with central ICE. Weak ICE signals were also detected during Ohmic and ECH plasmas but are not well understood. A thorough study [79] of the effect of beam characteristics on the ICE signal has been carried out including the influence of the plasma triangularity [80] on the ICE location: positive triangularity is associated with edge ICE; H-mode, whereas negative triangularity is associated with core ICE; which is weaker. Off-axis co-beamlines do not generate much central ICE, which is stronger with on-axis beams, particularly perpendicular beams. In addition, the central ICE fundamental is shifted between tangential and perpendicular co-beam operation. It is observed that counter-current tangential beams generate more core ICE power than counter-perpendicular beams due to possible voltage difference. In addition, counter-current beams excite more ICE, located more centrally, than co-current beams. The central ICE magnitude appears to increase with voltage (as a possible consequence of increased \( v_{beam}/V_A \)) but is not affected by power. The magnitude of prompt losses amounts to 1.6% for co-tangential beams associated with the lowest ICE power, while it was 2.9% for co-perpendicular beams and was as high as 27.2% in counter-tangential beams. The dependence of ICE signals on beam characteristics was also investigated during H-mode plasmas in DIII-D. The spectral peaks are observed to broaden due to superposition when co-beam power increases. The addition of counter beam power further increases ICE intensity and harmonics [79]. In this case, prompt losses vary between 0.4 – 0.6% in the case of co-beams and reach 23% for counter-beams.
Earlier on, off-axis fishbones were detected and thoroughly characterized as $n = 1$ mode driven by neutral beam ions [38], and ICE was among the diagnostics in this study. The mode is responsible for the loss of fast ions, which peaks close to the maximum mode amplitude. This is demonstrated in Fig. 11 of Ref. [38] which shows phase space at the midplane at different major radii, and displays the associated passing, trapped and lost trajectories, along with the locations of the precession resonance and the central beam deposition. The trajectories are constructed by assuming conservation of magnetic moment $\mu$, of energy $W$ and of canonical angular momentum $P_\phi$ in a collisionless regime. The beam, resonance precession and lost trajectories locations in phase space can all intersect with little toroidal angular momentum change generated by the off-axis fishbone leading to the loss. The loss boundary exists for fast counter-going ions ($v_\parallel/v \approx -0.5$). This was confirmed by fast-ion lost detector measurements hit during fishbone activity, see Fig. 12 of [38]. The intense spots at gyroradius-pitch angle positions allowed the reconstruction of the fast ions trajectories which corresponded to the prompt losses of the counter-beam, being twice as intense during the fishbone. These figures show that the instability takes place in the edge, at major radius $R \approx 2.25\, \text{m}$. As many as seven diagnostics studied the instability, including ICE, which was measured by mean of a high-bandwith toroidal coil [38]. The signal was filtered by a lower-pass and a higher-pass filter. These respectively resolve the fundamental deuterium cyclotron frequency, and the second and third D harmonics. The ICE signal correlated with the beacon-like Mirnov bursts, particularly the higher-pass ICE signals, and occurred to be phase-shifted relative to the beacon mode, a general feature observed in several diagnostics. This reliable correlation was unexpected, due to the two stage nature of the phenomenon: initially the fast ions are expelled, then they drive the MCI in the edge, resulting in ICE signals which provide a good measure of the timing and magnitude of the losses [38].

In KSTAR

A substantial qualitative step forward in the measurement and understanding of ICE has followed from the recent application of new antenna diagnostics [37] with very high sampling rates $> 1\, \text{GSs}^{-1}$ and bandwidth $> 1\, \text{GHz}$. These second generation ICE measurements have been taken from the conventional aspect ratio KSTAR tokamak [23, 37] and from the Large Helical Device (LHD) heliotron-stellarator. In KSTAR [37, 81] deuterium H-mode plasmas heated by deuterium NBI, downward chirping ICE was driven by a small, hitherto unanticipated, sub-population of confined fusion-born protons passing through the edge plasma, whose density rapidly
declines during an ELM crash. The distribution of energy across successive fixed cyclotron harmonic frequencies in the ICE signal changes rapidly ("chirping"), on microsecond timescales, probably associated with ELM filament bursts. This striking phenomenology represents the final stage of ICE signal time evolution associated with an ELM. Previous stages include a pre-crash broadband and continued radiation at around 200MHz (possibly the lower hybrid frequency in KSTAR edge), followed by ICE intensification at harmonic spectral lines of D or H as well as wideband continuum RF excitations at the crash onset. ECE imaging at that time shows the burst of a non-modal filamentary structure which expels heat, then followed by rapid RF chirping. Multiple simulations of the MCI [82] for KSTAR edge plasma conditions with the relevant fusion-born proton population, spanning density values from $n_{pedestal}$ to a small fraction thereof, generated different spectra whose properties match those of the chirping ICE. The mapping from density in the simulations to time in the chirping ICE observations enables sub-microsecond time resolution of the evolving local edge density, probably reflecting ELM filament dynamics. The extremely high resolution time series from KSTAR have unveiled transient "ghost" ICE features [37], above the lower hybrid frequency, between 500MHz and 900MHz, slightly offset by 1µs from their ICE counterparts below 500MHz. These features evolve on microsecond time scales and have been demonstrated to result from non-linear wave-wave coupling [83] between the dominant RF waves in the ICE spectral feature below 500MHz. Importantly, these dynamical higher frequency ICE spectral structures also chirp downwards and PIC simulations show that this can be accounted for by the pedestal density collapse.

1.7 ICE theory

1.7.1 Analytical theory of the magnetoacoustic cyclotron instability (MCI)

The magnetoacoustic cyclotron instability (MCI) [19, 20, 54, 61, 84] is the most likely emission mechanism to account for ICE generation. The theory of the MCI was developed analytically between the 1970s and 1990s, and using large PIC numerical simulation from 2010 onwards [22, 85, 86]. At the plasma wave-particle resonant level of description, the MCI essentially involves the resonance of a fast Alfvén wave supported by the background plasma with negative-energy ion cyclotron harmonic waves sustained by minority fast ions whose non-Maxwellian velocity distribution incorporates a population inversion. The theory was originally developed by Belikov and Kolesnichenko [21] for purely perpendicular propagating waves satisfying
\( \omega \gg \Omega_i \), the background ion cyclotron frequency, including a ring beam distribution for fast ions. The theory was revisited and extended by Dendy et al. [76] to lower frequencies for perpendicular propagation, and MCI growth rates were further obtained for energetic ion distributions in velocity space that have the form of both a spherical and an extended-spherical shell [87], in addition to monoenergetic ring beams [21, 88]. The interest in the ring beam-type distribution arose from the subset of fast ions thought to be responsible for generating ICE from DT plasmas in JET as can be seen in Fig. 1.12 reproduced from Ref. [4] and, subsequently, TFTR. These ions, born in a very narrow range of pitch angles, undergo large drift orbit excursions from the core whose trajectories intersect the outer midplane edge. This leads to a local population inversion in velocity space, in the form of a thin cone shape which is limited by: the maximum energy of the \( \alpha \) particles; their narrow range of pitch angles; and, at the lower bound, the strong decrease of radial excursion with decreasing energy (Fig 15 of [4]). It was conjectured that the MCI could possibly explain the excitations at proton half-harmonics in JET DD plasmas [34] as a result of secondary \( \alpha \) emission with more detailed information on the relative edge \( \alpha \)-density.

The fully kinetic analytical treatment of the MCI [54, 61] included a finite background temperature. This contributes to a stabilisation of the MCI for fast ions having a mono-energetic ring-beam distribution, due to positive energy loading: background ions enter into cyclotron resonance with an initially unstable mode, preventing its further growth. A critical beam density exists, below which the \( \ell \)th cyclotron harmonic is stable [21, 76]. The impact of a drifting ring-beam with a finite perpendicular thermal spread was explored in [88] for perpendicular propagation at low \( \beta \) for parameters relevant to space plasmas. The associated growth
rates of the MCI can be greatly reduced and blend into a continuous band of ring beam speeds (or of $k$-values) see Fig. 3 [88], particularly when $ku/\Omega > u/v_r$, where $u$ and $v_r$ are the drift and spread velocities respectively. This is equivalent to stating that the finite spread of a ring could prevent cyclotron harmonic instability for frequencies $\omega > (V_A/v_r)\Omega$. In addition, it was shown that the MCI growth rates are robust against velocity spreads in the fast ion populations, up to velocity spreads which are of the order of the mean ring speed. The ion cyclotron instability dominated the MCI in the case of a non-drifting warm ring, having both finite parallel and perpendicular thermal spreads. In [89], linear theory at higher $\beta$ for conditions relevant to the terrestrial bow shock, showed that a moderate thermal spread in the ring suppressed instabilities at cyclotron harmonics, beginning with higher ones for perpendicular propagation. In the oblique propagation case, strong instability was deduced for a mono energetic ring in a bi-Maxwellian background plasma which can lead to a merging of adjacent harmonics into a continuum. This is reminiscent of Fig 1.a. in [19]. An extended ring was expected to reduce cyclotron harmonics instabilities as well. The Alfvén ion cyclotron instability ($k_{\parallel} \gg k_{\perp}$) was on the contrary most strongly driven by a extended-ring [89] (whose temperatures are equal in the perpendicular and parallel directions).

Ring-beam distributions are also appropriate for representing freshly ionised populations of neutral beam-injected (NBI) ions, and were used in initial studies of the excitation of electrostatic modes [90, 91] to interpret probe measurements of ICE in TFTR plasmas with deuterium and tritium NBI [61]. This instability did not rely on the fast Alfvén wave, and could be driven by significantly sub-Alfvénic fast ions given a very narrow distribution of speeds parallel to the magnetic field. Computed growth rates for tritium (T) harmonics degenerate with background D were relatively lower, providing a link with the observed reduced intensity of ICE spectral peaks at the of third and sixth T harmonics in some TFTR experiments, see Fig. 2 of [61]. The instability was shown to involve a sensitive trade-off between the relative beam ion density $\xi = n_{\text{beam}}/n_e$ and their parallel velocity spread $v_r$, with a large enough $v_r$ eventually halting the instability. This sensitivity could account for the edge location of the NBI-driven ICE, assuming a greater spread of beam ion velocities deeper in the plasma, whereas it was the narrow range of energy and pitch angles of fast ions that were driving ICE at the edge of JET. Additionally, it was suggested that waves propagating poloidally could be strongly amplified due to their small group velocity and the relatively constant magnetic field underwent for cyclotron resonance.

ICE from fusion-born alpha-particles was subsequently observed from TFTR DT su-
pershot plasmas \[60\]. As in JET, the emitting region was the outer midplane edge, populated by confined ions that had undergone large drift orbit excursions from their birth location in the plasma core, see Fig.2 of \[60\]. These ions had a much larger spread in parallel velocities than in previous instances of ICE, and this led to a re-examination of the MCI \[54\]. Its underlying theory \[92\] was further generalized \[54\] to include finite parallel propagation, incorporating cyclotron damping and electron transit time damping. Electron Landau damping was neglected as the wave electric field was assumed to be polarized perpendicular to the background magnetic field. The Doppler shifts arising from the finite \(k_\parallel\) in the resonance condition had the effect of decoupling the MCI drive from bulk ion cyclotron resonant damping, such that instability can take place at extremely low fusion ion densities. This extended theory \[92\] was applied for parameters matching the form of population inversion in velocity-space expected from fusion-born alpha-particles in the ICE-emitting region at the outer midplane edge in JET and TFTR (fig 16 of \[41\]). As noted in Section 1.6.1, in contrast to JET, where ICE driven by fusion-born ions tended to persist throughout the duration of the plasma, in typical TFTR NBI-heated discharges, ICE with spectral peaks at successive cyclotron harmonics of fusion-born ions was only observed during the first 100 ms of the plasma, followed by ICE with spectral peaks at NBI ion cyclotron harmonics. The fast ions in TFTR were sub-Alfvénic under the plasma conditions at the location of emission, whereas they were super-Alfvénic in JET. For the MCI, this distinction leads to different growth rates, higher in JET’s case, accounting for the different ICE phenomenology as follows. In JET, the maximal radial excursion \(r_{\text{max}}\) of \(\alpha\) particles was slightly smaller than \(a\), the minor radius, while in TFTR supershots \(r_{\text{max}} > a\). Therefore, the velocity inversion wedge for FP-driven ICE in TFTR was wider than in JET DT plasmas \[34, 41\], and than that for beam-driven ICE in TFTR. Before the TFTR core \(\alpha\) distribution broadens, the wedge gets evenly filled and a finite thermal spread exists before significant collisions occur. The fusion reactivity increased at the beginning of the supershot while the \(\alpha\) particle distribution simultaneously started to broaden, having been extremely narrow at the edge due the narrow range of pitch angles for which large orbits arise. Thus, there are at least two critical and competing time scales controlling the edge velocity distribution in terms of anisotropy and narrowness: the fusion ion slowing-down time \(\tau_{\text{slow}}\), and the neutron rise time \(\tau_N\) \[59\]. It was suggested that if \(\tau_{\text{slow}} < \tau_N\), broadening of the \(\alpha\) distribution dominates over its replenishment and prevents velocity space inversion \[58\]. These parameters inferred from measurements were used to feed Sigmar’s model for the approximate evolution of the \(\alpha\) distribution function \[54, 57\]. The associated spread in velocity space was
assumed to apply in the edge and, was used as the spread of a drifting bi-Maxwellian in the analytical calculation of the generalized MCI growth rates. A striking correlation was obtained between the maximum linear growth rates computed this way and the time evolution of the ICE amplitude averaged over six TFTR supershots, see Fig. 6 of [54]. The saturation process was not clearly identified but nonlinear wave-wave interaction was suggested. While linear theory makes no assumption on the initial wave amplitude, it appeared that growth rates were strong enough to amplify its power well above the noise level [41]. The growth rates scaled linearly with the fusion ion density and were very sensitive to thermal spread. It was further shown that under the MCI, a sub-Alfvénic fast ion population needs some degree of velocity-space anisotropy to be unstable while an infinitely thin isotropic shell of super-Alfvénic ions can be unstable, see equations 18 and 19 of [41]. ICE from the core of TFTR was not observed, even though fusion α ions are born super-Alfvénic in that region of the plasma. This absence was most probably due to the monotonically decreasing slowing-down α particle distribution in the core plasma [54, 93]. Other possible reasons include the attenuation of the wave whilst propagating, or a stabilizing spread [54]. In [94], the possibility of driving ICE through density gradients was considered, for $k_\parallel = 0$, by examining the instability threshold of the drift-cyclotron loss-cone anisotropy (DCLC). This is important since ICE originated in the edge of TFTR plasma, where a steep gradient exists. For $k_\parallel = 0$ it was concluded that DCLC was not destabilized in TFTR.

ICE measurements obtained in ICRH and NBI heated JET DT plasmas were analysed in [95]. A classical α-particle confinement model was applied to obtain an approximate slowing-down distribution of the core α particles [57], assuming that they interact with the plasma primarily via Coulomb collisions, and that prompt losses can be neglected. In addition, these assumptions neglected several physics aspects such as velocity-space anisotropies, effects arising from width orbits and time variation of the slowing-down time function $\tau_s$. The solution was also used to infer the parameters of an assumed drifting bi-Maxwellian distribution, with equal parallel and perpendicular velocity spreads. This model was adopted for the location of ICE emission at the JET edge, following the observation that the edge distribution was more peaked than the core distribution in velocity space. It was demonstrated [95] that the linear theory of the MCI applied to this specific distribution gave rise to instability. In addition, an unexplained spectral feature was accounted for by ion hybrid waves [96]; the tritium density at the ICE location leads to an ion hybrid frequency which matches the observed frequency. Further generalization in Ref. [97] included the curvature and grad B drifts in the linear analysis which was applied to
JET plasmas, and further applied to TFTR DT plasmas [97], resulting in increased growth rates. A distribution function calculated for the JET edge plasma based on trapped ion banana orbits was considered in Ref. [98] and shown to be unstable against the MCI. Additional work investigated consequences for the MCI of the finite size of the banana orbit [99]. A weak relativistic mass effect was introduced by Chen [100, 101] to explain the intensity of first proton and $\alpha$ harmonic based on electrostatic Bernstein waves. In JT-60 U, FP-$^3$He ICE was attributed to the MCI at oblique propagation [70, 92]. As part of an ongoing work, solution of the dispersion relation for Maxwellian D and $^3$He was solved for the relevant plasma parameters (simulations pertaining to $^3$He are reviewed in Section 1.7.3). It was demonstrated that Bernstein waves supported by FP $^3$He could resonate with the Alfvén wave supported by thermal D, at the second harmonic, and that resonance at $\Omega_{^3He}$ was not possible as the background density was increased supporting ICE measurements. ICE due to FP T was hypothesized to be a consequence of resonance with the slow Alfvén which can be unstable to temperature anisotropy. The reason was twofold: first its wavenumber was relatively larger, second the observed frequency was smaller than the T cyclotron frequency $\Omega_T$ at the outermost magnetic surface in JT-60 U plasma edge. $^3$He ions born in the center could not reach the edge and this could suggest why $^3$He-driven ICE needed very narrow and anisotropic distributions and was mostly observed to be transient.

### 1.7.2 Initial computational modelling of ICE

The advent of NBI as a heating scheme drove early interest in the instabilities that could be driven by the generated fast ions. This motivated an early study [102] of the MCI relevant to NBI heating scenarios in the tokamak Doublet-3. The distribution of the fast ions was computed using a bounce-averaged Fokker-Planck code for near perpendicular beam injection, for both Doublet-3 and PDX plasma parameters. The initial beam energies were the (expected) 80-40-27 keV for Doublet-3 and 38-19 keV in the case of PDX. The form of the distribution calculated in the equatorial plane allowed the evaluation of $T_\perp/T_\parallel$, the ratio of perpendicular and parallel temperatures lying in the range 2.3-7.7. An approximate dispersion relation for weak damping and growth was derived at arbitrary propagation angle with respect to the background magnetic field in slab geometry. Inversion in velocity space, $\partial F^a_0/\partial v_\perp > 0$, with $F^a_0$ the fast ion distribution function, was indicated to be the primary trigger of the MCI. It was found that it could also be driven with $\partial F^a_0/\partial v_\parallel > 0$ for an anisotropic Maxwellian, provided that the parallel resonance velocity was negative, or that $\omega$ is just below the $\ell$ harmonic. The focus for in-
stability was then placed on anisotropy. The dispersion relation was developed for a Maxwellian background distribution along with an anisotropic Maxwellian-like distribution for the fast ions $\propto v^2 n_\perp \exp\left(-\frac{v^2}{v_\perp^2,\alpha} - \frac{v^2}{v_\parallel^2,\alpha}\right)$ with $v_\perp,\alpha$ and $v_\parallel,\alpha$, characterising the perpendicular and parallel velocity spread. A density instability threshold $\xi_{\text{crit}}$ of 4% was obtained with an anisotropic Maxwellian for the fast ions when $n = 0$ and characterized by $T_\perp,\alpha / T_\parallel,\alpha \gg 1 \gg T_0 / T_\parallel,\alpha$, with $T_0$ the background temperature. The latter condition arises from the background stabilizing effect. A single Bessel function term was retained near resonance; all other terms in the series composing the dispersion relation were neglected. Expanding the Bessel function to first order, it was concluded that linear growth rate values could be up to a few percent of $\Omega_\alpha$ and was decreasing for increasing harmonic number. The study did not retain electron transit-time and Landau damping, which were two orders of magnitude below cyclotron damping. The interaction of the MCI with fast particles trapped in toroidal geometry and having $\partial F^0_\alpha / \partial v_\perp > 0$ was also investigated, assuming a background magnetic field parametrized with $B(\theta) = B_0 (1 + \epsilon \cos \theta)$, with the inverse aspect ratio $\epsilon = r/R$ and $\theta$ the poloidal angle. This expression was used in Fülöp and Lisak [97]'s generalized analytical treatment of the MCI. A dispersion relation with this inhomogeneous magnetic field was obtained by transforming time integrals into sums over particle orbits expressed by integrals over the angle $\theta$ instead.

Following this early investigation and anticipating the work in [38], bursty electromagnetic beam-driven signals were reported from PDX [33]. The spectral contents peaked at harmonics of fast ion cyclotron frequency evaluated at the edge of PDX. An increase of fast ion density in the edge was assumed to be a consequence of either fishbone activity or counter-beam injection, leading to an anisotropic fast ion velocity distribution. In the fishbone case, charge exchange measurements and neutron activity showed that the loss of confinement is due to the expulsion of fast ions. A fishbone, understood as the resonance of an internal kink mode ($m = 1, n = 1$) [33, 103] with fast ions, appears when the rotating speed of such a mode gets closer to the precession frequency of fast ions, eventually resulting in their expulsion from the plasma. The spectra were reported to peak at high cyclotron harmonics and in cases to be as low as the fundamental. High-frequency coil measurements (placed on the inside wall) indicated that the emission originated beyond the major radius. In order to study the instability, the background plasma was modelled with a Maxwellian while the fast ions were described by a Maxwellian-like distribution with a finite perpendicular drift which is related to their precession (and to the mode frequency for resonance). The parallel and perpendicular spread of the fast ions were
deduced from the initial distribution function at the time of injection in the case of counter-beam driven ICE. In the case of the fishbone-driven ICE, the spreads were estimated from the frequency resonance mismatch in the mode they create [103]. It was discussed that magnetosonic waves could not be destabilized at cyclotron harmonics, especially at low density and when fast ion parallel and perpendicular velocities are of the same magnitude. An instability involving electromagnetic Bernstein waves with the curvature grad B drift Doppler shift was treated. First, the warm plasma dispersion relation for purely perpendicular propagation waves [104] particularized to the aforementioned distributions was obtained. The fast ion contribution was then included, with a modified resonance condition to account for the curvature grad-B drift resulting from the initial fast ions drift velocity. It was further argued that for \( k \) values of interest in PDX, the Bernstein part of the solution could be more likely than the Alfvén wave, whence Bernstein instability. Instability was obtained at the fundamental as well as at successive cyclotron harmonics for parameters relevant to PDX with growth rates being \( 1 - 10\% \) of \( \Omega_c \).

1.7.3 ICE simulations

Direct numerical simulations of ICE scenarios were first reported in 2013 [20]. These used a particle-in-cell (PIC) [10, 13, 105] code [106] (see also chapter 2.2) to evolve the full orbit kinetics of millions of thermal ions and electrons, together with the self-consistent electric and magnetic fields, all governed by the Maxwell and Lorentz equations. The distribution of energetic ions in velocity space is typically initialised to reflect physics considerations relevant to the observations of ICE. PIC simulations motivated by ICE measurements from JET show [19, 20] that energetic minority ions relax, under Maxwell-Lorentz dynamics, in ways that replicate the linear MCI at early times and, at later times, produce power spectra capturing measured ICE features. It is hypothesized [81] that the drive at \( D \) harmonics could be due to \( \mathbf{E} \times \mathbf{B} \) velocity shear expelling filaments further driving ICE. Beam-driven ICE, including parallel injection under strong radial electric field, is another potential mechanism to account for the radiations such as reported in the present paper. Multiple simulations with different concentrations of energetic ions give rise to a linear scaling at spectral peaks intensity that matches the observed linear scaling of ICE intensity with fusion activity [22] in JET. An ICE-related scenario relevant to \( \alpha \)-channelling [107] has been proposed on the basis of PIC simulations [86]. It rests on a process that can arise when a radially inward propagating fast Alfvén wave, unstable against the MCI in the outer edge plasma, thereby extracts energy from a fast ion population and transfers it to the bulk plasma. While the relaxation of fast
ions through the MCI spontaneously excites electromagnetic waves at sequential cyclotron harmonics of these ions, which is ICE, alpha channeling based on the MCI exploits a stimulated emission mechanism: a wave in the ion cyclotron range of frequency such as the fast Alfvén wave is sent into the plasma, and if it is unstable against the MCI, could extract energy from fast ions and could be amplified before it damps on the bulk plasma. In a nutshell, the mechanism described would enforce ICE in the plasma with the goal to channel energy from an energetic ion population to the thermal plasma.

In [108], the velocity distribution of fusion-born ions in JT-60U was evolved with an orbit following code. This required tracing the orbits of NBI deuterons and integrating the associated beam-thermal deuteron-deuteron fusion reactivity over time to obtain a spatial distribution profile for the resulting $^3$He ions, assumed to be born isotropically in velocity space with energy 0.82MeV. The drift orbits of the $^3$He ions were then evolved, and shown to give rise to a population inversion in velocity space near the JT-60U plasma edge, as required to excite the MCI and hence drive ICE. In addition, Ref. [108] solves the generalized analytical dispersion relation for a thermal D plasma with a ring-beam $^3$He distribution (eq.1 of [84]), yielding toroidal wavenumbers $k_\parallel \sim 3\text{m}^{-1}$ and frequencies $\omega$ that match experiments. The negative $k_\parallel$ value computed accounts for the observed frequency $\omega$ of the $^3$He ICE spectral peak being slightly lower than the $^3$He cyclotron frequency $\Omega_{^3\text{He}}$ in the plasma edge. This is interpreted by the Doppler-shifted resonance condition for the MCI [30], $\omega - k_\parallel V - \ell\Omega_{^3\text{He}} = 0$ where $V$ is the drift velocity along the local magnetic field and $\ell$ an integer. The transient behaviour of FP $^3$He ICE, which disappeared during JT-60U N-NBI, was accounted for in [109] by numerically solving the full dielectric tensor, in the uniform case, for non-Maxwellian velocity distributions that represent $^3$He behaviour as obtained from a full-orbit code. Growth rates of the MCI were calculated for relative densities of $^3$He FP compared to thermal deuterons as low as $10^{-8}$ in the edge plasma, capturing FP $^3$He ICE characteristics.

Modelling capabilities undergo a rapid growth and this includes hybrid models, coupling various PIC models (full, $\delta f$) to various fluid MHD models [19]. Recent multiphase simulations coupling nonlinear bulk MHD equations to gyrokinetic-PIC have been applied to the study of energetic particle driven instabilities in both tokamak [110, 111] as well as in the LHD heliotron-stellarator [112]. In LHD, in addition to have been detected during NBI [8], ICE has also been observed during toroidal Alfvén eigenmodes (TAE’s) [113, 113–116] and measured by magnetic loops. Related numerical studies have been performed in Ref. [116].
Chapter 2

The hybrid modelling approach

2.1 Introduction

The hybrid approximation, which we introduce in this chapter, has been used to study physical phenomena unfolding on very large scales typical of space plasmas and yet retaining kinetic ion effects such as in shocks [117–119], in collisionless reconnection [120], in turbulence in the solar wind [121, 122]. It has also been used in heavy-ion driven ICF [123]. Hybrid simulations have been used in the study of coherent filament structures (blob formation) for edge tokamaks by Gingell et al. [124, 125] and in the simulation of ICE under JT DT plasma conditions by Carbajal et al. [19], Carbajal et al. [22]. All these show the importance of retaining a kinetic description of the ions. Additionally, hybrid-PIC codes are still being developed due to the trade-off they offer between increased resolution (kinetic ions) and tractability (lower timescales removed) Gargaté et al. [126], Valentini et al. [127], Amano et al. [128], Muñoz et al. [129] including Adaptive Mesh Refinement (AMR) [130].

2.2 PROMETHEUS++, 1D3V hybrid-PIC code

The simulation results presented in this thesis have been obtained with the one dimension and three velocity space dimensions (1D3V) PROMETHEUS++ HPC (High Performance Computing) code developed by L. Carbajal Gomez [131], [19]. This hybrid particle-in-cell (PIC) code [124, 132] evolves the full velocity-space distribution of an arbitrary number of ion species self-consistently with the electromagnetic fields and the fluid electrons in a collisionless plasma. The ions, typically consisting of an energetic and of a thermal population, are represented by a large collection of macroparticles which contain millions of physical particles. The code
incorporates all three vector components of the electric and magnetic fields, and of each particle’s velocity, and represents the electrons as a massless (or inertial) neutralising fluid. It self-consistently solves and iterates the Lorentz force equation for each particle while Maxwell’s equations in the Darwin approximation [23, 124] are evolved on a fixed grid with the fluid electrons. The code makes use of periodic boundary conditions. It resolves ion gyromotion, which is necessary to simulate phenomena such as ICE where key physical length scales and time scales are of the order of the ion gyro-radius (and ion skin depth) and ion cyclotron frequency, as shown on Fig. 2.1. In particular, the code captures the full spatial and gyrophase dynamics of resonant particle-field interactions close to the ion cyclotron frequency and its harmonics. This level of approximation is known to sustain fluid waves such as the fast Alfvén and whistler waves, as well as kinetic waves such as electrostatic Bernstein and ion cyclotron modes. Weakly magnetized plasmas, $\Omega_H \ll \omega_p$, in the ion-cyclotron range of frequencies with $\beta$ between $\sim 10^{-4}$ and $\sim 1$ make the code suitable to study both fusion and astrophysical plasmas. PROMETHEUS++ is written in C++, and benefits from the object-oriented programming which makes it modular. In addition, the 1D grid is decomposed into subdomains which communicates thanks to the Message Passing Interface (OpenMPI), thus the code is highly parallelized. Each subdomain, which corresponds to an MPI process, is further parallelized by use of shared-memory parallel programming (OpenMP). Each process is assigned a set of macroparticles when the simulation is initialized. The architecture is described in details in the work of Carbajal Gomez [131]. New features have been added including a quiet-start, see Section 2.4.5 which launches the majority thermal ions in phase space to reduce noise [133] and the code now incorporates electron inertia. We present the equations solved by the code before giving a derivation of the generalized Ohm’s law in next section. We then present the numerical implementation of the hybrid set of equations. We end the chapter with the presentation of simulation benchmarks: resonant electromagnetic ion-ion instability and the warm-magnetized plasma dispersion relation (Bernstein modes).
The code assumes quasineutrality

\[ \sum_{l=1}^{N} Z_l n_l = n_e \]  

with \( n_e \) the number density of electrons and \( n_l, Z_l \) the number density and electric charge of each ion species \( l \). We also have

\[ \nabla \cdot \mathbf{B} = 0 \Rightarrow B_x = 0 \]  

where \( x \) denotes distance along the 1D slab-geometry spatial domain of our code. We solve for \( \mathbf{B} \) using Faraday’s law

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \]  

while Ampère’s law in the Darwin approximation [134] combines with the massless electron momentum equation to give the generalized Ohm’s law [135],

\[ \mathbf{E} = \frac{1}{\mu_0 e n_e} (\nabla \times \mathbf{B}) \times \mathbf{B} - \mathbf{V}_i \times \mathbf{B} - \frac{\nabla p_e}{e n_e} \]  

Here the charge-weighted mean ion velocity \( \mathbf{V}_i \) is defined by

\[ \mathbf{V}_i = \frac{\sum_{l=1}^{N} Z_l n_l \mathbf{u}_l}{\sum_{l=1}^{N} Z_l n_l} \]

where \( \mathbf{u}_l \) denotes the bulk ion velocity of species \( l \). We assume an isothermal pressure law, \( p_e = n_e k_B T_e \), with \( T_e \) the electron temperature.

Before moving to the next section, we add that quasineutrality removes the need to resolve the Debye length in the model and allows the use of bigger cell sizes in the simulations, by a factor varying from 10 to more than 100 (as shown in Tables 3.1 and 3.2). Thus, less grid points for a given domain length are necessary (for example to resolve the fundamental ion cyclotron wavenumber). It also means that the characteristic scales of the problem we are looking at must be much greater than the Debye length. This assumption yields a direct expression for the electric field through the generalised Ohm’s law (modulo Darwin’s approximation which neglects the displacement current). There is no need to solve \( \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \) or alternatively to use an advanced numerical scheme to collect the current on the grid to solve the continuity equation exactly such as the Buneman [136] or the Esirkepov [137] numerical schemes. It should still be possible to set up quasineutrality on its own
and to keep the displacement current. It would still require to tighten the time step in the simulation in order to satisfy the Courant-Friedrichs-Lewy (CFL) condition which states that $v_{max} \times \Delta t < \Delta x$, where $\Delta t$, $\Delta x$ and $v_{max}$ are the time step, the cell size and the maximum speed in the simulation. When $v_{max}$ is the speed of light (inclusion of the displacement current), this considerably lowers the required time step for stable simulations. The algorithm to advance the electric and magnetic fields over time is not more complicated when the displacement current is retained. It is usually based on the Boris pusher which leapfrogs the electric and magnetic fields over time [105, 138]. Another way to enforce this approximation, given the ions are treated kinetically and the electrons as a massless fluid, is to take the limit for $\epsilon_0$ going to 0. This yields quasineutrality and removes the displacement current at once. Equivalently this sets the speed of light to be equal to infinity [139, 140].

2.3 The generalized Ohm’s law

This section is devoted to a detailed derivation of the generalized Ohm’s law, taking electrons’ inertia effects into account following Ref. [128]. In Ref. [141], electron inertia was introduced in the one-fluid generalized Ohm’s law. The method does not require to evolve the electron flow in time but necessitates the solution of a linear system for the electric field. This is the path taken in [128] that we loosely follow here. With this approach, the electron physical quantities do not appear explicitly in the inertial hybrid kinetic model since $n_e$ is obtained via quasineutrality and the electron momentum equation is encapsulated in Ohm’s law. We incorporate resistivity in the derivations. The density appears explicitly in [128] Ohm’s law and this allows to treat regions of vanishing density as Ohm’s law reduces to Laplace’s equation. This generalizes the work of [142, 143] which treated regions of low density separately. However the assumptions of the hybrid approximation would be violated in that scenario since as the ion density $n_i \to 0$ leads $V_A \to \infty$, while model relies on $V_A \ll c$.

2.3.1 Ohm’s law derivation

We use Ampère’s law in Darwin approximation (no displacement current, the time variation is neglected in that context),

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$  \hspace{1cm} (2.6)
along with Faraday’s law,
\[ \frac{\partial B}{\partial t} = -\nabla \times E \] (2.7)

Taking the time derivative of both sides of Eq. 2.6 and permuting \( \nabla \) with the time derivative, we obtain,
\[ \nabla \times \left( \frac{\partial B}{\partial t} \right) = \frac{\partial}{\partial t} (\mu_0 J) \] (2.8)

If we substitute Eq. 2.7 in Eq. 2.8, we obtain the following,
\[ -\nabla \times \nabla \times E = \frac{\partial}{\partial t} (\mu_0 J) \] (2.9)

where the total current density \( J \) is defined as
\[ J = \sum_s q_s \int v f_s (v) \, dv \] (2.10)

and \( q_s \) and \( f_s (x, v) \) are respectively the charge and the velocity distribution function of species \( s \) (including electrons at this stage), summed over all species. The particles velocity distribution function obeys Vlasov equation
\[ \frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + a_s \cdot \frac{\partial f_s}{\partial v} = 0 \] (2.11)

We consider for now Boltzmann equation, to include resistivity effects in the electron momentum equation (or generalized Ohm’s law). The collisions are neglected in the simulations relevant to ICE throughout this thesis. The only reason to keep the resistivity in Eq. 2.12 is to align our generalized Ohm’s law expression for the dispersion relation calculation of Chapter 5 for inertial electrons to that of Ref. [26] in which it is carried in the massless electron case. This leads to a Vlasov equation for the ions where an effective electric field will be introduced in order to conserve momentum, see Section. 2.3.2. The particles’ dynamics evolved along with the electromagnetic fields under Boltzmann equation
\[ \frac{\partial f_s}{\partial t} + v \cdot \frac{\partial f_s}{\partial x} + a_s \cdot \frac{\partial f_s}{\partial v} = \left( \frac{\delta f_s}{\delta t} \right)_c \] (2.12)

and
\[ f_s = f_s (x, v, t) \] (2.13)

and since the particles evolve under the electromagnetic fields, the acceleration is given by
\[ a_s = \frac{q_s}{m_s} (E + v \times B) \] (2.14)
Observe that because of the Lorentz force,
\[ \frac{\partial}{\partial v} a_s = 0 \quad (2.15) \]
and by the divergence theorem applied to rank 2 tensors,
\[ \int_V \nabla \cdot \sigma dV = \int_S \sigma \cdot ndS \quad (2.16) \]
which in tensor notation gives,
\[ \int_V \sigma_{ij,j} dV = \int_S \sigma_{ij} n_j dS \quad (2.17) \]
Let us apply the divergence theorem 2.16 to \( \sigma = va_s f_s \), ie \( \sigma_{ij} = v_i a_s f_s \), we get,
\[ \int_V \nabla_v \cdot (va_s f_s) dV = \int_S (va_s f_s) \cdot ndS = 0 \quad (2.18) \]
The right hand side of Eq. 2.18 is identically zero because the distribution function \( f_s \) is assumed to rapidly decay, while its left hand side can be developed to give, using Eq. 2.15
\[
\int \nabla_v \cdot (va_s f_s) dV = \int \frac{\partial}{\partial v} \cdot (va_s f_s) \\
= \int \left[ \frac{\partial v}{\partial v} a_s f_s + v \cdot \frac{\partial a_s}{\partial v} f_s + va_s \cdot \frac{\partial f_s}{\partial v} \right] dv \\
= \int \left[ a_s f_s + va_s \cdot \frac{\partial f_s}{\partial v} \right] dv \\
= 0 \\
(2.19)
\]
or equivalently,
\[
\int va_s \cdot \frac{\partial f_s}{\partial v} dv = - \int a_s f_s dv \\
(2.20)
\]
Starting with Eq. 2.10 in which the time derivative is taken and use of the Vlasov equation 2.12 along with Eqs. 2.20 and 2.14 is made,

\[
\frac{\partial J}{\partial t} = \sum_s q_s \int v \frac{\partial f_s}{\partial t} dv
\]

\[
= -\sum_s q_s \int v \left[ v \cdot \frac{\partial f_s}{\partial x} + a_s \cdot \frac{\partial f_s}{\partial v} - \left( \frac{\delta f_s}{\delta t} \right)_c \right] dv
\]

\[
= -\sum_s q_s \int vv \cdot \frac{\partial f_s}{\partial x} dv + \sum_s q_s \int va_s \cdot \frac{\partial f_s}{\partial v} dv + \sum_s q_s \int v \left( \frac{\delta f_s}{\delta t} \right)_c dv
\]

\[
= -\sum_s \frac{\partial}{\partial x} \left( q_s \int vv f_s dv \right) + \sum_s \frac{q_s^2}{m_s} E \int f_s dv + \sum_s \frac{q_s^2}{m_s} \int (v \times B) f_s dv
\]

\[
+ \sum_s q_s \int v \left( \frac{\delta f_s}{\delta t} \right)_c dv
\]

\[ (2.21) \]

We reexpress Eq. 2.21 as

\[
\mu_0 \frac{\partial J}{\partial t} = \sum_s \left[ \Lambda_s E + \Gamma_s \times B - \nabla \cdot \Pi_s + \frac{\mu_0 q_s}{m_s} A_s \right] \quad (2.22)
\]

and where we have define the following,

\[
\begin{align*}
\Lambda_s & \triangleq \frac{\mu_0 q_s^2}{m_s} \int f_s dv \\
\Gamma_s & \triangleq \frac{\mu_0 q_s^2}{m_s} \int vf_s dv \\
A_s & \triangleq m_s \int v \left( \frac{\delta f_s}{\delta t} \right)_c dv \\
\Pi_s & \triangleq \mu_0 q_s \int vv f_s dv 
\end{align*} \quad (2.23)
\]

At this point no approximation other than the quasineutrality assumption has been made. Substituting Eq. 2.9 in Eq. 2.22, we obtain a generalized Ohm’s law

\[
-\nabla \times \nabla \times E = \sum_s \left[ \Lambda_s E + \Gamma_s \times B - \nabla \cdot \Pi_s + \frac{\mu_0 q_s}{m_s} A_s \right] \quad (2.24)
\]

which combined with Eqs. 2.6, 2.7 and 2.12 leads to neutral Vlasov theory [139]. This is a consistent set of equations and the electric field could have otherwise been obtained from the first moment of any of the Vlasov equation for species s (either
ions or electrons). We will now rewrite the last line of 2.23, for that purpose, we reexpress \( v_s \) as follows, with \( n_s, V_s, P_s \), the mean density, the bulk velocity and the pressure tensor of species \( s \) respectively,

\[
v_s = V_s + u_s, \quad \langle u_s \rangle = 0, \quad V_s = \langle v_s \rangle = \frac{\int v f_s dv}{\int f_s dv}, \quad n_s V_s = \int v f_s dv, \quad P_s = m_s \langle u_s u_s \rangle
\]  

(2.25)

We will get, using the former,

\[
\int (V_s + u_s) (V_s + u_s) f_s dv = V_s V_s \int f_s dv + 2 V_s \int u_s f_s dv + \int u_s u_s f_s dv
\]

\[
= V_s V_s \int f_s dv + \int u_s u_s f_s du
\]

(2.26)

and so \( \Pi_s \) becomes,

\[
\Pi_s = \mu_0 q_s \left( n_s V_s V_s + \frac{1}{m_s} P_s \right)
\]

(2.27)

From 2.27, it follows that a significant part of the gradient of the pressure tensor comes from the electrons unless the ions are much hotter. The first term is generally small except for electrons where it can be significant.

One way to express \( A_s \), the mean rate of momentum change per unit volume resulting from collisions between different particle species (which will be between electrons and ions in the following) is as follows:

\[
A_s = -n_s m_s \sum_{j \neq s} \nu_{sj} (V_j - V_s)
\]

(2.28)

where \( \nu_{sj} \) is the collision frequency between the \( s \)th and \( j \)th particle species. The conservation of momentum imposes

\[
\sum_s A_s = \sum_s -n_s m_s \sum_{j \neq s} \nu_{sj} (V_j - V_s) = 0
\]

(2.29)

and rearranging the sum in 2.29 yields:

\[
\frac{1}{2} \left( \sum_s \sum_{j \neq s} -n_s m_s \nu_{sj} (V_j - V_s) + \sum_j \sum_{s \neq j} -n_j m_j \nu_{js} (V_s - V_j) \right) = 0
\]

(2.30)

This has to hold true for any value of \( V_j \) and of \( V_s \) and implies

\[
n_s m_s \nu_{sj} = n_j m_j \nu_{js}
\]

(2.31)
From 2.23, we observe that the main contributions come from the electrons, having the smallest mass and therefore, in the hybrid regime considered, we keep the electron contributions and neglect the ions ones:

\[
\begin{align*}
\Lambda_e & \triangleq \frac{\mu_0 e^2}{m_e} n_e = \frac{1}{c^2} \omega_{pe}^2 \\
\Gamma_e & \triangleq \frac{\mu_0 e^2}{m_e} V_e = \frac{1}{c^2} \omega_{pe}^2 V_e = -\mu_0 \frac{e J_e}{m_e} \\
A_e & \triangleq -n_e m_e \sum_{j \neq e} \nu_{ej} (V_e - V_s) \\
\Pi_e & \triangleq -e \mu_0 \left( n_e V_e V_e + \frac{1}{m_e} P_e \right) 
\end{align*}
\] (2.32)

We now obtain the generalized Ohm’s law using Eq. 2.9 and 2.22 by only retaining the contributions coming from the electrons Eq. 2.32 contrary to Ref. [127] which keeps the ion pressure,

\[
-\nabla \times \nabla \times E = \frac{\partial}{\partial t} (\mu_0 J)
\]

\[
= \sum_s \left[ \Lambda_s E + \Gamma_s \times B - \nabla \cdot \Pi_s + \frac{\mu_0 q_s}{m_s} A_s \right]
\]

\[
\approx \left[ \Lambda_e E + \Gamma_e \times B - \nabla \cdot \Pi_e - \frac{\mu_0 e}{m_e} A_e \right]
\]

\[
= \left[ \frac{\omega_{pe}^2}{c^2} E - \frac{e \mu_0}{m_e} J_e \times B + e \mu_0 \nabla \cdot \left( n_e V_e V_e + \frac{1}{m_e} P_e \right) + \mu_0 \epsilon_0 n_e \sum_{j \neq e} \nu_{ej} (V_e - V_s) \right]
\] (2.33)

Rearranging Eq. 2.33, we find a Ohm’s law

\[
- \left[ \frac{\omega_{pe}^2}{c^2} + \nabla \times \nabla \times \right] E = \left[ -\frac{e \mu_0}{m_e} J_e \times B \right. - \left. \mu_0 \nabla \cdot (V_e J_e) + \frac{e \mu_0}{m_e} \nabla \cdot P_e + \mu_0 \epsilon_0 n_e \sum_{j \neq e} \nu_{ej} (V_e - V_s) \right]
\]

which can be calculated solely in terms of ion quantities since

\[
J_e = \frac{1}{\mu_0} \nabla \times B - J_i
\] (2.35)
where $\mathbf{J}_i$ is the contribution to the current from every ion species. By casting the LHS of 2.34 as:

$$\varepsilon_0 m_e c^2 \left[ \frac{\omega_{pe}^2}{c^2} + \nabla \times \nabla \times \right] \mathbf{E} = \varepsilon_0 m_e c^2 \left[ \frac{c^2 n_e}{\varepsilon_0 m_e c^2} + \nabla \times \nabla \times \right] \mathbf{E}$$

$$= \left[ \varepsilon n_e + \frac{\varepsilon_0 m_e c^2}{e} \nabla \times \nabla \times \right] \mathbf{E} \quad (2.36)$$

and the RHS accordingly,

$$\varepsilon_0 m_e c^2 \left[ - \frac{e\mu_0}{m_e} \mathbf{J}_e \times \mathbf{B} - \mu_0 \nabla \cdot (\mathbf{V}_e \mathbf{J}_e) + \frac{e\mu_0}{m_e} \nabla \cdot \mathbf{P}_e + \mu_0 e n_e \sum_{j \neq e} \nu_{ej} (\mathbf{V}_e - \mathbf{V}_s) \right] =$$

$$= \left[ - \mathbf{J}_e \times \mathbf{B} - \frac{m_e}{e} \nabla \cdot (\mathbf{V}_e \mathbf{J}_e) + \nabla \cdot \mathbf{P}_e + m_e n_e \sum_{j \neq e} \nu_{ej} (\mathbf{V}_e - \mathbf{V}_s) \right] \quad (2.37)$$

leads to

$$- [\varepsilon n_e + \nabla \times \nabla \times] \mathbf{E} = \left[ - \mathbf{J}_e \times \mathbf{B} - \frac{m_e}{e} \nabla \cdot (\mathbf{V}_e \mathbf{J}_e) + \nabla \cdot \mathbf{P}_e + m_e n_e \sum_{j \neq e} \nu_{ej} (\mathbf{V}_e - \mathbf{V}_s) \right] \quad (2.38)$$

In [128], assuming $\nabla \cdot \mathbf{E} = 0$ they have $- \nabla^2 \mathbf{E}$ instead of $\nabla \times \nabla \times \mathbf{E}$ in Eq. 2.38. In the code, all the terms in the generalised Ohm’s law with electron inertia are kept since Equation 2.38 has been implemented, as described in Section 2.4.4. The following part of the presentation covers the expression met in the literature. An interesting fact arises from this equation, as described in [128], the terms are proportional to $n_e$ in Eq. 2.38. This allows one to treat regions of 0 density at once as this equation reduces to Laplace’s equation in that specific case. This avoids the need to separate the domain into vacuum regions and regions of finite densities. Assuming that on the ion scales,

$$\frac{\partial n_e}{\partial t} \approx 0, \quad \frac{\partial n_e}{\partial \mathbf{x}} \approx 0 \quad (2.39)$$

and using the electron continuity equation,

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{V}_e) = \frac{\partial n_e}{\partial t} + \nabla n_e \cdot \mathbf{V}_e + n_e \nabla \cdot \mathbf{V}_e = 0 \quad (2.40)$$

we obtain

$$\nabla \cdot \mathbf{V}_e \approx 0 \quad (2.41)$$

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such that
\[ \nabla \cdot (V_e J_e) = (\nabla \cdot V_e) J_e + (V_e \cdot \nabla) J_e \approx (V_e \cdot \nabla) J_e \] (2.42)

and 2.38 becomes,
\[ - \left[ \frac{\omega_{pe}^2}{c^2} + \nabla \times \nabla \times \right] E = \left[ -\frac{e\mu_0}{m_e} J_e \times B - \mu_0 (V_e \cdot \nabla) J_e + \frac{e\mu_0}{m_e} \nabla \cdot P_e \right] \] (2.43)

Although Eq. 2.39 is quite crude, the assumptions described thereafter apply to the massless electron case in particular. The ICE physics turns out to be captured as it mainly involves an ion kinetic instability. The electron gyroradius is about forty times smaller than the proton Larmor radius, and even smaller compared with the Larmor radius of the energetic ions. As such, on their spatial and time scales these ions would not "see" much variation of the electron macroscopic quantities of Eq. 2.39.

In the literature 2.43 is rewritten, noting
\[ \frac{c^2}{\omega_{pe}^2} e\mu_0 = \frac{m_e}{e n_e}, \quad \lambda_e^2 \triangleq \frac{c^2}{\omega_{pe}^2} \] (2.44)
as
\[ - [1 + \lambda_e^2 \nabla \times \nabla \times] E = V_e \times B + \frac{m_e}{e} (V_e \cdot \nabla) V_e + \frac{1}{e n_e} \nabla \cdot P_e \] (2.45)

Let \( \tilde{\nabla}^2 E \triangleq -\nabla \times \nabla \times E \) and let us define,
\[
\begin{align*}
\hat{E} &\triangleq [1 - \lambda_e^2 \tilde{\nabla}^2] E \\
\hat{B} &\triangleq [1 - \lambda_e^2 \tilde{\nabla}^2] B
\end{align*}
\] (2.46)

Since \( \nabla \times (\tilde{\nabla}^2 E) = \tilde{\nabla}^2 (\nabla \times E) \), we reexpress \( \hat{E} \), with Eq. 2.39, as
\[ \hat{E} = [1 + \lambda_e^2 \nabla \times \nabla \times] E = E - \lambda_e^2 \nabla \times \frac{\partial B}{\partial t} \approx E - \frac{\partial}{\partial t} (\lambda_e^2 \nabla \times B) \] (2.47)

What is usually found in the literature is then:
\[
\begin{align*}
\hat{E} &\triangleq E - \frac{\partial}{\partial t} (\lambda_e^2 \nabla \times B) \\
\hat{B} &\triangleq B + \nabla \times (\lambda_e^2 \nabla \times B)
\end{align*}
\] (2.48)
The following is automatically satisfied

\[ \frac{\partial \hat{B}}{\partial t} = -\nabla \times \hat{E} \]  

(2.49)

One can obtain the generalized electric field \( \hat{E} \) from 2.47 and push it to obtain the generalized magnetic field \( \hat{B} \). The original electromagnetic fields result from inverting 2.48. It happens that in practice, one assumes that modified and original electric fields are about the same and ignores the electron spatial scale variation which has the advantage to be set up as a process:

\[
\begin{cases}
\hat{E} &= \text{E} \\
\hat{B} &= (1 - \lambda_e^2 \nabla^2) \text{B}
\end{cases}
\]  

(2.50)

Neglecting the electric field correction term arises from computational convenience: the variation of the electron scales are ignored and \( \text{E} \) is assumed to be the same as \( \hat{E} \). Recovering \( \text{E} \) and \( \text{B} \) from Eq. 2.48 is indeed more difficult than in the case of Eq. 2.50.

Another way to proceed is to consider the generalized electric field and avoid invoking the solution to the generalized magnetic field (with the extra step of inverting \( \text{E} \), regardless of the method)

\[
- \left[ \frac{\omega_{pe}^2}{c^2} + \nabla \times \nabla \times \right] \text{E} = -\frac{\omega_{pe}^2}{c^2} \text{E} - \nabla \cdot (\nabla \cdot \text{E}) + \nabla^2 \text{E}
\]  

(2.51)

by invoking quasineutrality, one can get rid of \( \nabla \cdot \text{E} \sim O(V_A/c)^2 \). This way, equation 2.34 takes the form:

\[
- \left( \frac{\omega_{pe}^2}{c^2} - \nabla^2 \right) \text{E} = \left[ -\frac{e\mu_0}{m_e} \text{J} \times \text{B} - \mu_0 (\text{V} \cdot \nabla) \text{J}_e + \frac{e\mu_0}{m_e} \nabla \cdot (\text{P}_e) \right]
\]  

(2.52)

The advantage of this expression relies on the right hand side being proportional to the density. In a near vacuum region, equation 2.43 boils down to the Laplace’s equation. The derivation presented here had the main purpose of introducing electron inertia without advancing the electron flow in time. It is interesting to see how the equation we obtain compares with the electron fluid momentum equation. Let
us use 2.33:
\[-\nabla \times \nabla \times \mathbf{E} = \frac{\partial}{\partial t}(\mu_0 \mathbf{J}) = \left[ \frac{\omega_p^2}{c^2} \mathbf{E} - \frac{e \mu_0}{m_e} \mathbf{J}_e \times \mathbf{B} + e \mu_0 \nabla \cdot \left( n_e \mathbf{V}_e \mathbf{V}_e + \frac{1}{m_e} \mathbf{P}_e \right) \right] \tag{2.53} \]

where \( \mathbf{J} \) is expressed as
\[(q_i n_i \mathbf{V}_i - e n_e \mathbf{V}_e) \tag{2.54} \]

with \( \mathbf{V}_i \) given in expression 2.5.

\[
\mu_0 \frac{\partial}{\partial t} (e n_e \mathbf{V}_i - e n_e \mathbf{V}_e) = \left[ \frac{\omega_p^2}{c^2} \mathbf{E} - \frac{e \mu_0}{m_e} \mathbf{J}_e \times \mathbf{B} + e \mu_0 \nabla \cdot \left( n_e \mathbf{V}_e \mathbf{V}_e + \frac{1}{m_e} \mathbf{P}_e \right) \right] \tag{2.55} \]

Using 2.32, 2.55 gives
\[
\frac{\partial}{\partial t} (e n_e \mathbf{V}_i - e n_e \mathbf{V}_e) = \left[ \frac{e^2 n_e}{m_e} \mathbf{E} + \frac{e^2 n_e}{m_e} \mathbf{V}_e \times \mathbf{B} + e \nabla \cdot \left( n_e \mathbf{V}_e \mathbf{V}_e + \frac{1}{m_e} \mathbf{P}_e \right) \right] \tag{2.56} \]

Namely,
\[
\frac{\partial}{\partial t} (n_e \mathbf{V}_e) + \nabla \cdot \left( n_e \mathbf{V}_e \mathbf{V}_e + \frac{1}{m_e} \mathbf{P}_e \right) + \frac{e n_e}{m_e} (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) = \frac{\partial}{\partial t} (e n_e \mathbf{V}_i) \tag{2.57} \]

The left handside corresponds to electron fluid momentum equation. Note that multiplying 2.57 by \( m_e \) and taking the inertialess limit gives the Ohm’s law of a hybrid code.

There is a hierarchy in the development of Ohm’s law which is presented in Ref. [127]. If we had treated the ions as a fluid, then Eq. 2.12 and 2.57 would lead to the two-fluid MHD set of equations with suitable pressure laws. For long wavelengths and low frequency, Ohm’s law in ideal MHD reduces to \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} \), with \( \mathbf{u} \) the plasma bulk velocity. Addition of the Hall term is necessary when characteristic lengths go down to ion inertia scales: \( \mathbf{E} = -\mathbf{u} \times \mathbf{B} + 1/n_e \mathbf{J} \times \mathbf{B} - 1/n_e \nabla \cdot \mathbf{P}_e \) (HMHD) [138, 144]. When electron and ion dynamics decouple, electron inertia scales are to be included, equation 2.33 is used and is also known as electron MHD (EMHD) [127]. In Ref. [145], a derivation is given and the model relies on immobile neutralizing ions. The different models are a trade-off between accuracy and computational tractability: MHD allows one to capture the physics of systems that vary more slowly (below the cyclotron frequency for example) on very large time scales such as the solar wind and can be improved with various degrees of precision. For example, the magnetic field lines are frozen in a fluid with infinite conductivity. This is not always the case and requires refined generalised Ohm’s laws. Refined models are necessary
to account for complicated transport, such as the study of turbulence [146, 147].
The other extreme is the fully kinetic approach which requires the resolution of the
Debye length and limits its applicability to large systems (and/or long time scales).

2.3.2 Inclusion of resistivity, momentum conservation

In this part we seek to obtain a Vlasov like equation for the ions, having started
with a Boltzmann equation, to include resistivity. In the case where collisions are
not treated (which is the case of our simulations of the MCI), we simply use Vlasov
equation. Since momentum is exchanged macroscopically, we investigate what the
first 2 moments of Eq. 2.12 are,

\[ \int d\mathbf{v} m_s \left( \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \mathbf{a}_s \cdot \frac{\partial f_s}{\partial \mathbf{v}} \right) = \int m_s d\mathbf{v} \left( \frac{\delta f_s}{\delta t} \right) \quad (2.58) \]

Since particles are neither created nor destroyed, the RHS 2.58 is 0 and using 2.15

\[ m_s \int d\mathbf{v} \left( \mathbf{a}_s \cdot \frac{\partial f_s}{\partial \mathbf{v}} \right) = m_s \int d\mathbf{v} \frac{\partial}{\partial \mathbf{v}} (f_s \mathbf{a}_s) - m_s \int d\mathbf{v} f_s \frac{\partial}{\partial \mathbf{v}} \mathbf{a}_s = 0 \quad (2.59) \]

using 2.59, 2.58 gives the continuity equation:

\[ m_s \frac{\partial n_s}{\partial t} + m_s \mathbf{V}_s \cdot (n_s \mathbf{V}_s) = 0 \quad (2.60) \]

Similarly,

\[ \int d\mathbf{v} m_s \left( \frac{\delta f_s}{\delta t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \mathbf{a}_s \cdot \frac{\partial f_s}{\partial \mathbf{v}} \right) = \int d\mathbf{v} m_s \left( \frac{\delta f_s}{\delta t} \right) \quad (2.61) \]

Recalling 2.20 and 2.27, 2.61 can be written as

\[ m_s \frac{\partial n_s}{\partial t} V_s + m_s \nabla \cdot \left( n_s \mathbf{V}_s \mathbf{V}_s + \frac{1}{m_s} \mathbf{P}_s \right) - n_s \mathbf{F}_s = \mathbf{A}_s \quad (2.62) \]

and

\[ n_s \mathbf{F}_s = \int d\mathbf{v} f_s (m_s \mathbf{a}_s) \quad (2.63) \]

with \( \mathbf{a}_s \) defined in 2.14. Developing 2.62, we obtain

\[ m_s \frac{\partial n_s}{\partial t} \mathbf{V}_s + n_s m_s \frac{\partial \mathbf{V}_s}{\partial t} + m_s \mathbf{V}_s \nabla \cdot (n_s \mathbf{V}_s) + m_s n_s (\mathbf{V}_s \cdot \nabla) \mathbf{V}_s + \nabla \cdot \mathbf{P}_s - n_s \mathbf{F}_s = \mathbf{A}_s \quad (2.64) \]
or,
\[
(m_s \frac{\partial n_s}{\partial t} + m_s \nabla \cdot (n_s \mathbf{V}_s)) \mathbf{V}_s + n_s m_s \frac{\partial \mathbf{V}_s}{\partial t} + m_s n_s (\mathbf{V}_s \cdot \nabla) \mathbf{V}_s + \nabla \cdot \mathbf{P}_s - n_s \mathbf{F}_s = \mathbf{A}_s
\]  
(2.65)

Using the continuity equation 2.59 and the material derivative Eq. 2.65 becomes,
\[
n_s m_s \frac{D \mathbf{V}_s}{Dt} + \nabla \cdot \mathbf{P}_s = n_s \mathbf{F}_s + \mathbf{A}_s = n_s \left( \mathbf{F}_s + \frac{1}{n_s} \mathbf{A}_s \right) \]  
(2.66)

and
\[
\mathbf{F}_s = q_s (\mathbf{E} + \mathbf{V}_s \times \mathbf{B}) \]  
(2.67)

We could have obtained 2.66 starting from a Vlasov equation instead of a Boltzmann equation in 2.12 and 2.61 if we had redefined \( \mathbf{F}_s \) as
\[
\mathbf{F}'_s = n_s \left( \mathbf{F}_s + \frac{1}{n_s} \mathbf{A}_s \right) = n_s q_s \left( \mathbf{E} + \frac{1}{q_s n_s} \mathbf{A}_s + \mathbf{V}_s \times \mathbf{B} \right) \]  
(2.68)

and \( \mathbf{a}_s \) accordingly as,
\[
\mathbf{a}_s = \frac{q_s}{m_s} \left( \mathbf{E} + \frac{1}{q_s n_s} \mathbf{A}_s + \mathbf{v} \times \mathbf{B} \right) = \frac{q_s}{m_s} (\mathbf{E}' + \mathbf{v} \times \mathbf{B}) \]  
(2.69)

with
\[
\mathbf{E}' = \mathbf{E} + \frac{1}{q_s n_s} \mathbf{A}_s \]  
(2.70)

as stated in [148] and ensures momentum is conserved. If we consider that an ion species \( s \) only collides with electrons with an effective frequency \( \nu_{se} \),
\[
\mathbf{A}_s = -m_s n_s \nu_{se} (\mathbf{V}_s - \mathbf{V}_e) \]  
(2.71)

which gives
\[
\mathbf{E}' = \mathbf{E} - \frac{m_s n_s \nu_{se}}{n_s q_s} (\mathbf{V}_s - \mathbf{V}_e) = \mathbf{E} - \frac{m_s \nu_{se}}{q_s} (\mathbf{V}_s - \mathbf{V}_e) \]  
(2.72)

If we further assume that only one ion species is present, using 2.31, and \( Z_s \) is the atomic number of the ion species, Eq. 2.72 is
\[
\mathbf{E}' = \mathbf{E} - \frac{m_e n_e \nu_{se}}{n_s Z_s e} (\mathbf{V}_s - \mathbf{V}_e) \]  
(2.73)
and since by quasineutrality, $Z_s n_s = n_e$,

$$E' = E - \frac{m_e v_{se}}{e^2 n_e} J$$

(2.74)

we obtain by setting $\eta = \frac{m_e v_{se}}{e^2 n_e}$,

$$E' = E - \eta J$$

(2.75)

in agreement with Lipatov [148, p.29] and Ref. [13]. To summarize, if resistivity is included in the generalized Ohm’s law, one has to solve

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{x}} + \frac{q_s}{m_s} (E' + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

(2.76)

for the ions with $E'$ defined in equation 2.72. We will this expression in Chapter 5 when computing the hybrid kinetic dispersion relation. The resistivity is not considered in the simulations presented throughout this thesis otherwise.

## 2.4 Numerical implementation

Vlasov equation is expressed in a 6-dimensional phase-space, 3 for the positions and 3 for the velocities. Its solution is thus demanding both in terms of computer memory and of computing time. The particle-in-cell paradigm avoids the use of a grid (or a cube) to represent velocity-space. The drawback comes from the noise suffered by the simulations due to the use of a finite number of macroparticles. This is addressed with shape functions: macroparticles which have a finite-spatial extent. We present the numerical schemes used to solve the hybrid equations using the PIC method. The steps leading to the macroparticles equations of motions are given. We introduce the charge assignment scheme which allows the calculation of smoothed macroscopic density and current from the ions which provide the source terms to solve the equations for the electromagnetic fields. This goes hand in hand with the force interpolation scheme which permits to calculate the Lorentz force at each ion position when the fields values are known on a discrete fixed grid. The numerical scheme used to advance the ions in time is described followed by the spatial and time discretizations providing the electric and magnetic fields in the massless and inertial electron case. The inclusion of electron inertia in our simulations allows the resolution of the electron skin depth which can be of the order of the thermal gyroradius. The ratio $\lambda_e / r_{L,i} = m_e / n_i (V_A / V_{T,i})^2$ can be close to unity when $V_{T,i}$ is small such that resolution of higher wavenumber can be achieved by retaining the
electron mass in the code. The methods presented follow the Finite Difference Time Domain (FDTD) approach.
The code is 1D3V, the notation $\mathbf{x}$ is used in the next sections to be general but it actually corresponds to $\mathbf{x}\mathbf{x}$ throughout the rest of the thesis. The variations are only permitted along the $x$-direction and the Larmor gyration can be reconstructed by performing integration over time of each of the three velocity components that the model keeps. However gradients (in the Larmor radius if there was any for example) can only appear along the $x$-direction.

2.4.1 Charge assignment scheme and interpolation.

In the hybrid particle-in-cell approach, the ions are represented by Lagrangian macroparticles (labelled with the index $p$) and they consist of an important number $N_j$ of physical particles ($j$ labelling the different ion species). The electrons are represented as a massless or inertial neutralising fluid. Each macroparticle produces a contribution $f^p_j (\mathbf{x}, \mathbf{v}, t) = N_j^p S(\mathbf{x} - \mathbf{x}^p_j(t)) \delta(\mathbf{v} - \mathbf{v}^p_j(t))$, where $S(\mathbf{x} - \mathbf{x}^p_j(t))$ is the shape function and $\mathbf{x}^p_j(t)$ and $\mathbf{v}^p_j(t)$ are the position and velocity of the $p$th macroparticle of ion species $j$ as detailed in Ref. [129, 148, 149]. The use of the shape function, which gives macroparticle a finite-spatial size, allows to distribute the current and density from ions more smoothly across different cells to avoid sudden jumps of charge and of momentum when a particle moves from a cell to another. This generates large fluctuations at small spatial scales which are the source of noise. The distribution function is obtained by summing these contributions: $f_j (\mathbf{x}, \mathbf{v}, t) = \sum_p N_j^p S(\mathbf{x} - \mathbf{x}^p_j(t)) \delta(\mathbf{v} - \mathbf{v}^p_j(t))$. Plugging $f^p_j (\mathbf{x}, \mathbf{v}, t)$ into Vlasov equation 2.11 and taking the first spatial-moment and velocity-space moment over phase-space $\int d\mathbf{x} d\mathbf{v}$ gives the Newton equations for the macroparticles [129]

$$\frac{d\mathbf{x}^p_j(t)}{dt} = \mathbf{v}^p_j(t)$$
$$M_j \frac{d\mathbf{v}^p_j(t)}{dt} = e Z_j N_j^p \left( \mathbf{E}^p_j(t) + \mathbf{v}^p_j \times \mathbf{B}^p_j(t) \right)$$

where $M_j = N_j^p m_j$ and $\mathbf{x}^p_j$ and $\mathbf{v}^p_j$ are the first $\mathbf{x}$- and $\mathbf{v}$-moments of $f^p_j$ respectively and

$$\mathbf{E}^p_j(t) = \int d\mathbf{x} S\left( \mathbf{x} - \mathbf{x}^p_j(t) \right) \mathbf{E}(\mathbf{x}, t)$$
$$\mathbf{B}^p_j(t) = \int d\mathbf{x} S\left( \mathbf{x} - \mathbf{x}^p_j(t) \right) \mathbf{B}(\mathbf{x}, t)$$

The finite spatial extent of the macroparticles in PIC simulations resulting from the use of the shape function allows reduction of the noise at short length scales. The
smoothed macroscopic variables such as ion current densities \( J_j(x,t) \) and charge densities \( \rho_j(x,t) \) are given by integration of \( f_j(x,v,t) \) over velocity space:

\[
\rho_j(x,t) = Z_je \sum_p N_j^p \int S(x - x^p_j(t)) dV
\]

\[
J_j(x,t) = Z_je \sum_p N_j^p v^p_j(t) \int S(x - x^p_j(t)) dV
\]

(2.79)

and the shape function is normalized to unity

\[
\int S(x - x') dx = 1
\]

(2.80)

and centered at \( x' \) as shown in Fig. 2.2. The grid consists of cells identified by \( x_k \) with volume \( V_{x_k} \). The charge density and current density deposited in that cell from the ion species \( j \) is

\[
\rho_j(x_k,t) = \frac{1}{V_{x_k}} Z_je \sum_p \int_{V_{x_k}} dx N_j^p S(x - x^p_j(t))
\]

\[
J_j(x_k,t) = \frac{1}{V_{x_k}} Z_je \sum_p \int_{V_{x_k}} dx N_j^p v^p_j(t) S(x - x^p_j(t))
\]

(2.81)
In one dimension, the grid consists of equally spaced points \( x_k \), the nodes, lying at the middle of a cell of width \( \Delta x \), so the charge and current density assignment Eq. 2.81 reduces to

\[
\rho_j(x_k,t) = \frac{1}{\Delta x} Z_j e \sum_p N_p^j \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} S(x-x_p^j(t)) \, dx \\
J_j(x_k,t) = \frac{1}{\Delta x} Z_j e \sum_p N_p^j v_{p}^j(t) \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} S(x-x_p^j(t)) \, dx
\]  

As shown in on the left panels of Fig. 2.2, the fraction of the charge assigned to a cell centered at \( x_k \) from a macroparticle whose center is at \( x' = x_p^j(t) \) is the area of the shape function which overlaps the cell. This is a function of the distance between \( x' \) and \( x_k \) such that we can define the assignment function \( W \)

\[
W (x_k - x') = W_k (x') = \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} S(x-x') \, dx
\]  

Introducing the Top-Hat function

\[
\Pi (x) = \begin{cases} 
0 & |x| > 1/2 \\
1/2 & x = 1/2 \\
1 & |x| < 1/2 
\end{cases}
\]  

and considering even shape function, the weight function \( W \) from Eq. 2.83 is expressed as a convolution

\[
W (x) = \Pi \left( \frac{x}{\Delta x} \right) * S(x)
\]  

and must satisfy \( \sum_{k=1}^{N_g} W(x_k - x) = 1 \) with \( N_g \) the number of grid points. Charge and current are reexpressed as

\[
\rho_j(x_k,t) = Z_j e \sum_p N_p^j W \left( x_k - x_p^j (t) \right) \\
J_j(x_k,t) = Z_j e \sum_p N_p^j v_{p}^j (t) W \left( x_k - x_p^j (t) \right)
\]  

with

\[
W \left( x_k - x_p^j (t) \right) = \frac{1}{\Delta x} \int_{x_k-\Delta x/2}^{x_k+\Delta x/2} S \left( x - x_p^j (t) \right) \, dx
\]

Similarly, the electric fields and magnetic fields in Eq. 2.78 are specified on the grid (more precisely they are staggered on a Yee lattice as described in Section 2.4.4) and need to be extrapolated at the particle position. We reexpress \( \mathbf{E}(x,t) \) by means of
its value on the grid $E(x_k, t) = E_k(t)$ and of an interpolation function $R$:

$$E(x, t) = \sum_{k=1}^{N_g} E_k(t) R(x - x_k)$$  \hspace{1cm} (2.88)

with $\sum_{k=1}^{N_g} R(x_k - x) = 1$ and the electric field at the particle position Eq. 2.78 becomes

$$E^p_j(t) = \int \sum_{k=1}^{N_g} E_k(t) R(x - x_k) S(x - x^p_j(t)) \, dx$$  \hspace{1cm} (2.89)

However, in order to conserve momentum, the interpolation function $R(x)$ has to be equal to the particle weighting assignment function $W(x)$, as shown in Ref. [148]:

$$E^p_j(t) = \sum_{k=1}^{N_g} E_k(t) W(x^p_j(t) - x_k)$$  \hspace{1cm} (2.90)

and also applies to the magnetic field. The hybrid code incorporates the TSC assignment function $W(x)$ which ensures continuity along with continuity of the first derivative of the currents and densities and is defined as

$$W(x) = \begin{cases} 
3 & |x| \leq \frac{\Delta x}{2} \\
\frac{1}{2} \left( \frac{3}{2} \frac{|x|}{\Delta x} \right)^2 & \frac{\Delta x}{2} \leq x \leq \frac{3\Delta x}{2} \\
0 & \text{otherwise}
\end{cases}$$  \hspace{1cm} (2.91)

where $x$ is the distance between the center of the shape functions and the different nodes. We have presented the self-consistent coupling between the PIC-ions and the electromagnetic fields. The ion advancement in time is introduced in the following section.

### 2.4.2 Ion time advancement

Each macroparticle must be advanced in time by integrating their equations of motion 2.77. A time-centered finite difference is adopted [10] where derivatives are evaluated at the half-time step. We denote $t_{n+\frac{1}{2}} = (n + 1/2) \Delta t$, where $\Delta t$ is the timestep of the simulation and the positions and velocities are leapfrog: the positions are determined at integer time step $x^n$ and velocities at half-integer time steps $v^{n+1/2}$, as illustrated in Fig. 2.3.

To see the second order accuracy, we assume $f$, an unknown function whose derivative is known, its true solution at time $t_n$ is denoted $f(t_n)$ and its numerical solution
is written as $f^n$ such that

$$\frac{df}{dt}(t_{n+\frac{1}{2}}) = \frac{f(t_{n+1}) - f(t_n)}{\Delta t} + O(\Delta t)^2 \approx \frac{f^{n+1} - f^n}{\Delta t} \approx \frac{df^{n+\frac{1}{2}}}{dt} \quad \text{(2.92)}$$

provides a second order accurate estimation of the derivative of $f$ at $t_{n+\frac{1}{2}}$ by Taylor expanding $f(t_{n+1})$ and $f(t_n)$ around $t_{n+\frac{1}{2}}$. In the case of Eq. 2.77 $f$ corresponds to the position and to the velocity. We drop the index $p$ to set $E^n_j(t_n) = E^n(x^n_j)$, and we have

$$x^{n+1}_j - x^n_j = \Delta t v^{n+\frac{1}{2}}_j$$

$$v^{n+\frac{1}{2}}_j - v^{n-\frac{1}{2}}_j = \Delta t Z J eN_j \left( E^n(x^n_j) + v^n \times B^n(x^n_j) \right) \quad \text{(2.93)}$$

where $x^n_j$ and $v^n_j$ are the position and velocity of an ion of species $j$ at $n$th iteration respectively. The velocity at the $n$th step appears on the RHS of Eq. 2.93 and is estimated with $v^n = (v^{n+\frac{1}{2}}_j + v^{n-\frac{1}{2}}_j)/2$. A detailed account of the computation of $v^{n+\frac{1}{2}}_j$ by the code, which appears on both sides of Eq. 2.93, is provided in Winske and Omidi [13]. Their method consists in taking the dot and cross products of the velocity equation with $v^{n+\frac{1}{2}}_j$. The value of $\Delta t$ must be chosen small enough in order to satisfy the Courant-Friedrichs-Lewy (CFL) condition. For numerical stability, one particle cannot cross more than a cell (of length $\Delta x$) every time step: $v_{max} \Delta t < \Delta x$ [150] with $v_{max}$ the maximum ion velocity which is regularly checked throughout the simulation.

### 2.4.3 Iterative methods for linear system

The introduction of electron inertia in PROMETHEUS++ requires the solution of a symmetric tridiagonal system arising from the underlying spatial-finite differences. In this section, we introduce three important iterative solution methods for linear
problems: the *Jacobi*, *Gauss-Seidel* and SOR methods and principally follow Ref. [151]. If we consider the linear system $Ax = b$, $A \in \mathbb{R}^{n \times n}$ and $x, b \in \mathbb{R}^n$ with $x$ to be determined. We can rewrite $A = M - N$ to have the linear system $Mx = Nx + b$ reexpressed as a fixed point problem and devise the iteration

$$Mx^{(k+1)} = Nx^k + b$$ (2.94)

If we take the difference of the 2 former equations,

$$M \left( x - x^{(k+1)} \right) = N \left( x - x^{(k)} \right)$$ (2.95)

Setting $e^{(k)} = x - x^{(k)}$

$$e^{(k+1)} = M^{-1}N e^{(k)}$$ (2.96)

**Theorem** [152, p. 115]: $\lim_{k \to \infty} e^{(k)} = 0 \ \forall e_0$ if and only if the spectral radius $\rho \left( M^{-1}N \right) < 1$. We split the matrix $A$ into three distinct matrices, its upper diagonal part $U$, lower diagonal part $L$ and main diagonal $D$ such that $A = L + D + U$.

**Jacobi Method**

We can write the linear system in components

$$\sum_{j=1}^n a_{ij} x_j = b_i, \quad i = 1, \ldots, n,$$ (2.97)

then

$$a_{ii} x_i = b_i - \sum_{j \neq i} a_{ij} x_j$$ (2.98)

or

$$x_i = \frac{1}{a_{ii}} \left( b_i - \sum_{j \neq i} a_{ij} x_j \right)$$ (2.99)

which leads to the iteration

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j \neq i} a_{ij} x_j^{(k)} \right) \quad (2.100)$$

The related matrices $M$ and $N$ are from Eq. 2.98: $M = D$ and $N = - (L + U)$. The iteration is

$$x^{(k+1)} = - D^{-1} (L + U) x^{(k)} + D^{-1} b$$ (2.101)
and converges if $A$ is *strictly diagonally dominant*:

$$\max_{1 \leq i \leq n} \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right| < 1$$  \hspace{1cm} (2.102)

**Gauss Seidel**

The *Jacobi* method updates $x^{(k+1)}$ from the values of $x^{(k)}$. However at the time of computing $x^{(k+1)}_i$, all the values $x^{(k+1)}_j$, $j < i$ are already available and could be used to accelerate convergence. A new iteration can be written as

$$a_{ii} x^{(k+1)}_i = b_i - \sum_{j=1}^{i-1} a_{ij} x^{(k+1)}_j - \sum_{j=i+1}^{n} a_{ij} x^{(k)}_j$$  \hspace{1cm} (2.103)

In a matrix form this becomes,

$$D x^{(k+1)} = b - L x^{(k+1)} - U x^{(k)}$$  \hspace{1cm} (2.104)

which gives

$$x^{(k+1)} = -(D + L)^{-1} U x^{(k)} + (D + L)^{-1} b$$  \hspace{1cm} (2.105)

**SOR**

The method of *successive over-relaxation* (SOR) relies on accelerating the convergence by updating $x^{(k+1)}_i$ with a linear combination of the $k$th iterate $x^k$ and of the Jacobi iteration:

$$x^{(k+1)}_i = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij} x^{(k+1)}_j - \sum_{j=i+1}^{n} a_{ij} x^{(k)}_j \right) + (1 - \omega) x^{(k)}_i$$  \hspace{1cm} (2.106)

The parameter $\omega$ is the over-relaxation parameter and when equal to one, we retrieve the *Gauss-Seidel* method. In a matrix form, we have

$$(I + \omega D^{-1} L) x^{(k+1)} = \omega D^{-1} b - \omega D^{-1} U x^{(k)} + (1 - \omega) I x^{(k)}$$  \hspace{1cm} (2.107)

namely

$$(D + \omega L) x^{(k+1)} = \omega b + ((1 - \omega) D - \omega U) x^{(k)}$$  \hspace{1cm} (2.108)

or otherwise

$$x^{(k+1)} = \left( \frac{1}{\omega} D + L \right)^{-1} b + \left( \frac{1}{\omega} D + L \right)^{-1} \left( \left( \frac{1}{\omega} - 1 \right) D - U \right) x^{(k)}$$  \hspace{1cm} (2.109)
The matrix multiplying $x^{(k)}$ in Eq. 2.109 is upper-triangular and its determinant reduces to the product of its diagonal elements, calling this matrix $L_\omega$:

$$
\det L_\omega = \det \left( \frac{1}{\omega} D + L \right)^{-1} \det \left( \frac{1}{\omega} - 1 \right) D - U \\
= \frac{\omega^n}{\prod_{i=1}^{n} a_{ii}} \frac{(1 - \omega)^n}{\omega^n} \\
= (1 - \omega)^n
$$

(2.110)

If the eigenvalues of $L_\omega$ are $\lambda_1$, $\lambda_2$, ..., $\lambda_n$ with $|\lambda_1| \geq ... \geq |\lambda_n|$, it follows that $\prod_{i=1}^{n} \lambda_i = (1 - \omega)^n$. This implies $|\lambda_1|^n \geq (1 - \omega)^n$. Convergence is ensured provided $|\lambda_1| < 1$, so a necessary condition for SOR to be converging is to require $0 < \omega < 2$. It is possible to determine the optimal over-relaxation parameter $\omega_{OPT}$.

**Definition** [153, p.20]: A symmetric matrix is consistently-ordered if the eigenvalues of

$$
D^{-1} \left( \alpha L + \frac{1}{\alpha} U \right)
$$

are independent of $\alpha$ for all real $\alpha \neq 0$. This holds for the discrete Laplacian operator by using row and column operations.

**Theorem** [153, p.23]: Let $A$ be a symmetric and consistently-ordered matrix, then the optimal relaxation parameter satisfies

$$
\omega_{OPT} = \frac{2}{1 + \sqrt{1 - \rho^2 (M_J)}}
$$

(2.112)

where $\rho (M_J)$ is the spectral radius of the Jacobi iteration matrix associated to $A$.

### 2.4.4 Electric and magnetic fields time advancement

The electric and magnetic fields are advanced in time by solving Faraday’s law for the magnetic field and the generalized Ohm’s law for the electric field. The spatial derivatives are evaluated using a first-order forward finite difference:

$$
\frac{df (x_k)}{dx} = \frac{f (x_k + \Delta x) - f (x_k)}{\Delta x} + O (\Delta x) \approx \frac{f_{k+1} - f_k}{\Delta x}
$$

(2.113)

with the specificity that the electric and magnetic fields are staggered. They are defined on a Yee lattice which offsets them as seen in Fig. 2.4. The code introduces a one-dimensional version of this lattice and uses periodic boundary conditions. The advantage results in $\nabla \times \mathbf{E}$ being second-order accurate at the $\mathbf{B}$-field position and conversely for $\nabla \times \mathbf{B}$ at the $\mathbf{E}$-field position: the staggered-grid has changed.
the forward-finite spatial difference to a centered-finite spatial difference and has a
$O(\Delta x^2)$ error. The ion density and ion bulk velocity are calculated at the E-field
locations \[131, 154\]. The staggered E- and B-fields are defined at integer time steps

To advance the E-field, the ion densities are available at step $n + 1$ following the ion
time advancement Eq. 2.93 but the ion bulk velocities are known at $n + 1/2$ and
not at $n + 1$. A fourth-order Bashford Adams extrapolation is employed to obtain
the ion bulk velocity $V_i$ at step $n + 1$

$$V_i^{n+1} = 2V_i^{n+\frac{1}{2}} - \frac{3}{2}V_i^{n-\frac{1}{2}} + \frac{1}{2}V_i^{n-\frac{3}{2}}$$  \hspace{1cm} (2.114)

For massless electrons, this leads to

$$E_i^{n+1} = \frac{1}{\mu_0 e n_e^{n+1}} (\nabla \times B)^{n+1} \times B^{n+1} - V_i^n \times B^{n+1} - \frac{1}{en_e^{n+1}} \nabla P_e^{n+1}$$ \hspace{1cm} (2.115)

and the RHS is evaluated at the locations of the different E-field components. In
the case of inertial electrons, we need to solve

$$- \left[ en_e^{n+1} + \frac{e_0 m_e c^2}{e} \nabla \times \nabla \times \right] E^{n+1} = F^{n+1}$$ \hspace{1cm} (2.116)

56
with
\[ F^{n+1} = -J_e^{n+1} \times B^{n+1} - \frac{m_e}{e} \nabla \cdot (V_e J_e)^{n+1} + \nabla \cdot P_e^{n+1} \] (2.117)

and
\[ J_e^{n+1} = (\nabla \times B)^{n+1} - \mu_0 e n_e^{n+1} V_i^{n+1} \] (2.118)
The \( x \)-component of Eq. 2.116, calculated at the edge of each cell, is directly obtained:
\[ E_x^{n+1} = \frac{1}{e n_e^{n+1}} F_x^{n+1} \] (2.119)

For the \( y \)- and \( z \)-component of \( E \), a linear system needs to be solved
\[
\frac{\epsilon_0 m_e c^2}{\epsilon \Delta x^2} E_{k+1}^{n+1} - \left( e n_{e,k}^{n+1} + \frac{2 \epsilon_0 m_e c^2}{\epsilon \Delta x^2} \right) E_k^{n+1} + \frac{\epsilon_0 m_e c^2}{\epsilon \Delta x^2} E_{k-1}^{n+1} = F_k^{n+1}
\] (2.120)

where \( k = 1, \ldots, N \) indicates the node of the grid. Thus the \( y \)- and \( z \)-component of \( E \) at step \( n + 1 \) are obtained by solving a symmetric tridiagonal linear system:
\[
\begin{bmatrix}
  a_1 & b & 0 & & \\
  b & a_2 & b & & \\
  & \ddots & \ddots & \ddots & \\
  & b & a_k & \ddots & \\
  0 & & \ddots & \ddots & b \\
\end{bmatrix}
\begin{bmatrix}
  E_1 \\
  E_2 \\
  \vdots \\
  E_k \\
  \vdots \\
  E_N 
\end{bmatrix}
= \begin{bmatrix}
  F_1 \\
  F_2 \\
  \vdots \\
  F_k \\
  \vdots \\
  F_N 
\end{bmatrix}
\] (2.121)

with
\[ a_k = - \left( e n_{e,k}^{n+1} + \frac{2 \epsilon_0 m_e c^2}{\epsilon \Delta x^2} \right), \quad b = \frac{\epsilon_0 m_e c^2}{\epsilon \Delta x^2} \] (2.122)

We use the successive-over relaxation (SOR) iterative method [152, 153] described in Section 2.4.3 to find \( E \). The method converges rapidly [153], typically 5 - 20 steps and is suited for a parallel implementation [155]. The iteration 2.106 for the system 2.121 is
\[ E^{(l+1)}_k = (1 - \omega) E^{(l)}_k + \frac{\omega}{a_k} \left( F_k - b E^{(l+1)}_{k-1} - b E^{(l)}_{k+1} \right) \] (2.123)

The optimal parameter \( \omega = \omega_{OPT} \) is computed from Eq. 2.112 with
\[ \rho (M_J) = \max_{1 \leq k \leq N} \lambda_k 2 \cos \left( \pi / N \right) \] (2.124)
as in Section 2.4.3, with

$$\lambda_k = \frac{(\lambda_{e,k}/\Delta x)^2}{1 + 2(\lambda_{e,k}/\Delta x)^2}$$

(2.125)

and $\lambda_{e,k}$ is the electron skin depth evaluated at position $k$ on the grid. To allow parallelization, a red-black ordering is adopted [155]. To update $E_{2k}$, only the neighbouring values $E_{2k+1}$ and $E_{2k-1}$ are necessary. Therefore we call red the cells with even-index and black the cells with odd-index. Each process initially works by updating their red cells, using the known black quantities and then update the black cells with the newly updated red cells. To summarize, a sub-domain (or MPI process) executes a first sweep to update the red

$$E_{2k}^{(l+1)} = (1 - \omega) E_{2k}^{(l)} + \frac{\omega}{a_{2k}} \left( F_{2k} - bE_{2k-1}^{(l)} - bE_{2k+1}^{(l)} \right)$$

(2.126)

The different processes exchange their updated values at their boundary before they carry out with a second sweep to update the black cells from iteration $l$ to $l + 1$

$$E_{2k+1}^{(l+1)} = (1 - \omega) E_{2k+1}^{(l)} + \frac{\omega}{a_{2k+1}} \left( F_{2k+1} - bE_{2k}^{(l+1)} - bE_{2k+2}^{(l+1)} \right)$$

(2.127)

and $1 \leq ... \leq 2k - 1 \leq 2k \leq 2k + 1 \leq ... \leq N$.

The $B$-field is advanced by using a fourth order explicit Runge-Kutta method. The CFL condition for the whistler wave is usually more constraining than the CFL condition for the ions due to the massless electron. The dispersion relation goes as $\omega/\Omega_{ci} = (kc)^2/w_{pi}^2$ and requires that $\Omega_{ci} \Delta t < (\Delta x \omega_{pi}/c)^2/\pi$ [138, 150] which is satisfied by subcycling the time advancement for the $B$-field. This allows to use a higher time step for the ion time advancement (which satisfies their CFL condition). Assuming the original time step is $\Delta t$, and the time step for the $B$-field time advance is $\Delta t'$ with $\theta = \Delta t/\Delta t'$, an integer, its evolution is computed as [131, 132, 148]:

$$B^{n+\theta} = B^n + \frac{\Delta t'}{6} (K_1^n + 2K_2^n + 2K_3^n + K_4^n)$$

(2.128)

$$K_1^n = -\nabla \times E (B^n)$$

$$K_2^n = -\nabla \times E \left( B^n + \frac{\Delta t'}{2} K_1^n \right)$$

$$K_3^n = -\nabla \times E \left( B^n + \frac{\Delta t'}{2} K_2^n \right)$$

$$K_4^n = -\nabla \times E (B^n + \Delta t' K_3^n)$$

(2.129)
and
\[ E(B) = -n^{n+\frac{1}{2}} \times B - \nabla n^\mu e_{n+\frac{1}{2}} - B \times \nabla \times B \] (2.130)
in the case of massless electrons. We have introduced the PIC-modelling paradigm which makes use of finite-sized macroparticles. We have presented the numerical scheme to advance the ions in time before detailing the time advance of the electric fields for both massless and inertial electrons. We focused on the SOR method used to solve the linear system resulting from the inertial electron Ohm’s law and described the RK4 method implemented to advance the magnetic field. In the following section we present benchmarks of the code.

2.4.5 Quiet start

The code is equipped with a quiet start routine that loads the particles more uniformly in physical and in velocity space in order to reduce the noise in the simulations. While a random loading uses random numbers to place the macroparticles on the grid and to sample their velocity distribution (a Maxwellian or a ring-beam for example), a quiet start uses uniform sequences of numbers instead, which are non-random, such as Hammersley sequences [133, 156]. This results in a decrease of the noise that goes as $1/N$, where $N$ is the number of macroparticles in the simulation, contrary to random loading which leads to a scaling of the noise in $1/\sqrt{N}$. The quiet start improves the resolution of linear physics in PIC simulations [133]. However non-uniformities generated in phase-space can lead to correlations responsible for non-physical instabilities [105]. The implementation follows [105, Chapter 16] which uses Hammersley sequences [133, 156] to invert the velocity distribution function of the thermal ions (instead of uniform random sequences for random starts). For $N$ particles, this sequence $X$ is generated as follows:

\[ X = \{ (j - 1/2), \Phi_2(j), \Phi_3(j), ..., \Phi_r(j), ..., \Phi_p(j) \}, \quad j = 1...N \] (2.131)

where $r$ is a prime number and $\phi$ is the radical inversion of $j$ in base $r$:

\[ j = a_0 + a_1 r^1 + a_2 r^2 + ... \Rightarrow \Phi_r(j) = a_0 r^{-1} + a_1 r^{-2} + ... \] (2.132)

For example, $\Phi_3(1) = 1/3, \Phi_3(8) = 2/3 + 2/5 = 8/5$.

It is then possible to generate multiple sequences $X$ by varying $r$ for the positions and the different velocity components of the ion species to load in phase space. Figure 2.5 shows a 2D random sequence along with a 2D Hammersley sequence used as input numbers to place particles on the grid and to generate the velocity distribution.
function using for example Box-Muller algorithm [157]. The Hammersley sequence is coined as a maximally self-avoiding sequence as the points are generated to be as uniformly spread as possible in the given volume in which they belong. As an illustration, we have performed the hybrid simulations pertaining to JET DT plasma 26148, as originally presented in Ref. [19]. The physical parameters at the ICE emission location were as follows: background magnetic field $B_0 = 2.1T$, the density of the thermal electrons and deuterium $n_e = n_D = 10^{19}\,\text{m}^{-3}$ while their temperatures were equal to 1keV. A minority population of 3.5MeV alpha particles was initialised with a ring-beam distribution $1/(2\pi) \delta(v_\parallel)(v_\perp - u_\perp)$ with a relative density $\xi = n_\alpha/n_e = 0.001$. The PIC-hybrid simulations used 1000 macroparticles per cell for each ion species and 8192 cells during 10 deuterium gyroperiods. The background magnetic field is set perpendicularly to the 1D simulation domain. The spatiotemporal Fourier transform of $\delta B_z$ is shown in Fig. 2.6.
2.5 Code benchmarks

2.5.1 Resonant electromagnetic ion-ion instability

When a low-density beam of fast ions \( (n_b \ll n_e) \) is drifting along a magnetic field \( B_0 = B_0 \hat{x} \) at relative speed \( V_r \gg V_A \) with respect to a bulk ion population, the resonant ion-ion electromagnetic instability can occur and corresponds to the ions being in resonance with the wave. The Alfvén speed is given as \( V_A = B_0 / \sqrt{\mu_0 n_e m_i} \) and \( m_i \) and \( n_i \) are the mass and density of the bulk ions respectively, assumed to be protons. The electron density \( n_e = n_i + n_b \) when the beam ions are also protons. There is no net current in the system by choosing a frame of reference in which the electrons are at rest. In this frame of reference, we denote the background and beam ion velocities along \( B_0 \), \( V_i \) and \( V_b \) respectively which satisfy \( e n_b V_b + e n_i V_i = 0 \) and \( V_b = V_i + V_r \). This gives a drift velocity for the background ions of

\[
\frac{V_i}{V_A} = -\xi V_b
\]

(2.133)

where \( \xi = n_b / n_e \), \( V = V_r / V_A \) and \( \hat{b} = B_0 / B_0 \), while the beam ions have a velocity expressed as

\[
\frac{V_b}{V_A} = (1 - \xi) V b
\]

(2.134)
The cold plasma dispersion relation for electromagnetic waves propagating along the background magnetic field is

$$\omega^2 - c^2k^2 - \sum_j \frac{\omega_j^2 (\omega - kV_j)}{\omega - kV_j \pm \Omega_j} = 0$$  \hspace{1cm} (2.135)

The index $j$ includes electron (e) and beam (b) and bulk (i) ion species, $\omega_j = \sqrt{n_j q_j^2/\epsilon_0 m_j}$ and $\Omega_j$ are their associated plasma and cyclotron frequencies respectively ($\Omega_i = \Omega_b$). The waves are right hand (+) or left hand (-) polarized. The REIII condition for the right hand polarized wave, propagating in the beam direction, is

$$\omega - kV_b + \Omega_b \approx 0$$  \hspace{1cm} (2.136)

In the electron rest frame, these waves are right hand polarized but they are left hand polarized in the frame of the beam (ion cyclotron). In the low density ($n_b/n_e \ll 1$), super-Alfvénic ($V_r/V_A \gg 1$) regime, the growth rate of the instability can be obtained analytically. More specifically, in the case of relative beam densities of the order of a few percents $\gamma/\Omega_b > V_b/V_A$, the maximum growth rate with $\omega = \omega_r + \gamma$ is [158]

$$\frac{\gamma_m}{\Omega_b} \approx \left(\frac{\xi}{2}\right)^{1/3}$$  \hspace{1cm} (2.137)

which is independent of $V_b$, at wavenumber

$$\frac{k_m V_A}{\Omega_b} \approx \frac{1}{V}$$  \hspace{1cm} (2.138)

We have run a hybrid-PIC simulation of REIII with the identical parameters to those of Ref. [13, 19] as a first benchmark of the code, initializing the background ions with a random start and then with a quiet start. Both beam and bulk ions are protons, the relative beam density is $\xi = 0.015$ and we set $V_b/V_A = 10$. The plasma has parameters $\beta_j = n_j k_B T_j/(B_0^2/2\mu_0) = 1$ while $\omega_i/\Omega_i = 10^4$. In our simulation, this implies $B_0 = 1.364 \times 10^{-4}$T, $n_i = 10^{16}$m$^{-3}$ and $T_i = 5.362 \times 10^4$K. Numerical smoothing of the field and bulk quantities is added. The grid consists of 512 cells whose size is set to be the ion skin depth $\lambda_i = 2.294$m ($\lambda_e = 0.053$m in comparison), and the time step satisfies $\Delta t = 0.031\Omega_c$. A number of 90 macroparticles per cell are used to initialize both proton species. The bulk and beam protons follow an original drifting-Maxwellian distribution with respective drifts $V_i$ and $V_b$. We load the bulk ions in phase-space with a random start in one simulation which is repeated with the use of quiet start while the beam protons are initialized with a random start in
Fig. 2.7: Time evolution of the spectral density of the spatial Fourier transform of $\delta B_z$ plotted on a log$_{10}$ scale, obtained from a hybrid-PIC simulation (left panel). A wide range of intense modes with wavenumbers $0.05 \leq kc/\omega_i \leq 0.15$ are excited and peak at $0.11 \pm 0.01$ (dark red trace). Beyond $t\Omega_i = 35$, wavenumbers with $kc/\omega_i \geq 0.15$ start to be excited. For comparison, the numerical dispersion relation for the same simulated parameters is reproduced from Winske and Omidi [13] on the right panel and show the real frequency $\omega_r$ (continuous line) and the positive growth rates $\gamma$ (dashed line) as a function of wavenumber. The instability is maximum at $kc/\omega_i = 0.12$ with $\gamma_m/\Omega_b \simeq 0.2$, $\omega_r/\Omega_b \simeq 0.2$ and the simulation result is in close agreement.

Fig. 2.7 shows the time evolution of the power spectral density in the spatial Fourier transform of $\delta B_z$, computed from our simulation on the left panel. Important instability takes place at wavenumbers $0.05 \leq kc/\omega_i \leq 0.15$ and the most intense mode has $kc/\omega_i = 0.11 \pm 0.01$. The numerical solution of the dispersion relation on the right panel of Fig. 2.7 shows $\gamma/\Omega_b$ and $\omega_r/\Omega_b$ as a function of $k_c/\omega_i$ and predicts maximum instability at $kc/\omega_i \sim 0.2$ with $\gamma_m/\Omega_b \sim 0.2$ and $\omega_r/\Omega_b \simeq 0.2$ as well.

Fig. 2.8 is again a plot of the power spectral time evolution of the spatial FFT of $\delta B_z$ on a log$_{10}$ scale for protons initialized with a random start (left panel) and with a quiet start (right panel). The power density is much lower in the early stage of the simulation with the quiet start, as well as at low $k$ wavenumbers.

The time evolution of the most unstable mode $\delta B_m^2(t)/B_0^2$ is plotted on the top panels of Fig. 2.9. The instability saturates at the same energy at $t\Omega_i \approx 40$ in the randomly-loaded bulk simulation but not until $t\Omega_i \approx 60$ in the quietly-loaded bulk simulation. The best linear fits of $\delta B_m^2(t)/B_0^2$ are computed during the growth phase as shown on the bottom panels. The maximum growth rate $\gamma_m/\Omega_b = 0.189 \pm 0.001$ in both simulations is in good agreement with linear theory which predicts a value of $\gamma_m/\Omega_b \approx 0.2$ (Fig. 2.7). We observed on these bottom panels that the initial power density is smaller by two order of magnitudes in the quiet simulation. The behaviour of the simulations differ in the non linear regime and could be due to the
Figure 2.8: Time evolution of the power spectral density of the spatial FFT of $\delta B_z$ plotted on a log$_{10}$ scale. The left panel (which reproduces the left panel of Fig. 2.7) corresponds to a random start initialisation of the bulk protons in phase-space and the right panel to a quiet start loading. The spectral power density is much lower at the beginning of the simulation with the quiet start and the noise is especially reduced at lower $k$ values. The quiet start sees saturation of the instability at a later stage and both simulations show excitation of wavenumbers $kc/\omega_i \geq 0.15$ at the end of their linear phase.

...fact that as the bulk ions relax towards a Maxwellian, they interact with the beam producing the different time evolutions. Energy conservation varies between $-0.002$ and $0.014$ in the randomly loaded simulation and between $-0.004$ and $0.008$ in the quietly-loaded simulations.

We have run two hybrid simulations of a known instability as a first benchmark of the code. The linear physics is well captured and the quiet start reduces the noise at the beginning of the simulations although the non linear regime somewhat differs with the different particles loading.

### 2.5.2 Ion Bernstein waves

Bernstein waves [159] or hot plasma waves occur in magnetized plasmas as a result of their finite temperature. Such waves exist for both ions and electrons and are kinetic in nature. In the case of purely perpendicular wave propagation, $k_\parallel = 0$, Landau damping is absent and the dispersion relation is given by [160]

$$\sum_s \frac{1}{\kappa_s} e^{-\kappa_s} \sum_{n=1}^{\infty} \frac{2n^2 \omega_{ps}^2}{\omega^2 - n^2 \Omega_{cs}^2} I_n (\kappa_s) = 1$$

(2.139)

with $s$ denoting the electrons and the different ion species, $\omega_{ps}$ and $\Omega_{cs}$ are their respective plasma and cyclotron frequencies. The perpendicular thermal velocity is
Figure 2.9: Time evolution of the most unstable mode $\delta B_m^2(t)/B_0^2$ which corresponds to $\delta B^2(\kappa_c/\omega_i = 0.11, t)/B_0^2$ in the hybrid simulation (top panels) and best fit calculation of $\log_{10}(\delta B_m^2(t)/B_0^2)$ (bottom panels). The bulk protons are loaded with a random start (quiet start) on the left (right) panel. The top panels show that the initial power is lower in the quiet start simulation while the instability saturates at the same power in both instances as seen on the bottom panels.
denoted \( v_{\perp,0} \) such that \( \kappa_s = \frac{k^2 v_{\perp,0}^2}{2 \Omega_{ci}^2} \). We have performed simulations of a warm plasma, for parameters almost identical to those related to ICE in LHD explored in this thesis, resolving multiple cyclotron harmonics as compared to the REIII simulations of the previous section. With some variations, the parameters correspond to LHD 79126 edge plasma conditions given in Table 3.1. More precisely, we set \( B_0 = 0.46 \text{T} \), the plasma consists of thermal protons and massless (inertial) electrons with a temperature of \( T_H = T_e = 2 \text{keV} \) and of a density \( n_H = n_e = 10^{19} \text{m}^{-3} \). The Debye length \( l_D = 1.0511 \times 10^{-4} \text{m} \), the proton and electron skin depths are \( \lambda_i = 0.0720 \text{m} \) and \( \lambda_e = 0.0017 \text{m} \) respectively. The Larmor radius \( r_H \) and plasma \( \beta \) are increased to 0.140m and 0.0380 due to the high temperature in comparison with Table 3.1, while \( \omega_H/\Omega_H = 94.5 \) is unchanged. The simulation domain contains 24576 cells, each of length 0.0024m and each cell is initially quietly-loaded with 4000 macroparticles. The angle between the simulation domain and the background magnetic field is 89.5°. Twelve proton gyroperiods are followed. We compare the code outputs with the solution of Eq. 2.139 restricted to the ion species only (so one species only in this case). The spectral power of the spatiotemporal fast Fourier transform of \( \delta B_z \) is plotted on a log\(_{10} \) scale in Fig. 2.10. The dark dots are the solution of the ion Bernstein dispersion 2.139. Good agreement is obtained at higher harmonics (\( \geq 5 \Omega_H \)). Some modes seem to occur at low \( \omega \) at higher \( k \) values, which locate at \( \Omega_H \leq \omega \leq 5 \Omega_H \) and \( 20 \Omega_H/V_A \leq k \leq 50 \Omega_H/V_A \). They correspond to the red traces which increase with \( \omega \) as \( k \) increases in the aforementioned (\( \omega,k \))-range. The features appear with and without Hanning window in the temporal Fourier transform. They could be cyclotron harmonics or differ from the Bernstein dispersion as the propagation angle is 89.5° and 90°. In Ref. [161], a distinction is made between pure ion Bernstein waves characterized by \( k_{\parallel} = 0 \) and neutralized Bernstein waves which have a small finite \( k_{\parallel} \) and allow electrons if they flow fast enough to cancel charge. In that case, the dispersion takes the form

\[
k_{\parallel}^2 \left[ 1 + \frac{k_{\parallel}^2}{k_{\parallel}^2} - \frac{\Omega_{ci}^2}{\omega^2} e^{-\kappa_i}, I_0 (\kappa_i) \right] + k_{\perp}^2 \left( 1 - \frac{\omega_p^2}{\omega^2 - \Omega_{ce}^2} + \sum_{n=1}^{\infty} \frac{2n^2 \omega_{ci}^2}{\omega^2 - n^2 \Omega_{ci}^2} I_n (\kappa_i) \right) = 0
\]

(2.140)

whose solution is superimposed to the simulated dispersion relation on the bottom panel of Fig. 2.10. The agreement is good but the same discrepancy is observed. To further assess the results presented in Fig. 2.10, we have performed additional hybrid simulations looking at pure perpendicular wave propagation. The simulations were run with both massless and inertial electrons. We used the aforementioned physical parameters. The simulation parameters for the massless calculations are
the same except that 2000 macroparticles/cell are used. The inertial electron simulations are set up with 4000 macroparticles/cell over 8192 cells which have a length of $6.32 \times 10^{-4} \, \text{m} (0.38 \times \lambda_e)$. In this case, as seen from Fig. 2.11, we observe an excellent agreement for both massless and inertial electron hybrid calculations with the Bernstein modes predicted by Eq. 2.139. The Bernstein modes are electrostatic and are therefore obtained from Poisson’s equation whereas our hybrid calculations are electromagnetic but quasineutral such that the electric field follows from a generalised Ohm’s law and not from Poisson’s equation. At 90°, no damping mechanism is involved which is no longer true for the finite $k_\parallel$ scenario from Fig. 2.10 at 89.5°. The mass of the electron does not extensively influence the Bernstein modes contrary to quasineutrality which seems the most likely explanation for the discrepancies observed at 89.5°. Another possibility would be the fact that the electrons are not kinetic in our model and it is rather difficult to entangle these assumptions. The physics at high $k$ must therefore be treated carefully and ideally should not be taken into account while the modes at lower $k$ are well reproduced.
Figure 2.10: Spectral power of the spatiotemporal FFT of $\delta B_z$ taken over the whole simulation domain and averaged over the 12 $\tau_H$ of the simulation run and displayed on a log$_{10}$. The black dots overlaid correspond to the solution of Eq. 2.139, the Bernstein dispersion relation (top) and of Eq. 2.140, the so-called neutralised Bernstein dispersion relation (bottom).
Figure 2.11: Spectral power from the spatiotemporal FFT of $\delta B_z$ averaged over $10\tau_H$ and plotted on a log$_{10}$ yielding the warm plasma dispersion relation. The red (roughly horizontal traces) feature the ion-Bernstein modes using the physical parameters of LHD edge thermal plasma 79126 given in Table 3.1 except for the temperature which is increased to 2keV. The propagation angle between $k$ and $B_0$ is 90$^\circ$. The left (right) panels pertain to massless (inertial) hybrid simulations. Top and bottom and panels represent the same simulation outputs but with different dispersion relations overlaid with the black dots. The analytical dispersion relation for Bernstein modes Eq. 2.139 (Eq. 2.140 ) is superimposed on the top (bottom) panels. We observe a very good agreement between the modes output from simulations and predicted Bernstein modes of Eq. 2.139.
2.6 Limitations of the Hybrid model

As presented in Sections 2.2 and 2.3, the hybrid approximation rests on several assumptions such as quasineutrality which requires that the typical lengths $L$ (the cell size $\Delta x$, or the Larmor radii $r_L$ of the different ion species) in our simulations be greater than the Debye length $\lambda_D$.

Since the electrons are treated as a fluid, kinetic electron effects cannot be modelled and are therefore assumed to remain close to a Maxwellian. Massless electrons set the additional constraint that $L$ needs to be greater than the electron skin depth $\lambda_e$ and that the lower hybrid frequency

$$\omega_{lh} = \left[ (\Omega_i \Omega_e)^{-1} + \omega_i^{-2} \right]^{-1/2}$$

(2.141)

might not be well resolved when treating high values of $k$, for perpendicular wave propagation. Here $\Omega_i$ and $\Omega_e$ are the ion and electron cyclotron frequencies and $\omega_i$ is the ion plasma frequency.

When it comes to parallel propagation, it can be seen that the whistler wave in the hybrid approximation with massless electrons is responsible for the severe constraint on the CFL condition. The frequency scales like $\omega/\Omega_i = (kc)^2/\omega_i^2$ and so diverges as $k^2$. The biggest wavenumber on the grid is given by $k_{max} = \pi/\Delta x$ so that $v_{max} = \omega_{max}/k_{max} = \Omega_i k_{max} (c/\omega_i)^2$. The CFL translates into $v_{max} \Delta t < \Delta x$ or $\Omega_i k_{max} (c/\omega_i)^2 \Delta t = \Omega_i \pi/\Delta x (c/\omega_i)^2 \Delta t < \Delta x$, namely, $\Omega_i \Delta t < (\Delta x \omega_i/c)^2 / \pi$ [138].

A small fraction of $\Omega_i$ must be taken to resolve scales of the order $c/\omega_i$, in addition to resolving the usual particles CFL condition. Sensitivity to low cell sizes is reported in Fig. 8 of Ref. [162].

In addition, neglecting the displacement current implies restrictions on the set of physical parameters being simulated. We seek some conditions on these parameters by first estimating a value for the electric field using Coulomb’s law (which the hybrid model does not use and is only used here to obtain some estimations)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

(2.142)

assuming a 1D slab along $\hat{x}$ such that $\rho \approx Z_b \xi n_e \delta x$, where $Z_b$ and $\xi$ are the atomic number and the relative density of the energetic ions,

$$|\mathbf{E}| = E \approx \frac{Z_b \xi n_e}{\epsilon_0} \delta x$$

(2.143)
Next, using Ampère’s law
\[
\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.144}
\]
and assuming that \(|\nabla \times \mathbf{B}|\) roughly varies as \(B_0/r_L\), where \(B_0\) is the magnitude of the background magnetic field in our simulations and \(r_L\) is the Larmor radius of the energetic ions (NBI or fusion-born). Expressing \(|\mathbf{J}| = Z_b e \xi n_e v_b\) with \(v_b\) the speed of the energetic ions, we can write Eq. 2.144 as
\[
1 \approx \frac{r_L}{B_0} \mu_0 Z_b e \xi n_e v_b + \frac{r_L}{B_0} \mu_0 \epsilon_0 \left| \frac{\partial \mathbf{E}}{\partial t} \right| \tag{2.145}
\]
Taking the time derivative of Eq. 2.143 gives
\[
\frac{\partial \mathbf{E}}{\partial t} \approx \frac{Z_b e \xi n_e}{\epsilon_0} v_b \tag{2.146}
\]
and along with \(r_L = v_b/\Omega_b\) and \(\Omega_b = Z_b e B_0/m_b\), Eq. 2.145 becomes
\[
1 \approx \frac{1}{B_0} \frac{v_b}{\Omega_b} \mu_0 \frac{Z_b e}{m_b} m_b \xi n_e v_b + \mu_0 \epsilon_0 \frac{1}{B_0} \frac{v_b}{\Omega_b} \frac{Z_b e \xi n_e}{\epsilon_0} v_b \tag{2.147}
\]
Introducing the Alfvén speed \(V_A\) and supposing that the energetic ions are of the same species as the thermal plasma, we find
\[
V_A = \sqrt{\frac{B_0^2}{\mu_0 m_b n_e}} \tag{2.148}
\]
and together with the plasma frequency \(\omega_b\),
\[
\omega_b^2 = \frac{(Z_b e)^2 \xi}{\epsilon_0 m_b} n_e \tag{2.149}
\]
this allows us to reexpress Eq. 2.147 as
\[
1 \approx \frac{v_b^2}{V_A^2} + \xi \frac{\omega_b^2}{c^2} \frac{\omega_b^2}{\Omega_b^2} \tag{2.150}
\]
Since \(\Omega_b/\omega_b = V_A/c\), this translates into
\[
1 \approx \frac{v_b^2}{V_A^2} (1 + \xi) \tag{2.151}
\]
which is satisfied provided $\xi \ll 1$ when $v_b \sim V_A$ as in our simulations, see Table 3.1. This estimate is somewhat crude and the presence of electrons should soften that estimate. Finally, the equations of motion being non relativistic for the fields and for the particles, it is finally necessary to ensure that

$$\frac{V_A}{c} \ll 1$$  \hspace{1cm} (2.152)

which implies that the plasma is weakly magnetised.

We turn back to the sensitivity of the simulations with respect to the value of the cell size by looking at the underlying dispersion relation [128, 129, 163]. We have run massless and inertial hybrid simulations of a thermal plasma characterised by LHD 79003 edge background parameters, given in Table 3.1. To that purpose, the simulation parameters are different to those of Table 3.2 as the cell size is comparatively lowered and corresponds to $0.10\lambda_e$. The calculations use 512 cells and 200 macroparticles per cell and are run for 2 gyroperiods. The angle between the simulation domain direction $\hat{x}$ and the background magnetic field $B_0$ is $90^\circ$.

Fig. 2.12 shows the spatiotemporal Fourier transform of $\delta B_z$ in both instances. The dark dashed line superimposed corresponds to the cold plasma dispersion relation while the horizontal trace is the lower hybrid frequency $\omega_{lh}$. At such a small cell size, the inertial dispersion approaches $\omega_{lh}$ while the massless dispersion blows up well beyond. The smaller the cell size, the stronger the behaviour. This implies that massless simulations must have cell sizes $\Delta x$ close to, or above $\lambda_e$ but not below it. Theoretically, $\omega_{lh}$ in the massless case should be equal to $\omega_i$, as seen from Eq. 2.141. We have verified this when solving the hybrid dispersion relation as presented in Chapter 5. We have repeated the hybrid simulations with massless electrons for several different cell sizes to observe the variation in the dispersion relation. These simulations correspond to a proton electron thermal plasma with $T_e = T_H = 4$keV, the density $n_e = 10^{19}$m$^{-3}$ and the magnetic field $B_0 = 1.7$T. The number of cells is varied such that the different simulations have the same domain length. The simulation with the smallest cell size has 2048 cells and every calculation uses 250 macroparticles per cell. We have plotted the maximum frequency $\omega$ as a function of the cell size in Fig. 2.14. The value of this frequency satisfies $d\omega/dk = 0$ (the maximum of the dispersion, as in Fig. 2.13 which is found at $(k, \omega) \approx (500, 300)$ for example). The trend depicted in Fig. 2.13 is reproduced in Fig. 2.14: as the cell size is lowered, the dispersion blows up and the numerical maximal mode frequency $\omega$ increases and and vis versa. Quantitatively, it is interesting to observe that this maximal numerical mode frequency $\omega$ goes with the inverse of the cell size $\Delta x$. As
Figure 2.12: Spatiotemporal Fourier transform of $\delta B_z$ resulting from 2 hybrid simulations of a thermal plasma whose background parameters are those of the edge of LHD 79003, Table 3.1. The left panel displays the result for massless electrons and the right panel for inertial electrons. The cell size is small and corresponds to $\Delta x = 0.10 \lambda_e$. The superimposed dark dashed curve is the cold plasma dispersion relation and the horizontal line is the lower hybrid frequency $\omega_{lh}$. The massless dispersion departs very strongly from the former, contrary to the inertial dispersion which approaches $\omega_{lh}$. Note that there is no aliasing in the massless dispersion as shown in Fig. 2.13.

Figure 2.13: Spatiotemporal Fourier transform of $\delta B_z$ obtained from a hybrid simulation with massless electrons. The graph corresponds to the left panel of Fig. 2.12 which has been unzoomed.
Figure 2.14: Maximal frequency $\omega$ obtained from the spatiotemporal Fourier transforms of several massless hybrid simulations. The numerical "lower hybrid frequency", as it represents the maximal mode frequency in the simulation, is plotted against the cell size normalised to the electron skin depth $\lambda_e$.

$\Delta x$ is reduced, the timestep is adapted accordingly to satisfy the CFL condition and to avoid any aliasing in the simulation outputs. The physical $\omega_{lh}$ is marked with the dark star in Fig 2.14 as well. The ideal cell size $\Delta x$ is one that lies above $\lambda_e$ and gives a correct dispersion relation in the case of the massless hybrid model, provided the additional assumptions of the model are satisfied.

### 2.7 Summary

We have presented PROMETHEUS++ and described its core algorithms. A derivation of the generalized Ohm’s law was given before presenting the methods to represent the ions and the numerical schemes to evolve their dynamics self-consistently with the electric and magnetic fields by mean of the shape functions. We briefly presented some iterative methods to solve linear problems which appears for the solution of the electric field in the presence of inertial electrons. We describe the SOR method and how it is parallelized in PROMETHEUS++. The numerical scheme to advance the magnetic field was given. We ran two test simulations. The first one simulated the resonant electromagnetic ion-ion instability (REIII) using both a quiet start and a random start. Good agreement with the linear theory was obtained. We also obtained the warm plasma dispersion relation encompassing Bernstein waves.
They show the potential of the code to simulate phenomena in the ion cyclotron range of frequencies up to several cyclotron harmonics. It is also important to remember the limitations of the code: quasineutrality is required which means that the smallest length scale (the cell size $\Delta x$) needs to be much larger than the Debye length $\lambda_D$, for massless electrons $\Delta x \geq \lambda_e$ and the plasma needs to be weakly magnetized $\omega_p/\Omega_c \gg 1$. Unlike a full-PIC simulation which must resolve the Debye length, a massless hybrid simulation cannot resolve the electron skin depth $\lambda_e$. In particular this implies limitations in the smallest scales or the highest wavenumbers attainable in a simulation. The cell size must be chosen carefully [162]: small enough to capture the ion physics, but not too small to remain within the approximations. One way forward is the use of electron inertia. Further work should focus on adding fast particle replenishment, on treating the ion pressure tensor in the generalized Ohm’s law [127] and on making the code-2D. This could require modifications of the parallelism: so far each process evolves the dynamics of the same set of macroparticles, which can a priori be anywhere on the grid. For density and current deposition, each MPI process thus possesses its own copy of the whole grid, even though each process updates the electric and magnetic fields on its dedicated subdomain. A final long term goal regards the addition of more geometry in the code.
Chapter 3

Interpreting observations of ion cyclotron emission from beam-injected ions in Large Helical Device plasmas

3.1 Introduction

Measurements of ion cyclotron emission (ICE) spectra have been obtained from heliotron-stellarator plasmas in the Large Helical Device (LHD), both with an ICRH antenna during NBI heated plasmas [7] and by magnetic probes [39] during toroidal Alfvén eigenmodes (TAE’s) [113–116]. Related numerical studies can be found in [116]. In combination with other advanced diagnostics, notably for MHD activity, these ICE measurements from LHD, which we analyse here, can yield fresh insights into the physics of energetic ions in magnetically confined fusion (MCF) plasmas, and into the interaction between these ions and MHD activity. Exploitation of these advanced, high resolution ICE measurements requires a correspondingly advanced modelling capability for its most likely emission mechanism, which is the magnetoacoustic cyclotron instability (MCI). Here we analyse ICE spectra which we attribute to neutral beam injected (NBI) proton population at energies $\approx 40$keV in the outer midplane edge regions of plasmas in LHD [7, 39], where the local electron temperature $T_e \approx 20 – 150$eV, number density $n_e \approx 10^{19}$m$^{-3}$ and magnetic field strength $B \approx 0.5$T. These spectra were measured with an ICRF heating antenna in receiver mode. Importantly, these spectra span plasma regimes where the ratio of the velocity of the energetic ions $V_{NBI}$ to the local Alfvén speed $V_A$ in the ICE-emitting
region of the LHD edge plasma takes values that can be both smaller or larger than unity. The transition between super-Alfvénic and sub-Alfvénic energetic ion phenomenology is of fundamental interest in MCF plasma physics. Here, in particular, we examine LHD plasmas 79126 and 79003 where $V_{NBI}/V_A = 0.872$ and $1.125$, respectively, in the ICE-emitting region. We follow the full velocity-space trajectories of tens to hundreds of millions of fully kinetic energetic and thermal ions, and all three vector components of the evolving electric and magnetic fields, with a massless neutralising electron fluid, using a fully nonlinear 1D3V PIC-hybrid particle-in-cell code [19]. The kinetic ions, fluid electrons, and fields are coupled self-consistently through the Lorentz force and Maxwell’s equations [164] in Darwin’s approximation [134]. This hybrid scheme [19] requires less computational resource as the Debye length needs not be resolved contrary to the full-PIC scheme implemented in EPOCH [106], retaining electron kinetics, that is also used for contemporary theoretical studies of ICE [20, 82, 83, 85, 86]. We follow these PIC-hybrid simulations through the linear phase of an instability that we identify as the MCI, and then deeply into its nonlinear saturated phase. The Fourier transforms of the excited fields yield frequency spectra that match the observed ICE spectra from these LHD plasmas. These simulation results for heliotron-stellarator plasmas complement and confirm earlier interpretation of ICE driven by sub-Alfvénic NBI ions in TFTR tokamak plasmas [42, 61].

ICE was previously measured in the W7-AS stellarator [40] by broadband loop antenna during hydrogen NBI. The emission correlated with induced turbulence peaking at the lower hybrid frequency $\omega_{LH}$, observed using Collective Thomson Scattering (CTS). The excitations at cyclotron harmonics were interpreted in terms of destabilized Bernstein waves.

### 3.2 ICE measurements from LHD

Highly resolved ICE signals were measured in LHD hydrogen plasmas using an ICRF antenna in receiver mode [7] during perpendicular neutral beam injection (NBI) of hydrogen ions. The large antenna loop area ($\approx 600\text{cm}^2$) enhances the quality of the data which was recorded at a maximum sampling rate of $5\text{GSas}^{-1}$ and processed via fast Fourier transform, with a rectangular window of typical duration $100\mu\text{s}$. Examples of time profiles of heating with four proton NBI sources, together with ICE spectra, are shown in Figures 3.1 and 3.2. These ICE signals are detected shortly after the turn-on of the perpendicular positive-ion based NB injector #4, displayed as the purple traces in Fig. 3.1, by an antenna located close to it, at
the outer midplane of LHD. This suggests that the observed ICE results from these energetic perpendicular NBI ions. In sub-Alfvénic LHD plasma 79126, NB injector #4 is operated between $t = 0.90s$ and $t = 1.21s$, while it is operated between $t = 1.817s$ and $t = 2.110s$ in super-Alfvénic LHD plasma 79003. The ICE spectra are measured at $t = 1.21s$ and $t = 1.817s$ respectively. The fundamental ICE frequency $f_0$ is defined by the measured interval between successive spectral peaks, and also typically corresponds to the frequency of the first measured spectral peak. A linear relation was obtained between $f_0$ and the magnitude of the magnetic field on axis (at a major radius of 3.6m) across several LHD plasmas, confirming the cyclotronic character of the detected signal since the cyclotron frequency is proportional to the magnitude of the magnetic field at one location. It follows that the location at which this ICE signal is generated in LHD lies along a magnetic field line on which the proton cyclotron frequency $f_{cH}$ corresponds to the measured fundamental ICE frequency $f_0$, see Figs. 3 and 4 of [7]; this is found to be at the LHD plasma outer edge and the precise locations for LHD plasmas 79126 and 79003 are shown in Fig. 3.2. The temperatures and densities at the ICE emission location are determined from Fig. 3.4. The measurements of temperature and density from LHD were taken using Thomson scattering and the signals turned out to have been rather noisy in the case of LHD plasma 79003.

The observation of ICE from high density ($n_e > 5 \times 10^{20}m^{-3}$) LHD plasmas, into which NBI cannot penetrate deeply, further supports the interpretation that ICE originates near the NBI #4 injection point in LHD. The ICE signal disappeared roughly 0.1ms after the turn-off of the perpendicular NBI (see Fig 5 of [7]). This synchronization suggests that ICE is driven by the fast injected protons. Particle orbit calculations [7] for the relevant LHD plasma and magnetic field show that NBI protons are lost in a few tens of microseconds, consistent with the observed decay time of ICE.

Two possible explanations for the width of the peaks in Fig. 3.2 include non zero parallel velocities of the fast ions which would shift the peaks as is investigated in the next chapter. Another one rests on the emission location which may have a finite width such that the resonant frequencies would satisfy $\omega = \Omega_c + \Delta\Omega_c = q_i B_0/m_i + q_i \Delta B_0/m_i$, namely gradients of magnetic fields. The time duration of the measured signals also influence the resolution obtained in the frequency domain (modulo stationarity of the signal in that time interval).
Figure 3.1: Time histories of NBI evolution in sub-Alfvénic LHD plasma 79126 (top) and in super-Alfvénic LHD plasma 79003 (bottom). In both cases, the ICE is concurrent with perpendicular NBI #4, at $t \approx 1.210\text{s}$ (top) and at $t \approx 1.817\text{s}$ (bottom).

Figure 3.2: ICE spectra measured on LHD. Left panel: $B$ field power spectrum for perpendicular 40 keV sub-Alfvénic NBI proton from LHD plasma 79126 at $t \approx 1.21\text{s}$. The radiation originates at $R_{ax} = 4.62\text{m}$ corresponding to $\Omega_{H} = 6.75\text{MHz}$, which we identify with the observed separation $f_0$ between successive spectral peaks, see Fig. 3.3 (left). The local plasma temperature $T_e \approx 150\text{eV}$ and density $n_e \approx 10^{19}\text{m}^{-3}$. Right panel: $B$ field power spectrum for perpendicular 36.5 keV super-Alfvénic NBI proton from LHD plasma 79003 at $t \approx 1.817\text{s}$. The radiation originates at $R_{ax} = 4.65\text{m}$ corresponding to $\Omega_{H} = 3.67\text{MHz}$, the observed separation between successive spectral peaks, see Fig. 3.3 (right). The local plasma temperature $T_e \approx 25\text{eV}$ and density $n_e \approx 0.5 \times 10^{19}\text{m}^{-3}$. These ICE measurements were obtained with an ICRF antenna in receiver mode with 5GHz sampling rate.
Figure 3.3: Dependence of the local value of the proton cyclotron harmonic frequencies on major radial position in LHD plasmas 79126 (left) and 79003 (right). The measured spectral peak separations $f_0$ in Fig. 3.2 are identified with the proton cyclotron frequencies $\Omega_H = 6.75\, \text{MHz}$ and $\Omega_H = 3.67\, \text{MHz}$ respectively. The corresponding magnetic fields have magnitude 0.46T and 0.24T. We infer that ICE is emitted at $R_{ax} = 4.62\, \text{m}$ (left) for LHD plasma 79126, and at $R_{ax} = 4.65\, \text{m}$ (right) for LHD plasma 79003.

Figure 3.4: Radial temperature (top) and density (bottom) profiles in LHD hydrogen 40 keV sub-Alfvénic plasma 79126 (left) and 36.5keV super-Alfvénic NBI-heated 79003 plasma (right). The horizontal axis corresponds to the major radius of LHD. The vertical lines at $R = 4.62\, \text{m}$ (left) and $R = 4.65\, \text{m}$ (right) show the location of the ICE source, see Fig. 3.3. These plasmas are heated with a total of $\sim 1\, \text{MW}$ of hydrogen NBI, both tangential (NBI #1,2,3) and perpendicular (NBI #4).
3.3 Estimation of NBI fast ion distribution.

Kinetic modelling [14] has previously been used to obtain the steady-state distribution function of NBI fast ions in stellarators, for both TJ-II and LHD. The LHD case focused on hydrogen plasmas heated by 40keV perpendicular NBI, relevant to our study. The orbit code ISDEP (Integrator of Stochastic Differential Equations for Plasmas) [14] is a Monte Carlo orbit-following code which solves the Fokker-Planck equation in 5D phase space, namely: \((x, y, z, v^2, \lambda)\). The three spatial coordinates \((x, y, z)\) are the guiding centre position, \(v^2\) is a normalised kinetic energy, and \(\lambda = v \cdot B/vB\) is a pitch. ISDEP includes collisions of fast ions with background ions and electrons, and treats re-entering particles. The initial NBI ion distribution function is calculated with HFREYA [15] which simulates the evolution of fast neutral particles by modelling their propagation, charge exchange and ionization processes. The resulting distribution in Bustos et al. [14]’s work, which is relevant to perpendicular NBI for the LHD case, is displayed on the left panel of Fig. 3.5. This presents distinct features, including the pitch angle \(\lambda\) peaking near zero for NBI perpendicular injection, whose \(\sim 34\text{keV}\) component appears in the upper middle panel of Fig. 3.5. This energy is close to the 36.5 keV super-Alfvénic fast proton population which, as we shall show, drives the ICE. The distribution is localized toroidally and increases with \(\rho\), the dimensionless plasma radius defined in terms of toroidal flux surfaces. This NBI proton population is thus expected to arise close to NBI # 4, and therefore close to the ICRF receiver antenna. We shall thus incorporate this form of minority energetic proton population in our first principles modelling of ICE in LHD.

The top right and bottom right panels of Fig. 3.5, from Ref. [14], show the time evolution of the average energy and of the persistence of the fast ions. The persistence \(P\) corresponds to the probability that fast ions are still in the plasma at a specific time \(t\) following the injection at \(t = 0\). There is a prompt loss phase shown by the sudden decrease of \(P\) around \(t \sim 0.5 \times 10^{-4}\text{s}\), which is very close to the time scale on which ICE decays following the NBI turn-off [7]. The average kinetic energy, shown at the top right of Fig. 3.5, does not significantly change until later. A second phase inferred from the persistence graph corresponds to the slowing-down time of the fast protons, taking place on millisecond time scales, between \(t \sim 10^{-3}\text{s}\) and \(t \sim 10^{-2}\text{s}\). For \(t > 10^{-2}\text{s}\), fast NBI ions undergo loss of confinement. The slowing-down time as a function of plasma radius was also inferred in Ref. Bustos et al. [14] from ISDEP calculations, yielding good agreement. Additional comparisons with the decay time of neutral flux intensities obtained from fast neutral analyzers (FNA) during LHD
radial pulsed NBI experiments (NB-blip) [17, 165] was reported. Good agreement was obtained, and variations owing to the different energy range and time scales were identified. In [17], losses were attributed to pitch-angle scattering. The ion lifetime suggested good confinement properties, particularly in the inward-shifted configuration, and evaluated close to unity when normalized to the 90° pitch-angle scattering time (and to about 10% at $R_{ax} = 3.75 m$). Measurements of the real time neutral particle angular distribution were then performed through two angularly resolved multi-sightline neutral particle analysers (ARMS-NPA) [166–168] during neutral beam injection or cyclotron resonance heating. Spatial resolution was obtained in [168], while the signals were integrated over the line of sight in [17]. The ARMS were set up to simultaneously resolve the tangential and perpendicular directions, which was not the case originally [167], so as to identify and measure the loss-cone in LHD plasmas. The neutrals, variously originating from either the wall, NBI or fuel/impurity pellet, provide information on the velocity distribution of fast ions due to the sensitivity of the particle loss with respect to pitch angle. The neutral flux is then expected to reflect the distribution of the remaining fast ions and thus to provide information on the loss. The left panel in Fig. 3.6 (reproduced from Ref. [168]) shows the particle loss phenomenology at different magnetic field values, during perpendicular NBI #4, on a colour map generated in polar coordinates: radius corresponds to energy and the white circles relate to 15keV, 30keV and 45keV fast ions. The pitch is measured clockwise and bright colouring indicates higher flux. The signal was acquired for 0.25 ~ 0.50s with a 5ms time resolution. The distribution has narrow pitch and losses are observed around a pitch angle of 80°, which decrease with increasing magnetic field. The first panel (top left) corresponds to an absolute magnetic field 0.425T, close to the 0.46T corresponding to the 40keV sub-Alfvénic LHD plasma 79126, see Fig. 3.3 (left). The right panel of Fig. 3.6 (reproduced from Ref. [14]) shows the velocity distribution function of the fast ions computed with ISDEP [14], sampled at points where they hit the vacuum vessel, at three different times associated with the phases identified on the persistence graph at the bottom right corner of Fig. 3.5. We infer from these that ICE occurs before significant slowing-down has taken place, on sub millisecond time scales due to prompt losses of NBI ions in LHD. These losses result in a strongly anisotropic distribution of the lost ions in velocity space which, as we shall show in this chapter, is capable of driving ICE at the LHD plasma edge. The modelled and measured distribution functions of fast NBI ions in the LHD plasma edge thus provide compelling evidence for this strongly anisotropic character. This guides the choice of an approximated ring-beam distribution to represent the velocity distribution of the
Figure 3.5: Panels reprinted with permission from A. Bustos et al., 2011 Nucl. Fusion 51 083040 [14]. Left and middle panels: Initial fast ion distribution computed with HFREYA [15] represented as a function of the parameters $\lambda$, $E$, $\rho$ and $\phi$ corresponding to pitch, energy, plasma radius (expressed in terms of normalized toroidal flux $(\Psi/\Psi_{LFS})^{1/2}$) and toroidal angle respectively. Right panels: Time evolution of the average energy (top) and persistence (bottom) of perpendicular NBI hydrogen in LHD plasmas computed with the code ISDEP [16]. The persistence is defined as the probability that fast ions have remained in the plasma at a given time. Three phases are identified: prompt losses ($t \sim 0.5 \times 10^{-4}$ to $10^{-4}$s), slowing-down phase ($t \sim 10^{-4}$ to $10^{-2}$) and loss of confinement $t > 10^{-2}$, computed with ISDEP.

NBI protons in our PIC-hybrid simulations.
Figure 3.6: Left: Panel reprinted with permission from T. Ozaki; E. Veshchev; T. Ido; A. Shimizu; P. Goncharov; S. Sudo; *Review of Scientific Instruments* 2012, 83, DOI: 10.1063/1.4742925, Copyright ©2012 American Institute of Physics [17]. Angularly-resolved velocity distribution of fast neutral particles measured (trapped) with ARMS-Neutral Particle Analyser showing particle loss characteristics at different magnetic field values. The graph is expressed in polar coordinates. Positive angle is measured from magnetic axis to a given sightline. The minimum measurable energy is 15keV, which appears as the lowest white half-circle, followed by 30 and 45keV. The energy resolution is a few keV. Colouring is on a log scale. Right: Panel reprinted with permission from A. Bustos *et. al.* 2011 *Nucl. Fusion* 51 083040 doi:10.1088/0029-5515/51/8/083040 [14]. Velocity space distribution computed with the code ISDEP [16] and representing the escape of perpendicular NBI hydrogen in LHD when they hit the vacuum vessel. The different graphs correspond to different times, early times at the top and later times at the bottom. A wedge-shaped, strongly anisotropic distribution is visible, which diffuses in velocity space at later times.
3.4 Physical and computational parameters for PIC-hybrid simulations of observed LHD ICE

The NBI protons are sub (super)-Alfvénic in the emitting region of LHD hydrogen plasmas 79126 (79003), whose measured ICE spectra are shown in Fig.3.2 and again in Fig. 3.10. Their initial energies are 40 keV (36.5 keV), corresponding to a proton perpendicular injection velocity \( v_{NBI} = 2.77 \times 10^6 \text{ms}^{-1} \) (2.64 \( \times \) 10^6 ms\(^{-1} \)). To approximately represent LHD edge plasma conditions in the simulations reported here, the thermal electrons and ions both have a temperature of 0.150 (0.025) keV, the electron density is 10\(^{19}\) m\(^{-3}\) (5 \( \times \) 10\(^{18}\) m\(^{-3}\)), see top and bottom of Fig.3.4 respectively; and the background magnetic field has strength \( B_0 = 0.46 \) (0.24) T. The local Alfvén speed \( V_A \) is therefore 3.17 (2.35) \( \times \) 10^6 ms\(^{-1}\), hence the NBI protons are sub (super)-Alfvénic with \( v_{NBI}/V_A \approx 0.870 \) (1.125).

In our simulations, the 40 (36.5) keV NBI proton population is taken to have particle concentration \( \xi = 5 \times 10^{-4} \) (7.5 \( \times \) 10\(^{-4}\)) relative to the thermal ions. It is represented by an initial ring-beam distribution in velocity space, \( f_{NBI}(v_{\parallel}, v_{\perp}) = \delta(v_{\parallel})\delta(v_{\perp} - v_{NBI}) \) where the NBI protons are initially uniformly and randomly distributed in gyro-angle. Denoting the spatial component of the 1D3V simulation domain by \( \mathbf{x} \), the wavevector \( \mathbf{k} = k\hat{x} \); in our simulations, the angle between \( \mathbf{B}_0 \) and \( \mathbf{k} \) is 89.5\(^{\circ}\). There are 22080 (10560) computational cells and 4000 (500) macroparticles per cell. The cell size is 0.61 (1.06) times the thermal ion Larmor radius \( r_L = 0.0031 \) (0.0025) m; for neutral beam ions \( r_{L,NBI} = 0.063 \) (0.115) m. This implies that the simulations resolve 1.35 (1.12) \( \times \) 10\(^4\) background proton gyroradii or 670 (240) NBI fast proton gyroradii.

Once the cell size has been chosen for each instance (usually of the order of the electron skin depth), the different number of cells used leads to grids that resolve the same number of background plasma gyro radii. The increased number of particles per cell in the sub-Alfvénic calculations reduces the noise in the simulations, and facilitates the calculation of the growth rates presented in Section 3.6. The simulations at 85\(^{\circ}\) between \( \omega \) and \( \mathbf{k} \) use fewer particles per cell and a larger grid length, which enables us to run the simulations for longer as shown in Fig. 3.7. The grid size is set to resolve enough wavenumbers, particularly those having small values of \( k \). The time step is 0.00025 (0.0001) \( \times \) \( \tau_H \), where \( \tau_H = 0.143 \mu\text{s} \) (\( \tau_H = 0.273 \mu\text{s} \)) is the proton gyroperiod in each simulation. Cyclotron motion is thus highly resolved, in space and time, for the energetic ions whose cyclotron resonant collective relaxation underlies the observed ICE signals. The simulations run for 10 to 100\( \tau_H \); this is determined by the time taken for the instability driven by the NBI ions to saturate, which for a given set of plasma parameters depends on \( \xi \) and on propagation an-
Table 3.1: Summary of the physical parameters used in the PIC-hybrid ICE simulations for LHD plasmas 79126 and 79003. The quantity $\lambda_D$ corresponds to the Debye length while $\lambda_e$ and $\lambda_H$ are the electron and proton skin depth respectively. The thermal plasma Larmor radii are expressed by $r_L$ and the plasma beta by $\beta$ while $\Omega_H$ and $\omega_H$ are the proton cyclotron and plasma frequencies respectively. The electron thermal speed $v_{Te} = \sqrt{2k_B T_e/m_e}$ and $c$ represents the speed of light.

gle. Physical and computational parameters for our simulations are summarized in Tables 3.1 and 3.2. When necessary, subcycling for the electric and magnetic fields is used to satisfy the CFL condition for the Alfvén wave [138]. In our simulations, energy is conserved within 0.2%.
Table 3.2: Summary of the computational parameters used in the PIC-hybrid ICE simulations for LHD plasmas 79126 and 79003. The cell size $\Delta x$ is greater or equal to $\lambda_e$ in all simulations.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Plasma 79126</th>
<th>Plasma 79003</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regime</td>
<td>sub-Alfvénic</td>
<td>super-Alfvénic</td>
</tr>
<tr>
<td>$\cos^{-1}\left(\hat{k} \cdot \mathbf{B}_0\right)$</td>
<td>$89.5^\circ$</td>
<td>$85.0^\circ$</td>
</tr>
<tr>
<td>Simulation duration</td>
<td>$[6, 20] \times \tau_{gp}$</td>
<td>$360 \times \tau_{gp}$</td>
</tr>
<tr>
<td></td>
<td>$[12, 25] \times \tau_{gp}$</td>
<td>$200 \times \tau_{gp}$</td>
</tr>
<tr>
<td>Number of grid cells</td>
<td>22080</td>
<td>1380</td>
</tr>
<tr>
<td>Number of part. per cell</td>
<td>4000</td>
<td>500</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>0.0019m</td>
<td>0.0038m</td>
</tr>
<tr>
<td></td>
<td>0.0026m</td>
<td>0.0054m</td>
</tr>
<tr>
<td>Length of simul. domain</td>
<td>$670 \times r_{NBI}$</td>
<td>$84 \times r_{NBI}$</td>
</tr>
<tr>
<td></td>
<td>$13500 \times r_L$</td>
<td>$1690 \times r_L$</td>
</tr>
<tr>
<td></td>
<td>$240 \times r_{NBI}$</td>
<td>$11175 \times r_L$</td>
</tr>
<tr>
<td></td>
<td>$125 \times r_{NBI}$</td>
<td>$5810 \times r_L$</td>
</tr>
<tr>
<td>$\xi = n_{NBI}/n_e$</td>
<td>$[0.10, 0.60] \times 10^{-3}$</td>
<td>$5.0 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td>$[0.10, 1.90] \times 10^{-3}$</td>
<td>$0.5 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

3.5 Results of PIC-hybrid simulations of observed LHD ICE

Figure 3.7 plots the time evolution, in our simulations, of the energy density of the electric and magnetic fields, and of the kinetic energy density of the bulk proton and NBI proton populations. The four panels of Fig. 3.7 are for sub-Alfvénic (left) and super-Alfvénic (right) NBI protons, and for propagation angles $\theta$ of $\mathbf{k}$ with respect to $\mathbf{B}_0$ of $89.5^\circ$ (top) and $85^\circ$ (bottom). It is evident that the time taken for the NBI fast protons, which are not replenished, to relax, and for the instability which we identify below with the MCI to unfold accordingly, saturates on time scales of between $10\tau_H$ and $100\tau_H$. This corresponds to a few microseconds to a few 100 microseconds, and is broadly consistent with the observed decay time of ICE at the NBI turn-off in LHD [7]. Spatio-temporal Fourier transforms of the excited fields are shown in Fig. 3.8, where the four panels correspond to the cases in Fig. 3.7. The simulated power spectra shown in Fig. 3.9 correspond again to the four panels in Figs. 3.7 and 3.8. These spectra, including also those in the bottom panels of Fig. 3.10, are constructed from Fig. 3.8 as follows: we compute the spatio-temporal fast Fourier transform (FFT) of the $z$-component of the fluctuating part of the magnetic field, $\delta B_z$, (Fig. 3.8), and then sum over wavenumbers between $k = 0$ and $k = 25\Omega_H/V_A$.

Fig. 3.9 shows good qualitative agreement between the spectra generated by relaxation of the NBI ion population in our first principles Maxwell-Lorentz computations.
with the 1D3V PIC-hybrid code, and the observed ICE spectra from LHD shown in Fig. 3.2, in both the sub-Alfvénic and super-Alfvénic NBI regimes in LHD. In the PIC-hybrid computations, the simulated emission is most strongly driven for propagation angles that are close to perpendicular to the magnetic field, for which the MCI is most unstable and saturates quickly as seen from the simulations at 89.5°. The measured ICE signals from LHD studied here were obtained in the same way as the early JET ICE measurements [4, 34], that is, using an ICRH antenna in receiver mode. The antenna has broad directionality, and in consequence can receive radiation across a wide range of incident angles. This is the reason for considering the emission at an angle of 85° whose corresponding simulated spectra are also shown at the bottom of Fig. 3.10. They show that ICE is still emitted away from pure perpendicular propagation under LHD edge plasma conditions and suggest the robustness of the MCI to drive ICE. In particular we infer that simulations at intermediate angles would generate ICE as well. Although the antenna cannot distinguish between incoming waves with a propagation angle of 89.5° and 85° separately, these would be summed together in forming the observed ICE spectrum. All the simulated spectra in Fig. 3.9 show strong excitation at multiple successive proton cyclotron harmonics; and the ICE signal in the simulations is a hundred to a thousand times more intense than the thermal noise. Decreasing the angle between $k$ and $B_0$ shows preferential excitation at lower harmonics, as well as longer timescales for mode excitations, see Fig. 3.8. Since the power spectra at the bottom panels of Fig. 3.9 are averaged over much longer time periods, high spectral resolution is achieved that translates into the narrow/thick curves. The similar intensities on the top right and bottom right panels are due to comparable NBI proton densities. Conversely, the proton beams densities differ by an order of magnitude on the top left and bottom left panels to saturate the MCI in a tractable computing time, whence the different intensities. The strong spike $\omega = 0$ in Fig. 3.9 could be due to a numerical artifact during the post processing. This spike could also result from three-wave interaction between oppositely propagating waves such that $\omega_1 + \omega_2 = 0$ and $k_1 + k_2 = 0$. The ICE spectra measured with the antenna are integrated over a range of angles. Here we compare the measured ICE spectra with our simulated spectra at 85°, which share a similar qualitative rise and then fall in the distribution of power across successive cyclotron harmonic peaks. At 89.5°, there is a monotonic increase of the power with harmonic number for the range of frequencies considered. In Fig. 3.11, we plot the time evolution of the spatial FFT, $\Delta B_z, \delta B_z^2 (k, t) / B_0^2$. There is a one-to-one mapping between the excited $k$-values and the cyclotron harmonics $\ell \Omega_H$, inferred from the dispersion relation of the fast
Alfvén wave using information in Fig. 3.8. Thus by plotting the time evolution of the distribution of energy across wavenumbers, as in Figure 3.11, we can identify the time sequence in which specific cyclotron harmonics in the simulated ICE spectrum are excited. A notable feature of Fig. 3.11 is the late excitation of the spectral peak at the fundamental cyclotron frequency of the protons in the nonlinear phase.
Figure 3.7: Time evolution (horizontal axis, in units of proton gyroperiods $\tau_H$) of the energy density of different field components and particle populations in four PIC-hybrid simulations. Left panels are for LHD hydrogen plasma 79126 with sub-Alfvénic 40 keV NBI protons, for which $v_{\text{NBI}}/V_A = 0.872$. Right panels are for LHD hydrogen plasma 79003 with super-Alfvénic 36.5 keV NBI protons for which $v_{\text{NBI}}/V_A = 1.125$. Top panels are for propagation angle of $k$ with respect to $B_0$ of 89.5°. Bottom panels are for propagation angle 85°. Concentration $\xi = n_{\text{NBI}}/n_e$ is chosen at a level that gives rise to saturation of the MCI within the simulation run time: $\xi = 5 \times 10^{-4}$ at top left and bottom right; $\xi = 7.5 \times 10^{-4}$ at top right; and $\xi = 5 \times 10^{-3}$ at bottom left. Red trace: bulk protons. Cyan: NBI protons. Green $z$-component and magenta $y$-component of the fluctuating part of the magnetic field. Dark blue: $x$-component of the electric field. The excitation of the $y$-component of the magnetic field is due to the NBI protons that have a velocity component along the $z$ direction, while $k$ is along the 1D simulation domain ($x$ direction). The NBI protons typically release 10-15% of their energy to the background protons and to the electric and magnetic fields. Saturation of the instability occurs after $\approx 1\mu s$ to 100$\mu s$. This time is similar to the time window used to compute the LHD power spectra (see Fig 2. of [7]).
Figure 3.8: Spatio-temporal Fourier transform of the $z$-component of the fluctuating part of the magnetic field $\delta B_z$ in the four PIC simulations whose energy evolution is shown in the corresponding panels of Fig. 3.7. The transform is calculated across the spatial domain and averaged over the simulation duration; magnitude is plotted on a log$_{10}$ colour scale. The left and right pairs of panels correspond to simulations that have different values of NBI proton injection speed $v_{NBI}$ compared to the Alfvén speed $V_A$, in the LHD plasma near the NBI injection point: (left) $v_{NBI}/V_A = 0.870$, (right) $v_{NBI}/V_A = 1.125$. Bright spots at sequential proton cyclotron harmonics along the fast-Alfvén branch result from the MCI, driven by NBI protons, for waves propagating in the $\hat{x}$ direction, almost perpendicular to the background magnetic field. The dark trace corresponds to $\omega = kv_{NBI}$ and lies either above or below $\omega = kV_A$, which defines the boundary between regions of $(\omega, k)$ space that can in principle resonate with sub-Alfvénic or super-Alfvénic NBI protons. The excitations occur along the fast Alfvén branch and preferentially close to modes satisfying $\omega = kv_{NBI}$.
Figure 3.9: Power spectra of the fluctuating z-component of the magnetic field in the four PIC-hybrid simulations whose energy evolution and spatiotemporal Fourier transforms are shown in the corresponding panels of Fig. 3.7 and Fig 3.8, respectively. Power spectra are obtained using different orientations of the background magnetic field with respect to the spatial domain of the 1D3V simulation, which defines the direction of k. Power spectra are obtained by taking the spatio-temporal Fourier transform $\delta B_z(t) = B_z(t) - B_0$ averaged over the simulation duration and summed between $k = 0$ and $k = 25\Omega_H/V_A$. Peaks at multiple successive proton cyclotron harmonics are captured.
Figure 3.10: Comparison of measured and simulated ICE spectra, plotted on dB scales. Top panels: measured LHD ICE power spectra during sub-Alfvénic (left) and super-Alfvénic (right) perpendicular proton NBI, reproducing Fig. 3.2. Bottom panels: power spectra of $\delta B_z^2 / B_0^2$ obtained from our PIC-hybrid simulations for parameters corresponding to LHD plasmas 79126 (left) and 79003 (right), see Table 3.1, reproducing the bottom two panels of Fig. 3.9.
3.6 Identification of the excitation process for NBI proton-driven ICE in LHD

The frequencies of the modes excited by ICE in our simulations typically range from $\omega \approx 5\Omega_H$ to $\approx 45\Omega_H$. We focus primarily on modes up to $\omega \approx 15\Omega_H$, marginally stable and unstable, because this is the upper frequency limit of the experimental measurements. These modes are electromagnetic and lie on the fast Alfvén branch, so that $\omega$ and $k$ are related by $\omega \approx V_A k$, where $V_A$ is the Alfvén speed. In our simulations, we consider waves propagating nearly perpendicular to the local background magnetic field. Such waves can leave an MCF plasma, propagating radially, and be detected beyond it. The simulation outputs encapsulated in Fig. 3.11 enable us to infer the rate at which the energy in a given cyclotron harmonic spectral peak grows over time. It is particularly helpful to calculate this during the early phase of growth, because this enables quantitative comparison with counterpart linear growth rates obtained from analytical theory. We denote the early phase growth rate inferred from the simulations at the $\ell$th harmonic by $\gamma_\ell$. This we shall compare with the corresponding scaling of analytical linear growth rate for the MCI, $\gamma_{lin}(\ell)$, defined [4] by equations 3.1 and 3.2 below. Equation 3.3 shows that $\gamma_{lin} \sim \xi^{1/2}$; hence Figs. 3.12 and 3.13 compare early phase simulation outputs with linear theory by plotting $\gamma_\ell$ versus $\xi^{1/2}$ for multiple simulations, focusing on $\ell = 11$ and $\ell = 12$ for sub-Alfvénic NBI LHD plasma 79126 and super-Alfvénic NBI LHD plasma 79003 respectively. The agreement shown is good; this further confirms the role of the MCI in our simulations and, by extension, the LHD experiments. Decreasing the angle between $k$ and $B_0$ shows preferential excitation at lower harmonics, as well as longer timescales for mode excitation, see Fig. 3.11.

Figures 3.12 and 3.13 are obtained as follows. The Alfvén dispersion relation provides a one-to-one mapping between the excited $\omega$ modes and the excited $k$ modes. In addition, our simulations use an initially uniform density for both the NBI protons and the background plasma, and the domain has periodic boundary conditions. This means there is neither loss of information, nor need for windowing, when taking spatial Fourier transforms of the electric and magnetic fields. We may therefore compute the growth rates of $k$-modes by taking the spatial Fourier transform of $B_z(x,t)$, leading to $B_z(k,t)$ as in Fig. 3.11. One selects an $\omega$-mode at $\omega = \ell\Omega_H, 5 \leq \ell \leq 15$, to which a unique $k$-mode, $k = k_{\ell\Omega_H}$ is associated through the dispersion relation as in Fig. 3.8. The time evolution of $B_z(k_{\ell\Omega_H},t)$ is then plotted, and best fits are constructed to extract the empirical growth rate $\gamma_\ell$ of this mode, as described below. This approach is convenient because it does not
Figure 3.11: Time evolution (vertical axis, in units of proton gyroperiods $\tau_H$) of the spectral power of the spatial Fourier transform of the $z$-component of the fluctuating magnetic field $\delta B_z$ in the four hybrid-PIC simulations whose energy evolution, spatiotemporal Fourier transforms, and ICE spectra are shown in the corresponding panels of Fig. 3.7, Fig. 3.8 and Fig. 3.9 respectively. Magnitude is plotted on a log 10 colour scale. Wavenumbers are normalized to the proton skin depth $\Omega_H/V_A$. The vertical bands correspond to the most strongly excited wavenumbers, which can be mapped to successive proton cyclotron harmonics using information in Fig. 3.8. The modes excited earliest in the simulations are the most strongly linearly unstable ones, starting at cyclotron harmonic $\ell = 5$, approximately.

require transformations from the time domain. Once $B_z(k, \Omega_H, t)$ is calculated, we identify the interval $[t_0, t_1]$ over which the initial exponential growth phase takes place in our simulations. A primary objective is to quantify the scaling of the initial growth rate inferred from simulations, $\gamma$, with NBI proton number density. This can be compared to the scaling given by the corresponding analytical expression $\gamma_{lin}(t)$ obtained from linear instability theory, specifically the MCI [76], see below. We find the duration of this initial exponential growth phase in the simulations to be $\approx 1.0\tau_H$ for the fastest-growing modes, while the slowest-growing ones unfold over $\approx 20\tau_H$. We perform multiple fits of $B_z(k, \Omega_H, t)$ between $[t_{1/2} - n\Delta t, t_{1/2} + n\Delta t]$ where: $t_{1/2}$ is at the centre of $[t_0, t_1]$, which are the start and end times of the initial exponential growth; $\Delta t \approx 0.001\tau_H$; and $n$ varies between 1 and $n_{max}$, such that
\[ t_{1/2} - n_{\text{max}} \Delta t, t_{1/2} + n_{\text{max}} \Delta t \] is the smallest interval to contain \([t_0, t_1]\). This yields a family of \(n\) growth rates \(\gamma_{\ell,n}, 1 \leq n \leq n_{\text{max}}\) for a given mode at \(\omega = \ell \Omega_H\). We take the mean of the individual best fits as the growth rate value \(\gamma_{\ell} = \bar{\gamma}_{\ell,n}\), and define the associated error \(\Delta \gamma_{\ell} = \sigma(\gamma_{\ell,n})\), where the bar and sigma respectively represent the average and the standard deviation. The average and variance are taken between \(n_{\text{min}} \leq n \leq n_{\text{max}}\), where \(n_{\text{min}}\) satisfies \(n_{\text{min}} \Delta t \geq 0.5 \tau_H / \ell\). That is, computation of the average starts from \(\Delta t\) corresponding to half an oscillation of the unstable \(\ell\)th mode. This enables us to use the same value of \(\Delta t\) for each cyclotron harmonic \(\ell\).

The procedure described above enables us to calculate growth rates denoted \(\gamma_{\ell}\) for the \(\ell\)th harmonic during the early phase of simulations. These are next compared with the scaling of the analytical expressions for the corresponding growth rate \(\gamma_{\text{lin}}(\ell)\) of the MCI, notably Eq. 36 of [76] which reads:

\[
\frac{\gamma_{\text{lin}}(\ell)}{\Omega_H} = \frac{\Delta \omega}{\omega_0} = \frac{1}{\sqrt{2} \nu_{\text{NBI}}} \frac{V_A}{\omega_{\text{pi}}} \chi_0 = \frac{1}{\sqrt{2} \nu_{\text{NBI}}} \sqrt{\frac{n_{\text{NBI}}}{n_e}} \chi_0
\]  

(3.1)

Here \(\omega_{\text{p,NBI}}\) and \(\omega_{\text{pi}}\) are the plasma frequencies of the NBI protons and of the bulk protons respectively, and \(\nu_{\text{NBI}}\) is again the initial velocity of NBI protons. We define

\[
\chi_0^2 = \Pi_{x,x} - \frac{2i \omega \Omega_H}{V_A^2 k^2} \Pi_{x,y} + \frac{\Omega_H^2}{V_A^2 k^2} \Pi_{y,y}
\]

(3.2)

where \(\Pi_{x,x}, \Pi_{x,y}\) and \(\Pi_{y,y}\) are the functions of \(z_{\text{NBI}} = k \nu_{\text{NBI}} / \Omega_H\) of given in the appendix of [76]. Near resonance, \(\omega_0 = V_A k = \ell \Omega_H\) and \(z_{\text{NBI}} = \ell \nu_{\text{NBI}} / V_A\). Near resonance, \(\omega_0 = V_A k = \ell \Omega_H\) and \(z_{\text{NBI}} = \ell \nu_{\text{NBI}} / V_A\). For a given mode \(\ell\), if all parameters are kept fixed except for the NBI proton density ratio \(\xi = n_{\text{NBI}} / n_e\), Eq. 3.1 yields the scaling

\[
\frac{\gamma_{\text{lin}}(\ell)}{\Omega_H} = \frac{\Delta \omega}{\omega_0} = \alpha_{\ell} \sqrt{\xi}
\]  

(3.3)

as in Fig. 3.13, where

\[
\alpha_{\ell} \equiv \frac{1}{\sqrt{2} \nu_{\text{NBI}}} \frac{V_A}{\chi_0}
\]

(3.4)

which depends on \(\ell\) solely.

We test the hypothesis that \(\gamma_{\ell} \simeq \gamma_{\text{lin}}(\ell)\) by running multiple simulations for different values of the beam proton density ratio \(\xi\), with all other parameters kept fixed. The computed growth rates at early times \(\gamma_{\ell}\) for a given mode \(\ell\) are then plotted against \(\sqrt{\xi}\), in line with the analytical scaling of \(\gamma_{\text{lin}}(\ell)\) [34, 76] in Eq. 3.3 as shown in Figure 3.12. This shows congruence between the early phase of collective relaxation of the NBI ion population in our first principles PIC-hybrid simulations, and the analytical theory of the linear stage of the MCI. It tends to confirm that our simulated ICE
spectra can be understood in terms of the MCI, as extended into the analytically inaccessible nonlinear regime, for both sub-Alfvénic and super-Alfvénic NBI proton populations.

The procedure is now applied across multiple harmonic modes, with the results shown in Fig. 3.13.

![Graphs showing linear fit of early phase growth rate γ_ℓ against the square root of ξ = n_{NBI}/n_e.](image)

Figure 3.12: Linear fit of early phase growth rate γ_ℓ inferred from the simulations, normalized to Ω_H, against the square root of ξ = n_{NBI}/n_e, subject to the constraint that for a zero density beam, the line should intersect the origin. It is evident that the ICE growth rate γ_ℓ/Ω_H scales as √ξ. Left panel is for cyclotron harmonic mode ℓ = 11 of the excited wave in the ICE emitting region of LHD plasma 79126 with locally sub-Alfvénic 40keV NBI protons, and each point is for a value of ξ between 0.5 × 10^{-4} and 7.5 × 10^{-4}. Right panel is for cyclotron harmonic mode ℓ = 12 of the excited wave in the ICE emitting region of LHD plasma 79003 with locally super-Alfvénic 36.5keV NBI protons, and the values of ξ are between 1 × 10^{-4} and 2 × 10^{-3}.

As in Fig. 3.12, in Fig. 3.13 we plot the dependence of the growth rate γ_ℓ of the fields, calculated at early times in multiple simulations, on NBI concentration ξ in each simulation. The computations are performed for parameters corresponding to LHD plasmas 79126 and 79003. In both scenarios, the propagation angle between B_0 and k is 89.5°. The left pair of panels in Fig. 3.13 corresponds to the collapsed plot which recasts Eq. 3.3 as

\[
(\gamma_{lin}(\ell)/\Omega_H)/\alpha_\ell = \sqrt{\xi}
\]

Namely, the quantity \((\gamma_{lin}(\ell)/\Omega_H)/\alpha_\ell\) is proportional to \(\sqrt{\xi}\), whence the straight line obtained independently of the mode number ℓ. The right pair of panels in Fig. 3.13 instead reformulates Eq. 3.3 as

\[
(\gamma_{lin}(\ell)/\Omega_H)/\sqrt{\xi} = \alpha_\ell
\]

This implies that for a fixed mode value ℓ, the quantity \((\gamma_{lin}(\ell)/\Omega_H)/\sqrt{\xi}\) is in-
dependent of $\xi$ and equals a constant that depends on $\ell$ solely. We translate that constant such that it equals $\alpha'_{\ell} = \ell$. Figure 3.13 strongly suggests that in general $\gamma_\ell \sim \xi^{1/2}$ in our simulations, across an extended range of modes. This dependence is the same as for growth rates from linear analysis of the MCI [76], and from previous simulations e.g Fig. 3 of [22]. Additional linear and cubic root scaling tests have been performed, and F-test statistics [169] applied with a 99% significance further confirm the square root scaling.

Figure 3.13: Dependence of early phase growth rates $\gamma_\ell$ in simulations with different NBI ion concentration $\xi$, obtained across multiple cyclotron harmonics $\ell$. Top and bottom pairs of panels correspond to LHD plasmas 79003 and 79126 respectively. Left: collapsed plot of $(\gamma/\Omega_H)/\alpha_\ell$ versus $\xi$. Here $\gamma = \gamma(\xi)$ is the growth rate inferred from the simulations for mode number $\ell$, and depends on the relative NBI density $\xi$ at the ICE location, and $\alpha_\ell$ is the calculated slope from the best linear fit of $\gamma_\ell$ as a function of $\sqrt{\xi}$ for a given mode $\ell$ (see Fig. 3.12). Right: translated compensated plot of the quantity $(\gamma/\Omega_H)/\sqrt{\xi}$ versus $\xi$, with $\gamma_\ell$ inferred from simulations. If the simulation outputs match the linear theory of the MCI, we expect $\gamma/\Omega_H = \alpha_\ell \sqrt{\xi}$ as in Eq. 3.3. In this case, it follows that $(\gamma/\Omega_H)/\sqrt{\xi} = \alpha_\ell$, a quantity that does not depend on $\xi$, but on the mode value $\ell$ only. This outcome is reflected by the sequence of horizontal lines in to the compensated plot. The values of $\xi$ span one order of magnitude, between $10^{-4}$ and $10^{-3}$. Together, these graphs show that, within modest error bars, $\gamma_\ell \propto \sqrt{\xi}$. 

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3.7 Enhanced sub- and super-Alfvénic behaviour

We have also investigated, through PIC-hybrid simulations, the collective relaxation of NBI proton populations that have artificially enhanced sub- and super-Alfvénic characteristics under the LHD plasma conditions already considered. We have run multiple simulations of a sub-Alfvénic fast proton population of 25keV, for which \( v_{\text{NBI}}/V_A = 0.7 \) under LHD plasma 79126 conditions as shown in Fig. 3.14, as well as simulations with a 56keV super-Alfvénic fast proton beam, for which \( v_{\text{NBI}}/V_A = 1.4 \) under LHD 79003 plasma edge conditions, see Fig. 3.15. Corresponding simulation inputs were used as described in Table 3.3. Figs. 3.14 and 3.15 combine the energy density time evolution of the fields and of the protons, both thermal and NBI, along with power spectrum, the spatial and the spatio temporal fast Fourier transform of the \( z \)-component of \( \delta B_z \), which appear in Figs. 3.7, 3.9, 3.8 and 3.11 respectively. The energy released by the beam NBI protons to the electric and magnetic fields and to the bulk protons decrease with lower beam energy and increases with the more energetic, enhanced super-Alfvénic beam. We note that the time for saturation is different and is faster in the (enhanced) sub-Alfvénic than in the (enhanced) super-Alfvénic. This must be related to the energy available, in terms of the fast ion concentration and of its initial energy, of its Alfvénic character, and ultimately on the background plasma parameters and propagation angle. Typically, fast protons leave up to \( \approx 15\% \) of their energy to the bulk plasma and to the electric and magnetic fields, as shown in Fig. 3.16. The short term oscillation in the energy density of the majority thermal plasma is not specific to hybrid-PIC simulations and are also observed in full-PIC simulations of the MCI [20]. In the simulations of the MCI (with energetic ions), these oscillations are observed to appear at the most strongly excited ion cyclotron harmonic. In pure thermal plasma simulations (no energetic ions present), oscillations are also present. In both cases, they result from the self-consistent energy exchange between particles and fields. The fluctuation dissipation theorem quantifies this in a thermal plasma: the intensity of the fluctuating fields is proportional to the temperature of the plasma [170]. We have tested that the hybrid code satisfies this relation.

From the dispersion relation panels in Figs. 3.14 and 3.15, we observe again that the excitations preferentially take place at values of \( (\omega, k) \) close to \( v_{\text{NBI}} \). The enhanced super-Alfvénic power spectrum displays cyclotron harmonics strongly excited separated by cyclotron harmonics which show comparatively weaker excitation as seen from the top right panel of Fig. 3.15. The large difference in intensities between these power spectra result from the very different energies of the NBI fast ions: the
sub-Alfvénic simulation is set up with 25keV NBI protons and the super-Alfvénic calculations follow the evolution of 56keV NBI protons. In addition their densities differ to saturate the MCI in less than $15\tau_H$. The sub- and super-Alfvénic calculations are characterised by $\xi = n_{NBI}/n_e = 0.0005$ and 0.0025 respectively.

Multiple simulations with increasing simulation domain length have confirmed the effect does not come from the simulation grid. We infer the results to be physical, at least in the current setting: 1D spatial simulation, slab geometry. Fully kinetic and hybrid kinetic linear dispersion solutions presented in Chapter 6 display gaps between unstable proton cyclotron harmonics. As developed in Section 3.6, we have computed the growth rate scaling of the unstable and of the marginally stable modes with NBI proton beam density $\xi$ in Fig. 3.17. We reach the same conclusion as before: within error bar, growth rates $\gamma_\ell$ calculated from simulation are proportional to $\sqrt{\xi}$. The linear growth rates inferred from simulations are displayed on the bottom left panels of Figs. 3.14 and 3.15.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Plasma 79126</th>
<th>Plasma 79003</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regime</td>
<td>sub-Alfvénic</td>
<td>super-Alfvénic</td>
</tr>
<tr>
<td>Simulation duration</td>
<td>$[6, 20] \times \tau_H$</td>
<td>$[30, 80] \times \tau_H$</td>
</tr>
<tr>
<td>$T_{NBI}$</td>
<td>25 keV</td>
<td>56keV</td>
</tr>
<tr>
<td>$v_{NBI}/V_A$</td>
<td>0.700</td>
<td>1.400</td>
</tr>
<tr>
<td>Length of simulation domain</td>
<td>$847 \times r_{NBI}$</td>
<td>$192 \times r_{NBI}$</td>
</tr>
<tr>
<td></td>
<td>$13500 \times r_L$</td>
<td>$11175 \times r_L$</td>
</tr>
<tr>
<td>$\cos^{-1} (\hat{k} \cdot \hat{B}_0)$</td>
<td>$89.5^\circ$</td>
<td>$89.5^\circ$</td>
</tr>
<tr>
<td>$\xi = n_{NBI}/n_e$</td>
<td>$[0.05, 0.65] \times 10^{-3}$</td>
<td>$[0.10, 1.90] \times 10^{-3}$</td>
</tr>
<tr>
<td>Number of grid cells</td>
<td>22080</td>
<td>10560</td>
</tr>
<tr>
<td>Number of part. per cell</td>
<td>4000</td>
<td>500</td>
</tr>
<tr>
<td>$T_e = T_H$</td>
<td>150eV</td>
<td>25eV</td>
</tr>
<tr>
<td>$n_e$</td>
<td>$1.0 \times 10^{19} m^{-3}$</td>
<td>$0.5 \times 10^{19} m^{-3}$</td>
</tr>
<tr>
<td>$B_0$</td>
<td>0.46T</td>
<td>0.24T</td>
</tr>
</tbody>
</table>

Table 3.3: Summary of the input data used in the ICE simulations from LHD plasmas 79126 and 79003 simulations where the NBI protons sub and super-Alfvénic ICE regimes have been enhanced, respectively.
Figure 3.14: Enhanced sub-Alfvénic NBI proton relaxation via the magnetoacoustic cyclotron instability under LHD plasma 79126 edge conditions. The NBI protons have an initial energy of 25keV corresponding to $v_{\text{NBI}}/v_A = 0.7$. Top left: Energy density time evolution of the different components of the electric and magnetic fields and of the background and NBI beam protons, see Fig. 3.7. Top right: Power spectrum of the $z$-component of the fluctuating part of the magnetic field in our PIC-hybrid simulations, on a dB scale, obtained as in Fig. 3.11. Bottom left: spatio-temporal fast Fourier transform of $\delta B_z$, the fluctuating part of the $z$-component of the magnetic field represented on a log$_{10}$ scale. The dark straight line shows $\omega/k = v_{\text{NBI}}$, as in Fig. 3.8. The grey curves are the growth rates $\gamma$ computed from the simulations with associated dark error bars. All fits of the growth rates are computed between $0.20\tau_H$ and $3.3\tau_H$. Bottom right: time evolution of the spatial fast Fourier transform of $\delta B_z$, on a log$_{10}$ scale, whose original LHD 79126 40keV NBI proton wavenumber time evolution appears in Fig. 3.11. The beam density $\xi = 0.0005$. The angle between the background magnetic field and the $k$ vector is 89.5°.
Figure 3.15: Enhanced super-Alfvénic NBI proton relaxation via the magnetoacoustic cyclotron instability under LHD plasma 79003 edge conditions. The NBI protons have an initial energy of 56keV corresponding to $v_{NBI}/V_A = 1.4$. Top left: Energy density time evolution of the different components of the electric and magnetic fields and of the background and NBI beam protons, see Fig. 3.7. Top right: Power spectrum of the $z$-component of the fluctuating part of the magnetic field in our PIC-hybrid simulations, on a dB scale, obtained as in Fig. 3.11. Bottom left: spatio-temporal fast Fourier transform of $\delta B_z$, the fluctuating part of the $z$-component of the magnetic field represented on a log scale. The dark straight line shows $\omega/k = v_{NBI}$, as in Fig. 3.8. The grey curves are the growth rates $\gamma$ computed from the simulations with associated dark error bars. All fits of the growth rates are computed between $0.20\tau_H$ and $2.6\tau_H$. Bottom right: time evolution of the spatial fast Fourier transform of $\delta B_z$, on a log scale, whose initial 36keV NBI proton wavenumber time evolution appears in Fig. 3.11. The beam density $\xi = 0.0025$. The angle between the background magnetic field and the $k$ vector is 89.5°. Top right and bottom right graphs suggest some gaps exist the spectrum between strongly excited modes.
Figure 3.16: Absolute kinetic energy density time evolution of the bulk and NBI protons in the enhanced sub-Alfvénic simulation under LHD 79126 edge plasma parameters (left) and the enhanced super-Alfvénic simulation under LHD 79003 edge plasma parameters (right). The fast ions release $\approx 10\%$ of their initial energy to the electric and magnetic fields and to the bulk protons which experience a relative kinetic energy increase of $\approx 0.1\%$ to $\approx 15\%$. 
Figure 3.17: Dependence of early phase growth rates in the simulations on NBI ion concentration $\xi$, obtained across multiple cyclotron harmonics $\ell$. Top and bottom panels correspond to LHD plasmas 79003 and 79126 respectively. Left: collapsed plot of $(\gamma_\ell/\Omega_H)/\alpha_\ell$, versus $\xi$. Here $\gamma_\ell = \gamma_\ell(\xi)$ is the growth rate inferred from the simulations for mode number $\ell$, and depends on the relative NBI density $\xi$ at the ICE location, and $\alpha_\ell$ is the calculated slope from the best linear fit of $\gamma_\ell$ as a function of $\sqrt{\xi}$. Right: translated compensated plot of the quantity $(\gamma_\ell/\Omega_H)/\sqrt{\xi}$ versus $\xi$, with $\gamma_\ell$ inferred from simulations. From the linear theory of the MCI, $\gamma_\ell/\Omega_H = \alpha_\ell\sqrt{\xi}$, which means that $(\gamma_\ell/\Omega_H)/\alpha_\ell = \sqrt{\xi}$ such that the plot $(\gamma_\ell/\Omega_H)/\alpha_\ell$ with respect to $\sqrt{\xi}$ should bring the identity. This is the collapsed plot. Similarly, $(\gamma_\ell/\Omega_H)/\sqrt{\xi} = \alpha_\ell$, a quantity that does not depend on $\xi$, but on the mode value $\ell$ only, corresponds to the compensated plot. The values of $\xi$ typically span one order of magnitude, between $10^{-4}$ and $10^{-2}$. These graphs show that within error bars, $\gamma_\ell \propto \sqrt{\xi}$. 

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3.8 Finite thermal spread

Looking at the measured and simulated NBI proton distribution functions in LHD, Fig. 3.6, one observes the existence of a spread in $v_{\perp}$. It is then desirable to explore the robustness of the MCI against a thermal spread in the NBI proton velocity distribution function. Its anisotropy and narrowness in $v_{\perp}$ are essential to trigger the MCI as was the case in JET and in TFTR. Depending on a range of parameters such as the energy of the fast ions, the temperature of the thermal plasma, cyclotron damping etc, drifting rings with a finite thermal spread can indeed also be shown to be unstable against the MCI.

Due to the narrowness of the ring-beam distribution function, it can be argued that our simulations are only relevant to the very beginning of the NBI discharge although particles are not replenished in the code. It is suggested in Section 3.2 that ICE stops quickly after the NBI turn-off. Therefore a ring-beam distribution with some degree of thermalisation, which is not replenished during the simulations could be used to model the end of the NBI discharge.

This motivates the present section to study the relaxation of a somewhat more realistic distribution function and to compare it to the idealised ring-beam case. We are interested in the time required to saturate the MCI as well as in the shape of the power spectra in the two instances and particularly at low cyclotron harmonics. This is why we have run one simulation with the cold ring beam $f_{\text{NBI}} = 1/2\pi \delta (v_{\perp} - v_{\text{NBI}})$ and a second one with $f_{\text{NBI}} = \delta (v_{\parallel}) 1/2\pi v_{\perp} \exp \left[ -\left( v_{\perp} - v_{\text{NBI}} \right)^2 / v_{\perp}^2 \right]$, with $v_{\perp}^2 / v_{\text{NBI}}^2 = 0.1$ under LHD plasma 79126 conditions with 40keV NBI protons and massless electrons. Neither parallel drift nor parallel thermal spread are used. The MCI takes a longer time to saturate when thermal spread is included and energy exchanged between the beams and bulk plasma and fields at saturation is decreased as seen on the top panels of Fig. 3.18. The power in the spatiotemporal fast Fourier transform of $\delta B_z$ is evened out in the presence of thermal spread in the ring as compared with the cold ring on the right panel where the power mostly concentrates at higher harmonics. This is further observed in the power spectra on the top panels of Fig. 3.19 and on its middle panels which show the time evolution of the wavenumbers $k$ power density along with various $k$-modes displayed on the bottom panels. For the parameters considered, we see that the addition of thermal spread influences the simulation results, particularly the shape of the power spectra.

A thorough study of the MCI could investigate the shape of the power spectra as a function of the initial distribution function for given plasma parameters (and vice versa) [171].

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Figure 3.18: (Top) Time evolution of the change in the electric and magnetic field energy density and in the beam and background proton kinetic energy in 2 massless electron PIC-hybrid simulations. (Bottom) Spectral power of the fast Fourier transform of $\delta B_z$ taken over the whole simulations domain over $20 \tau_H$. The graphs are obtained from simulations initialized with LHD plasma parameters 79126 with sub-Alfvénic 40 keV NBI proton distributed in velocity-space as $f_{NBI} = \delta (v_\|) \frac{1}{2\pi v_\perp} \exp \left[ - \left( v_\perp - v_{NBI} \right)^2/v_\perp^2 \right]$, with $v_\perp^2/v_{NBI}^2 = 0.1$ (left panels) and $f_{NBI} = \frac{1}{2\pi} \delta (v_\|) \delta (v_\perp - v_{NBI})$ (right panels), both with $\xi = 0.0005$. The saturation energy of the NBI is decreased with the inclusion of a finite spread. However, examination of the power spectra in Fig. 3.19 averaged over the whole simulation show an increase in intensity at lower harmonics up to approximately the lower hybrid frequency. The power is distributed more evenly in the finite spread case than in the delta peaked distribution scenario.
Figure 3.19: Top: Power spectra of the $z$-component of the fluctuating part of the magnetic field generated by simulations initialized with LHD plasma parameters 79126 with sub-Alfvénic 40 keV NBI proton. The left (right) panel corresponds to a simulation which has 10% (0%) perpendicular thermal spread $v^2_{\perp}/v^2_{\text{NBI}}$ in the proton beam distribution as described in Fig. 3.18. The relative beam density is $\xi = 0.0005$. The power spectra are obtained by taking the spatio-temporal Fourier transform of $\delta B_z$, averaged over 20 proton gyrations and summed between $k = 0$ and $k = 25\Omega_H/V_A$. Middle: $(k,t)$ plots showing the time evolution of the spectral power of wavenumbers on a log scale. They are obtained by taking the square modulus of the spatial Fourier transform of $\delta B_z$. Bottom: Time evolution of the power of several wavenumbers $kV_A/\Omega_H$ on a log scale corresponding to a specific proton cyclotron harmonic through the dispersion relation, as shown on the bottom panels of Fig. 3.18.
From this exercise, we see that the MCI still unfolds when a thermal spread is added to the NBI proton velocity distribution. In addition, lower harmonics are excited to even higher intensity, starting from $\omega \sim 4\Omega_H$. The intensity of the spectral peaks is more homogeneous across a wide range of cyclotron harmonics compared to the pure ring-beam case. This could reflect some sort of equipartition of energy between these cyclotron harmonics. The study of the growth rates and their scaling with density $\xi = n_{\text{NBI}}/n_e$ for drifting-Maxwellian should be the object of future additional studies A.8.

The physics of ICE is usually well resolved within a few gyroperiods in the simulations, provided the MCI has reached saturation and the non linear stage has been entered. When the simulations are run for a very long time, a stationary state is reached where the energy of the different fields and particles stop changing as shown in the energy density plot on the top right panel of Fig. 3.18. At these time scales, the ICE physics is well resolved. However we are not able to simulate the interaction of ICE with MHD modes which is possible (in JET) with the code HALO [172] and as such we cannot capture Cyclotron Alfvénic Eigenmodes (CAEs) as is done in [173] which requires geometry and possibly longer time scales. Longer simulations would then necessitate the inclusion of these effects while this work is zooming in space and in time instead to follow the self-consistent evolution of the interaction of electric and magnetic fields with the thermal and energetic ions and the fluid electrons.

### 3.9 Convergence

Plasma simulations using the PIC method usually suffers from the detrimental effects of noise due to the finite number of macroparticles used which can lead to artificial heating in the simulation [10, 133]. We have run two sets of simulations of LHD plasma 79126 with all parameters kept constant, as given in Table 3.1 except for the number of macroparticles per cell which was varied to investigate the noise level. The first set is characterized by $\xi = 0$ corresponding to a pure thermal proton plasma and the second set has a NBI proton beam with relative density $\xi = 0.0005$. The power spectra of $\delta B_z$ are computed over the duration of the simulation ($15\tau_H$) and averaged between $k = 0$ and $k = 30\Omega_H/V_A$. Fig. 3.20 shows the resulting power spectra for the thermal plasma on the left panel and the thermal + NBI protons on the right panel. The spectra including the energetic protons are well above the noise level. It amounts with 250 and 8000 macroparticles per cell to about $-90\text{dB}$ and $-120\text{dB}$ respectively for the simulation with background protons only. This im-
plies that $10 \log_{10} \left( \frac{\delta B_{250}^2}{\delta B_{8000}^2} \right) \approx 30$ or equivalently that $\delta B_{250}/\delta B_{8000} \approx \sqrt{1000}$ while $8000/250 = 32$ and thus we have a decrease of the noise in the simulations that goes as $1/N$, characteristic of the use of a quiet start where $N$ is the number of macroparticles used in the simulation. Both panels show that as the number of macroparticles is increased, the difference between successive power spectra decrease which suggests numerical convergence, although it is not clear whether it has been reached. The peaks at multiple cyclotron harmonics with the NBI protons lie above the noise level.

![Figure 3.20: Convergence test for simulations relevant to plasma 79126, 40 keV NBI, 22080 cells for various ppc. Left: power averaged between the beginning of the simulations up $15\tau_H$, right up to $15\tau_H$. The $k$-values are integrated around a small interval. The graphs show convergence and suggest that lower cyclotron harmonics are equally as intense.](image)

We have calculated the time evolution of several cyclotron harmonics and plotted their associated $k$-mode dynamics (as obtained from the mapping by mean of the dispersion relation) in Fig. 3.21 on a log$_2$ scale. The initial power of each mode is decreased, and the power at saturation does not change for the modes on the bottom panels.
Figure 3.21: Time evolution of several $k$-modes of $\delta B_z$ for sub-Alfvénic NBI proton LHD plasma 79126 plot. The modes are integrated between $k = 5.10$ and $k = 5.65 \Omega_H/V_A$ (top left) and between $k = 7.20$ and $k = 7.90 \Omega_H/V_A$ corresponding to harmonics $\ell = 5$ and $\ell = 7$ respectively along with $k$-modes integrated between $k = 34.0$ and $k = 36.4 \Omega_H/V_A$ and between $k = 36.4$ and $k = 38.6$ which correspond to harmonics $\ell = 27$ and $\ell = 28$ respectively. These graphs suggest that the growth rates do not change much by varying the number of particles per cell and lead to numerical convergence.

3.10 Relevance of hybrid simulations

It would have been possible to run fully-kinetic simulations, such as with EPOCH [106] as done in [20, 82] instead of running the hybrid code and to obtain very good results without approximation of the set of equations solved. This raises the question of ”why” using hybrid. The physics of the MCI related to ICE mainly involves kinetic ions and as such the results obtained in the hybrid model are rather good given the assumptions that are made (massless fluid electrons, quasineutrality, Darwin’s approximation). While 1D3V simulations are doable with full-PIC, moving to 2D3V simulations of ICE might require the adoption of a hybrid model. As such, studying
it in 1D, including its limits is an interesting exercise. In addition, the hybrid approximation, by focusing on a smaller region of phase space gives up on some physics but allows more resolution on that same region at the same time. Finally, giving up on some physics speeds up the simulations which is an advantage when parameter scans are performed such as for the calculation of the growth rates and their scaling with beam density which is a lot more difficult due to noise constraints with a full-PIC scheme. In LHD plasmas 79126 and 79003, the ratios $V_A/c$ are 0.0106 and 0.0078 respectively and the associated $\gamma = 1/\sqrt{1 - (V_A/c)^2}$ are equal to 1.000056 and 1.00003 respectively, which indicates that Darwin’s approximation is acceptable. In addition, $v_{NBI}/V_A$ equals 0.870 and 1.125 for LHD plasmas 79126 and 79003 respectively as summarised in Table 3.1. The related ratios for the electron thermal speed yield $v_{Te}/V_A$ 1.868 and 1.030 and are mobile enough to follow the beams to maintain quasineutrality, despite the discrepancy in the hybrid dispersion relations at higher values of $k$, as shown in Fig. 6.2 in Chapter 6.

### 3.11 Summary

We have presented the first 1D3V PIC-hybrid simulations of ICE where the minority energetic ion population arises from NBI, hence with kinetic energy more than an order of magnitude lower than in previous simulations [19, 22, 82, 83, 86] relating to super-Alfvénic fusion-born ions in JET. Second, first principles kinetic simulations of ICE and MCI physics in the sub-Alfvénic regime for energetic ions have not previously been carried out, with one exception. This exception is the set of multiple PIC simulations [82, 83] of ICE driven by fusion-born protons under rapidly evolving edge plasma conditions in KSTAR where, for local electron densities below $n_e \sim 1.05 \times 10^{19} m^{-3}$, corresponding to the lower panels of Fig. 4 of Ref. [82], the perpendicular velocity of the protons is sub-Alfvénic. We have found that, in the LHD-relevant context, the transition between sub-Alfvénic and super-Alfvénic ICE phenomenology is extremely smooth, both in observations and simulations.

A third novel aspect of this work is that it is the first to report first principles kinetic ICE simulations with wavevectors inclined more than one degree from perpendicular to the magnetic field direction. While more challenging computationally, this leads to better congruence of simulation outputs with the observed ICE spectra, as in Fig. 3.9, and is helpful in a context where antenna sensitivity may not be known in detail as a function of $k_\parallel$ in the relevant range. Fourth, we have carried out the first study of early-time growth rates inferred from simulations that span sub-Alfvénic and super-Alfvénic energetic ion regimes. Of particular interest is how these growth

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rates depend on energetic ion concentration $\xi = n_{\text{NBI}}/n_e$; Fig. 3.13 establishes square root scaling, in line with prediction from the corresponding linear analytical MCI theory [76]. For simulations relevant to the super-Alfvénic regime, this scaling was established in Ref. [22].

In summary, the measured ion cyclotron emission (ICE) spectra (Fig. 3.2) from LHD hydrogen plasmas with both sub-Alfvénic and super-Alfvénic perpendicular proton NBI have been successfully simulated (Fig. 3.9) using a first principles approach. Direct numerical simulation of kinetic ions (bulk protons and minority energetic NBI protons) and fluid electrons using a 1D3V PIC-hybrid code captures the self-consistent Maxwell-Lorentz dynamics of the plasma and fields. It is found from the Fourier transforms and time evolution of the energy and field components in the PIC-hybrid code outputs that the dominant physical process in our first principles Maxwell-Lorentz computations is the magnetoacoustic cyclotron instability (MCI).

The many correlations between our code outputs and the measured ICE spectra suggest that an emission mechanism, which corresponds essentially to the nonlinear MCI, is responsible for the main features of ICE in these LHD stellarator plasmas. In the context of the extensive prior research on the role of the MCI in ICE from tokamak plasmas, this outcome suggests a significant degree of commonality across tokamak and stellarator ICE physics. This appears to be a consequence of the strongly spatially localised character of ICE physics. The spontaneously excited electric and magnetic fields in our simulations, which are carried out in local slab geometry, correspond to the fast Alfvén wave. This work helps establish a baseline for future energetic particle experiments in LHD, where magnetic geometry and toroidal eigenfunctions [68] may play a larger role. ICE links beam ion physics in LHD to fusion-born ion physics in tokamaks, and has significant diagnostic potential [23].
Chapter 4

Ion cyclotron emission in LHD Deuterium plasmas

4.1 Introduction

During the 2017 LHD campaign, deuterium plasma experiments have been conducted for the first time, during which ICE was detected both from deuterium perpendicular NBI and during MHD transient events. In this chapter we study the dependence of deuterium NBI-driven ICE on the background electron density. We are able to interpret the qualitative characteristics and distribution of the ICE spectral peaks, whose separation corresponds to integer multiples of the cyclotron frequency of deuterium in the edge of LHD, in terms of the variable ratio between $v_{NBI}$ and $V_A$ respectively, the NBI deuteron speed and the Alfvén speed at the emission location. We then turn to the ICE that arises during MHD transient events. Brief intense RF signals are detected in the hundreds of megahertz range during which spectral peaks identifiable as Doppler-shifted ICE are found with a typical spacing of 20 – 25MHz. It has been suggested that these abrupt MHD events are caused by the helically trapped energetic-ion-driven resistive interchange mode (EIC), characterized by the mode numbers $m = 1$ and $n = 1$ (poloidal and toroidal respectively) [25, 174]. We study the possibility that the ICE signal during this transient is driven by fusion-born protons with significant parallel velocity. We present the measurements and use parameters at the emission location to run hybrid-PIC simulations. We perform cepstrum analysis on the measured spectra. A schematic representation of the acquisition system is displayed in Fig. 4.1. It has been developed in partnership with KSTAR. [175–177]. The measurement system is made of a dipole antenna located in the 10-O port of LHD, inside the vacuum vessel.
A fast digitizer performs direct sampling of the radiofrequency measurements at a frequency of 1.25 GS/s. The time evolution of the RF signal intensity is collected by a 14-channel filter bank spectrometer in the range of 70 – 2800 MHz, with intermediate spectral resolution and with µs time resolution for duration spanning the whole plasma discharge [177].

### 4.2 ICE during perpendicular Deuterium NBI

During perpendicular deuterium NBI experiments in LHD, it was observed that the ICE spectral character is significantly modified between plasmas whose parameters in the emitting edge region differ only in their electron density. As the electron density increases:

- the amplitude of the signal increases
- the number of harmonics increases
- the harmonics width increase

as is observed in Fig. 4.2. The antenna that detected the ICE is located close to
Figure 4.2: Observed ICE power spectra for three LHD plasmas 138439, 138458 and 138433 which differ primarily in their edge plasma densities. The peak-to-peak frequency separation $\Delta f$ is equal to 12MHz in all three cases. Identifying $\Delta f$ with the local deuteron cyclotron frequency $\Omega_D$ implies a magnetic field strength $B = 1.581$T, and this corresponds to an outer midplane edge location in LHD.
NBI #5 and opposite NBI #4 such that no ICE was observed when NBI #5 was turned off and NBI #4 was operated instead, see Fig. 4.3. Since the Alfvén speed is given by

\[ V_A = \frac{B_0}{\sqrt{\mu_0 n_e m_D}} \]  (4.1)

with \( n_e \) and \( m_D \) the electron density and the deuterium mass respectively, it follows that the ratio of the NB deuterium injection speed to the Alfvén speed, \( v_{NBI}/V_A \), also increases as \( n_e \) increases. We present hybrid-PIC simulation results in relation to ICE measurements taken during deuterium NBI experiments. The 12.05MHz spacing between the peaks suggests that the emission location is situated at \( R = 4.651 \text{m} \) such that the background magnetic field value is 1.581T as illustrated in Fig. 4.4. The plasma parameters at the ICE location for LHD plasmas 138439, 138458 and 138433 are summarized in Table 4.1.

<table>
<thead>
<tr>
<th>LHD plasma (eV)</th>
<th>( 10^{19} \text{m}^{-3} )</th>
<th>NBI#1</th>
<th>NBI#2</th>
<th>NBI#3</th>
<th>NBI#4a</th>
<th>NBI#4b</th>
<th>NBI#5a</th>
<th>NBI#5b</th>
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<tr>
<td>138439</td>
<td>70</td>
<td>1.2</td>
<td>172</td>
<td>-</td>
<td>174</td>
<td>30.5</td>
<td>45.2</td>
<td>66.7</td>
</tr>
<tr>
<td>138458</td>
<td>80</td>
<td>1.8</td>
<td>172</td>
<td>-</td>
<td>-</td>
<td>30.5</td>
<td>46.2</td>
<td>66.7</td>
</tr>
<tr>
<td>138433</td>
<td>50</td>
<td>3.4</td>
<td>172</td>
<td>163</td>
<td>174</td>
<td>35.4</td>
<td>51.1</td>
<td>68.7</td>
</tr>
</tbody>
</table>

Table 4.1: Table giving the edge plasma parameters and NBI energies, either H or D. The simulations use the deuterium energies of NBI #5a.

The hybrid-PIC simulations use the deuterium injection energies of NBI #5a of Table 4.1 to define initial perpendicular velocity of the energetic minority deuteron.

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Observations in LHD are consistent with features of Ion Cyclotron Emission

Necessary conditions for ICE

- Fast ions with $\nu_i \ll \nu_\perp$
- Population of fast ions

satisfied at the edge where is near perpendicular beam.

Observation:
stronger emission during NB#5 injection, which is neighboring NBI

Figure 4.4: Description of ICE in LHD (left) and approximate location of the ICE emission during perpendicular deuterium NBI (right).
population. There are represented in velocity space as a ring-beam \( \sim \delta (v_\perp - v_{\text{NBI}}) \). In contrast, the majority thermal deuterons have \( T_i = 280 \text{eV} \) and associated thermal velocity \( v_{Ti} = 1.64 \times 10^5 \text{ms}^{-1} \) characterising their Maxwellian distribution. It follows that \( v_{NBI}/v_{Ti} = 15.43 \) which also corresponds to the ratio of ion Larmor radii \( r_{L,NBI}/r_{Ti} \). We use 32798 particles per cell for the thermal background deuterons and 8192 particles per cell for the NBI deuterons. The 1.581T background magnetic field is perpendicular to the 1D simulation domain which consists of 4096 cells.

The cell sizes are 0.0028m, 0.0022m and 0.0014m for LHD plasma 138439, 138458 and 138433 respectively, which corresponds to an associated fraction of 1.31, 1.00 and 0.65 of the background gyroradius. The different densities give rise to different electron skin depths (0.0015m, 0.0013m and 0.0009m, respectively) and the cell sizes are chosen to be slightly bigger in our massless electron hybrid-PIC simulations. The beam absorption depends on the cross-sections for charge exchange and ionization by ions and on the electron ionization rate [178] and is proportional to the electron density. This suggests to set a constant relative beam density across the simulations rather than a constant absolute beam density. The relative beam densities are set equal to \( \xi = 0.0005 \) in all simulations. The ratios of the deuterium NBI speed to the local Alfvén speed are 0.36, 0.44 and 0.60 for LHD plasmas 138439, 138458 and LHD plasma 138433 respectively. The NBI deuterons are thus in a significantly sub-Alfvénic regime for all three plasmas. In our PIC-hybrid simulations, this energetic deuteron minority population is initialized with a ring-beam distribution in velocity space, of the form \( f_{\text{NBI}} (v_\perp, v_\parallel) = 1/(2\pi v_{\text{NBI}}) \delta (v_\parallel) \delta (v_\perp - v_{\text{NBI}}) \), where \( v_{\text{NBI}} \) is the injection speed associated with the NBI #5a beam energies given in Table 4.1. We focus on the lower deuteron cyclotron harmonics \( \ell \leq 10 \), for which the observed ICE spectra in Fig. 4.2 display significant variation between these three LHD plasmas. Fig. 4.5 (upper panel) re-plots the measured spectra extracted from Fig. 4.2, while Fig. 4.5 (lower panel) provides a corresponding plot of the simulated power spectra. The red, green and blue traces correspond to LHD plasmas 138439, 138458 and 138433 respectively. In the lower panel of Fig. 4.5, they are obtained by taking the spatiotemporal fast Fourier transform of the values of \( \delta B_z \) output from the PIC-hybrid computations between \( t = \tau_D \) and \( t = 9\tau_D \) which is summed up to \( k = 40\Omega_D/V_A \). The number of excited cyclotron harmonics qualitatively increases along with their amplitude as \( v_{\text{NBI}}/V_A \) increases. We do not capture the broadening of the peaks which could be a consequence of magnetic field gradients that our simulations do no incorporate. We have checked the noise level of our results by running simulations with a sole deuterium thermal background plasma, all other parameters being kept equal, for each 3 plasmas. The top panel of Fig.
Experiments

Superimposed LHD plasma ICE power spectra. The blue trace corresponds to LHD plasma 138433, the green trace to LHD plasma 138438 and the red trace is associated to LHD plasma 138439.

Simulations

Corresponding simulated ICE power spectra, obtained by averaging the power of the \( z \)-component of the perturbed magnetic field between \( t = \tau_D \) and \( t = 9\tau_D \).

Figure 4.5: Experimental and simulated power spectra.

4.6 displays the simulated power spectra with NBI and thermal deuterium shown by continuous lines and the spectra obtained from the simulations run only with thermal deuterium plasmas correspond to the dashed curves. From the bottom panel of Fig. 4.6, we see that the noise level is similar in all simulations, at \(-145\) dB, although slightly higher for LHD plasma 138433.

These simulated power spectra, shown as dark traces in Fig. 4.7 are superimposed to the measured spectra in coloured traces for comparison (red: LHD plasma 138439, green: LHD plasma 138458, blue: LHD plasma 138433). The simulated power spectra reach their maximum around the lower hybrid frequency at the \( \approx 30 \)th harmonics. This corresponds to a frequency of \( \approx 360 \) MHz since the deuterium cyclotron frequency \( \Omega_D/2\pi \) is 12 MHz in these plasmas. The highest spectral peaks between 300 and 400 MHz in Fig. 4.2 could then correspond to the lower hybrid frequency. The spatiotemporal fast Fourier transform of \( \delta B_z \) plotted in Fig. 4.8 shows that the wave amplitude in \( (\omega, k) \) space is stronger along a region that satisfies \( \omega/k = v_{NBI} \), represented by the dark line in each panel with \( (\omega/k)/V_A = V_{NBI}/V_A = 0.36, 0.44 \) and 0.60 from left to right. The red curve is the fast Alfvén satisfying \( \omega = kV_A \). The beam deuterons saturate faster, releasing more energy to the fields and bulk plasma as \( v_{NBI}/V_A \) increases as shown in the energy density change time evolution for the 3 plasmas in Fig. 4.9. In this section, we have attempted to interpret the ICE spectra obtained during different deuterium
Figure 4.6: Simulated power spectra with NBI and thermal deuterium (top panel) and thermal deuterium only for parameters corresponding to the ICE location in LHD plasmas 138439, 138458 and 138433 (bottom panel).

NBI heated plasmas in LHD including perpendicular-NBI thought to be responsible for the ICE. The edge parameters mostly differ by their electron density implying that the ratio of $v_{NBI}/V_A$ was changing as $v_{NBI}$ was kept constant across the three plasmas we have studied. Hybrid-PIC simulations show that as this ratio varies, the number of harmonics in the ICE signal increases along with their amplitude, although we do not explain the broadening of the peaks. The results are reminiscent of the experiments carried out in TFTR when gas was puffed resulting in a change of the edge plasma density, transitioning the fusion-born $\alpha$ particles from a sub- to a super-Alfvénic regime in that region [60] as discussed in Section 1.6.1.
Figure 4.7: Red, green and blue traces: experimental ICE power spectrum of LHD plasma 138439, 138458 and 138433 at $t = 4.92$ sec respectively. Dark traces: corresponding simulated power spectrum of the $z$-component of the perturbed magnetic field averaged between $t = \tau_D$ and $t = 9\tau_D$.

Figure 4.8: Spatio temporal Fourier transform of the fluctuating part the $z$-component of the magnetic field, averaged over the entire simulations and plotted on a log scale for the 3 simulations. The excitations preferentially occur along $(\omega/k)/V_A = V_{NB1}/V_A$ which increases across simulations.

Figure 4.9: Energy density change time evolution of the different fields and particle species; red: background deuterium, blue: NBI deuterium, green: $z$-component of the fluctuating part of the magnetic field, dark: $x$-component of the electric field.
4.3 ICE during LHD plasma 133979 transient event

In the following sections, we attempt to interpret ICE which is in a frequency range comprised between 200 and 300MHz and is emitted during a transient event. More specifically, we focus on LHD plasma 133979 heated with 59keV of perpendicular D-NBI and 170keV tangential NBI. The spectral content of the measured perturbed magnetic field time series has spectral peaks whose separations could be consistent with cyclotron frequencies of energetic fast ions, possibly fusion-born protons. These spectra are different than previously investigated: they are Doppler shifted. We make the assumption that a finite parallel velocity of the energetic ions driving ICE is responsible for these shifts. The top left panel of Fig. 4.10 shows the time series of the bursting ICE at \( t \approx 4.443s \) in LHD plasma 133979 and the top right panel shows the spectrogram with intense radio frequency activity in the hundreds of megahertz, which are shown in more details in the bottom panels to provide higher resolution in time and in frequency to further appreciate the time and frequencies at play.

![Image of time series and spectrograms](image)

Figure 4.10: Time series of the magnetic field fluctuation encompassing LHD plasma 133979 bursty event (top left). Corresponding windowed FFT (top right) showing the time evolution of the frequency content in the hundreds of MHz. The spectrograms are zoomed in both time and frequency (bottom panels).

In this plasma, three tangential H-NBI and the two perpendicular D-NBI heated the plasma. Two power spectra, just before the bursty event at \( t = 4.44s \) and during
the event at $t = 4.444s$ are presented in Fig. 4.11. The major difference consists in the three peaks labelled $b, c, d$ whose frequencies are $f_b = 255.3$, $f_c = 281.7$ and $f_d = 308.2MHz$. This gives an average spacing $\Delta f = 26.6MHz$. Fig. 4.12 gives the time evolution of the measured neutron flux between $t = 3.0s$ and $t = 5.5s$ during LHD plasma 133979. This neutron flux reaches its maximum shortly before the burst and decreases following it, which could indicate some relation [179]. The top panel of Fig. 4.13 is a zoom of the time evolution around the burst, between 4.4s and 4.6s of the NBI power while the middle and bottom panels show the time evolution of the radio frequency power radiated by the plasma. Intense activity takes place at several hundreds of megaHertz, including at 600 and 880MHz in the bottom panel. It is not clear what is the origin of the activity. In the case of hydrogen plasmas, it has been hypothesized that the signal with frequency 880MHz corresponds to high cyclotron harmonics excitations driven by 40keV perpendicular NBI hydrogen undergoing a reversal in their velocity distribution during tongue events in LHD [180, 181]. Tongue events are magnetic surface deformations in magnetised plasmas
Figure 4.12: Neutron activity is peaking slightly before the bursty event (bottom).

(see Figs. 1 and 2 of [174] for LHD). They are difficult to understand as a result of the "trigger problem": the growth rates of these deformations abruptly increase without significant changes in the underlying parameters likely to control their linear stability. The tongue-shaped topology is reported in [174] and is toroidally and poloidally localised. It is responsible for the collapse of the electron temperature along with the phase-space deformation of the distribution of carbon ions present in the plasma. These tongue-like structures do not correspond to modes and were predicted since the 1960s but were only recently observed [174].

We are interested in the radio frequency bursts in the range 200-300MHz. The burst might be a consequence of the resistive interchange mode destabilized by trapped energetic ions when their precession motion resonates with the mode [25, 174]. The redistribution of these initially helically-trapped ions could generate very anisotropic, transient, distribution in velocity space either by the expelled ions or by freshly trapped ions. Our simulations do not take geometrical effects into account and as a consequence, no MHD modes present in which fast ions could be trapped.

We hypothesise a possible origin that would result in the creation of anisotropic fast ion distribution functions able to generate ICE and an interesting discussion in that direction is presented in Section 3.4 of Ref. [182] as part of the study of Energetic Interchange Mode (EIC) mode in LHD. As a note, particle trapping in the well of the waves can take place in our PIC hybrid simulations but are not further investigated. This should be the object of future studies.

We make the assumption that 3.02MeV fusion-born protons could be responsible for the peaks spaced by 26.6MHz. Our goal is to perform multiple hybrid-PIC simulations of the MCI with those highly energetic protons, using the plasma parameters at the ICE emission location and investigate whether the simulated power spectra can be compatible with the measurements in Fig. 4.11. If the radiation is generated by hydrogen, the magnetic field value associated to the 26.6MHz spacing in the spectrum of the bursty event is 1.75T, see Fig. 4.14. The corresponding radius $a_{1.75T}$ is 4.521m, which is around the last closed magnetic surface. The electron and deuterium temperatures at the emission location are 846eV and 907eV respectively, while the density is $8.8 \times 10^{18} \text{m}^{-3}$ and are used as input parameters in
Figure 4.13: Time evolution of NBI power and radio frequency intensity emitted from the plasma. The two blue dashed lines indicate the time just before the burst at $t = 4.44\,s$ and during the burst at $t = 4.444\,s$. Radio frequency radiations also appear at 600 and 880MHz.
the hybrid-PIC simulations. Table [183] in Fig. 4.15 gives the time after which 3MeV protons are lost as a function of their radial birth location and of their initial pitch angle. We look at this before running local hybrid PIC simulations of the MCI to evaluate whether fusion-born protons could efficiently drive ICE in LHD by studying if the ICE emission location could lie on the path of confined fusion-born proton trajectories. The red boxes are the initial locations corresponding to unconfined fusion-born protons. There is thus the possibility for fusion-born proton confined trajectories [184]. In Ref. [185], a study of 15MeV protons resulting from the D + ^3_He → ^4_He (3.67 MeV) + p (15 MeV) reaction shows that they can be confined over the chaotic field line region. By computing the magnetic moment of the hypothetical fusion-born protons at the burst location

\[ \mu_{ICE} \triangleq \frac{mv^2_{\perp,burst}}{2B} \]  

(4.2)

assuming \( v_{\perp,burst} \) is known, and invoking the conservation of magnetic moment, it allows us to compute the perpendicular proton velocities at various major radii \( R \), considered to be proton birth locations, whose corresponding magnetic field is denoted \( B_R \):

\[ v_{\perp,R} = \sqrt{\frac{2B_R\mu_{ICE}}{m}} \]  

(4.3)

If we further use energy conservation,

\[ \frac{1}{2}mv^2_R = \frac{1}{2}mv^2_{\perp,R} + \frac{1}{2}mv^2_{||,R} = 3\text{MeV} \]  

(4.4)
Figure 4.15: The figure shows the time after which 3 MeV protons are lost in LHD. The red box is the lost particle (initial pitch angle and radial position on equatorial plane) and the number shows time (ms) to loss. This calculation tracks the guiding center and collisions are not considered.

we can then compute the pitch angles at the inferred birth locations

$$\alpha = \arcsin\left(\frac{v_{\perp} R}{v R}\right)$$

(4.5)

for $v_{\perp} = v_{\perp,\text{burst}} = (0.8, 0.9, 1.0, 1.1, 1.2) V_A$ at $R = 4.651 \text{m}$ and compare these values with Fig. 4.15 to find out whether these fast protons are born on confined trajectories. The values are summarized in Table 4.2 and they suggest that the protons that intersect the burst location in the range of velocities considered were originally born on confined trajectories as shown in Fig. 4.15.

<table>
<thead>
<tr>
<th>$R$ (m)</th>
<th>3.10</th>
<th>3.20</th>
<th>3.30</th>
<th>3.40</th>
<th>3.50</th>
<th>3.60</th>
<th>3.70</th>
<th>3.80</th>
<th>4.00</th>
<th>4.10</th>
<th>4.20</th>
<th>4.30</th>
<th>4.40</th>
<th>4.50</th>
<th>4.60</th>
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<td>$B_R$ (T)</td>
<td>2.28</td>
<td>2.42</td>
<td>2.56</td>
<td>2.64</td>
<td>2.68</td>
<td>2.69</td>
<td>2.66</td>
<td>2.61</td>
<td>2.53</td>
<td>2.43</td>
<td>2.33</td>
<td>2.20</td>
<td>2.06</td>
<td>1.92</td>
<td>1.78</td>
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<tr>
<td>$0.9V_A$</td>
<td>22.96</td>
<td>23.71</td>
<td>24.42</td>
<td>24.82</td>
<td>25.01</td>
<td>25.07</td>
<td>24.91</td>
<td>24.68</td>
<td>24.26</td>
<td>23.76</td>
<td>23.22</td>
<td>22.53</td>
<td>21.75</td>
<td>20.97</td>
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<td>$1.0V_A$</td>
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<td>26.54</td>
<td>27.35</td>
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<td>25.19</td>
<td>24.31</td>
<td>23.44</td>
<td>22.51</td>
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<td>30.87</td>
<td>31.12</td>
<td>31.19</td>
<td>30.98</td>
<td>30.68</td>
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<td>28.81</td>
<td>27.92</td>
<td>26.93</td>
<td>25.94</td>
<td>24.90</td>
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<td>32.42</td>
<td>33.45</td>
<td>34.04</td>
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<td>34.16</td>
<td>33.83</td>
<td>33.22</td>
<td>32.49</td>
<td>31.72</td>
<td>30.72</td>
<td>29.60</td>
<td>28.51</td>
<td>27.35</td>
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</table>

Table 4.2: Table giving the initial pitch angle in degrees at different $R$ locations leading to proton perpendicular velocities of $v_{\perp} = v_{\perp,\text{burst}} = (0.8, 0.9, 1.0, 1.1, 1.2) V_A$ at $R=4.521 \text{m} (@1.75T)$.

Computing the pitch angles of 3MeV protons at the emission location in Table 4.3, again show they are confined since on the same trajectories as computed in Table 4.2.
Table 4.3: Table giving pitch angles in degrees of 3MeV protons having different $v_\perp$ in LHD plasma 133979 during the bursty event at 4.444 sec @1.75T.

<table>
<thead>
<tr>
<th>$v_\perp/V_A$</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pitch angle</td>
<td>17.69</td>
<td>19.99</td>
<td>22.32</td>
<td>24.69</td>
<td>27.11</td>
</tr>
</tbody>
</table>

4.4 Simulations of LHD plasma 133979 bursting ICE

Three intense peaks have been measured during a bursty event whose frequencies are 255.1, 287.1 and 308.2 MHz. The average spacing between the peaks is 26.6MHz which we infer correspond to fundamental H at the ICE location. If we normalize those frequencies to 26.6MHz, they would correspond to harmonics 9.6, 10.6 and 11.6 which are not integer multiples of the fundamental. If we evaluate $[308.2\ 281.7\ 255.1]$MHz$-26.6\times[11\ 10\ 9]$MHz, we obtain positive shifts that equate to $[15.6\ 15.6\ 15.7]$MHz, or $0.6 \times 26.6$MHz. The shifts could also be negative if harmonics 10, 11 and 12 were excited instead: $[308.2\ 281.7\ 255.1]$MHz$-26.6\times[12\ 11\ 10]$MHz$=\left[-11.0\ -10.90\ -10.90\right]$MHz. The negative shifts correspond to $-0.4 \times 26.6$MHz. Namely, the Doppler shift is 15.7MHz (harmonics 9-10-11) or -10.9MHz (harmonics 10-11-12).

We seek a constraint on $k_\parallel$ which would be compatible with the observed frequency shifts, assuming that the measured ICE results from the MCI. These waves should therefore propagate very close to perpendicular and we illustrate that this is feasible. Under the MCI, the speeds are $\approx V_A$, such that the Doppler shifts should satisfy $k_\parallel v_\parallel \sim \Omega_H \Rightarrow k_\parallel V_A \sim \Omega_H$. This is equivalent to $\left(k_\parallel/k_\perp\right) k_\perp V_A \sim \Omega_H$. In order to excite the MCI corresponding to quasi-perpendicular fast-Alfvén wave resonant with the $n$th (proton) cyclotron harmonic, the following condition is required: $\omega_{fast} \sim k_\perp V_A \sim n\Omega_H$. Therefore, it is possible to satisfy $k_\parallel v_\parallel \sim k_\parallel V_A \sim \Omega_H$ at $\omega \sim k_\perp V_A \sim n\Omega_H$ if $k_\parallel/k_\perp \sim 1/n$. If the emission was due to $^3$He, the magnetic field at the emission location would be $3/2 \times 1.75T$ which is 2.62T, the magnetic field in the core of LHD, where an inversion in velocity space would be less likely to occur than in the edge. The fast $^3$He would be sub-Alfvénic and estimation of the lower hybrid frequency in the plasma center $\approx 19\Omega_{3He}$ using Fig. 4.14 suggests that the emission would occur at $\approx$ the 20th harmonic of the fast $^3$He, instead of being around the 10th harmonic. It was also shown that a shell which is the expected distribution at fusion ions birth in the plasma center is required to be super-Alfvénic in order to drive the MCI [54].

Table 4.4 gives the perpendicular and parallel energies of protons and their corresponding velocities normalized to the local Alfvén speed at the emission location.
We use the hybrid-PIC simulations to explore the range of $k_\parallel$, which together with $v_\parallel$ could result in Doppler shifts consistent with the measured power spectrum, under the constraint that the MCI is most strongly driven around the 10th proton cyclotron harmonics. The last column shows that the Doppler shift arising from the curvature-grad $B$ drift [97] is one order of magnitude smaller than the Doppler shift inferred from the measured power spectrum in Fig. 4.11. The energies of the perpendicular deuterium-NBI are very sub-Alfvénic and could generate ICE near the lower hybrid frequency at high cyclotron harmonics but at higher $k$-number. Presently, no wavenumber measurements of ICE have been reported from LHD. Although the perpendicular D-NBI could contribute to the power spectrum, we will not consider it in what follows. In order to access higher $k$-values, it may also be necessary to run full-PIC simulations instead of hybrid-PIC, the latter being our approach. We run multiple hybrid-PIC simulations at a given angle between $k$, which is along the simulation domain, and the magnetic field $B_0$, as shown in Fig. 4.16. In these simulations, the fast proton distribution function is given by a ring beam $f_H = 1/(2\pi u_\perp) \delta (v_\parallel) \delta (v_\perp - u_\perp)$ and undergoes a collisionless relaxation via the MCI. Their velocity is set to $u_\perp = 0.1$ to $1.2V_A$ and related energies are indicated in Table 4.4. The simulations for these tests use 500 particles per cell for background deuterium and 500 particles per cell for the fast protons, which are not replenished, with massless electrons. The grid has 1024 cells and the cell size is

<table>
<thead>
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<th>Species</th>
<th>$v_\perp/V_A$</th>
<th>Energy $\perp$ (keV)</th>
<th>$v_\parallel/V_A$</th>
<th>Energy $\parallel$ (keV)</th>
<th>$\omega_{Drift}/\Omega_H$</th>
</tr>
</thead>
<tbody>
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<td>H</td>
<td>0.50</td>
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<td>2.59</td>
<td>2891.84</td>
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</tr>
<tr>
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<td>155.76</td>
<td>2.56</td>
<td>2844.24</td>
<td>0.0143</td>
</tr>
<tr>
<td>H</td>
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<td>170.00</td>
<td>2.63</td>
<td>2830.00</td>
<td>0.0142</td>
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<tr>
<td>H</td>
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<td>212.00</td>
<td>2.54</td>
<td>2788.00</td>
<td>0.0141</td>
</tr>
<tr>
<td>H</td>
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<td>276.90</td>
<td>2.51</td>
<td>2723.10</td>
<td>0.0140</td>
</tr>
<tr>
<td>H</td>
<td>0.90</td>
<td>350.45</td>
<td>2.47</td>
<td>2649.55</td>
<td>0.0138</td>
</tr>
<tr>
<td>H</td>
<td>1.00</td>
<td>432.66</td>
<td>2.44</td>
<td>2567.34</td>
<td>0.0136</td>
</tr>
<tr>
<td>H</td>
<td>1.10</td>
<td>523.52</td>
<td>2.39</td>
<td>2476.48</td>
<td>0.0134</td>
</tr>
<tr>
<td>H</td>
<td>1.20</td>
<td>623.03</td>
<td>2.34</td>
<td>2376.97</td>
<td>0.0131</td>
</tr>
</tbody>
</table>

D ($\perp$ NBI) | 0.26 | 58.00 | - | - | - |
D ($\perp$ NBI) | 0.28 | 69.00 | - | - | - |
D | 0.90 | 700.00 | - | - | - |
D | 1.00 | 865.00 | - | - | - |

Table 4.4: Table mapping fast ion energies and their normalized velocities to the local Alfvén speed in LHD plasma 133979 during the bursty event at 4.444 sec @1.75T. The last column gives the grad and curvature $B$-drift $\omega_{Drift} = -mv_\perp \nabla B/r$, with $v_\perp = (v_\perp^2 + v_\parallel^2/2)/\Omega_H R$, with $m$, the poloidal mode number that we set equal to 1 and $r$ and $R$ are the minor and major radii respectively.
determined such that the dispersion relation fits the cold plasma dispersion. The relative proton densities are chosen to saturate the MCI within $20\tau_H$: $\xi = 0.0025$ for $89.5^\circ$ and $89.0^\circ$ and $88.5^\circ$ and is increased to $\xi = 0.0050$ for $88.0^\circ$. We observe that as the perpendicular speed is increased, the magnitude of the power spectra increase towards the lower harmonics and the spectra intensity are dominated by a few harmonics as seen for speeds ranging between $0.9 - 1.1V_A$ at $89.5^\circ$ and $89.0^\circ$. Despite the use of a very simple distribution function, the features presented could capture some of the essential feature of the spectrum during the LHD transient: the spectrum is dominated by a few harmonics between $\omega = 8 - 12\Omega_H$. It is interesting to observe the sudden decrease in spectral power on the top panels of Fig. 4.16 after $\omega = 12\Omega_H$ (left) and $\omega = 13\Omega_H$ (right). These frequencies lie close to the lower hybrid frequency. There exists a region of $(\omega,k)$ space above the lower hybrid frequency where no modes are present. For the plasma parameters under consideration (thermal deuterium plasma), the lower hybrid frequency corresponds to $14.5\Omega_H$ which is consistent with the top panels of Fig. 4.16. This frequency increases as the angle goes below $90^\circ$. 
Figure 4.16: Power spectra of multiple simulations of the MCI for LHD plasma parameters at the time and location the bursting ICE event with protons having purely perpendicular velocity. Propagation between $k$ and $B_0$ of 89.5° (top left), 89.0° (top right), 88.5° (bottom left), 88.0° (bottom right).

Fig. 4.17 shows the sensitivity of the peak maxima to the propagation angle as well as to the perpendicular beam velocity. Fig. 4.18 results from a relaxation of the MCI for a fast proton distribution which incorporates perpendicular thermal spread: $f_H \propto \exp \left[ - (v_{\perp} - u_{\perp})^2 / v_{\perp,r}^2 \right]$ with $v_{\perp,r} = 0.05u_{\perp}$ for the same propagation angles.
Figure 4.17: Power spectra for a propagation angle of 89.0° (left), and 89.2° (right), for different perpendicular velocities. The peaks appear in the range of frequencies relevant to LHD plasma 133979.

Figure 4.18: Power spectra resulting from a simulation whose fast protons distribution include a perpendicular thermal spread \( v_{\perp,r} = 0.05v_{\perp} \) which can modify the peaks locations, for propagation angle of 89.0° (left) and of 89.2° (right).

For fusion-born protons, \( E = \frac{1}{2}m_H \left( u_{\perp}^2 + u_{\parallel}^2 \right) = 3.02 \text{MeV} \). We use the foregoing \( u_{\perp} = [0.900 : 0.025 : 1.050] \times V_A \) which gives \( u_{\parallel} \)

\[
   u_{\parallel} = \left[ \frac{2E}{m_H} - (u_{\perp})^2 \right]^{1/2}
\]

When \( u_{\perp} = 1.05V_A \), this provides, \( u_{\parallel} = 2.42V_A \). We use these velocities for the simple modelled distribution function of the fusion protons

\[
   f_H (v_{\perp}, v_{\parallel}) = \frac{1}{2\pi u_{\perp}} \delta(v_{\parallel} - u_{\parallel}) \delta(v_{\perp} - u_{\perp})
\]

Included in PIC-hybrid simulations, \( u_{\parallel} \) of this size can give rise to substantial frequency shifts in the saturated spectrum because \( k_{\parallel}v_{\parallel} = k_{\parallel}u_{\parallel} \) is significant, even for 89°, i.e. 1° from perpendicular propagating \( \mathbf{k} \), as the magnitude of \( v_{\parallel} \) is important.
for 3.02MeV under the considered perpendicular velocities. Two power spectra of waves propagating in the $-\hat{x}$ direction (the simulation domain is in the $\hat{x}$ direction) at an angle of 89.0° and 89.2° with respect to the background magnetic field is shown in Fig. 4.19 for 3.02MeV protons having the distribution function $f_0$.

The waves propagating both backward and forward are essentially Alfvén waves, lying at the intersection of ion cyclotron harmonics and the cold plasma dispersion relation, although some ion Bernstein modes might be excited, but to a lower extent, as shown on Fig. 4.21. Forward and backward can refer to energy flow directions, namely to the relative direction between the phase velocity and the group velocity but it is not what is meant here. The $k$ vector is aligned to the one dimensional simulation domain direction called $x$, but can either be positive or negative and so the waves can propagate either in the $+x$ or in the $-x$ direction. When taking the spatiotemporal Fourier transforms of the fields, we obtain four quadrants, characterised by their signs of $\omega$ and $k$. Since we analyse real signals, the values obtained in the top right and bottom left quadrants of the dispersion relation graphs (and in the top left and bottom right quadrants) are equal. These distinguish between waves that propagate along or in the opposite direction with respect to the simulation domain.

The dispersion yields an amplitude $A(\omega, k)$. If $\omega$ and $k$ are strictly positive, then,

$$\frac{1}{2} \left( A(\omega, k) \exp(i\omega t + ikx) + A(-\omega, -k) \exp(-i\omega t - ikx) \right) = A(\omega, k) \cos(\omega t + kx)$$

because $A(\omega, k) = A(-\omega, -k)$.

The simulations consist of 2000 particles per cell for each ion species and 8192 cells, run for $15\tau_H$. The green power spectra result from 3.02MeV protons whose parallel and perpendicular velocities are $u_\parallel = 2.415V_A = 2.199 \times 10^7\text{ms}^{-1}$ and $u_\perp = 1.050V_A = 0.956 \times 10^7\text{ms}^{-1}$, while the blue traces include no parallel velocities in the protons. The frequency resolution in the spectra is 0.07$\Omega_H$. Three intense peaks appear at $9-10-11\Omega_H$. These are shifted to $9.50 - 10.44 - 11.30\Omega_H$ at 89.0° propagation angle and to $9.57 - 10.50 - 11.30\Omega_H$ at 89.2°. The power spectra of the waves propagating along the simulation domain, in the $+\hat{x}$ direction are displayed in Fig. 4.20. The intense peaks are also observed from the spatiotemporal fast Fourier transform of $\delta B_z$, taken over the simulation domain and averaged over $15\tau_H$ as shown on the right panel of Fig.4.21. Summing the dispersion relation between $k = 0$ and $k = 24\Omega_H/V_A$ yield the power spectrum as on the left panel in Fig. 4.21.
Figure 4.19: Power spectra of $\delta B_z$ for waves propagating in the $\hat{x}$ direction at an angle of 89.0° (left) and 89.2° with respect to the background magnetic field. The Doppler-shifted peaks (green traces) locate at 9.50, 10.44 and 11.30$\Omega_H$ (left) and at 9.57, 10.50 and 11.30$\Omega_H$ (right). The proton distribution functions are $1/(2\pi u_\perp)\delta(v_\parallel - u_\parallel)$ (blue trace) and $1/(2\pi u_\perp)\delta(v_\parallel - u_\parallel)$ (green trace) respectively. The velocities are given by $u_\parallel = 2.199 \times 10^7$ ms$^{-1}$ and $u_\perp = 0.956 \times 10^7$ ms$^{-1}$ and satisfy $u_\perp = 1.05V_A$ and $u_\parallel = 2.42V_A$. They are such that $1/2m_H(u_\parallel^2 + u_\perp^2) = 3.02$ MeV.

Figure 4.20: Power spectra of $\delta B_z$ for waves propagating in the $+\hat{x}$ direction at an angle of 89.0° (left) and 89.2° with respect to the background magnetic field. The Doppler-shifted peaks (green traces) locate at 8.37, 9.44 and 10.37$\Omega_H$ (left) and at 8.30, 9.37 and 10.37$\Omega_H$ (right). The proton distribution functions are as described in Fig. 4.19.
Figure 4.21: Power spectrum of $\delta B_z$ and spatiotemporal Fourier transform of $\delta B_z$, on a dB and log$_{10}$ scale respectively for 3.02MeV protons initialized with $1/(2\pi u_\perp)\delta(v_\parallel - u_\parallel)\delta(v_\perp - u_\perp)$, $u_\perp = 1.05 V_A$ and $u_\parallel = 1.42 V_A$. The propagation angle between $k$ and $B_0$ is 89.0°.

There is nothing special about the backward propagating waves and we can find parameters for which the forward propagating waves (travelling in the +$\hat{x}$ direction) are qualitatively close to the measured power spectrum in Fig. 4.11. For protons with $u_\perp = 0.95 V_A = 0.865 \times 10^7 \text{ms}^{-1}$ and $u_\parallel = 2.244 \times 10^7 \text{ms}^{-1}$, the Doppler shifted spectrum of the forward propagating wave in Fig. 4.22 has major peaks at $9.49 - 10.55 - 11.55 \Omega_H$ while the backward propagating waves show intense peaks at $11.32 - 12.22 - 13.05 \Omega_H$. The former is closer to the $9.6 - 10.6 - 11.6 \Omega_H$ peaks of Fig. 4.11. The angle of propagation is 89.0°. We observe from the simulated power spectra that the Doppler-shifts are not significant at low harmonics. This could suggest that the measured spectral peaks at low harmonics could still be a good indication of the unshifted fundamental cyclotron frequencies which are necessary to infer the ICE emission location and the related local plasma parameters. We have obtained power spectra for the fluctuating part of the magnetic field $\delta B_z$ from multiple hybrid-PIC simulations which show intense peaks at a few successive cyclotron harmonics, between $8 - 12 \Omega_H$ and which are Doppler-shifted by a fraction of $\Omega_H$, featuring the three intense peaks from the measured LHD spectrum in Fig. 4.11. The simulated spectra result from the collisionless relaxation of 3.02MeV protons via the magnetoacoustic cyclotron instability for parameters relevant to a bursting ICE event during LHD plasma 133979, and for waves that propagate almost perpendicularly to the background magnetic field. The energy density time evolution of the fields and of the particles are very similar in shape to those of Figs. 3.7 and 4.29. The energetic protons excite the electric and magnetic fields and self-consistently and the bulk plasma. At these very short time scales, we do not expect
collisions to play any significant roles and this is the reason for calling the relaxation collisionless. The simulated results have been obtained using a simple drifting-ring and additional simulations have shown that the inclusion of thermal spread in the ring shifts the most strongly intense harmonics. The propagation angle also affects the calculated spectra. So, although the results are sensitive to the plasma parameters, the main conclusion would still hold: the measured LHD spectrum could be compatible with waves that propagate mostly perpendicular to the local magnetic field and are growing due to the relaxation of a population of 3.02MeV protons that drifts and has a narrow distribution in $v_\perp$ and in $v_\parallel$ inducing Doppler-shifts which are a fraction of $\Omega_H$.

4.5 Spectrum reexamined

We have focused on the three peaks between 250 and 310MHz in the power spectrum Fig. 4.11 of bursting ICE during LHD plasma 133979 which also displayed a spectral peak at lower frequency labelled $a$. Taking the difference of the power spectra before and during the bursts should help distinguish what peaks may result from the burst itself as seen in Fig. 4.23. The peak at 32.73MHz clearly appears while the three intense peaks are present at 252.3, 280.7 and 304.2MHz which would correspond to harmonics 7.71, 8.58 and 9.24 $\Omega_H$. Structures are present at higher frequencies whose origin are difficult to assess: the spacing between the peaks varies. These spectra could result from ICE being generated at different locations in the plasma, which we cannot model in a PIC simulation since the density and magnetic field are homogeneous. They could also be driven by different ion species. We make
the assumption that the peak at 32.73 MHz corresponds to the proton cyclotron frequency at the emission location and proceed as in the last section. We identify a location whose magnetic field leads to a proton cyclotron frequency of 32.73 MHz and obtain the local temperatures and electron density as in Fig. 4.24. We have run PIC-hybrid simulations for 3.02 MeV protons with the new parameters at multiple values of $u_\perp/V_A$ in the range $[0.8, 1.3]$ with no parallel velocity at an angle of 89.0° between $B_0$ and $k$. The relative proton density $\xi = 0.001$. As $u_\perp$ increases, we observe again from the power spectra of $\delta B_z$ in Fig. 4.25 that higher peaks are generated towards the lower proton cyclotron harmonics. From these simulations, we focus on $u_\perp/V_A = 1.2$. For fusion-born protons, this gives $u_\parallel = \left[\frac{2E/m_H - (1.2V_A)^2}{f_H(v_\perp, v_\parallel)}\right]^{1/2} = 2.175V_A$ in $f_H(v_\perp, v_\parallel) = 1/(2\pi u_\perp)\delta(v_\parallel - u_\parallel)\delta(v_\perp - u_\perp)$. The power spectrum of $\delta B_z$ obtained from the simulation with 3.02 MeV protons, which is the green trace in Fig. 4.26, display shifted cyclotron harmonics for the wave travelling backward whose main peaks are at $7.64 - 8.57 - 9.44\Omega_H$ while in the measurements they were found at $7.71 - 8.58 - 9.29\Omega_H$. An additional feature in comparison with the former section comes from the fundamental which looks more driven than subsequent other harmonics. The lower harmonics are linearly stable in these simulations but can be driven later in the nonlinear stage of the simulation as is shown in Fig. 4.27, where the fundamental seems to grow in the simulation and be very slightly more intense in the case of 3.02 MeV protons. This was discussed in Carbajal et al. [19].
Figure 4.24: Magnetic field and electron density evaluated at $R = 4.234$. The magnitude of the magnetic field 2.147T leads to a proton cyclotron frequency of 32.73MHz (top). Deuterium and electron temperature are displayed on a semilog scale and on a linear scale $R = 4.234$ are $T_i = 2.4337$keV and $T_e = 2.0466$T respectively (bottom).

Figure 4.25: Power spectra of $\delta B_z$ averaged over the $15\tau_H$ of simulation and summed over $k = 0$ and $k = 24\Omega_H/V_A$ for various fast proton energies initialized with a ring beam distribution in velocity space $1/(2\pi u_{\perp}) \delta(v_{||})\delta(v_{\perp} - u_{\perp})$. 

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Figure 4.26: 89.0°, power spectra of backward propagating wave. Shifted peaks (green trace) located at 7.636, 8.57 and 9.436 $\Omega_H$. The proton distribution functions are $1/(2\pi u_{\perp})\delta(v_{\parallel} - u_{\parallel})$ (blue trace) and $1/(2\pi u_{\perp})\delta(v_{\parallel} - u_{\parallel})$ (green trace) respectively. The velocities are given by

- $u_{\parallel} = 2.106 \times 10^7$ms$^{-1}$ and $u_{\perp} = 1.161 \times 10^7$ms$^{-1}$ and satisfy $u_{\perp} = 1.200V_A$ and $u_{\parallel} = 2.175V_A$.
- They are such that $1/2m_H(u_{\perp}^2 + u_{\parallel}^2) = 3.02$MeV. The blue trace corresponds to the power spectrum of the associated ring beam distribution $1/(2\pi u_{\perp})\delta(v_{\parallel} - u_{\parallel})\delta(v_{\perp} - u_{\perp})$.

Figure 4.27: Time evolution of power density of the spatial fast Fourier transform of $\delta B_z$, $k(t)$, as a function of time, shown on a log$_{10}$ scale. The intensity at the fundamental cyclotron frequency increases in the later stage of the simulation. The left panel corresponds to proton having $u_{\perp} = 1.2V_A$ and $u_{\parallel} = 0$ and the right panel is for 3.02MeV protons with $u_{\perp} = 1.2V_A$ and $u_{\parallel} = 1.175V_A$ (corresponding to the blue and green curves in Fig. 4.26 respectively).
Table 4.5: Location of peaks for several harmonics of the backward propagating waves.

<table>
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<tr>
<th>$\nu$</th>
<th>$\ell$</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
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<td>$u_\bot = 1.100V_A, u_\parallel = 2.228V_A$</td>
<td>5.769</td>
<td>6.769</td>
<td>7.703</td>
<td>8.570</td>
<td>9.57</td>
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</tr>
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<td>6.702</td>
<td>7.636</td>
<td>8.570</td>
<td>9.436</td>
<td></td>
</tr>
<tr>
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<td>6.702</td>
<td>7.636</td>
<td>8.570</td>
<td>9.436</td>
<td></td>
</tr>
<tr>
<td>$u_\bot = 1.300V_A, u_\parallel = 2.117V_A$</td>
<td>5.769</td>
<td>6.702</td>
<td>7.636</td>
<td>8.570</td>
<td>9.436</td>
<td></td>
</tr>
</tbody>
</table>

4.6 Relaxation of fusion-born protons and Helium-3

Additional power spectra relevant to LHD plasma 133979 during the ICE burst triggered at $\approx 4.44$s are superimposed in Fig. 4.28. They are calculated at different times spaced by 0.1ms, as obtained from the time series in Fig. 4.10 (of which we do not have the tabulated values). The left panel of Fig. 4.28 shows that although they are qualitatively similar, they are evolving over time and particularly at the lower frequencies (below 200MHz). The right panel highlights this variability by offsetting all the spectra with the spectrum in dark trace (at 4.442s, before the burst). A rich set of spectral peaks are present, particularly at the maximum of the burst which occurs at $t = 4.4440$s, and corresponds to the blue trace in each panel. It is difficult to identify the origin of such a broad range of spectral peaks. They could be driven by both fusion-born ions, either proton and Helium-3 (and/or Tritium [71]), and NBI fast ions including both 170keV tangential H and 59keV perpendicular D. Taking the difference of the spectra with respect to a given time (here before the burst) may therefore help to identify which of the spectral peaks are a consequence of the burst itself, namely peaks that were not present beforehand, or which were present and were magnified during the burst and possibly facilitate the identification of the harmonic structures. At the lowest frequencies, we focus on $f = 21.1$MHz and $f = 32.7$MHz. The peak at 32.7MHz was already observed in Fig. 4.11 and Fig. 4.23 (which is the orange curve in Fig. 4.28). Assuming again that the 32.7MHz peak is due to protons, it would be likely to infer that the 21.1MHz spectral peak is driven by $^3$He ($32.73$MHz $\times 2/3 = 21.1$MHz).
Figure 4.28: Power spectra from bursting ICE in LHD plasma 133979. They are obtained at several times, just before the burst (at \( t = 4.442 \) s), as indicated by the dark trace on the left panel. The subsequent different colored traces then show the evolution of the spectra during the burst. The corresponding spectra are offset by the spectrum measured at \( t = 4.442 \) s on the right panel. The dark trace is thus a flat horizontal line.

We have thus attempted to simultaneously relax fusion-born protons and Helium-3 assuming they were driving ICE at exactly the same location in the plasma. We have run several hybrid-PIC simulations with \( E_{3\text{He}} = 0.82 \text{MeV} \) whose energy lies entirely in the perpendicular direction, namely, their distribution function is initialized to be 
\[
\frac{1}{(2\pi u_{\perp,3\text{He}})} \delta \left( v_\perp - u_{\perp,3\text{He}} \right) \delta \left( v_\parallel - u_{\parallel,3\text{He}} \right) \delta \left( v_{\perp,3\text{He}} \right) \delta \left( v_{\parallel,3\text{He}} \right) \frac{2E_{3\text{He}}}{m_{3\text{He}}}. 
\]

The 3.02MeV protons are set up as before with 
\[
\frac{1}{2m_H} \left( u_{\perp,H}^2 + u_{\parallel,H}^2 \right) = 3.02 \text{MeV}. 
\]

With these parameters, we have \( u_{\perp,3\text{He}}/V_A = 0.80 \) and we vary again \( u_{\perp,H} \) (and \( u_{\parallel,H} \) accordingly). The simulations have used 500 particles per cell per ion species and 2048 cells, using otherwise the same physical parameters as described in last section. The protons and Helium-3 have identical relative concentration \( \xi = 0.001 \). The energy density time evolution of the bulk deuterium, the fusion-born protons and Helium-3 are shown on the left panels of Fig. 4.29, along with the \( x \)-component of the perturbed electric field, and the \( y \)- and \( z \)-components of the perturbed magnetic field. From top to bottom, the protons have \( u_{\perp,H} = [1.4, 1.5, 1.6, 1.7] \times V_A \). A novel feature is observed in the latter stage of the simulation when the \( ^3\text{He} \) ions gain net energy. Initially, the Helium-3 energy is transferred to the electromagnetic fields while the proton kinetic energy stays relatively constant (top panel) or looses a fraction of its kinetic before it reaches a plateau (second, third and fourth panels). This is followed by the protons releasing more energy, not only to the electromagnetic fields but also to the Helium-3. The absolute energy density time evolution is shown in Fig. 4.30 and show that Helium-3 have gained an overall \( 8 - 12\% \) of their initial energy after 30 proton gyroperiods (in the case of \( u_{\perp,H} = 1.5 \) and 1.7). Turning to the right panels of Fig. 4.29, we observe that the first and second harmonics of \(^3\text{He} \) are captured, along with the fundamen-
tal of $H$, although their magnitude is much lower than the subsequent harmonics, at least with our choice of distribution function. The noise level is at $\approx -105$ dB. Increasing the number of particles per cell would allow a better resolution of the peaks, especially the $^3\text{He}$ harmonics. The simulations are run at an angle of $89.0^\circ$ between $\mathbf{k}$ and $\mathbf{B}_0$ and the power spectra are for the wave moving forward with respect to the simulation domain. The Doppler-shifted proton cyclotron harmonic peaks of the wave moving backward hamper the peaks due to $^3\text{He}$ as seen on Fig. 4.31. We have run a hybrid-PIC simulation with inertial electrons along with its massless counterpart to check the potential effect of massless electrons on the simulation outputs. The behaviour is the same, the Helium-3 regain energy in both cases, see Fig. 4.32. A full-PIC simulation might be desirable to confirm these trends and it is likely that the outcome would be similar following the work in Carbajal et al. [19], Cook et al. [20] and more recently in McClements et al. [67] where hybrid- and full-PIC simulations of the MCI were run side by side.

Figure 4.32: Comparison of energy density change time evolution of fields and particles when the electrons are massless (left panel) with the case of inertial electrons (right panel). The 3.02 MeV protons have $u_{\perp,H} = 1.3V_A$. The time evolution has the same structure in both scenario.
Figure 4.29: Energy density time evolution of 3.02MeV protons and 0.82MeV Helium 3 (left panels), which both have relative densities $\xi = 0.001$ for various proton pitch angle with respect to the local magnetic field $B_0$. All the energy of $^3$He is in $\perp$-direction with respect to $B_0$. From top to bottom the protons are characterized by $u_\perp = 1.4 - 1.5 - 1.6 - 1.7 \times V_A$ and their associated parallel velocities are $2.22 - 2.16 - 2.08 - 2.00 \times V_A$. The energy range is different for all graphs. The corresponding averaged power spectrum of $\delta B_z$ of the forward propagating waves (right panels) summed between 0 and $2\Omega H/V_A$ show strong peaks at proton cyclotron harmonics. The first two harmonics of $^3$He are also captured.
Figure 4.30: Energy density time evolution of the different ion species. The bulk deuterons and Helium-3 ions gain 1% and 8% of their initial energy respectively while the protons loose 5% of their original energy when $u_{\perp,H} = 1.5$ after $30\tau_H$ (left panel). The bulk deuterons and Helium-3 ions kinetic energy increase by 4% and 12% respectively while the protons see an 11% decrease of their energy when $u_{\perp,H} = 1.7$ after $30\tau_H$ (right panel).

Figure 4.31: Power spectra for waves moving backward (left panel) and forward (right panel). The spectra are averaged over $50\tau_H$ and the relative density of the protons, whose velocity is characterized by $u_{\perp,H} = 1.5V_A$, and Helium-3 was decreased by a half to $\xi = 0.0005$. The harmonics of $\Omega_{He}$ are better resolved for the wave propagating forward.
4.7 Cepstra analysis

In the former section, we have assumed a priori which fast ions species could be responsible for the different spectral peaks that were observed during a bursting ICE event. We supposed that the peaks at low harmonics were not significantly Doppler-shifted. This allowed us to use the fundamental cyclotron frequencies to calculate the associated background magnetic field and to locate the ICE origin. Given the richness of the spectral peaks, driven by both neutral beam ions and possibly fusion-born ions, and given that they are likely to be Doppler-shifted, it becomes difficult to characterise them by eye only. Therefore, instead of postulating which specific cyclotron harmonics of an ion species is responsible for the occurrence of the spectral peaks with particular offsets due to Doppler-shifts, it would be easier to have a method that is able to classify the peaks and their shifts from integer cyclotron harmonics systematically. The calculation of the cepstrum, which corresponds to the inverse Fourier transform of the logarithm of the power spectrum of a signal (magnitude of the Fourier transform of the signal), allows this [186]. The cepstrum was motivated by speech processing. When a signal contains some echo, this results in additional periodic components which are determined by the size and delay of the echoes. Similarly with the power spectra we are analysing, we are interested in identifying one frequency (fundamental cyclotron frequency, possibly shifted). We use the real cepstrum, or simply cepstrum [187], defined for a discrete-time signal as

$$c[n] = \Re \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log|X(e^{i\omega})|e^{i\omega n} d\omega \right]$$

with $\log|X(e^{i\omega})|$ corresponding to the logarithm of the magnitude of the Discrete Time Fourier Transform (DTFT) of the signal. The MATLAB function which implements this, called rceps, returns $y = \text{real} (\text{ifft} (\text{log} (\text{abs} (\text{fft} (x)))))$; [188]. Since we have the power spectra, the first step of the calculation (the log of the magnitude of the fourier transform of the signal, or possibly the magnitude squared) is already done. We thus only need to compute the inverse fast Fourier transform of the data. Our aim is to identify harmonics of different spectral peaks from the LHD magnetic field time series. We simply think of the power spectrum as a new time series of which harmonics are periodic repetition of the fundamental cyclotron frequency. We thus expect to reduce the fundamental cyclotron frequency and all its harmonics in the power spectrum to only one peak in the power cepstrum, at the fundamental. Since the Fourier transform is a complex number, the phase will be retained and should indicate by which amount these peaks are offset if they are. Peaks with low cyclotron frequency and their harmonics are closely spaced in a power spectrum.
compared to peaks which are separated by a different higher cyclotron frequency (due to different ion species, or due to a species experiencing a different magnetic field magnitude in the plasma). However, thought as fast oscillating features of a time series, the closely separated peaks will correspond to a high *queFrequency* in the power *cepstrum*. The question of degenerate harmonics still arise. As test cases, we still focus on LHD plasma 133979 and we compute the power cepstra at two different times: at *t* = 4.4438s, just before the burst maximum, and at 4.4440s which is the time of maximum burst. Several peaks at various *queFrencies* can be observed on the top panels of Fig. 4.33. The *queFrequency* resolution is 0.30MHz⁻¹. The features at 12.26MHz and at 12.51MHz are probably due to perpendicular deuterium NBI and correspond to ICE emitted from the edge of LHD. We focus on two peaks, corresponding to frequencies of 24.05MHz (24.11MHz) and 26.06MHz (25.94MHz) respectively (the frequency resolution in the measured power spectra is 0.30MHz). The dark vertical lines superimposed to the measured power spectra on the middle panels of Fig. 4.33 correspond to the 24.94MHz fundamental frequency along with its successive harmonics which are Doppler-shifted by -10.88MHz before the burst (left panel) and by -9.62MHz at the burst maximum (right panel). We have applied the same method to the 25.94MHz fundamental cyclotron frequency (bottom panels of Fig. 4.33). We observe that these peaks principally intersect the major peaks between 200 and 300MHz but do not cross spectral peaks at lower frequencies. This means a priori that harmonics of such spectral peaks are absent in the spectrum, or that we are unable to detect them. Overall it complicates the interpretation of their origin. We have repeated the calculations by considering the difference of the spectrum at *t* = 4.4438s with a spectrum obtained before the burst (at *t* = 4.4420s) and likewise with the spectrum at *t* = 4.4440s. The resulting cepstra are shown on the top panels of Fig. 4.34. The most prominent peaks are the ones treated earlier (12.51MHz, 24.05MHz and 26.06MHz) and are likely to be accountable for the burst. The middle panels display the Doppler-shifted 24.11MHz fundamental cyclotron frequency peak along with its Doppler-shifted harmonics before the burst (left panel) and at the maximum of the burst (right panel) indicated by vertical dark lines superimposed to the relative spectra. The three major spectral peaks between 250MHz and 300MHz are still intersected and the same findings hold for the low frequencies: two extremely close and intense peaks appear at 20.75MHz and 21.36MHz. A frequency of 28.38MHz with a Doppler-shift of −6.84MHz in the relative spectrum at *t* = 4.4440s leads the first peak to be located at 21.54MHz which is fairly close to 21.36MHz but no subsequent harmonics are obvious, see Fig. 4.35, as the magnitude of the cepstrum is not prominent at the associated *queFrequency*.
We note that the 24.11MHz peaks have Doppler-shifts which are of the same order as those generated by the hybrid-PIC simulations during the relaxation of 3.02MeV protons under the MCI, for wave propagation almost perpendicular to the local magnetic field. In this section, we have tried to classify the spectral peaks and their harmonics, along with their Doppler-shifts by mean of the cepstrum analysis. We have shown preliminary results and confirmed the existence of Doppler-shifted spectral peaks in the neighbourhood of \(\approx 24\text{MHz}\), which we assign to protons, probably fusion-born considering the extent of the Doppler shifts. We cannot rule out contributions of 170keV H-NBI which were injected at the time of the burst. As they are sub-Alfvénic, they could drive ICE at higher cyclotron harmonics, at higher wavenumbers \(k\) near the lower hybrid frequency. We have not yet found the origin of the intense first peaks in the measured spectra of Figs. 4.23, 4.33, 4.34. In these measured spectra, ICE could have originated from different plasma locations. Wavenumber measurements could shed light on the ICE emission by confirming the existence of a finite \(k_\parallel\) and provide estimates on the value of \(k_\perp\). A systematic study of a collection of bursts under different plasma heating conditions (tangential H-NBI on and off) would further help to assess the robustness of the results presented here. In principles, both beating effects (non linear three wave interactions [83]) and Doppler shifts are captured self-consistently.
Just before the burst $t = 4.4438s$  

At the burst max $t = 4.4440s$

ν = 24.11MHz, Δν = −10.88MHz  

ν = 24.11MHz, Δν = −9.62MHz

ν = 25.94MHz, Δν = −5.71MHz  

ν = 25.94MHz, Δν = −5.77MHz

Figure 4.33: Power cepstra computed just before the burst at $t = 4.4438s$ (left) and at the maximum of the burst at $t = 4.4440s$. Several peaks are identified and three are selected:12.26MHz (12.51 MHz), 24.05 MHz and 26.06MHz. The frequency resolution is $0.30MHz^{-1}$. The middle panels show the reconstructed periodicity (dark vertical lines) superimposed on the measured spectra using the fundamental frequency of 24.11MHz (closest to 24.05MHz). The Doppler shifts are $\approx 10MHz$. The same operation is performed on the bottom panels for the fundamental cyclotron frequency of 25.94MHz (closest to 26.0MHz) which are the vertical lines superimposed on the measured spectrum.

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Spectrum at \( t = 4.4438s \) minus spectrum at \( t = 4.4420s \)

Spectrum at \( t = 4.4440s \) minus spectrum at \( t = 4.4420s \)

Figure 4.34: Power cepstra computed from the difference of the spectra at \( t = 4.4438s \) and at \( t = 4.4420 \) (top left) and from the difference of the spectra at \( t = 4.4440s \) and \( t = 4.4420s \) (top right). The major peaks are at 12.51MHz, 24.05 MHz and 26.06MHz and are less numerous than in Fig. 4.33. The queFrequency resolution is \( 0.30 \text{MHz}^{-1} \). The middle panels show the reconstructed periodicity (dark vertical lines) superimposed on the measured relative spectra using the fundamental frequency of 24.11MHz (closest to 24.05MHz) before and at the burst maximum. The Doppler shifts are \( \approx 10 \text{MHz} \). The same operation is performed on the bottom panels for the fundamental cyclotron frequency of 25.94MHz (closest to 26.6MHz) which are the vertical lines superimposed on the measured relative spectrum, before and at the maximum of the burst.
Figure 4.35: Relative spectrum at $t = 4.4440s$. The vertical dark lines show the fundamental frequency 28.38MHz and its harmonics, all Doppler-shifted by $-6.84$MHz. The first peak at 21.54MHz is close to the measured spectral peak at 21.36MHz (left panel).

4.8 Spectral frequency shift and fusion-born protons

We have studied the generation of transient ICE driven by fusion-born protons in LHD. We have carefully analysed the power spectra to extract the parameters of the underlying distribution function modelled by a drifting ring beam

$$\frac{1}{2\pi u_\perp} \delta (v_\parallel - u_\parallel) \delta (v_\perp - u_\perp)$$

The three major steps consist in, first, using the spectral peak spacing $\nu$ to calculate the magnetic field $B_0 = 2\pi \nu (m/q)$, assuming an ion species and hence a prescribed charge to mass ratio compatible with the magnetic field magnitude in LHD. This in turn provides the emission location: the local density and temperatures can therefore be obtained. Normalising the spectral peak frequencies to their spacing in a second step yield values which can be integer but not necessarily, if the peaks are shifted, due for example to finite parallel propagation. The integer parts of the normalised frequencies give the closest harmonic number of the emission that is excited. Under the assumption that this ICE is driven by the MCI, the harmonic numbers quantify the necessary perpendicular energy of the energetic ions which is close to the local Alfvén speed $V_A$ and provides $u_\perp$ for the distribution function model. In the third step, the parallel energy is obtained from the initial energy or birth energy of NBI and fusion-born ions respectively. Together with the inferred $u_\perp$, the value of $u_\parallel$ is computed. For consistency this $u_\parallel$ should provide a simulated spectral offset which is in agreement with the ICE measurements, and no offset if $u_\parallel$ is negligible. The methodology can be tested directly on measurements to rule
out the drive of 170keV tangential hydrogen NBI from the ICE transients in LHD. Figure 4.36 shows bursting ICE power spectra in three distinct LHD deuterium plasmas. The left panels are relevant to LHD plasma 142566 which is heated by pure hydrogen beams, both perpendicular and tangential. The central panels correspond to LHD plasma 133979, simulated in this paper, heated by perpendicular NBI deuterium and parallel NBI hydrogen. Pure deuterium NBI heat LHD plasma 139841 whose spectra appear on the right panels of Fig 4.36. The second rows show cepstra. They correspond to the modulus of the inverse Fourier transforms of the spectra shown on the top panels. They are useful in identifying periodicity in these spectra and hence to reduce harmonics of a given species to a single peak in the cepstrum. Several peaks in the cepstrum therefore indicates the presence of various ion species. Dominant components are between 20MHz and 25MHz and at 12.5MHz. The former are thought to originate from fusion-born protons and the latter from perpendicular NBI deuterium. For LHD 133979, the cepstrum is obtained from the average of the spectra between $t = 4.4438s$ and $t = 4.4445$ from which the spectrum before the burst at $t = 4.4420$ has been substracted. The base spectrum at $t = 4.4490s$ from which the remaining spectra averaged between $t = 4.4486s$ and $t = 4.4496s$ are substracted, yield the rightmost cepstrum for LHD plasma 138941 heated by pure deuterium. The phase encoded in the cepstra provide information regarding the offset of the peaks from integer harmonics. The power spectra are reconstructed on the last two bottom panels. The spectra on the third rows are the ones described to calculate the cepstra while the ones the last row are the average from $t = 4.4438$ to $t = 4.4445$ (thus excluding the spectrum at $t = 4.4420s$ before the burst) for LHD 133979 and the average of the maximal spectra at $t = 4.4488$ and $t = 4.4490$ for LHD 138941.

The important observation arises from the fact that a plasma heated by pure hydrogen beams has transient ICE spectra which are not offset from integer harmonics, contrary to the plasma heated with perpendicular deuterium NBI which appear on the middle and right panels of Fig. 4.36. The non offset spectra in LHD 142566 could result from tangential hydrogen NBI. This is reflected in the different total neutron rate given in Table 4.6 which indicate that the offset in the power spectra are likely to be the result of fusion-born ions and here fusion-born protons activity.
Figure 4.36: Power spectra measured during transient ICE events in three LHD deuterium LHD plasma using different deuterium and hydrogen combination for the perpendicular and tangential NBI heating. Left columns are for LHD 142566 heated by hydrogen for both perpendicular and tangential NBI. The central columns correspond to LHD 133979 heated with perpendicular deuterium NBI and hydrogen tangential NBI. The left column is associated to LHD 138941 heated by perpendicular and tangential deuterium NBI. The top panels show the raw power spectra of transient ICE at different times of the intense MHD activity. The second row represents cepstra calculated from averages of the spectra displayed just above. They allow to identify the periodicity of the power spectral peaks and hence to reduce them to their fundamental cyclotron frequencies. By taking the phase into account, the power spectra are displayed again on the third and fourth rows. The gray vertical lines intercept the most intense spectral peaks by offsetting the fundamental cyclotron frequencies.
Table 4.6: Beam ion species heating three LHD deuterium plasmas and associated total neutron per discharge.

| LHD D plasma | \( \perp \) beam \( \parallel \) beam \( Y_n \) (/shot) |
|--------------|------------------|------------------|
| 142566       | H                | H                | \( 1.2 \times 10^{11} \) |
| 133979       | D                | H                | \( 2.8 \times 10^{14} \) |
| 139841       | D                | D                | \( 1.8 \times 10^{15} \) |

## 4.9 Summary

We have explored ICE measurements from the 2017 LHD Deuterium campaign which were obtained as a result of perpendicular Deuterium NBI and during abrupt events which are currently identified as the EIC. The deuterium NBI-driven ICE varied in their spectral structure: the number of harmonics in the different measured ICE power spectra increase along with their amplitude and they also broaden. The main physical parameter identified to explain the ICE harmonics and amplitudes increase is the electron density \( n_e \). The Deuterium beam injection speed is almost constant in the plasmas explored such that the ratio of \( v_{NBI}/V_A \) is controlled by \( n_e \). Simulated ICE spectra using the parameters of three LHD plasmas characterized by \( v_{NBI}/V_A = 0.36, 0.44 \) and \( 0.60 \) captured the increase of ICE harmonics and their increased amplitude. We checked the noise level of our simulations to confirm this. The broadening of the peaks might require additional physics, such as magnetic field gradients, which our code is unable to simulate due to the periodic boundary conditions. The NBI-driven ICE in LHD has been attributed to prompt losses [7, 177]. Gas puff during the TFTR DT experiments modified the edge density. The fusion-born \( \alpha \) particles undergoing large drift orbit excursions transitioned from a sub- to a super-Alfvénic regime in this region resulting in the sustained generation of ICE, which was reported by Cauffman et al. [60].

In a second part, we have investigated ICE measured during LHD transients. Intense ICE is characterized by a spacing of \( \sim 26\text{MHz} \) in the range \( 200 \text{–} 300\text{MHz} \) and is Doppler-shifted. We have hypothesized that fusion-born protons could be responsible for this Doppler-shifted ICE. We have relaxed energetic protons initially distributed as a cold-ring beam for different initial (purely) perpendicular velocity in multiple hybrid-PIC simulations, under LHD plasma 133979 conditions. The velocities were chosen to be a multiple of the local Alfvén speed. This scan was repeated at different propagation angles with respect to the local magnetic field to identify at which proton cyclotron harmonics ICE is the most intense. Parallel drift was included to the cold-ring beam, such that the protons have a total initial en-
ergy of 3.02MeV, for propagation angle and initial perpendicular velocity giving rise to intense spectral peaks at $\sim 10\Omega_H$. The simulated Doppler-shifted most intense spectral peaks qualitatively resemble the intense measured ICE peaks between 250 and 300MHz, for our choice of proton initial velocity-distribution function. It is not excluded that 170keV H-NBI could be responsible for the bursting ICE although we would expect it at higher harmonics due to its sub-Alfvénic nature at the emission location. We have focused on the measured lower harmonics. It is difficult to characterize them as the emission location cannot reliably be estimated when no successive harmonics are present. In addition, both NBI or fusion-born ions could generate these spectral peaks. Simulations relaxing simultaneously fusion-born H and $^3$He showed an exchange of energy occurred between the two species and was confirmed with a simulation incorporating inertial electrons. A full-PIC simulation could be tested. Work in Ref. [67] and in Refs. [19, 20] suggest that the agreement between hybrid-PIC and full-PIC simulations of the MCI is usually good. Furthermore, both fundamental of H and $^3$He are captured, assuming that ICE takes place at the same plasma location but is not necessarily the case during an LHD abrupt event. We carried out cepstra analysis to examine the periodicity in the measured spectra. Cepstra analysis thus helps us infer which harmonics are present and how important it is. Harmonics with a spacing $\sim 25MHz$ are confirmed although more work is necessary to address additional spectral peaks, possibly underlying other ion species. Future work could include a systematic analysis of the cepstra obtained at different times during the abrupt events, for several such events and across different LHD plasma discharges (tangential H-NBI on and off). The drifting-ring beam distribution is an idealised choice for the energetic ions. However, when using a drifting-ring with some degree of thermalisation, as in Section 3.8, the ICE intensity appears to be more evenly shared between the cyclotron harmonics. In addition, mild thermal spread in the ring does not change the results as shown in Fig. 4.18. The limited number of very intense ICE spectral peaks observed during LHD transients might thus reflect a strong anisotropy in velocity space which is well approximated by a ring-beam whose parameters can then be determined by analysing the measured ICE power spectra.
Chapter 5

Inertial electron hybrid kinetic dispersion

5.1 Introduction

This part aims at deriving the linear dispersion relation of the hybrid kinetic model including electron inertia. We follow [26, 189–191] closely who have obtained the massless electron counterpart that we expand here in the pure perpendicular propagation case as well as in the finite angle propagation scenario. A hybrid kinetic dispersion relation for massless electrons considering purely perpendicular propagating waves in a magnetized plasma was obtained in Kazeminezhad et al. [191]. Their generalised Ohm’s law assumed cold electrons. In addition they solved for $\mathbf{V}_i \times \mathbf{B}_0$, of which underlying linear system is singular and they considered a particular solution. In our calculations, we include the kernel of the linear operator, such that the solution of the associated linear system is fully characterized (see. equation 5.30). Very recently, the hybrid kinetic dispersion relation was obtained by Told et al. [26] in the finite angle propagation case for massless electrons in a one ion species-magnetized plasma (excluding the purely perpendicular case). We extend the computations to retain electron inertial contributions in the generalized Ohm’s law. We also consider a one ion species-magnetized plasma whose distribution function is arbitrary which we then particularize to two gyrotropic distribution functions: $a(n)$ (anisotropic) Maxwellian and a ring beam distribution function for energetic minority ions (which have to be of the same species). As in [26], we treat the electrons as either isothermal or adiabatic and we consider the finite angle propagation case. We also retain the purely perpendicular propagation case since our initial goal is to compare simulated dispersion relations at that angle with their
linearized hybrid kinetic counterparts. In addition, it is at that angle, $90^\circ$ between $\mathbf{k}$ and $\mathbf{B}_0$, that the magneto-acoustic cyclotron instability is most strongly driven and at (around) that angle that most of the simulations were run in this project. More specifically, our prospect is to look at the behaviour of the fast Alfvén wave. In addition, exploration of the linear physics of electromagnetic cyclotron instabilities pertaining to both sub- and super-Alfvénic energetic ions in relation can be considered, see Chapter 6. Several generalizations are possible such as retaining multiple ion species or treating full electron pressure tensor obtained from the second moment of Vlasov equation in the generalized Ohm’s law. The latter is done in Camporeale and Tronci [192] (for inertial electron) in a study of the Weibel instability. Their linear treatment considers no background magnetic field. Linearization of the equations in more complicated geometries, or which retains gradients in background or densities are additional routes [97]. A growing set of plasma modelling approaches come along with their associated dispersion relation solvers. This includes in particular, fully kinetic and relativistic treatments [18], fully kinetic and non relativistic dispersion relation [193, 194], neutral Vlasov [139], hybrid kinetic with massless electrons [26, 190] with the different possible closure relation for the electrons or with their full pressure tensor. Several of these approximations have been compared [195, 196].

We retain electron inertia in the derivation of the hybrid kinetic dispersion relation. Our main goals are to revisit the MCI analytical results within the hybrid framework and to explore sub-super-Alfvénic regime for MCF edge plasma parameters where electromagnetic cyclotron waves are excited. The chapter is subdivided as follows, we linearize the electron inertial hybrid kinetic equations and isolate a relation from the generalized Ohm’s law, in 3 cases: purely perpendicular propagating case, purely parallel and finite propagation angle case. We obtain the solution to the linearised Vlasov equation in terms of $\delta n$ and of $\delta \mathbf{B}$, the perturbed density and magnetic field respectively, which permits to compute the dielectric tensor and the dispersion relation.

### 5.2 Linearisation of the inertial hybrid kinetic model

The set of equations to solve the inertial hybrid kinetic model is made up of the Vlasov equations for the ions, namely,

$$
\frac{\partial f_i}{\partial t} + \mathbf{v} \cdot \frac{\partial f_i}{\partial \mathbf{x}} + \frac{Z_i e}{m_i} \left( \mathbf{E}' + \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_i}{\partial \mathbf{v}} = 0
$$

(5.1)
with \( f_i = f_i(x, v, t) \) the distribution function of the ion species and \( M_i \), its mass. We introduced \( E' \) in order to include resistivity as defined in 2.72 as

\[
E' = E - \eta J
\]  

(5.2)

With the assumption of charge neutrality,

\[
n_e = Z_i n_i = Z_i \int f_i dv = Z n
\]  

(5.3)

Gauss’s law is not used in the Hybrid approximation and no statement regarding \( \nabla \cdot E \) (Eq. 2.34) is made as mentioned in [197]: it is possible to have \( n_e = Z_i n_i \) while \( \nabla \cdot E \neq 0 \) as discussed in [161, p.77], which is called the plasma approximation (and is distinguished from quasineutrality). It is valid for low frequency waves and slow motions such that both electrons and ions can move. This assumption results in an error proportional to \( 1/(1 + k^2 \lambda_D^2) \) [161, p.98], with \( \lambda_D \) the Debye length, which is indeed much smaller than the spatial scales of interest for the LHD plasma investigated in our simulations, contrary to the electron skin depth \( \lambda_e \) whose value can be close to the background ion gyroradius and the simulation cell size. In summary, instead of determining the electric fields from charge densities through Poisson’s equation, the electric field can be determined from quasineutrality by means of the equation of motion (a generalised Ohm’s law). Poisson’s equation can then be used to compute the charge density [161, p.77].

From the continuity equation, quasineutrality imposes the longitudinal part of the current to be identically zero.

We use Faraday’s law,

\[
\frac{\partial B}{\partial t} = -\nabla \times E
\]  

(5.4)

as well as Ampère’s law in the Darwin approximation (no displacement current)

\[
\nabla \times B = \mu_0 J
\]  

(5.5)

along with

\[
\nabla \cdot B = 0
\]  

(5.6)
The generalized Ohm’s law is given in 2.34, namely

\[
\left[ \frac{\omega_{pe}^2}{c^2} + \nabla \times \nabla \times \right] \mathbf{E} = \left[ \frac{e\mu_0}{m_e} \mathbf{J}_e \times \mathbf{B} + \mu_0 \nabla \cdot (\mathbf{V}_e \mathbf{J}_e) - \frac{e\mu_0}{m_e} \nabla \cdot \mathbf{P}_e + \nu \mu_0 \mathbf{J} \right]
\]

(5.7)

The state equation of the electron corresponds to either the adiabatic or to the isothermal law, such that the pressure tensor \( \mathbf{P}_e = 1p_e \) and \( p_e \) given by

\[
p_e = n_e^\gamma k_B T e n_e^{1-\gamma}
\]

(5.8)

which completes the set of equations of the Hybrid approximation.

The expression for the current is given as,

\[
\mathbf{J} = -en_e \mathbf{V}_e + e \int v f dv
\]

(5.9)

and is divergence free,

\[
\nabla \cdot \mathbf{J} = 0
\]

(5.10)

as can be observed by taking the divergence of Ampère’s law 5.5. As a very first step in our calculations, we linearize 5.1, assuming

\[
\begin{cases}
\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B} \\
f = f_0 + \delta f
\end{cases}
\]

(5.11)

where \( n_0 = \int f_{i,0} dv \) is spatially independent and we assume that \( \mathbf{V}_{i,0} \) and \( \mathbf{V}_{e,0} \) are both zero, as is \( \mathbf{E}_0 \). We also assume \( f_0 \) to be isotropic and stationary and \( \mathbf{B}_0 \) uniform, pointing in the \( \hat{y} \) direction. So \( \mathbf{B}_0 = (0, B_0, 0) \) and we set \( \mathbf{k} = (k_\perp, k_\parallel, 0) \).

This gives

\[
\frac{\partial \delta f}{\partial t} + v \cdot \frac{\partial \delta f}{\partial \mathbf{x}} + \frac{Z_i e}{M_i} (\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial \delta f}{\partial \mathbf{v}} + \frac{Z_i e}{M_i} \left( \delta \mathbf{E}' + \mathbf{v} \times \delta \mathbf{B} \right) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0
\]

(5.12)
Using $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$, and $\omega_c = Z_\iota e B_0 / M_i$, we get for the third term of 5.12

$$
\frac{Z_\iota e}{M_i} \frac{\partial f}{\partial \mathbf{v}} (\mathbf{v} \times \mathbf{B}_0) = \frac{Z_\iota e}{M_i} \mathbf{B}_0 \cdot \left( \frac{\partial \delta f}{\partial \mathbf{v}} \times \mathbf{v} \right) = -\omega_c (\mathbf{v} \times \nabla_v) \delta f = -\omega_c (\mathbf{v} \times \nabla_v)_y \delta f = \omega_c \frac{\partial \delta f}{\partial \theta}
$$

(5.13)

In [26], it is stated that $(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0$ if $f_0$ is gyrotropic, but it should not appear in the expansion 5.12 since it is a 0th order term.

Before expressing the linearised quantities in the Fourier domain, we look what conditions are necessary for the last term of 5.12 to disappear. One such condition is if $f_0$ is an isotropic distribution function as is observed by expressing the gradient in spherical coordinate,

$$ \frac{\partial f_0}{\partial \mathbf{v}} = \left( \frac{\partial f_0}{\partial v_\perp} \hat{\mathbf{v}} + \frac{1}{v} \frac{\partial f_0}{\partial \theta} \hat{\theta} + \frac{1}{v \sin \theta} \frac{\partial f_0}{\partial \phi} \hat{\phi} \right) = \frac{\partial f_0}{\partial \mathbf{v}} \hat{\mathbf{v}} $$

(5.14)

which is then orthogonal to $\mathbf{v} \times \partial \mathbf{B}$. Since $f_0$ is gyrotropic as a solution of the zeroth order Vlasov equation, $\frac{\partial f_0}{\partial \theta} \hat{\theta} = 0$, then using the gradient in cylindrical coordinates gives,

$$ (\mathbf{v} \times \partial \mathbf{B}) \cdot \frac{\partial f_0}{\partial \mathbf{v}} = \left[ (v_\perp + v_\parallel) \times \partial \mathbf{B} \right] \cdot \left( \frac{\partial f_0}{\partial v_\perp} \hat{v}_\perp + \frac{\partial f_0}{\partial v_\parallel} \hat{v}_\parallel \right) $$

(5.15)

and we note that the RHS of 5.15 can be zero if $\partial \mathbf{B}$ is parallel to $\mathbf{B}_0$ which should be the case for perpendicular propagation. However, we will keep this term to allow arbitrary propagation angle when $\partial \mathbf{B}$ is not necessarily along $\mathbf{B}_0$.

Next, we suppose that $\delta f = \delta \tilde{f} e^{i(kz - \omega t)} = \delta \tilde{f} (\omega, \mathbf{k}, v_\parallel, v_\perp, \theta) e^{i(kz - \omega t)}$ and $\partial \mathbf{B} = \partial \mathbf{B} e^{i(kz - \omega t)}$, such that, $\nabla \times (\delta \mathbf{B} e^{i(kz - \omega t)}) = e^{i(kz - \omega t)} \nabla \times (\delta \mathbf{B}) + (\nabla e^{i(kz - \omega t)}) \times \delta \mathbf{B} = \left( i \mathbf{k} \times \delta \mathbf{B} \right) e^{i(kz - \omega t)}$ bearing in mind that they are functions of $\omega$ and $\mathbf{k}$.

\[ ^1 \text{In cylindrical coordinates, the gradient is given as follows: } \nabla f = \left( \frac{\partial f}{\partial v_\perp} \hat{v}_\perp + \frac{1}{v_\perp} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{v_\perp \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \right), \text{ with } \hat{v}_\perp = \frac{x \hat{x} + z \hat{z}}{v_\perp}, \text{ and } \hat{\theta} = -\frac{z \hat{x} + x \hat{z}}{v_\perp}, \text{ and } \hat{v}_\parallel = (x^2 + z^2)^{1/2}, \]

\[ ^2 \text{With } \mathbf{v} = x \hat{x} + z \hat{z} + y \hat{y}, \text{ and } \mathbf{z} \times \hat{x} = \hat{y}, \text{ and } (\mathbf{v} \times \nabla_v)_y f = z \left( \frac{\partial f}{\partial v_\perp} \hat{v}_\perp + \frac{1}{v_\perp} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{v_\perp \sin \theta} \frac{\partial f}{\partial \phi} \hat{\phi} \right) \]

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Dropping the tilde in what follows, Vlasov equation 5.12 becomes

\[-i(\omega - \mathbf{k} \cdot \mathbf{v}) \delta f + \omega_e \frac{\partial \delta f}{\partial \theta} + Z_i e \frac{\delta \mathbf{E}'}{M_i} \left( \delta \mathbf{v} \times \delta \mathbf{B} \right) \frac{\partial f_0}{\partial \mathbf{v}} = 0 \quad (5.16)\]

Linearization and Fourier expansions of equations 5.4, 5.5 and 5.6 give respectively

\[-i \omega \delta \mathbf{B} = -i \mathbf{k} \times \delta \mathbf{E} \]

\[i \mathbf{k} \times \delta \mathbf{B} = \mu_0 \delta \mathbf{J} \quad (5.17)\]

while 5.7 gives using Ampère’s law 5.5 for the last term

\[\left[ \frac{\omega_{pe}^2}{c^2} - \mathbf{k} \times \mathbf{k} \right] \delta \mathbf{E} = \left[ \frac{e \mu_0}{m_e} \delta \mathbf{J}_e \times \mathbf{B}_0 - i \mathbf{k} \frac{e \mu_0}{m_e} \gamma k_B T_e \delta n_e + i \nu \mathbf{k} \times \delta \mathbf{B} \right] \quad (5.18)\]

We now eliminate the electric field from Vlasov equation. Setting \( \lambda_e^2 = c^2/\omega_{pe}^2 \), the operator acting on \( \delta \mathbf{E} \) can be rewritten as

\[\left[ (\lambda_e^{-2} + k^2) \mathbf{1} - \mathbf{k} \mathbf{k} \right] \cdot \delta \mathbf{E} \quad (5.19)\]

and its inverse is given by

\[\lambda_e^2 \left( 1 + (k \lambda_e)^2 \right)^{-1} \left[ 1 + \lambda_e^2 \mathbf{k} \mathbf{k} \right] \cdot \delta \mathbf{E} \quad (5.20)\]

such that with \( \lambda_e^2 (e \mu_0) / m_e = 1/(en_e) \) and \( \eta = \nu m_e / (e^2 n_e) \), we have

\[\delta \mathbf{E} = \left( 1 + (k \lambda_e)^2 \right)^{-1} \left[ 1 + \lambda_e^2 \mathbf{k} \mathbf{k} \right] \left[ \frac{1}{en_e,0} \delta \mathbf{J}_e \times \mathbf{B}_0 - i \mathbf{k} \frac{1}{en_e,0} \gamma k_B T_e \delta n_e + i \eta \mathbf{k} \times \delta \mathbf{B} \right] \quad (5.21)\]

Developing \( \delta \mathbf{J}_e \times \mathbf{B}_0 \) using Ampère’s law 5.5 (5.17) and the ion current,\n
\[\frac{1}{en_e,0} \delta \mathbf{J}_e \times \mathbf{B}_0 = \left( \frac{1}{\mu_0 en_e,0} i \mathbf{k} \times \delta \mathbf{B} - \frac{1}{en_e,0} \delta \mathbf{J}_i \right) \times \mathbf{B}_0 \]

\[= \frac{1}{\mu_0 en_e,0} i \left( (\mathbf{k} \cdot \mathbf{B}_0) \delta \mathbf{B} - (\delta \mathbf{B} \cdot \mathbf{B}_0) \mathbf{k} \right) - \frac{1}{en_e,0} \delta \mathbf{J}_i \times \mathbf{B}_0 \quad (5.22)\]

Before proceeding further, if we set \( \lambda_e = 0 \) in 5.21 after substituting the first line of 5.22, we find

\[\delta \mathbf{E} = -\frac{1}{en_e,0} \delta \mathbf{J}_i \times \mathbf{B}_0 + \frac{1}{\mu_0 en_e,0} i (\mathbf{k} \times \delta \mathbf{B}) \times \mathbf{B}_0 - i \mathbf{k} \frac{1}{en_e,0} \gamma k_B T_e \delta n_e \]

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\[ + \frac{i}{\mu_0} k \times \delta \mathbf{B} \quad (5.23) \]

and we recover equation 8 of [26] with the change from SI to cgs units, by changing \( \mu_0 \rightarrow 4\pi/c^2 \) and \( \mathbf{B}_0, \delta \mathbf{B} \rightarrow \mathbf{B}_0/c, \delta \mathbf{B}/c \).

Introducing the second line of 5.22 into 5.21, remembering that \( k \cdot \delta \mathbf{B} = 0 \), one finds

\[ \delta \mathbf{E} = \frac{1}{1 + (k\lambda_e)^2} \left( \frac{1}{\mu_0 e n_{e,0}} - i \left( k \cdot \mathbf{B}_0 \right) \delta \mathbf{B} + i \frac{\eta}{\mu_0} k \times \delta \mathbf{B} - \frac{1}{e n_{e,0}} \left( 1 + \lambda_e^2 k \mathbf{k} \right) \cdot \delta \mathbf{J_i} \times \mathbf{B}_0 \right) \]

\[ - i k \frac{1}{e n_{e,0}} \gamma k_B T_e \delta n_e - \frac{1}{\mu_0 e n_{e,0}} i (\delta \mathbf{B} \cdot \mathbf{B}_0) k \quad (5.24) \]

We can insert the expression 5.24 of \( \delta \mathbf{E} \) in Faraday's law 5.4 (5.17). Noting that the terms along \( k \) will not contribute in \( k \times \delta \mathbf{E} \), we have

\[ \omega \delta \mathbf{B} = k \times \delta \mathbf{E} \]

\[ \omega \delta \mathbf{B} = \frac{1}{1 + (k\lambda_e)^2} \left( \frac{1}{\mu_0 e n_{e,0}} i \left( k \cdot \mathbf{B}_0 \right) k \times \delta \mathbf{B} + i \frac{\eta}{\mu_0} k \times \delta \mathbf{B} - k \times \left( \delta \mathbf{V_i} \times \mathbf{B}_0 \right) \right) \]

\[ \text{(5.25)} \]

Developing \( k \times k \times \delta \mathbf{B} \) yields,

\[ k \times k \times \delta \mathbf{B} = (k \cdot \delta \mathbf{B}) k - k^2 \delta \mathbf{B} = -k^2 \delta \mathbf{B} \quad (5.26) \]

To find the relation of interest, we take the cross product of both sides of 5.25 with \( k \), apply 5.26, to give:

\[ \omega k \times \delta \mathbf{B} = \frac{1}{1 + (k\lambda_e)^2} \left( - \frac{1}{\mu_0 e n_{e,0}} i \left( k \cdot \mathbf{B}_0 \right) k^2 \delta \mathbf{B} - i \frac{\eta}{\mu_0} k^2 k \times \delta \mathbf{B} - k \times \left( \delta \mathbf{V_i} \times \mathbf{B}_0 \right) \right) \]

or equivalently,

\[ \frac{1}{k^2} \left( \omega \left( 1 + (k\lambda_e)^2 \right) + i \frac{\eta}{\mu_0} k^2 \right) k \times \delta \mathbf{B} = \frac{1}{\mu_0 e n_{e,0}} i \left( k \cdot \mathbf{B}_0 \right) k^2 \delta \mathbf{B} \]

\[ \text{(5.27)} \]

while

\[ k \times k \times (\delta \mathbf{V_i} \times \mathbf{B}_0) = - \omega \left( 1 + (k\lambda_e)^2 \right) + i \frac{\eta}{\mu_0} k^2 \]

\[ \text{(5.28)} \]

\[ \text{while} \]

\[ k \times k \times (\delta \mathbf{V_i} \times \mathbf{B}_0) = (k \cdot (\delta \mathbf{V_i} \times \mathbf{B}_0)) k - k^2 \delta \mathbf{V_i} \times \mathbf{B}_0 \]

\[ \text{(5.29)} \]

allows us to rewrite 5.28 as,

\[ \delta \mathbf{V_i} \times \mathbf{B}_0 = \frac{1}{k^2} \left( k \cdot (\delta \mathbf{V_i} \times \mathbf{B}_0) \right) k + \frac{\omega}{k^2} \left( 1 + (k\lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \] k \times \delta \mathbf{B} + \frac{1}{\mu_0 e n_{e,0}} i \left( k \cdot \mathbf{B}_0 \right) \delta \mathbf{B} \]

\[ \text{(5.30)} \]

Since \( \delta \mathbf{V_i} \times \mathbf{B}_0 \) appears on both sides of 5.30, one can take the dot product of both
sides of the equality with $B_0$ to get rid of the left hand side in order to find an expression for $k \cdot (\delta V_i \times B_0)$:

$$
\frac{1}{k^2} (k \cdot (\delta V_i \times B_0)) (k \cdot B_0) + \frac{1}{\mu_0 e n_e,0} i (k \cdot B_0) (\delta B \cdot B_0)
$$

$$
+ \left( \frac{\omega}{k^2} \left( 1 + (k\lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) (k \times \delta B) \cdot B_0 = 0
$$

(5.31)

5.3 $\delta V_i \times B_0$ in different propagation angle cases

In [26, 190], an expression for $\delta V_i \times B_0$ is found. We will operate in a similar way but we will distinguish three different cases,

- Case 1: $k \perp B_0$
- Case 2: $k \parallel B_0$
- Case 3: $k \cdot B_0 \neq 0$

Case 1: in the perpendicular case, we must satisfy $k \cdot B_0 = 0$ which leads to satisfy either of

$$
(k \times \delta B) \cdot B_0 = 0
$$

(5.32)

or

$$
\frac{\omega}{k^2} = -\frac{i \eta}{\mu_0 \left( 1 + (k\lambda_e)^2 \right)}
$$

(5.33)

For non trivial solutions of $\omega$ and $k$, we keep the former, 5.32, which implies that the magnetic perturbation is along the background magnetic field $B_0$ as can be seen by rewriting the first line of 5.32 as $(B_0 \times k) \cdot \delta B = (-kB_0) \delta B_z = 0$ while $k \cdot \delta B = 0$ leads to $\delta B_x = 0$. As mentioned earlier (equation 5.15), this implies that for perpendicular propagation, $\delta B$ is along $B_0$. We do not obtain a solution for $\delta V_i \times B_0$ but can still obtain its component along the $\hat{z}$ direction by taking the dot product of both sides of 5.30 with $B_0 \times k(\propto \hat{z})$. The only contribution on the RHS comes from the term $\propto k \times \delta B_0$ in 5.30 which satisfies

$$
(k \times \delta B) \cdot (B_0 \times k) = (k \cdot B_0) (\delta B \cdot k) - (\delta B \cdot B_0) (k \cdot k) = -k^2 \delta B \cdot B_0
$$

(5.34)

and the LHS gives

$$
(\delta V_i \times B_0) \cdot (B_0 \times k) = -kB_0 (\delta V_i \times B_0)_z
$$

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\[ = (\delta \mathbf{V}_i \cdot \mathbf{B}_0) (\mathbf{B}_0 \cdot \mathbf{k}) - (\delta \mathbf{V}_i \cdot \mathbf{k}) (\mathbf{B}_0 \cdot \mathbf{B}_0) = -k B_0^2 \delta V_{i,x} \quad (5.35) \]

Together with 5.30, 5.34 and 5.35, we obtain,

\[ -k B_0^2 \delta V_{i,x} = -k^2 \delta B_y B_0 \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \quad (5.36) \]

that is

\[ \delta V_{i,x} = \delta B_y \frac{B_0}{k} \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta k}{\mu_0} \right) \quad (5.37) \]

which could be also be obtained by directly developing 5.25. We use 5.37 when substituting for \( \delta \mathbf{E} \) in Vlasov equation. Equivalently,

\[ (\delta \mathbf{V}_i \times \mathbf{B}_0)_z = \delta V_{i,z} B_0 = k \delta B_y \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta k}{\mu_0} \right) \quad (5.38) \]

The quantity \( k \cdot (\delta \mathbf{V}_i \times \mathbf{B}_0) \) is equal to \( \delta \mathbf{V}_i \cdot (\mathbf{B}_0 \times \mathbf{k}) \) which corresponds to the ion bulk velocity in the direction perpendicular to the wave vector and to the background magnetic field \( \mathbf{B}_0 \). For the consistency of the equations we have left it unknown as in 5.30. However, in the derivation of the Alfvén wave dispersion relation using MHD, the perturbed bulk velocity, which corresponds to both ions and electrons together, is along the wave vector \( \mathbf{k} \) [144, 190].

**Case 2**: in the parallel case, \( \mathbf{k} \times \mathbf{B}_0 = 0 \) needs to be satisfied. We can divide 5.31 by \( \mathbf{k} \cdot \mathbf{B}_0 \) to isolate \( \mathbf{k} \cdot (\delta \mathbf{V}_i \times \mathbf{B}_0) \) and simplify the expression with \( (\mathbf{k} \times \mathbf{D}) \cdot \mathbf{B}_0 = (\mathbf{B}_0 \times \mathbf{k}) \cdot \mathbf{D} = 0 \):

\[ \mathbf{k} \cdot (\delta \mathbf{V}_i \times \mathbf{B}_0) = -\frac{1}{\mu_0 e n_e,0} i k^2 (\delta \mathbf{B} \cdot \mathbf{B}_0) \quad (5.39) \]

This gives us the expression for \( \delta \mathbf{V}_i \times \mathbf{B}_0 \) in 5.30,

\[ \delta \mathbf{V}_i \times \mathbf{B}_0 = -\frac{1}{\mu_0 e n_e,0} i (\delta \mathbf{B} \cdot \mathbf{B}_0) \mathbf{k} + \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \mathbf{k} \times \delta \mathbf{B} + \frac{1}{\mu_0 e n_e,0} i (\mathbf{k} \cdot \mathbf{B}_0) \delta \mathbf{B} \quad (5.40) \]

Note that this case can naturally be obtained from the following one.

**Case 3**: this is the most general one, as treated in the massless electron case in [26], and we obtain directly from 5.31

\[ (\mathbf{k} \cdot (\delta \mathbf{V}_i \times \mathbf{B}_0)) = -\frac{1}{\mu_0 e n_e,0} i k^2 (\delta \mathbf{B} \cdot \mathbf{B}_0) - k^2 \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{(\mathbf{k} \times \delta \mathbf{B}) \cdot \mathbf{B}_0}{(\mathbf{k} \cdot \mathbf{B}_0)} \quad (5.41) \]

Inserting 5.41 in 5.30 yields,
\[
\delta V_i \times B_0 = -\frac{1}{\mu_0 e n_{\omega,0}} i (\delta B \cdot B_0) k - \left( \frac{\omega}{k^2} (1 + (k \lambda_e)^2) + i \frac{\eta}{\mu_0} \right) \frac{(k \times \delta B) \cdot B_0}{k \cdot B_0} k
\]
\[+ \left( \frac{\omega}{k^2} (1 + (k \lambda_e)^2) + i \frac{\eta}{\mu_0} \right) k \times \delta B + \frac{1}{\mu_0 e n_{\omega,0}} i (k \cdot B_0) \delta B \quad (5.42)\]

To verify the result, we set \( \lambda_e = 0 \) in 5.42 and obtain,
\[
\delta V_i \times B_0 = \left( i \frac{\eta}{\mu_0} + \frac{\omega}{k^2} \right) k \times \delta B + \frac{1}{\mu_0 e n_{\omega,0}} i (k \cdot B_0) \delta B
\]
\[- k \left( \left( i \frac{\eta}{\mu_0} + \frac{\omega}{k^2} \right) \frac{(k \times \delta B) \cdot B_0}{k \cdot B_0} + \frac{1}{\mu_0 e n_{\omega,0}} i (\delta B \cdot B_0) \right) \quad (5.43)\]

which recovers equation 10 of [26] with the change from SI to cgs units cfr equation 5.23.

### 5.4 Solution of Vlasov equation

Having explored which terms of the electric field could be eliminated, we solve Vlasov equation 5.16 which is recast as
\[
p (\theta) \delta f + \frac{\partial \delta f}{\partial \theta} = g (\theta) \quad (5.44)
\]
where
\[
p (\theta) \triangleq -i (\nu - \kappa \cos \theta) \quad (5.45)
\]
and
\[
g (\theta) \triangleq -\frac{1}{B_0} (\delta E^\prime + v \times \delta B) \cdot \frac{\partial f_0}{\partial v} \quad (5.46)
\]
with \( \nu = \frac{\omega}{\omega_c} - \frac{k \parallel v \parallel}{\omega_c} \) and \( \kappa = \frac{k \perp v \perp}{\omega_c} \). This equation is solved with the integrating factor method,
\[
\delta f (\theta) = \mu^{-1} (\theta) \int_{-\infty}^{\theta} d \theta' \mu (\theta') g (\theta') \quad (5.47)
\]
where the integrating factor is given by
\[
\mu (\theta) = e^{\int d \theta' p (\theta')} = e^{-i (\nu \theta - \kappa \sin \theta)} \quad (5.48)
\]
and 5.47 becomes with 5.48
\[
\delta f (\theta) = \int_{-\infty}^{\theta} d \theta' e^{i (\nu (\theta - \theta') - \kappa (\sin \theta - \sin \theta'))} g (\theta') = \int_{-\infty}^{\theta} d \theta' K (\nu, \kappa, \theta', \theta) g (\theta') \quad (5.49)
\]
We have introduced the variable \( \sigma \) at the lower bound which takes value +1 when \( \text{Im}(\omega) > 0 \) and -1 when \( \text{Im}(\omega) < 0 \), as introduced in [26], which guarantees convergence of the integrals in those two cases. The integrals will however diverge on the real axis. We will proceed to particularize \( g(\theta) \) in 5.49.

### 5.5 Perpendicular propagation

We now treat the perpendicular propagation case. In order for \( f_0 \) to be a solution of the time independent Vlasov equations with a sole background magnetic field and no background electric field, \( f_0 \) has to be homogeneous and gyrotropic. This implies that the last term of Vlasov equation 5.16 vanishes since \( \delta B \) is along \( B_0 \),

\[
(v \times \delta B) \cdot \frac{\partial f_0}{\partial v} = (v_\perp \times \delta B) \cdot \frac{\partial f_0}{\partial v_\parallel} \hat{v}_\parallel = 0
\]  

(5.50)

and

\[
\frac{\partial \delta f}{\partial t} + v \cdot \frac{\partial \delta f}{\partial x} + \omega_c \frac{\partial \delta f}{\partial \theta} + \frac{Z_i e M_i}{M_i} (\delta E' + v \times \delta B) \cdot \frac{\partial f_0}{\partial v} = 0
\]

(5.51)

The effective electric field becomes using 5.2 and 5.24,

\[
\delta E' = \delta E - \eta \delta J = \frac{1}{1 + (k \lambda e)^2} \left( i \frac{\eta}{\mu_0} k \times \delta B - \frac{1}{e n_e,0} \left[ 1 + \lambda_e^2 k k \right] \cdot \delta J \times B_0 \right)

- \eta \delta J - i k \frac{1}{e n_e,0} \gamma k_B T_e n_e - \frac{1}{\mu_0 e n_e,0} i (\delta B \cdot B_0) k
\]

(5.52)

and reexpressing \( \delta J \) with Ampere’s law 5.17, we have

\[
\delta E' = \frac{- (k \lambda e)^2}{1 + (k \lambda e)^2} \left( i \frac{\eta}{\mu_0} k \times \delta B \right) - \frac{1}{1 + (k \lambda e)^2} \left[ 1 + \lambda_e^2 k k \right] \cdot \delta V_i \times B_0

- i k \frac{1}{e n_e,0} \gamma k_B T_e n_e - \frac{1}{\mu_0 e n_e,0} i (\delta B \cdot B_0) k
\]

(5.53)

To express 5.53 in cartesian coordinate, remembering \( B_0 = (0, B_0, 0) \), we will use

\[
\begin{cases}
\delta B = (0, \delta B_y, 0) = (0, \delta B, 0) \\
k = (k_\perp, 0, 0) = (k, 0, 0) \\
k \times \delta B = (0, 0, k_\perp \delta B_y) = (0, 0, k \delta B) \\
[1 + \lambda_e^2 k k] \cdot \delta V_i \times B_0 = B_0 \left( - \left( 1 + (\lambda_e k)^2 \right) \delta V_z, 0, \delta V_x \right)
\end{cases}
\]

(5.54)
For a gyrotropic distribution,

\[ \nabla f_0 = \left( \cos \theta \frac{\partial f_0}{\partial v_\perp}, \frac{\partial f_0}{\partial v_\parallel}, \sin \theta \frac{\partial f_0}{\partial v_\perp} \right) \]  

(5.55)

Combining the gyrotropic assumption 5.50 along with the expression 5.53 for \( \delta E' \), 5.54,

\[
g(\theta) = i k \frac{1}{B_0} \left( \frac{1}{e n_{e,0}} \gamma k B e \delta n_e + \frac{1}{\mu_0 e n_{e,0}} \delta BB_0 \right) \cos \theta \frac{\partial f_0}{\partial v_\perp} - \delta V_z \cos \theta \frac{\partial f_0}{\partial v_\perp} \\
+ \frac{\delta V_x}{1 + (\lambda_e k)^2} \sin \theta \frac{\partial f_0}{\partial v_\perp} + \frac{(k \lambda_e)^2}{1 + (k \lambda_e)^2} \left( \frac{i}{\mu_0 k} \frac{\delta B}{B_0} \right) \sin \theta \frac{\partial f_0}{\partial v_\perp}
\]

(5.56)

Defining,

\[
g \begin{cases} 
\gamma k B e &= m_e \ \frac{\gamma k B e}{e B_0} = \frac{v_{te}^2}{\omega_{ce}} \\
\frac{B_0}{\mu_0 e n_{e,0}} &= \frac{e B_0}{m_e} = \frac{m_e}{\mu_0 e^2 n_{e,0}} = \omega_{ce} \frac{e^2 \mu_0 n_{e,0}}{e^2} = \omega_{ce} \lambda_e^2 \\
\end{cases}
\]

(5.57)

and setting

\[
A \equiv \left[ ik \left( \frac{v_{te}^2}{\omega_{ce} n_{e,0}} + \omega_{ce} \lambda_e^2 \delta B - B_0 \delta V_z \right) \right] \\
B \equiv \frac{1}{1 + (k \lambda_e)^2} \left[ \delta V_z + (k \lambda_e)^2 \frac{i}{\mu_0 B_0} \right]
\]

(5.58)

Using 5.56 and 5.37 to eliminate \( \delta B \) from 5.58 yields for \( A \)

\[
A = \left[ ik \frac{v_{te}^2}{\omega_{ce} n_{e,0}} + \frac{i k \omega_{ce} \lambda_e^2}{\omega \left( 1 + (k \lambda_e)^2 \right)} \left( \frac{1}{\mu_0 B_0} \right) \delta V_z - \delta V_z \right]
\]

(5.59)

and likewise for \( B \),

\[
\delta V_x \left( \frac{1}{1 + (\lambda_e k)^2} \right) + \frac{(k \lambda_e)^2}{1 + (k \lambda_e)^2} \frac{i}{\mu_0} \left( \frac{\omega}{k} \left( 1 + (k \lambda_e)^2 \right) \right) \delta V_z
\]

(5.60)
\[ B = \frac{\delta V_x}{(1 + (\lambda_e k)^2)} \left[ \frac{\omega/k (1 + (k\lambda_e)^2) + i\eta k/\mu_0}{\omega/k (1 + (k\lambda_e)^2) + i\eta k/\mu_0} \right] \tag{5.61} \]

This reduces to
\[ B = \frac{\delta V_x}{\left( \frac{\omega}{k} (1 + (k\lambda_e)^2) + i\frac{\eta k}{\mu_0} \right)} \tag{5.62} \]

and simplifies to
\[ \frac{\delta V_x}{1 + (k\lambda_e)^2} \tag{5.63} \]

when \( \eta = 0 \). The expression 5.56 for \( g(\theta) \) becomes
\[ g(\theta) = A \cos \theta \frac{\partial f_0}{\partial v_{\perp}} + B \sin \theta \frac{\partial f_0}{\partial v_{\perp}} \tag{5.64} \]

With 5.49 and 5.56, the solution to Vlasov equation in the perpendicular case is
\[ \delta f(\theta) = A \frac{\partial f_0}{\partial v_{\perp}} \int_{-\infty}^{\theta} d\theta' \cos \theta' K(\nu,\kappa,\theta,\theta') + B \frac{\partial f_0}{\partial v_{\perp}} \int_{-\infty}^{\theta} d\theta' \sin \theta' K(\nu,\kappa,\theta,\theta') \tag{5.65} \]

since \( f_0 \) is independent of \( \theta \).

The solution \( \delta f(\theta) \) depends on its zeroth and first moments as \( \delta n_e, \delta V_x \), and \( \delta V_z \) enter the expressions of \( A \) and \( B \) in 5.65. In order to obtain the dispersion relation, we will successively integrate 5.65 over velocity space to bring relations that will give the inertial hybrid kinetic dispersion relation. In cylindrical coordinates, the volume element is given by \( d^3v = v_{\perp} dv_{\perp} dv_{||} d\theta \). Applying quasineutrality to zeroth and to first order, we have
\[ \frac{\delta n_e}{n_{e,0}} = Z \frac{\delta n_i}{Z n_{i,0}} = \frac{\delta n_i}{n_{i,0}} \tag{5.66} \]

and assuming in what follows that \( f_0 \) is normalized to unity instead of being normalized to \( n_{i,0} \), then the integral of \( \delta f(\theta) \) over velocity space gives
\[
\begin{align*}
\frac{\delta n_e}{n_{e,0}} &= A \int_{0}^{\infty} v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} dv_{||} \frac{\partial f_0}{\partial v_{\perp}} \int_{0}^{2\pi} d\theta \int_{-\sigma}^{\theta} d\theta' \cos \theta' K(\nu,\kappa,\theta,\theta') \\
&\quad + B \int_{0}^{\infty} v_{\perp} dv_{\perp} \int_{-\infty}^{\infty} dv_{||} \frac{\partial f_0}{\partial v_{\perp}} \int_{0}^{2\pi} d\theta \int_{0}^{2\pi} d\theta' \int_{-\sigma}^{\theta} d\theta' \sin \theta' K(\nu,\kappa,\theta,\theta')
\end{align*}
\]
The second relation is obtained by multiplying 5.65 by $v_\perp \cos \theta$ and by integrating over velocity space, such that

$$\delta V_x = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel \int_0^{2\pi} d\theta v_\perp \cos \theta \delta f (\theta)$$

$$\quad = A \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \frac{\partial f_0}{\partial v_\perp} \int_0^\theta d\theta \int_{-\sigma\infty}^{\theta} d\theta' \cos \theta \cos \theta' K (\nu, \kappa, \theta, \theta')$$

$$\quad + B \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \frac{\partial f_0}{\partial v_\perp} \int_0^\theta d\theta \int_{-\sigma\infty}^{\theta} d\theta' \cos \theta \delta f (\theta)$$

and the last relation is similarly obtained by multiplying 5.65 by $v_\perp \sin \theta$, namely,

$$\delta V_z = \int_0^\infty v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel \int_0^{2\pi} d\theta v_\perp \sin \theta \delta f (\theta)$$

$$\quad = A \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \frac{\partial f_0}{\partial v_\perp} \int_0^\theta d\theta \int_{-\sigma\infty}^{\theta} d\theta' \sin \theta \cos \theta' K (\nu, \kappa, \theta, \theta')$$

$$\quad + B \int_0^\infty v_\perp^2 dv_\perp \int_{-\infty}^{\infty} dv_\parallel \frac{\partial f_0}{\partial v_\perp} \int_0^\theta d\theta \int_{-\sigma\infty}^{\theta} d\theta' \sin \theta \sin \theta' K (\nu, \kappa, \theta, \theta')$$

By making repetitive use of the identity,

$$\int_0^{2\pi} d\theta \int_{-\sigma\infty}^\theta d\theta' e^{i\nu(\theta-\theta')} - i\kappa (\sin \theta - \sin \theta') e^{i\kappa \theta} e^{i\ell \theta'} = \sum_{m=-\infty}^{\infty} \frac{2i\pi}{(\nu - \ell - m)} J_m (\kappa) J_{m+k+\ell} (\kappa)$$

(5.70)
closed expressions which are derived in the appendix can be obtained for the following angular integrals, defined as

\[
L_1(\kappa, \nu) = L_{c\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \cos \theta \int_{-\infty}^{\theta} d\theta' \cos \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_2(\kappa, \nu) = L_{s\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \sin \theta \int_{-\infty}^{\theta} d\theta' \cos \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_3(\kappa, \nu) = L_{c\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \cos \theta \int_{-\infty}^{\theta} d\theta' \sin \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_4(\kappa, \nu) = L_{s\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \sin \theta \int_{-\infty}^{\theta} d\theta' \sin \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_5(\kappa, \nu) = L_{c\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \int_{-\infty}^{\theta} d\theta' \cos \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_6(\kappa, \nu) = L_{s\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \int_{-\infty}^{\theta} d\theta' \sin \theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_7(\kappa, \nu) = L_{c\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \cos \theta \int_{-\infty}^{\theta} d\theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_8(\kappa, \nu) = L_{s\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \sin \theta \int_{-\infty}^{\theta} d\theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

\[
L_9(\kappa, \nu) = L_{c\ell}(\kappa, \nu) = \int_0^{2\pi} d\theta \int_{-\infty}^{\theta} d\theta' e^{i\nu(\theta-\theta')-i\kappa(\sin \theta-\sin \theta')}
\]

(5.71)

The expressions of the integrals 5.71 and the derivation of 5.70 are found in the appendix. We need to compute integrals of the form,

\[
E_{\ell,m} \equiv \int_{0}^{\infty} dv_{\parallel} \int_{-\infty}^{\infty} v_{\parallel}^{n} dv_{\perp} L_{\ell} \left( \frac{k_{\perp} v_{\perp}}{\omega_{c}}, \frac{\omega}{\omega_{c}} \right) \frac{\partial f_{0}}{\partial v_{\perp}} (v_{\perp}, v_{\parallel}) \quad m = 1, 2, \ldots, 6 \quad \ell = 1, 2, \ldots, 6
\]

(5.72)

where \( L_{\ell} \) are defined in 5.71 for the particular case of \( f_{0} \) being the sum of an anisotropic Maxwellian,

\[
f_{M}(v_{\perp}, v_{\parallel}) = \frac{1}{\sqrt{\pi v_{\parallel 0}} \pi v_{\parallel 0}} e^{-v_{\parallel}^2/2v_{\parallel 0}^2} e^{-v_{\perp}^2/2v_{\perp 0}^2}
\]

(5.73)

with

\[
v_{\perp 0} = \sqrt{\frac{2k_{B} T_{\perp}}{m}}
\]

(5.74)
and a ring beam distribution

\[ f_B (v_\perp, v_\parallel) = \frac{1}{2\pi u_\perp} \delta (v_\parallel) \delta (v_\perp - u_\perp) \] (5.75)

We heavily rely on Watson formula [198] and derived expressions which we obtain in the appendix, and are summarized here,

\[ \int_0^\infty t \exp \left( -p^2 t^2 \right) J_n^2 (at) \, dt = \frac{1}{2p^2} \exp \left( -a^2 / 2p^2 \right) I_n \left( a^2 / 2p^2 \right) \] (5.76)

along with

\[ \int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_n (at) J_n' (at) \, dt = \frac{a}{8p^4} \exp \left( -a^2 / 2p^2 \right) \left( I_{n+1} \left( a^2 / 2p^2 \right) - 2I_n \left( a^2 / 2p^2 \right) + I_{n-1} \left( a^2 / 2p^2 \right) \right) \] (5.77)

and

\[ \int_0^\infty t^3 \exp \left( -p^2 t^2 \right) J_n^2 (at) \, dt = \frac{1}{2p^4} \exp \left( -a^2 / 2p^2 \right) \left[ I_n \left( a^2 / 2p^2 \right) \left( 1 - \frac{a^2}{2p^2} \right) + \frac{a^2}{4p^2} \left( I_{n-1} \left( a^2 / 2p^2 \right) + I_{n+1} \left( a^2 / 2p^2 \right) \right) \right] \] (5.78)

With the use of 5.70 along Watson formula 5.76 and derived expressions 5.77 and 5.78, the different element integrals can be calculated. The expressions are in the appendix. We can rewrite 5.67, 5.68 and 5.69

\[
\begin{align*}
\delta n_e / n_{e,0} &= A E_{5,1} + B E_{6,1} \\
\delta V_x &= A E_{1,2} + B E_{3,2} \\
\delta V_z &= A E_{2,2} + B E_{4,2}
\end{align*}
\] (5.79)

and unfolding the dependence on \( \delta n_e, \delta V_x \) and \( \delta V_z \) in \( A \) and \( B \) we have

\[
\begin{bmatrix}
M_{1,1} & M_{1,2} & M_{1,3} \\
M_{2,1} & M_{2,2} & M_{2,3} \\
M_{3,1} & M_{3,2} & M_{3,3}
\end{bmatrix} \begin{bmatrix}
\delta n_e / n_{e,0} \\
\delta V_x \\
\delta V_z
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\] (5.80)
from which the hybrid kinetic dispersion for perpendicular propagation is obtained by imposing

$$\det \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} = 0 \quad (5.81)$$

and the elements are given as

$$M_{1,1} = i k \frac{v_{te}^2}{\omega_{ce}} E_{5,1} - 1$$

$$M_{2,1} = i k \frac{v_{te}^2}{\omega_{ce}} E_{1,2}$$

$$M_{3,1} = i k \frac{v_{te}^2}{\omega_{ce}} E_{2,2}$$

$$M_{2,2} = \frac{ik \omega_{ce} \lambda_e^2 E_{5,1} + (\omega/k + i \eta k/\mu_0) E_{6,1}}{\omega/k \left(1 + (k \lambda_e)^2\right) + i \eta k/\mu_0}$$

$$M_{2,2} = \frac{ik \omega_{ce} \lambda_e^2 E_{1,2} + (\omega/k + i \eta k/\mu_0) E_{3,2}}{\omega/k \left(1 + (k \lambda_e)^2\right) + i \eta k/\mu_0} - 1 \quad (5.82)$$

$$M_{3,2} = \frac{ik \omega_{ce} \lambda_e^2 E_{2,2} + (\omega/k + i \eta k/\mu_0) E_{4,2}}{\omega/k \left(1 + (k \lambda_e)^2\right) + i \eta k/\mu_0}$$

$$M_{1,3} = -E_{5,1}$$

$$M_{2,3} = -E_{1,2}$$

$$M_{3,3} = -(E_{2,2} + 1)$$

## 5.6 Arbitrary propagation angle

In order to solve the arbitrary propagation angle scenario, we substitute the expression 5.42 for $\delta V_i \times B_0$ in the general expression for the electric field $\delta E'$, expressed by equation 5.53. The result is then used in the solution in Vlasov equation. We rewrite $\delta E'$

$$\delta E' = \frac{-(k \lambda_e)^2}{(1 + (k \lambda_e)^2)} \left( i \frac{\eta}{\mu_0} k \times \delta B \right) - \frac{1}{(1 + (k \lambda_e)^2)} \left[ 1 + \lambda_e^2 k k \right] \cdot \delta V_i \times B_0$$

$$- i k \frac{1}{e n_{e,0}} \gamma k_B T_e \delta n_e - \frac{1}{\mu_0 e n_{e,0}} i (\delta B \cdot B_0) k \quad (5.83)$$

and $\delta V_i \times B_0$: 

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\[ \delta V_i \times B_0 = - \frac{1}{\mu_0 e n_e, 0} i (\delta B \cdot B_0) k - \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{(k \times \delta B) \cdot B_0}{(k \cdot B_0)} k \]

\[ + \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) k \times \delta B + \frac{1}{\mu_0 e n_e, 0} i (k \cdot B_0) \delta B \quad (5.84) \]

Remembering \( k \cdot \delta B = 0 \), we find

\[ [1 + \lambda_e^2 k k] \cdot \delta V_i \times B_0 = \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) k \times \delta B + \frac{1}{\mu_0 e n_e, 0} i (k \cdot B_0) \delta B \]

\[ - k \left( 1 + (k \lambda_e)^2 \right) \left( \frac{1}{\mu_0 e n_e, 0} i (\delta B \cdot B_0) + \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{(k \times \delta B) \cdot B_0}{(k \cdot B_0)} \right) \]

(5.85)

Plugging expression 5.85 into 5.83 gives

\[ \delta E' = - \frac{(k \lambda_e)^2}{1 + (k \lambda_e)^2} \left( \frac{i \eta}{\mu_0} k \times \delta B \right) - i k \frac{1}{\mu_0 e n_e, 0} \gamma T_e k B \delta n_e - \frac{1}{\mu_0 e n_e, 0} i (\delta B \cdot B_0) k \]

\[ + k \left( \frac{1}{\mu_0 e n_e, 0} i (\delta B \cdot B_0) + \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{(k \times \delta B) \cdot B_0}{(k \cdot B_0)} \right) \]

\[ - \left( \frac{\omega}{k^2} + i \frac{1}{1 + (k \lambda_e)^2} \frac{\eta}{\mu_0} \right) k \times \delta B - \frac{1}{\mu_0 e n_e, 0} i (k \cdot B_0) \delta B \quad (5.86) \]

The term in \((\delta B \cdot B_0)\) cancel out and the term involving \((k \times \delta B)\) simplifies reducing 5.87 to

\[ \delta E' = - \left( i \frac{\eta}{\mu_0} + \frac{\omega}{k^2} \right) k \times \delta B - i k \frac{1}{\mu_0 e n_e, 0} \gamma T_e k B \delta n_e \]

\[ + k \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{(k \times \delta B) \cdot B_0}{(k \cdot B_0)} \]

\[ - \frac{1}{1 + (k \lambda_e)^2} \frac{1}{\mu_0 e n_e, 0} i (k \cdot B_0) \delta B \quad (5.87) \]
We set again $B_0 = (0, 0, 0)$, we will use 

$$
\begin{align*}
\mathbf{v} &= (v_\perp \cos \theta, v_\parallel, v_\perp \sin \theta) \\
\delta \mathbf{B} &= (\delta B_x, \delta B_\parallel, \delta B_z) \\
\mathbf{k} &= (k_\perp, k_\parallel, 0) \\
\mathbf{k} \times \delta \mathbf{B} &= (k_\parallel \delta B_z, -k_\perp \delta B_z, k_\perp \delta B_\parallel - k_\parallel \delta B_x) \\
\frac{(\mathbf{k} \times \delta \mathbf{B}) \cdot \mathbf{B}_0}{(\mathbf{k} \cdot \mathbf{B}_0)} &= -\frac{k_\parallel}{k_\parallel} \delta B_z
\end{align*}
$$

(5.88)

We supplement relation 5.87 with $\mathbf{k} \cdot \delta \mathbf{B} = 0$ in cartesian coordinate,

$$
(k_\perp, k_\parallel, 0) \cdot (\delta B_x, \delta B_\parallel, \delta B_z) = k_\perp \delta B_x + k_\parallel \delta B_\parallel = 0
$$

(5.89)

which can be used to eliminate $\delta B_x$ from $\mathbf{k} \times \delta \mathbf{B}$ in 5.88:

$$
\mathbf{k} \times \delta \mathbf{B} = \left( k_\parallel \delta B_z, -k_\perp \delta B_z, \frac{k^2}{k_\perp} \delta B_\parallel \right)
$$

(5.90)

with $k^2 = k_\perp^2 + k_\parallel^2$. The expression for $\mathbf{v} \times \delta \mathbf{B}$ using 5.88 is given by

$$
\mathbf{v} \times \delta \mathbf{B} = (v_\parallel \delta B_z - v_\perp \sin \theta \delta B_\parallel, -v_\perp \cos \theta \delta B_z + v_\perp \sin \theta \delta B_x, v_\perp \cos \theta \delta B_\parallel - v_\parallel \delta B_x)
$$

(5.91)

and with eq. 5.89 reduces to

$$
\mathbf{v} \times \delta \mathbf{B} = \left( v_\parallel \delta B_z - v_\perp \sin \theta \delta B_\parallel, -v_\perp \cos \theta \delta B_z - v_\perp \sin \theta \frac{k_\parallel}{k_\perp} \delta B_\parallel, v_\perp \cos \theta \delta B_\parallel + v_\parallel \frac{k_\parallel}{k_\perp} \delta B_\parallel \right)
$$

(5.92)

We unfold $\delta \mathbf{E}'$ using 5.88 and 5.90 in a similar way as we did in expression 5.54 when treating the perpendicular propagation case. The components of $\delta \mathbf{E}'$ are

$$
\begin{align*}
\delta E'_x &= -i \frac{1}{en_e,0} \gamma_k B_T e k_\perp \delta n_e + i \frac{k^2}{\mu_0 e n_e,0 k_\perp} \frac{B_0}{1 + (k \lambda_e)^2} \delta B_\parallel - \left[ i \frac{\eta}{\mu_0} \frac{k}{k_\parallel} + \frac{\omega}{k} \left( 1 + (\lambda_e k_\perp)^2 \right) \right] \delta B_z \\
\delta E'_y &= -i \frac{1}{en_e,0} \gamma_k B_T e k_\parallel \delta n_e - i \frac{k_\parallel B_0}{\mu_0 e n_e,0 \left( 1 + (k \lambda_e)^2 \right)} \delta B_\parallel - \frac{\omega}{k} \left( \lambda_e k_\perp \right)^2 \frac{k_\perp}{k} \delta B_z \\
\delta E'_z &= \left( i \eta k + \frac{\omega}{k} \right) \frac{k_\parallel}{k_\perp} \delta B_\parallel - i \frac{k_\parallel B_0}{\mu_0 e n_e,0 \left( 1 + (k \lambda_e)^2 \right)} \delta B_z
\end{align*}
$$

(5.93)
We recast eq. 5.93 as

\[ \delta E' = A_{x,1} B_0 \delta n_{e,0}^e + A_{x,2} \delta B + A_{x,3} \delta B_z \]

\[ \delta E'_y = A_{y,1} B_0 \delta n_{e,0}^e + A_{y,2} \delta B + A_{y,3} \delta B_z \]

\[ \delta E'_z = A_{z,1} B_0 \delta n_{e,0}^e + A_{z,2} \delta B + A_{z,3} \delta B_z \]

(5.94)

from which the coefficients \( A_{i,j}, 1 \leq i,j \leq 3 \) are identified. We can solve Vlasov equation using 5.44-5.49. The function \( g(\theta) \) 5.46 becomes with 5.92 and 5.93 along with the gradient of the distribution function 5.55

\[ g(\theta) = -\frac{1}{B_0} \left( \delta E_x \cos \theta \frac{\partial f_0}{\partial v_{\perp}} + \delta E_y \frac{\partial f_0}{\partial v_{\parallel}} + \delta E'_z \sin \theta \frac{\partial f_0}{\partial v_{\perp}} + v_{\parallel} \delta B_z \cos \theta \frac{\partial f_0}{\partial v_{\perp}} - v_{\perp} \cos \theta \frac{\partial f_0}{\partial v_{\parallel}} \delta B_z - v_{\perp} \sin \theta \frac{k_{\parallel}}{k_{\perp}} \delta B + v_{\parallel} \sin \theta \frac{k_{\parallel}}{k_{\perp}} \delta B \right) \]

(5.95)

One necessary relation to solve the hybrid kinetic dispersion is obtained by integrating the perturbed distribution function 5.49 over velocity space, as this leaves us with our first relation between \( \delta n_{e}, \delta B_{\parallel} \) and \( \delta B_z \), using 5.66.

\[ \frac{\delta n_{e}}{n_{e,0}} = \int_0^\infty v_{\perp} dv_{\perp} \int_{-\infty}^\infty dv_{\parallel} \int_0^{2\pi} d\theta \int_{-\sigma}^\infty d\sigma' K(\nu, \kappa, \theta, \theta') g(\theta') \]

(5.96)

We find two additional relations by using the solution of \( \delta V_i \times B_0 \), namely after developing 5.84 we find

\[ \delta V_{i,x} = \frac{i B_0}{\mu_0 e n_{e,0}} k_{\parallel} B_0 k_{\perp} \delta B + \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{k_{\parallel}^2 \delta B_{\parallel}}{k_{\perp} B_0} \]

(5.97)

\[ \delta V_{i,z} = \frac{i B_0}{\mu_0 e n_{e,0}} k_{\perp} B_0 k_{\parallel} \delta B - \left( \frac{\omega}{k^2} \left( 1 + (k \lambda_e)^2 \right) + i \frac{\eta}{\mu_0} \right) \frac{k_{\parallel}^2 \delta B_{\parallel}}{k_{\perp} B_0} \]

(5.98)

As in 5.93, we recast eq. 5.97 and 5.98:

\[ \delta V_{i,x} = B_{x,2} \frac{\delta B_{\parallel}}{B_0} + B_{x,3} \frac{\delta B_z}{B_0} \]

\[ \delta V_{i,z} = B_{z,2} \frac{\delta B_{\parallel}}{B_0} + B_{z,3} \frac{\delta B_z}{B_0} \]

(5.99)
and \( \delta V_{i,x} \) and \( \delta V_{i,z} \) are known by computing the first moments 5.49 of Vlasov equation in the \( x \) and \( z \) directions. Equations 5.96, 5.97 and 5.98 leave us with 3 relations for the three variables \( \delta n_\varepsilon/n_\varepsilon,0, \delta V_{i,x} \) and \( \delta V_{i,z} \). To compute the matrix elements of the linear relations, we define:

\[
E_{m,n;\perp}^{k,\ell}(\omega,k) \triangleq \int_{-\infty}^{\infty} dv^m \int_{0}^{\infty} v^p dv^m L_{k,\ell}(\nu,\kappa) \frac{\partial f_0}{\partial v^\perp}(v_\perp,v^\parallel) 
\]

with \( \nu \) and \( \kappa \) defined in Section 5.4. Expressions of \( L_{k,\ell}(\nu,\kappa) \) are given by relations 5.71 with \( k, \ell \) taking all the values of 1, c, s and of 1', c', s' as a reminder of 1, \( \cos \theta, \sin \theta \) and 1, \( \cos \theta', \sin \theta' \) respectively, in the associated integrals 5.71. The definition of \( E_{m,n;\parallel}^{k,\ell}(\omega,k) \) follows accordingly. This translates the linear relation into:

\[
\begin{bmatrix}
M_{1,1} & M_{1,2} & M_{1,3} \\
M_{2,1} & M_{2,2} & M_{2,3} \\
M_{3,1} & M_{3,2} & M_{3,3}
\end{bmatrix}
\begin{bmatrix}
\delta n_\varepsilon/n_\varepsilon,0 \\
\delta B_\parallel/B_0 \\
\delta B_\parallel/B_0
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(5.101)

whose elements are given by

\[
M_{1,1} = 1 + A_{x,1}E_{0,1;1,1}^{1,1} + A_{y,1}E_{0,1;1,1}^{1,1} \\
M_{1,2} = A_{x,2}E_{0,1;1,1}^{1,1} + A_{y,2}E_{0,1;1,1}^{1,1} + A_{z,2}E_{0,1;1,1}^{1,1} - \frac{k_\parallel}{k_\perp} (E_{1,1;2,1}^{1,1} - E_{1,1;1,1}^{1,1}) \\
M_{1,3} = A_{x,3}E_{0,1;1,1}^{1,1} + A_{y,3}E_{0,1;1,1}^{1,1} + A_{z,3}E_{0,1;1,1}^{1,1} - E_{c,c,1}^{1,1} + E_{1,c,c,1}^{1,1} \\
M_{2,1} = A_{x,1}E_{0,2;2,1}^{c,1} + A_{y,1}E_{0,2;2,1}^{c,1} \\
M_{2,2} = A_{x,2}E_{0,2;2,1}^{c,1} + A_{y,2}E_{0,2;2,1}^{c,1} + A_{z,2}E_{0,2;2,1}^{c,1} - \frac{k_\parallel}{k_\perp} (E_{c,c,3}^{c,c,1} - E_{1,2,1;1}^{c,c,1} + B_{x,2} \\
M_{2,3} = A_{x,3}E_{0,2;2,1}^{c,c,1} + A_{y,3}E_{0,2;2,1}^{c,c,1} + A_{z,3}E_{0,2;2,1}^{c,c,1} - E_{c,c,1}^{c,c,1} + E_{1,2,1;1} + B_{x,3} \\
M_{3,1} = A_{x,1}E_{0,2;2,1}^{c,c,1} + A_{y,1}E_{0,2;2,1}^{c,c,1} \\
M_{3,2} = A_{x,2}E_{0,2;2,1}^{c,c,1} + A_{y,2}E_{0,2;2,1}^{c,c,1} + A_{z,2}E_{0,2;2,1}^{c,c,1} - \frac{k_\parallel}{k_\perp} (E_{c,1,2;3}^{c,c,1} - E_{1,2,1;1}^{c,c,1} + B_{x,2} \\
M_{3,3} = A_{x,3}E_{0,2;2,1}^{c,c,1} + A_{y,3}E_{0,2;2,1}^{c,c,1} + A_{z,3}E_{0,2;2,1}^{c,c,1} - E_{c,c,1}^{c,c,1} + E_{1,2,1;1} + B_{x,3} \\
\]

(5.102)
Non trivial solutions leading to the inertial hybrid kinetic dispersion for finite propagation angle is obtained by forcing

\[
\det \begin{bmatrix}
M_{1,1} & M_{1,2} & M_{1,3} \\
M_{2,1} & M_{2,2} & M_{2,3} \\
M_{3,1} & M_{3,2} & M_{3,3}
\end{bmatrix} = 0 \tag{5.103}
\]

5.7 Summary

We have derived a hybrid kinetic dispersion relation which treats a one ion species plasma kinetically and includes an inertial electron fluid under the assumption of quasineutrality. Light waves are excluded of the model which is reasonable provided the speeds at play are much lower in comparison. We focused on purely perpendicular propagating wave as initially done in [191] and move on to propagation with finite \( k_\parallel \) as originally considered for massless electrons in [26]. The different derivation occurred from the division by \( \mathbf{k} \cdot \mathbf{B}_0 \) in the expression for \( \delta \mathbf{V}_i \times \mathbf{B}_0 \). We were interested in treating the perpendicular case given that the MCI is most unstable at that propagation angle. In addition, hybrid simulation benchmarks in the literature do not necessarily show the behaviour of the Alfvén wave at higher cyclotron frequency which is important for a hybrid treatment of the MCI.

Our approach directly enforced quasineutrality in Maxwell’s equations, it should be possible to consider the susceptibilities from an arbitrary number of species treated kinetically to which the susceptibilities corresponding to fluid species are added. The quasineutral limit would be obtained by imposing \( \epsilon_0 \) to be 0.

The expressions for the dispersion relations are standard, as found in the Appendix, and involve infinite sums of products of Bessel functions as a result of the identity 5.70 of Section 5.5. Interesting work has shown that the calculation of those infinite sums of products of Bessel functions to obtain the dielectric tensor of a hot magnetized plasma can be circumvented by the use of Newberger sum’s rules [199, 200]. We include the results in the Appendix. Recently, it was pointed that since the dispersion unfolds with the use Eq. 5.70, which brings expressions involving infinite sums which are later simplified with Newberger’s sum rule, it would be feasible to get around the infinite sums expansion and this was done in Qin et al. [201]. The authors obtained the kinetic dispersion relation for a hot magnetized plasma without infinite sums in their final result and no infinite sums appears in their derivation either [202, 203]. The matrix elements have been computed in closed forms for different distribution functions including ring-beams, drifting ring-beams, drifting-Maxwellian, and Maxwellian ring-beams [204]. Our sets of equations could
complement and fit in the hybrid-kinetic dispersion relation solver (HYDROS) [26].
Chapter 6

Solution of hybrid, quasineutral and fully kinetic dispersion relations

6.1 Introduction

Simulations of the relaxation of energetic ion distributions as a result of an inversion in velocity space for ring beams have been extensively carried out in both electromagnetic [19, 20, 22, 86, 205–209] and electrostatic cases [210] for parameters relevant to both fusion and space plasmas for frequencies ranging from the cyclotron frequency $\Omega_c$ up to the lower hybrid frequency $\omega_{lh}$. Related analytical dispersion relations were studied in [88, 89, 204, 211–215]. It is important to have models able to capture (low-frequency) electromagnetic plasma instabilities involving for example the aforementioned fast ion physics, on the timescales on which they unfold with enough spatial resolution. This is facilitated if the CFL condition for light waves is relaxed. The Darwin approximation [134] offers such a reduced model by neglecting the solenoidal part (with no divergence) of the displacement current [216]. From the continuity equation, the irrotational part of the displacement current cancels the irrotational part the total current which reduces Ampère’s law to [216]

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{sol} \quad (6.1)$$

Darwin’s model is light-waves free and retains both electrostatic and low-frequency inductive electric fields and requires to resolve $\omega_{pe}$ and the Debye length which is still a heavy constraint. This can be alleviated by enforcing quasineutrality which
gets rid of these scales to allow the use of much bigger grid size [197]. Plasma simulations are now combining a variety of models allowing to treat the electrons and the ions on very different ground. The hybrid approach is one example and a more recent one is the compelling combination of kinetic ions with gyro-kinetic electrons [217], applied to both space [218] and fusion plasmas [219]. Here we investigate the reliability of our hybrid simulations by solving the dispersion relation relevant to ICE in the edge plasma of LHD using different levels of description for a background proton Maxwellian and an energetic cold-ring proton beam. Comparative studies of different classes of plasma model for the same phenomenology are helpful, see for example the treatment of gyrokinetic, hybrid-kinetic and fully kinetic wave physics carried out in [195, 196]. Our treatment includes the solution of the massless and inertial electron hybrid kinetic dispersion relations, see equations 5.80 and 5.101 in chapter 5. We also solve the dispersion relation using the full dielectric tensor in the non relativistic case, for the same plasma parameters, in the quasineutral and non quasineutral cases. The diagram 6.1 illustrates the 3 models considered in this section: the fully kinetic and hybrid kinetic description along with an intermediate model called neutral Vlasov [139]. Alternatively, we could have considered a non quasineutral treatment with kinetic ions and fluid electrons. However, neutral Vlasov could turn to be a useful compromise between the kinetic and hybrid kinetic treatment aforementioned. Fully kinetic studies of the

![Figure 6.1: Diagram of the 3 models considered to solve the linearized ICE physics for parameters relevant to the edge of LHD.](image-url)
MCI in two spatial dimensions may still be computationally challenging especially if the full mass ratio between the electrons and ions is considered. Since a kinetic treatment of the electrons to retain electron Landau damping could be more adequate for a self-consistent 2D treatment resolving waves at every propagation angle, we include the neutral Vlasov treatment. This approach would relax the need to resolve the Debye length and focus on the resolution of the electron cyclotron motions (and beyond): the ratio of the Debye length to the electron Larmor radius \( \lambda_D/r_{L,e} = \Omega_e/\omega_{pe} \) is typically small for plasma parameters relevant to ICE. The neutral Vlasov description could also help us to distinguish the implications of the two main assumptions of the hybrid models adopted in this thesis: the plasma is quasi-neutral and the electrons are treated as a fluid (massless or inertial).

6.2 Hybrid, quasineutral and fully kinetic dispersion solutions

The top panels of Fig. 6.2 follows from the linearization of the hybrid kinetic set of equations 2.1, 2.2, 2.3 together with a generalized Ohm’s law 2.4 which has the following form:

\[
\omega_{pe}^2/c^2 + \nabla \times \nabla \times \sqrt{E} = \left[ \frac{e\mu_0 J_e \times B + \mu_0 \nabla \cdot (V_e J_e) - \frac{e\mu_0}{m_e} \nabla \cdot P_e + \nu \mu_0 J}{m_e} \right]
\]  

(6.2)

as derived in chapter 5. Here \( \omega_{pe}/c \) is the inverse electron skin depth and \( J_e, m_e, e, P_e \), are the current, mass, charge and (isothermal) pressure of electrons, respectively. The last term neglected hereafter corresponds to a drag, where \( \nu \) is the resistivity and \( J \) the plasma current. We have treated purely perpendicular propagating waves. Equations 2.1, 2.2, 2.3 and 6.2 allow the inclusion of finite electron inertia without the need for pushing the electron flow over time [141], which we derived in chapter 2, as its time evolution is all encapsulated in equation 6.2 [128]. Our work closely follows [26] in which a hybrid kinetic dispersion was recently obtained using the massless electron fluid momentum equation while our work uses a generalized expression 6.2 which retains electron inertia. Their hybrid kinetic dispersion relation has been applied to an anisotropic Maxwellian distribution in the finite \( k_\parallel \) propagation case. They have implemented a dedicated dispersion relation solver HYDROS which is open sourced [26] and benefits from advanced features such as following root evolution in parameter scans. We have only needed to treat a single ion species plasma since we are studying NBI protons in a hydrogen plasma.
The full kinetic dispersion relation is given by [220]

\[ \mathbf{n} \times (\mathbf{n} \times \mathbf{E}) + \mathbf{e} \cdot \mathbf{E} = 0 \quad (6.3) \]

of which underlying determinant is set to zero, with \( \mathbf{n} = \mathbf{k}c/\omega \) where the dielectric tensor \( \mathbf{e} \) is given in terms of the susceptibilities \( \chi_s \), \( s \) denoting the different species, as

\[ \mathbf{e}(\omega, \mathbf{k}) = 1 + \sum_s \chi_s(\omega, \mathbf{k}) \quad (6.4) \]

themselves found to be

\[ \chi_s = \frac{\omega_{ps}^2}{\omega} \int_0^\infty 2\pi v_{\perp} dv_{\perp} \int_{-\infty}^\infty dv_{\parallel} \left[ \mathbf{e}_\parallel \mathbf{e}_\parallel \frac{\Omega_s}{\omega} \left( \frac{1}{v_{\parallel}} \frac{\partial f_0}{\partial v_{\parallel}} - \frac{1}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} \right) v_{\parallel}^2 \right. \\
+ \left. \sum_{n=-\infty}^\infty \frac{v_{\perp} U}{\omega - k_{\parallel} v_{\parallel} - n\Omega} T_n \right] \quad (6.5) \]

with \( J_n = J_n(z) \), the Bessel function of the first kind and \( z = k_{\perp} v_{\perp}/\Omega_c \),

\[ T_n = \begin{bmatrix} \frac{n J_n^2}{z} & \frac{in J_n J_n'}{z} & \frac{n J_n^2}{z} \\
\frac{in J_n J_n'}{z} & \frac{(J_n')^2}{z} & -\frac{in J_n J_n'}{z} \\
\frac{n J_n^2}{z} & \frac{in J_n J_n'}{z} & \frac{J_n^2}{z^2} \\
\end{bmatrix} \quad (6.6) \]

with

\[ v_{\perp} U = v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} + \frac{k_{\parallel} v_{\perp}}{\omega} \left( v_{\parallel} \frac{\partial f_0}{\partial v_{\parallel}} - \frac{\partial f_0}{\partial v_{\perp}} \right) \quad (6.7) \]

In the specific case of perpendicular propagation, Eq. 6.3 reduces to a 2 × 2 determinant (top left corner of \( T_n \)). The susceptibilities 6.4 for a non-drifting isotropic background ions and electrons Maxwellian \( 1/(\sqrt{\pi}v_{0,\parallel}) 1/(\pi v_{0,\perp}^2) e^{-v_{\perp}^2/v_{0,\perp}^2 - v_{\parallel}^2/v_{0,\parallel}^2} \) are expressed with \( \lambda = k_{\perp} v_{0,\perp}^2 / 2 \Omega_{cs}^2 \) as

\[ \chi_s = \frac{\omega_{ps}^2}{\omega} \sum_{n=-\infty}^\infty \exp(-\lambda) \left[ \frac{n^2 I_n}{\lambda} A_n + \frac{-in (I_n - I_n')}{\lambda} A_n \right] \quad (6.8) \]
with \( I_n = I_n(\lambda) \) the modified Bessel function of the first kind while for a cold ion energetic ring beam \( 1/(2\pi u_\perp) \delta(v_\parallel) \delta(v_\perp - u_\perp) \) denoted as species \( r \), we have

\[
\chi_r = -\frac{\omega^2}{\omega} \sum_{\ell=-\infty}^{\infty} \frac{1}{\omega - i\Omega_r z_r} \begin{bmatrix}
-\ell^2 \frac{dJ_\ell^2}{dz_r}
\frac{d}{dz_r} (z_r J_\ell J'_\ell)
-i\ell \frac{d}{dz_r} (z_r^2 J'_\ell^2)
\end{bmatrix}, z_r = k_\perp u_\perp/\Omega_{cr}
\]

(6.9)

with \( J_\ell = J_\ell(z_r) \). More general expressions for anisotropic drifting-Maxwellians at arbitrary propagation angles can be found in [220] and for ring beams in [89]. The bottom panels of Fig. 6.2 correspond to the solution of Eqs. 6.3, 6.4 and of Eqs. 6.8 and 6.9.

The kinetic dispersion expression in the quasineutral approach is directly obtained by dropping the unit term from the dielectric tensor in expression 6.4 [139]. Quasineutrality can be found by taking the limit \( \epsilon_0 \to 0 \) in Maxwell’s equations: \( \lim_{\epsilon_0 \to 0} \epsilon_0 \nabla \cdot E = \rho = 0 \), and also leads to Darwin’s approximation since Ampère’s law reduces to \( \nabla \times B = \mu_0 J \). Taking a closer look at Eq. 6.3 shows that the first term \( \mathbf{n} \times (\mathbf{n} \times \mathbf{E}) \) is proportional to \( 1/\epsilon_0 \) due to the \( c^2 \) factor while the second term \( \mathbf{e} \cdot \mathbf{E} \) also contains a \( 1/\epsilon_0 \) factor due to the plasma frequency dependence \( \omega_{ps} \) in \( \chi_s \). Thus, factoring \( 1/\epsilon_0 \) out of Eq. 6.3 and taking its limit to 0 is equivalent to solving the same equation 6.3 with \( \epsilon \) reduced to [139]

\[
\epsilon(\omega, \mathbf{k}) = \sum_s \chi_s(\omega, \mathbf{k})
\]

(6.10)

Hence the middle panels of Fig. 6.2 are solutions of Eqs. 6.3, 6.10 and of Eqs. 6.8 and 6.9.

The aforementioned equations are solved numerically in the simplest possible way. We discretize \((\omega_R + i\omega_I, k)\) space in units of \( \Omega_c \) for \( \omega \) and in units of the ion skin depth \( \Omega_c/V_A \), where \( \omega_R \) and \( \omega_I \) denote the real and imaginary parts of \( \omega \). We are concerned with modes unstable under the MCI in the range \( \omega_R = \Omega_c \) up to \( \omega_R \approx 15 - 25\Omega_c \) and beyond, up to the lower hybrid frequency \( \omega_{lh} \approx 35 - 40\Omega_c \). Typically \( \Delta\omega_R = 0.05\Omega_c \) and \( \Delta k = 0.05\Omega_c/V_A \). Thus, the real frequencies are gridded: \((m\Delta k, n\Delta\omega_R), 0 \leq m \leq M_{max}, 0 \leq n \leq N_{max}\), with \( M_{max} = 45/0.05, N_{max} = 35/0.05 \). We also increment the imaginary part \( \omega_I \), whose values need to be guessed to obtain solutions to the dispersion relations as these solutions strongly depend on the plasma parameters, such as the beams relative density \( \xi \) etc. The range of \( \omega_I \) usually lies \( \in [0, 0.4]\Omega_c \) and is above \( \Omega_c \) for extreme beam densities. We set \( \Delta\omega_I = 0.002\Omega_c \) which leads us to evaluate \( \epsilon(m\Delta k, n\Delta\omega_R + \ell\Delta\omega_I) \),

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$0 \leq \ell \leq 0.4/0.002$. All the operations involving $m$ and $n$ are performed at once using matrix operations in MATLAB, and repeated for each different value $\ell$ over which a loop is performed. Once the susceptibilities are computed, they are plugged into the matrix elements of 6.3, 5.80 and 5.101. The matrix determinant is computed and must evaluate to zero to bring a solution. Since the determinant is a complex expression, we calculate its modulus $M = M_{m,n,\ell}$. We set up a threshold value $\text{thres}$ which depends on the problem and is equal to 0.005 for the graphs presented here. When $M_{m,n,\ell} < \text{thres}$, we plot $(m\Delta k, n\omega_R)$ for the real solution in blue, along with $(m\Delta k, \ell\Delta \omega_I)$ for the imaginary part in red. The latter corresponds to the growth rate and tells us where instability occurs and with which magnitude. Computing the full dispersion $D(\omega, k) = 0$ with expressions 5.80 or 5.101, which corresponds to the determinant of the matrix 6.3, allows us to choose arbitrary beam densities $\xi$ values since we do not assume that $\gamma = \omega_I < \omega_R$ to use the approximation that

$$D(\omega_R + i\omega_I, k) = 0$$

and then to take the imaginary part of Eq. 6.11 which yields [102, 221]

$$\gamma = \omega_I = -D_I (\omega_R, k) \left[ \frac{\partial D_R}{\partial \omega} (\omega_R, k) \right]^{-1} \quad (6.12)$$

Hence resonances take place at arbitrary locations of $(\omega, k)$ space. In all three panels of Fig. 6.2, we observe the same structures for the unstable modes. The sub-Alfvénic dispersion (LHD plasma 79126 plasma parameters, left panels) feature no linear instability over a wide range of $k$ values which are followed by 5 unstable modes starting at $k \approx 31-33\Omega_H/V_A$. The maximum growth rate values agree well in all three models, increasing first and then decreasing with the maximum peak taking value $\gamma = 0.35\Omega_H$ at $k = 37\Omega_H/V_A$ for the inertial hybrid and neutral dispersion. The fully kinetic dispersion has its maximum growth at $k = 35\Omega_H/V_A$ and the discrepancy seems to be a result of the departure from the cold plasma dispersion relation in the 2 former cases, which is indicated on all graphs by the thick dark crosses. Thus the mode structure is the same in all 3 models but is slightly offset in the quasineutral cases (top and middle left panels). The unstable mode at the bottom left of Fig. 6.2 has an unstable mode at $k = 44\Omega_H/V_A$ which is also captured by the hybrid kinetic and neutral dispersions at $k = 46\Omega_H/V_A$ outside of the range of the graphs. The right panels of Fig. 6.2 show the same graphs for a super-Alfvénic ring proton beam (LHD 79003 plasma parameters). The structure
of the unstable modes is the same for all 3 right panels. Some gaps appear in the unstable mode structure and a closer look show that modes $\ell = 21, 22$ are linearly stable and this is verified in simulations, see Fig. 6.5. As for the sub-Alfvénic results, we observe again a mild discrepancy between the cold plasma dispersion relation and the real solution of the dispersions in the quasineutral instances (top and middle right panels) leading to a small shift of the unstable modes in $k$ space: while the fully kinetic dispersion (bottom right) feature 2 unstable modes at $k = 42\Omega_H/V_A$ and $k = 43\Omega_H/V_A$ at $\omega = 29\Omega_H$ and $\omega = 30\Omega_H$ respectively, the two top right panels show that the corresponding unstable modes occur at $k = 43\Omega_H/V_A$ and $k = 44\Omega_H/V_A$ with $\omega = 30\Omega_H$ and $\omega = 31\Omega_H$ respectively. Finally the top right graph includes an unstable mode at $k = 22\Omega_H/V_A$ ($\omega = 20\Omega_H$) which is not well resolved in the quasineutral fully kinetic dispersion (middle right panel) and in the fully kinetic (bottom right), but if it is unstable, it is at a smaller value of $\gamma$. An inertial hybrid kinetic simulation run with the same parameters confirm growth, but indeed smaller than predicted by the linear hybrid kinetic dispersion solution, see top left corner in Fig. 6.5. The mode with $k = 29\Omega_H/V_A$ has a higher $\gamma$ in the quasineutral graphs than in the fully kinetic bottom right panel of Fig. 6.5. Those plots were obtained by summing all Bessel function terms with index between $n = -100$ and $n = 100$.

Figure 6.3 illustrates the influence of the number of terms summed in the dispersion solution. In order to capture the growth (or stability) of a mode $\ell$, the indices $m$ of the terms summed must lie in the neighbourhood of $\ell$. The appearance of stable modes (gaps) can be thought to result from the summation of complex numbers which interfere constructively or destructively and as such, it might be important to sum enough terms that encapsulates this behaviour. Having examined the results of the inertial hybrid kinetic dispersion relation (solution of Eq. 5.80 in chapter 5) with the fully kinetic descriptions, we now compare the growth rates computed from inertial hybrid-PIC simulations with their analytical counterparts presented on the top panels of Fig. 6.2. The simulations are both run with a magnetic field $B_0$ perpendicular to the 1D simulation domain. The right panels of Fig. 6.5 are the spatiotemporal fast Fourier transform of $\delta B_z$ over the simulation duration ($12\tau_H$ and $10\tau_H$ for LHD plasma 79003 and 79126 respectively), after the instability saturates, plotted on a $\log_{10}$ scale. The growth rates have been computed from the simulations, using the technique described in Section 3.6. For both simulations, 500 fits are taken over the intervals $[2.5 - n \times 2.5/500, 2.5 + n \times 2.5/500] \tau_H$ and $[2.25 - n \times 1.25/500, 2.25 + n \times 1.25/500] \tau_H$, $1 \leq n \leq 500$, for LHD plasma 79003 and 79126 respectively. These fits are performed on the spatial Fourier trans-
Figure 6.2: Solution of kinetic dispersion relations in different approximations for the case of a Maxwellian thermal proton distribution with a fast minority NBI-proton ring beam. The top panels are solutions of the linearized inertial electron hybrid kinetic equations 2.1, 2.2, 2.3 and 6.2. The middle panels are the solution of the quasineutral kinetic dispersion given by equations 6.3, 6.4 and of eqs. 6.8 and 6.9 and likewise for the bottom panels which use Eq. 6.4, $\epsilon = 1 + \sum s \chi s$ instead of Eq. 6.10. The left panels are for sub-Alfvénic LHD plasma 79126 edge plasma parameters, $\xi = 0.0005$ and the right panels are for super-Alfvénic LHD plasma 78803, $\xi = 0.00075$ (right). The propagation angle between $B_0$ and $k$ is $90^\circ$ in both cases. The real part $\omega$ corresponds to the blue trace while the red curves are the associated growth rates $\gamma$, both normalized to the proton cyclotron frequency. In each case, the Bessel function (for the ring-beam) and the modified Bessel function terms (for the Maxwellian) are summed between $m = -100$ and $m = 100$. 

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Figure 6.3: Solution of the linearized electron inertial hybrid kinetic equations 2.1, 2.2, 2.3 and 6.2 for the case of a Maxwellian thermal proton distribution with a fast minority NBI-proton ring beam, for super-Alfvénic LHD plasma 79003, $\xi = 0.00075$ (right). The propagation angle between $B_0$ and $k$ is $90^\circ$ for each graph. The real part $\omega$ corresponds to the blue trace while the red curves are the associated growth rates $\gamma$, both normalized to the proton cyclotron frequency. For each panel, a variable finite number of terms are retained in the dispersion relation series: the bessel function (for the ring-beam) and the modified bessel function terms (for the Maxwellian) are summed between $m = -n$ and $m = n$. 

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form of $\delta B_z$ for every resolved $k$-mode lying between 0 and $45\Omega_H/V_A$. The average and variance of these empirically computed growth rates define their assigned value and their error which correspond to the grey and black traces of the right panels. They are superimposed to the inertial hybrid dispersion relation on the left panels. The numerically computed dispersion relations can be altered by change of the cell size, this is the reason why the numerically computed growth rates are slightly offset by $\Delta k = 1.15\Omega_H/V_A$ for LHD plasma 79126 and $\Delta k = 0.22\Omega_H/V_A$ for LHD plasma 79003. The cell size of these simulations are $\Delta x = 4.4 \times 10^{-4}m$ ($3.1 \times 10^{-4}m$) for LHD plasma 79126 (79003), and the electron skin depths are $16.8 \times 10^{-4}m$ ($23.8 \times 10^{-4}m$). For comparison, the massless simulations presented in Table 3.1 had cell sizes $\Delta x = 19.0 \times 10^{-4}m$ (79126) and $\Delta x = 25.9 \times 10^{-4}m$ (79003), namely just above the electron skin depth, while cell sizes are 4 to 8 times smaller in electron inertia simulations. The cell sizes of the simulations with electron inertia are 4 to 8 times smaller, enabling them to resolve higher $k$-values along with electron skin depth.

We have compared the growth rates in the super-Alfvénic regime of LHD plasma 79003 for 2 additional beam concentrations as shown in Fig. 6.4. Excitations of Bernstein modes appear for $\xi = 7.5 \times 10^{-3}$ and the graph suggests that $k$-modes become unstable at more than one cyclotron harmonics: at cyclotron harmonics supported by the background plasma along the Alfvén waves or by Bernstein modes directly supported by the energetic ring beams. Additional work would be necessary to identify and to map to which cyclotron harmonics different growth rates belong when they occur at the same $k$. The maximum growth rates for $\xi = 7.5 \times 10^{-5}$, $\xi = 7.5 \times 10^{-4}$ and $\xi = 7.5 \times 10^{-3}$ are $\sim 0.08\Omega_H$, $\sim 0.26\Omega_H$ and $\sim 0.85\Omega_H$ respectively. This corresponds to a relative increase of about $\sqrt{10}$ when $\xi$ increases by a factor of 10 and suggests that we verify the MCI growth rate beam density scaling $\propto \sqrt{\xi}$ [76] as in Section 3.6.

Figure 6.4: Inertial hybrid kinetic dispersion relation for LHD 79003 plasma parameter. The relative beam densities $\xi$ are (from left to right), $\xi = 7.5 \times 10^{-5}$, $\xi = 7.5 \times 10^{-4}$ and $\xi = 7.5 \times 10^{-3}$.
Figure 6.5: Inertial electron hybrid kinetic dispersion relation of a proton plasma consisting of a background Maxwellian and of a minority energetic ring-beam for purely perpendicular wave propagation. Top and bottom panels correspond to super-Alfvénic LHD 79003 ($\xi = 0.00075$) and to sub-Alfvénic LHD 79126 ($\xi = 0.0005$) edge plasma parameters respectively, as found in Table 3.1. The left panels show the numerically computed analytical solutions of the linearized equations of the inertial electron hybrid kinetic model, contained in equation 5.80 and for which matrix elements are given in Section A.9 and A.10. Both the real (blue trace) and imaginary part (red trace) of $\omega$ are computed and show linear instability at sequential cyclotron harmonics. The right panels display the power of the spatio-temporal fast Fourier transform of $\delta B_z$ obtained from inertial electron hybrid-PIC simulations, plotted on a log_{10} scale. The yellow curves (real part of $\omega$) are supplemented by grey curves which correspond to the linear growth rates of the MCI inferred from the same simulations. The numerically computed growth rates have been superimposed to the linear dispersions on left panels and plotted with their errors in dark.
The top panels of Fig. 6.6 correspond to the full dielectric tensor solution [18] of the linearised Vlasov equation for magnetized plasmas [194, 220] for the case of a background thermal proton and electron Maxwellian population and a fast-ion minority ring beam distribution. This solution was obtained numerically by S.A. Irvine using a new code he devised [18]. There is an unrestricted choice of the distribution functions, and the code is able to sample any prescribed one for each of an arbitrary number of ion species composing the plasma, and for the electrons. The code is expressed in the local approximation and performs all the integrals numerically. To obtain the dispersion relation, the wavenumber space is marginalised (kept as a parameter) and the dielectric tensor is evaluated multiple times at discretized values of the 2D (complex) ω-space, for a given propagation angle between \( \mathbf{k} \) and \( \mathbf{B}_0 \). The minima of the moduli output are located and a steepest gradient method algorithm is applied to precisely locate the roots of the dielectric tensor. The procedure is repeated over the range of \( k \)-values assigned to cover the required values of (\( \omega, k \)) -space. The code is written in C++, is parallel and possesses a python wrapper. A fully-relativistic version is available. The full-dielectric tensor has been solved for the JT-60U edge FP \(^3\)He distribution and growth for the MCI was obtained. The left panel of Fig. 6.6 corresponds to edge parameters of sub-Alfvénic LHD plasma 79126 and the right panel corresponds to the edge parameters of super-Alfvénic LHD plasma 79003 for proton neutral beam densities \( \xi = 0.5 \times 10^{-4} \) and \( \xi = 0.75 \times 10^{-4} \) respectively. The angle between \( \mathbf{B}_0 \) and \( \mathbf{k} \) is 89.5°.

The lower panels are solutions of the hybrid dispersion with finite \( k_\parallel \), Eq. 5.101 in chapter 5. As discussed in Fig. 6.2, a discrepancy still exists between the computed dispersion and the cold plasma dispersion, represented by the dark crosses. The electron skin depth \( \lambda_e \) explicitly appears in Eq. 5.101 and is set to its exact value \( c/\omega_{pe} \) on the bottom panels of Fig. 6.6. On the contrary, \( \lambda_e \) is artificially altered on the middle panels of Fig. 6.6 to \( 1.10 \times c/\omega_{pe} \). The computed dispersion follows the cold plasma dispersion and the unstable mode locations are in good agreement with the full kinetic numerically-computed solutions. We mentioned that the cell size influences the simulated dispersion relation. It could then also be affected by modifying the skin depth in the electric field solver of the hybrid code, which is interesting if this slight modification reproduces exactly the full kinetic dispersion. The growth rates are higher in the sub-Alfvénic regime in the hybrid case. This can be due to the fact that 40 Bessel function terms were summed in the fully kinetic dispersion while the hybrid one included 100 terms. Conversely, the growth rates peak at higher values between \( k = 12 \) and \( k = 25\Omega_c/V_A \) for the full kinetic
solution and again at lower values for $k$ above $40\Omega_H/V_A$ when compared with the hybrid solutions. The small thermal spread amounting to the background plasma temperature set in the ring used in the code [18] could also account for these different observed features.
Figure 6.6: Top: Solution of the full dielectric tensor generated by S.I. Irvine obtained with the code [18] for sub-Alfvénic LHD plasma 79126, $\xi = 0.0005$ (left) and super-Alfvénic LHD plasma 79003, $\xi = 0.00075$ (right). The propagation angle between $B_0$ and $k$ is 89.5° in both cases. The black and red curves correspond to the real and imaginary part of $\omega$ respectively, each normalized to the proton cyclotron frequency. The middle and bottom panels are solutions to Eq. 5.101 for the same parameters. The middle panels use an artificially modified electron skin depth $\lambda_e = 1.10 \times c/\omega_{pe}$, while the bottom panels use $\lambda_e = 1.00 \times c/\omega_{pe}$. The blue and red traces are the real and imaginary parts of $\omega$. The black crosses correspond to the cold plasma dispersion.
6.3 Summary

We have computed the dispersion solutions in 3 different models: hybrid-kinetic, quasineutral fully kinetic and fully kinetic in two scenarios: the sub- and super-Alfvénic LHD plasma 79126 and 79003, whose parameters are described in Table 3.1. The solutions were obtained for perpendicular propagation (or very close to it: $89.5^\circ$) which involve the two different calculations carried out in chapter 5 leading to equations 5.80 and 5.101. The locations and values of the growth rates agreed well between the different approaches, with slight discrepancies at higher cyclotron harmonics where the dispersion branch curves down towards $\omega_{ih}$. All 3 models capture the gaps in the growth rate structure of the super-Alfvénic plasma (harmonics 21 and 22 were found to be stable). This is confirmed by inertial hybrid kinetic simulations. We checked that a sufficient number of Bessel function terms need to be summed to obtain reliable solutions, which has to be of the order the cyclotron harmonics being studied. We then briefly looked at the growth rate behaviour with increased and decreased beam density. We observed that several harmonics at the same $k$-value can go unstable at high beam density $\xi$. The hybrid dispersions were calculated at $89.5^\circ$ and compared with the fully kinetic dispersion solver developed by S.A. Irvine [18]. The hybrid solutions mildly deviate from the cold plasma dispersion which might be a consequence of quasineutrality and this can be circumvented by artificially altering the electron skin depth parameter used in the hybrid solution. The agreement is good overall. Additional work should involve solutions away from purely perpendicular propagation along with beams which have a finite spread. The results show that the quasineutral kinetic model could be a promising way to incorporate Landau damping whilst still be able to model ICE at cheaper computational cost than required by a fully kinetic scheme.
Chapter 7

Conclusion

The study and interpretation of ion cyclotron emission (ICE) measurements reported from the Large Helical Device (LHD) is the focus of the present thesis. In hydrogen plasmas, ICE was reported during toroidal Alfvén eigenmodes and during perpendicular neutral beam injections. Part of this work focused on modelling the latter. In addition, the first deuterium plasmas were created during the LHD 2017 campaign with the possibility of observing fusion products. Fourier analysis of ICE data collected during neutral beam injection showed variation in their spectral peaks when compared between different LHD discharges. During bursty events, the ICE spectral peaks were Doppler-shifted. We hypothesize that it be could accounted for by fusion-born protons having a non-negligible parallel velocity along the magnetic field.

We reviewed the ICE measurements as reported over the years in large tokamaks and described their origins: fusion products ICE, NBI-driven along with mICE which results from minority ICRH accelerated fast ions. Central ICE has recently been observed in ASDEX-U [46] and in DIII-D [45] and its origin is not clear. We presented the analytical theory of the magneto acoustic cyclotron instability (MCI), thought to be the underlying mechanism responsible for ICE. This was followed by an account of the modelling work which includes solution of the (full)-dielectric tensor and related dispersion relations for linear growth rate analysis, Fokker-Planck calculation of energetic ion distributions and simulation of its time evolution along with fully self-consistent particle-in-cell (PIC) simulations.

We presented the hybrid approximation which treats the ions fully kinetically while the electrons are modelled as a neutralising fluid. A derivation of the generalized Ohm’s law was given for massless and inertial electrons. The numerical schemes of the hybrid code PROMETHEUS++, developed by Carbajal Gomez
are described: evolution of the ions dynamics along with the charge assignment scheme. We detailed the SOR linear solver used for the electric field calculation which allows to retain electron inertia, implemented during this thesis. The code assumes quasineutrality, $V_A^2/c^2 \ll 1$, is highly parallelized and is well suited to simulate plasma phenomena which unfold in the ion cyclotron range of frequencies.

We have studied two LHD plasmas characterized by NBI protons that are sub-Alfvénic and super-Alfvénic in the outer plasma edge at the ICE emission location. We have run hybrid simulations in both instances. The NBI proton populations were initialized with a ring beam in velocity space at their birth velocity $v_{NBI}$, 40keV and 36.5KeV respectively, perpendicularly to the local magnetic field. These relaxed in a background Maxwellian plasma and saturated the magneto acoustic cyclotron instability. The spectral power in the fluctuating part of the magnetic field showed peaks at multiple proton cyclotron harmonics, for waves propagating at $85^\circ$ as well as close to perpendicular with respect the local magnetic field, yielding initial proxy of the measured ICE spectra. The linear growth rates inferred from simulations display square root scaling with the NBI proton density as obtained in the linear analytical theory of the MCI. We have also tested the sensitivity of the MCI by includings thermal spread in the ring-beam and found that instead of peaking near the lower hybrid frequency, the ICE power was more evenly distributed across proton cyclotron harmonics.

We have turned to ICE observed during the deuterium campaign. We first looked at power spectra displaying peaks at successive cyclotron harmonics, whose spacing corresponds to the deuterium cyclotron frequency calculated in the edge plasma, during neutral beam injection. The spectral features are different in different plasmas: the number of excited harmonics, their amplitude and their width increase with the edge electron density. We reinterpret this as a variable ratio $v_{NBI}/V_A$. Several hybrid PIC simulations have been run with the relevant LHD edge plasma parameters to relax the fast NBI deuterium population. The simulations, which ultimately differ in their $v_{NBI}/V_A$, produce spectra which capture the qualitative changes.

During abrupt events in LHD, intense peaks at higher cyclotron harmonics, thought to be the signature of fast protons were reported. They display a significant Doppler-shift and we have tested the hypothesis that 3.02MeV fusion-born protons with a large parallel velocity along the magnetic field could drive the MCI and generate similar spectra. We have thus inferred that these protons have Alfvénic speeds perpendicular to the local magnetic fields to induce ICE. Consequently the remaining energy is set in the parallel direction $v_\parallel$. Hybrid simulations of the MCI for near
perpendicular wave propagation, with 3.02MeV protons initialized as ring beams,
display the strongest excitations at cyclotron harmonics in broad agreement with
the measurements along with appropriate Doppler-shift frequencies. The measured
spectra are extremely rich as they combine numerous discrete peaks with continu-
ous structures. We think that several ion species generate these spectra. As such it
is difficult to identify which ion species is responsible for any given spectral peak,
especially when few harmonics are present. When the power spectrum results from
multiple ion species, it is not clear if they drive ICE at the same plasma position. We
have performed cepstrum analysis to identify the periodicity in the measured power
spectrum which in principle can tell us which ion species are present. Again, this
holds true assuming the emission occurs at locations characterized by the same back-
ground magnetic field magnitude from which cyclotron harmonics is inferred. We
mostly identified harmonics which we assume are from protons, along with harmon-
ics of deuterium due to NBI. More work is necessary to rule out 170keV tangential
neutral beam protons and to identify the various ion species at play.

In the remaining chapters, we have obtained the hybrid kinetic dispersion
relation retaining electron inertia with a simple isotropic pressure, adiabatic or
isothermal. We have solved the equations for a Maxwellian background plasma
along with a proton ring beam and compared the results with the fully-kinetic and
with the neutral-kinetic dispersions. Good agreement between the three approaches
is obtained with slight deviations from the cold plasma dispersion relations at higher
cyclotron harmonics in the neutral models. We suggested that neutral-kinetic could
be a good avenue for future simulations of the MCI, particularly 2D. The equations
could be introduced in the HYDROS code [26].

7.1 Future work

Before embarking on a 3D3V extension of PROMETHEUS++ with additional ge-
ometry, an intermediate step would be its extension to the 2D3V case with inclusion
of particle replenishment [131] and ion pressure tensor. Simulations of the MCI in
2 spatial dimensions would encompass more physics, allowing interactions between
parallel and perpendicular wave modes. Neutral-kinetic, which consists kinetic elec-
trons and kinetic ions while enforcing quasineutrality, would be interesting to use
as it would retain electron physics and still alleviate resolution of the Debye length.

Inspection of additional LHD spectra (and time series) from transient events
using cepstrum analysis might help create a database and give more insight into
ICE and the underlying ion species that drive it.
To use ICE as a non intrusive diagnostic of energetic ions, a broader range of parameter space would need to be explored. For example, the study could look at multiple wave propagation angles, which 2D simulations could do at once, while studying the effects of different fast ion distribution, jointly with the impact of varying magnetic field and background density. A map between the fast ion distribution and the simulated power spectra could be built. The measured spectra could then infer properties of the underlying distribution function as suggested by Salewski et al. [171]. Modelling of ICE could be boosted by the HALO code developed at Culham which tracks full-particle orbits in toroidal geometry [172].
Appendix A

Integral computations

A.1 Maxwellians and ring beams

This section aims at computing the integrals \( E_{\ell,m}(\omega,k) \), in the case of perpendicular propagation which were defined in chapter 5 and that we rewrite here,

\[
E_{\ell,m} \triangleq \int_{-\infty}^{\infty} dv_\parallel \int_{0}^{\infty} v_\perp^m dv_\perp L_\ell \left( \frac{k_\perp v_\perp}{\omega_\parallel}, \frac{\omega}{\omega_\parallel} \right) \frac{\partial f_0}{\partial v_\perp} (v_\perp, v_\parallel)
\]  

(A.1)

when \( f_0 \) is the sum of an anisotropic Maxwellian,

\[
f_M(v_\perp, v_\parallel) = \frac{1}{\sqrt{\pi v_0,\parallel}} \frac{1}{\pi v_0^2,\perp} e^{-v_\perp^2/v_0^2,\perp - v_\parallel^2/v_0^2,\parallel}
\]  

(A.2)

where

\[
v_{0,\perp,\parallel} = \sqrt{\frac{2kB T_{\perp,\parallel}}{m}}
\]  

(A.3)

and a ring beam distribution

\[
f_B(v_\perp, v_\parallel) = \frac{1}{2\pi u_\perp} \delta(v_\parallel) \delta(v_\perp - u_\perp)
\]  

(A.4)

where \( m = 1, 2 \) and \( \ell = 1, 2, ..., 6 \).

We also compute the integrals 5.100

\[
E_{m,n;\perp}(\omega,k) \triangleq \int_{-\infty}^{\infty} dv_\parallel v_\parallel^m \int_{0}^{\infty} v_\perp^n dv_\perp L_{k,\ell}(\kappa,\nu) \frac{\partial f_0}{\partial v_\perp} (v_\perp, v_\parallel)
\]  

(A.5)

and \( E_{m,n;\parallel}(\omega,k) \) defined in Chapter 5 (see 5.6) for waves propagating with a finite parallel component with respect to the background magnetic field for unperturbed
distribution functions given by
\[ f_M (v_{\perp}, v_{\parallel}) = \frac{1}{\sqrt{\pi v_{0,\parallel}}} \frac{1}{\pi v_{0,\perp}^2} e^{-v_{\perp}^2/v_{0,\perp}^2 - (v_{\parallel} - u_{\parallel})^2/v_{0,\parallel}^2} \] (A.6)

as well as for
\[ f_B (v_{\perp}, v_{\parallel}) = \frac{1}{\sqrt{\pi v_{0,\parallel}}} e^{-(v_{\parallel} - u_{\parallel})^2/v_{0,\parallel}^2} \frac{1}{2\pi u_{\perp}} \delta (v_{\perp} - u_{\perp}) \] (A.7)

with \( v_{0,\perp,\parallel} \) defined as in A.3 and \( u_{\parallel} \) and \( u_{\perp} \) the drift speed in the parallel and perpendicular directions respectively. We first derive 5.70 in next section,
\[ \int_{2\pi}^{2\pi} \int_{-\sigma\infty}^{\sigma\infty} d\theta' e^{i(\nu - \nu') (\theta - \theta')} e^{i(k\theta + \ell\theta')} = \sum_{m=-\infty}^{\infty} \frac{2i\pi}{(\nu - \ell - m)} J_m (\kappa) J_{m+k+\ell} (\kappa) \] (A.8)

which allows to compute the angular part of all the integrals considered. We then give results used to perform the \( v_{\perp} \) integrals. Linear combinations of angular integrals multiplied by A.2 and A.4 integrated over velocity space give us the final results and the procedure is applied to the calculation of \( E_{1,2} \) as an example. All the integral results are then given for perpendicular propagation. Integrals involving \( v_{\parallel} \) are examined when \( k_{\parallel} \neq 0 \) before giving the results.

### A.2 Identity for angular integrations

The procedure to compute A.8 is given in [26, 190] which we present here, for completeness. The lower bound in A.8 follows from [26] which differs from [190]. As a consequence, the integrals converge for both positive and negative imaginary parts of \( \nu \), namely modes that grow or decay over time. The integrals diverge on the real axis. In order to integrate
\[ \int_{2\pi}^{2\pi} d\theta \int_{-\sigma\infty}^{\sigma\infty} d\theta' e^{i(\nu - \nu') (\theta - \theta')} e^{i(k\theta + \ell\theta')} \] (A.9)

with \( \sigma = \pm 1 \). We set the change of variable \( \sigma \phi = \theta - \theta' \), such that \( \sigma d\phi = -d\theta' \), and modify the integral bounds accordingly:
\[ \sigma \int_{0}^{2\pi} d\theta \int_{0}^{\infty} d\phi e^{i(\nu - \nu) (\theta - \sigma\phi)} e^{i(k\theta + \ell\theta') e^{-i(\nu - \nu) \phi}} \] (A.10)
namely,
\[ \sigma \int_0^{2\pi} d\theta e^{-i\kappa \sin \theta} e^{i(k+\ell)\theta} \int_0^\infty d\phi e^{i(\nu-\ell)\sigma \phi} e^{i\sin(\theta-\sigma \phi)} \] (A.11)

or
\[ \sigma \int_0^{2\pi} d\theta e^{-i\kappa \sin \theta} e^{i(k+\ell)\theta} \sum_{n=0}^\infty \int_0^{2\pi(n+1)} d\phi e^{i(\nu-\ell)\sigma \phi} e^{i\sin(\theta-\sigma \phi)} \] (A.12)

The function
\[ h(\phi) \triangleq e^{i\sin(\theta-\sigma \phi)} \] (A.13)
is 2\pi periodic such that A.12 is equivalent to
\[ \sigma \int_0^{2\pi} d\theta e^{-i\kappa \sin \theta} e^{i(k+\ell)\theta} \sum_{n=0}^\infty \int_0^{2\pi} d\phi e^{i(\nu-\ell)\sigma \phi} e^{i\sin(\theta-\sigma \phi)} h(\phi + 2\pi n) \] (A.14)

and becomes
\[ \sigma \int_0^{2\pi} d\theta e^{-i\kappa \sin \theta} e^{i(k+\ell)\theta} \sum_{n=0}^\infty e^{2i\pi n\sigma} \int_0^{2\pi} d\phi e^{i(\nu-\ell)\sigma \phi} h(\phi) \] (A.15)

Since \( \ell \) is an integer, we have
\[ \sigma \int_0^{2\pi} d\theta e^{-i\kappa \sin \theta} e^{i(k+\ell)\theta} \sum_{n=0}^\infty e^{2i\pi n\sigma} \int_0^{2\pi} d\phi e^{i(\nu-\ell)\sigma \phi} h(\phi) \] (A.16)

The value of \( \sigma \) is chosen to ensure convergence of the series
\[ Q(\nu) = \sum_{n=0}^\infty e^{2i\pi n\sigma} = \frac{1}{1 - e^{2i\pi n\sigma}} \] (A.17)

and takes the value +1 for a growing mode and −1 for a damped mode. Swapping the \( \theta \) and \( \phi \) integrals leads to
\[ Q(\nu) \sigma \int_0^{2\pi} d\phi e^{i(\nu-\ell)\sigma \phi} \int_0^{2\pi} d\theta e^{i(k+\ell)\theta} e^{i\sin(\theta-\sigma \phi)} e^{-i\kappa \sin \theta} \] (A.18)

The generating function of \( J_n(x) \) gives \[222\]
\[ e^{\frac{1}{2}z(t-t^{-1})} = \sum_{m=-\infty}^{\infty} t^m J_m(z) \] (A.19)
and particularized to $t = e^{i\theta}$ in A.18 leads to

$$Q(\nu) \sigma \int_0^{2\pi} d\phi e^{i(\nu-\ell)\sigma \phi} \int_0^{2\pi} d\theta e^{i(k+\ell+\theta)\theta} \sum_{-\infty}^{\infty} e^{im(\theta-\sigma \phi)} J_m(\kappa) \sum_{-\infty}^{\infty} e^{-in\theta} J_n(\kappa)$$

(A.20)

or swapping the sum with the integrals, factoring all the $\theta$ dependence and separating it from the $\phi$ dependence results in

$$Q(\nu) \sigma \sum_{m,n} \int_0^{2\pi} d\phi e^{i(\nu-\ell-m)\sigma \phi} \int_0^{2\pi} d\theta e^{i(k+\ell+m-n)\theta} J_m(\kappa) J_n(\kappa)$$

(A.21)

and after introducing the Kronecker delta symbol and performing the integral over $\phi$, A.21 reduces to

$$Q(\nu) \sigma \sum_{m,n} \frac{1}{\sigma (\nu-\ell-m)} e^{i(\nu-\ell-m)\sigma \phi} \int_0^{2\pi} 2\pi \delta_{k+\ell+m,n} J_m(\kappa) J_n(\kappa)$$

(A.22)

After evaluation we find,

$$Q(\nu) \sigma \sum_{m} \frac{2\pi}{i\sigma (\nu-\ell-m)} \left( e^{2i\pi \sigma (\nu-\ell-m)} - 1 \right) J_m(\kappa) J_{k+\ell+m}(\kappa)$$

(A.23)

and the term within bracket is defined to a $2\pi$ phase since $\ell$ and $m$ are integers,

$$Q(\nu) \sigma \left( e^{2i\pi \nu} - 1 \right) \sum_{m} \frac{2\pi}{i\sigma (\nu-\ell-m)} J_m(\kappa) J_{k+\ell+m}(\kappa)$$

(A.24)

Using

$$Q(\nu) \left( 1 - e^{2i\pi \nu} \right) = 1$$

(A.25)

we obtain the sought result

$$\int_0^{2\pi} d\theta \int_{-\infty}^{\theta} d\theta' e^{i\nu (\theta-\theta') - i\kappa (\sin \theta - \sin \theta')} e^{ik\theta} e^{i\ell \theta'} = \sum_{m=-\infty}^{\infty} \frac{2i\pi}{(\nu-\ell-m)} J_m(\kappa) J_{k+\ell+m}(\kappa)$$

(A.26)

### A.3 Newsberger sum rule

Extensive integral results involving kinetic plasma dispersion relations are presented in [200] which are of use for our hybrid kinetic dispersion relations. By using Newsberger’s sum rule [199], it is possible to reduce the series containing products of Bessel functions to only one single term. We reproduce some derived identities
taken from [200] which can be directly applied to the calculation of the ring beam
dispersions matrix elements, given in Section A.10. However, as mentioned in [200,
p.184], the closed forms for the integrals involving Maxwellians still result in an in-
finite sum. One might still use these formulae if the integrals were to be computed
numerically, for a Maxwellian, or an arbitrary gyroptropic distribution function.

\[ \sum_{m=-\infty}^{\infty} \frac{1}{a-m} J_m(z) J_{m-\ell}(z) = (-1)^\ell \frac{\pi}{\sin(\pi a)} J_{\ell-a}(z) J_a(z), \quad \ell \geq 0 \quad (A.27) \]

This identity leads to the following useful results:

\[ \sum_{n=-\infty}^{\infty} \frac{n^2 J_n^2(z)}{a-n} = \frac{\pi a^2}{\sin(\pi a)} J_a(z) J_{-a}(z) - a \quad (A.28) \]

\[ \sum_{n=-\infty}^{\infty} \frac{|J_n'(z)|^2}{a-n} = \frac{\pi}{\sin(\pi a)} J_a'(z) J_{-a}'(z) + \frac{a}{z^2} \]

\[ \sum_{n=-\infty}^{\infty} \frac{n J_n(z) J_n'(z)}{a-n} = \frac{\pi a}{\sin(\pi a)} J_a(z) J_{-a}'(z) + \frac{a}{z} \]

\[ \sum_{n=-\infty}^{\infty} \frac{J_n(z) J_n'(z)}{a-n} = \frac{\pi a}{\sin(\pi a)} J_a(z) J_{-a}'(z) + \frac{1}{z} \]

\[ \sum_{n=-\infty}^{\infty} \frac{n J_n^2(z)}{a-n} = \frac{\pi a}{\sin(\pi a)} J_a(z) J_{-a}(z) - 1 \quad (A.29) \]

\[ \sum_{n=-\infty}^{\infty} \frac{J_n^2(z)}{a-n} = \frac{\pi}{\sin(\pi a)} J_a(z) J_{-a}(z) \]

For example, the series with the factors \( n J_n^2 \) can be calculated from Newberger sum
rule by using the identity \( J_{n+1}(z) + J_{n-1}(z) = 2n J_n(z) / z \), and manipulating the
expressions such that they contain \( \ell \geq 0 \) in order to apply A.27. The identity is
obtained after showing

\[ -\frac{z}{2} \frac{\pi}{\sin(\pi a)} (J_{1-a}(z) J_a(z) + J_{a-1}(z) J_{-a}(z)) = -1 \quad (A.30) \]

One possible way is to first compute the derivative with respect to \( z \) and observe
that it is identically 0 such that A.30 is a constant. This constant value can then be
determined by applying the limiting form of the Bessel function for small \( z \) which
is known to be:

\[ J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)} \quad (A.31) \]
and the gamma function disappears by use of the identity

\[ \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(\pi z)}, \quad z \not\in \mathbb{Z} \]  

(A.32)
A.4 Angular integral results

\begin{align}
L_1(\kappa, \nu) &= \frac{\pi i}{\kappa} \sum_{n=-\infty}^{\infty} \left( \frac{(2n+1)\nu - 2n(n+1)}{(\nu - n)(\nu - n - 1)} \right) \frac{1}{\kappa} J_n(\kappa) J_{n+1}(\kappa) \\
&= 2i\pi \sum_{n=-\infty}^{\infty} \frac{n^2}{(\nu - n)^2} \frac{1}{\kappa^2} J_n^2(\kappa) \\
L_2(\kappa, \nu) &= L_{sc'}(\kappa, \nu) = -\frac{\pi \nu}{\kappa} \sum_{n=-\infty}^{\infty} \frac{1}{\kappa} \frac{J_n(\kappa) J_{n+1}(\kappa)}{(\nu - n)(\nu - n - 1)} \\
&= -\frac{\pi}{\kappa} \sum_{n=-\infty}^{\infty} \frac{J_n^2(\kappa)}{(\nu - n - 1)(\nu - n + 1)} \\
&= -2\pi \sum_{n=-\infty}^{\infty} \frac{n}{(\nu - n)^2} \frac{1}{\kappa} J_n(\kappa) J_n'(\kappa) \\
L_3(\kappa, \nu) &= L_{cc'}(\kappa, \nu) = -L_2(\kappa, \nu) \\
&= -L_{sc'} \\
L_4(\kappa, \nu) &= L_{ss'}(\kappa, \nu) = 2\pi i \sum_{n=-\infty}^{\infty} \frac{J_n^2(\kappa)}{\nu - n} \\
L_5(\kappa, \nu) &= L_{1c}(\kappa, \nu) = 2\pi i \sum_{n=-\infty}^{\infty} \frac{n}{\nu - n} \frac{1}{\kappa} J_n^2(\kappa) \\
&= i\pi \sum_{n=-\infty}^{\infty} \frac{2\nu - 2n - 1}{(\nu - n)(\nu - n - 1)} J_n(\kappa) J_{n+1}(\kappa) \\
L_6(\kappa, \nu) &= L_{1s'}(\kappa, \nu) = \pi \sum_{n=-\infty}^{\infty} \frac{J_n(\kappa) J_{n+1}(\kappa)}{(\nu - n)(\nu - n - 1)} \\
&= 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{(\nu - n)^2} J_n(\kappa) J_n'(\kappa) \\
L_7(\kappa, \nu) &= L_{c1'}(\kappa, \nu) = L_5(\kappa, \nu) \\
&= L_{1c}(\kappa, \nu) \\
L_8(\kappa, \nu) &= L_{s1'}(\kappa, \nu) = -L_6(\kappa, \nu) \\
&= -L_{1s'}(\kappa, \nu) \\
L_9(\kappa, \nu) &= L_{11'}(\kappa, \nu) = 2i\pi \sum_{n=-\infty}^{\infty} \frac{J_n^2(\kappa)}{\nu - n}
\end{align}

(A.33)
A.5 Identities for $v_{\perp}$ integrals

We need several identities in order to carry out integrations over velocity space of which some the results can be found in Umeda [223]. The functions $J_n$ and $I_n$ are the Bessel and the modified Bessel functions of the first kind respectively. The integrals we use are:

1. $\int_0^\infty t \exp\left(-p^2 t^2\right) J_n^2(at) \, dt$
2. $\int_0^\infty t^3 \exp\left(-p^2 t^2\right) J_n^2(at) \, dt$
3. $\int_0^\infty t^2 \exp\left(-p^2 t^2\right) J_n J'_n(at) \, dt$ (A.34)
4. $\int_0^\infty t^{\lambda - 1} \exp\left(-p^2 t^2\right) J_n^2(at) \, dt$
5. $\int_0^\infty t^{\lambda} \exp\left(-p^2 t^2\right) J_n J'_n(at) \, dt$

Case 1

The first integral of A.34 is known as Watson formula [198] and is given by

$$\int_0^\infty t \exp\left(-p^2 t^2\right) J_n^2(at) \, dt = \frac{1}{2p^2} \exp\left(-a^2/2p^2\right) I_n\left(a^2/2p^2\right)$$ (A.35)

Case 2

The derivative of the integral A.35 with respect to $p^2$ gives us the second integral in A.34 as follows

$$\int_0^\infty t^3 \exp\left(-p^2 t^2\right) J_n^2(at) \, dt = -\frac{\partial}{\partial p^2} \left[\int_0^\infty t \exp\left(-p^2 t^2\right) J_n^2(at) \, dt\right]$$ (A.36)

or

$$\int_0^\infty t^3 \exp\left(-p^2 t^2\right) J_n^2(at) \, dt = \frac{1}{2p^4} \left[I_n\left(a^2/2p^2\right) - \frac{a^2}{2p^2} I_n\left(a^2/2p^2\right) + \frac{a^2}{2p^2} I'_n\left(a^2/2p^2\right)\right]$$ (A.37)

using the identity,

$$I_{n+1}(x) + I_{n-1}(x) = 2I_n'(x), \quad I'_0(x) = I_1(x)$$ (A.38)

we obtain,
\[
\int_0^\infty t^3 \exp \left(-p^2 t^2\right) J_n^2(at) \, dt
= \frac{1}{2p^4} \exp \left(-a^2/2p^2\right) \left[ I_n \left(a^2/2p^2\right) \left[ 1 - \frac{a^2}{2p^2}\right] + \frac{a^2}{4p^2} \left[I_{n-1} \left(a^2/2p^2\right) + I_{n+1} \left(a^2/2p^2\right)\right]\right] \quad (A.39)
\]

If we further apply
\[
I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x) \quad (A.40)
\]
and set \( \kappa_c^2 \triangleq a^2/2p^2 \), this leads to
\[
\int_0^\infty t^3 \exp \left(-p^2 t^2\right) J_n^2(at) \, dt
= \frac{2\kappa_c^4}{a^4} \exp \left(-\kappa_c^2\right) \left[ \kappa_c^2 I_{n+1} \left(\kappa_c^2\right) + \left[n + 1 - \kappa_c^2\right] I_n \left(\kappa_c^2\right)\right] \quad (A.41)
\]

**Case 3**

Likewise we calculate the third integral of A.34 by differentiating Watson formula A.35 with respect to the parameter \(a\):
\[
\int_0^\infty t^2 \exp \left(-p^2 t^2\right) J_n \left(at\right) J'_n \left(at\right) \, dt = \frac{1}{2} \frac{\partial}{\partial a} \left[ \int_0^\infty t \exp \left(-p^2 t^2\right) J_n^2 \left(at\right) \, dt \right] \quad (A.42)
\]
and we find with A.35
\[
\int_0^\infty t^2 \exp \left(-p^2 t^2\right) J_n \left(at\right) J'_n \left(at\right) \, dt
= \frac{a}{4p^4} \exp \left(-a^2/2p^2\right) \left( -I_n \left(a^2/2p^2\right) + 2I'_n \left(a^2/2p^2\right) \right)
= \frac{a}{8p^4} \exp \left(-a^2/2p^2\right) \left(I_{n+1} \left(a^2/2p^2\right) - 2I_n \left(a^2/2p^2\right) + I_{n-1} \left(a^2/2p^2\right)\right) \quad (A.43)
\]

**Case 4**

Although the fourth integral can be computed by taking successive derivatives of the third integral with respect to \(p^2\) for odd powers of \(t\), even powers require the generalized hypergeometrical functions [198]:
\[
\int_0^\infty t^{\lambda-1} \exp \left(-p^2 t^2\right) J_n^2 \left(at\right) \, dt
\]
\[
\begin{align*}
&= a^{2n} \frac{1}{2^{2n} p^{2n+\lambda}} \frac{1}{\Gamma^2 (n+1)} 3F_3 \left( n + \frac{1}{2}, n + 1, n + \frac{\lambda}{2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right) \\
&= a^{2n} \frac{1}{2^{2n} p^{2n+\lambda}} \frac{1}{\Gamma^2 (n+1)} 3F_3 \left( n + \frac{1}{2}, n + 1, n + \frac{\lambda}{2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right)
\end{align*}
\]  

(A.44)

**Case 5**

The last integral of A.34 can be obtained by taking the derivative of integral A.44 with respect to \(a\). The derivative of a generalized hypergeometric function \(F_n\) is given [224] by:

\[
\frac{d}{dz} mF_n (a_1, ..., a_m; b_1, ..., b_n; z) = \frac{\prod_{i=1}^{m} (a_i)}{\prod_{j=1}^{n} (b_j)} mF_n (a_1 + s, ..., a_m + s; b_1 + s, ..., b_n + s; z)
\]  

(A.45)

and \((z)_n\) is defined by:

\[
(z)_n = \frac{\Gamma (z + 1)}{\Gamma (z)} = \begin{cases} 
1, & n = 0 \\
(z + 1) \cdots (z + n - 1), & n > 0
\end{cases}
\]  

(A.46)

When \(n = 1\), \((z)_n\) simply equals \(z\). Taking the derivative of both sides of A.44 with respect to \(a\) leads to

\[
\int_0^\infty t^\lambda \exp \left( -p^2 t^2 \right) 2J_n (at) J_n' (at) dt = 2n a^{2n-1} \frac{1}{2^{2n} p^{2n+\lambda}} \frac{1}{\Gamma^2 (n+1)} 3F_3 \left( n + \frac{1}{2}, n + 1, n + \frac{\lambda}{2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right)
\]

\[
+ a^{2n} \frac{1}{2^{2n} p^{2n+\lambda}} \frac{1}{\Gamma^2 (n+1)} 3F_3 \left( \frac{n + \lambda/2 + n + 1/2 + n + \lambda/2}{n + 1/2 + n + 1/2 + n + 1/2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right)
\]

\[
\times 3F_3 \left( n + \frac{3}{2}, n + 2, n + 1 + \frac{\lambda}{2}; n + 2, n + 2, 2n + 2; -\frac{a^2}{p^2} \right)
\]

\[
= a^{2n-1} \frac{1}{2^{2n-1} p^{2n+\lambda}} \frac{1}{\Gamma^2 (n+1)} \left[ (n) 3F_3 \left( n + \frac{1}{2}, n + 1, n + \frac{\lambda}{2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right) \right]
\]

\[
- \frac{a^2}{p^2} \frac{(n + 1/2)(n + \lambda/2)}{(n + 1/2)(2n + 1)} 3F_3 \left( n + \frac{3}{2}, n + 2, n + 1 + \frac{\lambda}{2}; n + 2, n + 2, 2n + 2; -\frac{a^2}{p^2} \right)
\]

\[
= \frac{a}{2p^2} \left( \frac{a^2}{4p^2} \right)^{n-1} \frac{1}{p^{\lambda} \Gamma^2 (n+1)} \left[ (n) 3F_3 \left( n + \frac{1}{2}, n + 1, n + \frac{\lambda}{2}; n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2} \right) \right]
\]

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\[
- \frac{a^2 (n + 1/2) (n + \lambda/2)}{p^2} (n + 1/2) (2n + 1) \binom{3}{n + 1/2, n + 1 + \lambda/2, n + 2, 2n + 2; -\frac{a^2}{p^2}}
\] (A.47)

We divide both sides by 2 and obtain the last integral A.34

\[
\int_0^\infty t^\lambda \exp \left( -p^2 t^2 \right) J_n(\lambda t) J'_n(\lambda t) \, dt
\]

\[
= \frac{a}{4p^2} \left( \frac{a^2}{4p^2} \right)^{n-1} \frac{1}{p^\lambda} \Gamma(n + \lambda/2) \Gamma^2(n + 1) \left( \binom{n + 1/2, n + 1 + \lambda/2, n + 2, 2n + 2; -\frac{a^2}{p^2}}{n + 1, n + 1, n + 1, 2n + 1; -\frac{a^2}{p^2}} \right) - \frac{a^2 (n + 1/2) (n + \lambda/2)}{p^2} (n + 1/2) (2n + 1) \binom{3}{n + 1/2, n + 1 + \lambda/2, n + 2, 2n + 2; -\frac{a^2}{p^2}}
\] (A.48)

**Sidetrack**

As a little side track, integrals 1 and 2 in A.34 are linked. This can be seen by integrating Watson formula A.35 by part, raising the power of t and taking the derivatives of the remaining factors. We find

\[
\left( 1 + \frac{\partial}{\partial \alpha} + 2p^2 \frac{\partial}{\partial p^2} \right) \left[ \int_0^\infty t \exp \left( -p^2 t^2 \right) J_n^2(\alpha t) \, dt \right] = 0 \] (A.49)

A second side track involves rewriting A.42 by remembering the identity

\[
J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x), \quad J'_0(x) = -J_1(x)
\] (A.50)

which gives

\[
\int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_n(\alpha t) J_{n+1}(\alpha t) \, dt
\]

\[
= \int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_n(\alpha t) J_{n-1}(\alpha t)
\]

\[
- \frac{\partial}{\partial \alpha} \left[ \int_0^\infty t \exp \left( -p^2 t^2 \right) J_n^2(\alpha t) \, dt \right]
\] (A.51)

and leads to the recurrence relation,

\[
\int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_n(\alpha t) J_{n+1}(\alpha t) \, dt
\]

\[
= \int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_1(\alpha t) J_0(\alpha t)
\]
\[ - \sum_{k=1}^{n} \frac{\partial}{\partial a} \left[ \int_0^\infty t \exp \left( -p^2 t^2 \right) J_k^2 \left( at \right) dt \right] \]
\[ = - \frac{\partial}{\partial a} \left[ \int_0^\infty t \exp \left( -p^2 t^2 \right) \left( \frac{1}{2} J_0^2 \left( at \right) + \sum_{k=0}^{n} J_k^2 \left( at \right) \right) dt \right] \quad \text{(A.52)}\]

The last line of A.52 holds because \( J_0' = -J_1 \) and we obtain
\[ \int_0^\infty t^2 \exp \left( -p^2 t^2 \right) J_n \left( at \right) J_{n+1} \left( at \right) dt \]
\[ = - \frac{a}{4p^3} \exp \left( -a^2/2p^2 \right) \left[ -I_0 \left( a^2/2p^2 \right) + I_0' \left( a^2/2p^2 \right) \right. \]
\[ + \sum_{k=0}^{n} \left( I_{k+1} \left( a^2/2p^2 \right) - 2I_k \left( a^2/2p^2 \right) + I_{k-1} \left( a^2/2p^2 \right) \right) \right] \quad \text{(A.53)}\]
since \( I_0' = I_1 \), the telescopic sum reduces to give the final expression for the integral A.53
\[ \frac{a}{4p^3} \left( I_{n+1} \left( a^2/2p^2 \right) - I_n \left( a^2/2p^2 \right) \right) \exp \left( -a^2/2p^2 \right) \quad \text{(A.54)}\]

### A.6 Computation of \( E_{1,2} (\nu, \kappa_0) \)

In order to compute \( E_{1,2} \) as defined in A.1, in the cases when \( f_0 \) is a Maxwellian and a ring beam distribution function, we first compute
\[ L_1 (\kappa, \nu) = \int_0^{2\pi} d\theta \cos \theta \int_{-\infty}^{\theta} d\theta' \cos \theta' e^{i\nu (\theta - \theta')} - i\kappa (\sin \theta - \sin \theta') \quad \text{(A.55)} \]
which captures the angular dependence and in a second time integrates the product of A.55 with \( \frac{\partial f_0}{\partial v_{\perp}} (v_{\parallel}, v_{\perp}) \) over \( v_{\parallel} \) and \( v_{\perp} \) for \( f_0 \) defined in A.2 and A.4. We can find an expression for \( L_1 (\kappa, \nu) \) by taking linear combinations of A.26 as follows,
\[ \cos \theta \cos \theta' = \frac{1}{4} \left( e^{i(\theta+\theta')} + e^{i(\theta-\theta')} + e^{i(-\theta+\theta')} + e^{i(-\theta-\theta')} \right) \quad \text{(A.56)} \]
namely, we need to average the results of A.25 when \( (k, \ell) \) are \((1,1), (1,-1), (-1,1) \) and \((-1,-1) \) which brings
\[ L_1 (\kappa, \nu) = \frac{2i\pi}{4} \sum_{m=-\infty}^{\infty} \left[ \frac{J_m (\kappa) J_{m+2} (\kappa)}{(\nu - 1 - m)} + \frac{J_m (\kappa) J_{m+2} (\kappa)}{(\nu + 1 - m)} \right. \]
\[ + \frac{J_m (\kappa) J_{m} (\kappa)}{(\nu - 1 - m)} + \frac{J_m (\kappa) J_{m-2} (\kappa)}{(\nu + 1 - m)} \right] \quad \text{(A.57)} \]
Factoring term 1 with term 3 and term 2 with term 4 gives
\[
L_1(\kappa, \nu) = \frac{2i\pi}{4} \sum_{m=-\infty}^{\infty} \left[ \frac{1}{\nu - (m + 1)} \left( J_m(\kappa) J_{m+2}(\kappa) + J_m(\kappa) J_m(\kappa) \right) + \frac{1}{\nu - (m - 1)} \left( J_m(\kappa) J_m(\kappa) + J_m(\kappa) J_{m-2}(\kappa) \right) \right] \tag{A.58}
\]
Since \( m \) is a dummy index which runs from \(-\infty\) to \( \infty \) we can shift it by an arbitrary integer provided the relative integer difference of each term is preserved. We change the first term to \( m' \equiv m + 1 \) and the second to \( m'' \equiv m - 1 \) and dropping \( ' \) and \( '' \) leads to
\[
L_1(\kappa, \nu) = \frac{2i\pi}{4} \sum_{m=-\infty}^{\infty} \left[ \frac{1}{\nu - m} \left( J_{m-1}(\kappa) J_{m+1}(\kappa) + J_{m-1}(\kappa) J_{m-1}(\kappa) \right) + \frac{1}{\nu - m} \left( J_{m+1}(\kappa) J_{m+1}(\kappa) + J_{m+1}(\kappa) J_{m-1}(\kappa) \right) \right] \tag{A.59}
\]
and factoring \( J_{m-1} \) on the first line of \( A.59 \) and \( J_{m+1} \) on its second line, we obtain
\[
L_1(\kappa, \nu) = \frac{2i\pi}{4} \sum_{m=-\infty}^{\infty} \left[ \frac{1}{\nu - m} J_{m-1}(\kappa) \left( J_{m+1}(\kappa) + J_{m-1}(\kappa) \right) + \frac{1}{\nu - m} J_{m+1}(\kappa) \left( J_{m+1}(\kappa) + J_{m-1}(\kappa) \right) \right] \tag{A.60}
\]
and factoring once more
\[
L_1(\kappa, \nu) = \frac{2i\pi}{4} \sum_{m=-\infty}^{\infty} \frac{1}{\nu - m} \left( J_{m+1}(\kappa) + J_{m-1}(\kappa) \right)^2 \tag{A.61}
\]
Applying the identity
\[
J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x} J_m(x) \tag{A.62}
\]
leads to
\[
L_1(\kappa, \nu) = 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{1}{\kappa^2} J_m^2(\kappa) \tag{A.63}
\]
At this point, we can properly tackle the computation of
\[
E_{1,2} = \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} v_\perp^2 dv_\perp L_1 \left( \frac{k_\perp v_\perp}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_\perp}(v_\perp, v_\parallel) \tag{A.64}
\]
and specify the distribution functions.

Case 1: Starting with the (anisotropic) Maxwellian, we have with A.2 and A.63

\[ E_{1,2} = \int_{-\infty}^{\infty} dv_\parallel \frac{1}{\sqrt{\pi} v_{0,\parallel}} e^{-v_\parallel^2 / v_{0,\parallel}^2} \int_0^\infty v_\perp^2 dv_\perp 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{1}{\kappa^2} J_m^2(\kappa) \frac{\partial}{\partial v_\perp} \frac{1}{\pi v_{0,\perp}^2} e^{-v_\perp^2 / v_{0,\perp}^2} \]

(A.65)

The parallel integral is unity as it is normalized and does not necessarily need to be Maxwellian as it is completely factored out of the integral elements in the perpendicular propagation case. Developing \( \kappa = \frac{k_\perp v_\perp}{\omega_c} \)

\[ E_{1,2} = 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \int_0^\infty v_\perp^2 dv_\perp \frac{\omega_c^2}{v_\perp^2 k_\perp^2} J_m^2(\kappa) \frac{-2v_\perp}{\pi v_{0,\perp}^2} e^{-v_\perp^2 / v_{0,\perp}^2} \]

(A.66)

\[ E_{1,2} = \frac{-2\omega_c^2}{\pi k_\perp^2 v_{0,\perp}^4} 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \int_0^\infty v_\perp dv_\perp J_m^2 \left( \frac{k_\perp v_\perp}{\omega_c} \right) e^{-v_\perp^2 / v_{0,\perp}^2} \]

(A.67)

and with Watson [198], defining

\[ \kappa_0^2 = \frac{k_\perp^2 v_{0,\perp}^2}{2\omega_c^2} \]

(A.68)

we find for A.67 using equation A.35

\[ E_{1,2} = \frac{-2\omega_c^2}{\pi k_\perp^2 v_{0,\perp}^4} 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{v_{0,\perp}^2}{2} I_n(\kappa_0^2) e^{-\kappa_0^2} \]

(A.69)

Finally,

\[ E_{1,2} = -i \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{1}{\kappa_0^2} I_m(\kappa_0^2) e^{-\kappa_0^2} \]

(A.70)

Case 2: Following with the ring beam A.4

\[ E_{1,2} = \int_{-\infty}^{\infty} dv_\parallel \delta(v_\parallel) \int_0^\infty v_\perp^2 dv_\perp 2i\pi \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{1}{\kappa^2} J_m^2(\kappa) \frac{\partial}{\partial v_\perp} \frac{1}{2\pi u_\perp} \delta(v_\perp - u_\perp) \]

(A.71)

We use the fact that for a distribution \( \delta \), the following holds for any test functions \( \phi \),

\[ \langle \delta', \phi \rangle = -\phi'(0) \]

(A.72)
and the parallel integral is unity,

\[ E_{1,2} = 2i\pi \frac{\omega^2}{k_\perp^2} \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \int_{0}^{\infty} dv_\perp \frac{-1}{2\pi u_\perp} \delta (v_\perp - u_\perp) \frac{\partial}{\partial v_\perp} J_m^2 \left( \frac{k_\perp v_\perp}{\omega_c} \right) \] (A.73)

So we obtain,

\[ E_{1,2} = 2i\pi \frac{\omega^2}{k_\perp^2} \sum_{m=-\infty}^{\infty} \frac{m^2}{(\nu - m)} \frac{2J_m \left( \frac{k_\perp v_\perp}{\omega_c} \right)}{2\pi u_\perp} \frac{J_m' \left( \frac{k_\perp v_\perp}{\omega_c} \right)}{\omega_c} \] (A.74)

Defining,

\[ \kappa_b \equiv \frac{k_\perp u_\perp}{\omega_c} \] (A.75)

equation A.74 reduces to

\[ E_{1,2} = \sum_{m=-\infty}^{\infty} \frac{-2im^2}{(\nu - m)} \frac{1}{\kappa_b} J_m (\kappa_b) J_m' (\kappa_b) \] (A.76)

A.7 Integrals involving \( v_\parallel \) and the plasma dispersion function

The integrals of the parallel component of the distribution functions A.6 and A.7 cannot be separated from the angular integrals \( L_{a,b} (\kappa, \nu) \), \( a, b = 1, c, s; 1', c', s' \) given from section A.4 due to their dependance in \( v_\parallel \) through

\[ \frac{1}{(\nu - n)} \] (A.77)

with \( n \in \mathbb{Z} \) and \( \nu = \omega / \omega_c - k_\parallel v_\parallel \) since \( k_\parallel \) is finite. We denote

\[ f_{0,\parallel} (v_\parallel) \equiv \frac{1}{\sqrt{\pi v_0,\parallel}} e^{- (v_\parallel - u_\parallel)^2 / v_0,\parallel} \] (A.78)

and make use of the plasma dispersion function \( Z (\xi_n) \) as defined by Fried and Conte [225]:

\[ Z (\xi_n) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} dt \] (A.79)

with

\[ \xi_n = \frac{\omega - k_\parallel u_\parallel - n\omega_c}{k_\parallel v_\parallel} \] (A.80)
The plasma dispersion [9] function is a solution of the following differential equation

\[ Z'(\xi_n) = -2 (1 + \xi_n Z(\xi_n)) \]  \hspace{1cm} (A.81)

From expressions A.77, A.78, A.79 and A.80, we find [223], with \( \kappa_0, || = \frac{klv_0, ||}{\omega_c} \)

\[
\int_{-\infty}^{\infty} \frac{1}{(\nu - n)} f_{0,||}(v_{||}) \, dv_{||} = -\frac{1}{\kappa_{0,||}} Z(\xi_n)
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(\nu - n)} \frac{\partial f_{0,||}}{\partial v_{||}} (v_{||}) \, dv_{||} = -\frac{1}{v_{0,||} k_{0,||}} Z'(\xi_n)
\]  \hspace{1cm} (A.82)

\[
\int_{-\infty}^{\infty} \frac{v_{||}}{(\nu - n)} f_{0,||}(v_{||}) \, dv_{||} = -\frac{\omega_c}{k_{||}} \left(1 + \frac{\omega - n\omega_c}{k_{||}v_0,||} Z(\xi_n)\right)
\]

In the limit of large argument \( \xi_n = x + iy, x > 0 \), the asymptotic expansion of the plasma dispersion function is expressed by [9, 220]

\[ Z(\xi_n) \sim i\sqrt{\pi} \sigma e^{-\xi_n^2} - \xi_n^{-1} \left(1 + \frac{1}{2\xi_n^2} + \frac{3}{4\xi_n^4} + \frac{15}{8\xi_n^6} + \ldots \right) \]  \hspace{1cm} (A.83)

and \( \sigma = 0, 1, 2 \) when \( y > 1/|x|, |y| < 1/|x| \) and \( y < -1/|x| \) respectively. We obtain the limiting cases of the integrals A.82 when \( v_{0,||} \rightarrow 0 \) by using A.83 which corresponds to a parallel distribution function given by \( f_{0,||}(v_{||}) = \delta (v_{||} - u_{||}) \), with \( v_{b} = \omega/\omega_c - k_{||}u_{||} \):

\[
\int_{-\infty}^{\infty} \frac{1}{(\nu - n)} f_{0,||}(v_{||}) \, dv_{||} = \frac{1}{v_{b} - n}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(\nu - n)} \frac{\partial f_{0,||}}{\partial v_{||}} (v_{||}) \, dv_{||} = \frac{k_{||}}{\omega_c (v_{b} - n)^2}
\]  \hspace{1cm} (A.84)

\[
\int_{-\infty}^{\infty} \frac{v_{||}}{(\nu - n)} f_{0,||}(v_{||}) \, dv_{||} = \frac{u_{||}}{(v_{b} - n)}
\]

A.8 Integrals w.r.t \( v_{\perp} \) involving Maxwellian with \( \perp \) drift

We are interested in computing integral elements of a distribution function whose perpendicular component \( f_{0,\perp}(v_{\perp}) \) satisfies

\[ f_{0,\perp}(v_{\perp}) = Ae^{-(v_{\perp} - u_{\perp,\perp})^2/v_{0,\perp}^2} \]  \hspace{1cm} (A.85)
where \( u_\perp \) is the drift velocity and the normalisation constant is given by [88]

\[
A^{-1} = \frac{v_{0,\perp}^2}{2} \left[ e^{-u_\perp^2/v_{0,\perp}^2} + \frac{2u_\perp}{v_{0,\perp}\sqrt{\pi}} \left( 1 - \frac{1}{2\sqrt{\pi}} \Gamma \left( \frac{1}{2}, \frac{u_\perp^2}{v_{0,\perp}^2} \right) \right) \right] \quad (A.86)
\]

with \( \Gamma(s,x) \), the upper incomplete gamma function.

The integral elements involve

\[
A \int_0^\infty v_\perp^\ell e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} J_n^2(\kappa) \, dv_\perp
\]

\[
A \int_0^\infty v_\perp^\ell e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} J_n^2(\kappa) J_n'(\kappa) \, dv_\perp
\]

with \( \ell \in \mathbb{N} \) and \( \kappa = k_\perp v_\perp/\omega_c \). To this purpose, we expand the exponential arguments:

\[
e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} = e^{-(v_\perp^2-2v_\perp u_\perp+u_\perp^2)/v_{0,\perp}^2} = e^{-v_\perp^2/v_{0,\perp}^2+2v_\perp u_\perp/v_{0,\perp}^2-u_\perp^2/v_{0,\perp}^2} \quad (A.88)
\]

and Taylor expand the double product

\[
e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} = e^{-u_\perp^2/v_{0,\perp}^2} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2v_\perp u_\perp}{v_{0,\perp}^2} \right)^k e^{-v_\perp^2/v_{0,\perp}^2} \quad (A.89)
\]

By substituting the expansion A.89 into integrals A.87, they become

\[
R'(n, \ell, \kappa_{0,\perp}) \triangleq A e^{-u_\perp^2/v_{0,\perp}^2} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2u_\perp}{v_{0,\perp}^2} \right)^k \int_0^\infty v_\perp^k e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} J_n^2(\kappa) \, dv_\perp
\]

\[
S'(n, \ell, \kappa_{0,\perp}) \triangleq A e^{-u_\perp^2/v_{0,\perp}^2} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2u_\perp}{v_{0,\perp}^2} \right)^k \int_0^\infty v_\perp^k e^{-(v_\perp-u_\perp)^2/v_{0,\perp}^2} J_n(\kappa) J_n'(\kappa) \, dv_\perp
\]

(A.90)

and we can apply the identity A.47 for the top integral A.90 with \( \lambda = k + \ell + 1 \) as well as the identity A.48 for the bottom integral A.90 with \( \lambda = k + \ell \). We note that in A.47 \( p \) corresponds to \( 1/v_{0,\perp} \) while \( a = k_\perp/\omega_c \).

We set again \( \kappa_{0,\perp}^2 = \frac{1}{2} \frac{k_\perp^2 v_{0,\perp}^2}{\omega_c^2} \) and define the scaled normalisation constant \( A' = v_{0,\perp}^2 A \) such that integrals A.90 unfold as

\[
R'(n, \ell, \kappa_{0,\perp}) = \frac{A'}{v_{0,\perp}^2} e^{-u_\perp^2/v_{0,\perp}^2} \left( \frac{\kappa_{0,\perp}^2}{2} \right)^n v_{0,\perp}^{\ell+1} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{2u_\perp}{v_{0,\perp}^2} \right)^k
\]

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Finally we define $R(n, \ell, \kappa_{0,\perp})$ and $S(n, \ell, \kappa_{0,\perp})$ such that they satisfy

\[ R'(n, \ell, \kappa_{0,\perp}) = v_{0,\perp}^{\ell-1} R(n, \ell, \kappa_{0,\perp}) \]
\[ S'(n, \ell, \kappa_{0,\perp}) = v_{0,\perp}^{\ell-2} S(n, \ell, \kappa_{0,\perp}) \]  

(A.92)

A.9 Integral elements for Maxwellian, $\perp$ propagation

We collect the results of all 6 integrals in the case of an (anisotropic) Maxwellian, the calculations closely following the steps from the derivation of $E_{1,2}$, with $\kappa_0^2 = \frac{k^2 v_\perp^2}{2\omega_c^2}$.
we have:

\[
E_{1,2} = \int_{-\infty}^{\infty} dv_{\parallel} \int_{0}^{\infty} v_{\perp}^2 dv_{\perp} L_1 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_{\parallel})
\]

\[= \sum_{m=-\infty}^{\infty} \frac{-i m^2}{(\nu - m) \kappa_0^2} I_m (\kappa_0^2) e^{-\kappa_0^2} \]

\[
E_{2,2} = \int_{-\infty}^{\infty} dv_{\parallel} \int_{0}^{\infty} v_{\perp}^2 dv_{\perp} L_2 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_{\parallel})
\]

\[= \frac{1}{2} \sum_{m=-\infty}^{\infty} \frac{m}{(\nu - m)} e^{-\kappa_0^2} \left[ I_{m+1} (\kappa_0^2) - 2 I_m (\kappa_0^2) + I_{m-1} (\kappa_0^2) \right] \]

\[
E_{3,2} = -E_{2,2}
\]

\[
E_{4,2} = \int_{-\infty}^{\infty} dv_{\parallel} \int_{-\infty}^{0} v_{\perp}^2 dv_{\perp} L_2 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_{\parallel})
\]

\[= \sum_{m=-\infty}^{\infty} \frac{i}{(\nu - m)} e^{-\kappa_0^2} \left[ \left( \frac{m^2}{\kappa_0^2} - \kappa_0^2 \right) I_m (\kappa_0^2) - (I_{m-1} (\kappa_0^2) + I_{m+1} (\kappa_0^2)) \left( 1 - \kappa_0^2 \right) \right]
\]

\[= -\kappa_0^2 \left( I_{m-2} (\kappa_0^2) + I_{m+2} (\kappa_0^2) \right) \]

\[
E_{5,1} = \int_{-\infty}^{\infty} dv_{\parallel} \int_{0}^{\infty} v_{\perp} dv_{\perp} L_5 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_{\parallel})
\]

\[= -\frac{i k_{\perp}}{\omega_c} \sum_{m=-\infty}^{\infty} \frac{m}{(\nu - m)} \frac{1}{\kappa_0^2} e^{-\kappa_0^2} I_m (\kappa_0^2) \]

\[
E_{6,1} = \int_{-\infty}^{\infty} dv_{\parallel} \int_{0}^{\infty} v_{\perp} dv_{\perp} L_6 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_{\parallel})
\]

\[= -\frac{k_{\perp}}{2 \omega_c} \frac{1}{(\nu - m)} e^{-\kappa_0^2} \left[ I_{m+1} (\kappa_0^2) - 2 I_m (\kappa_0^2) + I_{m-1} (\kappa_0^2) \right] \]

(A.93)
A.10 Integral elements for a ring beam, \( \perp \) propagation

Likewise, we express the results of all 6 integrals in the case of a ring beam distribution, the calculations still proceeding as in the case of \( E_{1,2} \)

\[
E_{1,2} = \int_{-\infty}^{\infty} dv_\| \int_{0}^{\infty} v_{\perp}^2 dv_{\perp} L_1 \left( \frac{k_{\perp} v_{\perp}}{\omega_c}, \frac{\omega}{\omega_c} \right) \frac{\partial f_0}{\partial v_{\perp}} (v_{\perp}, v_\|)
\]

\[
= \sum_{m=-\infty}^{\infty} \frac{-2im^2}{(\nu - m) \kappa_b} J_m (\kappa_b) J'_m (\kappa_b)
\]

\[
E_{2,2} = \sum_{m=-\infty}^{\infty} \frac{-m}{(\nu - m) \kappa_b} \left[ J_m (\kappa_b) J'_m (\kappa_b) + \kappa_b J''_m (\kappa_b) + \kappa_b J_m (\kappa_b) J''_m (\kappa_b) \right]
\]

\[
E_{3,2} = -E_{2,2}
\]

\[
E_{4,2} = \sum_{m=-\infty}^{\infty} \frac{-2i}{(\nu - m)} \left[ J''_m (\kappa_b) + \kappa_b J'_m (\kappa_b) J''_m (\kappa_b) \right]
\]

\[
E_{5,1} = -\frac{i k_{\perp}}{\omega_c} \sum_{m=-\infty}^{\infty} \frac{m}{(\nu - m) \kappa_b} J_m (\kappa_b) J'_m (\kappa_b)
\]

\[
E_{6,1} = -\frac{1}{u_\perp} \sum_{m=-\infty}^{\infty} \frac{m}{(\nu - m)} \left( J_m (\kappa_b) J'_m (\kappa_b) + \kappa_b J''_m (\kappa_b) + \kappa_b J'_m (\kappa_b) J''_m (\kappa_b) \right)
\]

\[\text{(A.94)}\]
A.11 Integral elements for Maxwellian, arbitrary angle propagation

\begin{align}
E^{1,c'}_{0,1;\perp} &= \frac{k_\perp}{\omega_c} \frac{i}{\kappa_0 \parallel \kappa_0 \perp} \sum_{m=-\infty}^{\infty} mZ\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{1,1}_{0,1;\parallel} &= -i \frac{1}{\nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} Z'\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{1,s'}_{0,1;\perp} &= \frac{1}{k_\perp \omega_c} \sum_{m=-\infty}^{\infty} \left(1 + \frac{\omega - m\omega_c}{k_\parallel \nu_0 \parallel} Z\left(\xi_m\right)\right) I_{m+1}\left(\kappa_0^2\right) - 2I_m\left(\kappa_0^2\right) + I_{m-1}\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{1,s'}_{1,1;\perp} &= \frac{1}{2} \frac{\omega_c^2}{k_\perp \nu_0 \parallel} \sum_{m=-\infty}^{\infty} \left(1 + \frac{\omega - m\omega_c}{k_\parallel \nu_0 \parallel} Z\left(\xi_m\right)\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{1,c'}_{0,2;\parallel} &= -i \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} mZ'\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{1,c'}_{0,2;\perp} &= -i \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} mZ'\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{0,2;\perp} &= \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} m^2 Z\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{0,2;\parallel} &= -i \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} m^2 Z'\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{0,3;\parallel} &= -i \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} m^2 Z'\left(\xi_m\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{0,3;\perp} &= -i \frac{1}{4} \frac{\omega_c}{\kappa_0 \parallel \nu_0 \parallel} \sum_{m=-\infty}^{\infty} mZ'\left(\xi_m\right) I_{m+1}\left(\kappa_0^2\right) - 2I_m\left(\kappa_0^2\right) + I_{m-1}\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{1,2;\perp} &= \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} \left(1 + \frac{\omega - m\omega_c}{k_\parallel \nu_0 \parallel} Z\left(\xi_m\right)\right) I_{m+1}\left(\kappa_0^2\right) - 2I_m\left(\kappa_0^2\right) + I_{m-1}\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{1,2;\perp} &= i \frac{1}{k_\perp \nu_0 \parallel \kappa_0 \parallel} \sum_{m=-\infty}^{\infty} m^2 \left(1 + \frac{\omega - m\omega_c}{k_\parallel \nu_0 \parallel} Z\left(\xi_m\right)\right) I_m\left(\kappa_0^2\right) e^{-\kappa_0^2 \perp} \\
E^{c,c'}_{0,2;\perp} &= -E^{c,c'}_{0,2;\perp} \\
\end{align}

(A.95)
\[ E_{0,2;\parallel}^{s,s'} = -E_{0,2;\parallel}^{1,s'} \]

\[ E_{0,2;\perp}^{s,s'} = -i \frac{1}{\kappa_{0;\parallel}} \sum_{m=-\infty}^{\infty} Z(\xi_m) e^{-\kappa_{0;\perp}^2} \left[ \left( \frac{m^2}{\kappa_{0;\perp}^2} - \kappa_{0;\perp}^2 \right) I_m(\kappa_{0;\parallel}^2) \right. \]
\[ - \left( I_{m-1}(\kappa_{0;\parallel}^2) + I_{m+1}(\kappa_{0;\parallel}^2) \right) \left( 1 - \kappa_{0;\perp}^2 \right) - \frac{1}{2} \kappa_{0;\perp}^2 \left( I_{m-2}(\kappa_{0;\parallel}^2) + I_{m+2}(\kappa_{0;\parallel}^2) \right) \right] \]

\[ E_{0,3;\parallel}^{s,s'} = -E_{0,3;\parallel}^{c,s'} \]

\[ E_{0,3;\perp}^{s,s'} = -\frac{i v_{0;\perp}^2}{v_{0;\parallel}} \frac{1}{\kappa_{0;\parallel}} \sum_{m=-\infty}^{\infty} Z'(\xi_m) e^{-\kappa_{0;\perp}^2} \left[ \left( \frac{m^2}{\kappa_{0;\perp}^2} - \kappa_{0;\perp}^2 \right) I_m(\kappa_{0;\parallel}^2) \right. \]
\[ - \left( I_{m-1}(\kappa_{0;\parallel}^2) + I_{m+1}(\kappa_{0;\parallel}^2) \right) \left( 1 - \kappa_{0;\perp}^2 \right) - \frac{1}{2} \kappa_{0;\perp}^2 \left( I_{m-2}(\kappa_{0;\parallel}^2) + I_{m+2}(\kappa_{0;\parallel}^2) \right) \right] \]

\[ E_{1,2;\parallel}^{s,s'} = -E_{1,2;\parallel}^{c,s'} \]

\[ E_{1,2;\perp}^{s,s'} = -\frac{\omega_c}{k_{\parallel}} \sum_{m=-\infty}^{\infty} \left( 1 + \frac{\omega - n\omega_c}{k_{\parallel} v_{0;\parallel}} Z(\xi_n) \right) e^{-\kappa_{0;\perp}^2} \left[ \left( \frac{m^2}{\kappa_{0;\perp}^2} - \kappa_{0;\perp}^2 \right) e^{-\kappa_{0;\perp}^2} I_m(\kappa_{0;\parallel}^2) \right. \]
\[ - \left( I_{m-1}(\kappa_{0;\parallel}^2) + I_{m+1}(\kappa_{0;\parallel}^2) \right) \left( 1 - \kappa_{0;\perp}^2 \right) - \frac{1}{2} \kappa_{0;\perp}^2 \left( I_{m-2}(\kappa_{0;\parallel}^2) + I_{m+2}(\kappa_{0;\parallel}^2) \right) \right] \]

(\text{A.96})

\section*{A.12 Integral elements for a ring beam, arbitrary angle propagation}

In what follows, \( \kappa_{0;\parallel} = \frac{k v_{0;\parallel}}{\omega_c} \) and \( \kappa_{b;\perp} = \frac{k_{\perp} u_{\perp}}{\omega_c} \).
\[
E_{0,1}^{1,1} = \frac{2i}{\kappa_0} \frac{1}{\nu_0} \sum_{m=-\infty}^{\infty} m Z(\xi_m) J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp})
\]

\[
E_{0,1}^{1,1} = -i \frac{1}{\nu_0} \sum_{m=-\infty}^{\infty} Z'(\xi_m) J_m^2(\kappa_{b,\perp})
\]

\[
E_{0,1}^{1,1} = \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} Z(\xi_m) \left( J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) + \kappa_{b,\perp} \left[ J_m^2(\kappa_{b,\perp}) + J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) \right] \right)
\]

\[
E_{1,1}^{1,1} = \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} \left( 1 + \frac{\omega - m\omega_c}{k_v v_0} \right) Z(\xi_m) \left( J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) + \kappa_{b,\perp} \left[ J_m^2(\kappa_{b,\perp}) + J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) \right] \right)
\]

\[
E_{0,2}^{1,0} = \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} m Z'(\xi_m) J_m^2(\kappa_{b,\perp})
\]

\[
E_{0,2}^{1,0} = 2i \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} m^2 Z(\xi_m) J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp})
\]

\[
E_{0,2}^{1,0} = -i \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} m Z'(\xi_m) J_m^2(\kappa_{b,\perp})
\]

\[
E_{0,2}^{1,0} = \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} \left( 1 + \frac{\omega - m\omega_c}{k_v v_0} \right) Z(\xi_m) \left( J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) + \kappa_{b,\perp} \left[ J_m^2(\kappa_{b,\perp}) + J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) \right] \right)
\]

\[
E_{1,2}^{1,0} = \frac{1}{\kappa_{b,\perp}} \sum_{m=-\infty}^{\infty} m \left( 1 + \frac{\omega - m\omega_c}{k_v v_0} \right) Z(\xi_m) \left( J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) + \kappa_{b,\perp} \left[ J_m^2(\kappa_{b,\perp}) + J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp}) \right] \right)
\]

\[
E_{1,2}^{1,0} = 2i \frac{1}{\kappa_{b,\perp}} \sum_{m=-\infty}^{\infty} \left( 1 + \frac{\omega - m\omega_c}{k_v v_0} \right) Z(\xi_m) J_m(\kappa_{b,\perp}) J'_m(\kappa_{b,\perp})
\]

\[
E_{0,2}^{1,0} = -E_{0,2}^{1,0}
\]

\[
E_{0,2}^{1,0} = -E_{0,2}^{1,0}
\]

(A.97)
\[ E_{s',s''}^{s,c'} E_{0,2,\perp} = 2i \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} Z(\xi_m) J'_m(\kappa_{b,\perp}) \left( J'_m(\kappa_{b,\perp}) + \kappa_{b,\perp} J''_m(\kappa_{b,\perp}) \right) \]

\[ E_{s',s''}^{s,c'} E_{0,3,\parallel} = -E_{s',s''}^{s,c'} \]

\[ E_{s',s''}^{s,c'} E_{1,2,\perp} = 2i \frac{1}{\kappa_0} \sum_{m=-\infty}^{\infty} Z(\xi_m) J'_m(\kappa_{b,\perp}) \]
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