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HECKE ALGEBRAS AND THE LUSZTIG ISOMORPHISM

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INTRODUCTION

Let $G$ be a Chevalley group over a finite field with $q$ elements and let $B$ be a Borel subgroup of $G$. Let $H(G,B)$ be the Hecke algebra of the pair $(G,B)$. J. Tits showed that the Hecke algebra over $C$ is isomorphic to the group algebra over $C$ of the Weyl group. N. Iwahori conjectured that the Hecke algebra over $Q$ of the pair $(G,B)$ is isomorphic to the group algebra over $Q$ of the Weyl group. Benson and Curtis proved that this conjecture is true whenever $G$ is simple of type different from $E_7$, $E_8$. With the help of Springer they proved that the conjecture is no longer true when $G$ is of type $E_7$, $E_8$. G. Lusztig constructed an explicit isomorphism from the Hecke algebra over $Q(q^{1/2})$ to the group algebra over $Q(q^{1/2})$ of the Weyl group.

The main purpose of this thesis is to investigate the general properties of this isomorphism. As a consequence of our investigation we introduce a way of obtaining orthogonal primitive idempotents inside the Hecke algebra.

This thesis has been divided into six chapters.

In Chapter 1 we recall some auxiliary results about the structure of Coxeter groups and their associated Hecke algebras. We also recall the Kazhdan-Lusztig decomposition of a Coxeter group into left, right and two sided cells and we explain how the cells give rise to representations of the Coxeter groups and of the corresponding Hecke algebras.

Let $W$ be a finite indecomposable Coxeter group satisfying a certain property (property (A)) for the structure of its two sided cells. We recall an explicit isomorphism from $H_{Q(q^{1/2})}(W)$ to $Q(u^{1/2})(W)$ constructed by G. Lusztig, where $Q(u^{1/2})$ is the field of fractions of the polynomial ring $Q[u^{1/2}]$. 
The subsequent chapters are our own work.

In Chapter 2 we find an explicit formula for Lusztig's isomorphism in the case where \( W = D_{2n} \), the dihedral groups. It turns out that these groups satisfy the required property (A). Here we achieve our results using classical properties of the Chebyshev polynomials of the second kind.

In Chapter 3 we investigate the centre of the Hecke algebra over the polynomial ring \( \mathbb{Q}[u] \), following some ideas of R.W. Carter. These ideas give a natural basis for the centre of the Hecke algebras of dihedral groups and they lead to an interesting conjecture for the form of a basis of the centre of the Hecke algebra in the general case.

In Chapter 4 we find the images of the central basis elements of the Hecke algebra of dihedral type determined in the previous chapter, under Lusztig's isomorphism. Here we show that the images of these elements no longer involve \( u^\frac{1}{2} \).

In Chapter 5 we prove results valid for arbitrary Hecke algebras. Here we show that the images of the generators \( T_s \) of the Hecke algebra under Lusztig's isomorphism \( \phi \) are given by \( \phi(T_s) = \frac{u-1}{2} - \frac{u+1}{2} - s + (u^2-1)^2 F_s \) for some \( F_s \in \mathbb{Q}W \).

We give two independent proofs of this result. The second one is based on some conjectures made by R.W. Carter and uses the results of A. Gyoja for the irreducible representations of Coxeter groups and Hecke algebras.

We also show that if \( c = \sum_{W \in W} a_W W \) is an element in the centre of the Hecke algebra with \( a_W \in \mathbb{Q}[u] \), then in most of the cases the image of \( c \) under Lusztig's isomorphism \( \phi \), belongs to \( \mathbb{Q}[u](W) \).

In Chapter 6 we deal with the construction of orthogonal primitive idempotents inside the Hecke algebra. These idempotents are obtained naturally from the decomposition of a maximal commutative subalgebra inside
the Hecke algebra. We shall achieve this decomposition in some special cases.

Finally we discuss some open questions which arise naturally from our work, and we make some conjectures which would allow these questions to be settled.
1.1 Coxeter groups

We shall first state some well known results about Coxeter groups. A group $W$ given by generators and relations as follows,

\[ W = (W,S) = \langle s_i, i \in I, s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, m_{ij} \in \mathbb{Z}, m_{ij} \geq 2 \quad i \neq j \text{ or } m_{ij} = \infty \]  

is called a Coxeter group. When $m_{ij} = \infty$ we omit the relation $(s_is_j)^{m_{ij}} = 1$. We put $S = \{s_i, i \in I\}$.

The Coxeter graph $\Gamma$ associated to a given Coxeter group $(W,S)$ is by definition a set of nodes labelled by the elements of $S$, together with a set $Y$ of edges. An edge is a subset of $S$ consisting of two elements, such that for $s_i, s_j \in S \{s_is_j \in Y, \text{ if and only if } m_{ij} \geq 3.\}$. In this case the node $i$ is joined to the node $j$ by $m_{ij} - 2$ bonds. If $m_{ij} = 2$, then $s_is_j = s_js_i$ and $s_is_j$ are not joined by a bond.

A Coxeter group $W$ is entirely determined up to isomorphism by its associated Coxeter graph.

A Coxeter group $(W,S)$ is defined to be indecomposable if its Coxeter graph is connected, i.e. for any $s, t \in S$, there exists a sequence $s_0 = s, s_1, s_2, \ldots s_r = t$ in $S$ for some $r \geq 0$ such that $(s_{i-1}, s_i)$ is an edge for every $1 \leq i \leq r$.

Theorem 1.1 If $(W,S)$ is an indecomposable Coxeter group of finite order, then its Coxeter graph has one of the following forms.
Assume that $W = \langle s_1, \ldots, s_k | s_i^2 = 1, (s_is_j)^{m_{ij}} = 1, 1 \leq i, j \leq k \neq j \rangle$ and $m_{ij} \geq 2$ if $i \neq j$, $m_{ii} = 1$ is a finite Coxeter group. Then $W$ can be described as a group generated by reflections in a finite dimensional...
Euclidean space. Let $V$ be a vector space over the real field $\mathbb{R}$ of dimension $t$ with basis $\{a_1, \ldots, a_t\}$. We define a bilinear form on $V$ by $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that $\langle a_i, a_j \rangle = \cos \frac{\pi}{m_{ij}}$ and extend by linearity. This form is symmetric since $m_{ij} = m_{ji}$, and $\langle a_i, a_i \rangle = 1$.

Let $H_1$ be the subspace of $V$ defined by $H_1 = \{v \in V : \langle a_1, v \rangle = 0\}$. Then, $\dim H_1 = t-1$ and $V = \mathbb{R}a_1 \oplus H_1$. We define a linear map $\tau_1 : V \to V$ by $\tau_1(v) = v - 2 \langle a_1, v \rangle a_1$. Then $\tau_1(a_1) = -a_1$ and $\tau_1(v) = v$ for all $v \in H_1$.

Thus $\tau_1$ is the reflection in the hyperplane $H_1$, and so $\tau_1^2 = 1$. We also have that $(\tau_1 \tau_j)^{m_{ij}} = 1$ if $i \neq j$, and that $\langle \tau_1(v), \tau_1(v') \rangle = \langle v, v' \rangle$ for all $v, v' \in V$. So there exists a homomorphism $\theta : W \to \langle \tau_1, \ldots, \tau_t \rangle$ from $W$ into the group of isometries of $V$ given by $\theta(s_i) = \tau_i$.

The form $\langle v, v' \rangle$ on $V$ can be shown to be non-singular and positive definite and so $V$ may be regarded as Euclidean space. It can also be shown that $W$ acts faithfully on $V$ and that $V$ is an irreducible $W$-module.

We call $V$ the reflection representation of $W$.

For any element $w$ of a Coxeter group $(W, S)$ we define the length of $w$, denoted by $\ell(w)$, to be the smallest integer $q \geq 0$ such that $w$ is a product of a sequence $(s_1, \ldots, s_q)$ of $q$ elements of $S$. We define a reduced expression of $w$ to be an expression $w = s_{r_1} \cdots s_{r_q}$ where $q = \ell(w)$.

Let $\Pi = \{a_1, \ldots, a_t\}$. $\Pi$ is called the set of simple roots.

Let $\phi = W(\Pi) = \{w(a_i), w \in W, a_i \in \Pi\}$. $\phi$ is called the set of roots.

It can be shown that each $a \in \phi$ has the form $a = \sum_{i=1}^{t} \lambda_i a_i$ with each $\lambda_i \geq 0$ or each $\lambda_i \leq 0$.

Let $\phi^+ = \{a \in \phi : a = \sum_{i=1}^{t} \lambda_i a_i, \lambda_i \geq 0 \ \forall \ i = 1, \ldots, t\}$, and let
\( \Phi^- = \{ \alpha \in \Phi : \alpha = \sum_{i=1}^{k} \lambda_i \alpha_i \leq 0 \ \forall i = 1, \ldots, k \} \),

\( \Phi^+ \) is called the set of positive roots, \( \Phi^- \) is called the set of negative roots, and clearly \( \Phi = \Phi^+ \cup \Phi^- \).

For any \( w \in W \) we denote by \( n(w) \) the number of positive roots made into negative by \( w \).

The following proposition provides some well known results about the \( \pi(w) \).

**Proposition 1.2**

(i) For any \( w \in W \), \( n(w) = \pi(w) \)

(ii) Let \( s_i \in S \), \( w \in W \) and \( w = s_1 \ldots s_q \) a reduced expression of \( w \). Then there are two possibilities.

(a) \( \pi(s_iw) = \pi(w) + 1 \) and \((s_1, s_2, \ldots, s_q)\) is a reduced expression of \( s_iw \).

In this case \( w^{-1}(a_i) \in \Phi^+ \).

(b) \( \pi(s_iw) = \pi(w) - 1 \) and there exists a \( 1 \leq j \leq q \) such that

\( (s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_q) \) is a reduced expression of \( s_iw \) and

\( (s_1 s_2 \ldots s_{j-1} s_{j+1} s_{j+2} \ldots s_q) \) is a reduced expression of \( w \). In this case

\( w^{-1}(a_i) \in \Phi^- \).

(iii) If \( W \) is finite there exists a unique element of maximal length, denoted by \( w_0 \).

(iv) Let \( s, t \in S \), \( w \in W \) such that \( \pi(sw) = \pi(wt) \) and \( \pi(swt) = \pi(w) \).

Then \( sw = wt \). (For a proof see [6], §2.2, and [6], pages 15, 18).

The Bruhat order relation on \( W \) is defined by: For any \( y, w \in W \) we say \( y \preceq w \) if there exist reduced expressions \( y = s_{j_1} s_{j_2} \ldots s_{j_q} \) and
\[ w = s_1 s_2 \ldots s_q \] with all \( s_i \) lying in \( S \), such that \((j_1, j_2, \ldots, j_3)\) is a subsequence of \(1, 2, \ldots, q\).

A Coxeter group is called crystallographic if, for all \(1 \neq j\)
\[ m_{ij} \in \{2, 3, 4, 6\}. \]
Thus the indecomposable Coxeter groups which are crystallographic are of type \(A_n, B_n, D_n, E_6, E_7, E_8, F_4, G_2\).

1.2 The Hecke algebra of a finite Chevalley Group

Let \(L\) be a simple Lie algebra of finite dimension over \(\mathbb{C}\). Then, there is a finite crystallographic Coxeter group \(W\) associated to \(L\), called the Weyl group of \(L\). For each such finite indecomposable crystallographic Coxeter group \(W\), there is just one simple Lie algebra which has \(W\) as its Weyl group except when \(W\) has type \(B_k\), \(k \geq 3\), when there are two such Lie algebras, called \(B_k, C_k\).

Any such Lie algebra has a Cartan decomposition \(L = H \oplus \Sigma L_a\) where 
\[ H \text{ is a maximal commutative subalgebra called a Cartan subalgebra of } L \text{ and } L_a \text{ is a 1-dimensional } H\text{-module of } L. \]
The set \(\phi\) of 1-dimensional representations of \(H\) arising in this way is called the set of roots of \(L\). \(\phi\) has a subset \(\Pi\), called a set of simple roots, such that each root in \(\phi\) is uniquely expressible as a linear combination of elements of \(\Pi\) with coefficients in \(\mathbb{Z}\) which are either all non-negative or all non-positive. The set \(\phi\) decomposes in this way as \(\phi = \phi^+ \cup \phi^-\) where \(\phi^+, \phi^-\) are the positive and negative roots respectively. In this root system we no longer assume that roots are unit vectors, so in general this root system is different from the one defined in 1.1. (Proof of these results can be found in [16]).

We can choose a basis for \(L\) relative to a Cartan decomposition, called a Chevalley basis, whose elements are \(\{h_a, a \in \Pi, e_a, a \in \phi\}\).
The Lie product of any two elements in the basis is a linear combination of basis elements with coefficients in \( Z \).

For each \( x \in L \) we define \( \text{ad} x : L \to L \) by \( \text{ad} x y = [x, y] \). This is a derivation. For each \( \lambda \in \mathfrak{c} \) and for each \( a \in \mathfrak{q} \), the map \( \text{ad}(\lambda e_a) : L \to L \) is nilpotent. Thus we can form \( \exp \text{ad}(\lambda e_a) \) which is an automorphism of \( L \). We write \( \chi_e(\lambda) = \exp \text{ad}(\lambda e_a) \). The Chevalley basis has the property that the matrix \( M_a(\lambda) \) of each \( \chi_e(\lambda) \) with respect to this basis has entries which are of the form \( b \lambda^i \) for \( i \geq 0 \) and \( b \in \mathbb{Z} \).

Now for any field \( K \) we can define a Lie algebra \( L_K \) over \( K \) by taking all \( K \)-combinations of elements in a Chevalley basis and taking Lie multiplication as before, interpreting the integers as elements of the prime subfield of \( K \). \( L_K \) has a basis \( \{ h_a, a \in \mathfrak{p}, e_a, a \in \mathfrak{q} \} \), where \( h_a = h_a \otimes 1_K, e_a = e_a \otimes 1_K \). For each \( a \in \mathfrak{q}, t \in K \), let \( \chi_e(t) \) be the matrix obtained by replacing the entry \( b \lambda^i \) by \( b \lambda^{1_i} \) where \( \lambda^i \) is the element of the prime subfield of \( K \) corresponding to \( b \in \mathbb{Z} \).

Define \( \chi_e(t) \) to be the linear map of \( L_K \) into itself represented by the matrix \( M_a(t) \) with respect to the basis \( \{ h_a, a \in \mathfrak{p}, e_a, a \in \mathfrak{q} \} \). Then it can be shown that \( \chi_e(t) \) is an automorphism of \( L_K \) for each \( a \in \mathfrak{q}, t \in K \). The group of automorphisms of \( L_K \) generated by \( \chi_e(t) \) for all \( a \in \mathfrak{q}, t \in K \) is called the adjoint Chevalley group of type \( L \) over \( K \). In particular if \( K \) is the finite field with \( q \) elements \( \mathbb{F}_q \), we obtain a finite Chevalley group which will be denoted by \( G(q) \).

For example if \( L \) is the simple Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \) of all \( n \times n \) matrices of trace 0 over \( \mathbb{C} \), \( G(q) \) will be the group \( \text{PSL}_n(q) \) of all \( n \times n \) matrices of determinant 1, factored by its centre. Let \( U(q) \) be the subgroup of \( G(q) \) generated by \( \chi_e(\lambda) \) for all \( a \in \mathfrak{q}^+ \) and all \( \lambda \in \mathbb{F}_q \). Let
B(q) = N_{G(q)}(U(q)), the normalizer of U(q) in G(q). In the previous example U(q) is the subgroup of all upper unitriangular matrices and B(q) is the subgroup of all upper triangular matrices. Let \(1_{B(q)}\) be the trivial representation of B(q) over \(\mathbb{C}\), and let \(p\) be the representation of G(q) induced by \(1_{B(q)}\), i.e. \(p = 1_{B(q)}(g)\). Let \(e = \frac{1}{|B(q)|} \sum_{x \in B(q)} x\). Then \(e \in \mathbb{E}B(q)\), \(e^2 = e\), and \(\mathbb{E}B(q) \cdot e\) is a left \(B(q)\)-module affording \(1_{B(q)}\) while \(\mathbb{E}G(q) \cdot e\) is a left \(G(q)\)-module affording \(1_{B(q)}\) (see [7], Proposition 11.21). We define the Hecke algebra of the pair \((G(q), B(q))\) to be the endomorphism algebra of the module \(\mathbb{E}G(q) \cdot e\). We write \(H_q(G) = \text{End}_{\mathbb{E}}(1_{B(q)})\). Recall that the group algebra \(\mathbb{E}G(q)\) is isomorphic as \(\mathbb{E}\)-algebra with the algebra of \(\mathbb{E}\)-valued functions \(f: G(q) \to \mathbb{E}\) under convolution product, with the element \(\sum_{x \in G(q)} a_x x\) corresponding to the function \(f\), defined by \(f(x) = a_x\), \(x \in G(q), a_x \in \mathbb{E}\). If \(f, g\) are \(\mathbb{E}\)-valued functions on \(G(q)\), their convolution product is defined as the function \(f \cdot g: G(q) \to \mathbb{E}\) given by \((f \cdot g)(x) = \sum_{y \in G(q)} f(xy^{-1})g(y)\).

Let \(A = B(q)gB(q), g \in G(q)\) and consider the \(\mathbb{E}\)-algebra of all formal linear combinations of \((B(q), B(q))\) double cosets, \(\sum_{A \in \mathbb{E}} \alpha_A A\) under the following multiplication:

Let \(A = B(q)gB(q) = \mathbb{U} B(q)g_i, A' = B(q)g'B(q) = \mathbb{U} B(q)g'_j\).

Define \(A \cdot A' = \sum_C \sum_{i,j} \mu_{A,A'}^{C} C\) where the summation is taken over all double cosets \(C = B(q) c B(q)\) and \(\mu_{A,A'}^{C}\) is the number of pairs \((i,j)\) such that \(B(q)g_i g'_j = B(q) \cdot c\). It can be shown that this is a well defined multiplication, in other words the number \(\mu_{A,A'}^{C}\) is independent of the choice of the coset representatives \(g_i, g'_j, c\) (see [14]). Moreover,
Theorem 1.2.1: The following four descriptions of the Hecke algebra of a finite Chevalley group $G(q)$ with respect to the subgroup $B(q)$ are equivalent:

(i) $\text{End}_G(\mathcal{C}_B)$

(ii) $e \cdot \mathcal{C}_G(q) \cdot e$ where $e = \frac{1}{|B(q)|} \sum_{x \in B(q)} x$

(iii) Functions constant on $(B(q), B(q))$ double cosets, under convolution product.

(iv) The set of all formal linear combinations of the form $\sum A_n A$, the summation being taken over the $(B(q), B(q))$ double cosets, with multiplication defined as above (Proof, see [10]).

Theorem 1.2.2: The dimension of $H_\xi(q)$ is $\dim H_\xi(q) = |W|$, where $W$ is the Weyl group of $L$. $H_\xi(q)$ has a basis $\{v_w, w \in W\}$ such that if $w = s_1 \ldots s_q$ is a reduced expression of $w$, then $v_w = v_{s_1} \ldots v_{s_q}$. We write $v_i$ for $v_{s_i}$, $i = 1, \ldots, \xi$. Each $v_i$ satisfies the quadratic relation

$$v_i^2 = q \cdot 1 + (q-1)v_i \quad (v_1 = 1 \text{ the identity of } H_\xi(q)).$$

Finally $H_\xi(q)$ has a presentation as $\mathbb{C}$-algebra given by

$$H_\xi(q) = \langle v_1, \ldots, v_\xi \mid v_i^2 = q \cdot 1 + (q-1)v_i, v_i v_j = v_j v_i \ldots \rangle$$

$m_{ij}$ being the order of $s_i s_j$.

(For the nature of the elements $v_w, w \in W$ and for a proof see [10]).
Theorem 1.2.3. (J. Tits' deformation Theorem)

The Hecke algebra $H_q(q)$ over $\mathbb{C}$ is isomorphic to the group algebra over $\mathbb{C}$ of the Weyl group: $H_q(q) = \mathbb{C}W$. (For a proof see [15] page 249).

We note that the product of any two basis elements $v_w$ of $H_q(q)$ is a $\mathbb{Z}$-combination of basis elements. So we have a subalgebra $H_z(q)$ of all $\mathbb{Z}$-combinations of the basis elements $v_w$. However in this subalgebra the element $v_w$ is not invertible since $v_i^{-1} = q^{-1}v_i + (q^i - 1) - 1$. We therefore extend the ground ring to include $q^{-1}$.

Moreover, although Tits showed that $H_q(q) = \mathbb{C}W$, he did not give any explicit isomorphism between $H_q(q)$ and $\mathbb{C}W$. Iwahori conjectured that $H_q(q) = \mathbb{C}W$ (see [15]). Benson and Curtis proved that this is true whenever the Chevalley group $G(q)$ is simple of type $E_7, E_8$. (See [3]). It is not true when $G(q)$ has type $E_7, E_8$. (See [4]).

G. Lusztig constructed an explicit isomorphism between $H_q(q)$ and $Q(q^2)(W)$. (See [12]).

The construction makes essential use of the Kazhdan-Lusztig theory.

1.3 The Kazhdan-Lusztig Theory

In this section we deal with the theory developed by Kazhdan and Lusztig in order to study representations of Hecke algebras. (See [11]).

Let $(W,S) = \langle s_1, \ldots, s_k | s_i^2 = 1, s_is_j \ldots = s_js_i \ldots \rangle$ be a finite Coxeter group. With such a Coxeter system we associate an algebra $\mathbb{H}$ over the polynomial ring $\mathbb{Z}[u], u$ being an indeterminate over $\mathbb{Z}$, as follows. $\mathbb{H}$ has basis elements $\{T_w, w \in W\}$ and multiplication defined by the rules:
It can be shown that $H$ has a presentation as an associative $\mathbb{Z}[u]$-algebra given by generators and defining relations:

$$T_{s_i}^2 = u.T_i + (u-1)T_i.s_i \quad s_i \in S.$$

We extend the ground ring and we define the generic Hecke algebra $H(u)$ as follows: $H(u) = H \otimes \mathbb{Z}[u^\frac{1}{2}, u^{-\frac{1}{2}}]$. We put $A = \mathbb{Z}[u^\frac{1}{2}, u^{-\frac{1}{2}}]/\mathbb{Z}[u]$.

Our first step is to define representations of $H(u)$ with respect to a special basis. These representations are defined in terms of certain graphs.

**Definition 1.3.0.** Let $\mathbb{Z}$ be the ring of integers. A $W$-graph over $\mathbb{Z}$ is a set of vertices $X$ together with a set of edges $Y$, an edge being a subset of $X$ consisting of two elements and with two additional requirements:

1. For each vertex $x \in X$ we are given a subset $I_x$ of $S$ and for each ordered pair of vertices $y, x$ such that $\{y, x\} \in Y$ we are given an integer $u(y, x)$.
2. Let $E$ be the free $A$-module with basis $X$. Then for any $s \in S$

$$\tau_s(x) = \begin{cases} -x & \text{if } s \in I_x \\ u_x + u_j & \sum_{y \in Y} u(y, x)y \quad \text{if } s \notin I_x \end{cases}$$
defines an endomorphism of $E$, and there is a representation
\[ \phi(u) : H(u) \rightarrow \text{End}(E) \] such that $\phi(T_s) = \tau_s$, for each $s \in S$.

For any Coxeter group $W$ we shall construct such a graph. Let
\[ a \rightarrow \overline{a} \] be the involution of the ring $A = \mathbb{Z}[u^\frac{1}{2}, u^{-\frac{1}{2}}]$ defined by $u^\overline{T} = u^{-\frac{1}{2}}$. Now for every $s \in S$ the element $T_s$ is invertible and $T_s^{-1} = u^{-1}T_s + (u^{-1} - 1)T_1$.

Therefore it makes sense to extend the involution above to a map
\[ H(u) \rightarrow H(u) \] defined by $E \rightarrow E^{\overline{a}} = E \tau_s$. This map preserves addition and multiplication but not scalar multiplication.

For any $w \in W$ we define $u_w = u^{\varepsilon(w)}$, $c_w = (-1)^{\varepsilon(w)}$. Let $s$ be the Bruhat ordering on $W$, defined in §1.1. The following results are valid for an arbitrary Coxeter group $W$, however we restrict ourselves to a finite Coxeter group. For a proof of these results (see [11]).

**Theorem 1.3.1.** For any $w \in W$, there is a unique element $C_w \in H(u)$ such that:

1. $C_w = C_w$
2. $C_w = \sum_{y < w} c_{y,w} u_y^{\frac{1}{2}} u_y^{-1} P_{y,w} T_y$

where $P_{y,w} \in A$ is a polynomial in $u$ of degree $\leq \frac{1}{2} (\varepsilon(w) - \varepsilon(y) - 1)$ for $y \prec w$ and $P_{w,w} = 1$.

**Definition 1.3.2.** Given $y,w \in W$ we say $y \prec w$ if the following conditions are satisfied: $y < w$, $c_w = -c_y$ and $P_{y,w}$ is a polynomial in $u$ of degree exactly $\frac{1}{2} (\varepsilon(w) - \varepsilon(y) - 1)$. In this case the leading coefficient of $P_{y,w}$ is denoted by $\mu(y,w)$. It is a non-zero integer.
The polynomials $P_{y,w}$ defined in Theorem 1.3.1 are given by the following inductive formula:

$$P_{y,w} = u^{c} P_{s_{y},v} + u^{c} P_{y,v} - \sum_{y \leq z < v \text{ and } sz < z} \mu(z,v) u^{-1} u_{v}^{\frac{1}{2}} u_{y}^{\frac{1}{2}} P_{y,z} (y \leq w)$$

where $w = sv$, $s \in S$ with $\ell(w) = \ell(v) + 1$, $c = \begin{cases} 1 & \text{if } sy < y \\ 0 & \text{if } sy > y \end{cases}$ and we make the convention that $P_{x,v} = 0$ if $x \not\in v$. (See Theorem 1 in [11]).

Remark. The elements $(C_{w} | w \in W)$ defined in Theorem 1.3.1 provide a new basis for $H(u)$.

Lemma 1.3.3.

(i) For each $y < w$ with $\ell(w) = \ell(y) + 1$, we have $P_{y,w} = 1$. In particular we have $y < w$ and $\mu(y,w) = 1$.

(ii) For each $y < w$ with $\ell(w) = \ell(y) + 2$ we have $P_{y,w} = 1$.

For further properties of the $P_{y,w}$ see Lemma 2.6 in [11].

Lemma 1.3.4. Let $(C_{w} | w \in W)$ as defined in Theorem 1.3.1, and let $s \in S$. Then:

$$T_{s} C_{v} = \begin{cases} -C_{v} & \text{if } sv < v \\ u C_{v} + u^{\frac{1}{2}} C_{sv} + u^{\frac{1}{2}} \sum_{z < v \text{ and } sz < z} \mu(z,v) C_{z} & \text{if } sv > v \end{cases}$$
(11) Similarly interchanging left and right we have

\[ C_v T_s = \begin{cases} 
-C_v & \text{if } vs < v \\
-\sum_{v} u_{C_v} + u_{C_{vs}} + u_{C_{v}} & \sum_{v} u_{C_z} C_z & \text{if } vs > v \\
& \text{if } z < v \\
& \text{if } z < z 
\end{cases} \]

(See Proof of Theorem 1.3 in [11]).

Lemma 1.3.5. (1) Let \( x, y \in W, s \in S \) be such that \( x < y, sy < y, sx > x \). Then \( P_{x,y} = P_{sx,y} \). Moreover \( x < y \) if and only if \( y = sx \), and this implies that \( u(x,y) = 1 \).

(11) Let \( x, y \in W, s \in S \) be such that \( x < y, ys < y, xs > x \). Then \( P_{x,y} = P_{xs,y} \). Moreover \( x < y \) if and only if \( y = xs \), and this implies that \( u(x,y) = 1 \).

(See Proof of Theorem 1.3 in [11]).

Now, Lemma 1.3.4(1) and 1.3.5(1) enable us to prove the following:

Theorem 1.3.6. Let \( \Gamma \) be the graph whose vertices are the elements of \( W \) and whose edges are the subsets of \( W \) of the form \( \{y,w\} \) with \( y < w \). For each \( w \in W \), let \( I_w = E(w) = \{ s \in S : sw < w \} \). Then \( \Gamma \), together with the assignment \( w \mapsto I_w \) and with the function \( u \) given in Definition 1.3.2 is a \( W \)-graph. (See Theorem 1.3 in [11]).

Since Lemma 1.3.4 provides the action of the generators \( T_s \) on the basis \( \{ C_w | w \in W \} \), it is natural to ask what is the effect of any \( T_w \) on this basis.

The following result is stated for future purposes:
Lemma 1.3.7. Let $x, w \in W$. Then

$$T_w C_x = \sum_{z \in W} k_z(u) C_z + \sum_{z \in W} u^3 \lambda_z(u) C_z$$

where $k_z(u), \lambda_z(u) \in \mathbb{Z}[u]$.

Proof. We shall use induction on $\ell(w)$.

If $\ell(w) = 0$ then $w = 1$ and our assertion holds. Assume by induction that our assertion holds for all elements $w'$ with $\ell(w') < \ell(w)$.

Let $w \in W$. We write $w = sv$ for some $v$ with $\ell(v) = \ell(w) - 1$. We can apply our induction hypothesis on $T_v C_x$. So

$$T_w C_x = T_s(T_v C_x) = T_s \left( \sum_{z \in W} k_z(u) C_z + \sum_{z \in W} u^3 \lambda_z(u) C_z \right)$$

where $k_z(u), \lambda_z(u) \in \mathbb{Z}[u]$.

Consider a $C_z$ for which $\ell(z) = \ell(x) \mod 2$. Then

$$k_z(u) T_s C_z = \begin{cases} -k_z(u) C_z & \text{if } sz < z \\ k_z(u)[u C_z + u^3 C_{sz} + u^3 \sum_{\gamma < z, \gamma < \gamma} \mu(\gamma, z) C_{\gamma}] & \text{if } sz > z. \end{cases}$$

Now when $\gamma < z$, then $\ell(z) \neq \ell(\gamma) \mod 2$, therefore $\ell(\gamma) \neq \ell(x) \mod 2$ and the coefficient of $C_\gamma$ is $u^3 g(u)$ for some $g(u) \in \mathbb{Z}[u]$. The coefficient of $C_{sz}$ is also $u^3 k_z(u)$ and $\ell(sz) \neq \ell(x) \mod 2$, while the coefficient of $C_z$ is $u k_z(u)$. 
Next consider a \( C \) for which \( \ell(z) \not\equiv \ell(x) \mod 2 \). Then from the action of \( T \) on the summand \( \sum_{z \in W} u_{\Lambda}^z(u)C_z \) we obtain a certain linear combination of \( C \)'s as above and it is easy to check that those \( C \) for which \( \ell(b) \not\equiv \ell(x) \mod 2 \) appear with coefficient \( u f(u) \) for some \( f(u) \in \mathbb{Z}[u] \), while those \( C \) for which \( \ell(b) \equiv \ell(x) \mod 2 \) appear with coefficient \( h(u) \), for some \( h(u) \in \mathbb{Z}[u] \).

Therefore our lemma is now proved.

Corollary 1.3.7. The diagonal entries of the matrix which represents \( T \) with respect to the basis \( \{ C_x | x \in \mathcal{W} \} \) consist only of polynomials in \( \mathbb{Z}[u] \).

The following ideas are due to Kazhdan and Lusztig. (See [11]).

**Definitions 1.3.8.** Let \( x, x' \in \mathcal{W} \). We say that \( x, x' \) are joined, \( (x, x') \), if either \( x < x' \) or \( x' < x \). We define a left preorder \( \preceq \) on \( \mathcal{W} \) by saying \( x \preceq x' \) if there exists a sequence of elements of \( \mathcal{W} \): \( x = x_0, x_1, \ldots, x_n = x' \) such that for each \( i, 1 \leq i \leq n \), we have \( x_{i-1} - x_i \) and there exists an \( s \in S \) such that \( sx_{i-1} < x_{i-1} \) but \( sx_i > x_i \). We may then define an equivalence relation on \( \mathcal{W} \) by saying \( x \sim x' \) if \( x \preceq x' \preceq x \). The equivalence classes with respect to the relation \( \sim \) are called the left-cells of \( \mathcal{W} \). Similarly we define right cells by replacing the condition on \( s \in S \), by \( x_{i-1} < x_i \) and \( x_{i} > x_i \). The notation \( x \sim x' \) means that \( x, x' \) are in the same right cell. Finally we can define 2-sided cells by replacing the condition on \( s \in S \), by: either \( sx_{i-1} < x_{i-1} \) and \( sx_i > x_i \) or \( x_{i-1} < x_{i-1} \) and \( x_i > x_i \).
The notation $\preceq$ means the 2-sided preorder and $x \preceq x'$ means that $x, x'$ are inside the same 2-sided cell. Evidently, every left cell lies in a unique 2-sided cell and the same holds for every right cell. Each left cell, regarded as a full subgraph of the graph $\Gamma$ (Theorem 1.3.6) with the same sets $I_x$ and the same function $\mu$ is itself a $W$-graph. Therefore it gives rise to a representation of $H(u)$. Nevertheless, this representation is not always irreducible. However,

**Theorem 1.3.9.** Let $X$ be a left cell of $W = S_n$, and let $\Gamma$ be the $W$-graph associated to $X$ and let $\rho$ be the representation of $H(u)$ over the quotient field of $A$. Then $\rho$ is irreducible, and the isomorphism class of the $W$-graph $\Gamma$ depends only on the isomorphism class of $\rho$ and not on $X$. (See Theorem 1.4 in [10]).

### 1.4 The Lusztig Isomorphism

Let $W$ be a Coxeter group and $S$ its set of reflections. Let $E$ be the free $\mathbb{Q}[u^\pm]$ module with basis $\{e_w, w \in W\}$. Let $H$ be the generic Hecke algebra over $\mathbb{Q}[u^\pm]$. We know that the formulae

$$T_s e_w = \begin{cases} -e_w & \text{if } sw < w \\ ue_w + u^{-1} \sum_{y \prec w} \bar{\mu}(y,w)e_y & \text{if } sw > w \\ \end{cases}$$

$$e_w T_t = \begin{cases} -e_w & \text{if } wt < w \\ ue_w + u^{-1} \sum_{y \prec w} \bar{\mu}(y,w)e_y & \text{if } wt > w \\ \end{cases}$$
define an \((H,H)\) bimodule structure on \(E\). \(\tilde{\mu}(y,w) = \mu(y,w)\) if \(y < w\) and \(\tilde{\mu}(y,w) = \mu(w,y)\) if \(w < y\).

We also define a left \(W\)-module structure on \(E\) by

\[
se_w = \begin{cases} 
-e_w & \text{if } sw < w \\
e_w + \sum_{y-w, sy < y} \tilde{\mu}(y,w)e_y & \text{if } sw > w 
\end{cases}
\]

and a right \(W\)-module structure on \(E\) by

\[
e_w t = \begin{cases} 
-e_w & \text{if } wt < w \\
e_w + \sum_{y-w, yt < y} \tilde{\mu}(y,w)e_y & \text{if } wt > w 
\end{cases}
\]

We shall refer to the basis \(\{e_w \mid w \in W\}\) of \(E\) as the canonical basis of \(E\).

Now, the left and right \(W\)-module structures on \(E\) commute with each other. However, the left \(H\)-module structure does not necessarily commute with the right \(W\)-module structure. For each 2-sided cell \(X\) of \(W\) we shall construct an \(H\)-module \(M_X\) of dimension \(|X|\) over \(\mathbb{Q}[u^\pm]\). In fact, for each 2-sided cell \(X\) of \(W\), we consider the \(\mathbb{Q}[u^\pm]\) submodule \(E_X\) of \(E\), spanned by \(\{e_w : w \leq x \text{ for some } x \in X\}\). Inside \(E_X\) we consider \(E'_X\), the submodule spanned by \(\{e_w : w \in X, w \leq x \text{ for some } x \in X\}\). Put \(M_X = E_X/E'_X\). It is clear that \(M_X\) is an \(H\)-module of dimension \(|X|\). Let \(\text{grad}(E) = \sum_{X} M_X\), summed over all 2-sided cells \(X\) of \(W\). It has a canonical basis, the images \(\bar{e}_w\) of the elements \(e_w\) of \(E\). It is clear that both \(E_X\), \(E'_X\) are left \(H\)-submodules, left \(W\)-submodules, and right \(W\)-submodules. Therefore \(\text{grad}(E)\) is in a natural way a left \(H\)-module,
a left $W$-module and a right $W$-module. It is clear that the left $H$
action on the graded module is given by

$$T_s \tilde{e}_w = \begin{cases} 
-\tilde{e}_w & \text{if } sw < w \\
\omega_w + u_\frac{1}{2} \sum_{y \in \mathcal{L} \cap \mathcal{R}} u(y,w)\tilde{e}_y & \text{if } sw > w \\
\text{LR} & \text{sy} \leq y 
\end{cases}$$

and by specializing $u_\frac{1}{2} = 1$ we obtain the action of the generators $s \in W$
on the graded module, and therefore the action of any $w \in W$.

**Definition 1.4.1.** We say that the Coxeter group $W$ satisfies the property (A)
if given $y,w \in W$ such that:

1. $y - w$,
2. $(s \in S : sy < y) \neq (s \in S : sw < w)$,
3. $(s \in S : ys < y) \neq (s \in S : ws < w)$, then $y,w$ are not inside the
   same 2-sided cell of $W$.

**Lemma 1.4.2.** Assume that $W$ has the property (A). Then the left $H$-module
structure and the right $W$-module structure on grad($E$) commute. (Proof:
see Lemma 2.3 in [12].)

**Lemma 1.4.3.** Assume that $W$ is a finite Weyl group. Then $W$ satisfies the
property (A). (Proof: See Lemma 4.1 in [12].)

The only known proof of this result uses the theory of primitive
ideals in enveloping algebras. No elementary proof of this result is known.
Theorem 1.4.4. (Lusztig's isomorphism theorem). Assume that $W$ is a finite indecomposable Weyl group.

(a) There is a unique homomorphism of $\mathbb{Q}[u^\pm]$-algebras $\phi: H \rightarrow \mathbb{Q}[u^\pm](W)$ such that for any $h \in H$ and any $w \in W$, the difference $h e_w - \phi(h)e_w$ is a linear combination of $e_y$, $y$ not in the same 2-sided cell with $w$.

(b) Let $K$ be any field of characteristic zero, and $\chi$ any homomorphism of $\mathbb{Q}[u^\pm]$ into $K$, such that the specialized algebra $H_\chi = H \otimes K$ is semisimple. Then the specialized homomorphism of $K$-algebras $\phi_\chi: H_\chi \rightarrow KW$ is an isomorphism.

A more general version and a proof of this theorem is given in Theorem 3.1 in [2].

Remarks (1) The theorem above applies to any finite indecomposable Coxeter group $W$ provided that $W$ satisfies the property (A). In fact by taking $\chi: \mathbb{Q}[u^\pm] \rightarrow \mathbb{Q}(u^\pm)$ the natural inclusion, then $H_\chi(u^\pm) = \mathbb{Q}(u^\pm)(W)$.

(2) When $W$ is a finite Weyl group and $q$ is a prime power and $\mathbb{Q}(q^{1/2})$ is a field extension of $\mathbb{Q}$ of degree 1 or 2, then by taking $\chi: \mathbb{Q}[u^\pm] \rightarrow \mathbb{Q}(q^{1/2})$ such that $\chi(u^\pm) = q^{1/2}$ (the positive square root of $q$), we obtain $H_\chi(q^{1/2}) = \mathbb{Q}(q^{1/2})(W)$.

We next describe a procedure for an explicit construction of Lusztig's isomorphism.

Procedure: Let $W$ be a finite indecomposable Coxeter group which satisfies the property (A). The graded module over $\mathbb{Q}(u^\pm)$ when viewed as a left $H_\chi(u^\pm)$ module is semisimple and affords the left regular representation of $H_\chi(u^\pm)$. Let $L$ be a field containing $\mathbb{Q}(u^\pm)$ and assume that $L$ is a splitting
field for $H_{Q(u^1)}$. In fact such a field $L$ can be chosen of the form $F(u^1)$ where $F = Q$ if $W$ is crystallographic, (see [3], [4]), or $F = Q(\sqrt{5})$ if $W$ is of type $H_3$ (see [12]), or $F = Q(\frac{1 + \sqrt{5}}{2})$ if $W$ is of type $H_4$ (see [2]), or $F = Q(2 \cos \frac{2\pi}{n})$ if $W$ is a dihedral group $D_{2n}$ $n = 5$ or $n \geq 7$.

It is well known that $H_L$ is also semisimple and therefore the graded module over $L$ has a decomposition into a direct sum of left absolutely irreducible $H_L$ submodules. Let

$$\text{(D)} : \text{grad}(E) = V_{11} \oplus \ldots \oplus V_{1d_1} \oplus \ldots \oplus V_{t1} \oplus \ldots \oplus V_{td_2}$$

be one such decomposition where $V_{ij} = V_{rs}$ if and only if $i = r$, and for each $i \in \{1, \ldots, t\}$, $V_{ij}$ has dimension $d_i$, for every $j = 1, \ldots, d_i$. We choose a basis of each constituent $V_{1r}$ and in this way we obtain a basis adapted to this decomposition.

We also choose a full set of irreducible constituents, namely $X = (V_{11}, V_{21}, \ldots, V_{t1})$.

In order to determine Lusztig's isomorphism $\phi$, it is enough to determine the images under $\phi$ of the generators $T_s$, $s \in S$. By part (a) of the Theorem 1.4.4 we have that $T_s$ and $\phi(T_s)$ act on the same way on the graded module. Fix an $s \in S$ and let $\phi(T_s) = \sum_{w \in W} c_w w$, $c_w \in Q(u^1)$. We wish to determine the $c_w$, $w \in W$.

If $V_{ii} \in X$, $1 \leq i \leq t$ and has chosen basis $(v^{(1)}_1, \ldots, v^{(1)}_{d_1})$ then $T_s v^{(1)}_\lambda = \phi(T_s) v^{(1)}_\lambda$, for every $\lambda = 1, \ldots, d_1$. Moreover, for every $s \in S$,

$$T_s v^{(1)}_\lambda = \sum_{\mu=1}^{d_1} \gamma^{(s)}_{\mu,\lambda} (u^{\lambda}) v^{(1)}_\mu,$$

with $\gamma^{(s)}_{\mu,\lambda} (u^{\lambda}) \in F(u^1)$, $1 \leq \mu \leq d_1$.

When we consider the graded module as a left $W$-module with $W$-action obtained by specializing $u^{\lambda} = 1$ in the action of $T_s$, $s \in S$, then $i_s$ affords the left
regular representation of \( W \). Now each \( V_{ij}, 1 \leq i \leq t \) becomes an absolutely irreducible \( FW \) module.

By specializing \( u^1 = 1 \) in the matrices which represent \( T_s, s \in S \) with respect to the basis adapted to the constituent \( V_{ij} \in X, 1 \leq i \leq t \), we obtain the matrices which represent the generators \( s \) and therefore the matrices which represent every \( w \in W \).

Let \( w \in W \) and let \( (f_{k\lambda}^{(1)}(w)) \), \( k, \lambda \in \{1, \ldots, d_i\} \) be the matrix which represent \( w \) on the \( FW \) module \( V_{ij}, 1 \leq i \leq t \).

Let \( \text{m} \lambda ; (i) = \sum_{\mu=1}^{d_i} f_{\lambda \mu}^{(1)}(w) v_{\mu}^{(1)}, 1 \leq i \leq d_i \). Then, \( \Phi(T_s) v_{\lambda}^{(1)} = \sum_{w \in W} c_w (\sum_{\mu=1}^{d_i} f_{\lambda \mu}^{(1)}(w) v_{\mu}^{(1)}) \).

Therefore by comparing coefficients of the basis elements \( v_{\lambda}^{(1)} \) on both sides of the equation \( T_s v_{\lambda}^{(1)} = \Phi(T_s) v_{\lambda}^{(1)} \) we obtain \( d_i \) equations in the unknowns \( c_w, w \in W \) and therefore from every \( V_{ij} \in X \) we obtain \( d_i^2 \) such equations.

Hence from the full set of irreducible constituents \( \{V_{11}, \ldots, V_{tt}\} \)

we obtain \( \sum_{i=1}^{t} d_i^2 = |W| \) equations in the \(|W|\) unknowns \( c_w, w \in W \).

These equations are linearly independent (see 3.41 in [7]), and the solution of the system of these equations determines the \( c_w, w \in W \) and therefore the image of \( T_s \) under \( \Phi \).

Example. Let \( W \) be the Weyl group of type \( A_2, W = S_3 \). \( W \) has a presentation:

\( \langle s_1, s_2 | s_1^2 = s_2^2 = 1, (s_1 s_2)^3 = 1 \rangle \). The 2-sided cells are

\( X_1 = \{1\}, X_2 = \{s_1, s_2, s_1 s_2, s_2 s_1\}, X_3 = \{s_1, s_2, s_1 s_2\} \).

The graded module has canonical basis \( (\tilde{e}_w, w \in W) \). The module \( M_{X_2} \) obtained by the 2-sided cell \( X_2 \) has an easy decomposition into a direct sum of two left irreducible \( H_{q(u^3)} \) submodules, namely
MY = <\mathbf{e}_1, 5. \mathbf{e}_2>, \text{ hence}

\text{grad}(E) = <\mathbf{e}_1> 0 <\mathbf{e}_{s_1}s_{s_2}s_1> 0 <\mathbf{e}_{s_2}s_{s_1}s_2> 0 <\mathbf{e}_{s_1}s_2s_1>

It is easy to check that the two summands appearing in the decomposition of MY afford equivalent representations of $\mathcal{H}(u^\lambda)$.

Let $\phi(T_{s_1}) = \sum_{w \in W} c_w w$. Then from the equations $T_{s_1}w = \phi(T_{s_1})_w$, $w \in \{1, s_1, s_2, s_1s_2, s_1s_2s_1\}$ we obtain 6 equations in the 6 unknowns $c_w$, $w \in W$. The solution of these equations gives:

$$\phi(T_{s_1}) = \frac{u-1}{2}w + \frac{u+1}{2}s_1 + \frac{(u^2-1)^2}{6}(-s_2 + s_1s_2 - s_2s_1 - s_1s_2s_1)$$

Similar work for $T_{s_2}$ gives

$$\phi(T_{s_2}) = \frac{u-1}{2}w + \frac{u+1}{2}s_2 + \frac{(u^2-1)^2}{6}(-s_1 + s_2s_1 - s_1s_2 + s_1s_2s_1)$$

(This example appears in [12]).
CHAPTER 2

The Lusztig isomorphism for Hecke algebras of dihedral type.

§2.1 General properties of the dihedral groups

The dihedral group of order 2n, n > 1 is defined by
\[ D_{2n} = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1s_2)^n = 1 \rangle. \] We put \( S = \{s_1, s_2\}. \)

The example at the previous chapter demonstrates the Lusztig isomorphism for the Weyl group of type \( A_2 \) which is isomorphic to \( D_6 \). Therefore it is natural to ask the question, whether we can find explicitly this isomorphism for the Hecke algebras of dihedral type. In order to do so, we must check that the dihedral groups satisfy the property (A) of the definition 1.4.1.

For any \( y, w \in D_{2n} \) we have: \( y \leq w \) if and only if \( y = w \) or \( \ell(w) > \ell(y). \)

We also have that if \( \ell(w) - \ell(y) > 0 \) with \( \ell(w) - \ell(y) = \text{odd} \), then either \( s \in S : sw < w \neq s \in S : sy < y \) or \( s \in S : ws < w \neq s \in S : ys < y \), but not both conditions hold. By lemma 1.3.3(1) and (1i) we have: \( y < w \) if and only if either \( w = s_y y \) or \( w = ys_y \), for some \( i = 1, 2 \).

Lemma 2.1.1. For any \( y \leq w \) in \( D_{2n} \), we have \( P_y w = 1 \).

Proof. We use induction on \( \ell(w) \). The result is obvious if \( \ell(w) = 0 \).

Assume that our lemma holds for all elements \( w' \) with \( \ell(w') < \ell(w) \). Let \( w \in D_{2n} \) and we may assume that \( s_1w < w \), so \( w = s_1v \), \( \ell(v) = \ell(w) - 1 \).

We may also assume that \( y \in D_{2n} \) is such that \( \ell(w) - \ell(y) > 2 \) (Lemma 1.3.3). The inductive formula which defines the polynomials \( P_y w \) gives
\[ P_{y,w} = \begin{cases} u P_{s_1 y, v} + P_{y,v} - u & \text{if } s_1 y > y \\ P_{s_1 y, v} + u P_{y,v} - u & \text{if } s_1 y > y \end{cases} \]

By induction we have \( P_{y,v} = P_{s_1 y,v} = 1 \) and so in both cases we obtain \( P_{y,w} = 1 \).

We can now easily determine the 2-sided cells of \( W \). These are:

- \( X_1 = \{1\} \)
- \( X_2 = D_{2n} - \{1, w_0\} \)
- \( X_3 = \{w_0\} \)

where \( w_0 \) is the longest element of \( W \). The cell \( X_2 \) contains two left cells which are

- \( L_1 = \{w \in X_2 : \ell(w s_1) < \ell(w)\} \)
- \( L_2 = \{w \in X_2 : \ell(w s_2) < \ell(w)\} \)

and also contains two right cells

- \( R_1 = \{w \in X_2 : \ell(s_1 w) < \ell(w)\} \)
- \( R_2 = \{w \in X_2 : \ell(s_2 w) < \ell(w)\} \).

**Lemma 2.1.2**

\( D_{2n} \) satisfies the property (A).

**Proof**

Let \( y, w \) be two elements inside the cell \( X_2 \) such that \( y \) is joined to \( w \), \( y \prec w \). Then \( \ell(w) - \ell(y) \) is odd or \( \ell(y) - \ell(w) \) is odd. In this case we have that either \( \{s \in S : sw < w\} \neq \{s \in S : sy < y\} \) or \( \{s \in S : ws < w\} \neq \{s \in S : ys < y\} \), but not both conditions hold. So our lemma is proved.

Therefore the construction of Lusztig's isomorphism makes sense for the finite Coxeter groups of dihedral type.

The graded module in this case has canonical basis \( \{e_w, w \in D_{2n}\} \). The left action of the generators \( T_{s_1} \), \( i = 1, 2 \) on the canonical basis is as follows:

\[ T_{s_1} e_w = \begin{cases} -e_w & \text{if } s_1 w < w \\ u e_w + u^\perp e_{s_1 w} + u^\perp e_{s_2 w} & \text{if } s_1 w > w \end{cases} \]
and \( w \notin \{ s_2, s_1 w_0 \} \).

If \( w = 1 \), then 
\[
T_{s_1} e_1 = u e_1.
\]

If \( w = s_2 \), then 
\[
T_{s_1} e_{s_2} = u e_{s_2} + u^2 e_{s_1 s_2}.
\]

If \( w = s_1 w_0 \), then 
\[
T_{s_1} e_w = u e_w + u^2 e_{s_2 w}.
\]

Similarly, by interchanging the role of \( s_1, s_2 \) above, we obtain the action of \( T_{s_2} \) on the canonical basis.

By specializing \( u^2 + 1 \) we obtain the action of \( s_1, s_2 \) respectively.

\section{2.2 A set of polynomials \( S_n(x) \in \mathbb{Z}[x] \).}

We shall now introduce a set of polynomials \( S_n(x) \in \mathbb{Z}[x] \) called the Chebyshev polynomials of the second kind, which play an essential role in the decomposition of Lusztig's graded module of dihedral type into a direct sum of left \( H \) irreducible submodules, and also in the determination of Lusztig's isomorphism in this case. Many properties of these polynomials and their relation with other families of Chebyshev polynomials can be found in [1], pages 774-8. These polynomials are defined as follows:

\[
S_{-1}(x) = 0, \quad S_0(x) = 1, \quad S_1(x) = x, \quad S_{k+1}(x) = xS_k(x) - S_{k-1}(x) \quad \forall k \geq 1.
\]

The first few of these polynomials are:

\[
S_2(x) = x^2 - 1, \quad S_3(x) = x^3 - 2x, \quad S_4(x) = x^4 - 3x^2 + 1,
\]

\[
S_5(x) = x^5 - 4x^3 + 3x, \quad S_6(x) = x^6 - 5x^4 + 6x^2 - 1, \quad S_7(x) = x^7 - 6x^5 + 10x^3 - 4x.
\]

\textbf{Lemma 2.2.1} \( S_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} x^{n-2k} \) if \( n \) is even and

\[
S_n(x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} x^{n-2k} \) if \( n \) is odd.

Proof.

We assume that $n$ is odd and that $S_k(x)$ is given by the formula above for all $k \leq n$.

Then $S_{n+1}(x) = xS_n(x) - S_{n-1}(x) =$

$$
\frac{n-1}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} x^k = \frac{(n-1)/2}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-1-k}{k} x^k.
$$

$$
\frac{n-1}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-k}{k} x^k = \frac{(n-1)/2}{2} \sum_{k=0}^{n-1} (-1)^k \binom{n-1-k}{k} x^k.
$$

Therefore $S_{n+1}(x)$ has also the desired form. The argument is similar if in our inductive hypothesis we assume that $n$ is even.

Lemma 2.2.2. (i) The numbers $p_j = 2 \cos \frac{j\pi}{m}$, $1 \leq j \leq m-1$, are the zeros of the polynomial $S_{m-1}(x)$.

(ii) The numbers $p_j = 2 \cos \frac{2j\pi}{2m+1}$, $1 \leq j \leq m$, are the zeros of the polynomial $S_m(x) + S_{m-1}(x)$. 
Proof. (1) Let \( n = 2m \), and consider the polynomial \( y^{n-1} = (y^2-1)Q(y) \)
where \( Q(y) = y^{2m-2} + y^{2m-4} + \ldots + y^2 + 1 \). We write

\[
Q(y) = y^{m-1} \left[ \left( y^{m-1} - \frac{1}{y^{m-1}} \right) + \left( y^{m-3} - \frac{1}{y^{m-3}} \right) + \ldots \right]
\]

where \( R_{m-1}(y) \) finishes either with \( y + \frac{1}{y} \) if \( m \) is even or with 1 if \( m \) is odd. We put \( y + \frac{1}{y} = z \).

The zeros of \( Q(y) \) are the numbers \( y_j = e^{\frac{2\pi j}{m}} \), \( j = 1, \ldots, 2m-1 \), \( j \neq m \),
and for each such \( y_j \) we have \( Q(y_j) = y_j^{m-1} R_{m-1} \left( y_j + \frac{1}{y_j} \right) = 0 \).
Therefore the zeros of the polynomial \( R_{m-1}(z) \) are the numbers \( y_j + \frac{1}{y_j} = 2 \cos \frac{\pi j}{m} \), \( 1 \leq j \leq m-1 \).

Now there is a recurrence relation which generates the polynomial \( R_{m-1}(z) \). For,

\[
y^2 - \frac{1}{y^2} = (y^{m-2} + \frac{1}{y^{m-2}}) (y^2 + \frac{1}{y^2}) - (y^{m-4} + \frac{1}{y^{m-4}})
\]

for every \( \lambda = 3, 4, \ldots, m-1 \).

Hence \( R_{m-1}(z) = (z^2-2)R_{m-3}(z) - R_{m-5}(z) \), with \( R_0(z) = 1 \). Now we claim that for all integers \( n \), \( R_n(z) = S_n(z) \). In fact for \( n = 1 \) it is true since \( R_1(z) = z = S_1(z) \). Assume that \( R_\lambda(z) = S_\lambda(z) \) for all integers \( \lambda \leq k \).

Then,

\[
R_{k+1}(z) = (z^2-2)R_{k-1}(z) - R_{k-3}(z) = (z^2-2)S_{k-1}(z) - S_{k-3}(z) =
\]

\[
z S_k(z) + (zS_{k-2}(z) - S_{k-3}(z)) - 2S_{k-1}(z) = zS_k(z) - S_{k-1}(z) = S_{k+1}(z).
\]
In particular \( R_{m-1}(z) = S_{m-1}(z) \), and so the zeros of the polynomial 
\( S_{m-1}(z) \) are the numbers 
\[
p_j = 2 \cos \frac{j\pi}{m} \quad 1 \leq j \leq m-1.
\]

(11) Let \( n = 2m+1 \) and consider the polynomial \( y^{n-1} = (y-1)Q(y) \) where 
\[
Q(y) = y^{2m} + \ldots + y + 1.
\]
The zeros of \( Q(y) \) are the numbers 
\[
y_j = e^{2ij\pi/n}, \quad 1 \leq j \leq 2m.
\]
We write 
\[
Q(y) = y^m \left[ (y^m + \frac{1}{y^m}) + \ldots + (y + \frac{1}{y}) + 1 \right]
\]
We put \( y + \frac{1}{y} = z \) and we deduce as before that the zeros of \( R_m(z) \) are the numbers 
\[
y_j = \frac{1}{y_j} = 2 \cos \frac{2j\pi}{n} \quad 1 \leq j \leq m.
\]
Now the recurrence relation which generates \( R_m(z) \) is 
\[
y^\lambda + \frac{1}{y^\lambda} = (y^{\lambda-1} + \frac{1}{y^{\lambda-1}}) (y + \frac{1}{y}) - (y^{\lambda-2} + \frac{1}{y^{\lambda-2}}), \quad \forall \lambda = 2, \ldots, m.
\]
Hence \( R_m(z) = zR_{m-1}(z) - R_{m-2}(z) \), with \( R_0(z) = 1, R_1(z) = z + 1 \). By induction we can prove that for all integers \( n \), we have \( R_n(z) = S_{n-1}(z) + S_n(z) \).
For \( n = 1 \) it is true since \( R_1(z) = z+1 = S_0(z) + S_1(z) \).
Assume it is true for all integers \( \lambda \leq k \). Then 
\[
R_{k+1}(z) = zR_k(z) - R_{k-1}(z) = z(S_k(z) + S_{k-1}(z)) - (S_{k-1}(z) + S_{k-2}(z)) \\
= S_{k+1}(z) + S_k(z).
\]
In particular \( R_m(z) = S_m(z) + S_{m-1}(z) \), and hence the zeros of the polynomial
$S_m(z) + S_{m-1}(z)$ are the numbers $\rho_j = 2 \cos \frac{2j\pi}{2m+1}$, $1 \leq j \leq m$.

Our lemma is now proved.

**Lemma 2.2.3.** (1) Let $n = 2m+1$, then
\[
(x-2) \sum_{k=0}^{m-1} (n-2(k+1))S_k(x) = S_m(x) + S_{m-1}(x) - n.
\]

(1) Let $n = 2m$, then
\[
(x-2) \sum_{k=0}^{m-1} (n-2(k+1))S_k(x) = 2S_{m-1}(x) - n.
\]

**Proof.** (1) We have
\[
(x-2) \sum_{k=0}^{m-1} (n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+1))xS_k(x) - \sum_{k=0}^{m-1} 2(n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+1))(S_{k+1}(x) + S_{k-1}(x)) - \sum_{k=0}^{m-1} 2(n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+1))S_{k+1}(x) + \sum_{k=0}^{m-1} (n-2(k+1))S_{k-1}(x) - \sum_{k=0}^{m-1} 2(n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+1))S_{k+1}(x) + \sum_{k=0}^{m-1} (n-2(k+1))S_{k-1}(x) - \sum_{k=0}^{m-1} 2(n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+2))S_k(x) - \sum_{k=0}^{m-1} 2(n-2(k+2))S_k(x) - \sum_{k=0}^{m-1} 2(n-2(k+1))S_k(x) =
\]
\[
- \sum_{k=0}^{m-1} (n-2(k+1))S_k(x) - n + 3S_{m-1}(x) + S_{m}(x) + \sum_{k=0}^{m-1} (n-2(k+2))S_k(x) -
\]
\[
- (2(n-2(k+1))S_k(x) - 2S_{m-1}(x) = S_m(x) + S_{m-1}(x) - n
\]
(11) \((x-2)^m \sum_{k=0}^{m-2} (n-2(k+1))S_k(x) = \sum_{k=0}^{m-2} (n-2(k+1))xS_k(x)\)

\[- \sum_{k=0}^{m-2} 2(n-2(k+1))S_k(x) = \]

\[- \sum_{k=0}^{m-2} (n-2(k+1))S_{k+1}(x) + \sum_{k=0}^{m-2} (n-2(k+1))S_{k-1}(x) - \sum_{k=0}^{m-2} 2(n-2(k+1))S_k(x) - \]

\[- \sum_{k=0}^{m-3} (n-2k)S_k(x) - \sum_{k=0}^{m-3} (n-2(k+2))S_k(x) - \sum_{k=0}^{m-3} 2(n-2(k+1))S_k(x) - \]

\[- \sum_{k=0}^{m-3} 2(n-2(k+1))S_k(x) - 4S_{m-2}(x) = 2S_{m-1}(x) - n.\]

Our lemma is now proved.

§2.3. The determination of Lusztig’s isomorphism for Hecke algebras of dihedral type.

Our first step is to find a decomposition of Lusztig’s graded module into a direct sum of left \(H\)-irreducible submodules.

First case: \(n = 2m+1\). In this case we have two one-dimensional representations of \(H\), namely \(\sigma_0: T_{s_1} \rightarrow u, \sigma_1: T_{s_1} \rightarrow -1\) and \(\sigma_2: T_{s_1} \rightarrow -1\). These representations are afforded by the left \(H\)-submodules \(M_0 = \langle \tilde{e}_1 \rangle\) and \(M_1 = \langle \tilde{e}_w \rangle\).
respectively. Let $N$ be the subspace spanned by $\langle \tilde{e}_w, w \in W, w \neq 1, w_0 \rangle$.

It is easy to verify that $N$ is a left $H$-submodule complementary to $\langle \tilde{e}_1 \rangle \oplus \langle \tilde{e}_{w_0} \rangle$. We wish to decompose $N$ into a direct sum of $2m$ 2-dimensional left $H$-submodules.

First we note that $N$ has an obvious decomposition into the direct sum of two $(n-1)$-dimensional left $H$-submodules, namely

$$N_1 = \langle \tilde{e}_{s_1}, \tilde{e}_{s_2}, \tilde{e}_{s_1s_2s_1}, \ldots, \tilde{e}_{(s_1s_2)^{m-1}}, \tilde{e}_{(s_2s_1)^{m-1}} \rangle$$

and

$$N_2 = \langle \tilde{e}_{s_2}, \tilde{e}_{s_1}, \tilde{e}_{s_1s_2s_1}, \ldots, \tilde{e}_{s_2s_1^{m-1}}, \tilde{e}_{s_1s_2^{m-1}} \rangle.$$

We shall split $N_1$ (similarly $N_2$) into the direct sum of $m$ 2-dimensional left $H$-modules. Consider the numbers $p_j = 2 \cos \frac{2\pi j}{n}$, $1 \leq j \leq m$ and define the following sequence of real numbers:

$$a_0(j) = -1, a_1(j) = S_0(p_j) = 1, \ldots, a_{\lambda+1}(j) = S_\lambda(p_j) + S_{\lambda-1}(p_j) \lambda \in \{1, 2, \ldots, m-1\}$$

(for the definition of the polynomials $S_n(x)$ see §2.2).

Next consider the following elements of the graded module:

$$v_j = a_1(j)\tilde{e}_{s_1} + a_2(j)\tilde{e}_{s_1s_2s_1} + \ldots + a_m(j)\tilde{e}_{(s_1s_2)^{m-1}}$$

$$v_j = (a_1(j) + a_2(j))\tilde{e}_{s_2s_1} + (a_2(j) + a_3(j))\tilde{e}_{s_2s_1s_2s_1} + \ldots +(a_{m-1}(j) + a_m(j))\tilde{e}_{(s_2s_1)^{m-1}}$$

$$+ a_m(j)\tilde{e}_{(s_2s_1)^{m-1}} \quad 1 \leq j \leq m.$$

The number of these elements is $2m$ and they all lie inside the submodule $N_1$.

We shall denote by $Q_n$ the field $\mathbb{Q}(2 \cos \frac{2\pi j}{n})$. 
Proposition 2.3.1. The elements \( u_j, v_j \) defined above are all linearly independent over \( \mathbb{Q}_n(u^\lambda) \) and for each \( 1 \leq j \leq m \), the pair \((u_j, v_j)\) spans a 2-dimensional left \( \mathbb{Q}_n(u^\lambda) \)-submodule, namely \( H_j \). Moreover each \( H_j \) is irreducible and distinct \( j \) give rise to non-isomorphic such submodules.

Proof. From the definition of the numbers \( a_\lambda(j) \) above, we see that for every \( 1 \leq k \leq m-1 \),
\[
\sum_{j=1}^m a_k(j) = S_{k-1}(\rho_j) + S_{k-2}(\rho_j) + S_k(\rho_j) + S_{k-1}(p_j).
\]
We also note that if \( \lambda = m-1 \) then
\[
a_m(j) = S_{m-1}(\rho_j) + S_{m-2}(\rho_j) = (2+p_j)S_{m-1}(\rho_j),
\]
for \( S_m(\rho_j) + S_{m-1}(\rho_j) = 0 \), i.e. \( p_jS_{m-1}(\rho_j) - S_{m-2}(\rho_j) + S_{m-1}(\rho_j) = 0 \), i.e. \( S_{m-2}(\rho_j) = (1+p_j)S_{m-1}(\rho_j) \).

Assume that \( \lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_m \) are elements in \( \mathbb{Q}_n(u^\lambda) \) such that
\[
\sum_{j=1}^m \lambda_j u_j + \sum_{j=1}^m \mu_j v_j = 0. \quad (\mathbb{R})
\]
We shall show that \( \lambda_j = u_j = 0 \) \( \forall j = 1, \ldots, m \).

Since each \( u_j \) is by definition a linear combination of basis elements \( \varepsilon_i \) which do not appear in the expression of \( v_j \), we can concentrate separately on the coefficients \( \lambda_j \) and \( \mu_j \). For simplicity we put \( w_1 = \varepsilon_{s_1} \),
\[
w_2 = \varepsilon_{s_1s_2}, \ldots, w_m = \varepsilon_{s_1s_2 \cdots s_{m-1}s_1}.
\]
Then, in the expression \( \sum_{j=1}^m \lambda_j u_j \)
the coefficient of a typical element \( w_k, 1 \leq k \leq m \) is of the form
\[
\sum_{j=1}^m a_k(j). \quad \text{Since the elements } w_k \text{ are linearly independent, the relation } (\mathbb{R})
\]
above implies that \( \sum_{j=1}^m a_k(j) = 0 \) for every \( 1 \leq k \leq m \). Thus, we have a system of \( m \)-homogeneous linear equations of the form
\[
\sum_{j=1}^m a_k(j) = 0.
\]
In the \( m \)-unknowns \( \lambda_k \). I claim that any solution of these homogeneous
equations say \((\Sigma)\): \( \{E_k : \Sigma \lambda_j a_{k,}^{(j)} = 0 \mid 1 \leq k \leq m\}\) is also a solution of
the system of homogeneous equations \((\Sigma')\): \( \{E_k : \Sigma \lambda_j a_{k,}^{(j)} = 0 \mid 1 \leq k \leq m\}\).

In fact if \( k = 1 \), the equation \( E_1 \) of the system \((\Sigma)\) is \( \lambda_1 \ldots + \lambda_m = 0 \), because we have defined \( a_{1,}^{(j)} = 1, \forall 1 \leq j \leq m \). Assume that
the \( \lambda_j \) satisfy the equations \( E_{1}', E_{2}',\ldots,E_{k}' \). Then we look at the
equation \( E_{k+1} : \sum_{j=1}^{m} \lambda_j a_{k+1,j}^{(j)} = 0 \). We recall that \( a_{k+1,j}^{(j)} = S_k(\rho_j) + S_{k-1}(\rho_j) = \rho_j + \)
linear combination of lower powers of \( \rho_j \), for every \( 1 \leq j \leq m \). Therefore by
induction the \( \lambda_j \) satisfy the equation \( E_{k+1}' : \sum_{j=1}^{m} \lambda_j \rho_j^k = 0 \).

Now, the determinant of the coefficients of the \( \lambda_j \) in the homogeneous
system \((\Sigma')\) is the Vandermonde determinant

\[
\Delta = \begin{vmatrix}
1 & 1 & \ldots & 1 \\
\rho_1 & \rho_2 & \ldots & \rho_m \\
\rho_1 & \rho_2 & \ldots & \rho_m \\
\rho_1 & \rho_2 & \ldots & \rho_m \\
\end{vmatrix}
= \prod_{1 \leq i < j \leq 1} (\rho_i - \rho_j) \neq 0, \text{ since } \rho_i \neq \rho_j
\]

Therefore \( \lambda_j = 0 \quad \forall \ j = 1,\ldots,m \).

The argument is similar for the coefficients \( \mu_j \), \( j = 1,\ldots,m \).

We put \( z_1 = \bar{a}_{s_2}, z_2 = \bar{a}_{s_2 s_1, s_2} \ldots, z_m = \bar{a}_{s_2 s_1, s_2} \ldots s_2 \)

In the expression \( \sum_{j=1}^{m} \mu_j v_j \), the coefficient of a typical element \( z_k \), \( 1 \leq k \leq m \)
is of the form \( \sum_{j=1}^{m} \mu_j (a_k + a_{k+1}) \), \( 1 \leq k \leq m \) and we showed that for each
\( 1 \leq k \leq m \), \( a_k + a_{k+1} = (2+\rho_j)S_{k-1}(\rho_j) \). (Note that \( a_{k+1} = S_m(\rho_j) + S_{m-1}(\rho_j) = 0 \). Since the elements \( z_k \) are linearly independent, the relation \((R)\) implies that
\[ \sum_{j=1}^{m} \mu_j (a_k(j) + a_{k+1}(j)) = 0 \quad \forall \ 1 \leq k \leq m. \]

Now, we have a system of \( m \)-homogeneous equations of the form \((\Sigma)\) : \((E_k = \sum_{j=1}^{m} \mu_j (a_k(j) + a_{k+1}(j)) = 0, 1 \leq k \leq m)\), in the unknowns \( \mu_j \).

I claim that any solution of \((\Sigma)\) is also a solution of \((\Sigma')\) : \((E'_k = \sum_{j=1}^{m} \mu_j (2 + \rho_j)\rho_j^{-1} = 0, 1 \leq k \leq m)\). In fact if \( k = 1 \) this is true since \( a_1(j) + a_2(j) = 2 + \rho_j \). Assume that for all integers \( \lambda \leq k-1 \), the \( \mu_j \) satisfy the equations \( E'_\lambda \). Then we look at the equation \( E_k : \sum_{j=1}^{m} \mu_j (a_k(j) + a_{k+1}(j)) = 0 \). We have \( a_k(j) + a_{k+1}(j) = (2 + \rho_j)\rho_{k-1}(\rho_j) = (2 + \rho_j)(\rho_j^{-1} + \text{linear combination of lower powers of } \rho_j) \). Therefore, by induction we obtain that \( \mu_j \) satisfy the equation \( E'_k = \sum_{j=1}^{m} \mu_j (2 + \rho_j)\rho_j^{-1} = 0 \). Now the determinant of the unknowns \( \mu_j \) in the system \((\Sigma')\) is

\[
\begin{vmatrix}
2 + \rho_1 & 2 + \rho_2 & \cdots & 2 + \rho_m \\
(2+\rho_1)\rho_1 & (2+\rho_2)\rho_2 & \cdots & (2+\rho_m)\rho_m \\
\cdots & \cdots & \cdots & \cdots \\
(2+\rho_1)\rho_1 & (2+\rho_2)\rho_2 & \cdots & (2+\rho_m)\rho_m
\end{vmatrix}
= \prod_{j=1}^{m} (2+\rho_j) \prod_{j=1}^{m} (\rho_j-p_j) \neq 0
\]

since \( \rho_j \neq -2 \forall j = 1, \ldots, m \) and \( \rho_1 \neq \rho_j \text{ if } i \neq j \). Therefore we obtain \( \mu_j = 0 \forall j = 1, \ldots, m \).

So, the elements \( \{\mu_j, \ \nu_{ij}, 1 \leq j \leq m\} \) are linearly independent over \( \mathbb{F}_n(\mathbb{U}_\lambda) \) and hence they form a basis for the submodule \( N_1 \).
Next we show that for each \( j \), the pair \((u_j, v_j)\) spans a 2-dimensional left \( H_{q,n}(u^i)\)-module. We shall show that the 2-dimensional subspace spanned by \((u_j, v_j)\) is invariant under the action of \(T_{s_1}\) and \(T_{s_2}\).

By recalling the action of the generators \(T_{s_1}\) on the graded module (see §2.1) we have that,

\[
T_{s_1} u_j = -u_j, \quad T_{s_2} u_j = u u_j + u^i v_j
\]

\[
T_{s_2} v_j = -v_j
\]

The following relations hold:

\[
T_{s_1} v_j = u v_j + u^i (a_1(j) + a_2(j)) S_1 s_1 + u^i (a_1(j) + 2 a_2(j) + a_3(j)) S_2 s_1 + \ldots
\]

\[
+ \ldots + u^i (a_{m-2}(j) + a_{m-1}(j)) S_1 s_1 + u^i (a_{m-1}(j) + 2 a_{m}(j)) S_2 s_1 + u^i (a_{m}(j) + 2 a_{m+1}(j)) S_3 s_1 + \ldots
\]

The following relations hold:

\[
a_{1}(j) + a_{2}(j) = 2 + \rho_j \quad \forall 1 \leq j \leq m
\]

\[
a_{k}(j) + 2 a_{k+1}(j) + a_{k+2}(j) = a_{k}(j) + a_{k+1}(j) + a_{k+2}(j) = (2 + p_j) S_{k-1}(\rho_j) + (2 + p_j) S_k(\rho_j)
\]

\[
= (2 + p_j)(S_k(\rho_j) + S_{k-1}(\rho_j)) = (2 + p_j) a_{k+1}(j) \quad \text{for every } 1 \leq k \leq m-2.
\]

\[
a_{m-1}(j) + 2 a_{m}(j) = a_{m-1}(j) + a_{m}(j) + a_{m}(j) = (2 + p_j) S_{m-2}(\rho_j) + a_{m}(j)
\]

\[
= (2 + p_j) S_{m-2}(\rho_j) + (2 + p_j) S_{m-1}(\rho_j) = (2 + p_j)(S_{m-2}(\rho_j) + S_{m-1}(\rho_j))
\]

\[
= (2 + p_j) a_{m}(j).
\]

Therefore \(T_{s_1} v_j = u v_j + (2 + p_j) u^i u_j\).

So, the subspace spanned by \((u_j, v_j)\) is a left \( H_{q,n}(u^i)\)-submodule of the graded module.
Let $M_j = \langle u_j, v_j \rangle$. Then the submodule $N_1$ decomposes as:

$$N_1 = \oplus_{j=1}^{m} \langle u_j, v_j \rangle = \oplus_{j=1}^{m} M_j.$$ 

Moreover each $M_j$ is an irreducible left $H$-module, because otherwise it would have a non-trivial one-dimensional left $H$-submodule. However, the graded module affords the regular representation of $H$. We have two one-dimensional representations of $H$, afforded by the submodules $\langle e_1 \rangle$, and $\langle e_w \rangle$ respectively, and they can appear only once.

Finally the matrix which represents $T_{s_1}$ on $M_j$ with respect to the basis $\{ u_j, v_j \}$ is

$$\begin{pmatrix} -1 & (2 + \rho_j)u^{\frac{1}{2}} \\ 0 & u \end{pmatrix}$$

and the matrix which represents $T_{s_2}$ on $M_j$ with respect to the same basis is

$$\begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{pmatrix}$$

Therefore, the matrix which represents $T_{s_1s_2}$ on $M_j$ is

$$\begin{pmatrix} u(1 + \rho_j) & -u^{\frac{1}{2}}(2 + \rho_j) \\ u^{3/2} & -u \end{pmatrix}.$$ 

Hence the character of this element is $u\rho_j$, $1 \leq j \leq m$. Since $\rho_j \neq \rho_{j'}$ if $j \neq j'$ we conclude that $M_j \neq M_{j'}$ if $j \neq j'$. Our proposition is now proved.
The set \( \{M_0, M_s, M_j, 1 \leq j \leq m \} \) is a full set of left irreducible \( \mathcal{H} \) modules since the sum of the squares of the degrees is \( 4m + 2 - 2n \).

With a similar argument we can decompose the submodule \( N_2 \) into a direct sum of left irreducible \( \mathcal{H} \)-submodules, namely \( N_2 = \bigoplus_{j=1}^{m} \langle \hat{u}_j, \hat{v}_j \rangle \) where

\[
\hat{u}_j = a_1^{(j)} \hat{e}_{s_2} + a_2^{(j)} \hat{e}_{s_2}^2 s_2 + \ldots + a_m^{(j)} \hat{e}_{s_2}^{m-1} s_2,
\]

\[
\hat{v}_j = (a_1^{(j)} + a_2^{(j)} \hat{e}_{s_2} s_2 + (a_2^{(j)} + a_3^{(j)}) \hat{e}_{s_2}^2 s_2 + \ldots + (a_{m-1}^{(j)} + a_m^{(j)}) \hat{e}_{s_2}^{m-1} s_2 + a_m^{(j)} \hat{e}_{s_2}^m
\]

and the \( a_n^{(j)} \) are defined as before.

The action of \( T_{s_1}, T_{s_2} \) on \( \hat{u}_j, \hat{v}_j \) is given by:

\[
\begin{align*}
T_{s_1} \cdot \hat{u}_j &= u \hat{u}_j + u^1 \hat{v}_j, & T_{s_1} \cdot \hat{v}_j &= -\hat{v}_j \\
T_{s_2} \cdot \hat{u}_j &= -\hat{u}_j, & T_{s_2} \cdot \hat{v}_j &= u \hat{v}_j + u^1 (2 + \rho_j) \hat{u}_j
\end{align*}
\]

So, eventually we have a decomposition of \( \text{grad}(E) \) over \( \mathbb{Q}_n(u^1) \) into:

\[
\text{grad}(E) = M_0 \oplus M_s \oplus \hat{M}_j \oplus M_s \text{ of left irreducible } \mathcal{H} \text{-submodules. In fact, for every } 1 \leq j \leq m, M_j = \hat{M}_j.
\]

This will become clearer in the lines below.

We note that \( 2 + \rho_j = 2(1 + \cos \frac{2j\pi}{n}) = 4 \cos^2 \frac{j\pi}{n}, j = 1, \ldots, m \).

Thus:

\[
\begin{align*}
T_{s_1} \cdot u_j &= -u_j, & T_{s_1} \cdot v_j &= u v_j + 4u^1 \cos^2 \frac{j\pi}{n} u_j \\
T_{s_2} \cdot u_j &= u u_j + u^1 v_j, & T_{s_2} \cdot v_j &= -v_j.
\end{align*}
\]
We replace $u_j$ by $u_j' = 2 \cos \frac{j\pi}{n} u_j$ to obtain:

\[ T_{s_1} u_j' = -u_j', \quad T_{s_1} v_j = u v_j + 2u^3 \cos \frac{j\pi}{n} u_j \]

\[ T_{s_2} u_j' = u u_j' + 2u^3 \cos \frac{j\pi}{n} v_j, \quad T_{s_2} v_j = -v_j \]

Therefore the matrices which represent $T_{s_i}, i = 1, 2$ on each $M_j$ with respect to the basis $\{u_j', v_j\}$ are:

\[
T_{s_1} = \begin{pmatrix} -1 & 2u^3 \cos \frac{j\pi}{n} \\ 0 & 1 \end{pmatrix}, \quad T_{s_2} = \begin{pmatrix} 2u^3 \cos \frac{j\pi}{n} & 0 \\ 0 & 1 \end{pmatrix}
\]

Similarly, we replace $u_j$ by $\hat{u}_j = 2u^3 \cos \frac{j\pi}{n} u_j$ to obtain that the matrices which represent $T_{s_i}, i = 1, 2$ on $\hat{M}_j$, with respect to the basis $\{\hat{u}_j, \hat{v}_j\}$ are:

\[
T_{s_1} = \begin{pmatrix} u & 0 \\ 2u^3 \cos \frac{j\pi}{n} & 1 \end{pmatrix}, \quad T_{s_2} = \begin{pmatrix} 1 & 2u^3 \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix}
\]

for $1 \leq j \leq m$.

It is easy to verify that the matrices which represent $T_{s_i}, i = 1, 2$ on $\hat{M}_j$ are obtained by conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ from the matrices which represent $T_{s_i}, i = 1, 2$ on $M_j$.

Second case: $n = 2m$. In this case we have four 1-dimensional representations of $M_\mathbb{Q}(u^3)$ which are:
\[ \sigma_0 : T_{s_1} + u, \quad T_{s_2} + u, \quad \sigma_1 : T_{s_1} - 1, \quad T_{s_2} - u, \]
\[ \sigma_2 : T_{s_1} + u, \quad T_{s_2} + 1, \quad \sigma_3 : T_{s_3} - 1, \quad T_{s_2} - 1. \]

The representations \( \sigma_0, \sigma_3 \) are afforded by the \( H \)-submodules \( M_0 = \langle e_1 \rangle \) and \( M_3 = \langle e_3 \rangle \) respectively.

Let \( M_1 \) be the subspace spanned by the element \( e_1 = \sum_{k=1}^{m} (-1)^{k-1} e_{(s_1 s_2)^{k-1}} \).

By recalling the action of \( T_{s_i}, i = 1,2 \) on the graded module, we have that
\[ T_{s_1} \theta_1 = -\theta_1. \]

We also have that for every \( k \in \{2, \ldots, m-1\}, \)
\[ T_{s_2} \tilde{e}_{(s_1 s_2)^{k-1}} = u \tilde{e}_{(s_1 s_2)^{k-1}} + u^k \tilde{e}_{(s_2 s_1)^{k-1}}, \]
while for \( k = 1 \) we have \( T_{s_2} \tilde{e}_{s_1} = u \tilde{e}_{s_1} + u^k \tilde{e}_{s_2 s_1} \), and for \( k = m \) we have
\[ T_{s_2} \tilde{e}_{(s_1 s_2)^{m-1}} = u \tilde{e}_{(s_1 s_2)^{m-1}} + u^k \tilde{e}_{(s_2 s_1)^{m-1}}. \]

Thus, \( T_{s_2} \theta_1 = \)
\[ = u \sum_{k=1}^{m} (-1)^{k-1} \tilde{e}_{(s_1 s_2)^{k-1}} + u^k \tilde{e}_{s_2 s_1} + u^m \tilde{e}_{(s_2 s_1)^{k-1}} + \tilde{e}_{(s_2 s_1)^{m-1}} \]
\[ + (-1)^{m-1} u^m \tilde{e}_{(s_2 s_1)^{m-1}} = u \theta_1 + u^k \tilde{e}_{s_2 s_1} + \]
\[ + u \sum_{k=2}^{m} (-1)^{k-1} \tilde{e}_{(s_2 s_1)^{k-1}} + u^m \sum_{k=3}^{m} (-1)^{k-2} \tilde{e}_{(s_2 s_1)^{k-1}} + (-1)^{m-1} u^k \tilde{e}_{(s_2 s_1)^{m-1}} = u \theta_1. \]
Therefore $M_1 = \langle \theta_1 \rangle$ affords the representation $\sigma_1$. We put
\[ \theta_2 = \sum_{k=1}^{m} (-1)^{k-1} \tilde{e}(s_2 s_1)^{k-1}, \]
and a similar argument (interchanging the role of $s_1, s_2$) shows that $M_2 = \langle \theta_2 \rangle$ affords the representation $\sigma_2$.

Let $a_k = \tilde{e}(s_1 s_2)^{k-1} + \tilde{e}(s_1 s_2)^k s_k$, $b_k = \tilde{e}(s_2 s_1)^{k-1} + \tilde{e}(s_2 s_1)^k s_k$, and
\[ \gamma_k = \tilde{e}(s_1 s_2)^k s_k, \quad k = 1, \ldots, m-1. \]

It is clear that these elements are linearly independent over $\mathbb{Q}(u^\frac{1}{d})$.

Moreover,
\[ T_{s_1} a_k = -a_k, \]
\[ T_{s_2} a_k = u a_k + 2 u^\frac{1}{d} \tilde{e}(s_2 s_1)^k + u^\frac{1}{d} \tilde{e}(s_2 s_1)^k s_k, \]
for every $k \in \{1, \ldots, m-2\}$, while for $k = 1$ we have
\[ T_{s_2} a_1 = u a_1 + 2 u^\frac{1}{d} \tilde{e}(s_2 s_1)^2, \quad \text{and for } k = m-1 \text{ we have,} \]
\[ T_{s_2} a_{m-1} = u a_{m-1} + 2 u^\frac{1}{d} \tilde{e}(s_2 s_1)^{m-1} + u^\frac{1}{d} \tilde{e}(s_2 s_1)^{m-2}. \]

Similarly, by interchanging the role of $s_1, s_2$ we obtain the action of $T_{s_1}$ on $b_k$, $i = 1, 2, k = 1, \ldots, m-1$. Finally $T_{s_1} \gamma_k = -\gamma_k$, $k = 1, \ldots, m-1$, and $T_{s_2} \gamma_k = u^\frac{1}{d} \tilde{e}(s_2 s_1)^k s_k$, $k = 1, \ldots, m-1$, i.e.
\[ T_{s_2} \gamma_k = u^\frac{1}{d} b_k, k = 1, \ldots, m-1. \]
By interchanging the role of $s_1, s_2$ we obtain that

$$T_{s_2} \delta_k = -\delta_k, \quad T_{s_1} \delta_k = u \delta_k + u^k a_k, \quad k = 1, \ldots, m-1.$$ 

Therefore $V = \langle a_k, \beta_k, \gamma_k, \delta_k \rangle$, $k = 1, \ldots, m-1$, is an $H$-submodule of the graded module, with dimension $4m-4$, and it is readily seen that $V$ is a complementary submodule inside the graded module of the direct sum $H_0 \oplus H_1 \oplus H_2 \oplus H_3$, where the 1-dimensional $H$-submodules $H_0, H_1, H_2, H_3$ are defined above.

We shall next decompose $V$ into a direct sum of $2m-2$ 2-dimensional left $H$-submodules.

Firstly $V$ has an obvious decomposition into the direct sum of two $(n-2)$-dimensional submodules, namely $V = \tilde{V}_1 \oplus \tilde{V}_2$, where

$$\tilde{V}_1 = \langle \tilde{e}_{s_1} + \tilde{e}_{s_2} s_1 s_2, \ldots, \tilde{e}_{s_1^2 s_2^2}^{m-2}, \tilde{e}_{s_2 s_1}^{m-1}, \tilde{e}_{s_2 s_1}^{0}, \tilde{e}_{s_2 s_1}^{s}, \tilde{e}_{s_2 s_1}^{s_2 s_1 m-1}, \ldots \rangle$$

and

$$\tilde{V}_2 = \langle \tilde{e}_{s_2} + \tilde{e}_{s_2} s_1 s_2, \ldots, \tilde{e}_{s_2 s_1}^{m-2}, \tilde{e}_{s_2 s_1}^{s_2 s_1 m-1}, \tilde{e}_{s_2 s_1}^{s_2 s_1 0}, \tilde{e}_{s_2 s_1}^{s_2 s_1 s}, \tilde{e}_{s_2 s_1}^{s_2 s_1 s_2 s_1 m-1}, \ldots \rangle$$

We shall decompose $\tilde{V}_1$ (similarly $\tilde{V}_2$) into a direct sum of $m-1$ 2-dimensional left $H$-submodules.

Consider the real numbers $p_j = 2 \cos \frac{j\pi}{m}$, $1 \leq j \leq m-1$, and for every $j$ define the following sequence of real numbers:

$$a_0^{(j)} = 0, \quad a_1^{(j)} = 1 = S_0(p_j), \quad a_2^{(j)} = S_1(p_j), \ldots, a_{\lambda}^{(j)} = S_{\lambda-1}(p_j),$$

$$\lambda \in \{1, 2, \ldots, m-1\}.$$ 

We also consider the following elements inside the graded module.
for every $1 \leq j \leq m-1$.

The number of these elements is $2m-2$ and they lie inside the submodule $V_1$.

We shall show the following.

**Proposition 2.3.2:** The elements $u_j$, $v_j$ defined above, are all linearly independent over $Q_n(u^j)$, and for each $1 \leq j \leq m-1$, the pair $(u_j, v_j)$ spans a 2-dimensional left $H_n(u^j)$-submodule, namely $V_j$. Moreover each $V_j$ is irreducible and distinct $j$ give rise to non-isomorphic such submodules.

**Proof:** From the definition of $a_i^{(j)}$ we have the following relations:

$$2a_1^{(j)} + a_2^{(j)} = 2 + \rho_j$$

$$a_{m-2}^{(j)} + 2a_{m-1}^{(j)} = S_{m-3}(\rho_j) + 2S_{m-2}(\rho_j). \text{ Recall that } S_{m-1}(\rho_j) = 0 = \rho_jS_{m-2}(\rho_j) - S_{m-3}(\rho_j). \text{ Hence } a_{m-2}^{(j)} + 2a_{m-1}^{(j)} = (2+\rho_j)S_{m-2}(\rho_j)$$

and for every $1 \leq k \leq m-3$, we also have that $a_k^{(j)} + 2a_{k+1}^{(j)} + a_{k+2}^{(j)} = S_{k-1}(\rho_j) + 2S_k(\rho_j) + S_{k+1}(\rho_j) = 2S_k(\rho_j) + \rho_jS_k(\rho_j) = (2+\rho_j)S_k(\rho_j)$.

From now on, the argument is entirely similar with the case when $n$ is odd.
We assume that \( \sum_{j=1}^{m-1} \lambda_j u_j + \sum_{j=1}^{m-1} u_j v_j = 0 \) (R), \( \lambda_j, u_j \in \mathbb{Q}(u^2) \) and we shall show that \( \lambda_j = u_j = 0 \) for every \( 1 \leq j \leq m-1 \). We put again

\[ w_1 = \tilde{a}_s + \tilde{a}_1 s_1 + \tilde{a}_2 s_1 s_2 + \cdots + \tilde{a}_{m-1} s_1 s_2 \cdots s_{m-1} = \tilde{a}_1 s_1 s_2 \cdots s_{m-2} + \tilde{a}_2 s_1 s_2 \cdots s_{m-3} s_{m-1} + \cdots + \tilde{a}_{m-1} s_1 s_2 \cdots s_{m-2} s_{m-1} \].

Then, in the expression \( \sum_{j=1}^{m-1} \lambda_j u_j \) the coefficient of a typical element \( w_k \), \( 1 \leq k \leq m-1 \), has the form

\[ \sum_{j=1}^{m-1} \lambda_j a_k^{(j)} , \quad 1 \leq k \leq m-1 \].

Since the elements \( w_k \) are linearly independent, the relation (R) implies that \( \sum_{j=1}^{m-1} \lambda_j a_k^{(j)} = 0 \) for every \( 1 \leq k \leq m-1 \). So, we obtain a system of \( m-1 \) homogeneous equations in \( m-1 \) unknowns the \( \lambda_j \) whose equations are of the form

\[ \sum_{j=1}^{m-1} \lambda_j a_k^{(j)} = 0, \quad 1 \leq k \leq m-1 \].

With an argument similar to the one when \( n \) is odd, we can show that any solution of these homogeneous equations is also a solution of the system of homogeneous equations

\[ (E') = \left\{ E_k' : \sum_{j=1}^{m-1} \lambda_j^p a_j^{(j)} = 0, \quad 1 \leq k \leq m-1 \right\} \].

The determinant of the coefficients in the later system is the Vandermonde determinant \( \Delta = \prod_{1 \leq i < j \leq m-1} (\rho_i - \rho_j) \neq 0 \), since \( \rho_i \neq \rho_j \) if \( i \neq j \). Hence \( \lambda_j = 0 \) \( \forall 1 \leq j \leq m-1 \).

We also put \( z_1 = \tilde{a}_s s_1, \ldots, z_{m-1} = \tilde{a}_1 s_1 s_2 \cdots s_{m-1} \). Then in the expression

\[ \sum_{j=1}^{m-1} u_j v_j \],

the coefficient of a typical element \( x_k \) has the form

\[ \sum_{j=1}^{m-1} u_j (a_{j-1}^{(j)} + 2a_k^{(j)} + a_{k+1}^{(j)}), \quad 1 \leq k \leq m-1 \],

with \( a_0^{(j)} = 0, a_{m}^{(j)} = S_{m-1}(\rho_j) = 0 \).
Hence we obtain again a system \((\Sigma)\) = 
\[
\begin{align*}
E_k &= \sum_{j=1}^{m-1} u_j (a_k^{(j)} + 2a_k^{(j)}a_{k+1}^{(j)}) = 0 \\
&\quad \forall 1 \leq k \leq m-1
\end{align*}
\]

and by induction we can prove that any solution of the system \((\Sigma)\) is also a solution of the system

\[(\Sigma') = \left( E_k' = \sum_{j=1}^{m-1} u_j (2+\delta_j) a_k^{(j)} \right) \forall k \leq m-1 \).

The determinant of the coefficients in the latter system is

\[
\prod_{j=1}^{m-1} (2+\delta_j) \prod_{k>j=1}^{m-1} (\delta_k - \delta_j) \neq 0 \text{ since } \delta_j \neq -2 \forall j = 1, \ldots, m-1, \text{ and } \delta_1 \neq \delta_j \text{ if } 1 \neq j.
\]

Hence \(u_j = 0 \forall 1 \leq j \leq m\).

Therefore the elements \((u_j, v_j) \forall j = 1, \ldots, m-1\) are linearly independent and so they form a basis for the submodule \(V_j\). Let \(V_j = \langle u_j, v_j \rangle, 1 \leq j \leq m-1\).

Then it can be shown that \(V_j\) is a left \(H_{n+1}(u_j^2)\) module. In fact by recalling the left action of \(T_{s_i}, i = 1, 2\) on the graded module, we have that

\[
\begin{align*}
T_{s_1} u_j &= -u_j, \\
T_{s_2} u_j &= u u_j + u^2 v_j \\
T_{s_2} v_j &= -v_j, \\
T_{s_1} v_j &= uv_j + u^2 (2+\delta_j)u_j
\end{align*}
\]

By replacing \(u_j\) by \(u_j = 2 \cos \frac{j\pi}{n} u_j\) we obtain

\[
\begin{align*}
T_{s_1} u'_j &= -u'_j, \\
T_{s_2} u'_j &= u u'_j + 2u^2 \cos \frac{j\pi}{n} v_j, \\
T_{s_2} v'_j &= -v'_j, \\
T_{s_1} v'_j &= uv'_j + 2u^2 \cos \frac{j\pi}{n} u'_j
\end{align*}
\]
For the same reason as in the case $n$ odd we have that each $V_j$ is an irreducible left $H_n(u)$ submodule, that $V_j \neq V_j$, and that the set $(M_0, M_1, M_2, M_3, V_j, 1 \leq j \leq m-1)$ is a full set of left irreducible $H_n(u)$ modules.

Our proposition is now proved.

The matrices which represent $T_{s_1}$, $1 = 1, 2$ on $V_j$ with respect to the basis $(u_j, v_j)$ are

$$T_{s_1} = -1 \begin{pmatrix} 2u \cos \frac{j\pi}{n} & 0 \\ 0 & u \end{pmatrix}, \quad T_{s_2} = u \begin{pmatrix} 0 & 1 \\ 2u \cos \frac{j\pi}{n} & -1 \end{pmatrix}$$

With a similar argument we can find a decomposition of the submodule $V_j$, namely $V_j = \bigoplus_{j=1}^{m-1} \langle \hat{u}_j, \hat{v}_j \rangle$, where $\hat{u}_j, \hat{v}_j$ are obtained by interchanging the role of $s_1, s_2$ in $u_j, v_j$ respectively. Thus

$$\hat{u}_j = a_1(j)(\bar{e}_{s_2} + \bar{e}_{s_2}s_1s_2) + \ldots + a_{m-1}(j)(\bar{e}_{s_2s_1s_2})^m - 2\bar{e}_{s_2s_1s_2}^m$$

$$\hat{v}_j = (2a_1(j) + a_2(j))\bar{e}_{s_1s_2} + \ldots + (a_{m-1} + 2a_{m-2} + a_{m-1})\bar{e}_{s_1s_2}^{m-2}$$

$$+ (a_{m-2} + 2a_{m-1})\bar{e}_{s_1s_2}^{m-1}$$

Each $\hat{V}_j = \langle \hat{u}_j, \hat{v}_j \rangle$ is irreducible left $H_n(u)$ module, and $\hat{V}_j = V_j$, $1 \leq j \leq m-1$. The matrices with represent $T_{s_1}$, $1 = 1, 2$ on $\hat{V}_j$ with respect to the basis $(\hat{u}_j, \hat{v}_j)$, where $\hat{u}_j = 2 \cos \frac{j\pi}{n} \hat{u}_j$, are
\[ T_{s_1} = \begin{pmatrix} u & 0 \\ 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} & -1 \end{pmatrix}, \quad T_{s_2} = \begin{pmatrix} -1 & 2u^{\frac{1}{2}} \cos \frac{j\pi}{n} \\ 0 & u \end{pmatrix} \]

Eventually we have \( \text{grad}(E) = M_0 \oplus M_1 \oplus M_2 \oplus M_3 \oplus \bigoplus_{j=1}^{m-1} V_j \oplus \bigoplus_{j=1}^{m-1} \hat{V}_j \).

**Theorem 2.3.1**: Let \( W \) be a finite Coxeter group of dihedral type, \( W = D_{2n} \) and let \( \phi \) be the Lusztig isomorphism between the generic Hecke algebra \( H_\mathbb{Q}(u^{\frac{1}{2}}) \) of dihedral type and the group algebra over \( \mathbb{Q}(u^{\frac{1}{2}}) \) of \( D_{2n} \). Define

\[ \Xi_k = -(s_2s_1)^{k-1}s_2 + (s_1s_2)^k - (s_2s_1)^k + (s_1s_2)^k s_1 \quad \text{and} \]

\[ \hat{\Xi}_k = -(s_1s_2)^{k-1}s_1 + (s_2s_1)^k - (s_1s_2)^k + (s_2s_1)^k s_2. \]

(i) If \( n = 2m+1 \), then the images of the generators \( T_{s_i}, i = 1,2 \) of the generic Hecke algebra under \( \phi \) are given by:

\[ \phi(T_{s_1}) = \frac{u-1}{2}.1 + \frac{u+1}{2}s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)\Xi_k \]

\[ \phi(T_{s_2}) = \frac{u-1}{2}.1 + \frac{u+1}{2}s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)\hat{\Xi}_k \]

(ii) If \( n = 2m \), then

\[ \phi(T_{s_1}) = \frac{u-1}{2}.1 + \frac{u+1}{2}s_1 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m-1} (n-2k)\Xi_k \]

\[ \phi(T_{s_2}) = \frac{u-1}{2}.1 + \frac{u+1}{2}s_2 + \frac{(u^{\frac{1}{2}}-1)^2}{2n} \sum_{k=1}^{m-1} (n-2k)\hat{\Xi}_k \]
Proof: In the previous pages we have established a decomposition of
the graded module over $\mathbb{Q}(u^{1/n})$ into a direct sum of left irreducible
$H$-submodules, for both the cases when $n$ is odd and $n$ is even.

(1) Let $n = 2m+1$. Then $\text{grad}(E) = \bigoplus_{j=1}^{n} M_{j} = \bigoplus_{j=1}^{m} M_{j} \oplus M_{0}$, where $M_{0} = \langle e_{1} \rangle$
affords the representation $T_{s_{1}}: u \mapsto 1, 2$. $M_{s} = \langle e_{0} \rangle$ affords the
representation $T_{s_{1}}: u \mapsto 1, 2$. $\dim M_{j} = 2$ and $M_{j} \leq M_{j}$, if $j \neq j'$, $M_{j} = \hat{M}_{j}$
$1 \leq j \leq m$. Moreover we established that the matrices which represent $T$
on $M_{j}$ with respect to the basis $\{u_{j}, v_{j}\}$ are

$$
\begin{pmatrix}
2u^{1/n} \cos \frac{j\pi}{n} & 0 \\
0 & u
\end{pmatrix}
$$

and that the matrices which represent $T_{s_{1}}$ on $M_{j}$ with respect to the basis
$\{\hat{u}_{j}, \hat{v}_{j}\}$ are

$$
\begin{pmatrix}
u & 0 \\
2u^{1/n} \cos \frac{j\pi}{n} & -1
\end{pmatrix}
$$

$$
\begin{pmatrix}
-1 & 2u^{1/n} \cos \frac{j\pi}{n} \\
0 & u
\end{pmatrix}
$$

$1 \leq j \leq m$.

It is enough to show that $T_{s_{1}}$ and $\phi(T_{s_{1}})$ act on the same way on each $M_{j}$, $\hat{M}_{j}$,
and on each 1-dimensional submodule. By specializing $u \mapsto 1$ in the matrices
which represent $T_{s_{1}}$ above, we obtain the matrices which represent $s_{1}$, $1 = 1, 2$.

It is trivial to verify that $T_{s_{1}}$ and $\phi(T_{s_{1}})$ $1 = 1, 2$ act on the same
way on the submodules $M_{0}$ and $M_{s}$. 
In the sequel the terminology blocks of the first kind, means the matrices which represent a specific element of the group \( W = D_{2n} \) on \( M_j \), \( j = 1, \ldots, m \), and blocks of the second kind, the matrices of the same element on \( M_j \), \( j = 1, \ldots, m \). I claim that the blocks of the first kind which represent the element \((s_1, s_2)^k\), \( k = 1, \ldots, m \) are given by

\[
\begin{align*}
\chi_j &= \begin{bmatrix}
S_k(\rho_j) + S_{k-1}(\rho_j) & -2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & - (S_{k-1}(\rho_j) + S_{k-2}(\rho_j))
\end{bmatrix} \\
\gamma_j &= \begin{bmatrix}
-(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) & 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
-2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j)
\end{bmatrix},
\end{align*}
\]

and the blocks of the second kind by

\[
\begin{align*}
\chi_j &= \begin{bmatrix}
S_k(\rho_j) + S_{k-1}(\rho_j) & -2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & - (S_{k-1}(\rho_j) + S_{k-2}(\rho_j))
\end{bmatrix} \\
\gamma_j &= \begin{bmatrix}
-(S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) & 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
-2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j)
\end{bmatrix},
\end{align*}
\]

where \( \rho_j = 2 \cos \frac{2j\pi}{n} \), \( 1 \leq j \leq m \).

We concentrate on the blocks of the first kind.

For \( k = 1 \) we have \( s_1 = \begin{bmatrix} -1 & 2 \cos \frac{j\pi}{n} \\ 0 & 1 \end{bmatrix} \), \( s_2 = \begin{bmatrix} 1 & 0 \\ 2 \cos \frac{j\pi}{n} & -1 \end{bmatrix} \)

therefore \( s_1 s_2 \mapsto \begin{bmatrix} 4 \cos^2 \frac{j\pi}{n} - 1 & -2 \cos \frac{j\pi}{n} \\ 2 \cos \frac{j\pi}{n} & -1 \end{bmatrix} \)

Since \( 2 + \rho_j = 4 \cos^2 \frac{j\pi}{n} \), \( S_0(\rho_j) = 1 \), \( S_1(\rho_j) = \rho_j \), \( S_{-1}(\rho_j) = 0 \), the assertion holds.
Assume, by induction, that the blocks of the first kind which represent 
\((s_1 s_2)^\lambda\) are of the form

\[
\begin{bmatrix}
S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) & -2 \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\
2 \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -(S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j))
\end{bmatrix}
\]

Then, \((s_1 s_2)^{\lambda+1} = (s_1 s_2)^\lambda (s_1 s_2)\) is represented by

\[
\begin{bmatrix}
S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) & -2 \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\
2 \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -(S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j))
\end{bmatrix}^{-1}
\begin{bmatrix}
1 + \rho_j \\
2 \cos \frac{j\pi}{n}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\]

\[
a_{11} = (1 + \rho_j) [S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - 4 \cos^2 \frac{j\pi}{n} S_{\lambda-1}(\rho_j) = S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j) + \rho_j S_\lambda(\rho_j) + \rho_j S_{\lambda-1}(\rho_j) - (2 + \rho_j) S_{\lambda-1}(\rho_j) = \rho_j S_\lambda(\rho_j) - S_{\lambda-1}(\rho_j) + S_\lambda(\rho_j) = S_\lambda(\rho_j) + S_{\lambda+1}(\rho_j) .
\]

\[
a_{12} = -2 \cos \frac{j\pi}{n} S_\lambda(\rho_j) .
\]

\[
a_{21} = 2 \cos \frac{j\pi}{n} (1 + \rho_j) S_{\lambda-1}(\rho_j) - 2 \cos \frac{j\pi}{n} (S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)) = 2 \cos \frac{j\pi}{n} [\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)] = 2 \cos \frac{j\pi}{n} S_\lambda(\rho_j) .
\]

\[
a_{22} = -4 \cos^2 \frac{j\pi}{n} S_{\lambda-1}(\rho_j) + S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j) = -(2 + \rho_j) S_{\lambda-1}(\rho_j) + S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)
\]

= - S_{\lambda-1}(\rho_j) - [\rho_j S_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)] = - (S_{\lambda-1}(\rho_j) + S_{\lambda}(\rho_j))

Hence our assertion holds for every k.

To obtain the matrices \( Y_j \) we conjugate the matrices \( X_j \) by \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

By symmetry, the blocks of the first kind which represent \((s_2s_1)^k, k = 1, \ldots, m\) are

\[
Y_j = \begin{bmatrix}
- (S_{k-1}(\rho_j) + S_{k-2}(\rho_j)) & 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
- 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j)
\end{bmatrix}
\]

and the blocks of the second kind are

\[
X_j = \begin{bmatrix}
S_k(\rho_j) + S_{k-1}(\rho_j) & - 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & - (S_{k-1}(\rho_j) + S_{k-2}(\rho_j))
\end{bmatrix}
\]

Simple matrix multiplication gives the blocks of the first kind of the element \((s_1s_2)^k, s_1, k = 0, \ldots, m\), which are

\[
Z_j = \begin{bmatrix}
- (S_k(\rho_j) + S_{k-1}(\rho_j)) & 2 \cos \frac{j\pi}{n} S_k(\rho_j) \\
- 2 \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & S_k(\rho_j) + S_{k-1}(\rho_j)
\end{bmatrix}
\]

and the blocks of the second kind which are
By symmetry, we obtain the blocks of the first kind which represent $(s_{2s_1})^k, s_{2s_1} k = 0, \ldots, m-1$ and which are

$$W_j = \begin{bmatrix}
S_k(p_j) + S_{k-1}(p_j) & -2 \cos \frac{j\pi}{n} S_{k-1}(p_j) \\
2 \cos \frac{j\pi}{n} S_k(p_j) & -(S_k(p_j) + S_{k-1}(p_j))
\end{bmatrix}$$

and the blocks of the second kind which are

$$Z_j = \begin{bmatrix}
-(S_k(p_j) + S_{k-1}(p_j)) & 2 \cos \frac{j\pi}{n} S_k(p_j) \\
-2 \cos \frac{j\pi}{n} S_{k-1}(p_j) & S_k(p_j) + S_{k-1}(p_j)
\end{bmatrix}$$

Now, for every $1 \leq k \leq m$, let $R_{k}^{(j)}$ be the matrix which represents the element of the group algebra $E_k$ on $M_j$. Then,

$$p_{k}^{(j)} = \begin{bmatrix}
-S_{k-1}(p_j) - S_{k-2}(p_j) & 2 \cos \frac{j\pi}{n} S_{k-2}(p_j) \\
-2 \cos \frac{j\pi}{n} S_{k-1}(p_j) & S_{k-1}(p_j) + S_{k-2}(p_j)
\end{bmatrix} + \begin{bmatrix}
S_k(p_j) + S_{k-1}(p_j) & -2 \cos \frac{j\pi}{n} S_{k-1}(p_j) \\
2 \cos \frac{j\pi}{n} S_{k-1}(p_j) & -(S_{k-1}(p_j) + S_{k-2}(p_j))
\end{bmatrix}$$
\[ R_k^{(j)} = \begin{bmatrix} 1 \end{bmatrix} A_k^{(j)} \begin{bmatrix} 1 \end{bmatrix} \]

where \( A_k^{(j)} = 2 \cos \frac{j \pi}{n} [S_k(p_j) + S_{k-2}(p_j)] - 4 \cos \frac{j \pi}{n} S_{k-1}(p_j). \)

Furthermore, \( p_j A_{k-1}^{(j)} - A_{k-2}^{(j)} = 2p_j \cos \frac{j \pi}{n} [S_{k-1}(p_j) + S_{k-3}(p_j)] - 4p_j \cos \frac{j \pi}{n} S_{k-2}(p_j) - \]

\[ - 2 \cos \frac{j \pi}{n} [S_{k-2}(p_j) + S_{k-4}(p_j)] + 4 \cos \frac{j \pi}{n} S_{k-3}(p_j) = \]

\[ = 2 \cos \frac{j \pi}{n} [p_j S_{k-1}(p_j) - S_{k-2}(p_j) + p_j S_{k-3}(p_j) - S_{k-4}(p_j)] - 4 \cos \frac{j \pi}{n} [p_j S_{k-2}(p_j) - S_{k-3}(p_j)] = A_k^{(j)}. \]

Therefore \( R_k^{(j)} = p_j R_{k-1}^{(j)} - R_{k-2}^{(j)} \quad 1 \leq k \leq m. \) When \( k = 1, \) then \( A_1^{(j)} = 2 \cos \frac{j \pi}{n} (p_j - 2). \)
Assume that for all integers \( \lambda \leq k \), we have \( A^{(j)}_\lambda = S_{\lambda-1}(\rho_j)A^{(j)}_1 \). Then
\[
A^{(j)}_{k+1} = \rho_j A^{(j)}_k - S_{k-1}(\rho_j)A^{(j)}_1 = \rho_j S_{k-1}(\rho_j)A^{(j)}_1 - S_{k-2}(\rho_j)A^{(j)}_1
\]
\[
= (\rho_j S_{k-1}(\rho_j) - S_{k-2}(\rho_j)) A^{(j)}_1 = S_k(\rho_j)A^{(j)}_1.
\]

Therefore, the matrix which represents the element of the group algebra
\[
\sum_{k=1}^{m} (n-2k)\Sigma \text{ on } M_j \text{ is: }
\begin{bmatrix}
0 & \sum_{k=1}^{m} (n-2k)A^{(j)}_k \\
0 & 0
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 & \sum_{k=1}^{m} (n-2k)S_{k-1}(\rho_j)A^{(j)}_1 \\
0 & 0
\end{bmatrix}
\]

Now \( \sum_{k=1}^{m} (n-2k)S_{k-1}(\rho_j)A^{(j)}_1 = 2\cos \frac{j\pi}{n} (\rho_j-2) \sum_{k=1}^{m} (n-2k)S_{k-1}(\rho_j) = 2\cos \frac{j\pi}{n} \cdot \sum_{k=1}^{m} (n-2k) = -2n \cos \frac{j\pi}{n} \) (by Lemma 2.2.2(11) and 2.2.3(1)).

Hence the matrix which represents the element of the group algebra:
\[
\frac{u-1}{2} \cdot \frac{u+1}{2} S_1 + \frac{(u^\frac{1}{2}-1)^2}{2n} \sum_{k=1}^{m} (n-2k)S_k \text{ on } M_j \text{ is: }
\]
\[
\begin{bmatrix}
\frac{u-1}{2} & \frac{u+1}{2} \left( 0 & 1 \right) -1 & 2\cos \frac{j\pi}{n} \left( 1 & 0 \right) + \frac{(u^\frac{1}{2}-1)^2}{2n} \left( 0 & -2n \cos \frac{j\pi}{n} \right) \\
-1 & 2u^\frac{1}{2} \cos \frac{j\pi}{n} & \frac{u+1}{2} \left( 1 & 0 \right)
\end{bmatrix}
\text{ for every } j = 1, \ldots, m.
Therefore $T_{s_1}$ acts on every $M_j$ in the same way as $\varphi(T_{s_1})$ does.

We next put $R_{k}^{(j)}$ to be the matrix which represents the element $E_{k}$ on $M_j$. It turns out that $R_{k}^{(j)} = \begin{bmatrix} \varphi(A_{k}^{(j)}) & 0 \\ 0 & 0 \end{bmatrix}$ and therefore the matrix which represents the element of the group algebra

$$u \frac{1}{2} + u \frac{1}{2} s_1 + \frac{(u^2 - 1)^2}{2n} \sum_{k=1}^{m} (n-2k)E_k$$
on $M_j$ is

$$= \begin{bmatrix} 1 & 0 \\ 2 \cos \frac{j\pi}{n} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \frac{-2n \cos \frac{j\pi}{n}}{2n} \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 2 \cos \frac{j\pi}{n} & 0 \end{bmatrix}.$$ So $T_{s_1}$ also acts on $M_j$ in the same way as $\varphi(T_{s_1})$ does. By symmetry (interchanging the role of $s_1, s_2$) we obtain the image $\varphi(T_{s_2})$ and so part (i) of our theorem is proved.

The proof of part (ii) of our theorem, is entirely similar, since the matrices which represent $T_{s_i}, i = 1, 2$ on the irreducible constituents of the graded module when $n = 2m$, have the same form as in the case $n = 2m + 1$.

In this case, the matrix which represents the element

$$\sum_{k=1}^{m-1} (n-2k)E_k$$
on $V_j, 1 \leq j \leq m-1$, is

$$= \begin{bmatrix} 0 & \sum_{k=1}^{m-1} (n-2k)A_k^{(j)} \\ 0 & 0 \end{bmatrix}$$

where $\rho_j = 2 \cos \frac{j\pi}{m}$, and

$$A_k^{(j)} = 2 \cos \frac{j\pi}{m} (\rho_j - 2)$$. 


Now Lemma 2.2.2(1) and 2.2.3(11) give

\[ \sum_{k=1}^{m-1} (n-2k)S_{k-1}(p_j)A_j^{(j)} = -2n \cos \frac{3\pi}{m} \quad 1 \leq j \leq m-1. \]

Therefore we can verify that \( T_{s_1}, 1 \leq 2 \) acts on every irreducible constituent \( \mathcal{V}_j \) and \( \mathcal{V}_j^\perp, 1 \leq j \leq m-1 \), on the same way as \( \varphi(T_{s_1}) \). This is also true when we consider each one of the four 1-dimensional representations, afforded by the submodules \( M_0, M_1, M_2, M_3 \). It is obvious when we consider the submodules \( M_0 \) and \( M_3 \).

For the representation \( \sigma_1 : s_1 \to -1, s_2 \to 1 \), we have that

\[
\varphi(T_{s_1}) = -1.
\]

For the same representation, we have that

\[
\varphi(T_{s_2}) = u.
\]

The argument is similar for the representation \( \sigma_2 : s_1 \to 1, s_2 \to -1 \). Thus our theorem is proved.
We illustrate the situation by giving some examples.

(1) \( W(A_2) = S_2 : \Theta(T_{s_1}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_1 \)

(2) \( W = D_4 : \Theta(T_{s_1}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_1 \)

\( \Theta(T_{s_2}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_2 \)

(3) \( W(A_2) = S_3 \). This example appears at the end of Chapter I.

(4) \( W(B_2) = D_8 \) \((n = 4, m = 2)\)

\( \Theta(T_{s_1}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_1 + \frac{(u^4-1)^2}{4} \cdot (-s_2 + s_1 s_2 - s_1 s_2 s_1 s_2 s_1) \)

\( \Theta(T_{s_2}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_2 + \frac{(u^4-1)^2}{4} \cdot (-s_1 s_2 s_1 - s_1 s_2 s_2 s_1 s_2) \)

(5) \( W = D_{10} \) \((n = 5, m = 2)\).

\( \Theta(T_{s_1}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_1 + 3 \cdot \frac{(u^4-1)^2}{10} \cdot (-s_2 + s_1 s_2 - s_1 s_2 s_1 s_2 s_1) + \frac{(u^4-1)^2}{10} \cdot (-s_2 s_1 s_2 + s_1 s_2 s_1 s_2 - s_2 s_1 s_2 s_1 s_2 + s_1 s_2 s_1 s_2 s_1) \)

\( \Theta(T_{s_2}) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s_2 + 3 \cdot \frac{(u^4-1)^2}{10} \cdot (-s_1 s_2 s_1 - s_1 s_2 s_2 s_1 s_2) + \frac{(u^4-1)^2}{10} \cdot (-s_1 s_2 s_1 + s_2 s_1 s_2 s_1 s_2 - s_1 s_2 s_2 s_1 s_2 + s_2 s_1 s_2 s_1 s_2) \)
(6) \(\mathcal{W}(G_2) = D_{12} : (n = 6, m = 3)\)

\[
\phi(T_{s_1}) = \frac{u^{-1}}{2} + \frac{u+1}{2} s_1 + \frac{(u^3-1)^2}{3} (-s_2 s_2 s_2 - s_2 s_2 s_1 + s_2 s_2 s_1) + \\
\quad + \frac{(u^4-1)^2}{6} (-s_2 s_2 s_2 - s_2 s_2 s_2 - s_2 s_2 s_1 + s_2 s_2 s_1)
\]

\[
\phi(T_{s_2}) = \frac{u^{-1}}{2} + \frac{u+1}{2} s_2 + \frac{(u^3-1)^2}{3} (-s_1 s_1 s_1 + s_2 s_2 s_2 + s_2 s_2 s_2) + \\
\quad + \frac{(u^4-1)^2}{6} (-s_1 s_1 s_1 - s_1 s_1 s_2 - s_1 s_1 s_2 + s_1 s_1 s_2)
\]

Remark: We note that when \(n\) is even, the images \(\phi(T_{s_i}), i = 1, 2\) do not involve all the elements of the group \(D_{2n}\). In fact, in the images of \(\phi(T_{s_i})\) in the examples above, when \(n\) is even the elements \(w_0\) and \(s_1 w_0\) do not appear at all.
CHAPTER 3

The centre of the generic Hecke algebra

Let $(W, S)$ be a finite Coxeter system and let $H$ be the generic Hecke algebra over the polynomial ring $\mathbb{Q}[u]$, associated to $(W, S)$. Let $Z(H)$ be the centre of $H$.

The following result is rather well known. We include a proof for the sake of completeness.

Lemma 3.1: Let $c = \sum_{w \in W} a_{wT} w$, be an element of $H$. Then $c$ lies in the centre of $H$ if and only if the following two conditions hold:

$C_1: a_{sws} = a_w$ if $\ell(sw) = \ell(w)$, $s \in S$, and

$C_2: a_w = u a_{sws} - (u-1)a_{sw}$ if $\ell(sw) = \ell(w) + 2$, $s \in S$.

Proof: It is clear that $c$ lies in the centre of $H$ if and only if $cT_s = T_s c$ for all $s \in S$. We fix a $s \in S$ and we denote by $C_w(s)$ the centralizer of $s$ in $W$. We can write

$c = \sum_{w \in W} a_{wT} w + \sum_{w \in W} a_{wT} w + \sum_{w \in W} a_{wT} w + \sum_{w \in W} a_{wT} w,$

where

$\ell(sw) > \ell(w)$, $\ell(sw) < \ell(w)$, $\ell(sw) = \ell(w)$, $w \in C_w(s)$

$\ell(ws) > \ell(w)$, $\ell(ws) < \ell(w)$, $\ell(ws) = \ell(w)$, $w \in C_w(s)$

Now $cT_s = T_s c$ if and only if

Now $cT_s = T_s c$ if and only if
\[ \sum a_w T_{ws} + \sum a_w (u T_{ws} + (u-1)T_w) + \sum a_w T_{ws} \]

\[ \ell(ws) \geq \ell(w), \quad \ell(ws) \leq \ell(w) \]

\[ \ell(ws) \geq \ell(w), \quad w \in C_M(s) \]

\[ \sum a_w (u T_{ws} + (u-1)T_w) \]

\[ \ell(ws) \leq \ell(w), \quad w \in C_M(s) \]

\[ \sum a_w T_{sw} + \sum a_w T_{sw} + \sum a_w (u T_{sw} + (u-1)T_w) + \sum a_w T_{sw} \]

\[ \ell(sw) \geq \ell(w), \quad \ell(sw) \geq \ell(w) \]

\[ \ell(sw) \leq \ell(w), \quad \ell(sw) \leq \ell(w) \]

\[ w \in C_M(s) \]

\[ w \in C_M(s) \]

\[ w \in C_M(s) \]

\[ w \in C_M(s) \]

This condition is equivalent to

\[ \sum a_w T_{ws} + \sum a_w T_{sw} + \sum a_w (u T_{ws} + (u-1)T_{sw}) + \sum a_w T_{sw} \]

\[ \ell(sws) \geq \ell(w) + 2, \quad \ell(sws) \leq \ell(w) + 2 \]

\[ \ell(sws) \geq \ell(ws) \]

\[ w \in C_M(s) \]

\[ w \in C_M(s) \]

\[ w \in C_M(s) \]
We now compare coefficients of the basis elements $T_w$, $w \in W$ on both sides of the relation $cT_s = T_s c$, to obtain

$$a_w = u a_{sw} - (u-1) a_{ws} \quad \text{if } \ell(sw) = \ell(w) + 2$$

$$a_w = u a_{sw} - (u-1) a_{sw} \quad \text{if } \ell(sw) = \ell(w) + 2$$

$$w a_{sw} = u a_{sw} \quad \text{if } \ell(sw) = \ell(w) + 2,$$

and for every $w \in \{w : w \not\in C_w(s) \text{ with } \ell(sw) = \ell(ws) \text{ and } \ell(ws) = \ell(w)\}$

$$a_{ws} = a_{sw}.$$

In the latter case, by replacing $w$ by $sw$ we obtain that $a_w = a_{sw}$ for $w$ such that $\ell(sw) > \ell(w)$ and $\ell(ws) = \ell(w)$, and by replacing $w$ by $ws$, we also obtain that $a_w = a_{sw}$ for $w$ such that $\ell(sw) < \ell(w)$ and $\ell(ws) > \ell(w)$. Thus $a_w = a_{sw}$ for $w$ such that $\ell(sw) = \ell(w)$.

Our lemma is now proved.
Definition 3.1: Let $w, w' \in W$. We say that $w, w'$ are strongly conjugate if there exists a sequence $x_1, \ldots, x_n, x_i \in W, i = 1, \ldots, n$ with $w = x_1 \cdot \ldots \cdot x_n = w'$ such that for all $i$, $x_{i+1} = sx_i s$ for some $s \in S$ with $\ell(x_{i+1}) = \ell(x_i)$.

Thus, if $c = \sum_{w \in W} a_w T_w \in Z(H)$ and if $w, w'$ are strongly conjugate then condition $C_1$ implies $a_w = a_{w'}$.

The relation of being strongly conjugate is an equivalence relation inside each conjugacy class and so each conjugacy class is a disjoint union of strong conjugacy classes.

If $C$ is a strong conjugacy class we write $a_C$ for the coefficient $a_w$ for some $w \in C$. Thus, if $c = \sum_{w \in W} a_w T_w \in Z(H)$, we have $c = \sum_{C \in \mathcal{C}} a_C T_C$, the summation being taken over all strong conjugacy classes $C$, with $T_C = \sum_{w \in C} T_w$.

Let $C, C'$ be two strong conjugacy classes inside a given conjugacy class. We say that $C'$ covers $C$ if, $\exists w \in C, s \in S$ such that $sws \in C'$ with $\ell(sws) = \ell(w) + 2$.

Suppose that $C'$ covers $C$. Let $D$ be a strong conjugacy class. We say that $D$ is an intermediate class for the pair $(C, C')$ if $\exists w \in C, s \in S$ such that $sw \in D$, and $sws \in C'$. It is clear that $D$ belongs to a different conjugacy class from the given one which contains $C$ and $C'$. If $C'$ covers $C$, and $D$ is an intermediate class for $(C', C)$, then condition $C_2$ gives $a_D = u a_C - (u-1) a_D$.

We define a partial ordering on strong conjugacy classes by saying that $C < C'$ if there exists a sequence of strong conjugacy classes $(C_1, \ldots, C_k)$
with \( C = C_1, \ldots, C_k = C' \) such that \( C_{i+1} \) covers \( C_i \) \( \forall 1 \leq i \leq k-1 \). We define the length of a strong conjugacy class \( s(C) = \ell(w) \) for all \( w \in C \). By \( C_M \) we denote a strong conjugacy class of maximal length inside a given conjugacy class. Thus every strong conjugacy class \( C \) can be joined to a \( C_M \) by means of a sequence of strong conjugacy classes \( C_1, \ldots, C_k \), with \( C_1 = C, C_k = C_M \) and with the property that either \( C_1 \) covers \( C_{i+1} \) or \( C_{i+1} \) covers \( C_i \) \( \forall 1 \leq i \leq k-1 \).

Therefore we can always express the coefficient \( a_C \) in terms of \( a_{C_M} \) for some maximal strong conjugacy class \( C_M \) and in fact \( a_C = u^e a_{C_M} + \text{linear combination of other } a_{C_i}, e \in \mathbb{Z} \). Nevertheless this can be done in many different ways.

Remarks: If \( w, w' \) are strongly conjugate, then \( a_w = a_{w'} \). The converse is not true. For instance if \( s, s' \in S \) and \( s, s \) are inside the same conjugacy class, then clearly \( s, s' \) are not strongly conjugate. However \( a_s = a_{s'} \). For \( s, s' \) are conjugate if and only if, there exists a sequence \( (s_1, \ldots, s_q) \) with \( s_1 = s, s_q = s' \) such that \( s_i s_{i+1} \) has finite odd order \( \forall 1 \leq i \leq q-1 \). Now, if \( s_i s_{i+1} \) has odd order \( n_i = 2e_i + 1 \), then by repeated application of the condition \( C_2 \) and using the fact that the elements \( (s_i s_{i+1})^\lambda, (s_{i+1} s_i)^\lambda \) are strongly conjugate for every \( \lambda = 1, \ldots, 6 \), we obtain \( a_{s_i} = a_{s_{i+1}} \) \( \forall \lambda = 1, \ldots, q-1 \). Thus \( a_s = a_{s'} \) \( \text{(11)} \). It is not true that inside a given conjugacy class, the elements of maximal length are strongly conjugate.

For instance, when \( W = S_6 \), the elements \( (34)(1526), (34)(1625), (16)(2435), (16)(2534) \), are conjugate and they have the same length 14, which is maximal length for their conjugacy class. However, they fall into two strong conjugacy classes namely
When $W = D_{2n}$, the theory above enables us to find a natural basis for the centre of the generic Hecke algebra.

**Proposition 3.1:** (1) Let $n = 2m + 1$, and let $H(D_{2n})$ be the generic Hecke algebra over the polynomial ring $\mathbb{Q}[u]$. Then, a basis for the centre $Z(H)$ is given by the following set

$$T_1, T(s_1s_2)^k + T(s_2s_1)^k - (u-1) \sum_{\lambda=1}^{k} u^{\lambda-1}[T(s_1s_2)^{k-\lambda}s_1 + T(s_2s_1)^{k-\lambda}s_2],$$

$$T_w + \sum_{\lambda=1}^{m} u^{\lambda}[T(s_1s_2)^{m-\lambda}s_1 + T(s_2s_1)^{m-\lambda}s_2], 1 \leq k \leq m$$

(11) Let $n = 2m$. Then, if $m$ is even, a basis for the centre of $H(D_{2n})$ is given by the set

$$T_1, T(s_1s_2)^k + T(s_2s_1)^k - (u-1) \sum_{\lambda=1}^{k} u^{\lambda-1}[T(s_1s_2)^{k-\lambda}s_2 + T(s_2s_1)^{k-\lambda}s_1],$$

$$T(s_1s_2)^{m-1}s_1 + \ldots + u^{m-1} T_{s_2}, T(s_2s_1)^{m-1}s_2 + \ldots + u^{m-1} T_{s_1}, T_w, 1 \leq k \leq m$$

and when $m$ is odd by the set
Proof: (1) When \( n = 2m + 1 \), the dimension of the centre of \( H \) is \( m + 2 \).

The conjugacy classes of \( W \) are given by \( A_0 = \{ 1 \} \).

\[
A_j = \{(s_1s_2)^J, (s_2s_1)^J\} \quad j = 1, \ldots, m, \quad A_{m+1} = \{s_1s_2s_1s_2, \ldots, w_0\}.
\]

Let \( c = \sum_{j} a_jT_j \in Z(H) \). We note that \( (s_1s_2)^J, (s_2s_1)^J \) are strongly conjugate \( \forall j = 1, \ldots, m \). So each \( A_j \) is itself a strong conjugacy class and therefore \( a(s_1s_2)^J = a(s_2s_1)^J, \quad j = 1, \ldots, m \).

The other strong conjugacy classes are: \( C_1 = \{s_1s_2 \}^{1-1}s_1 \) \( 1 = 1, \ldots, m \) \( C_1 = \{(s_2s_1)^{1-1}s_2 \} \quad 1 = 1, \ldots, m \) and \( C_M = \{w_0\} \). The partial ordering inside the conjugacy class \( A_{m+1} \) is given by the graph:
By repeated application of the relation $a_C = u a_C, - (u-1) a_D$ whenever $C'$ covers $C$ we obtain: (also using that $a_{(s_1 s_2)}^j = a_{(s_2 s_1)}^j$ $\forall 1 \leq j \leq m$)

$$a_{s_1} = u^m a_w - \sum_{j=1}^{m} u^m j (u-1) a_{(s_2 s_1)}^{m-j+1}$$

$$a_{s_1 s_2 s_1} = u^{m-1} a_w - f_{(121)}(m) \text{ where } f_{(121)}(m) = \begin{cases} 0 & \text{if } m = 1 \\ \sum_{j=2}^{m} u^m j (u-1) a_{(s_2 s_1)}^{m-j+2} & \text{if } m \geq 2 \end{cases}$$
\[ a_{s_1 s_2} s_{s_1} s_{s_2} = u^{-2} a_w - f^{(12121)} \text{ where } f^{(12121)} = \begin{cases} 0 & \text{if } m \leq 2 \\ m \sum_{j=1}^{m-1} (u-1) a_{s_2 s_1} u^{m-j-3} & \text{if } m \geq 3 \end{cases} \]

By interchanging the role of \( s_1, s_2 \) we obtain the coefficients

\[ a_{s_2 s_1} s_{s_1} s_{s_2} = u a_w - (u-1) a_{s_2 s_1}^m \]

Let \( c = \sum_{w \in D_{2n}} a_w T_w \in Z(H) \). Then

\[ c = a_1 T_1 + a_{s_1} [T_{s_1} + T_{s_2}] + a_{s_1 s_2} [T_{s_1 s_2} + T_{s_2 s_1}] + a_{s_1 s_2} [T_{s_1 s_2} + T_{s_2 s_1} + T_{s_2 s_1}]
+ \ldots + a_{s_1 s_2}^m [T_{s_1 s_2}^m + T_{s_2 s_1}^m] + a_w T_w \]

\[ = a_1 T_1 + (u a_w - \sum_{j=1}^{m-1} u^{m-j} (u-1) a_{s_1 s_2} (u-1)^{m-j-1}) [T_{s_1} + T_{s_2}] + a_{s_1 s_2} [T_{s_1 s_2} + T_{s_2 s_1} + T_{s_2 s_1}]
+ (u a_w - f^{(121)} [T_{s_1} s_{s_1} s_{s_2} + T_{s_2 s_1} s_{s_2} s_{s_1}] + a_{s_1 s_2}^2 [T_{s_1 s_2}^2 + T_{s_2 s_1}^2] +
+ \ldots + a_{s_1 s_2}^m [T_{s_1 s_2}^m + T_{s_2 s_1}^m] + a_w T_w \]

\[ = a_1 T_1 + a_{s_1 s_2} [T_{s_1 s_2} + T_{s_2 s_1} - (u-1) (T_{s_1} + T_{s_2})] + \ldots \]
Therefore part (1) of the proposition is proved.

(ii) Let \( W = D_{2n} \), \( n = 2m \), and say \( m \) is even. In this case, the conjugacy classes are: \( A_0 = \{1\} \).

\[
A_j = \{(s_1 s_2)^j, (s_2 s_1)^j\}, \quad 1 \leq j \leq m-1. 
\]

\[
A_m = \{s_1 s_2 s_1 s_2, (s_1 s_2)^2 s_1, \ldots, (s_2 s_1)^{m-1} s_2\}. 
\]

\[
A_{m+1} = \{s_2, s_1 s_2 s_1, (s_2 s_1)^2 s_2, \ldots, (s_1 s_2)^{m-1} s_1\}. 
\]

\[
A_{m+2} = \{w_0\}. 
\]

The dimension of \( Z(H) \) is \( m+3 \).

Each element inside the conjugacy class \( A_m \) forms by itself a strong conjugacy class, and each \( A_j \) \( 1 \leq j \leq m-1 \) is a strong conjugacy class.

The partial ordering inside the conjugacy class \( A_m \) is given by the graph:
By repeated application of the fundamental relation $a_C = ua_C - (u-1)a_D$, whenever $C'$ covers $C$, and using the fact that each $A_j$ is itself a strong conjugacy class, we obtain:

$$a_{s_1} = u^{m-1} a(s_2s_1)^{m-1}.s_2 - \sum_{j=1}^{m-1} u^{m-j-1}(u-1)a(s_1s_2)^{m-j}$$

$$a_{s_2s_1s_2} = u^{m-2} a(s_2s_1)^{m-1}.s_2 - f(212)$$

where $f(212) = \begin{cases} 0 & \text{if } m-1 < 2 \\ \sum_{j=1}^{m-1} u^{m-j-1}a(s_1s_2)^{m-j+1} & \text{if } m-1 \geq 3 \end{cases}$

$$a(s_1s_2)^{m-2}.s_1 = u a(s_2s_1)^{m-1}.s_2 - (u-1)a(s_1s_2)^{m-1}$$

By interchanging the role of $s_1$, $s_2$ we obtain similar relations from the partial ordering on strong conjugacy classes inside the conjugacy class $A_{m+1}$, for the coefficients $a_w$, $w \in A_{m+1}$. Now if $c = \sum_{w \in D_{2n}} a_w w \in Z(H)$, then we
replace the coefficients $a_w$ for $w \in A_m \cup A_{m+1}$ by the corresponding relations which express any $a_w$ in terms of $a(s_2s_1)^{m-1}s_2$ or $a(s_1s_2)^{m-1}s_1$. By gathering coefficients together in the expression $\sum_{w \in D_{2n}} a_w T_w$, we obtain a basis for $Z(H)$ of the desired form.

When $m$ is odd, then the conjugacy classes are:

$$A_0 = \{1\}, \quad A_j, \quad 1 \leq j \leq m-1, \quad A_m = \{s_1s_2s_1s_2, \ldots, (s_1s_2)^{m-1}s_1\},$$

$$A_{m+1} = \{s_2s_1s_2s_1, \ldots, (s_2s_1)^{m-1}s_2\}, \quad A_{m+2} = \{w_0\}.$$

With a similar argument we obtain a basis for $Z(H)$, and our proposition is now proved.

We note that under the specialization $u = 1$, the basis above specializes to the class sums.

Let $W = S_n$ the symmetric group and let $H$ be the generic Hecke algebra of symmetric type over the polynomial ring $\mathbb{Q}[u, u^{-1}]$. James and Dipper defined elements called generalized Murphy operators by

$$L_0 = T_1,$$

$$L_1 = \frac{1}{u} T_{s_1},$$

$$L_2 = \frac{1}{u} T_{s_2} + \frac{1}{u^2} T_{s_1s_2s_1},$$

$$\vdots$$

$$L_{n-1} = \frac{1}{u} T_{s_{n-1}} + \frac{1}{u^2} T_{s_{n-2}s_{n-1}s_{n-2}} + \cdots + \frac{1}{u^{n-1}} T_{s_1 \cdots s_{n-2}s_{n-1}s_1}.$$
By specializing \( u \to 1 \) we obtain the standard Murphy operators (See [13]).

James and Dipper proved (see [8]) the following:

(i) \( L_0, L_1, \ldots, L_{n-1} \) commute with each other

(ii) The algebra they generate contains the centre of \( H \). Using Murphy's construction for the centre of the group algebra of the symmetric group (see [13]) they showed that the centre of \( H \) consists of the symmetric polynomials in the \( L_0, L_1, \ldots, L_{n-1} \).

Moreover they showed that for any partition \( \lambda \) of \( n \), \( \lambda \vdash n \) say

\( \lambda = (\lambda_1, \lambda_2, \ldots) \) there exists a unique basis up to a scalar multiple of the centre of \( H \) say \( \{ y(\lambda), \lambda \vdash n \} \) with the following properties:

(i) The coefficients of the \( T_w \)'s involved in \( y(\lambda) \) belong to \( \mathbb{Z}[u] \)

(ii) \( y(\lambda) \) involves \( T_{u_\lambda} \), where \( u_\lambda = (1 2 \ldots \lambda_1) (\lambda_1 + 1 \ldots \lambda_1 + \lambda_2) \ldots \)

and does not involve \( T_{u_\mu} \) for \( \mu \neq \lambda \).

(iii) The coefficient of \( T_{u_\lambda} \) is a power of \( u \).

In fact their construction implies that \( y(\lambda) \) does not involve any other element \( T_w \) with \( s(w) \) minimal and \( w \) belongs to a different conjugacy class from the one determined by the partition \( \lambda \), and that the basis \( \{ y(\lambda), \lambda \vdash n \} \) specializes to the class sums under the specialization \( u \to 1 \).

Examples: \( W = S_3, s_1 = (12), s_2 = (23) \).

The conjugacy classes of \( W \) are:
A Dipper-James basis is given by the set of elements

\[ \lambda = 3 \quad \omega_\lambda = (123) = s_1 s_2 \]
\[ \lambda = 21 \quad \omega_\lambda = (12) (3) = s_1 \]
\[ \lambda = 111 \quad \omega_\lambda = (1) (2) (3) = 1 \]

The conjugacy classes of \( W \) are:

(4): \( \{s_1 s_2 s_3, s_2 s_1 s_3, s_3 s_2 s_1, s_1 s_2 s_3, s_1 s_2 s_1 s_3 s_2, s_2 s_3 s_1 s_2 s_1\} \)

(31): \( \{s_1 s_2, s_2 s_1, s_3 s_2, s_1 s_3 s_2 s_1, s_1 s_2 s_1 s_3, s_1 s_2 s_3 s_2, s_2 s_3 s_2 s_1\} \)

(22): \( \{s_1 s_3, s_2 s_1 s_3 s_2, s_1 s_2 s_1 s_3 s_2 s_1\} \)

(211): \( \{s_1, s_2, s_3, s_1 s_2 s_1, s_2 s_3 s_2, s_1 s_2 s_3 s_2 s_1\} \)

(1111): \( \{1\} \).
A Dipper-James basis is given by the set of elements \((y^{(\lambda)}, \lambda \vdash 4)\)

\[ y^{(4)} = u^3(T_{s_1s_2s_3} + T_{s_3s_2s_1} + T_{s_1s_3s_2} + T_{s_2s_1s_3}) + \\
+ (u^3-u^2+u)(T_{s_1s_2s_1s_3s_2} + T_{s_2s_3s_1s_2s_1}) + (2u^2-u-1)T_{s_1s_2s_1s_3s_2s_1} + \\
+ u^2(u-1)(T_{s_1s_2s_1s_3} + T_{s_2s_3s_2s_1} + T_{s_1s_2s_3s_2} + T_{s_1s_3s_2s_1} + T_{s_2s_1s_3s_2}) + \\
+ u(u-1)^2 T_{s_1s_2s_3s_2s_1} \]

\[ y^{(31)} = u^3(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) + \\
+ u^2(T_{s_1s_3s_2s_1} + T_{s_1s_2s_3s_2} + T_{s_2s_3s_2s_1} + T_{s_1s_2s_3s_1}) + u^2(u-1)(T_{s_1s_2s_1} + T_{s_2s_3s_2}) + \\
+ 2u(u-1)T_{s_1s_2s_3s_2s_1} + u(u-1)(T_{s_1s_2s_1s_3s_2} + T_{s_2s_3s_1s_2s_1}) + \\
+ (u-1)^2 T_{s_1s_2s_1s_3s_2s_1} \]
\[ y(22) = u^2 T_{s_1^2 s_2^3} + u T_{s_2^2 s_1^3 s_2^2} + T_{s_1^3 s_2^3 s_3^2 s_1^2} \]
\[ y(211) = u^2 (T_{s_1} + T_{s_2} + T_{s_3}) + u (T_{s_1 s_2 s_3} + T_{s_2 s_3^2 s_2}) + T_{s_1 s_2^2 s_3^3 s_2^1} \]
\[ y(1111) = T_{1^4} \]

We next provide an alternative basis for the centre of \( H(S_4) \) based on the partial ordering on strong conjugacy classes.

The strong conjugacy classes inside \( S_4 \) are: (indexed by the conjugacy classes inside which they occur)

\( \{1\} = C_1 \)
\( \{s_1\} = C_{21^2 A} \), \( \{s_2\} = C_{21^2 B} \), \( \{s_3\} = C_{21^2 C} \)
\( \{s_1 s_2, s_2 s_1\} = C_{31, A} \), \( \{s_2 s_3, s_3 s_2\} = C_{31, B} \)
\( \{s_1 s_3\} = C_{2^2 A}, \{s_1 s_2 s_1\} = C_{21^2, D} \), \( \{s_2 s_3 s_2\} = C_{21^2, E} \)
\( \{s_1 s_3 s_2, s_2 s_1 s_3, s_3 s_2 s_1, s_1 s_2 s_3\} = C_{4_1 A} \)
\( \{s_2 s_3 s_2, s_1 s_2 s_3, s_3 s_2 s_1, s_1 s_3 s_2 s_1, s_1 s_2 s_3 s_1, s_1 s_2 s_3 s_2, s_1 s_2 s_3 s_3\} = C_{31, C} \)
\( \{s_2 s_3, s_2 s_3 s_1\} = C_{2^2, B}, \{s_1 s_2 s_3 s_2 s_1\} = C_{21^2, F} \)
\( \{s_1 s_2 s_1 s_3 s_2, s_2 s_1 s_3 s_2 s_1\} = C_{4_1 B}, \{s_1 s_2 s_1 s_3 s_2 s_1\} = C_{2^2, C} \)

The partial ordering on the strong conjugacy classes is given by:
For the conjugacy class $C_{31}$

For the conjugacy class $C_{31}$

For the conjugacy class $C_{31}$

For the conjugacy class $C_{21}$

For the conjugacy class $C_{21}$

For the conjugacy class $C_{21}$

For the conjugacy class $C_{31}$

For the conjugacy class $C_{41}$

For the conjugacy class $C_{41}$
A basis for the centre of \( H(S_4) \) is given by the set of elements

\[
\begin{align*}
\nu(4) &= T_{s_1s_2s_1s_3s_2} + T_{s_2s_1s_3s_2s_1} + u(T_{s_1s_3s_2} + T_{s_2s_1s_3} + T_{s_1s_2s_3} + T_{s_3s_2s_1}) - \\
&\quad - (u-1)T_{s_2s_1s_3s_2} - 2u(u-1)T_{s_1s_3} - u(u-1)(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) + \\
&\quad + u(u-1)^2(T_{s_1} + T_{s_2} + T_{s_3}) \\
\nu(31) &= T_{s_1s_3s_2s_1} + T_{s_1s_2s_1s_3} + T_{s_1s_2s_3s_2} + T_{s_2s_3s_2s_1} + \\
&\quad + (u^2-u+1)(T_{s_1s_2} + T_{s_2s_1} + T_{s_2s_3} + T_{s_3s_2}) - \\
&\quad - (u-1)(T_{s_1s_3s_2} + T_{s_2s_1s_3} + T_{s_1s_2s_3} + T_{s_3s_2s_1}) - (u-1)^2T_{s_1s_3} - \\
&\quad - (u-1)(T_{s_1s_2s_1} + T_{s_2s_3s_2}) - (u^3-u^2 + u-1)(T_{s_1} + T_{s_2} + T_{s_3}) \\
\nu(22) &= T_{s_1s_2s_1s_3s_2s_3} + uT_{s_2s_1s_3s_2} + u^2T_{s_1s_3} \\
\nu(211) &= T_{s_1s_2s_3s_2s_1} + u(T_{s_1s_2s_1} + T_{s_2s_3s_2}) + u^2(T_{s_1} + T_{s_2} + T_{s_3}) \\
\nu(1111) &= T_1.
\end{align*}
\]

Remark: In contrast with the Dipper-James basis given above, this basis has the property that for each partition \( \lambda \in \mathcal{P} \), the element \( \nu(\lambda) \) has the form:

\[
\sum \nu_w T_w = \text{maximal inside the conjugacy class determined by } \lambda + \text{linear combination of } T_w \text{'s where } T_w \text{ is not of maximal length in any other conjugacy class different from the one determined by } \lambda.
\]
CHAPTER 4

The determination of Lusztig's isomorphism on the centre $Z(H)$ of the generic Hecke algebra of dihedral type

In Chapter 3, we found a basis for the centre $Z(H)$ of the generic Hecke algebra of dihedral type (see Proposition 3.1). In this chapter, we shall determine the images of the basis elements under the Lusztig isomorphism

The canonical basis of $H(D_{2n})$ is given by:

$$\{T_1, T(s_1s_2)^k, T(s_2s_1)^k, T(s_1s_2)^{\lambda}s_1, T(s_2s_1)^{\lambda}s_2, T_w\}$$

where $k = 1, \ldots, m$ if $n = 2m+1$ and $k = 1, \ldots, m-1$ if $n = 2m$, and $\lambda = 0, 1, \ldots, m-1$.

In §2.3 we achieved a decomposition of the graded module $\text{grad}(E)$ of dihedral type over $Q_n(u^\frac{1}{2})$ into a direct sum of left $H$-irreducible submodules, for both cases $n = 2m+1$ and $n = 2m$, by means of the Chebyshev polynomials of the second kind.

To these decompositions we adapt the basis $(\tilde{e}_1, u_j, v_j, \tilde{u}_j, \tilde{v}_j, 1 \leq j \leq m; \tilde{e}_w)$, for the case $n = 2m+1$ and $(\tilde{e}_1, \tilde{e}_2, u_j, v_j, \tilde{u}_j, \tilde{v}_j, 1 \leq j \leq m-1; \tilde{e}_w)$, for the case $n = 2m$. (See also §2.3, for the definition of the basis elements in both cases $n$ odd and $n$ even).

We first determine the matrices which represent each element of the canonical basis of $H(D_{2n})$ on each direct summand of the decompositions mentioned above with respect to the corresponding basis. We concentrate on the 2-dimensional summands whose number is $2m$ if $n = 2m+1$, and $2m-2$ if $n = 2m$. 
The terminology blocks of the first kind and blocks of the second kind is as in the proof of Theorem 2.3.1, and will be adopted here, for the elements of the canonical basis of $H(D_{2n})$.

Lemma 4.1 (i). The blocks of the first kind which represent $T(s_1 s_2)^k$, $k = 1, \ldots, m$, are given by:

$$x_j(u) = \begin{pmatrix}
  u^k [S_k(\rho_j) + S_{k-1}(\rho_j)] & -2u^{2k-1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
  2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & -u^k [S_{k-1}(\rho_j) + S_{k-2}(\rho_j)]
\end{pmatrix}$$

where $\rho_j = 2 \cos \frac{2j\pi}{n}$, $j = 1, \ldots, m$ if $n = 2m+1$, or $j = 1, \ldots, m-1$ if $n = 2m$, and the polynomials $S_k(x)$ are defined in §2.2.

The blocks of the second kind which represent $T(s_1 s_2)^k$, $k = 1, \ldots, m$, are given by:

$$y_j(u) = \begin{pmatrix}
  -u^k [S_{k-1}(\rho_j) + S_{k-2}(\rho_j)] & 2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) \\
  -2u^{2k-1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & u^k [S_k(\rho_j) + S_{k-1}(\rho_j)]
\end{pmatrix}$$

$j = 1, \ldots, m$ if $n = 2m+1$, or $j = 1, \ldots, m-1$ if $n = 2m$.

(ii) The blocks of the first kind which represent $T(s_2 s_1)^k$, $k = 1, \ldots, m$ if $n = 2m+1$, or $k = 1, \ldots, m-1$ if $n = 2m$, are given by the $y_j(u)$, and the blocks of the second kind by the $x_j(u)$. 
The blocks of the first kind which represent the element \( T_{(s_1s_2)^k s_1} \)
k = 0, 1, ..., m if \( n = 2m+1 \) or \( k = 0, 1, ..., m-1 \) if \( n = 2m \) are given by:

\[
Z_j(u) = \begin{bmatrix}
-u^k(S_k(\rho_j) + S_{k-1}(\rho_j)) & 2u^{2k+1/2} \cos \frac{j\pi}{n} S_k(\rho_j) \\
-2u^{2k+1/2} \cos \frac{j\pi}{n} S_{k-1}(\rho_j) & u^{k+1}[S_k(\rho_j) + S_{k-1}(\rho_j)]
\end{bmatrix}
\]

\( j = 1, ..., m \) if \( n = 2m+1 \) or \( j = 1, ..., m-1 \) if \( n = 2m \).

The blocks of the second kind which represent \( T_{(s_1s_2)^k s_1} \) are given by

\[
V_j(u) = \begin{bmatrix}
u^{k+1}[S_k(\rho_j) + S_{k-1}(\rho_j)] & -2u^{2k+1/2} \cos \frac{j\pi}{n} S_k(\rho_j) \\
2u^{2k+1/2} \cos \frac{j\pi}{n} S_k(\rho_j) & -u^{k+1}[S_k(\rho_j) + S_{k-1}(\rho_j)]
\end{bmatrix}
\]

\( j = 1, ..., m \) if \( n = 2m+1 \), or \( j = 1, ..., m-1 \) if \( n = 2m \).

The blocks of the first kind which represent the element \( T_{(s_2s_1)^k s_2} \)
k = 0, 1, ..., m-1 are the \( Z_j(u) \) and the blocks of the second kind are the \( V_j(u) \).

**Proof:** We concentrate on the blocks of the first kind. For \( k = 1 \), the result is true since \( T_{s_1} = \begin{bmatrix} -1 & 2u \cos \frac{j\pi}{n} \\
0 & u \end{bmatrix} \) and \( T_{s_2} = \begin{bmatrix} u & 0 \\
2u \cos \frac{j\pi}{n} & -1 \end{bmatrix} \).
Therefore \( T(s_1s_2) = \begin{bmatrix} u[4 \cos^2 \frac{j\pi}{n} - 1] & -2u^{3/2} \cos \frac{j\pi}{n} \\ 2u^{3/2} \cos \frac{j\pi}{n} & -u \end{bmatrix} \)

where \( 4 \cos^2 \frac{j\pi}{n} - 1 = (2 + \rho_j) - 1 = 1 + \rho_j = S_0(\rho_j) + S_1(\rho_j) \).

Assume by induction that the blocks of the first kind, which represent \( T(s_1s_2)^\lambda \) are of the form:

\[
\begin{bmatrix}
u^{\lambda}[S_{\lambda}(\rho_j) + S_{\lambda-1}(\rho_j)] & -2u^{2\lambda-1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\
2u^{2\lambda+1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -u^{\lambda}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)]
\end{bmatrix}
\]

Then \( T(s_1s_2)^{\lambda+1} = T(s_1s_2)^\lambda T(s_1s_2) \) is represented by

\[
\begin{bmatrix}
u^{\lambda}[S_{\lambda}(\rho_j) + S_{\lambda-1}(\rho_j)] & -2u^{2\lambda-1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) \\
2u^{2\lambda+1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) & -u^{\lambda}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)]
\end{bmatrix}
\begin{bmatrix}
u(1+\rho_j) -2u^{3/2} \cos \frac{j\pi}{n} \\
2u^{3/2} \cos \frac{j\pi}{n} & -u
\end{bmatrix}
\]

\[
= \begin{bmatrix} a(u) & a(u) \\ a(u) & a(u) \end{bmatrix}
\]

where
Thus the blocks of first kind which represent the element $T(s_1s_2)^{\lambda+1}$ have also the required form. To obtain the matrices $Y_j(u)$ we conjugate the matrices $X_j(u)$ by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. \[ a_{11}(u) = u^{\lambda+1}(1 + \rho_j)[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - 4u^{\lambda+1}\cos^2 \frac{j\pi}{n} S_{\lambda-1}(\rho_j) = \]

\[ = u^{\lambda+1}(1 + \rho_j)[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - u^{\lambda+1}(2 + \rho_j)S_{\lambda-1}(\rho_j) = \]

\[ = u^{\lambda+1}[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - u^{\lambda+1}(2 + \rho_j)S_{\lambda-1}(\rho_j) = \]

\[ = u^{\lambda+1}[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - 2\rho_jS_{\lambda-1}(\rho_j) - \rho_jS_{\lambda-1}(\rho_j) = \]

\[ = u^{\lambda+1}[S_\lambda(\rho_j) + S_{\lambda-1}(\rho_j)] - \rho_jS_{\lambda-1}(\rho_j) = \]

\[ = u^{\lambda+1}[S_{\lambda+1}(\rho_j) + S_{\lambda}(\rho_j)] = \]

\[ a_{12}(u) = -2u^{\lambda+1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) = \]

\[ = -2u^{\lambda+1/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) = \]

\[ a_{21}(u) = 2u^{\lambda+3/2} \cos \frac{j\pi}{n} (1 + \rho_j)S_{\lambda-1}(\rho_j) - 2u^{\lambda+3/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) = \]

\[ = 2u^{\lambda+3/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) - 2u^{\lambda+3/2} \cos \frac{j\pi}{n} S_{\lambda-2}(\rho_j) = \]

\[ = 2u^{\lambda+3/2} \cos \frac{j\pi}{n} S_{\lambda-1}(\rho_j) - 2u^{\lambda+3/2} \cos \frac{j\pi}{n} S_{\lambda}(\rho_j) = \]

\[ a_{22}(u) = -4 \cos^2 \frac{j\pi}{n} u^{\lambda+1} S_{\lambda-1}(\rho_j) + u^{\lambda+1}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)] = \]

\[ = -4 \cos^2 \frac{j\pi}{n} u^{\lambda+1} S_{\lambda-1}(\rho_j) + u^{\lambda+1}[S_{\lambda-1}(\rho_j) + S_{\lambda-2}(\rho_j)] = \]

\[ = -u^{\lambda+1}[\rho_jS_{\lambda-1}(\rho_j) - S_{\lambda-2}(\rho_j)S_{\lambda-1}(\rho_j)] = -u^{\lambda+1}[S_{\lambda}(\rho_j) + S_{\lambda-1}(\rho_j)]. \]
Part (ii) can be proved in a similar way. Finally matrix multiplication by $T_{s_1}$ and $T_{s_2}$ gives (iii) and (iv) respectively, so our lemma is now proved.

Remark (4.1). We note that the blocks of the first kind, which represent $T(s_1s_2)^k$ and $T(s_2s_1)^k$ are mutually obtained from one another by conjugation by the matrix $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. The situation is similar for the elements $T(s_1s_2)^{ks_1}$ and $T(s_2s_1)^{ks_2}$.

We shall now distinguish between two cases.

**First case:** $n = 2m + 1$.

In this case, a basis of the centre $\mathbb{Z}(H)$ of the generic Hecke algebra is given by the following set of elements:

$$v_0 = T_{s_1} \cdot v_k = T(s_1s_2)^k + T(s_2s_1)^k - (u-1) \sum_{\lambda=1}^{k} u^{\lambda-1}[T(s_1s_2)^{-\lambda}s_1 + T(s_2s_1)^{-\lambda}s_2],$$

$k = 1, \ldots, m$,

$$v_{m+1} = T_{s_0} + \sum_{\lambda=1}^{m} u^{\lambda}[T(s_1s_2)^{-\lambda}s_1 + T(s_2s_1)^{-\lambda}s_2].$$

We recall the decomposition of the graded module into a direct sum of irreducible left $H$-submodules. This decomposition is,

$$\text{grad}(E) = \bigoplus_{j=1}^{m} M_j \oplus \bigoplus_{j=1}^{m} \hat{M}_j \oplus \bigoplus_{j=1}^{m} N_j,$$

where $M_j, \hat{M}_j, N_j$ afford the 1-dimensional representations $\sigma_0 : T_{s_i} = u, i = 1, 2$, and $\sigma_{s_j} : T_{s_i} = -1, i = 1, 2$ respectively, and $M_j$ is a 2-dimensional $H$-submodule with basis $\{u_j, v_j\}$ which is isomorphic to the 2-dimensional $H$-submodule $\hat{M}_j$ which has basis $\{\hat{u}_j, \hat{v}_j\}$, for every $j = 1, \ldots, m$. Our first step is to establish the action of the basis elements of the centre on the irreducible submodules appearing in the decomposition of $\text{grad}(E)$, with respect to the above basis.
It is well known that each element of the centre is represented on an irreducible constituent of dimension say \( d \), by a scalar multiple of the identity matrix \( I_d \).

The elements \( v_k \) is represented on \( M_0 \) by:
\[
2u^{2k} - (u-1) \sum_{\lambda=1}^{k} u^\lambda \left( \sum_{\lambda=1}^{2(k-\lambda)+1} u^\lambda \right) = 2u^k, \quad k = 1, \ldots, m.
\]
and it is also represented on \( M_s \) by:
\[
2 - (u-1) \sum_{\lambda=1}^{k} u^\lambda (-2) = 2u^k, \quad k = 1, \ldots, m.
\]
The element \( v_{m+1} \) is represented on \( M_0 \) by:
\[
u_{m+1} = \sum_{\lambda=1}^{m} u^\lambda \sum_{\lambda=1}^{2(m-\lambda)+1} u^\lambda = 2u^m + 2 \sum_{\lambda=1}^{m} u^{2m+1} - 1 - 2 \sum_{\lambda=1}^{m} u^\lambda = -2u^m + 2u^{m-1} - \ldots - 2u - 1.
\]

Let \( \mu_{j}^{(k)} \times I_2 \) be the matrix which represents \( v_k \) on the 2-dimensional constituent \( M_j \), \( j = 1, \ldots, m \) (\( \mu_{j}^{(k)} \times I_2 \) the corresponding matrix on \( M_j \)).

We shall determine \( \mu_{j} \). We write \( v_k \) as:
\[
v_k = T_{s_1 s_2}^k + T_{s_2 s_1}^k - (u-1) \sum_{\lambda=0}^{k-1} u^{k-(\lambda+1)} [T_{s_1 s_2}^{s_1 s_2} + T_{s_2 s_1}^{s_2 s_1}] .
\]

We recall the matrices which represent the elements \( T_w \), \( w \in D_{2n} \) on the blocks of the first kind (see Lemma 4.1) and we concentrate on the diagonal entries of these matrices.

It turns out that the diagonal entries of the matrix which represents \( T_{s_1 s_2}^k + T_{s_2 s_1}^k \) have the form \( u^k [S_k(p_j) - S_{k-2}(p_j)] \) and the diagonal
entries of the matrix which represents $-(u-1)\sum_{k=0}^{k-1} u^{k-1} \binom{k}{1} T_s s_1 s_2 + T_s s_1 s_2$ have the form $-u^{k-1}(u-1)^2 \sum_{k=0}^{k-1} [S_k(p_j) + S_{k-1}(p_j)]$. Therefore

$$u_j^{(k)} = u^{k}(S_k(p_j) - S_{k-1}(p_j)) - u^{k-1}(u-1)^2 [S_{k-1}(p_j) + S_{k-1}(p_j) + \ldots + 2S_1(p_j) + 2S_0(p_j)].$$

The matrix $u_j^{(k)} I_2$ is obtained by conjugating $u_j^{(k)} I_2$ by $(0 1)^T$ and so $u_j^{(k)} = u_j^{(k)}$.

Similarly let $u_j^{(m+1)} I_2$ be the matrix which represents $v_{m+1}$ on $M_j$, $1 \leq j \leq m$ (or $u_j^{(m+1)} I_2$ the corresponding matrix on $M_j$). With a similar argument, and taking into account that $S_m(p_j) + S_{m-1}(p_j) = 0$, we find that

$$u_j^{(m+1)} = u_j^{(m)}(S_m(p_j) + 2S_{m-2}(p_j) + \ldots + 2S_1(p_j) + 2S_0(p_j)).$$

We shall now simplify the expressions $u_j^{(k)}$, $1 \leq j \leq m$.

In order to do so, we introduce certain families of polynomials. The Chebyshev polynomials of the first kind $T_k(x)$ are defined by:

$$T_0(x) = 1, T_1(x) = x, T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \forall k \geq 1$$

and they have the property that for every $\theta$, $T_k(\cos \theta) = \cos k\theta$. Clearly $T_k(1) = 1$ and so $T_k(x) - 1$ is divisible by $x-1$. The Chebyshev polynomials of the second kind have been already defined by $S_0(x) = 1, S_1(x) = x, S_{k+1}(x) = xS_k(x) - S_{k-1}(x), \forall k \geq 1$, and they have the property that $S_k(2 \cos \theta) = \frac{\sin(k+1}\theta}{\sin \theta}$ [see [1], p. 776-8].

Define $V_k(x)$ by $V_0(x) = 2$, $V_1(x) = x$, $V_k(x) = S_k(x) - S_{k-2}(x), \forall k \geq 2$.

We can easily show by induction that $V_k(x) = 2T_{k}(x/2)$. Define

$$R_k(x) = \frac{T_{k+1}(x) - 1}{x-1}$$

for every $k \geq 0$.

Let $\theta_j = \frac{2\sin \theta}{n}$, and let $R_k^{(j)} = S_{k-1}(2 \cos \theta_j) + 2S_{k-2}(2 \cos \theta_j) + \ldots + 2S_0(2 \cos \theta_j), 1 \leq j \leq m, 1 \leq k \leq m$. Then
Lemma 4.2: (1) \( V_k(2 \cos \theta_j) = 2 \cos k\theta_j \quad 1 \leq k \leq m, \; 1 \leq j \leq n \)

(11) \( \Delta_k^{(j)} = \sum_{\mu=1}^{j-1} 2\mu \cos(k-\mu)\theta_j + k, \; 1 \leq j \leq m, \; 1 \leq k \leq m. \)

Proof: (1) This follows from the fact that \( V_k(x) = 2T_k(x/2) \) and \( T_k(\cos \theta) = \cos k\theta_j. \)

(11) From the definition of \( R_k(x) \) and using the fact that \( T_{k+1}(x) = 2xT_k(x) - T_k(x) \), we can easily show that \( R_{k+1}(x) = 2xR_k(x) - R_k(x) + 2 \), for every \( k \).

By induction, we can also prove that (using also the fact that
\[ S_{k+1}(x) = x S_k(x) - S_{k-1}(x) \]
\[ R_{k-1}(x) = S_{k-1}(2x) + 2S_{k-2}(2x) + \ldots + 2S_1(2x) + 2S_0(2x), \quad 1 \leq k \leq m. \]

Therefore,
\[ \Delta_k^{(j)} = R_{k-1}(\cos \theta_j) = \frac{1 - \cos(k\theta_j)}{1 - \cos \theta_j} \]

Moreover \((1 - \cos \theta_j)(\sum_{\mu=1}^{j-1} 2\mu \cos(k-\mu)\theta_j + k) = \)
\[ \sum_{\mu=1}^{j-1} 2\mu \cos(k-\mu)\theta_j + k - k \cos \theta_j - \sum_{\mu=1}^{j-1} 2\mu \cos(k-\mu)\theta_j \cos \theta_j = \]
\[ \sum_{\mu=1}^{j-1} 2\mu \cos(k-\mu)\theta_j + k - k \cos \theta_j - \sum_{\mu=1}^{j-1} \mu \cos(k-\mu+1)\theta_j - \sum_{\mu=1}^{j-1} \mu \cos(k-\mu-1)\theta_j = \]
\[ = 1 - \cos(k\theta_j). \quad \text{Our lemma is now proved.} \]

We next show the central character table of the generic Hecke algebra of dihedral type for the case \( n = 2m + 1. \)
In the table above, $M_0, M_s, M_j, 1 \leq j \leq m$ is a full set of left irreducible $H$-modules, $\{v_0, v_k, 1 \leq k \leq m, v_{m+1}\}$ is a basis of the centre of the generic Hecke algebra,

$$\Delta^j_k = 2 \cos \frac{2kj\pi}{n}, \quad 1 \leq k \leq m, \quad 1 \leq j \leq m,$$

$$\Delta^j_k = R_{k-1}(\cos \theta_j), \quad 1 \leq k \leq m, \quad 1 \leq j \leq m.$$

The entries of this table represent the scalars according to which the basis elements of the centre of $H$ act on the irreducible constituents.
By specializing $u = 1$, we obtain the central character table of the dihedral group $D_{2n}$, $n = 2m + 1$.

<table>
<thead>
<tr>
<th></th>
<th>$[v_0]$</th>
<th>$[v_1]$</th>
<th>$[v_2]$</th>
<th>$\ldots$</th>
<th>$[v_m]$</th>
<th>$[v_{m+1}]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_0$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$\ldots$</td>
<td>2</td>
<td>$n$</td>
</tr>
<tr>
<td>$M_s$</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td></td>
<td>$-n$</td>
<td></td>
</tr>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>$2\cos\frac{2\pi}{n}$</td>
<td>$2\cos\frac{4\pi}{n}$</td>
<td>$2\cos\frac{2m\pi}{n}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td>1</td>
<td>$2\cos\frac{4\pi}{n}$</td>
<td>$2\cos\frac{6\pi}{n}$</td>
<td>$2\cos\frac{4m\pi}{n}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
<td>1</td>
<td>$2\cos\frac{6\pi}{n}$</td>
<td>$2\cos\frac{12\pi}{n}$</td>
<td>$2\cos\frac{6m\pi}{n}$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$M_m$</td>
<td>1</td>
<td>$2\cos\frac{2m\pi}{n}$</td>
<td>$2\cos\frac{4m\pi}{n}$</td>
<td>$2\cos\frac{2m^2\pi}{n}$</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Notation: $[v_0], [v_k], k=1,\ldots,m, [v_{m+1}]$ denote the class sums, and the entries of the table represent the scalars, according to which, the class sums act on the irreducible constituents. We now recall the following auxiliary result (see: [7], page 213).

Let $G$ be any finite group with $s$ conjugacy classes $K_j$, $1 \leq j \leq s$, and let $Z = (\chi_j^{(i)})$ be the character table of the group $G$. Let $C = (c_j^i)$ be the central character matrix, where $c_j^i = \frac{h_j\chi_j^{(i)}}{d_i}$, $d_i$ being the degree of $\chi_j^{(i)}$, $h_j$ the cardinality of the conjugacy class $K_j$, $i,j \in \{1,\ldots,s\}$. Then $Z$ is an invertible matrix with inverse matrix $Z$ whose $(i,j)$ entry is $\frac{h_i}{|G|} \bar{\chi}_j^{(i)}$, $\bar{\chi}_j^{(i)}$ being the complex conjugate of $\chi_j^{(i)}$. 
In our case all characters are real valued. We have two characters of degree 1 namely $\chi^{(1)}$, $\chi^{(2)}$ and $m$ characters of degree 2. With the aid of the orthogonality relations we can verify that the matrix $C$ determined by the central character table of the group $W$ is invertible with inverse matrix $C^{-1}$ where

$$
\begin{pmatrix}
1 & 1 & \frac{2}{n} & \frac{2}{n} & \frac{2}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n} \cos \frac{2\pi}{n} & \frac{2}{n} \cos \frac{4\pi}{n} & \frac{2}{n} \cos \frac{2m\pi}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n} \cos \frac{4\pi}{n} & \frac{2}{n} \cos \frac{6\pi}{n} & \frac{2}{n} \cos \frac{2m\pi}{n} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n} \cos \frac{6\pi}{n} & \frac{2}{n} \cos \frac{12\pi}{n} & \frac{2}{n} \cos \frac{6m\pi}{n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{1}{2n} & \frac{1}{2n} & \frac{2}{n} \cos \frac{2m\pi}{n} & \frac{2}{n} \cos \frac{4m\pi}{n} & \frac{2}{n} \cos \frac{2m^2\pi}{n} \\
\frac{1}{2n} & -\frac{1}{2n} & 0 & 0 & 0
\end{pmatrix}
$$

Every entry of the $i^{th}$ column $1 \leq i \leq m+2$ of the matrix $C$ is given by the corresponding entry in the $i^{th}$ row of the character table of the group $D_{2n}$ multiplied by the $d_i$ ($d_i = 2, 3 \leq i \leq m+2$) and divided by the order of the group.

Now if $z$ is a typical element of the basis $\{v_0, v_k, 1 \leq k \leq m, v_{m+1}\}$ of the centre of $H$, then $z$ acts on every irreducible constituent of the graded module according to the information given by the central character table of $H$. If $\phi$ is Lusztig's isomorphism, then $\phi(z)$ is a certain linear combination of the class sums, and each class sum acts on the irreducible constituents according to the information provided by the central character table of the group $D_{2n}$, $n = 2m + 1$. 
Furthermore $z$ and $\theta(z)$ act in the same way on the graded module and therefore the coefficients appearing in the expression of $\theta(z)$ as linear combination of the class sums are polynomials in the indeterminate $u$ with rational coefficients.

We shall determine these coefficients explicitly.

In fact if $A = (a_{ir})$, $i, r \in \{1, \ldots, m+2\}$ is the matrix describing Lusztig's isomorphism on the centre of $H$ with respect to the basis $(v_0, v_k, 1 \leq k \leq m, v_{m+1})$, then $A = \tilde{C}A$, where $\tilde{C}$ is the inverse of the central character matrix $C$ of the group $D_{2n}$ and $A$ is the matrix determined by the central character table of $H$.

We now compute the entries $a_{ir}$, $i, r \in \{1, 2, \ldots, m+2\}$. We shall make use of the trigonometric identities:

$$2 \cos a \cos b = \cos(a + b) + \cos(a - b)$$

$$\frac{1}{2} + \cos x + \cos 2x + \ldots + \cos px = \frac{1}{2} \frac{\sin(p+1)x}{\sin \frac{x}{2}}$$

Let $i = 2, r = 1$. Then $a_{11} = \frac{1}{2n} + \frac{1}{2n} + m \frac{2}{n} = \frac{2m+1}{n} = 1$. If $i \in \{2, 3, \ldots, m+1\}$ and $r = 1$, then $a_{ir} = \frac{1}{2n} + \frac{1}{2n} + \frac{2}{n} \sum_{j=1}^{m} \cos (i-1) \frac{2j}{n} = \frac{1}{n} + \frac{2}{n} (-\frac{1}{2}) = 0$.

If $i = m+2$ and $r < m+2$ then $a_{ir} = 0$.

If $i = m+2$ and $r = m+2$, then $a_{mr} = \frac{1}{2n} (u+1)(u^{n-1} + \ldots + u+1)$.

If $i = 1$ $r \in \{2, 3, \ldots, m+1\}$, then

$$a_{ir} = \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{2n} + \frac{2u^{r-1}}{2n} \sum_{j=1}^{m} \Delta_{r-1}^{j} - \frac{2}{n} u^{r-2}(u-1)^{2} \sum_{j=1}^{m} \Delta_{r-1}^{j} - \frac{2}{n} u^{r-2}(u-1)^{2} \sum_{j=1}^{m} \Delta_{r-1}^{j} - \frac{2}{n} u^{r-2}(u-1)^{2} \sum_{j=1}^{m} \Delta_{r-1}^{j}$$

$$\Delta_{r-1}^{j} = 2 \cos \frac{2(r-1)j}{n}, \quad \text{thus} \quad \sum_{j=1}^{m} \Delta_{r-1}^{j} = 2(-\frac{1}{2}) = -1$$

$$\Delta_{r-1}^{j} = 2 \cos (r-2) \frac{2j}{n} + 4 \cos (r-3) \frac{2j}{n} + \ldots + (2r-4) \cos \frac{2j}{n} + (r-1), \quad \text{hence}$$
\[
\sum_{j=1}^{m} \Delta_{r-1}^j = 2 \sum_{j=1}^{m} \cos(r-2) \frac{2j\pi}{n} + 4 \sum_{j=1}^{m} \cos(r-3) \frac{2j\pi}{n} + \ldots + (2r-4) \sum_{j=1}^{m} \cos \frac{2j\pi}{n} + m(r-1)
\]
\[
= (2 + 4 + \ldots + 2r-4) (-\frac{1}{2}) + m(r-1) = m(r-1) - \frac{(r-1)(r-2)}{2} - \frac{(r-1)(n-r+1)}{2}.
\]
Thus \( a_{i+r} = \frac{u-r-2}{u-1} \frac{(r-1)(n-r+1)}{n} \), \( i = 1, r \in (2, 3, \ldots, m+1) \).

Let \( i = 1, r = m+2 \). Then

\[
a_{1m+2} = \frac{1}{2n} [u^{m+1} + 2u^m + \ldots + 2u^2 - 2u - 2] + \frac{2}{n} u^m(u-1) \sum_{j=1}^{m} \Delta_j^m
\]

Now \( \sum_{j=1}^{m} \Delta_j^m = (2 + 4 + \ldots + 2m-2)(-\frac{1}{2}) + m^2 - \frac{m(m-1)}{2} = \frac{m(m+1)}{2} \).

So \( a_{1m+2} = \frac{1}{2n} [u^{m+1} + 2u^m + \ldots + 2u^2 - 2u - 2] + 2m(m+1)u^m(u-1)] =
\]
\[
= \frac{1}{2n} (u-1)[ \sum_{\lambda=0}^{m-1} (2\lambda+1)u^{m-\lambda} + (n+2m(m+1))u^m + \sum_{\lambda=1}^{m} (2m - 2\lambda + 1)u^{m-\lambda}].
\]

If \( i > 1, r \in (2, 3, \ldots, m+1) \), \( i = r \), then

\[
a_{1r} = \frac{2u^{r-1}}{zn} + \frac{2u^{r-1}}{zn} + \frac{2u^{r-1}}{zn} \sum_{j=1}^{m} 2 \cos \frac{2(r-1)j\pi}{n} \cos \frac{(r-1)2j\pi}{n} - \frac{2}{n} u^{r-2}u(1)^2 \sum_{j=1}^{m} \Delta_j^{r-1}.
\]

Now \( \sum_{j=1}^{m} 2 \cos \frac{2(r-1)j\pi}{n} \cos \frac{(r-1)2j\pi}{n} = \sum_{j=1}^{m} \cos \frac{4(r-1)j\pi}{n} + m - m - \frac{1}{2} = \frac{2m-1}{2} \).

Also \( \sum_{j=1}^{m} \cos \frac{2(r-1)j\pi}{n} \Delta_j^{r-1} =
\]
\[
= \sum_{j=1}^{m} \cos \frac{2(r-1)j\pi}{n} [2 \cos(r-2) \frac{2j\pi}{n} + 4 \cos(r-3) \frac{2j\pi}{n} + \ldots + (2r-4) \cos \frac{2j\pi}{n} + (r-1)]
\]
\[
= \sum_{j=1}^{m} \frac{r-2}{n} \sum_{\lambda=1}^{m} \lambda \cos \frac{2j\pi}{n} + \cos \frac{2j\pi}{n} + \sum_{j=1}^{m} \cos \frac{(r-1)2j\pi}{n}]
\]
\[
= - (1 + 2 + 3 + \ldots + r-2) - \frac{(r-1)}{2} - \frac{(r-1)(r-2)}{2} - \frac{(r-1)}{2}.
\]
\[
= - \frac{(r-1)^2}{2}.
\]
Hence \( a_{m+2} = \frac{2u^{m+1} + 2u^m + \ldots + 2u^1 - 2u^m - 2u^m - \ldots - 2u^1 + 2u^m(u-1) \cos(1-1) \sum_{j=1}^{m} \frac{2j\pi}{n} }{m} \).

Now \( \cos(1-1) \sum_{j=1}^{m} \frac{2j\pi}{n} = \cos(1-1) \sum_{j=1}^{m} \frac{2j\pi}{n} \left[ \sum_{k=1}^{m-1} (2k)\cos(m-k) \frac{2j\pi}{n} + m \right] \).

We can have \( 1 - 1 = m-k \) if \( k = m-1+1 \). Thus,

\[
\cos(1-1) \sum_{j=1}^{m} \frac{2j\pi}{n} = \cos(1-1) \sum_{j=1}^{m} \frac{2j\pi}{n} \left[ \sum_{k=1}^{m-1} (2k)\cos(m-k) \frac{2j\pi}{n} + m \right] \]

\[
= \sum_{k=1}^{m-1} k \left[ \cos(1 + m - k - 1) \frac{2j\pi}{n} + \cos(m - k + 1) \frac{2j\pi}{n} \right] + \sum_{k=m-1+2}^{m-1} k \left[ \cos(i + m - k - 1) \frac{2j\pi}{n} + \cos(m - k + 1) \frac{2j\pi}{n} \right] + \sum_{k=m-1+2}^{m} \cos(1-1) \frac{2j\pi}{n} \]

Thus, \( \sum_{j=1}^{m} \frac{2j\pi}{n} \).

\[
= \sum_{k=1}^{m-1} k + (m-1+1)(m - \frac{1}{2}) + \sum_{k=m-1+2}^{m-1} k + m(- \frac{1}{2}) \]

\[
= \sum_{k=1}^{m-1} k + (m-1+1)(m - \frac{1}{2}) + \sum_{k=m-1+2}^{m-1} k + m(- \frac{1}{2}) \]

\[
= \sum_{k=1}^{m-1} k + (m-1+1)(m - \frac{1}{2}) + \sum_{k=m-1+2}^{m-1} k + m(- \frac{1}{2}) \]

\[
= \sum_{k=1}^{m-1} k + (m-1+1)(m - \frac{1}{2}) + \sum_{k=m-1+2}^{m-1} k + m(- \frac{1}{2}) \]

\[
= \frac{m(m-1)}{2} \cdot (m-1+1)(m - \frac{1}{2}) - \frac{m}{2} = \frac{1}{2} \left[ n(m-1+1)-n^2 \right]. \]
Hence $a_{1m+2} =$

$$= \frac{1}{2n} (u-1)[u^{2m+3}u^{2m-1}+5u^{2m-2}+\ldots+(2m-1)\sum_{\lambda=0}^{m-1}(n+2n(m-1)+2m^2)u^m+(2m-1)^m-1+3u+1]$$

$$= \frac{1}{2n} (u-1)\sum_{\lambda=0}^{m-1}(2\lambda+1)u^{2m-\lambda}+(n+2n(m-1)+2m^2)u^m+\sum_{\lambda=1}^{m} (2m-2\lambda+1)u^{m-\lambda}$$

with $i > 1$, $1 \neq m + 2$.

Finally let $i > 1$, $r \in \{2,3,\ldots,m+1\}$ $1 \neq r$. Then,

$$a_{ir} = \frac{2u^r-1}{2n} + \frac{2u^r-1}{2n} + \frac{2u^r-1}{n} \sum_{j=1}^{m} 2 \cos\left(\frac{(i-1)2\pi}{n}\right) \cos\left(r-\frac{2\pi}{n}\right) -$$

$$- \frac{2}{n} u^{r-2}(u-1)^2 \sum_{j=1}^{m} \cos\left(1-\frac{2j\pi}{n}\right) \sin\left(\frac{r\pi}{n}\right) \Delta_{r-1}.$$

Now

$$= \frac{1}{n} \sum_{j=1}^{m} 2 \cos\left(1-\frac{2j\pi}{n}\right) \cos\left(r-\frac{2\pi}{n}\right) = -\frac{1}{2} - \frac{1}{2} = -1.$$
Now $\cos(1-l) \frac{2j\pi}{n} \Delta^{j}_{r-1} =$

\[
= \cos(1-l) \frac{2j\pi}{n} \left[ \sum_{\lambda=1}^{r-1-1} (2\lambda) \cos(r-\lambda-1) \frac{2j\pi}{n} + 2(r-1) \cos(1-l) \frac{2j\pi}{n} \right] \]

So \( \sum_{j=1}^{m} \cos(1-l) \frac{2j\pi}{n} \Delta^{j}_{r-1} = \)

\[
\sum_{j=1}^{m} \left[ \sum_{\lambda=1}^{r-1-1} \lambda \left[ \cos(r+1-\lambda-2) \frac{2j\pi}{n} + \cos(r-1-\lambda) \frac{2j\pi}{n} \right] \right] + \\
+ (r-1) \sum_{j=1}^{m} (1 + \cos \frac{4(1-l)j\pi}{n}) + \\
+ \sum_{j=1}^{m} \left[ \sum_{\lambda=r-1+1}^{r-2} \lambda \left[ \cos(r+1-\lambda-2) \frac{2j\pi}{n} + \cos(r-1-\lambda) \frac{2j\pi}{n} \right] \right] + \\
+ (r-1) \sum_{j=1}^{m} \cos \frac{(1-l)2j\pi}{n} = \\
= \sum_{\lambda=1}^{r-1-1} \lambda + (r-1) \left( m - \frac{1}{2} \right) - \sum_{\lambda=r-1+1}^{r-2} \lambda - \frac{(r-1)}{2} - \\
- \sum_{\lambda=1}^{r-2} \lambda + (r-1) + \frac{(r-1)}{2} (2m-1) - \frac{(r-1)}{2} = \\
- \frac{(r-2)(r-1)}{2} + \frac{(r-1)}{2} - n - \frac{(r-1)}{2} = \frac{(r-1)}{2} . n - \frac{(r-1)^2}{2} .
\]

Hence, when \( i \neq r, \ r > 1, \) then

\[ a_{ir} = - \frac{2}{n} u^{r-2} (u-1)^2 \left[ - \frac{(r-1)}{2} . n - \frac{(r-1)^2}{2} \right] = \\
= u^{r-2} (u-1)^2 \left[ \frac{(r-1)^2}{n} - (r-1) \right]. \]
For the convenience of the reader we summarize our calculations in the following matrix.

We put

\[ K(1,m+2) = \sum_{\lambda=0}^{m-1} (2\lambda+1)u^{2m-\lambda} + (n+2m(m+1))u^m + \sum_{\lambda=1}^{m} (2m-2\lambda+1)u^{m-\lambda}, \]

and

\[ K(1,m+2) = \sum_{\lambda=0}^{m-1} (2\lambda+1)u^{2m-\lambda} + (n+2n(m-1+1)-2m^2)u^m + \sum_{\lambda=1}^{m} (2m-2\lambda+1)u^{m-\lambda} \]

for every \( i = 2,3,\ldots,m+1. \)
When we specialize $u = 1$, the matrix above becomes the identity matrix as it should be.

$$
\begin{array}{cccccc}
1 & -\frac{1}{n}(u-1)^2(n-1) & -\frac{2}{n}u(u-1)^2(n-2) & -\frac{3}{n}u^2(u-1)^2(n-3) & \ldots & -\frac{m}{n}u^{m-1}(u-1)^2(n-m) & \frac{1}{2n}(u-1)K(1,m+2) \\
0 & u + \frac{1}{n}(u-1)^2 & u^{r-2}(u-1)^2\frac{(r-1)^2}{n} - (r-2) & \frac{1}{2n}(u-1)K(2,m+2) \\
0 & \frac{1}{n}(u-1)^2 & u^2 + \frac{4}{n}u(u-1)^2 & u^{r-2}(u-1)^2\frac{(r-1)^2}{n} - (r-3) & \frac{1}{2n}(u-1)K(3,m+2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
ith row & \frac{1}{n}u^{r-2}(u-1)^2(r-1)^2 & \frac{1}{n}u^{r-2}(u-1)^2(r-1)^2 & \frac{1}{n}u^{r-2}(u-1)^2(r-1)^2 & \frac{1}{2n}(u-1)K(i,m+2) \\
2 \leq i \leq m+1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(m+2) row & 0 & 0 & 0 & \frac{1}{2n}(u+1)u^{-1} + \ldots + u+1 & \frac{1}{2n}(u+1)u^{-1} + \ldots + u+1
\end{array}
$$
We give some examples to illustrate the situation

(i) \( W = D_6, n = 3 = 2 \cdot 1 + 1, m = 1 \).

A basis of the centre of the generic Hecke algebra \( H \) is given by:

\[
\begin{align*}
\nu_0 &= T_1, \\
\nu_1 &= T_s s_2 + T s_2 s_1 - (u-1)(T_s + T s_1), \\
\nu_2 &= T s_1 s_2 s_1 + u(T_s + T s_2).
\end{align*}
\]

The matrix which describes the Lusztig isomorphism on the centre of \( H(D_6) \) with respect to this basis is given by:

\[
\begin{bmatrix}
1 & -\frac{2}{3} (u-1)^2 & \frac{1}{6} (u-1)(u^2 + 7 u+1) \\
0 & u + \frac{1}{3} (u-1)^2 & \frac{1}{6} (u-1)(u^2 + u + 1) \\
0 & 0 & \frac{1}{6} (u+1)(u^2+u+1)
\end{bmatrix}
\]

The information given by this matrix is that:

\[
\begin{align*}
\phi(\nu_0) &= 1, \\
\phi(\nu_1) &= -\frac{2}{3} (u-1)^2 \cdot (u + \frac{1}{3} (u-1)^2) \cdot (s_1 s_2 + s_2 s_1) \\
\phi(\nu_2) &= \frac{1}{6} (u-1)(u^2+7u+1) \cdot 1 + \frac{1}{6} (u-1)(u^2+u+1)(s_1 s_2 + s_2 s_1) + \\
&\quad + \frac{1}{6} (u+1)(u^2+u+1)(s_1 + s_2 + s_1 s_2 s_1)
\end{align*}
\]

(ii) \( W = D_{10}, n = 5 = 2.2 + 1, m = 2 \).

A basis of the centre of \( H \) is given by:
The matrix which describes the Lusztig isomorphism $\phi$ on the centre of $H(D_{10})$ with respect to this basis is given by:

$$
\begin{bmatrix}
1 & -\frac{4}{5}(u-1)^2 & -\frac{6}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3+17u^2+3u+1) \\
0 & u + \frac{1}{5}(u-1)^2 & -\frac{1}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3+7u^2+3u+1) \\
0 & \frac{1}{5}(u-1)^2 & u^2 + \frac{4}{5}u(u-1)^2 & \frac{1}{10}(u-1)(u^4+3u^3-3u^2+3u+1) \\
0 & 0 & 0 & \frac{1}{10}(u+1)(u^4+u^3+u^2+u+1)
\end{bmatrix}
$$

Second case $n = 2m$:

When $m$ is even, a basis of the centre of $H(D_{2n})$ is given by:

$$
v_0 = T_1, \quad v_k = T(s_1s_2)^{k-1} + T(s_2s_1)^{k-1} - (u-1)(T(s_1s_2)^{k-1} + T(s_2s_1)^{k-1}), \quad k = 1, 2, \ldots, m-1,
$$

$$
v_m = T(s_1s_2)^{m-1} + u(T(s_2s_1)^{m-2} + \ldots + u^{m-1}T_{s_2})
$$

$$
v_{m+1} = T(s_2s_1)^{m-1} + u(T(s_1s_2)^{m-2} + \ldots + u^{m-1}T_{s_1})
$$

$$
v_{m+2} = T_{w_0}
$$

and when $m$ is odd by: $v_0, v_k, k = 1, \ldots, m-1,$
We recall the decomposition of the graded module into a direct sum of irreducible left $H$-submodules. This decomposition is,

$$\text{grad}(E) = M_0 \oplus M_0 \oplus M_1 \oplus M_2 \oplus V_j \oplus V_j, \quad \text{where } M_0, M_1, M_2, \text{ and } j = 1, \ldots, m-1$$

afford the 1-dimensional representations $\sigma_0 : T_{s_1} - u, 1 = 1, 2, \sigma_2 : T_{s_2} + 1$, respectively.

$V_j$ is a 2-dimensional $H$-submodule with basis $\{u_j^*, v_j\}$, which is isomorphic to the 2-dimensional $H$-submodule $\tilde{V}_j$ which has basis $\{u_j^*, \tilde{v}_j\}, j = 1, \ldots, m-1$ (see §2.3 for the definition of $u_j^*, v_j, \tilde{u}_j, \tilde{v}_j, j = 1, \ldots, m-1$).

We first establish the action of the central basis elements on the several irreducible constituents with respect to the corresponding basis adapted to them.

For the elements $v_k^* = k - 1, \ldots, m-1$, we already know that on both $M_0$ and $M_2$, they are respected by $2u^k$.

If $k$ is odd, then, on the $M_j$, $v_k^*$ is represented by $-(u^{k+1} + u^{k-1})$.

If $k$ is even, then on $M_j$, $v_k^*$ is represented by

$$2u^k - (u-1) \sum_{\lambda=1}^{k-1} u^{\lambda-1} (u^{k-\lambda} - u^{k-\lambda+1}) - (u-1) \sum_{\lambda=2}^{k} u^{\lambda-1} (u^{k-\lambda+1} - u^{k-\lambda}) =$$

$$= 2u^k - (u-1) \sum_{\lambda=1}^{k-1} (u^{k-\lambda} - u^k) + \sum_{\lambda=2}^{k} (u^k - u^{k-1}) = 2u^k.$$ 

Similarly we can verify that on $M_2$, $v_k^*$ is represented by $-(u^{k+1} + u^{k-1})$ if $k$ is odd, and by $2u^k$ if $k$ is even. The element $v_m^*$ is represented on $M_0$ by $u^{2m-1} + u^{2m-2} + \ldots + u^m$, and on $M_2$ by $-(u^{m-1} + u^{m-2} + \ldots + 1)$, for both cases $m$ even and $m$ odd.
When $m$ is even, the element

$$v_m = \sum_{\lambda=1}^{m-1} u^{\lambda-1} T(s_1 s_2)^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} T(s_2 s_1 s_2)^{m-\lambda}.$$  

Thus, $v_m$ is represented on $M_1$ by

$$\sum_{\lambda=1}^{m-1} u^{\lambda-1} u^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} u^{m-\lambda+1} = \frac{m}{2} (u^m + u^{-1})$$

and on $M_2$ by

$$\sum_{\lambda=1}^{m-1} u^{\lambda-1} u^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} u^{m-\lambda+1} = \frac{m}{2} (u^m + u^{-1})$$

When $m$ is odd then

$$v_m = \sum_{\lambda=1}^{m-1} u^{\lambda-1} T(s_1 s_2)^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} T(s_2 s_1 s_2)^{m-\lambda}$$

and now $v_m$ is represented on $M_1$ by

$$\sum_{\lambda=1}^{m-1} u^{\lambda-1} u^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} u^{m-\lambda+1} = \frac{m}{2} (u^m + u^{-1})$$

and on $M_2$ by

$$\sum_{\lambda=1}^{m-1} u^{\lambda-1} u^{m-\lambda} + \sum_{\lambda=2}^{m} u^{\lambda-1} u^{m-\lambda+1} = \frac{m}{2} (u^m + u^{-1})$$

The element $v_{m+1}$ is represented on $M_0$ by $u^{2m-1} + u^{2m-2} + \ldots + u^m$ and on $M_2$ by $-(u^{m+1} + u^{m+2} + \ldots + 1)$, for both cases $m$ even and $m$ odd.

With a similar argument as for the element $v_m$, we can verify that when $m$ is even $v_{m+1}$ is represented on $M_1$ by $\frac{m}{2} (u^m + u^{-1})$ and on $M_2$ by $\frac{m}{2} (u^m + u^{-1})$, while when $m$ is odd $v_{m+1}$ is represented by $\frac{m+1}{2} u^m + \frac{m-1}{2} u^{-1}$ and by $\frac{m+1}{2} u^m - \frac{m-1}{2} u^{-1}$ on $M_1$ and $M_2$ respectively.

Finally, the element $v_{m+2} = T_{w_0}$ is represented by:

$$u^{2m}, 1, u^m, u^{-m}$$
on $M_0, M_5, M_1, M_2$ respectively (when $m$ is even) and by:

$$u^{2m}, 1, -u^m, -u^{-m}$$
on the same modules, when $m$ is odd. We next determine the action of the central basis elements on each 2-dimensional irreducible
submodule. This has already been done for the elements \( v_k, k = 1, \ldots, m-1 \),
with the only difference that now \( \rho_j = 2 \cos \frac{2\pi j}{m} = 2 \cos \frac{\pi j}{m}, 1 \leq j \leq m-1 \).
We also recall that \( \rho_j \) are the zeros of the polynomial \( S_{m-1}(x) \) (see Lemma 2.2.2(1)).

Assume \( m \) is even and write \( v_m = \sum_{i=1}^{m-1} u^{-1} T(s_1 s_2)^{i-1} \sum_{j=1}^{m-1} u^{-1} T(s_2 s_1)^{i-1} \).

We recall the matrices which represent the elements \( T_w, w \in D_{2n} \) on the
blocks of the first kind and we concentrate on the diagonal entries. Let
\( \lambda_{m}(j) \times I_2 \) be the matrix which represents \( v_m \) on the 2-dimensional constituent
\( V_j, 1 \leq j \leq m-1 \) (\( \lambda_{m}(j) \times I_2 \) the corresponding matrix on \( V_j \)).

It turns out that the diagonal entries of the matrix which represents
\( v_m \) have the form

\[
u_m = \sum_{i=1}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{-1} \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)], \text{ and } \]

\[
u_m = \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{-1} \sum_{i=1}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] \]

Moreover,

\[
u_m = \sum_{i=1}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{-1} \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] =
\]

\[
u_m = \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{-1} \sum_{i=1}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] =
\]

\[
u_m = \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] - u^{-1} \sum_{i=1}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] =
\]

\[
u_m = u^{-1}(u-1) \sum_{i=2}^{m-1} S_{i-1}(\rho_j) + u^m \sum_{i=4}^{m} S_{i-1}(\rho_j) - u^{-1} \sum_{i=2}^{m-1} S_{i-1}(\rho_j)
\]

(since \( S_{-1}(x) = 0 \) and \( S_{m-1}(\rho_j) = 0 \))
= u^{m-1}(u-1) \sum_{i=2}^{m} S_{i-2}(\rho_j) + u^{m-1}(u-1) \sum_{i=2}^{m-2} S_{i-1}(\rho_j) = \\
= u^{m-1}(u-1)[S_0(\rho_j) + S_1(\rho_j) + \ldots + S_{m-3}(\rho_j) + S_{m-2}(\rho_j)], 1 \leq j \leq m-1.

Similarly we can calculate that

\[ u^m \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] = u^{m-1} \sum_{i=2}^{m-1} [S_{i-1}(\rho_j) + S_{i-2}(\rho_j)] \]

Therefore

\[ \lambda_m(j) = u^{m-1}(u-1)[S_0(\rho_j) + S_1(\rho_j) + \ldots + S_{m-3}(\rho_j) + S_{m-2}(\rho_j)], \text{ and since} \]

\[ \lambda_m(j) \times 1_2 \text{ is obtained by conjugating } \lambda_m(j) \times 1_2 \text{ by } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ we also have} \]

\[ \lambda_m(j) = \lambda_m(j), 1 \leq j \leq m-1. \]

An entirely similar calculation shows that when \( m \) is odd

\[ v_m = \sum_{i=1}^{m-1} u^{m-1} T(s_1 s_2)_{i-1-1} + \sum_{i=2}^{m-2} u^{m-1} T(s_2 s_1)_{i-1-1}, \text{ is represented on } V_j \]

and \( V_j \) by the same multiple of the identity matrix \( 1_2 \). The same multiple of the identity matrix \( 1_2 \) appears when we consider the element \( v_{m+1} \) in both cases \( m \) even and \( m \) odd.

Finally by Lemma 4.2 using the fact that \( S_{m-1}(\rho_j) = 0, 1 \leq j \leq m-1, \)
we obtain that

\[ S_0(\rho_j) + S_1(\rho_j) + \ldots + S_{m-2}(\rho_j) = \frac{1}{2} \frac{1 - \cos \left(\frac{m\theta_j}{2}\right)}{1 - \cos \theta_j}, \theta_j = \frac{4\pi}{m} \]
To summarize our calculations, we conclude that the elements $v_m$ and $v_{m+1}$ for both cases $m$ even and $m$ odd, are represented on each 2-dimensional constituent $V_j$, $1 \leq j \leq m-1$ by $\lambda_m^{(j)} = I_2$ and $\lambda_{m+1}^{(j)} = I_2$ respectively, where

$$
\lambda_m^{(j)} = \lambda_{m+1}^{(j)} = \begin{cases} 
0 & \text{if } j = \text{even} \\
\frac{u^{m-1}(u-1)}{1 - \cos \theta_j} & \text{if } j = \text{odd} 
\end{cases}
$$

Finally the matrix which represents $T_{w_0}$ on $V_j$ is (see Lemma 4.1)

$$
\begin{bmatrix}
u^m(S_m(\rho_j) + S_{m-1}(\rho_j)) & -2u^{m-1}\frac{\sqrt{2}}{\cos \frac{j\pi}{m}} S_{m-1}(\rho_j) \\
u^m(S_{m-1}(\rho_j) + S_{m-2}(\rho_j)) & -u^m(S_{m-1}(\rho_j) + S_{m-2}(\rho_j))
\end{bmatrix}
$$

Using the fact that $S_{m-1}(\rho_j) = 0$ and $S_m(\rho_j) = \rho_j S_{m-1}(\rho_j) - S_{m-2}(\rho_j) = -S_{m-2}(\rho_j)$, we obtain that this matrix above is $\lambda_{w_0}^{(j)} = I_2$ where $\lambda_{w_0}^{(j)} = -u^m S_{m-2}(\rho_j)$. Moreover $S_{m-2}(\rho_j) = \frac{\sin(m-1)\theta_j}{\sin \theta_j} = -\cos(m\theta_j) = \begin{bmatrix} 1 & \text{if } j = \text{odd} \\
-1 & \text{if } j = \text{even} \end{bmatrix}$

Therefore $\lambda_{w_0}^{(j)} = \begin{bmatrix} -u^m & \text{if } j = \text{odd} \\
u^m & \text{if } j = \text{even} \end{bmatrix}$

We next exhibit the central character table of $H(D_{2n})$, $n = 2m$, for both cases $m$ even and $m$ odd.
Central Character Table of $H(D_{2n})$ $n = 2m$, $m$ even

<table>
<thead>
<tr>
<th>$M_0$</th>
<th>$v_0$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_{m-1}$</th>
<th>$v_m$</th>
<th>$v_{m+1}$</th>
<th>$v_{m+2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2u</td>
<td>$2u^2$</td>
<td>...</td>
<td>$2u^{m-1}$</td>
<td>$u^{2m-1} + ... + u^m$</td>
<td>...</td>
<td>$u^{2m}$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>$-(u^2+1)$</td>
<td>$2u^2$</td>
<td>...</td>
<td>$-(u^m + u^{m-2})$</td>
<td>$\frac{m}{2}(u^{m} + u^{m-1})$</td>
<td>...</td>
</tr>
<tr>
<td>$M_2$</td>
<td>1</td>
<td>$-(u^2+1)$</td>
<td>$2u^2$</td>
<td>...</td>
<td>$-(u^m + u^{m-2})$</td>
<td>$-\frac{m}{2}(u^{m} + u^{m-1})$</td>
<td>...</td>
</tr>
<tr>
<td>$V_1$</td>
<td>1</td>
<td>$u_{2,1}^{1} - (u-1)^{2,1}_{2,1}$</td>
<td>$u_{2,2}^{2,1} - u(u-1)^{2,1}_{2,2}$</td>
<td>...</td>
<td>$u_{m-1,1}^{m-1} - u^{m-2}(u-1)^{2,1}_{m-1}$</td>
<td>$u_{m-1}^{m-1}(u-1) \frac{1}{1 - \cos \theta}$</td>
<td>...</td>
</tr>
<tr>
<td>$V_2$</td>
<td>1</td>
<td>$u_{2,1}^{3} - (u-1)^{2,3}_{2,1}$</td>
<td>$u_{2,2}^{2} - u(u-1)^{2,2}_{2,2}$</td>
<td>...</td>
<td>$u_{m-1,2}^{m-1} - u^{m-2}(u-1)^{2,2}_{m-1}$</td>
<td>0</td>
<td>...</td>
</tr>
<tr>
<td>$V_3$</td>
<td>1</td>
<td>$u_{2,1}^{3} - (u-1)^{2,3}_{2,1}$</td>
<td>$u_{2,2}^{3} - u(u-1)^{2,3}_{2,2}$</td>
<td>...</td>
<td>$u_{m-1,3}^{m-1} - u^{m-2}(u-1)^{2,3}_{m-1}$</td>
<td>$u_{m-1}^{m-1}(u-1) \frac{1}{1 - \cos 3\theta}$</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$V_{m-1}$</td>
<td>1</td>
<td>$u_{2,1}^{m-1} - (u-1)^{2,m-1}_{2,1}$</td>
<td>$u_{2,2}^{m-1} - u(u-1)^{2,m-1}_{2,2}$</td>
<td>...</td>
<td>$u_{m-1,m-1}^{m-1} - u^{m-2}(u-1)^{2,m-1}_{m-1}$</td>
<td>$u_{m-1}^{m-1}(u-1) \frac{1}{1 - \cos (m-1)\theta}$</td>
<td>...</td>
</tr>
<tr>
<td>$V_m$</td>
<td>1</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
In the table above \( \theta = \frac{\pi}{m} \), \( \Delta_j^k = 2 \cos \frac{k j \pi}{m} \), \( 1 \leq j \leq m-1 \), and
\( \Delta_k^j = R_{k-1}(\cos \theta_j), \ 1 \leq k \leq m-1, \ \theta_j = \frac{j \pi}{m} \). \( \{M_0, M_1, M_2, V_j, 1 \leq j \leq m-1\} \)

is a full set of left irreducible \( H \)-modules, \( \{v_j, 0 \leq j \leq m-2\} \) is a basis of the centre of \( H \), and the entries of the table represent the scalars according to which the basis elements of the centre, act on the irreducible constituents.

By specializing \( u \rightarrow 1 \) we obtain the central character table of \( W = D_{2n}, n = 2m, m \) even which determines the central character matrix \( C = \)

\[
\begin{array}{cccccc}
1 & 2 & 2 & \cdots & 2 & m & m & 1 \\
1 & 2 & 2 & \cdots & 2 & -m & -m & 1 \\
1 & -2 & 2 & \cdots & -2 & m & -m & 1 \\
1 & -2 & 2 & \cdots & -2 & -m & m & 1 \\
1 & 2 \cos \frac{\pi}{m} & 2 \cos \frac{2\pi}{m} & \cdots & 2 \cos \frac{(m-1)\pi}{m} & 0 & 0 & -1 \\
1 & 2 \cos \frac{2\pi}{m} & 2 \cos \frac{4\pi}{m} & \cdots & 2 \cos \frac{2(m-1)\pi}{m} & 0 & 0 & 1 \\
1 & 2 \cos \frac{(m-1)\pi}{m} & 2 \cos \frac{2(m-1)\pi}{m} & \cdots & 2 \cos \frac{(m-1)^2\pi}{m} & 0 & 0 & -1 \\
\end{array}
\]
The inverse of the central character matrix $C$ is given by $\overline{C} =$

$$
\begin{pmatrix}
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m} & \ldots & \frac{1}{m} \\
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m} \cos \frac{\pi}{m} & \ldots & \frac{1}{m} \cos \frac{(m-1)\pi}{m} \\
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m} \cos \frac{2\pi}{m} & \ldots & \frac{1}{m} \cos \frac{2(m-1)\pi}{m} \\
\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m} \cos \frac{3\pi}{m} & \ldots & \frac{1}{m} \cos \frac{3(m-1)\pi}{m} \\
\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & -\frac{1}{2n} & \frac{1}{m} \cos \frac{(m-1)\pi}{m} & \ldots & \frac{1}{m} \cos \frac{(m-1)\pi}{m} \\
\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & 0 & \ldots & 0 \\
\frac{1}{2n} & -\frac{1}{2n} & -\frac{1}{2n} & \frac{1}{2n} & 0 & \ldots & 0 \\
\frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{2n} & \frac{1}{m} & \ldots & \frac{1}{m}
\end{pmatrix}
$$

Every entry of the $i$th column of the matrix $\overline{C}$ is given by the corresponding entry in the $i$th row of the character table of the group $D_{2n}$, $n = 2m$, $m$ even, multiplied by the degree $d_i$ and divided by the order of the group.
Central character table of $H(D_{2n})$, $n = 2m$, $m$ odd.

<table>
<thead>
<tr>
<th></th>
<th>$v_0$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
<th>$v_4$</th>
<th>$v_5$</th>
<th>$v_6$</th>
<th>$v_7$</th>
<th>$v_{m-1}$</th>
<th>$v_m$</th>
<th>$v_{m+1}$</th>
<th>$v_{m+2}$</th>
</tr>
</thead>
</table>
| $M_0$ | 1 | $2u$ | $2u^{m-1}$ | $u^{2m-1} + \ldots + u^m$ | $u^{2m-1} + \ldots + u^m$ | 2
| $M_1$ | 1 | $2u$ | $2u^{m-1}$ | $-(u^{m-1} + \ldots + u^1)$ | $-(u^{m-1} + \ldots + u^1)$ | 1
| $M_2$ | 1 | $-(u^2 + 1)$ | $2u^{m-1}$ | $\frac{m+1}{2} u^{m-1} - \frac{m-1}{2} u^m$ | $\frac{m+1}{2} u^{m} + \frac{m-1}{2} u^{m-1}$ | $-u^m$
| $V_1$ | 1 | $u^{\Delta_1^1 - (u-1)^{2\Delta_1^1}}$ | $u^{\Delta_1^0 - (u-1)^{2\Delta_1^0}}$ | $0$ | $(u-1)^{\frac{1}{1-cos \theta}}$ | $-u^m$
| $V_2$ | 1 | $u^{\Delta_1^2 - (u-1)^{2\Delta_1^2}}$ | $u^{\Delta_1^1 - (u-1)^{2\Delta_1^1}}$ | $0$ | $(u-1)^{\frac{1}{1-cos \theta}}$ | $-u^m$
| $V_3$ | 1 | $u^{\Delta_1^3 - (u-1)^{2\Delta_1^3}}$ | $u^{\Delta_1^2 - (u-1)^{2\Delta_1^2}}$ | $0$ | $(u-1)^{\frac{1}{1-cos \theta}}$ | $-u^m$
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$
| $V_{m-1}$ | 1 | $u^{\Delta_1^{m-1} - (u-1)^{2\Delta_1^{m-1}}}$ | $u^{\Delta_1^{m-2} - (u-1)^{2\Delta_1^{m-2}}}$ | $0$ | $0$ | $u^m$
By specializing \( u + 1 \), we obtain the Central Character table of the group \( W = D_{2n} \), \( n = 2m \), \( m \) odd, which determines the central character matrix \( C' \):

\[
\begin{array}{cccc}
1 & 2 & 2 & = \\
1 & 2 & 2 & -m \\
1 & -2 & 2 & -m \\
1 & -2 & 2 & m \\
1 & 2 \cos \frac{\pi}{n} & 2 \cos \frac{2\pi}{m} & \ldots 0 \\
1 & 2 \cos \frac{2\pi}{m} & 2 \cos \frac{4\pi}{m} & \ldots 0 \\
1 & 2 \cos \frac{3\pi}{m} & 2 \cos \frac{6\pi}{m} & \ldots 0 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 2 \cos \frac{(m-1)\pi}{m} & 2 \cos \frac{2(m-1)\pi}{m} & \ldots 0 \\
\end{array}
\]
The inverse of the central character matrix $C'$, is $\tilde{C}' =$

<table>
<thead>
<tr>
<th>$\frac{1}{2n}$</th>
<th>$\frac{1}{2n}$</th>
<th>$\frac{1}{2n}$</th>
<th>$\frac{1}{2n}$</th>
<th>$\frac{1}{m}$</th>
<th>$\cdots$</th>
<th>$\frac{1}{m}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$\frac{1}{m} \cos \frac{\pi}{m}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{m} \cos \frac{(m-1)\pi}{m}$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{m} \cos \frac{2\pi}{m}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{m} \cos \frac{2(m-1)\pi}{m}$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$\frac{1}{m} \cos \frac{3\pi}{m}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{m} \cos \frac{3(m-1)\pi}{m}$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{m} \cos \frac{(m-1)\pi}{m}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{m} \cos \frac{(m-1)^2\pi}{m}$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$0$</td>
<td>$\cdots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$0$</td>
<td>$\cdots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\frac{1}{2n}$</td>
<td>$\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$-\frac{1}{2n}$</td>
<td>$\frac{1}{m}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{m}$</td>
</tr>
</tbody>
</table>
Let \( A = (a_{ir}) \) \( j, r \in \{1, 2, \ldots, m+3\} \) be the matrix describing Lusztig's isomorphism on the centre of the generic Hecke algebra \( H(D_{2n}) \) with respect to the basis \( \{v_0, v_k, 1 \leq k \leq m-1, v_m, v_{m+1}, v_{m+2}\} \). Then

\[ A = \tilde{C} A (\text{or } A = \tilde{C}' A') \]

where \( \tilde{C} \) is the inverse of the central character matrix \( C \) (when \( m \) is even), \( \tilde{C}' \) the inverse of the central character matrix \( C' \) (when \( m \) is odd), and \( A, A' \) the matrix determined by the central character table of \( H(D_{2n}) \) for the cases \( m \) even and \( m \) odd respectively.

The calculations are entirely similar as in the case \( n \) odd and they are omitted. We provide the result of our calculations in a matrix form in which we use the following notation.

\[
K(1,m+1) = K(1,m+2) = \frac{1}{2n} (u-1)[\sum_{\lambda=2}^{m} (\lambda-1)u^{2m-\lambda}+(m^2+m)u^{m-1} + \sum_{\lambda=2}^{m} (m-\lambda+1)u^{m-\lambda}] \\
K(i,m+1) = K(i,m+2) = \frac{1}{2n} (u-1)[\sum_{\lambda=2}^{m} (\lambda-1)u^{2m-\lambda}+(m^2-n(i-1)+m)u^{m-1} + \sum_{\lambda=2}^{m} (m-\lambda+1)u^{m-\lambda}] \\
\text{for every } i = 2, 3, \ldots, m.
\]

\[
K(m+1,m+1) = K(m+2,m+2) = \frac{1}{2n} [u^{n-1} + \ldots + 1] + \frac{1}{4} (u^m + u^{m-1}) \\
K(m+1,m+2) = K(m+2,m+1) = \frac{1}{2n} [u^{n-1} + \ldots + 1] - \frac{1}{4} (u^m + u^{m-1}) \\
K(m+3,m+1) = K(m+3,m+2) = \frac{1}{2n} (u-1)[\sum_{\lambda=2}^{m} (\lambda-1)u^{2m-\lambda}-(m^2-m)u^{m-1} + \sum_{\lambda=2}^{m} (m-\lambda+1)u^{m-\lambda}] \\
\]
When we specialize $u = 1$ the matrix above specializes to be identity matrix as it should be.
We illustrate the situation by giving some examples.

(1) \( W = D_8, \ n = 4, \ m = 2 \).

A basis for the centre of \( H(D_8) \) is given by the set of elements

\[
\begin{align*}
V_0 &= T_1, \\
V_1 &= T_{s_1 s_2} + T_{s_2 s_1} - (u-1)[T_{s_1} + T_{s_2}], \\
V_2 &= T_{s_1 s_2 s_1} + uT_{s_2}, \\
V_3 &= T_{s_2 s_1 s_2} + uT_{s_1}, \\
V_4 &= T_{s_1 s_2 s_1 s_2}
\end{align*}
\]

The matrix \( A \) is given by:

\[
\begin{bmatrix}
1 & -\frac{3}{4} (u-1)^2 & \frac{1}{8} (u-1)(u^2+6u+1) & \frac{1}{8} (u-1)(u^2+6u+1) & \frac{1}{8} (u^2-1)^2 \\
0 & u + \frac{1}{4} (u-1)^2 & \frac{1}{8} (u-1)(u+1)^2 & \frac{1}{8} (u-1)(u+1)^2 & \frac{1}{8} (u^2-1)^2 \\
0 & 0 & \frac{1}{8} (u+1)^3 & \frac{1}{8} (u+1)(u-1)^2 & \frac{u^4-1}{8} \\
0 & 0 & \frac{1}{8} (u+1)(u-1)^2 & \frac{1}{8} (u+1)^3 & \frac{u^4-1}{8} \\
0 & \frac{1}{4} (u-1)^2 & \frac{1}{8} (u-1)^3 & \frac{1}{8} (u-1)^2 & u^2 + \frac{1}{8} (u^2-1)^2
\end{bmatrix}
\]

The information given by this matrix is that

\[
\begin{align*}
\phi(v_0) &= 1 \\
\phi(v_1) &= -\frac{3}{4} (u-1)^2 - 1 + [u + \frac{1}{4} (u-1)^2](s_1 s_2 + s_2 s_1) + \frac{1}{4} (u-1)^2(s_1 s_2)^2 \\
\phi(v_2) &= \frac{1}{8} (u-1)(u^2+6u+1) - 1 + \frac{1}{8} (u-1)(u+1)^2(s_1 s_2 + s_2 s_1) + \\
&+ \frac{1}{8} (u+1)^3(s_1 s_2 s_1 + s_2) + \frac{1}{8} (u+1)(u-1)^2(s_2 s_1 s_2 + s_1) + \\
&+ \frac{1}{8} (u-1)^3(s_1 s_2)^2 \\
\phi(v_3) &= \frac{1}{8} (u-1)(u^2+6u+1) - 1 + \frac{1}{8} (u-1)(u+1)^2(s_1 s_2 + s_2 s_1) + \\
&+ \frac{1}{8} (u+1)(u-1)^2(s_1 s_2 s_1 + s_2) + \frac{1}{8} (u+1)^3(s_2 s_1 s_2 + s_1) + \frac{1}{8} (u-1)^2(s_1 s_2)^2 \\
\phi(v_4) &= \frac{1}{8} (u^2-1)^2 - 1 + \frac{1}{8} (u^2-1)^2(s_1 s_2 + s_2 s_1) + \frac{u^4-1}{8} (s_1 s_2 s_1 + s_2) + \\
&+ \frac{u^4-1}{8} (s_2 s_1 s_2 + s_1) + [u^2 + \frac{1}{8} (u^2-1)^2](s_1 s_2)^2
\end{align*}
\]
(ii) \( W = D_{12}, n = 6, m = 3 \).

A basis for the centre of \( H(D_{12}) \) is given by the set of elements:

\[
\begin{align*}
v_0 &= T_1, \\
v_1 &= T_{s_1 s_2} - T_{s_2 s_1} - (u-1)[T_{s_1} + T_{s_2}], \\
v_2 &= T_{s_1 s_2 s_1 s_2} + T_{s_2 s_1 s_2 s_1} - (u-1)[T_{s_1 s_2 s_1} + T_{s_2 s_1 s_2}] - u(u-1)[T_{s_1} + T_{s_2}], \\
v_3 &= T_{s_1 s_2 s_1 s_2} + u T_{s_2 s_1 s_2} + u^2 T_{s_1}, \\
v_4 &= T_{s_2 s_1 s_2 s_1 s_2} + u T_{s_2 s_1 s_2 s_1} + u^2 T_{s_2}, \\
v_5 &= T_{s_1 s_2 s_1 s_2 s_1 s_2}
\end{align*}
\]
\[
\begin{array}{cccc}
1 - \frac{5}{6} (u-1)^2 - \frac{4}{3} u(u-1)^2 & \frac{1}{12} (u-1)(u^4 + 2u^3 + 12u^2 + 2u + 1) & \frac{1}{12} (u-1)(u^4 + 2u^3 + 12u^2 + 2u + 1) & \frac{1}{12} (u^3-1)^2 \\
0 + \frac{1}{6} (u-1)^2 - \frac{1}{3} u(u-1)^2 & \frac{1}{12} (u-1)(u^4 + 2u^3 + 6u^2 + 2u + 1) & \frac{1}{12} (u-1)(u^4 + 2u^3 + 6u^2 + 2u + 1) & \frac{1}{12} (u^3-1)^2 \\
0 \frac{1}{6} (u-1)^2 u^2 + \frac{2}{3} u(u-1)^2 & \frac{1}{12} (u-1)(u^4 + 2u^3 + 2u + 1) & \frac{1}{12} (u-1)(u^4 + 2u^3 + 2u + 1) & \frac{1}{12} (u^3-1)^2 \\
0 0 0 & \frac{1}{12} (u^5+u^4+4u^3+4u^2 + u + 1) & \frac{1}{12} (u^5+u^4-2u^3-2u^2 + u + 1) & \frac{u^6 - 1}{12} \\
0 0 0 & \frac{1}{12} (u^5+u^4-2u^3-2u^2 + u + 1) & \frac{1}{12} (u^5+u^4+4u^3+4u^2 + u + 1) & \frac{u^6 - 1}{12} \\
0 \frac{1}{6} (u-1)^2 \frac{2}{3} u(u-1)^2 & \frac{1}{12} (u-1)(u^4+2u^3-6u^2+2u+1) & \frac{1}{12} (u-1)(u^4+2u^3-6u^2+2u+1) & u^3 + \frac{1}{12} (u^3-1)^2 \\
\end{array}
\]
The general form of Lusztig's isomorphism and its restriction to the centre of the generic Hecke algebra

In this chapter we shall generalize the results of Chapters 2 and 4. In §1.4, we gave a procedure for the determination of Lusztig's isomorphism. It is clear that the larger the order of the group $W$ becomes, the harder it is to find an explicit formula for this isomorphism. We wish to find some information which will simplify this procedure and enable us to establish a general formula for this isomorphism. The starting point of our investigation was the determination of Lusztig's isomorphism for the symmetric group $S_4$.

Let $W$ be the symmetric group $S_4$, given by a presentation

$$W = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = 1, (s_1s_2)^3 = (s_2s_3)^3 = 1, s_1s_3 = s_3s_1 \rangle.$$  

There are ten left cells in $W$, given by:

- $X_0 = \{1\}$
- $L_1 = \{(s_1s_2s_1, s_3s_2s_1)\}$
- $L_2 = \{(s_1s_2, s_2, s_3s_2)\}$
- $L_3 = \{(s_1s_2s_3, s_2s_3, s_3)\}$
- $M_1 = \{(s_1s_3, s_2s_1s_3)\}$
- $N_1 = \{(s_2s_3s_1s_2s_1, s_3s_1s_2s_1, s_1s_2s_1)\}$
- $N_2 = \{(s_2s_3s_2, s_1s_2s_3s_1, s_1s_2s_1s_3, s_1s_2s_1)\}$
- $N_3 = \{(s_2s_3s_2s_1, s_1s_3s_2s_1s_3, s_1s_2s_1s_3)\}$
- $X_4 = \{(s_1s_2s_1s_3s_2s_1)\}$

There are five two-sided cells in $W$, given by:

- $X_0 \cup X_1 = L_1 \cup L_2 \cup L_3$
- $X_2 = M_1 \cup M_2$
- $X_3 = N_1 \cup N_2 \cup N_3 \cup X_4$

We consider the free $\mathbb{Q}[u^{1/2}]$ module $E$ with basis $\{e_w : w \in S_4\}$, and we make it into left $H$-module with action described in §1.4 where $H$ is the generic Hecke algebra over the ring $\mathbb{Q}[u^{1/2}]$. Then we construct the graded module grad$(E)$ (see also §1.4) with canonical basis $\{e_w : w \in S_4\}$. We know that grad$(E)$ affords the left regular representation of $H$. 

Now, each left cell gives rise to a $W$-graph according to Theorem 1.3.6. The $W$-graphs arising from the left cells are

![Diagrams of $W$-graphs](attachment:image.png)

The circles represent the vertices, and inside each circle we describe the indices $i$ for which $s_i w < w$, $i = 1, 2, 3$, and $w$ is the vertex represented by the corresponding cycle. The function $u$ is identically 1 and it is omitted.

Each such $W$-graph gives rise to a representation of $H$ over $\mathbb{Q}(u^\frac{1}{2})$ which by Theorem 1.3.9 is irreducible.

For each left-cell say $C$, we consider the subspace $V_C$ of $\text{grad}(E)$ spanned by $\{e_w, w \in C\}$. Then $V_C$ is an irreducible left $H_{\mathbb{Q}(u^\frac{1}{2})}$ module.

We next provide the matrices which represent $T_{s_i}$, $i = 1, 2, 3$ on the various modules $V_C$, with bases $C$

$T_{s_1}$ is represented on $V_{x_0}, V_{L_1}, V_{N_1}, V_{N_1}, V_{x_4}$ by the matrices:

$$
\begin{bmatrix}
-1 & u^\frac{1}{2} & 0 \\
0 & u & 0 \\
0 & 0 & u
\end{bmatrix}, \begin{bmatrix}
-1 & u^\frac{1}{2} \\
0 & u \\
0 & 0 & u
\end{bmatrix}, \begin{bmatrix}
u^\frac{1}{2} & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}.
$$

respectively.

The matrices representing $T_{s_2}$ on these modules are

$$
\begin{bmatrix}
u & 0 & 0 \\
u^\frac{1}{2} & -1 & u^\frac{1}{2} \\
0 & 0 & u
\end{bmatrix}, \begin{bmatrix}
u & 0 \\
u^\frac{1}{2} & -1 \\
0 & 0 & u
\end{bmatrix}, \begin{bmatrix}
-1 & u^\frac{1}{2} & 0 \\
0 & u & 0 \\
0 & u^\frac{1}{2} & -1
\end{bmatrix}.
$$

and the matrices representing $T_{s_3}$ are

$$
\begin{bmatrix}
u & 0 & 0 \\
0 & u & 0 \\
0 & u^\frac{1}{2} & -1
\end{bmatrix}, \begin{bmatrix}
-1 & u^\frac{1}{2} \\
0 & u \\
0 & 0 & u
\end{bmatrix}, \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & u^\frac{1}{2} \\
0 & 0 & u
\end{bmatrix}.
$$
Let $\phi(T_{s_1})$ be the images of the generators $T_{s_1}$, $i = 1, 2, 3$, under the Lusztig isomorphism $\phi$.

Let $\phi(T_{s_1}) = \sum_{w \in S_4} c_w^{(i)} w, c_w^{(i)} \in \mathbb{Q}(u \bar{u}).$ The generators $s_1$ are represented on the various modules $V_C$ with bases $C$ according to the matrices obtained by the ones above by specializing $u \neq 1$. Therefore we can work out the matrices which represent every $w \in S_4$ with respect to these bases. Using the fact that $T_i$ and $\phi(T_{s_1})$ act on the same way on the graded module and comparing coefficients of the basis elements on both sides of the equation $T_\phi w = \phi(T_{s_1}) w$ we obtain from each irreducible representation of degree $d \in \{1, 3, 2, 3, 1\}$, equations in the $c_w^{(i)}$, $w \in S_4, i = 1, 2, 3$. Thus we obtain a total of 24 equations in 24 unknown $c_w^{(i)}$. These equations are linearly independent and the solution of the system of these equations gives

$$\phi(T_{s_1}) = \frac{u-1}{2} s_1 + \frac{u+1}{2} s_1 + \frac{(u^4-1)^2}{6} [-s_2 + s_1 s_2 - s_2 s_1 + s_1 s_2 s_1] +$$

$$+ \frac{(u^4-1)^2}{12} [-s_2 s_3 s_2 - s_1 s_2 s_3 s_2 + s_2 s_3 s_2 s_1 - s_1 s_2 s_3 s_1 ] +$$

$$+ \frac{(u^4-1)^2}{24} [-s_2 s_3 + s_3 s_2 - s_1 s_3 s_2 + s_3 s_2 s_1 - s_2 s_3 s_1 + s_1 s_2 s_3 - s_1 s_2 s_3 s_1 ] +$$

$$\phi(T_{s_2}) = \frac{u-1}{2} s_2 + \frac{u+1}{2} s_2 + \frac{(u^4-1)^2}{6} [-s_1 - s_3 - s_1 s_2 + s_2 s_1 + s_1 s_2 s_1 + s_2 s_3 s_1]$$

$$+ \frac{(u^4-1)^2}{12} [-s_3 s_2 s_1 - s_1 s_2 s_3 - s_1 s_2 s_1 + s_2 s_3 s_2 + s_2 s_3 s_2 s_1 + s_1 s_2 s_3 s_1 ] +$$

$$\phi(T_{s_3}) = \frac{u-1}{2} s_3 + \frac{u+1}{2} s_3 + \frac{(u^4-1)^2}{6} [-s_2 + s_3 s_2 - s_2 s_3 + s_2 s_3 s_2 ] +$$

$$+ \frac{(u^4-1)^2}{12} [-s_3 s_2 s_1 + s_1 s_3 s_2 s_1 - s_1 s_2 s_3 s_2 + s_3 s_2 s_1 s_2 s_3] +$$

$$+ \frac{(u^4-1)^2}{24} [-s_2 s_1 + s_1 s_2 - s_3 s_1 s_2 s_3 - s_2 s_1 s_2 s_3 - s_3 s_2 s_1 s_2 s_3 + -s_3 s_2 s_1 s_2 s_3 s_1 ] +$$

$$s_3 s_2 s_1 s_2 s_3 - s_1 s_2 s_3 s_2 s_1 + s_2 s_3 s_2 s_1 s_2 s_3]$$

$$= \frac{u^4-1}{6} [-s_1 - s_3 - s_1 s_2 + s_2 s_1 + s_1 s_2 s_1 + s_2 s_3 s_1]$$

$$+ \frac{(u^4-1)^2}{12} [-s_3 s_2 s_1 - s_1 s_2 s_3 - s_1 s_2 s_1 + s_2 s_3 s_2 + s_2 s_3 s_2 s_1 + s_1 s_2 s_3 s_1 ] +$$

$$+ \frac{(u^4-1)^2}{24} [-s_2 s_1 + s_1 s_2 - s_3 s_1 s_2 s_3 - s_2 s_1 s_2 s_3 - s_3 s_2 s_1 s_2 s_3 + -s_3 s_2 s_1 s_2 s_3 s_1 ] +$$

$$s_3 s_2 s_1 s_2 s_3 - s_1 s_2 s_3 s_2 s_1 + s_2 s_3 s_2 s_1 s_2 s_3]$$
We note that the coefficients of $1_{S_4}$ and $s_i$ in the images of $\varphi(T_{S_4})$ are $\frac{u-1}{2}$ and $\frac{u+1}{2}$ respectively and that $c_w^{(1)} = c_w^{(1)}$ while $c_w^{(1)} = -c_w^{(1)}$. Thus if $w \in C_w(S_4)$, $w \neq 1, s_i$, $i = 1, 2, 3$, then $c_w^{(1)} = 0$, where $C_w(S_4)$ denotes the centralizer of $s_i$ in $W$. (Compare these remarks with the formulae giving the isomorphism $\varphi$ for the dihedral case). Thus the coefficients of $s_3$, $s_4$, and $s_2$ in $\varphi(T_{S_4})$, and the coefficients of $s_1s_2s_3s_1$, $s_1s_2s_3s_2$, and $\varphi(T_{S_2})$ and the coefficients of $s_1$, $s_3$, and $\varphi(T_{S_3})$ are all zero. There are certainly other elements $w$ for which $c_w^{(1)} = 0$.

We now prove the following:

**Theorem 5.1:** Let $W$ be a finite indecomposable Weyl group, and $\varphi$ be the Lusztig isomorphism between the generic Hecke algebra over $\mathbb{Q}(u^1)$ and the group algebra of $W$ over $\mathbb{Q}(u^1)$. Let $\varphi(T_w) = \sum_{w \in W} c_w w$, $c_w \in \mathbb{Q}(u^1)$. Then

\begin{align*}
(1) & \quad c_1 = \frac{u-1}{2}, \quad c_s = \frac{u+1}{2}, \quad c_w = c_{sw}, \quad c_w = -c_{sw} \quad \forall w \neq 1, s \\
(11) & \quad \varphi(T_s) = \frac{u-1}{2} \cdot 1 + \frac{u+1}{2} \cdot s + (u^2-1)sF_s, \text{ where } F_s \in \mathbb{Q}W, F_s \text{ does not involve } 1, s, \text{ and } F_s \text{ satisfies the properties } F_s^2 = 0, sF_s = -F_s, F_s \text{ satisifies } F_s \text{ and } F_s = F_s.
\end{align*}

**Proof:** We consider the graded module over $\mathbb{Q}(u^1)$ with the canonical basis $(\hat{e}_w, w \in W)$. We fix an $s \in S$ and let $W_1 = \{w \in W : sw < w\}$, $W_2 = \{w \in W : sw > w\}$. Then $W = W_1 \cup W_2$, and if $d_1 = |W_1|$, $i = 1, 2$, then $d_1 = \frac{|W|}{2}$ since the map $W \to W$ such that $w \mapsto sw$ is clearly a bijection.

We order the canonical basis so that $(\hat{e}_w, w \in W) = (\hat{e}_w, w \in W_1) \cup (\hat{e}_w, w \in W_2)$, and then, the matrix which represents $T_s$ on grad(E) with respect to this ordering of the canonical basis is (considering $T_s$ acting on grad(E) by the left)

\[ A(T_s) = \begin{bmatrix} -I_{d_1} & u^1E_s \\ 0 & uI_{d_1} \end{bmatrix} \]
The action of $T_s$ on $\alpha_w$ is given in §1.4. $\text{Id}_d$ is the identity matrix of size $d$, $0$ is the zero matrix of the same size and $E_s$ is a matrix also of the same size whose entries are integers.

We consider the image of $T_s$ under $\phi$ and we write $\phi(T_s) = c_1 \cdot 1 + \sum_{w \neq 1} c_w \cdot w$. When we view the $\text{grad}(E)$ as a left $H$-module and as a left $W$-module with actions described in §1.4, then it affords the left regular representation of $H$ and $W$ respectively. We also recall that $T_s$ and $\Phi(T_s)$ act on $\text{grad}(E)$ in the same way. Since the trace of any $w \in W$, $w \neq 1$ for the regular representation of $W$ is zero, we obtain that the trace of the matrix $A(T_s)$ is $\frac{u^{-1}}{2} |W| = c_1 |W|$, therefore $c_1 = \frac{u^{-1}}{2}$. We next write $\Phi(T_s) = f_s(u) + u^{1/2} g_s(u)$, where $f_s(u)$, $g_s(u)$ belong to $\mathbb{Q}(u)(W)$.

We also write $A(T_s) = \begin{bmatrix} -\text{Id}_d & 0 \\ 0 & u\text{Id}_d \end{bmatrix} + u^{1/2} \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$.

Therefore the elements of the group algebra $f_s(u)$, $g_s(u)$ are represented on $\text{grad}(E)$ with respect to the ordered canonical basis as above, by the matrices:

$f_s(u) \rightarrow \begin{bmatrix} -\text{Id}_d & 0 \\ 0 & u\text{Id}_d \end{bmatrix}$, $g_s(u) \rightarrow \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$

By specializing $u^{1/2} = 1$ we obtain the matrix which represents the generator $s$, so

$s \rightarrow \begin{bmatrix} -\text{Id}_d & E_s \\ 0 & \text{Id}_d \end{bmatrix} = \begin{bmatrix} -\text{Id}_d & 0 \\ 0 & \text{Id}_d \end{bmatrix} + \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$.
Thus $g_s(u) = g_s(1) \in QW$.

Let $a$ be the following element inside $Q(u^1)(W)$.

$$a = \frac{u-1}{2} + \frac{u+1}{2} s - \frac{u+1}{2} g_s(1) + u^1 g_s(1).$$

Then, the matrix which represents $a$ on $\operatorname{grad}(E)$ with respect to the ordered canonical basis as above, is

$$\begin{bmatrix}
(u-1)^2 - \frac{u+1}{2} I_d & (u+1)^2 - \frac{u+1}{2} + u^3 E_s \\
0 & (u+1)^2 + \frac{u+1}{2} I_d
\end{bmatrix}$$

Therefore $\phi(T_s) = \frac{u-1}{2} + \frac{u+1}{2} s - \frac{u+1}{2} g_s(1) + u^1 g_s(1)$.

Put $F_s = -\frac{1}{2} g_s(1)$ to obtain

$$\phi(T_s) = \frac{u-1}{2} + \frac{u+1}{2} s + (u^3 - 1)^2 F_s,$$

where $F_s \in QW$. $F_s$ does not involve 1.

Now the matrix which represents $F_s$ on $\operatorname{grad}(E)$ with respect to the ordered canonical basis is $-\frac{1}{2} \begin{bmatrix} 0 & E_s \\ 0 & 0 \end{bmatrix}$. Thus, the matrix which represents $F_s^2$ is the zero matrix so $F_s^2 = 0$.

Suppose that $F_s = c_s^0 s + \sum_{w \neq 1} c_w^0 w$. Then, the matrix which represents $sF_s$ on $\operatorname{grad}(E)$ with respect to the ordered canonical basis is
Thus $sF_s = -F_s$ i.e. $c^1_s, 1 + \sum_{w \neq 1, s} c^i_{sw} w = -c^i_s, s - \sum_{w \neq 1, s} c^i_{sw} w$, and therefore we obtain that $c^i_s = 0$ and $c^i_{sw} = -c^i_{sw}, w \neq 1, s$. Therefore $F_s$ does not involve the element $s$ as well.

The matrix which represents the element $F_s, s$ is

\[
\begin{bmatrix}
-I_{d_1} & E_s \\
0 & I_{d_1}
\end{bmatrix}
\begin{bmatrix}
0 & -\frac{1}{2} E_s \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
-I_{d_1} & E_s \\
0 & I_{d_1}
\end{bmatrix}
= \begin{bmatrix}
0 & -\frac{1}{2} E_s \\
0 & 0
\end{bmatrix}
\]

Thus $F_s, s = F_s$ and so $c^i_w = c^i_{ws}, w \neq 1, s$.

Our theorem is now proved.

Now for every generator $s_1$ of the group $W$, we write

\[g(T_{s_1}) = \frac{u-1}{2} + \frac{u+1}{2} s_1 + (u^2-1)^2 F_{s_1},
\]

\[= \frac{u-1}{2} + \frac{u+1}{2} s_1 + (u+1 - 2u^2) F_{s_1}.
\]

We put $g_i = -2F_{s_1}$ and

\[f_i =: f_{s_1}(u) = \frac{u-1}{2} + \frac{u+1}{2} s_1 - \frac{u+1}{2} g_i.
\]

We note that $f_{s_1}(u)$ depends upon $u$, while $g_{s_1}$ does not.

**Proposition 5.2:**

1. $g_1^2 = 0$, $s_1g_1 = -g_1$, $g_1s_1 = g_1$
2. $f_1^2 = u_1 + (u-1)f_1$, $g_if_1 = f_1g_1 = (u-1)g_1$, $g_1f_1g_1 = 0$, $f_1g_1f_1 = -ug_1$
3. $g_1g_j = g_1s_j = s_jg_1 = s_1g_1 - g_1s_1 = s_1s_j$, for $i \neq j$. 

(4) \( f_i f_j f_i \ldots = f_j f_i f_j \ldots \), for \( m_{ij} \in \{2, 4, 6\}, i \neq j \) and 
\[ f_i f_j f_i = f_j f_i f_j = u(f_j f_i) \quad \text{if } i \neq j. \]

Proof: (1) It is clear from the fact that \( g_i = -2F_{s_1} \) and \( s_1 F_{s_1} = -F_{s_1}, \)
\( F_{s_1} s_1 = F_{s_1}, F_{s_1}^2 = 0. \)

(2) The matrix which represents \( f_i \) on grad(E) with respect to the canonical
basis (the basis elements being properly ordered), is
\[ \begin{bmatrix}
-I_{d_1} & 0 \\
0 & u I_{d_1}
\end{bmatrix} \]
and the matrix which represents \( g_i \) is
\[ \begin{bmatrix}
0 & E_{d_1} \\
0 & 0
\end{bmatrix} \]. Therefore, \( f_i^2 \) is
represented by
\[ \begin{bmatrix}
I_{d_1} & 0 \\
0 & u^2 I_{d_1}
\end{bmatrix} \]
and the element \( u - 1 + (u - 1)f_i \) is also represented by the same matrix, so \( f_i^2 = u - 1 + (u - 1)f_i \). The element \( g_i f_i + f_i g_i \) is represented by the matrix
\[ \begin{bmatrix}
0 & E_{d_1} \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
-I_{d_1} & 0 \\
0 & u I_{d_1}
\end{bmatrix} + \begin{bmatrix}
0 & E_{d_1} \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & (u - 1) E_{d_1} \\
0 & 0
\end{bmatrix}, \]
so \( g_i f_i + f_i g_i = (u - 1)g_i \),
and therefore \( g_i f_i g_i = 0 \) since \( g_i^2 = 0 \), while \( f_i g_i f_i = (u - 1)f_i g_i = f_i^2 g_i = (u - 1)f_i g_i - (u - 1)f_i g_i = (u - 1)g_i - (u - 1 + (u - 1)f_i) g_i = -u g_i. \)

(3) We write \( W = W_1 \cup W_2 \cup W_3 \cup W_4 \) where...
Let $d_i$ be the cardinality of $W_i$, $i = 1, 2, 3, 4$, and we order the elements of the canonical basis of $\text{grad}(\mathcal{E})$ according to the decomposition of $W$ above.

The matrices which represent $s_i$, $g_i$ with respect to this basis are respectively

$$
\begin{bmatrix}
-I_{d_1} & 0 & A^{(1)} & C^{(1)} \\
0 & -I_{d_2} & B^{(1)} & D^{(1)} \\
0 & 0 & I_{d_3} & 0 \\
0 & 0 & 0 & I_{d_4}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & A^{(1)} & C^{(1)} \\
0 & 0 & B^{(1)} & D^{(1)} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

where $A^{(1)}$, $B^{(1)}$, $C^{(1)}$, $D^{(1)}$ are matrices of size $d_1 \times d_3$, $d_2 \times d_3$, $d_1 \times d_4$, $d_2 \times d_4$ respectively.

Therefore the element $s_1 - g_1$ is represented by the matrix

$$
\begin{bmatrix}
-I_{d_1} \\
-I_{d_2} \\
I_{d_3} \\
I_{d_4}
\end{bmatrix}
$$

The matrices which represent $s_j$, $g_j$ are respectively

$$
\begin{bmatrix}
-I_{d_1} & A^{(j)} & 0 & C^{(j)} \\
0 & I_{d_2} & 0 & 0 \\
0 & B^{(j)} & -I_{d_3} & D^{(j)} \\
0 & 0 & 0 & I_{d_4}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & A^{(j)} & 0 & C^{(j)} \\
0 & 0 & B^{(j)} & D^{(j)} \\
0 & 0 & 0 & 0
\end{bmatrix}
$$
where $A(j)$, $B(j)$, $C(j)$, $D(j)$ are matrices of size $d_1 \times d_2$, $d_3 \times d_2$, $d_1 \times d_4$, $d_3 \times d_4$ respectively.

Therefore the element $s_j g_j$ is represented by the matrix

$$
\begin{bmatrix}
-I_{d_1} & I_{d_2} & -I_{d_3} & I_{d_4} \\
 & -I_{d_1} & I_{d_2} & -I_{d_3} \\
 & & -I_{d_1} & I_{d_2} \\
 & & & -I_{d_1}
\end{bmatrix}
$$

Hence the elements $s_i g_i$ and $s_j g_j$, $i \neq j$ commute, i.e.

$$(s_i g_i) (s_j g_j) = (s_j g_j) (s_i g_i),$$

so

$$s_i s_j - s_i g_j - g_i s_j + g_i g_j = s_j s_i - s_j g_i - g_j s_i + g_j g_i,$$

or

$$g_i g_j = g_j g_i = s_j s_i - s_i s_j.$$  

(4) Let $i \neq j$. The elements $f_i$, $f_j$ are represented by the matrices

$$
\begin{bmatrix}
-I_{d_1} & -I_{d_2} & uI_{d_3} & uI_{d_4} \\
 & -I_{d_1} & -I_{d_2} & uI_{d_3} \\
 & & -I_{d_1} & -I_{d_2} \\
 & & & -I_{d_1}
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
-I_{d_1} & uI_{d_2} & -I_{d_3} & uI_{d_4} \\
 & -I_{d_1} & uI_{d_2} & -I_{d_3} \\
 & & -I_{d_1} & uI_{d_2} \\
 & & & -I_{d_1}
\end{bmatrix}
$$

Hence $f_i f_j = f_j f_i$ for $i \neq j$ and $f_i f_j f_i f_j \ldots = f_j f_i f_j \ldots$ for $m_{ij} \in \{4, 6\}$,

$$m_{ij} + m_{ij} + m_{ij} +$$

while $f_i f_j f_i = f_j f_i f_j = f_j f_j f_i = f_i f_j f_i = (u1 + (u-1)f_i)f_j - (u1 + (u-1)f_j)f_i = u(f_j - f_i)$, $i \neq j$. 


We next provide an alternative proof of Theorem 5.1 based on some conjectures made by R.W. Carter.

This proof illuminates the procedure described in §1.4, for the determination of Lusztig's isomorphism. It has an interesting connection with the orthogonality relations for group characters, and it provides information which is going to be used in the next chapter.

We recall (see definition 1.3.0) the definition of a W-graph over $\mathbb{Z}$, for a Coxeter group W.

**Definition 5.0:** We say that a W-graph is even, if there is a map $\text{sgn} : X \to \{-1, 1\}$ such that $\mu(y,x)\text{sgn}(x)\text{sgn}(y) = -\mu(y,x)$ for any distinct $x, y \in X$.

Let $W$ be a finite crystallographic Coxeter group, and let $H$ be the generic Hecke algebra over the field $\mathbb{Q}(u^\frac{1}{2})$. We shall make use of the following:

**Theorem 5.3:**
1. Every irreducible $H$-module is afforded by a W-graph over $\mathbb{Z}$.
2. An irreducible $H$-module is afforded by an even W-graph over $\mathbb{Z}$, if and only if it is not exceptional.

(For the definition of an exceptional representation and also for a proof of this theorem, see [g]).

We now consider the graded module $\text{grad}(E)$ over $\mathbb{Q}(u^\frac{1}{2})$, associated to a finite crystallographic Coxeter group (i.e. a Weyl group), and we view it as a left $H$-module. We know that $\text{grad}(E)$ affords the left regular representation of $H$ and as it is also semisimple, it has a decomposition into a direct sum of left absolutely irreducible $H$-submodules ($H$ splits over $\mathbb{Q}(u^\frac{1}{2})$).

Let $\text{grad}(E) = V_{11} \oplus V_{1d_1} \oplus V_{21} \oplus V_{2d_1} \oplus \cdots \oplus V_{td} \oplus V_{td}$ be one such decomposition, where each $V_{ir}$ has dimension $d_i$, $1 \leq i \leq t$, $1 \leq r \leq d_i$ and $V_{ir} = V_{js}$ if and only if $i = j$. 
Let $X = \{V_1, \ldots, V_t\}$ be a full set of irreducible constituents. For the sake of simplicity we relabel the members of $X$ by putting $V_i = V_1 \quad \forall 1 \leq i \leq t$.

According to Theorem 5.3(1), for each $V_r \in X$, $1 \leq r \leq t$ we can choose a basis $B_r = \{v_1, \ldots, \mu v_k\}_{k=1}^{r_1}$, such that if $v_k \in B_r$, $1 \leq k \leq r_1$, and $T_s$ is a generator of $H$, then

$T_s v_k = -v_k \quad \text{if } s \in I(v_k)$

$\text{uv}_k + u^T \sum_{v \in B_r} u(v_1,v_k)v_1 \quad \text{if } s \notin I(v_k)$

with $u(v_1,v_k) \in \mathbb{Z}$ and $Y$ being the set of edges.

We fix an $s \in S$ and suppose that among the basis elements of $B_r$, $d_{r_1}$ elements $v_k$ are such that $s \in I(v_k)$, $1 \leq k \leq d_{r_1}$ and that $d_{r_2}$ elements $v_k$ are such that $s \notin I(v_k)$. Then $d_{r_1} + d_{r_2} = d_r$ and we can arrange the elements of $B_r$, in such a way so that the matrix which represents $T_s$, $s$ fixed, with respect to $B_r$, has the form

$A_r(T_s) = \begin{bmatrix} -I_{d_{r_1}} & u^T E_s \\ 0 & uI_{d_{r_2}} \end{bmatrix}$

 whose $I_{d_{r_1}}$, $I_{d_{r_2}}$ are the identity matrices of size $d_{r_1}$, $d_{r_2}$ respectively, $0$ is the zero matrix of size $d_{r_2} \times d_{r_1}$, and $E_s$ is a $d_{r_1} \times d_{r_2}$ matrix whose entries are integers.

Let $B_r = \{v_1, \ldots, v_{d_{r_1}}, v_{d_{r_1} + 1}, \ldots, v_{d_r}\}$, be this arrangement of the basis elements of $B_r$, with $s \in I(v_k)$, $1 \leq k \leq d_{r_1} \quad \text{and } s \notin I(v_k)$, $d_{r_1} < k \leq d_r$.

With respect to this arrangement of the basis elements of $B_r$, the matrix which represents $T_s$, $s' \neq s$ is not necessarily of the same form, but its entries
still involve only polynomials in \( \mathbb{Z}[u^\frac{1}{d}] \). By specializing \( u^\frac{1}{d} = 1 \) we obtain the matrices which represent the generators of the group \( W \), and therefore the matrices which represent every \( w \in W \) with respect to \( B_r \).

Let \( s \) fixed as above, and let \( (r_{ij}^{(r)}(w)), i, j \in \{1, \ldots, d_r\}, 1 \leq r \leq t \) be the matrix which represents \( w \) on \( \mathfrak{g}_r \) with respect to \( B_r \).

Let \( v_k \in B_r, 1 \leq k \leq d_r \) and let \( wv_k = \sum_{i=1}^{d_r} f_{ik}^{(r)}(w)v_i, f_{ik}^{(r)}(w) \in \mathbb{Z} \).

Suppose that \( \phi(T_s) = \sum_{w \in W} c_w w \) is the image of \( T_s \) under the Lusztig isomorphism \( \phi, c_w \in \mathbb{Q}(u^\frac{1}{d}) \). Then \( \phi(T_s)v_k = \sum_{w \in W} c_w (\sum_{i=1}^{d_r} f_{ik}^{(r)}(w)v_i) = T_s v_k \), since \( T_s \) and \( \phi(T_s) \) act on the same way on the graded module.

Therefore, by comparing coefficients of the basis elements \( v_i \) on both sides of the equation above, we obtain \( d_r \) equations in the unknowns \( c_w, w \in W \) of the form

\[
\lambda \sum_{w \in W} f_{ik}^{(r)}(w)c_w = \lambda, \quad \lambda \in \{-1, 0, u, ou^\frac{1}{d}, \theta \in \mathbb{Z}\}.
\]

Hence from the constituent \( V_r \) of \( X, 1 \leq r \leq t \) we obtain in this way \( d_r^2 \) equations in the unknowns \( c_w, w \in W \).

We now recall the form of the matrix \( A_r(T_s) \) which represents \( T_s \) on \( V_r \). All diagonal positions in this matrix are \( u \) or \( -1 \). Also \( u, -1 \) do not occur except on the diagonal. We also emphasize the fact that every position of this matrix, gives rise to a certain equation in the unknowns \( c_w, w \in W \) in a way we have described above.

The striking thing about these equations is that some of them behave better than the others.

The following definition and conjectures appearing in Proposition 5.4 are due to R.W. Carter.
Definition 5.1: An equation \( \sum_{W \in W} f_{ik}(w)c_w = \lambda \cdot \mathbf{1} \cdot k \in \{1, \ldots, d\} \) is called amenable with respect to \( s \), if the form
\[
\sum_{W \in W} f_{ik}(w)c_w
\]
is a linear combination of \( c_1, c_s, c_w + c_{sw}, c_w - c_{ws} \) for all \( w \in W \) with \( w \neq 1, s \).

Proposition 5.4: (1) Every position \( -1 \) on the diagonal gives rise to an amenable equation.
(2) Every position which occurs in the same column as a \( -1 \) on the diagonal gives rise to an amenable equation.
(3) Every position \( u \) on the diagonal gives rise to an amenable equation.
(4) Every position which occurs in the same column and row as a \( u \) on the diagonal gives rise to an amenable equation.

Proof: In order to prove (1) and (2) we consider an element \( v_k \in B_r \) such that \( s \in I(v_k) \). Then \( T_{svk} = -v_k \cdot 1 \leq k \leq d \). We write
\[
\theta(T_{svk}) = c_1 \cdot 1 + c_s \cdot s + \sum_{W \in W} c_w + \sum_{W \in W} c_{sw} - c_{ws},
\]
where \( W \) is the unknowns in \( W \). If \( w \in W \) with \( w \neq 1 \), then
\[
\theta(T_{svk})v_k = c_1 v_k + c_s (s,v_k) + c_{w,v_k} + \sum_{W \in W} c_{ws(w,v_k)}.
\]
Thus,
\[
\theta(T_{svk})v_k = c_1 v_k + c_s (s,v_k) + c_{w,v_k} + \sum_{W \in W} c_{ws(w,v_k)}.
\]
Therefore, by comparing coefficients on both sides of the equation
\[ T_s v_k = \phi(T_s) v_k = -v_k, \quad 1 \leq k \leq d_{r_1}, \]
we obtain
\[ c_1 - c_s + \sum_{w \in W, w > w} f(r)(w)(c_w - c_{ws}) = -1 \quad \text{and} \]
\[ \sum_{w \in W, w > w} f(r)(w)(c_w - c_{ws}) = 0 \quad \text{if } 1 \neq k, \quad 1 \leq i \leq d_r. \]

Both these equations are amenable with respect to \( s \).

In order to prove (3) and (4) we consider an element \( v_k \in B_r \) with \( d_{r_1} < k \leq d_r \), i.e., \( s \notin I(v_k) \). We now write
\[
\phi(T_s) = c_1 1 + c_s s + \sum_{w \in W, w > w} c_{sw} + \sum_{w \in W, w \neq s} c_w
\]
\[
= c_1 1 + c_s s + \sum_{w \in W, w > w} c_{sw} + \sum_{w \in W, w \neq s} c_w
\]

So \( \phi(T_s) v_k = c_1 v_k + c_s (s v_k) + \sum_{w \in W, w > w} c_{sw} (w v_k) + \sum_{w \in W, w \neq s} c_w (sw v_k) \).

We recall that when \( s \notin I(v_k) \), then
\[ T_s v_k = u v_k + \sum_{v_i \in B_r} \mu(v_i, v_k) v_i \]
\[ \sum_{s \in I(v_k)} (v_i, v_k) \in Y \]
So \( s^r v_k = v_k + \sum_{i=1}^{d_r} \mu(v_i, v_k)v_i \) and \( u(v_i, v_k) \in \mathbb{Z} \).

We put \( u(v_i, v_k) = u_{ik} \) so \( s^r v_k = v_k + \sum_{i=1}^{d_r} u_{ik} v_i \).

Let \( w \in \mathbb{W} \) with \( sw > w, w' \neq 1 \) and let \( w \cdot v_k = \sum_{i=1}^{d_r} f(r_i(w))v_i + \sum_{i=d_r+1}^{d_r} f(r_i(w))v_i \).

Then \( (sw)v_k = -\sum_{i=1}^{d_r} f(r_i(w))v_i + \sum_{i=d_r+1}^{d_r} f(r_i(w))v_i \).

Then \( \phi(T_s)v_k = c_1 v_k + c_0(v_k + \sum_{i=1}^{d_r} u_{ik} v_i) + \sum_{i=d_r+1}^{d_r} f(r_i(w))v_i \).

By comparing coefficients on both sides of the equation

\[ T_s v_k = \phi(T_s)v_k, \quad d_{r_1} < k \leq d_r \]

we obtain:

\[ c_1 + c_0 + \sum_{w \in \mathbb{W}, sw > w, w' \neq 1} f(r_{i_k(w)}(c_w + c_{sw}) = u \]

and

\[ \sum_{w \in \mathbb{W}, sw > w, w' \neq 1} f(r_{i_k(w)}(c_w + c_{sw}) = 0 \text{ if } 1 \neq k, d_{r_1} < 1 \leq d_r \]

Both these equations are amenable with respect to \( s \). Our proposition is now proved.
Remark 5.4: From the proof of (3) and (4) of Proposition 5.4 we see that if \( v_i, v_k \) are such that \( \{v_i, v_k\} \in Y \), \( Y \) being the set of edges, then by comparing the coefficient of the element \( v_i \) on both sides of \( T_s \cdot v_k = \theta(T_s) \cdot v_k \), we obtain the equation:

\[ c_s \cdot \nu_{ik} = \text{integral linear combination of } c_w \]

\( \nu_{ik} \neq 0 \), while if \( \{v_i, v_k\} \notin Y \), \( 1 \leq i \leq d \), we obtain the equation:

\[ \text{Integral linear combination of } c_w = 0. \]

We shall show that both types of these equations in the unknowns \( c_w \) are not amenable with respect to \( s \).

Proposition 5.5: The total number of equations of type (1), (2), (3), (4) of Proposition 5.4, is equal to \( |C_w(s)| \cdot |W| - |C_w(s)| \), and this number is the same as the total number of linearly independent combinations of \( c_1, c_s, c_w + c_{sw}, c_w - c_{sw} \), where \( s \) is fixed and \( w \in W, w \neq 1. \). \( C_w(s) \) denotes the centralizer of \( s \) in \( W \).

Proof: The number of entries of the matrix \( A_s(T_s) \) which represents \( T_s \) on the constituent \( V_r \) with respect to the ordered basis \( B_r \), which give rise to amenable equations of type (1), (2), (3), (4) is \( d_1^2 + d_2^2 + d_1 d_2 \). Thus, the total number of equations of these types obtained from all the inequivalent irreducible constituents is \( \sum_{r=1}^{t} d_r^2 + d_1^2 + d_1 d_2 \). The trace of the matrix \( A_s(T_s) \) is:

\[ \text{Trace } A_s(T_s) = u d_1. \]

By specializing \( u = 1 \) we obtain the trace of the matrix which represents \( s \) on \( V_r \) with respect to the ordered basis \( B_r \), \( 1 \leq r \leq t \). Let \( C_w(s) \) be the centralizer of \( s \) in \( W \), \( K_i \) the conjugacy class of \( s \), \( x_i^{(r)} \) be the irreducible character which corresponds to the constituent \( V_r \), defined by \( x_i^{(r)}(s) = x_i^{(r)}(x), x \in K_i \). The second orthogonality relation gives:

\[ |C_w(s)| = \sum_{r=1}^{t} \overline{x_i^{(r)}} x_i^{(r)}, \]

where \( \overline{x_i^{(r)}} \) is the complex conjugate of \( x_i^{(r)} \). Here we have \( \overline{x_i^{(r)}} = x_i^{(r)} \) because \( s^2 = 1 \).
\begin{align*}
\text{Hence } |C_w(s)| &= t \sum_{r=1}^{t} (x(r))^{2} = t \sum_{r=1}^{t} (d_{r_2} - d_{r_1})^{2} = t \sum_{r=1}^{t} d_{r_2}^{2} - 2 \sum_{r=1}^{t} d_{r_1} d_{r_2}.
\
\text{So, } -|C_w(s)| &= 2 \sum_{r=1}^{t} d_{r_1} d_{r_2} - \sum_{r=1}^{t} d_{r_2}^{2} - \sum_{r=1}^{t} d_{r_1}^{2}.
\
\text{So, } |W| - |C_w(s)| &= \sum_{r=1}^{t} d_{r_1}^{2} + 2 \sum_{r=1}^{t} d_{r_1} d_{r_2} - \sum_{r=1}^{t} d_{r_2}^{2} - \sum_{r=1}^{t} d_{r_1}^{2} = 4 \sum_{r=1}^{t} d_{r_1} d_{r_2}.
\
\text{Hence } |C_w(s)| + \frac{3}{4} (|W| - |C_w(s)|) &= \sum_{r=1}^{t} d_{r_1}^{2} + \sum_{r=1}^{t} d_{r_2}^{2} + \sum_{r=1}^{t} d_{r_1} d_{r_2}.
\end{align*}

Next we calculate the number of linearly independent combinations of $c_1, c_s, c_w + c_{sw}, c_w - c_{ws}, w \in W, w \neq 1,s$.

If $w \in C_w(s), w \neq 1,s$, then $w$ gives rise to two such combinations, namely $c_w + c_{sw}, c_w - c_{ws}$. For $sw = ws$ and the element $sw$ gives rise to the combinations $c_{sw} + c_w, c_{sw} - c_{ws} = c_{ws} - c_w$, and these are linearly dependent on those already obtained. Thus, for every $w \in C_w(s), w \neq 1,s$, the pair $(w, sw)$ contributes to two such linearly independent combinations, and therefore the whole of $C_w(s), w \neq 1,s$, contributes to $|C_w(s)|$ such linearly independent combinations (counting $c_1, c_s$ as well).

Now, if $w \in W, w \notin C_w(s)$, the quadruple $(w, sw, ws, sws)$ contributes to three such linearly independent combinations, namely $c_w + c_{sw}, c_w - c_{ws}, c_{ws} + c_{sws}$. For, the combination $c_{sw} - c_{sws} = (c_w + c_{sw}) - (c_w - c_{ws}) - (c_{ws} + c_{sws})$ and this is a linear combination on those already obtained. Therefore, the total number of linearly independent combinations of $c_1, c_s, c_w + c_{sw}, c_w - c_{ws}, c_{sw}, c_{sws}, w \in W, w \neq 1,s$, is exactly $|C_w(s)| + \frac{3}{4} (|W| - |C_w(s)|)$, and our proposition is now proved.

We recall that these equations of type (1), (2), (3), (4) are all linearly independent as they are obtained from inequivalent irreducible representations.
of the group \( W \) (see procedure in §1.4 for the determination of Lusztig's isomorphism \( \phi \)). Hence

Corollary 5.5: Both types of equations in the \( c_w, w \in W \), obtained by the inequivalent irreducible representations of \( W \), mentioned in the Remark 5.4, are not amenable with respect to \( s \). For, otherwise we would have a system of \( N \) say linearly independent equations in \( M \) unknowns the \( c_1, c_s, c_w + c_{sw}, c_w - c_{ws}, w \neq 1, s, \) with \( N > M \). The matrix of this system has rank at most \( M \). So the rank is exactly \( M \) and therefore the remaining \( N-M \) amenable equations with respect to \( s \), would be linearly dependent on the previous \( M \). This is a contradiction.

For example in case \( W = S_4, s = s_1 \), the first 0 in the third column of the first \( 3 \times 3 \) matrix which represents \( T_{s_1} \), is the right-hand side of the following equation:

\[
\begin{align*}
c_{s_1}s_2 + c_{s_1}s_2s_1 + c_{s_1}s_3s_2 - c_{s_1}s_2s_3 - c_{s_1}s_3s_2s_1 - c_{s_1}s_2s_1s_3 - c_{s_2}s_1s_3s_2
- c_{s_1}s_2s_3s_2 - c_{s_1}s_2s_3s_2s_1 - c_{s_1}s_2s_1s_3s_2 - c_{s_2}s_3s_2s_1s_2 - c_{s_1}s_2s_1s_3s_2s_1
\end{align*}
\]

This equation is not amenable with respect to \( s_1 \).

Also, the \( u^1 \) appearing in this matrix is the right-hand side of the equation:

\[
\begin{align*}
c_{s_1} - c_{s_1}s_2 + c_{s_2}s_1 + c_{s_2}s_1 - c_{s_3}s_3 + c_{s_3}s_3 + c_{s_2}s_1s_3
- c_{s_1}s_3s_2 + c_{s_2}s_3s_2 + c_{s_2}s_3s_2s_1 - c_{s_2}s_3s_2s_1s_2
\end{align*}
\]

This is also not amenable with respect to \( s_1 \).

We can now give an alternative proof to Theorem 5.1.

Second proof: Part (i) is now an immediate consequence of Propositions 5.4 and 5.5. For, we put \( A = |C_w(s)| + \frac{3}{4} (|W| - |C_w(s)|) \). Then, the system of \( A \)-amenable equations with respect to \( s \) in the \( A \)-unknowns \( c_1, c_s, c_w + c_{sw}, c_w - c_{ws}, w \neq 1, s \) has a unique solution. Such a solution is:
So this is the only one.

In order to prove part (ii) we consider the set of \(|W|\) equations in the \(|W|\) unknowns \(c_w, w \in W\), obtained from the several inequivalent irreducible representations of \(W\). From these equations we omit those coming from the irreducible constituents which afford the representations of \(H\) \(\sigma_0 : T_0 + u, vs \in S, \) and \(\sigma_s : T_s - 1 vs \in S\). These equations are

\[ \sum_{w \in W} c_w = u \text{ and } \sum_{w \in W} (-1)^{g(w)} c_w = -1 \text{ respectively.} \]

So we are left with \(|W| - 2\) linearly independent equations whose type is one of the following (see Proposition 5.4 and Remark 5.4)

\[ c_1 + c_s + \text{integral linear combination of } c_w = u \quad \text{if } 1 \neq s, \]

\[ c_1 - c_s + \text{integral linear combination of } c_w = -1 \quad \text{if } 1 \neq s, \]

\[ \lambda c_s + \text{integral linear combination of } c_w = \lambda u^1, \lambda \in \mathbb{Z}, \lambda \neq 0 \quad \text{if } 1 \neq s, \]

\[ \text{integral linear combination of } c_w = 0 \quad \text{if } 1 \neq s. \]

In these equations we replace \(c_1, c_s\) by \(\frac{u-1}{2}\) and \(\frac{u+1}{2}\) respectively and we divide the third type of equations by the non-zero integer \(\lambda\). Thus we obtain a system of \(|W| - 2\) linearly independent equations in the \(|W| - 2\) unknowns \(c_w, w \neq 1, s\).

The coefficients of the unknowns \(c_w, w \neq 1, s\) are rationals and the right-hand side of these equations is now either 0 or \((u^1 - 1)^2\). This system has a unique solution in the \(c_w, w \neq 1, s\) and therefore we obtain that each \(c_w, w \neq 1, s\) is a rational multiple of \((u^1 - 1)^2\). Thus
\[ \theta(T_s) = \frac{u^{-1}}{2}u + \frac{u+1}{2}s + (u^\frac{1}{2}-1)^2F_s \] where \( F_s \in \mathbb{Q}W \) and \( F_s \) does not involve \( 1,s \).

Moreover, let \( F_s = \sum_{w \neq 1,s} c_w w, c_w \in \mathbb{Q}. \) Then,

\[ sF_s = s(\sum_{w \neq 1,s} c_w w) = \sum_{w \neq 1,s} c_w w \]

and

\[ F_s' s = (\sum_{w \neq 1,s} c_w w) s = \sum_{w \neq 1,s} c_w w. \]

Thus \((sF_s)^2 = sFs sFs = -F_s^2\) and \((sF_s)^2 = F_s^2\), so \( F_s^2 = 0 \).

**Corollary 5.6:** (i) If \( w \in C_w(s), w \neq 1,s \) then \( c_w = 0 \)

(ii) \( \sum_{w} c_w = 0 \).

**Proof:** (i) is obvious and for (ii) we note that since \( c_w = c_{w s}, w \neq 1,s \)

we have \( \sum_{w} c_w = 0 \).

The equation which is obtained from the one dimensional constituent which

affords the representation \( T_s \to u \forall s \in S \), is

\[ c_1 + c_s + \sum_{w \neq 1,s} c_w = u, \text{ so } 2 \sum_{w \neq 1,s} c_w = 0, \text{ so } \sum_{w \neq 1,s} c_w = 0 \]

**Proposition 5.7:** Let \( W \) be a finite indecomposable Coxeter group not of type \( E_7, E_8, H_3, H_4 \) and let \( H \) be the generic Hecke algebra over the polynomial ring \( \mathbb{Q}[u] \) associated with \( W \). Let \( c = \sum_{w} a_w T_w, a_w \in \mathbb{Q}[u] \) be an element of the centre of \( H \). Then, the image \( \phi(c) \) of \( c \) under the Lusztig isomorphism \( \phi \) belongs to \( \mathbb{Q}[u](W) \).

**Proof:** The case where \( W \) is a Coxeter group of dihedral type has been treated in Chapter 4, so we may assume that \( W \) is a crystallographic Coxeter group.

We consider the graded module \( \text{grad}(E) \) over \( \mathbb{Q}(u^\frac{1}{2}) \) as a left \( H_{\mathbb{Q}(u^\frac{1}{2})} \)-module with action described in §1.4. Then it has a decomposition into a direct sum of left \( H_{\mathbb{Q}(u^\frac{1}{2})} \) irreducible submodules each one occurring with multiplicity equal
to its dimension. Let \( V \) be such an irreducible submodule. Then \( V \) can be afforded by an even \( W \)-graph over \( \mathbb{Z} \) (see Theorem 5.3(2)). In other words, there exists a basis \( X \) of \( V \) such that for any \( x \in X \) and \( s \in W \) we have

\[
T_s x = \begin{cases} 
- x & \text{if } s \in I(x) \\
ux + u^* \sum_{y \in X} \mu(y,x)y & \text{if } s \not\in I(x)
\end{cases}
\]

with \((y,x) \in \mathbb{Z} \) and there is also a map \( \text{sgn} : X \to \{-1,1\} \) such that \( \mu(y,x) \text{sgn}(y)\text{sgn}(x) = - \mu(y,x) \) for any distinct \( x,y \in X \). We shall show that for any \( w \in W \), \( T_{w,x} = \sum_{z \in X} \lambda_z(u)z + \sum_{z \in X} u^* k_z(u)z \), where \( \lambda_z(u), k_z(u) \in \mathbb{Z} \setminus \{u \} \), and there is a map \( \text{sgn} : X \to \{-1,1\} \) such that \( u(y,z) \text{sgn}(y)\text{sgn}(x) = - \mu(y,z) \) for any \( y,z \in X \). We shall show that for any \( w \in W \), \( T_{w,x} = \sum_{z \in X} \lambda_z(u)z + \sum_{z \in X} u^* k_z(u)z \) belong to \( \mathbb{Z}[u] \). As in Lemma 1.3.7 we argue by induction on \( \ell(w) \). When \( \ell(w) = 0 \), \( w = 1 \) and we have nothing to prove. We assume that our assertion holds for all elements \( w' \) with \( \ell(w') < \ell(w) \) and let \( w = sv \) with \( \ell(v) = \ell(w) - 1 \). Then by induction we have \( T_{v,x} = \sum_{z \in X} \lambda_z(u)z + \sum_{z \in X} u^* k_z(u)z \). Therefore \( T_{w,x} = \sum_{z \in X} \lambda_z(u)z + \sum_{z \in X} u^* k_z(u)z \).
Now we note that the coefficient of $y$ in the expression above is a polynomial of the form \( u^\frac{1}{2} f_y(u) \), where \( f_y(u) \in \mathbb{Z}[u] \) and these \( y \) have $\text{sgn}(y) = -\text{sgn}(x)$, so \( \text{sgn}(y) = -\text{sgn}(x) \). While the coefficient of \( y' \) is a polynomial \( g_{y'}(u) \in \mathbb{Z}[u] \) and these \( y' \) have $\text{sgn}(y') = \text{sgn}(x)$. So our induction is now complete.

In particular, the diagonal entries of the matrix which represents any $\mathcal{T}_w$, $w \in \mathcal{W}$ with respect to the basis $X$ are polynomials in $\mathbb{Z}[u]$. Hence if $c = \sum_{w \in \mathcal{W}} a_w \mathcal{T}_w$, $a_w \in \mathbb{Q}[u]$ is an element in the centre of $\mathcal{H}$, then $c$ is represented on the irreducible module $V$ of dimension $d$ with respect to the basis $X$, by a scalar multiple of the identity matrix $I_d$, and therefore this scalar is necessarily a polynomial in $\mathbb{Q}[u]$. Since $c$ and $\phi(c)$ act on the same way on the graded module we conclude that $\phi(c)$ belongs to $\mathbb{Q}[u](\mathcal{W})$.

Remark 5.8: The result above fails if $\mathcal{W}$ is one of the Coxeter groups $E_7$, $E_8$, $H_3$, $H_4$. For instance let $\mathcal{W}$ be the Coxeter group of type $H_3$. A decomposition of the graded module over $\mathbb{Q}(u^{\frac{1}{2}}, 5)$ into a direct sum of irreducible left $\mathcal{H}$-submodules and a $\mathcal{W}$-graph for each one of them is provided in [12] page 496-7. It can be shown that $H_3 = A_5 \times C_2$ (see [5] Ch. 6, page 231, exercise 11), where $A_5$ is the alternating group on 5 symbols. Therefore the centre of $H_3$ is a cyclic group of order 2, say $Z = \{1, a\}$, $a^2 = 1$.

The element $a$ is represented on every irreducible representation of $\mathcal{W}$ by a scalar multiple of the identity matrix, and since $a^2 = 1$ this scalar is either 1 or -1. When we consider the reflection module $V$ (see Ch. 1) which is a faithful module, then this scalar must be -1. So $a$ transforms every positive root into a negative one, and therefore $a = w_0$, the element of maximal length in $\mathcal{W}$, so $Z = \{1, w_0\}$.
There are three fundamental reflections $s_1, s_2, s_3$, and we know that for every $i = 1, 2, 3$, there exists an element $x_i \in W$ such that $w_0 = s_i x_i$ with $e(w_0) = e(s_i x_i) = e(x_i) + 1$. Since $w_0$ commutes with $s_i$ we also have $w_0 = x_i s_i$.

Now $T_0 T_w = T_s_0 T_s x_i = u T x_i + (u-1) T w_0$ and

$T w_0 T s_i = T x_i s_i T s_i = u T x_i + (u-1) T w_0$.

So $T w_0$ commutes with every $T s_i$ $i = 1, 2, 3$, therefore $T w_0$ belongs to the centre of $H$.

Nevertheless there exist irreducible representations of $H$ on which $T w_0$ is represented by $u^{15/2}$ or $-u^{15/2}$ (see [12], page 497).

The following result relates an algebra defined by Gyoja (see [9]) to the Lusztig isomorphism. We denote Gyoja’s algebra over the polynomial ring $Z[u^1]$, associated to a Coxeter group $W$ by $G(W)$, $u^1$ being an indeterminate over $Z$. This is an algebra given by the following presentation. For every generator $s$ of $W$, $G(W)$ has generators $s(0)$ and $s(1)$ subject to the relations $Z[u^1]$

\[ s(0)^2 = s(0) \quad (R) \]

\[ s(0)s'(0) = s'(0)s(0) \]

\[ s(0)s(1) = s(1) \]

\[ s(1)s(0) = 0 \]

together with additional relations given as follows:

Let $\bar{T}_s$ be an element of $G(W)$ defined by $\bar{T}_s = -s(0) + u(1-s(0)) + u^1 s(1)$.

Then, we require that the elements $\bar{T}_s$ satisfy the homogeneous Coxeter relations, i.e., $\bar{T}_s \bar{T}_s \bar{T}_s \cdots = \bar{T}_s \bar{T}_s \bar{T}_s \cdots$, $m_{ij}$ being the order of $s_i s_j$, $i \neq j$

$+ m_{ij} + \cdots + m_{ij} +$
The relations (R) above imply that $s(1)^2 = 0$ and this enables us to show that $\bar{T}_s$ also satisfies the quadratic relation i.e. $\bar{T}_s^2 = u1 + (u-1)\bar{T}_s$.

Thus, we have an algebra homomorphism $\phi_u : H[u^\frac{1}{2}] \to G(W)$ defined by $\phi_u(T_s) = \bar{T}_s$, $s \in S$.

where $H$ is the generic Hecke algebra over $Z[u^\frac{1}{2}]$ associated to $W$.

Let $E$ be Lusztig's graded module over $Z$, with canonical basis $\{e_w : w \in W\}$. Gyoja showed that $E$ can be made into a left (similarly right) $G(W)$ module by defining

\[ s(0)e_w = \begin{cases} e_w & \text{if } sw < w \\ 0 & \text{if } sw > w \end{cases} \quad (R_1) \]

and

\[ s(1)e_w = \begin{cases} \sum \mu(y,w)e_y & \text{if } sw > w \\ \sum e_y, sy < y & \text{if } sw < w. \end{cases} \quad (R_2) \]

The interpretation of $\bar{L}Y \bar{W}$ and $\tilde{\mu}(y,w)$ is given in §1.4. This action gives an action of $\bar{T}_s$ on $E_{Q(u^\frac{1}{2})}$ because we can easily verify that

\[ \bar{T}_s e_w = \begin{cases} -e_w & \text{if } sw < w \\ ue_w + u^\frac{1}{2} \sum \mu(y,w)e_y & \text{if } sw > w \end{cases} \]

Therefore the left and right $G(W)$ action on $E$ induces a left and right $H_{Q(u^\frac{1}{2})}$ action on $E_{Q(u^\frac{1}{2})}$ by defining $T_s e_w : \phi_u(T_s)e_w = \bar{T}_s e_w$.

We know that $E_{Q(u^\frac{1}{2})}$ affords the two-sided regular representation of $H_{Q(u^\frac{1}{2})}$. Hence the map $\phi_u$ is injective and so we can regard $H_{Q(u^\frac{1}{2})}$ as a subalgebra of $G(W)$. Let $\text{End}_Q(E_{Q(u^\frac{1}{2})})$ be the endomorphism of $E$ which commute with the right $H_{Q(u^\frac{1}{2})}$ action. Since $E_{Q(u^\frac{1}{2})}$ affords the two sided regular representation of $H_{Q(u^\frac{1}{2})}$, the left $H_{Q(u^\frac{1}{2})}$ action on $E_{Q(u^\frac{1}{2})}$ gives rise to an
algebra isomorphism \( \alpha : \mathcal{H}_q(u^\frac{1}{2}) \rightarrow \text{End}_Q(\mathcal{E}_q) \).

Moreover Gyoja showed that the left and right \( G(W) \) action on \( \mathcal{E}_q \) commute (see [9] Lemma 2.11).

Let \( b : G(W) \rightarrow \text{End}_Q(\mathcal{E}_q) \) be the algebra homomorphism defined by the left \( G(W) \) action on \( \mathcal{E}_q \). Then the map \( \alpha^{-1}b \) restricted to \( \mathcal{H}_q(u^\frac{1}{2}) \) is the identity map, i.e., \( \alpha^{-1}b|_{\mathcal{H}_q(u^\frac{1}{2})} = 1_{\mathcal{H}_q(u^\frac{1}{2})} \), and hence the map \( b \) is surjective.

Proposition 5.9: Let \( W \) be a finite crystallographic Coxeter group and let \( \phi(T_s) = f_s(u) + u^\frac{1}{2}g_s \), be the image of the generator \( T_s \) of the generic Hecke algebra over \( \mathbb{Q}(u^\frac{1}{2}) \) under the Lusztig isomorphism, where \( f_s(u) = u^\frac{1}{2} - 1 + \frac{u^1}{2}.s - \frac{u^1}{2}.g_s \) and \( g_s \in \mathbb{Q}W \). Then, there exists a surjective homomorphism of algebras \( \theta : G_q(W) \rightarrow \mathbb{Q}W \) such that

\[
\theta(s(0)) = \frac{1}{2}(1 - s + g_s) \quad \text{and} \quad \theta(s(1)) = g_s .
\]

Proof: By specializing \( u^\frac{1}{2} \rightarrow 1 \) we obtain an algebra homomorphism

\[
\phi : \mathbb{Q}W \rightarrow G_q(W) \quad \text{such that} \quad \phi(s(0)) = -s(0) + (1-s(0)) + s(1) .
\]

Therefore the left and right action of the element \( -s(0) + (1-s(0)) + s(1) \) on \( \mathcal{E}_q \) gives rise to a left and right action of the group algebra \( \mathbb{Q}W \) on \( \mathcal{E}_q \). In fact \( \mathcal{E}_q \) affords the two sided regular representation of \( W \). Therefore, the left \( W \) action on \( \mathcal{E}_q \) induces an algebra isomorphism \( \tilde{\alpha} : \mathbb{Q}W \rightarrow \text{End}_Q(\mathcal{E}_q) \) where \( \text{End}_Q(\mathcal{E}_q) \) are the endomorphisms of \( \mathcal{E}_q \) which commute with the right \( W \) action. Let \( \tilde{\delta} \) be the algebra homomorphism induced by the left \( G_q(W) \) action on \( \mathcal{E}_q \). Then \( \tilde{\delta} : G_q(W) \rightarrow \text{End}_Q(\mathcal{E}_q) \) is surjective, since \( \tilde{\alpha}^{-1}\tilde{\delta}|_{\mathbb{Q}W} = 1_{\mathbb{Q}W} \). Hence, under the surjective map \( \tilde{\alpha}^{-1}\tilde{\delta} : G_q(W) \rightarrow \mathbb{Q}W \), every element \( g \) of \( G_q(W) \) maps to an element of \( \mathbb{Q}W \) which is determined by the property that induces the same element of \( \text{End}_Q(\mathcal{E}_q) \) as the element \( g \). Such an element inside \( \mathbb{Q}W \) is unique. We put \( \delta = \tilde{\alpha}^{-1}\tilde{\delta} \).
Now we consider the endomorphisms of $E_q$ induced by the elements $s(0)$ and $s(1)$ (see relations $(R_1)$ and $(R_2)$ above).

Let $\theta(T_s) = f_s(u) + u^1 g_s$, $g_s \in QW$.

$$f_s(u) = \frac{u-1}{2} + \frac{u+1}{2} s - \frac{u+1}{2} g_s = u - 1 - \frac{u+1}{2} (1 - s + g_s).$$

We recall that with respect to a suitable arrangement of the elements of the canonical basis of $E$, the matrices which represent $f_s(u), g_s$ are respectively

$$\begin{pmatrix} -1 & 0 \\ 0 & uI \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & E_s \\ 0 & 0 \end{pmatrix}$$

(see, first proof of Theorem 5.1).

By specializing $u^1 = 1$ we obtain the matrix which represents $s \in W$ and which is

$$\begin{pmatrix} -1 & E_s \\ 0 & I \end{pmatrix}.$$

Now if $sw < w$ then $\frac{1}{2} (1-s+g_s).e_w = \frac{1}{2} e_w - \frac{1}{2} e_w = e_w$, while if $sw > w$, then $\frac{1}{2} (1-s+g_s).e_w = \frac{1}{2} e_w - \frac{1}{2} e_w = 0$.

In other words, the element $\frac{1}{2} (1-s+g_s)$ induces the same endomorphism of $E_q$ as the element $s(0)$ and hence $\theta(s(0)) = \frac{1}{2} (1-s+g_s)$.

We also have that if $sw < w$ then $g_s.e_w = 0 = s(1).e_w$, while if $sw > w$ then $T_s e_w = v e_w + u^1 \sum_{y \in \mathfrak{W}} u(y, w) e_y$ (see left action of $H$ on the graded module, $\sum_{y \in \mathfrak{W}} u(y, w) e_y$ in §1.4).

Hence $g_s e_w = \sum_{y \in \mathfrak{W}} u(y, w) e_y = s(1).e_w$, if $sw > w$.

In other words, the element $g_s$ induces the same endomorphism of $E_q^*$ as the element $s(1)$, and hence $\theta(s(1)) = g_s$. Our proposition is now proved.
CHAPTER 6

6.1 A maximal commutative subalgebra of the generic Hecke algebra.

Let $W$ be a finite crystallographic Coxeter group and let $H$ be the
generic Hecke algebra over $K = \mathbb{Q}(u^\lambda)$, which is a splitting field for $H$. Let $V_1$, $V_2$, \ldots, $V_s$ be a full set of left irreducible $H$-modules with
$K$-dimensions $d_1$, $1 = 1, \ldots, s$ respectively. According to Theorem 5.3(1),
every $V_i$ can be afforded by a $W$-graph over $\mathbb{Z}$. Such a $W$-graph determines
for each $V_i$ a $K$-basis with properties described in the previous chapter,
and therefore we obtain in this way a full set of irreducible matrix
representations for $H$, namely $A_1, A_2, \ldots, A_s$.

Since $H$ is semisimple we obtain a $K$-algebra isomorphism

$$
\pi : H \to \bigoplus_{i=1}^{s} M_{d_i}(K) \quad \text{such that}
$$

$$
\pi(h) = (A_1(h), \ldots, A_s(h)), \quad \forall h \in H.
$$

Inside $H$ we define $M(u) = \{ h \in H \text{ such that } A_i(h) \text{ is a diagonal matrix} \}

\forall i = 1, \ldots, s \}$. Then clearly $M(u)$ is a maximal commutative subalgebra of $H$.

It is clear that the $K$-dimension of $M(u)$ is $\sum_{i=1}^{s} d_i$. It is also
important to emphasize that the definition of $M(u)$ depends on the chosen
$W$-graph.

On the other hand we define inside $H$ a subset

$$
L(u) = \left\{ \sum_{w \in W} c_w T_w \in H \text{ such that } c_w - c_{ws} + uc_{sw} - u c_{sws} = 0 \right\}
$$

for all pairs $(w, s) \in W \times S$ such that $\lambda(sw) = \lambda(w) + 2$.

It is clear by its definition that $L(u)$ does not depend on any $W$-graph.

There seems to be an interesting connection between $M(u)$ and $L(u)$. 
Proposition 6.1: Let \( M(u) \), \( L(u) \) defined as above. Then \( M(u) \subseteq L(u) \).

Proof: Let \( h = \sum_{w \in W} c_w T_w \) be an element of \( M(u) \) and we fix a generator \( s \in S \).

We know that for any \( w \in W \) we have either \( g(sws) = g(w) \) or \( g(sws) = g(w) \pm 2 \).

If \( w \not\in C^u(s) \) and if \( g(sws) = g(w) \) then \( g(sw) \neq g(ws) \) (see Proposition 1.1 (iv)). Therefore we can write

\[
    h = \sum_{w \in W, w \not\in C^u(s)} c_w T_w + \sum_{w \in W, w \in C^u(s)} c_w T_w + \sum_{w \in W, w \not\in C^u(s)} c_w T_w
\]

\[
    g(sws) = g(w) + 2 \quad g(sws) = g(w) - 2
\]

Let \( B_r \) be the basis determined by a \( W \)-graph and adapted to \( V_r \), \( 1 \leq r \leq s \).

Then we can arrange suitably the basis elements (as in Chapter 5), so that the matrix which represents \( T_w, s \) fixed, with respect to \( B_r \) has the form

\[
    \begin{bmatrix}
        -I_{d_1} & u^2E_s \\
        0 & uI_{d_2}
    \end{bmatrix}
\]

where \( I_{d_1} \) and \( I_{d_2} \) are the identity matrices of size \( d_1 \times d_1 \) and \( d_2 \times d_2 \) respectively, \( E_s \) is a \( d_1 \times d_1 \) matrix and \( 0 \) is the \( d_2 \times d_2 \) zero matrix, and \( d_1 + d_2 = d_r \).

Let \( \begin{bmatrix} A_w^r & B_w^r \\ K_w^r & L_w^r \end{bmatrix} \) be the matrix which represents an element \( T_w, g(sws) = g(w) + 2 \) on \( V_r \) with respect
to $B_r$, where $A^r_w$ is a $d_{r_1} \times d_{r_1}$ matrix, $B^r_w$ is a $d_{r_1} \times d_{r_2}$ matrix, $K^r_w$ is a $d_{r_2} \times d_{r_1}$ matrix, and $L^r_w$ is a $d_{r_2} \times d_{r_2}$ matrix.

Then, the matrix which represents $T_{ws}$ is

$$
\begin{bmatrix}
-A^r_w & u^3A^r_wE_s + uB^r_w \\
-K^r_w & u^3K^r_wE_s + uL^r_w
\end{bmatrix}
$$

the matrix which represents $T_{sw}$ is

$$
\begin{bmatrix}
-A^r_w + u^3E_sK^r_w & -B^r_w + u^3E_sL^r_w \\
uK^r_w & uL^r_w
\end{bmatrix}
$$

and the matrix which represents $T_{sws}$ is

$$
\begin{bmatrix}
A^r_w - u^3E_sK^r_w & -u^3A^r_wE_s - uB^r_w + uE_sK^r_wE_s + u^{3/2}E_sL^r_w \\
-K^r_w & uE_sK^r_wE_s - u^2L^r_w
\end{bmatrix}
$$

If $w \in C_{w}(s)$ with $z(ws) > z(w)$ then since $T_{ws} = T_{sw}$, we have that

$$
\begin{bmatrix}
-A^r_w & u^3A^r_wE_s + uB^r_w \\
-K^r_w & u^3K^r_wE_s + uL^r_w
\end{bmatrix} = \begin{bmatrix}
-A^r_w + u^3E_sK^r_w & -B^r_w + u^3E_sL^r_w \\
uK^r_w & uL^r_w
\end{bmatrix}
$$

Therefore, if $w \in C_{w}(s)$, $z(ws) > z(w)$, then $K^r_w = 0$. Hence, the matrix which represents $h$ on $V_r$ with respect to $B_r$ has the form

$$
\begin{bmatrix}
I \\
\Sigma_{w \in W : z(ws) = z(w) + 2} (c_w - c_{ws} - u_{sws})K^r_w
\end{bmatrix}
$$
Having assumed that $h \in \mathcal{M}(u)$, we must have that

$$
\sum_{w \in W : \varepsilon(wsws) = \varepsilon(w)+2} (c_w c_{ws} + u c_{sw} - uc_{sws}) k_w^r = 0.
$$

As we have mentioned above, for each $w$ with $\varepsilon(wsws) = \varepsilon(w)+2$, $k_w^r$ is a $d_1 r_1 \times d_2 r_2$ matrix. We consider the quantities $c_w c_{ws} + u c_{sw} - uc_{sws}$ as unknowns, $w \notin C_w(s)$, $\varepsilon(wsws) = \varepsilon(w)+2$. From each $V_{s r}$, $1 \leq r \leq s$ we obtain $d_1 r_1 d_2 r_2$ homogeneous equations in these unknowns, hence from the full set of irreducible $H$-modules $\{V_1, \ldots, V_s\}$ we obtain $s d_1 r_1 d_2 r_2$ homogeneous equations in these unknowns.

On the other hand, the number of these unknowns is the same as the number of distinct quadruples $(w, ws, sw, sws)$, $w \notin C_w(s)$, which is equal to $|W| - |C_w(s)|$ . The latter number is equal to $\sum_{r=1}^s d_1 r_1 d_2 r_2$ (see Proposition 5.5).

Let $W^1$ be the subset of $W$ defined by $W^1 = \{w \in W : \varepsilon(wsws) = \varepsilon(w)+2\}$. Let $k_w^r = (f_{i j}^r(w)^T) w \in W^1$, $1 \leq r \leq s$. We shall show that the functions $f_{i j}^r : T_w \mapsto f_{i j}^r(T_w)$, $w \in W^1$ for all $r$, $i, j$ are linearly independent over $K$.

In fact, let $\sum_{i \geq 1, j \geq 1, r \geq 1} a_{i j}^r f_{i j}^r(T_w) = 0$ for all $w \in W^1$. We note that

$$
-f_{i j}^r(T_{ws}) = f_{i j}^r(T_{sw}) - f_{i j}^r(T_{sws}), \quad u f_{i j}^r(T_w) = f_{i j}^r(T_{sws})
$$

for $w \in W^1$. We recall that if $w \in C_w(s)$ then $k_w^r = 0$, while if $w \notin C_w(s)$ then it gives rise to quadruple $(x, xs, sx, sxs)$ with $\varepsilon(sxs) = \varepsilon(x) + 2$ and $x \in (w, ws, sw, sws)$.

Hence if $\sum_{i \geq 1, j \geq 1, r \geq 1} a_{i j}^r f_{i j}^r(T_w) = 0$ for all $w \in W^1$, then $\sum_{i \geq 1, j \geq 1, r \geq 1} a_{i j}^r f_{i j}^r(T_h) = 0$ for all $w \in W$, and so $\sum_{i \geq 1, j \geq 1, r \geq 1} a_{i j}^r f_{i j}^r(h) = 0$ for all $h \in H$.

So, (by 3.41 in [7]) we obtain $a_{i j}^r = 0$, for all $i, j, r$. 

Thus our system of \( \sum_{r=1}^{s} d_r e_r \) homogeneous equations in the same number of unknowns, the \( c_w - c_{ws} + uc_{sw} - uc_{sws} \), \( g(sws) = g(w) + 2 \) has only the trivial solution.

In other words \( c_w - c_{ws} + uc_{sw} - uc_{sws} = 0 \) for all \( w \) with \( g(sws) = g(w) + 2 \).

Our proposition is now proved.

Remark: The argument above applies to any finite Coxeter group, not necessarily crystallographic, provided that every irreducible \( H_K \) module can be afforded by a \( W \)-graph, and \( K \) is a splitting field of \( H \).

Now let \( l \) be the maximum number of linearly independent expressions of the form \( c_w - c_{ws} + uc_{sw} - uc_{sws} \) for all pairs \( (w,s) \in W, s \in S \) such that \( g(sws) = g(w) + 2 \).

In other words \( l \) is the rank of the matrix determined by a certain number of homogeneous equations of the form \( c_w - c_{ws} + uc_{sw} - uc_{sws} = 0 \), for all pairs \( (w,s) \) such that \( g(sws) = g(w) + 2 \), in the \( |W| \) unknowns \( c_w, w \in W \).

Then, the \( K \) dimension of \( L(u) \) is \( |W| - l \geq \sum_{i=1}^{s} d_i \). By specializing \( u^1 \rightarrow 1 \), every \( V_i \) becomes a left irreducible \( QW \) module, and we can similarly define \( M(1) = \{ R \in QW \text{ such that } A_i(R) \text{ is a diagonal matrix } \forall i = 1, \ldots, s \} \), and \( L(1) = \{ \sum_{W \in W} c_w w, c_w \in Q \text{ such that } c_w c_{ws} + c_{sw} - c_{sws} = 0 \text{ for all pairs } (w,s) \in W, s \in S \text{ such that } g(sws) = g(w) + 2 \} \).

An entirely similar argument as in Proposition 6.1 shows (by specializing \( u^1 \rightarrow 1 \)) that \( M(1) \subseteq L(1) \).

Let \( \bar{l} \) be the rank of the matrix determined by a certain number of homogeneous equations of the form \( c_w - c_{ws} + c_{sw} - c_{sws} = 0 \) for all pairs \( (w,s) \) such that \( g(sws) = g(w) + 2 \), in the \(|W| \) unknowns \( c_w, w \in W \). Then the dimension of \( L(1) \) is \( |W| - \bar{l} \geq \sum_{i=1}^{s} d_i \). We next recall that the group algebra \( QW \) is isomorphic as a \( Q \)-algebra with the algebra of \( Q \)-valued functions \( f : W \rightarrow Q \), under convolution product, with the element \( \sum_{W \in W} c_w w \) corresponding to the
function $f$, defined by $f(w) = c_w$, $w \in W, c_w \in \mathbb{Q}$.

If $f, g$ are $\mathbb{Q}$-valued functions on $W$, their convolution product is defined as the function $f \cdot g : W \to \mathbb{Q}$ given by

$$(f \cdot g)(w) = \sum_{x \in W} f(wx^{-1})g(x).$$

**Proposition 6.2:** (1) $L(1)$ is a subalgebra of $\mathbb{Q}W$.

(2) $\dim L(u) \leq \dim L(1)$.

**Proof:** (1) It is enough to show that $L(1)$ is closed under multiplication.

We note that every element of $L(1)$ determines a function $f : W \to \mathbb{Q}$ such that $f(w) - f(ws) + f(sw) - f(sws) = 0$ for all $w \in W, s \in S$.

Let $f, g$ be two such functions. We shall show that the function $h : W \to \mathbb{Q}$, defined by $h(w) = \sum_{x \in W} f(wx^{-1})g(x)$ has also the property $h(w) - h(ws) + h(sw) - h(sws) = 0$

for all $w \in W, s \in S$.

In fact $h(w) - h(ws) + h(sw) - h(sws) =$

$$= \sum_{x \in W} f(wx^{-1})g(x) - \sum_{x \in W} f(wsx^{-1})g(x) + \sum_{x \in W} f(sw^{-1})g(x) - \sum_{x \in W} f(swx^{-1})g(x)$$

$$= \sum_{x \in W} (f(wx^{-1}) + f(sw^{-1}))g(x) - g(xs)) =$$

$$= \sum_{x \in W} (f(wx^{-1}) + f(sw^{-1}))g(x) - g(xs)) + \sum_{x \in W} f(wx^{-1}) + f(sw^{-1}))g(x) - g(xs)$$

$$\leq \sum_{x \in W} f(w^{-1})g(x) - g(xs) + \sum_{x \in W} f(sw^{-1})g(x) - g(xs) +$$

$$\sum_{x \in W} f(w^{-1}s) + f(sw^{-1}s))g(sx) - g(sxs) =$$

$$= \sum_{x \in W} (f(wx^{-1}) - f(wx^{-1}s) + f(sw^{-1}s) - f(sw^{-1}s))(g(x) - g(xs)) = 0.$$
(2) In order to prove (2) we observe that the matrix say $A$ determined by
the homogeneous equations $c_w - c_{ws} + c_{sw} - c_{sws} = 0$ for all pairs $(w, s)$ such
that $\varepsilon(sws) = \varepsilon(w) + 2$ in the $|W|$ unknowns $c_w, w \in W$ has entries 1, -1 or 0.
Let $r$ be the rank of $A$. Then there exists an $r \times r$ minor whose determinant
$D$ is non zero. Let $B$ be the $r \times r$ matrix with determinant $D$.

On the other hand, the matrix say $A$ determined by the homogeneous
equations $c_w - c_{ws} + uc_{sw} - uc_{sws} = 0$ for all pairs $(w, s)$ such that
$\varepsilon(sws) = \varepsilon(w) + 2$, has entries 1, -1, $u$, -$u$. Let $B$ be the $r \times r$ matrix
inside $A$ which under the specialization $u = 1$ specializes to the matrix $B$.
If $D$ is the determinant of $B$, then $D$ is a polynomial in $u$, namely $\phi(u)$.
Moreover $\phi(u) \neq 0$ since $\phi(1) \neq 0$. Therefore if $r$ is the rank of $A$, we have
$r \geq r$. Hence $\dim L(u) = |W| - r \leq |W| - r = \dim L(1)$.

Our proposition is now proved.

In the case where $L(1)$ is a set of commutative elements then $M(1) = L(1)$
and so by Proposition 6.1 and 6.2 (2) we also have $M(u) = L(u)$ and so $M(u)$
does not depend on a chosen $W$ graph.

It is also clear that $M(u) \cong K^\lambda$, ($K$-algebra isomorphism) where
$\lambda = \sum_{i=1}^s d_i$.

Thus $M(u)$ is a semisimple $K$-algebra and the identity $1_{M(u)}$ has a unique
decomposition into a sum of orthogonal primitive idempotents, namely
$1 = \sum_{i=1}^s e_i$, $e_i \in M$. We shall determine this orthogonal idempotent decomposition
in some special cases.

In the cases to follow it turns out that $M(u) = L(u)$.
Let $W$ be the symmetric group $S_4$. A $W$-graph for each irreducible $H$ module is given in
Chapter 5, and the sum of the degrees of the irreducible representations is 10.
The order $|C_w(s_4)|$ of the centralizer of $s_4$ in $W_4 = 1, 2, 3$ is 4 and therefore
the number of all expressions of the form $c_w - c_{ws} + uc_{sw} - uc_{sws} = 0$
$\varepsilon(s_4 w) = \varepsilon(w) + 2$, $i = 1, 2, 3$ is equal to $\sum_{i=1}^4 \frac{3}{|W| - |C_w(s_4)|} = 15$.
There is one non trivial relation between these expressions namely,
Hence there are 14 linearly independent expressions of the form:

\[ c_w - c_{w1} - u c_{s1w} - u c_{s1ws} = \mathbb{E}(s_is_j) = \mathbb{E}(w) + 2, \quad 1 = 1, 2, 3 \quad \text{and so}
\]

the dimension of \( L(u) \), is 10. Hence \( M(u) = L(u) \). A basis for \( L(u) \) is given by the following set of elements:

\[ v_0 = T_1 \]

\[ v_1 = T_{s1} - u T_{s3} + T_{s1s2} - u T_{s1s3} + T_{s2s3} - u T_{s1s2} + T_{s1s3} \]

\[ v_2 = -u T_{s1} + u^2 T_{s2} - u T_{s3} - u T_{s1s2} - u T_{s3s2} + T_{s1s3} + T_{s1s2s3} \]

\[ v_3 = (1+u) T_{s1} - (1+u^2) T_{s2} + (1+u) T_{s1s2} - u T_{s2s3} + T_{s1s2s3} - u T_{s2s3s2} + T_{s1s2s3} \]

\[ v_4 = -(u^2+3) T_{s1} + (u+u^2) T_{s2} + (u+u^2) T_{s1s2} - u T_{s3s2} + T_{s2s3s2} - u T_{s3s2s1} + T_{s1s2s2s1} \]

\[ v_5 = (u^2+u) T_{s2} + (u^2+u^3) T_{s3} - u T_{s1s2} + (u+u^2) T_{s2s3} + T_{s1s2s1} - u T_{s1s2s3} + T_{s1s2s1s3} \]

\[ v_6 = -(u+u^2) T_{s1} + (1+u) T_{s3} - u T_{s2s1} + (1+u) T_{s3s2} - u T_{s1s2s1} + T_{s3s2s1} + T_{s1s3s2s1} \]
\[ v_7 = u^2 T_{s_1} + u^2 T_{s_2} + u^2 T_{s_3} + u T_{s_2 s_1} + u T_{s_2 s_1^2} + u T_{s_2^2 s_1^2} + T_{s_1 s_2 s_3} \]
\[ v_8 = (u^3 + u^3) T_{s_1} - 2 u^2 T_{s_2} + (u^3 + u^3) T_{s_3} + u T_{s_1 s_2} - u^2 T_{s_2 s_1} - u^2 T_{s_2 s_3} + T_{s_1 s_2 s_3} \]
\[ + u T_{s_1 s_2 s_1} + (u - u^2) T_{s_1 s_2} + u T_{s_3 s_1^2 s_1} + u T_{s_1 s_2 s_3} + (1 - u) T_{s_1 s_2 s_3} + T_{s_1 s_2 s_3} \]
\[ v_9 = T_{s_1 s_2 s_1} + u T_{s_2 s_1} + u T_{s_1 s_2} + u^2 T_{s_1 s_3} \]

In order to find this basis, we consider a typical element \( h = \sum_{w \in S_4} c_w T_w \) of \( L(u) \) and with the aid of the relations \( c_w - c_{w s_1} + uc_{s_1 w} - uc_{s_1 w s_1} = 0 \), \( i = 1, 2, 3 \), we express the coefficients \( c_w \) in terms of \( c_w \), where \( w' \) have bigger length than \( w \). We also make use of the fact that \( c_{s_2 s_1 s_3 s_2} = c_{s_1 s_2 s_1 s_3} \) which is a consequence of the relations:

\[ c_{s_2 s_1 s_3 s_2} - c_{s_2 s_1 s_3 s_1} + uc_{s_1 s_2 s_1 s_3} - uc_{s_1 s_2 s_2 s_1} = 0 \]
\[ c_{s_2 s_1 s_3 s_2} - c_{s_1 s_2 s_1 s_3} + uc_{s_1 s_2 s_1 s_3} - uc_{s_1 s_2 s_1 s_3} = 0. \]

Then, we substitute the expressions of the \( c_w \) obtained in this way, in \( h \) to obtain \( h = \sum_{i=0}^9 c_{w_i} v_i \), for certain \( c_w \).

Finally we determine a system of orthogonal primitive idempotents for the Hecke algebra of the group \( S_4 \). This system arises from the decomposition of \( M(u) \), as described above, and consists of the following set of elements:

\[ e_0 = \frac{1}{(u+1)(u^2+u+1)(u^3+u^2+u+1)} \sum_{w \in S_4} T_w \]

\[ e_1^{(1)} = \frac{1}{(u+1)(u^3+u^2+u+1)} (u T_{s_1} + T_{s_2} + T_{s_3} + T_{s_2 s_1} + T_{s_2 s_3} + T_{s_2 s_3}) \]

\[ e_2^{(1)} = \frac{1}{u(u+1)(u^3+u^2+u+1)} (u T_{s_1} + T_{s_2} + T_{s_3} + T_{s_1 s_3} + T_{s_3 s_2} + T_{s_3 s_1} + T_{s_2 s_1}) \]

\[ e_3^{(1)} = \frac{1}{(u+1)(u^3+u^2+u+1)} (u T_{s_1} + T_{s_2} + T_{s_3} + T_{s_3 s_3} + T_{s_2 s_3} + T_{s_3 s_2} + T_{s_1 s_2} + T_{s_1 s_3}) \]
The element $e_0$ is determined by the fact that it is represented by $(1)$ on the one dimensional submodule which affords the representation $T_{s_1}$, $i = 1, 2, 3$, and by the zero matrix on every other irreducible submodule.

Similarly the element $e_9$ is represented by $(-1)$ on the irreducible submodule which affords the representation $T_{s_1}$, $i = 1, 2, 3$, and by the zero matrix on every other irreducible submodule.

The elements $e_j^{(1)}$, $j = 1, 2, 3$ are represented on the first three dimensional irreducible submodule $V_L$ (see Chapter 5 for the definition of $V_L$) by the diagonal matrix which has 1 in the $j$ entry and 0 elsewhere, and by the zero matrix on every other irreducible submodule.

The elements $e_j^{(2)}$, $j = 1, 2, 3$ are determined similarly by their action on the second three dimensional irreducible submodule $V_N$ (see also Chapter 5).

Finally the elements $e_4$, $e_5$ are represented respectively on the two dimensional irreducible submodule $V_M$ by the matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and by the zero matrix on every other irreducible submodule.
6.2 A system of orthogonal primitive idempotents inside the generic Hecke algebra of dihedral type.

In this paragraph we shall make use of the results in Chapter 2. In that Chapter we had established a decomposition of Lusztig's graded module over $Q_n(u^\frac{1}{2})$ associated to the dihedral group $D_{2n}$, into a direct sum of left irreducible $H$-submodules, where $H$ is the generic Hecke algebra of dihedral type over $Q_n(u^\frac{1}{2})$, each one of them being afforded by a $W$-graph.

When $n = 2m+1$ we had obtained a decomposition of $\text{grad}(E)$ as

$$\text{grad}(E) = \bigoplus_{j=1}^{m} M_j \oplus \bigoplus_{j=1}^{m} V_j \oplus \bigoplus_{j=1}^{m-1} \hat{V}_j$$

where $(M_j, V_j, \hat{V}_j, \tilde{V}_j)$ is a full set of left irreducible $H$-modules, and to this decomposition we adapted the basis $B_0 = \{ e_i, u_j, v_j, \hat{u}_j, \hat{v}_j, \tilde{e}_0 \}$ where $M_j = \langle u_j, v_j \rangle$, $\hat{M}_j = \langle \hat{u}_j, \hat{v}_j \rangle$, $j = 1, \ldots, m$.

When $n = 2m$ we had achieved a decomposition of $\text{grad}(E)$ as

$$\text{grad}(E) = \bigoplus_{j=1}^{m} M_j \oplus \bigoplus_{j=1}^{m-1} V_j \oplus \bigoplus_{j=1}^{m-1} \hat{V}_j$$

where $(M_j, V_j, \hat{V}_j, \tilde{V}_j)$ is a full set of left irreducible $H$-submodules, and to this decomposition we had adapted a basis $B_1 = \{ e_i, e_0, \theta_1, \theta_2, u_j, v_j, \hat{u}_j, \hat{v}_j, \tilde{e}_0 \}$ where $V_j = \langle u_j, v_j \rangle$, $\tilde{V}_j = \langle \hat{u}_j, \hat{v}_j \rangle$, $j = 1, \ldots, m-1$.

For the definition of the elements of $B_0$ and $B_1$ see also Chapter 2.

By defining $M(u)$ and $L(u)$ for the case $W = D_{2n}$, and since $Q_n(u^\frac{1}{2})$ is a splitting field of $H$ and every irreducible $H$ submodule appearing in these decompositions above is being afforded by a $W$-graph, we have that $M(u) \leq L(u)$.

Let $h = \sum_{w \in D_{2n}} c_w T_w$ be an element of $L(u)$. Now for each $w \in ((s_2 s_1)^{k-1})$

$k = 1, \ldots, m$ if $n = 2m + 1$ or $k = 1, \ldots, m-1$ if $n = 2m$ we consider the quadruples

$$s_1 w \quad \quad w s_1$$

and for each $w \in ((s_1 s_2)^{k-1})$ $k = 1, \ldots, m$
if \( n = 2m+1 \), or \( k = 1, \ldots, m-1 \) if \( n = 2m \) we consider the quadruples

Each of these \( w \) appears at the bottom of such a quadruple only once and it is related to elements of length bigger than the length of \( w \). Therefore it is clear that the expressions \( c_w - c_{w_1} + uc_{s_1} - uc_{s_2}w \) for \( w \in \{(s_2s_1)^{k-1}\} \)

and \( c_w - c_{w_2} + uc_{s_2}w - uc_{s_2s_2w} \) for \( w \in \{(s_1s_2)^{k-1}\} \) are all linearly independent.

The number of these expressions is \( 2m \) if \( n = 2m+1 \) or \( 2m-2 \) if \( n = 2m \). Thus the dimension of \( L(u) \) is \( 4m+2 - 2m = 2m+2 \) if \( n = 2m+1 \) or \( 4m - (2m-2) = 2m+2 \) if \( n = 2m \). In both cases the dimension of \( L(u) \) is equal to the sum of the degrees of the irreducible representations of \( H \) and hence \( M(u) = L(u) \).

In order to establish a decomposition of \( M(u) \) into a sum of orthogonal primitive idempotents, we need some properties of the polynomials \( S_n(x) \).

We recall that the polynomials \( S_n(x) \) are defined by

\[
S_0(x) = 1, \quad S_1(x) = x, \quad S_{n+1}(x) = xS_n(x) - S_{n-1}(x), \quad \forall n \geq 1.
\]

In Chapter 2 we showed that the numbers \( \rho_j = 2 \cos \frac{j\pi}{m} 1 \leq j \leq m-1 \) are the roots of the polynomial \( S_{m-1}(x) \), while the numbers \( \rho_j = 2 \cos \frac{2j\pi}{2m+1} 1 \leq j \leq m \) are the roots of the polynomial \( S_m(x) + S_{m-1}(x) \).

Lemma 6.3: (Sum formulae):

1. Let \( \rho_k = 2 \cos \frac{2k\pi}{2m+1} 1 \leq k \leq m \). Then \( \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(\rho_k) = \prod_{j=1}^{m} (\rho_k - \rho_j) \).

2. Let \( \rho_k = 2 \cos \frac{k\pi}{m} 1 \leq k \leq m-1 \), and let \( m \) be odd. Then

\[
\sum_{\lambda=1}^{m-1} (\lambda+1)S_\lambda(\rho_k) = \prod_{j=1}^{m-1} (\rho_k - \rho_j). \quad \text{If} \ m \text{ is even, then}
\]

\[
\sum_{\lambda=\text{odd}}^{m-2} (\lambda+1)S_\lambda(\rho_k) = \prod_{j=1}^{m-2} (\rho_k - \rho_j).
\]
\[ m-1 \sum_{\lambda=1}^{m-1} \lambda S_{\lambda-1}(\rho_k) = \prod_{j=1}^{m-1} (\rho_k - \rho_j) \]

\[ \lambda \text{ odd} \]

**Proof:** (1) We know that the numbers \( \rho_k \) are the roots of the polynomial \( S_m(x) + S_{m-1}(x) \).

If \( m \) is even, then
\[
S_m(x) + S_{m-1}(x) = \sum_{k=0}^{m/2} (-1)^k \binom{m-k}{k} x^{m-2k} + \sum_{k=0}^{(m-2)/2} (-1)^k \binom{m-1-k}{k} x^{m-1-2k},
\]
while if \( m \) is odd, then
\[
S_m(x) + S_{m-1}(x) = \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-k}{k} x^{m-2k} + \sum_{k=0}^{(m-1)/2} (-1)^k \binom{m-1-k}{k} x^{m-1-2k}.
\]

Let \( \sigma_\lambda \) be the \( \lambda \) elementary symmetric function on the \( \rho_k \), \( 1 \leq k \leq m \), given by
\[
\sigma_\lambda = \sum_{1 \leq i_1 < i_2 < \ldots < i_\lambda \leq m} \rho_{i_1} \rho_{i_2} \ldots \rho_{i_\lambda}
\]

Then
\[
\sigma_\lambda = \begin{cases} 
(-1)^k \binom{m-k}{k} & \text{if } \lambda = 2k \quad k \geq 1 \\
(-1)^{k+1} \binom{m-1-k}{k} & \text{if } \lambda = 2k + 1 \quad k \geq 0.
\end{cases}
\]

We know that there exists at most one polynomial of degree \( \leq m-1 \), say \( f_m(x) \) which at the \( m \) different points \( \rho_1, \ldots, \rho_m \) assumes given values \[ f_m(\rho_k) = \prod_{j=1}^{m} (\rho_k - \rho_j). \] By Lagrange's interpolation formula there is always one polynomial of degree \( \leq m-1 \) which assumes the given values at these points.

It is the polynomial
\[
f_m(x) = \sum_{k=1}^{m} f_m(\rho_k) \frac{(x-\rho_1) \ldots (x-\rho_{k-1})(x-\rho_{k+1}) \ldots (x-\rho_m)}{(\rho_k - \rho_1) \ldots (\rho_k - \rho_{k-1})(\rho_k - \rho_{k+1}) \ldots (\rho_k - \rho_m)}.
\]

By putting \[ f_m(\rho_k) = \prod_{j=1}^{m} (\rho_k - \rho_j), \] we obtain \[ f_m(x) = \sum_{k=1}^{m} \prod_{j=1}^{m} (x-\rho_j). \] We
shall show that \( f_m(x) = \sum_{\lambda=0}^{m-1} (\lambda+1) S_\lambda(x) \) and then (1) follows. Let

\[ 1 \leq i_1 < i_2 < \ldots < i_r \leq m, \ r \in \{1, \ldots, m-1\}. \]

Then every summand of \( f_m(x) \) contributes to the expression \((-1)^r \rho_{i_1} \rho_{i_2} \ldots \rho_{i_r}\) except from the summands which correspond to \( k = i_1, i_2, \ldots, i_r \).

Thus \( f_m(x) = \max_{m=1}^{m-1} + \sum_{\lambda=1}^{m-1} (-1)^\lambda (m-\lambda) c_\lambda x^{m-\lambda-1}. \) We assume that \( m \) is even (the argument is entirely similar if \( m \) is odd). We substitute the expressions for \( c_\lambda \) in \( f_m(x) \) to obtain

\[
f_m(x) = \sum_{\lambda=1}^{m-1} (-1)^\lambda (m-\lambda) (-1)^k (m-k)^x^{m-\lambda-1} +
\]

\[
+ \sum_{\lambda=1}^{m-1} (-1)^\lambda (m-\lambda) (-1)^{k+1} (m-1-k)^x^{m-\lambda-1} +
\]

\[
+ \sum_{\lambda=2}^{m/2} (-1)^\lambda (m-\lambda) (-1)^{k+1} (m-1-k)^x^{m-\lambda-1} +
\]

\[
+ \sum_{\lambda=2k+1}^{m/2} (-1)^\lambda (m-\lambda) (-1)^{k+1} (m-1-k)^x^{m-\lambda-1} +
\]

\[
+ \sum_{k=1}^{m/2} (-1)^k (m-2k+1)(m-k+1)x^{m-2k+1} +
\]

\[
+ \sum_{k=1}^{m/2} (-1)^k (m-2k+1)(m-k)x^{m-2k} +
\]

Therefore \( f_{m-1}(x) = \max_{k=1}^{m/2} (-1)^{k+1} (m-2k+1)(m-k)x^{m-2k} + \)

\[
+ \sum_{k=1}^{(m-2)/2} (-1)^{k+1} (m-2k+1)(m-k)x^{m-2k-1}.
\]

Thus \( f_m(x) - f_{m-1}(x) = \max_{k=1}^{(m-2)/2} (-1)^{k+1} (m-2k+1)(m-k)x^{m-2k+1} - \)

\[
+ \sum_{k=1}^{(m-2)/2} (-1)^{k+1} (m-2k+1)(m-k)x^{m-2k-1} = mS_{m-1}(x).
\]
Therefore $f_m(x) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x)$.

Thus part (1) is now proved.

(2) The argument is similar. We assume that $m$ is odd, the proof being similar if $m$ is even.

We recall that the numbers $p_j = 2 \cos \frac{j\pi}{m}$, $1 \leq j \leq m-1$ are the zeros of the polynomial $S_{m-1}(x) = \sum_{k=0}^{m-1/2} (-1)^k (m-1-k)x^{m-1-2k}$. We know that there exists at most one polynomial of degree $\leq m-2$, say $H_{m-2}(x)$, which at $m-1$ different points $\rho_1, \ldots, \rho_{m-1}$ assumes given values $H_{m-2}(\rho_k) = \prod_{j=1, j \neq k}^{m-1} (\rho_k - \rho_j)$.

By Lagrange's interpolation formula there is always one polynomial of degree $\leq m-2$ which assumes the given values at these points. It is the polynomial

$$H_{m-2}(x) = \sum_{k=1}^{m-1} H_{m-2}(\rho_k) \frac{(x-\rho_1) \ldots (x-\rho_{k-1})(x-\rho_{k+1}) \ldots (x-\rho_{m-1})}{(p_k - \rho_1) \ldots (p_k - \rho_{k-1})(p_k - p_{k+1}) \ldots (p_k - p_{m-1})} =$$

$$= \sum_{k=1}^{m-1} \prod_{j=1, j \neq k}^{m-1} (x-\rho_j).$$

Let $\sigma_\lambda = \sum_{1 \leq i_1 < i_2 < \ldots < i_{\lambda-1} \leq m-1} \rho_{i_1} \rho_{i_2} \ldots \rho_{i_{\lambda-1}}$ be the $\lambda$ elementary symmetric function on the $p_j$. Then

$$\sigma_\lambda = \left\{ \begin{array}{ll}
0 & \text{if } \lambda = 2k+1 \quad k \geq 0 \\
(-1)^k \frac{m-1-k}{k} & \text{if } \lambda = 2k.
\end{array} \right.$$ 

If $1 \leq i_1 < i_2 < \ldots < i_r \leq m-1$, $r \in \{1, 2, \ldots, m-2\}$, then every summand of $H_{m-2}(x)$ contributes to $(-1)^r \rho_{i_1} \rho_{i_2} \ldots \rho_{i_r}$ except from the summands which correspond to $k = 1, 2, \ldots, i_r$. Thus, $H_{m-2}(x) = (m-1)x^{m-2} + \sum_{\lambda=1}^{m-2} (-1)^{\lambda(m-\lambda-1)} \sigma_\lambda x^{m-\lambda-2} = (m-1)x^{m-2} + \sum_{\lambda=2}^{m-3} (-1)^{\lambda(m-\lambda-1)} \sigma_\lambda x^{m-\lambda-2}$. (by
Theorem is now proved.

\[(x)^{z-w} S(1-w) =
\]

\[= \frac{(1-kz-w)(1-k-1)}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)} i(1-\gamma) z^{-z} w^{-z}(1-w) =
\]

\[= \chi_{Z-w} x(t-2) + \frac{i(kz-w)(i(1-k))}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)}  i(1-\gamma) z^{-z} w^{-z}(1-w) =
\]

\[= \chi_{Z-w} x(t-2) + \frac{i(kz-w)(i(1-k))}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)}  i(1-\gamma) z^{-z} w^{-z}(1-w) =
\]

\[= \chi_{Z-w} x(t-2) + \frac{i(kz-w)(i(1-k))}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)}  i(1-\gamma) z^{-z} w^{-z}(1-w) =
\]

\[= \chi_{Z-w} x(t-2) + \frac{i(kz-w)(i(1-k))}{i(1-k-w)} - \frac{i(kz-w)(i(1-k))}{i(1-k-w)}  i(1-\gamma) z^{-z} w^{-z}(1-w) =
\]

In fact, we have that \((x)^{z-w} H - (x)^{z-w} H = (x)^{z-w} S(1-w)\) and then the following.

We also have that \((x)^{z-w} S(1-w) = (x)^{z-w} H - (x)^{z-w} H\). Hence, we obtain by taking into account that \(w\) is odd and for the odd values of \(y\).
Lemma 6.4: (Orthogonality formulae)

(1) Let \( \rho_k = 2 \cos \frac{2k \pi}{2m+1} \), \( 1 \leq k \leq m \). Then
\[
\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_{\lambda}(\rho_j) + S_{\lambda-1}(\rho_j)) = 0,
\]
for every \( 1 \leq j \leq m, j \neq k \).

(2) Let \( \rho_k = 2 \cos \frac{k \pi}{m} \), \( 1 \leq k \leq m-1 \). Then
\[
\sum_{\lambda=0}^{m-2} S_{m-2-\lambda}(\rho_k)S_{\lambda}(\rho_j) = 0,
\]
for every \( 1 \leq j \leq m-1, j \neq k \).

Proof: (1) We recall that the roots of the polynomial \( S_m(x) + S_{m-1}(x) \) are the numbers \( \rho_j = 2 \cos \frac{2j \pi}{2m+1} \), \( 1 \leq j \leq m \). We fix a \( k, 1 \leq k \leq m \) and we shall show that
\[
\frac{S_m(x) + S_{m-1}(x)}{x-\rho_k} = \sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_{\lambda}(x) + S_{\lambda-1}(x)).
\]
In fact \( (x-\rho_k) \sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_{\lambda}(x) + S_{\lambda-1}(x)) =
\]
\[
(x-\rho_k)S_{m-1}(\rho_k) + (x-\rho_k) \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x)).
\]
Now for every \( \lambda \geq 2 \) we have \( S_{\lambda}(x) = xS_{\lambda-1}(x) - S_{\lambda-2}(x) \), and so
\[
x[S_{\lambda-1}(x) + S_{\lambda-2}(x)] = S_{\lambda}(x) + S_{\lambda-1}(x) + S_{\lambda-2}(x) + S_{\lambda-3}(x). \]
We also have
\[
-\rho_k S_{m-\lambda}(\rho_k) = -S_{m-\lambda+1}(\rho_k) - S_{m-\lambda-1}(\rho_k) \quad \forall \lambda = 2, 3, \ldots, m. \]
Therefore,
\[
\sum_{\lambda=0}^{m-1} S_{m-1-\lambda}(\rho_k)(S_{\lambda}(x) + S_{\lambda-1}(x)) =
\]
\[
(x-\rho_k)S_{m-1}(\rho_k) + \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda}(x) + S_{\lambda-1}(x)) + \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-2}(x) + S_{\lambda-3}(x))
\]
\[
- \sum_{\lambda=2}^{m} S_{m-\lambda+1}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x)) - \sum_{\lambda=2}^{m} S_{m-\lambda-1}(\rho_k)(S_{\lambda-1}(x) + S_{\lambda-2}(x))
\]
\[
= (x-\rho_k)S_{m-1}(\rho_k) + \sum_{\lambda=2}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda}(x) + S_{\lambda-1}(x)) + S_{m}(x) + S_{m-1}(x) +
\]
\[
+ S_{m-2}(\rho_k) + \sum_{\lambda=3}^{m} S_{m-\lambda}(\rho_k)(S_{\lambda-2}(x) + S_{\lambda-3}(x)) - S_{m-1}(\rho_k)(1 + x) -
\]
Now part (1) is proved.
(2) We recall that the numbers \( \rho_j = 2 \cos \frac{j\pi}{m} \) for \( 1 \leq j \leq m-1 \) are the roots of the polynomial \( S_{m-1}(x) \). We fix a \( k, 1 \leq k \leq m-1 \) and we shall show that

\[
S_{m-1}(x) = \sum_{\lambda=0}^{m-2} S_\lambda(x)S_{m-2-\lambda}(\rho_k).
\]

In fact

\[
(x-\rho_k) \sum_{\lambda=0}^{m-2} S_\lambda(x)S_{m-2-\lambda}(\rho_k) = (x-\rho_k) \sum_{\lambda=2}^{m} S_{\lambda-2}(x)S_{m-\lambda}(\rho_k) =
\]

\[
= x \sum_{\lambda=2}^{m} S_{\lambda-2}(x)S_{m-\lambda}(\rho_k) - \rho_k \sum_{\lambda=0}^{m} S_{\lambda-2}(x)S_{m-\lambda}(\rho_k).
\]

Moreover, \( xs_{\lambda-2}(x) = S_{\lambda-1}(x) + S_{\lambda-3}(x) \) \( \forall \lambda = 2, \ldots, m \) and

\[
-\rho_k s_{m-\lambda}(\rho_k) = -s_{m-\lambda+1}(\rho_k) - s_{m-\lambda-1}(\rho_k).
\]

Therefore,

\[
(x-\rho_k) \sum_{\lambda=0}^{m-2} S_\lambda(x)S_{m-2-\lambda}(\rho_k) =
\]

\[
= \sum_{\lambda=2}^{m} s_{m-\lambda}(\rho_k)(s_{\lambda-1}(x) + s_{\lambda-3}(x)) - \sum_{\lambda=2}^{m-2} s_{\lambda-2}(x)(s_{m-\lambda+1}(\rho_k) + s_{m-\lambda-1}(\rho_k)) =
\]

\[
= s_{m-1}(x) + s_{m-3}(x) + \sum_{\lambda=2}^{m-1} s_{m-\lambda}(\rho_k)(s_{\lambda-1}(x) + s_{\lambda-3}(x)) - s_{m-2}(x)s_1(\rho_k) - s_{m-3}(x)(s_2(\rho_k) + 1)
\]

\[
- \sum_{\lambda=2}^{m-2} s_{\lambda-2}(x)(s_{m-\lambda+1}(\rho_k) + s_{m-\lambda-1}(\rho_k)) =
\]

\[
= s_{m-1}(x) + s_{m-3}(x) + \sum_{\lambda=2}^{m-1} s_{m-\lambda}(\rho_k)s_{\lambda-1}(x) + \sum_{\lambda=2}^{m-1} s_{m-\lambda}(\rho_k)s_{\lambda-3}(x) + s_{m-2}(x)s_1(\rho_k) - s_{m-3}(x)s_2(\rho_k) + 1).
\]
- $S_{m-3}(x)S_2(\rho_k) - S_{m-3}(x) - \sum_{\lambda=1}^{m-3} S_{\lambda-1}(x)S_{m-\lambda}(\rho_k) - \sum_{\lambda=2}^{m-1} S_{\lambda-3}(x)S_{m-\lambda}(\rho_k) = \\
- S_{m-1}(x) + \sum_{\lambda=3}^{m-3} S_{m-\lambda}(\rho_k)S_{\lambda-1}(x) + S_2(\rho_k)S_{m-3}(x) + S_1(\rho_k)S_{m-2}(x) + \\
+ \sum_{\lambda=3}^{m-1} S_{m-\lambda}(\rho_k)S_{\lambda-3}(x) - S_{m-2}(x)S_1(\rho_k) - S_{m-3}(x)S_2(\rho_k) - \sum_{\lambda=2}^{m-3} S_{\lambda-1}(x)S_{m-\lambda}(\rho_k) - \\
- S_{m-1}(\rho_k) - \sum_{\lambda=3}^{m-1} S_{\lambda-3}(x)S_{m-\lambda}(\rho_k) = S_{m-1}(x).

Now our lemma is proved.

**Lemma 6.5:** (Multiplication formula)

$$S_i(x)S_j(x) = \sum_{k=0}^{m} S_{j-1+2k}(x) \forall 1 \leq j.$$

**Proof:** We use induction on $i$. If $i = 0$, it is obvious. Assume it is true for all integers $\leq 1$. Then

$$S_i(x)S_j(x) = [xS_{i-1}(x) - S_{i-2}(x)]S_j(x) = xS_{i-1}(x)S_j(x) - S_{i-2}(x)S_j(x) =$$

$$= x \sum_{k=0}^{1-1} S_{j-1+1+2k}(x) - \sum_{k=0}^{1-2} S_{j-1+2+2k}(x).$$

Moreover $xS_{j-1+1+2k}(x) = S_{j-1+2+2k}(x) + S_{j-1+2k}(x)$. Hence

$$S_i(x)S_j(x) = \sum_{k=0}^{1-1} S_{j-1+1+2k}(x) + \sum_{k=0}^{1-2} S_{j-1+2+2k}(x) = \sum_{k=0}^{i} S_{j-1+2k}(x).$$

**Lemma 6.6:**

$$m^m S_{m-\lambda}(x)S_\lambda(x) + S_{\lambda-1}(x) = \sum_{\lambda=0}^{m-1} (\lambda+1)S_\lambda(x).$$

**Proof:** Assume $m$ is odd, the argument being similar if $m$ is even. We have

$$\sum_{\lambda=0}^{m-1} S_{m-\lambda}(x)(S_\lambda(x) + S_{\lambda-1}(x)) =$$

$$= \sum_{\lambda=0}^{m-1} S_\lambda(x)S_{m-\lambda}(x) + \sum_{\lambda=1}^{m-1} S_{\lambda-1}(x)S_{m-1-\lambda}(x) =$$
Now, the number of pairs \((i, j)\), \(1 < j, i + j = m-1, i \in \{0, 1, \ldots, \frac{m-3}{2}\}\) is clearly \(\frac{m-1}{2}\). Each such pair gives the number \(j-i \in \{2, 4, \ldots, m-3, m-1\}\) and therefore, (by means of 6.5), for a given \(\lambda \in \{1, 3, \ldots, m-2\}\), the polynomial \(S_{m-\lambda}(x)\) appears in the product \(S_i(x)S_j(x)\), unless if \(j-i > m-\lambda\). The number of pairs \((i, j)\) such that \(j-i > m-\lambda\) is clearly \(\frac{m-\lambda}{2}\). Hence the number of pairs \((i, j)\), \(1 < j, i + j = m-2, i \in \{0, 1, \ldots, \frac{m-3}{2}\}\), for which the corresponding product \(S_i(x)S_j(x)\) contributes to \(S_{m-\lambda}(x), \lambda \in \{1, 3, \ldots, m-2\}\) is \(\frac{m-\lambda}{2}\).

Similarly, the number of pairs \((i, j)\), \(1 < j, i + j = m-2, i \in \{0, 1, \ldots, \frac{m-3}{2}\}\) for which the corresponding product \(S_i(x)S_j(x)\) contributes to \(S_{m-\lambda}(x), \lambda \in \{2, 4, \ldots, m-1\}\) is \(\frac{m-\lambda+1}{2}\). Hence,

\[
\sum_{\lambda=0}^{m-1} S_\lambda(x)S_{m-1-\lambda}(x) + \sum_{\lambda=1}^{m-1} S_{\lambda-1}(x)S_{m-1-\lambda}(x) = \frac{m-\lambda+1}{2}. 
\]

\[
\sum_{\lambda=0}^{m-1} (m-\lambda)S_{m-\lambda}(x) + \sum_{k=0}^{(m-1)/2} S_{2k} + \sum_{\lambda=2}^{m-1} (m-\lambda+1)S_{m-\lambda}(x) = \sum_{\lambda=0}^{(\lambda+1)} S_{\lambda}(x). 
\]

Lemma 6.7:

\[
\sum_{k=0}^{m-2} \left( \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k}S_{m-\lambda}(x) \right) \left( S_{k-1}(x) + S_k(x) \right) + \sum_{\lambda=2}^{m} (-1)^{\lambda}S_{m-\lambda}(x) = \sum_{1 \leq 0}^{m-1-2l} S_{m-2-2l}(x) \begin{cases} \text{finishes with } S_0(x) \text{ if } m \text{ is even,} \\
\text{or with } 2S_1(x) \text{ if } m \text{ is odd.}
\end{cases}
\]

Proof: For the sake of simplicity we call \(C_{m-2}(x)\) the left hand side of the equality we wish to show.
We also put \( D_r(x) = \sum_{\lambda=r}^{m} (-1)^{\lambda-r} S_{m-\lambda}(x) \) \( \forall r \in (3,4,\ldots,m). \)

Hence \( C_{m-2}(x) \) becomes

\[
C_{m-2}(x) = \sum_{r=3}^{m} D_r(x) (S_{r-2}(x) + S_{r-3}(x)) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + D_3(x)S_1(x) + \sum_{r=4}^{m-1} D_r(x) S_{r-2}(x) + S_{r-3}(x) + \sum_{r=4}^{m-1} D_r(x) S_{r-3}(x) + S_{r-2}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + D_3(x)S_1(x) + \sum_{r=4}^{m-2} D_r(x) S_{r-2}(x) + S_{r-3}(x) + (S_1(x) - 1) S_{m-3}(x) + S_{m-2}(x) + D_4(x) S_1(x) + \\
+ \sum_{r=4}^{m-2} D_{r+1}(x) S_{r-2}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + S_1(x) (D_3(x) + D_4(x)) + \sum_{r=4}^{m-2} S_{r-2}(x) (D_r(x) + D_{r+1}(x)) + (S_1(x) - 1) S_{m-3}(x) + \\
+ S_{m-2}(x) + S_{m-3}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + 2S_1(x) S_{m-3}(x) + S_{m-2}(x) + \sum_{r=4}^{m-2} S_{r-2}(x) S_{m-r}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x).
\]

Assume \( m \) is odd, then

\[
C_{m-2}(x) = D_3(x) + 2S_1(x) S_{m-3}(x) + S_{m-2}(x) + \sum_{r=4}^{m-2} S_{r-2}(x) S_{m-r}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + 2S_1(x) S_{m-3}(x) + S_{m-2}(x) + \sum_{r=1}^{(m-5)/2} S_{r+1}(x) S_{m-3-r}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \\
= D_3(x) + S_{m-2}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) + 2 \sum_{1 \leq j \leq \frac{m-2}{2}} S_1(x) S_{j}(x) \\
1 \leq j \leq \frac{m-2}{2}
\]

\( \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) \)

if \( m \) is even, then

\[
C_{m-2}(x) = D_3(x) + S_{m-2}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) + 2 \sum_{1 \leq j \leq \frac{m-2}{2}} S_1(x) S_{j}(x) + \left[ S_{\frac{m-2}{2}}(x) \right]^2 \\
1 \leq j \leq \frac{m-2}{2}
\]

\( \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) \)
Moreover \( D_3(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = \sum_{\lambda=3}^{m} (-1)^{\lambda-3} S_{m-\lambda}(x) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(x) = S_{m-2}(x). \)

Therefore

\[
C_{m-2}(x) = 2 \sum_{1 < j, \; 1 + j = m-2, \; 1 \in \{0, 1, \ldots, \frac{m-3}{2}\}} S_i(x) S_j(x) \quad \text{if } m \text{ is odd}
\]

or

\[
C_{m-2}(x) = 2 \sum_{1 < j, \; 1 + j = m-2, \; 1 \in \{0, 1, \ldots, \frac{m-4}{2}\}} S_i(x) S_j(x) + \left[ \frac{S_{m-2}(x)}{2} \right]^2 \quad \text{if } m \text{ is even}.
\]

When \( m \) is odd, then the number of pairs \((1, j)\) such that \(1 < j, \; 1 + j = m-2, \; 1 \in \{0, 1, \ldots, \frac{m-3}{2}\}\), is \(\frac{m-1}{2}\).

Each such pair gives the number \(j-1 \in \{1, 3, \ldots, m-4, m-2\}\) and therefore for a given \(\lambda \in \{2, 4, \ldots, m-1\}\), the product \(S_i(x) S_j(x)\) (see Lemma 6.5) contributes to \(S_{m-\lambda}(x)\), unless if \(j-1 > m-\lambda\). Clearly the number of pairs \((1, j)\) such that \(j-1 > m-\lambda\) is \(\frac{\lambda-2}{2}\), and therefore the number of pairs \((1, j)\), \(1 < j, \; 1 + j = m-2, \; 1 \in \{0, 1, \ldots, \frac{m-3}{2}\}\), for which the corresponding product \(S_i(x) S_j(x)\) contributes to \(S_{m-\lambda}(x)\) is \(\frac{m-1}{2} - \frac{\lambda-2}{2} = \frac{m-\lambda+1}{2}, \; \lambda \in \{2, 4, \ldots, m-1\}\).

Therefore, when \( m \) is odd

\[
C_{m-2}(x) = 2 \sum_{\lambda=2}^{m-1} \frac{(m-\lambda+1)}{2} S_{m-\lambda}(x) = \sum_{\lambda=2}^{m-1} (m-\lambda+1) S_{m-\lambda}(x).
\]

When \( m \) is even, a similar argument shows that the number of pairs \((1, j), \; 1 < j, \; 1 + j = m-2, \; 1 \in \{0, 1, \ldots, \frac{m-4}{2}\}\), such that the corresponding product \(S_i(x) S_j(x)\) contributes to \(S_{m-\lambda}(x), \; \lambda \in \{2, 4, \ldots, m-1\}\) is \(\frac{m-2}{2} - \frac{\lambda-2}{2} = \frac{m-\lambda}{2}\). In this case

\[
C_{m-2}(x) = \sum_{\lambda=2}^{m-2} (m-\lambda) S_{m-\lambda}(x) + \left[ \frac{S_{m-2}(x)}{2} \right]^2 = \sum_{\lambda=2}^{m-2} \frac{(m-\lambda)}{2} S_{m-\lambda}(x) + \sum_{k=0}^{m/2} S_{2k}(x) = \sum_{\lambda=2}^{m} (m-\lambda+1) S_{m-\lambda}(x).
\]
We are going to apply the results stated in the previous lemmas in order to obtain a system of orthogonal primitive idempotents of the generic Hecke algebra of dihedral type. This system of idempotents is obtained from the decomposition of $M(u)$, which in this case is the same as $L(u)$, as we have shown before. It was easier to conjecture this system of idempotents in the case $n$ is odd, $n = 2m+1$, where we get irrational values for the $\rho_j = 2 \cos \frac{2j\pi}{n}$, $1 \leq j \leq m$ rather quickly. The conjecture for the case $n = 2m$ was made after we had worked out the case $n = 101$. We adopt the following notation:

$$Q_n =: Q(2 \cos \frac{2\pi}{n})$$

$\rho_j = 2 \cos \frac{2j\pi}{n}$, $j \in \{1, \ldots, m\}$ if $n = 2m+1$, or $j \in \{1, \ldots, m-1\}$ if $n = 2m$

$$Q_k = uT(s_2 s_1^{k-1} - T(s_1 s_2)^{k-1} + uT(s_2 s_1^{k-1} - T(s_1 s_2)^k)$$

$1 \leq k \leq m$

$$\hat{Q}_k = uT(s_1 s_2^{k-1} - T(s_2 s_1)^{k-1} - uT(s_1 s_2^{k-1} - T(s_2 s_1)^k)$$

$1 \leq k \leq m$.

We shall denote the generic Hecke algebra of dihedral type over the field $\mathbb{Q}_n(u^\frac{1}{2})$, by $H(D_{2n})$. We now state the following.

**Theorem 6.8:** (1) Let $n = 2m+1$. Then the following set of elements inside $H(D_{2n})$ forms a system of orthogonal primitive idempotents whose sum is $1_{H(D_{2n})}$.

$$e_\sigma = \frac{1}{(u+1)(u^{n-1} + \ldots + u+1)} \sum_{w \in D_{2n}} T_w$$

$$e_\sigma = \frac{1}{(u+1)(u^{n-1} + \ldots + u+1)} \sum_{w \in D_{2n}} (-1)^{\varepsilon(w)} u^{n-\varepsilon(w)} T_w$$

and for every $1 \leq j \leq m$, the pair $\{e_1(j), e_2(j)\}$ where

$$e_1(j) = \frac{1}{u^{m-1} \prod_{i=1}^{m} (\rho_j - \rho_i)(u^2 - u\rho_j + 1)} \sum_{k=1}^{m} u^{m-k} S_{m-k}(\rho_j) Q_k$$

and $e_2(j)$ is constructed similarly.
(2) Let $n = 2m$. Then the following set of elements inside $H(D_{2n})$ forms a system of orthogonal primitive idempotents whose sum is $1_{H(D_{2n})}$:

$$e_0 = \frac{1}{(u+1)(u^{-1}+\ldots u+1)} \sum_{w \in D_{2n}} e_{T_w},$$

$$e_o = \frac{1}{(u+1)(u^{-1}+\ldots u+1)} \sum_{w \in D_{2n}} (-1)^{\ell_1(w)} u^{-\ell_2(w)} e_{T_w},$$

where $\ell_1(w)$ is the number of $s_1$'s in a reduced expression of $w \in D_{2n}$,

$$e_1 = \frac{1}{mu^{-1}(u+1)^2} \sum_{w \in D_{2n}} (-1)^{\ell(w)} u^{-\ell_Z(w)} e_{T_w},$$

where $\ell_Z(w)$ is the number of $s_2$'s in a reduced expression of $w \in D_{2n}$, and for every $1 \leq j \leq m-1$, the pair $(e_1(j), e_2(j))$ where

$$e_1(j) = \frac{1}{mu^{-1}(u+1)^2} \prod_{i=1}^{m} (p_j-p_i)(u_i^2-u_i^{j+1}),$$

$$\times \sum_{k=0}^{m-1} u^{-k-1} \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{\lambda}(p_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda-1} S_{\lambda}(p_j) J_{Q_{k+1}}$$

and

$$e_2(j) = \frac{1}{mu^{-1}(u+1)^2} \prod_{i=1}^{m} (p_j-p_i)(u_i^2-u_i^{j+1}),$$

$$\times \sum_{k=0}^{m-1} u^{-k-1} \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{\lambda}(p_j) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda-1} S_{\lambda}(p_j) J_{Q_{k+1}}.$$
Proof: We shall make use of the decomposition of the graded module grad(E) into a direct sum of left irreducible H-modules (see at the beginning of §6.2), and we consider the basis \( B_0 \) adapted to this decomposition. We also recall that the submodules \( M_0 = \langle v_1 \rangle, M_s = \langle w_0 \rangle, M_j = \langle w_j, v_j \rangle, \) \( 1 \leq j \leq m \) form a full set of left irreducible H-modules. Let \( A_0, A_s, A_j, 1 \leq j \leq m \) be the irreducible matrix representations obtained in this way, with degrees \( d_0 = 1, d_s = 2, 1 \leq j \leq m. \) Then, under the isomorphism \( \Pi \) (see beginning of §6.1), we have

\[
\Pi(h) = (A_0(h), A_s(h), \ldots, A_m(h), A_s(h)), \quad \forall h \in H.
\]

We first consider the elements \( e^{(j)}_j, 1 \leq j \leq m \) and we shall show that \( e^{(j)}_j \) is represented on \( M_j \) by the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and by the zero matrix on every other irreducible constituent. It is clear by their definition that \( Q_k \) belongs to \( L(u) = M(u), \) and so \( e^{(j)}_j \) belongs to \( L(u). \) Therefore \( e^{(j)}_j \) is represented by a diagonal matrix on every irreducible constituent. Now

\[
Q_k = uT(s_2 s_1)^{k-1} - T(s_1 s_2)^{k-1} + uT(s_2 s_1)^{k-1} - T(s_1 s_2)^{k-1} \quad k \geq 1.
\]

Thus, on the module \( M_0, \) which affords the representation \( T_{s_i} \rightarrow u, i = 1, 2, \) each \( Q_k \) is represented by: \( uu^{2k-2} - u^{2k-1} + uu^{2k-1} - u^{2k} = 0. \) So \( e^{(j)}_1 \) is represented by \( (0) \) on \( M_0. \) Similarly each \( Q_k \) is represented on \( M_s \) which affords the representation \( T_{s_i} \rightarrow -1, i = 1, 2, \) by:

\[
u(-1)^{2k-2} - (-1)^{2k-1} + v(-1)^{2k-1} - (-1)^{2k} = 0, \] and therefore \( e^{(j)}_1 \) is also represented by \( (0) \) on \( M_s. \)

By recalling the matrices which represent the elements \( T_{s_i} \) on \( M_s, \)

\[
1 \leq \lambda \leq m (\text{see Lemma 4.1}) \quad \text{we obtain that each } Q_k \text{ is represented on } M_s \text{ by the diagonal matrix } \begin{pmatrix} A_k^{(\lambda)} & 0 \\ 0 & B_k^{(\lambda)} \end{pmatrix}
\]
while the $A_k^{(\lambda)}$ are given by the following relations

$$
A_1^{(\lambda)} = u^2 - u\rho_\lambda + 1, \quad A_2^{(\lambda)} = u(1 + \rho_\lambda)(u^2 - u\rho_\lambda + 1),
$$

$$
A_k^{(\lambda)} = u(-u^{k-1}(S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda))) + u^{k-1}(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) + u(u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda))) - u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) =
$$

$$
= -u^k(S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda)) + (u^{k-1} + u^{k+1})(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) - u^k(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)), \text{ for } k = 3, 4, \ldots, m.
$$

Moreover $S_k(\rho_\lambda) = \rho_\lambda S_{k-1}(\rho_\lambda) - S_{k-2}(\rho_\lambda)$ for all $k = 3, \ldots, m$ and therefore

$$
S_{k-2}(\rho_\lambda) + S_{k-3}(\rho_\lambda) = \rho_\lambda(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) - S_k(\rho_\lambda) = S_{k-1}(\rho_\lambda).
$$

Hence

$$
A_k^{(\lambda)} = -u^k(\rho_\lambda)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) + (u^{k-1} + u^{k+1})(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) =
$$

$$
= u^{k-1}(u^2 - u\rho_\lambda + 1)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)), \text{ for } k = 3, \ldots, m, \text{ and so we eventually have }
$$

$$
A_k^{(\lambda)} = u^{k-1}(u^2 - u\rho_\lambda + 1)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)) \text{ for } 1 \leq k \leq m.
$$

Therefore the element $e^1_{(j)}$ is represented on $M_\lambda$, $1 \leq \lambda \leq m$ by the matrix $\left[ \begin{array}{cc} F(\lambda) & 0 \\ 0 & 0 \end{array} \right]$, where

$$
F(\lambda) = \frac{1}{u^{m-1} \prod_{s=1}^{m} (\rho_j - \rho_i)(u^2 - u\rho_i + 1)} \sum_{S_{m-k}(\rho_j)(S_{k-1}(\rho_\lambda) + S_{k-2}(\rho_\lambda)).}
$$

Now if $\lambda \neq j$, then Lemma 6.4(1) implies $F(\lambda) = 0$. If $\lambda = j$ then Lemmas 6.6 and 6.3(1) imply $F(\lambda) = 1$. We next consider the elements $e^2_{(j)}$, $1 \leq j \leq m$ and we shall show that $e^2_{(j)}$ is represented on $M_j$ by the matrix $\left[ \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right]$, and by the zero matrix on every other irreducible constituent.
It is easy to verify that it is represented by the zero matrix on \( M_0, M_s \), simply by looking at the action of \( Q_k \) on these modules, \( 1 \leq k \leq m \), which turns out to be the zero action. We recall (see Remark 4.1), that the matrices which represent the elements \( T_{(s_1 s_2)^\mu} \) and \( T_{(s_2 s_1)^\mu} \), \( \mu = 1, \ldots, m \) on the constituent \( M_\lambda, 1 \leq \lambda \leq m \), are obtained from one another by conjugation by the matrix \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

The same is true for the elements \( T_{(s_1 s_2)^\mu} \) and \( T_{(s_2 s_1)^\mu} \), \( \mu = 0, 1, \ldots, m-1 \).

Therefore the matrix which represents \( Q_k \) on \( M_\lambda \) is given by conjugating the matrix which represents \( Q_k \) on \( M_\lambda \) by \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \).

In other words the matrix which represents \( Q_k \) on \( M_\lambda \) is

\[
\begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} \begin{bmatrix}
F(\lambda) & 0 \\
0 & 0 \\
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
1 & 0 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
0 & F(\lambda) \\
\end{bmatrix}
\]

Thus \( e_2^{(0)} \) is represented by the matrix \( \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \) on \( M_j \) and by the zero matrix on every other irreducible constituent.

Finally we consider the elements \( e_0 \) and \( e_o \). We put \( P(u) = (u+1)(u^{n-1} + \cdots + u+1) \)

We write \( e_0 = \frac{1}{P(u)} \sum_{w} (T_w + T_{\overline{s_1 w}}) \) for \( i = 1, 2 \). Then

\[
T_{s_1} e_0 = \frac{1}{P(u)} \sum_{w \in \mathcal{W}} (T_{\overline{s_1 w}} + uT_w + (u-1)T_{\overline{s_1 w}}) = u e_0 \text{ and therefore } He_0
\]

affords the representation \( T_{s_1} = u, i = 1, 2 \).

Moreover \( e_0 \) belongs to the centre of \( H \) and since \( T_w e_0 = u^w e_0 \) for every \( w \in \mathcal{W} \), we have \( e_0^2 = e_0 \). We similarly write the element \( e_0 \) as
Thus \( T_{s_i}e_0 = \frac{1}{P(u)} \left[ \sum_{w} (-1)^{E(w)}u^{n-E(w)} T_w + \sum_{w} \sum_{s_i \neq w} (-1)^{E(w)}u^{n-E(w)-1}(uT_w+(u-1)T_{s_i}w) \right] \)

\[ = -e_0, \text{ for } i = 1, 2. \]

Therefore \( H e_0 \) affords the representation \( T_{s_i} \rightarrow -1, \ i = 1, 2. \) We also have that \( e_0 \) belongs to the centre of \( H \) and since \( T_w e_0 = (-1)^{E(w)} e_0 \) for every \( w \in W, \)

we have \( e_0^2 = e_0. \) It is now clear that the simple components of \( H \) are given by \( H e_0, H e_{(j)} \oplus H e_2(j), 1 \leq j \leq m \) and \( H e_0. \)

Moreover \( H e_0 \) annihilates every other component different from itself.

Therefore \( e_0 \) is represented by the zero matrix on every irreducible module different from \( M_0, \) which affords the representation \( T_{s_i} u, \ i = 1, 2. \)

The argument is similar for the element \( e_0. \)

So part (1) of our theorem is now proved.

(2). Let \( n = 2m. \) We now make use of the decomposition of \( \text{grad}(E) \) as

\[ \text{grad}(E) = \bigoplus_{j=1}^{m-1} M_0 \oplus M_j \oplus M_1 \oplus M_2 \oplus V_j \oplus V_j, \]

where \( (M_0, M_j, M_1, M_2, V_j, 1 \leq j \leq m-1) \)

is a full set of left irreducible \( H \)-modules. To this decomposition we adopt a basis \( B_1 = \{e_0, e_0', v_1, v_2, u_1, u_2, v_1', v_2', 1 \leq j \leq m-1\} \) with

\[ V_j = \langle u_j, v_j \rangle, V_j = \langle u_j', v_j' \rangle, j = 1, \ldots, m-1. \]

Let \( \Lambda_0, \Lambda_j, R_1, R_2, A_j, 1 \leq j \leq m-1, \)

be the irreducible matrix representations obtained in this way with underlying modules \( M_0, M_j, M_1, M_2, V_j \) and degrees \( d_0 = d_3 = d_4 = d_2 = 1, f_j = 2, 1 \leq j \leq m-1. \)

Then under the isomorphism \( \pi \) we have \( \pi(h) = (\Lambda_0(h), \Lambda_j(h), R_1(h), R_2(h), A_1(h), \ldots, A_{m-1}(h)), \) for every \( h \in H. \) We first consider the element \( e_1(j), 1 \leq j \leq m-1 \) and we shall show that \( e_1(j) \) is represented on \( V_j \) with respect to the adopted basis \( \{u_j, v_j\} \) by the matrix \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and by the zero matrix on
every other irreducible module \( V_{j}, j \neq j' \), \( M_{0}, M_{s}, M_{1}, M_{2} \). In order to do so, we need the action of the quantities \( Q_{k}, 1 \leq k \leq m \) on the several irreducible modules. This has already been established for the 2-dimensional modules \( V_{\lambda}, 1 \leq \lambda \leq m-1 \) (see case \( n \) odd) and for the modules \( M_{0}, M_{s} \) which afford the representations \( T_{s_{1}} \rightarrow u \) and \( T_{s_{2}} \rightarrow -1, 1 = 1,2 \) respectively. We have to consider the action of \( Q_{k} \) on \( M_{1}, M_{2} \) which afford the representations \( T_{s_{1}} \rightarrow -1, T_{s_{2}} \rightarrow u \) and \( T_{s_{1}} \rightarrow u, T_{s_{2}} \rightarrow -1 \) respectively.

On the module \( M_{1} \), \( Q_{k} \) is represented by: \((-1)^{k-1} u^{k} - (-1)^{k} u^{k-1} + (-1)^{k-1} u^{k+1} - (-1)^{k} u^{k} = (-1)^{k-1} u^{k-1}(u+1)^{2} \), and on the module \( M_{2} \) is represented by: \((-1)^{k-1} u^{k} - (-1)^{k-1} u^{k} + (-1)^{k} u^{k} - (-1)^{k} u^{k} = 0. \)

To summarize the action of \( Q_{k}, 1 \leq k \leq m \) on the several irreducible modules we have:

\[
\begin{align*}
M_{0} & : Q_{k} \rightarrow 0, \\
M_{s} & : Q_{k} \rightarrow 0, \\
M_{1} & : Q_{k} \rightarrow (-1)^{k-1} u^{k-1}(u+1)^{2}, \\
M_{2} & : Q_{k} \rightarrow 0, \quad \text{and on } V_{\lambda}, 1 \leq \lambda \leq m-1, \quad Q_{k} \text{ is represented by the matrix} \\
\begin{bmatrix}
A_{1}(k) \\
0 \\
0
\end{bmatrix}, \quad \text{where} \\
A_{1}(k) = u^{k-1}[S_{k-1}(\rho_{k}) + S_{k-2}(\rho_{k})](u^{2} - u \rho_{k} + 1), 1 \leq k \leq m, \quad \text{with} \quad A_{m}(z) = u^{m-1}S_{m-2}(\rho_{k})(u^{2} - u \rho_{k} + 1) \quad (\text{since} \quad S_{m-1}(\rho_{k}) = 0). \\
We adopt the notation \( G_{j} = : m^{m-1} \prod_{i=1}^{m-1} (\rho_{j} - \rho_{i})(u^{2} - u \rho_{j} + 1). \)

We must investigate the action of \( e_{j}^{(j)} \) only on the modules \( M_{1}, V_{\lambda}, 1 \leq \lambda \leq m-1. \)

By taking into account the way in which \( Q_{k} \) acts on \( M_{1}, 1 \leq k \leq m \), we obtain that \( e_{j}^{(j)} \) is represented on \( M_{1} \) by:

\[
\frac{u^{m-1}(u+1)^{2}}{G_{j}} \prod_{k=0}^{m-1} (-1)^{k} \sum_{\lambda=0}^{\lambda=k+2} (-1)^{\lambda} S_{m-\lambda}(\rho_{j}) + (-1)^{k-1} \sum_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_{j}) -
\]

\[
\frac{u^{m-1}(u+1)^{2}}{G_{j}} \prod_{k=0}^{m-1} (-1)^{k} \sum_{\lambda=0}^{\lambda=k+2} (-1)^{\lambda} S_{m-\lambda}(\rho_{j}) + \sum_{k=0}^{m-1} (-1)^{2k-1} \prod_{\lambda=2}^{m} (-1)^{\lambda}(\lambda-1) S_{m-\lambda}(\rho_{j})]
\]

\[\]

\]
In the last expression the coefficient of $S_{m-\lambda}(\rho_j)$, $\lambda \in \{2, \ldots, m\}$ is
\[
\frac{m!}{G_j} \left[ \sum_{k=0}^{\lambda-2} (-1)^{\lambda-k} \sum_{\lambda=k+2}^{m} \frac{(-1)^{\lambda-1} S_{m-\lambda}(\rho_j)}{\lambda=2} \right] = 0.
\]

Hence $e^{(j)}_1$ is represented by the zero matrix on the one dimensional module $M_1$.

Using a similar argument as in the case $n$ odd, we find that $e^{(j)}_1$ is represented on the two dimensional module $V_{\lambda}$, $1 \leq \lambda \leq m-1$ by the diagonal matrix
\[
\begin{bmatrix}
F(\varepsilon) & 0 \\
0 & 0
\end{bmatrix}
\]
where $F(\varepsilon) = \frac{1}{G_j}$.

Moreover
\[
\begin{bmatrix}
\sum_{\lambda=2}^{m-1} (-1)^{\lambda-1} S_{m-\lambda}(\rho_j)[(-1)^{k-1}(S_{k-1}(\rho_j)+S_k(\rho_j))] \\
\sum_{\lambda=2}^{m-2} (-1)^{\lambda-1} S_{m-\lambda}(\rho_j) S_{m-\lambda}(\rho_j) + S_{m-2}(\rho_j)
\end{bmatrix}
\]

Moreover
\[
\begin{bmatrix}
\sum_{\lambda=2}^{m-3} (-1)^{k-1} S_{k-1}(\rho_j) + S_k(\rho_j) + \sum_{k=2}^{m-2} (-1)^{k-1} S_k(\rho_j) + 1 \cdot (-1)^{m-3} S_{m-2}(\rho_j)
\end{bmatrix}
\]

So, $F(\varepsilon) = \frac{u^{m-1}(u^2-w\rho_\lambda+1)}{G_j}$.
Hence \( F(\xi) = \frac{\mu u^{m-1}(u^2-u_\xi+1)}{G_j} \times \)

\[
\begin{align*}
& m-2 \sum_{k=1}^{m} (-1)^{k+2} S_{m-\lambda}(\rho_j) (S_{k-1}(\rho_{\lambda}) + S_k(\rho_{\lambda})) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j).
\end{align*}
\]

Now if \( \xi = j \), then Lemmas 6.7 and 6.3(2) imply \( F(\xi) = 1 \). Next we assume that \( \xi \neq j \), and let \( D_r(\rho_j) = \sum_{\lambda=r}^{m} (-1)^{\lambda-r} S_{m-\lambda}(\rho_j) \) for every \( r \in \{3,4,\ldots,m\} \).

(See Lemma 6.7). Then

\[
\begin{align*}
& \sum_{k=1}^{m-2} \sum_{\lambda=k+2}^{m} (-1)^{\lambda-k} S_{m-\lambda}(\rho_j) (S_{k-1}(\rho_{\lambda}) + S_k(\rho_{\lambda})) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& \sum_{r=3}^{m} D_r(\rho_j) (S_{r-2}(\rho_{\lambda}) + S_{r-3}(\rho_{\lambda})) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& \rho_{\lambda} D_3(\rho_j) + D_3(\rho_j) + \sum_{r=4}^{m-1} D_r(\rho_j) S_{r-2}(\rho_{\lambda}) + S_{m-2}(\rho_{\lambda}) + \sum_{r=3}^{m-2} D_r(\rho_j) S_{r-2}(\rho_{\lambda}) + S_{m-3}(\rho_{\lambda}) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& = \rho_{\lambda} D_3(\rho_j) + D_3(\rho_j) + \sum_{r=4}^{m-1} D_r(\rho_j) S_{r-2}(\rho_{\lambda}) + S_{m-2}(\rho_{\lambda}) + \sum_{r=3}^{m-2} D_r(\rho_j) S_{r-2}(\rho_{\lambda}) + S_{m-3}(\rho_{\lambda}) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& = D_3(\rho_j) + \rho_{\lambda} D_3(\rho_j) + D_4(\rho_j) + \sum_{r=4}^{m-1} S_{r-2}(\rho_{\lambda})(D_r(\rho_j) + D_{r+1}(\rho_j)) + \\
& + D_{m-1}(\rho_j) S_{m-3}(\rho_{\lambda}) + S_{m-2}(\rho_{\lambda}) + S_{m-3}(\rho_{\lambda}) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& = \sum_{\lambda=3}^{m} (-1)^{\lambda-3} S_{m-\lambda}(\rho_j) + \rho_{\lambda} S_{m-3}(\rho_{\lambda}) + \sum_{r=4}^{m-2} S_{r-2}(\rho_{\lambda}) S_{m-r}(\rho_j) + (\rho_j^{-1}) S_{m-3}(\rho_{\lambda}) + \\
& + S_{m-2}(\rho_{\lambda}) + \sum_{\lambda=2}^{m} (-1)^{\lambda} S_{m-\lambda}(\rho_j) = \\
& = \sum_{r=2}^{m} S_{r-2}(\rho_{\lambda}) S_{m-r}(\rho_j) = 0 \text{ (by Lemma 6.4(2))}.
\end{align*}
\]

Hence if \( \xi \neq j \), \( 1 \leq \xi \leq m-1 \), then \( F(\xi) = 0 \).
Therefore we have established that $e_1^{(j)}$ is represented on the constituent $V_j$, $1 \leq j \leq m-1$ by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and by the zero matrix on every other irreducible constituent $V_\ell$, $\ell \neq j$, $M_0$, $M_3$, $M_1$, $M_2$.

The argument is entirely similar when we consider the elements $e_2^{(j)}$, $1 \leq j \leq m-1$. The action of the quadruples $Q_k$ on the four one dimensional submodules $M_0$, $M_3$, $M_1$, $M_2$ is given by:

$$
M_0 : \hat{Q}_k \to 0, M_3 : \hat{Q}_k \to 0, M_1 : \hat{Q}_k \to 0, M_2 : \hat{Q}_k \to (-1)^{k-1} u^{k-1}(u+1)^2,
$$

$1 \leq k \leq m$, and therefore $e_2^{(j)}$ is represented by the zero matrix on every one of them.

On the other hand, $\hat{Q}_k$ is represented on each $V_\ell$, $1 \leq \ell \leq m-1$ by a matrix which is obtained by conjugating the matrix which represents $Q_k$ on $V_\ell$, by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Hence $\hat{Q}_k$ is represented on $V_\ell$ by

$$
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & F(x) \end{bmatrix}
$$

Thus $e_2^{(j)}$ is represented by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ on $V_j$ and by the zero matrix on every $V_\ell$, $\ell \neq j$, $1 \leq j \leq m-1$.

We finally consider the elements $e_0$, $e_0^1$, $e_0^2$, $e_0^2$. We have already shown (see the case $\nu$ odd), that $H_0$, $H_{\nu}$ afford the one dimensional representations $T_{s_1} \to u$ and $T_{s_1} \to -1 + 1 = 1, 2$ respectively. Thus we have to consider only the elements $e_0^1$, $e_0^2$.

For every $w \in D_{2n}$ let $\ell_1(w)$, $\ell_2(w)$ be the number of $s_1$'s (respectively $s_2$'s) in a reduced expression of $w$.

We write
Therefore,
\[ T_{s_1} \sigma_1 = \frac{1}{m \mu^{m-1}(u+1)^2} \left[ \sum_{w: s_1 \not\in w} (-1)^1 \left( u^m z_1(w) \right) T_w + \sum_{w: s_1 \in w} (-1)^1 \left( u^m z_1(w) \right) T_{s_1 w} \right] \]
\[ (u T_w + (u-1) T_{s_1 w}) \]

\[ = -\sigma_1 \]
and,
\[ T_{s_2} \sigma_1 = \frac{1}{m \mu^{m-1}(u+1)^2} \left[ \sum_{w: s_2 \not\in w} (-1)^1 \left( u^m z_1(w) \right) T_w + \sum_{w: s_2 \in w} (-1)^1 \left( u^m z_1(w) \right) T_{s_2 w} \right] \]
\[ (u T_w + (u-1) T_{s_2 w}) \]

\[ = u \sigma_1 \]

Hence \( H e_{\sigma_1} \) affords the representation \( T_{s_1} = -1, T_{s_2} = u \). Similarly by considering pairs \((w, w_{s_1})\) or \((w, w_{s_2})\) we can verify that \( e_{\sigma_1} T_{s_1} = -e_{\sigma_1} \),
\( e_{\sigma_1} T_{s_2} = u e_{\sigma_1} \), and so \( e_{\sigma_1} \) is a central element inside \( H \). Similarly we write \( e_{\sigma_2} \)
\[ e_{\sigma_2} = \frac{1}{m \mu^{m-1}(u+1)^2} \left[ \sum_{w: s_2 \not\in w} (-1)^1 \left( u^m z_2(w) \right) T_w + \sum_{w: s_2 \in w} (-1)^1 \left( u^m z_2(w) \right) T_{s_2 w} \right] \]
\[ (u T_w + (u-1) T_{s_2 w}) \]

and we can easily show that \( T_{s_1} e_{\sigma_2} = e_{\sigma_2} T_{s_1} = u e_{\sigma_2} \), and \( T_{s_2} e_{\sigma_2} = e_{\sigma_2} T_{s_2} = -e_{\sigma_2} \).

Hence \( e_{\sigma_2} \) belongs to the centre of \( H \) and \( H e_{\sigma_2} \) affords the representation \( T_{s_1} = u, T_{s_2} = -1 \). Moreover
\[ T_w e_{\sigma_1} = (-1)^1 \left( u T_{s_2} \right) \]
and
\[ T_w e_{\sigma_2} = (-1)^1 \left( u T_{s_1} \right) \]
for every \( w \in D_{\mathbb{Z} n} \).
We also write \( W = \{1\} \cup A_1 \cup A_2 \cup A_3 \cup (\{s_1s_2\}^m) \), where

\[
A_1 = (\{s_1s_2\}^j, 1 \leq j \leq m-1),
A_2 = (s_1s_2)^{-1}s_1, 1 \leq j \leq m,
A_3 = (s_2s_1)^{-1}, 1 \leq j \leq m.
\]

Thus

\[
e_2 = \frac{1}{m(m-1)} \sum_{w \in W} (s_1s_2)(w)e_1(w),
\]

\[
\sum_{w \in W} (s_1s_2)(w)e_1(w) = u^m(2m-2)u^m-1+mu^m-1+u^m = mu^m(u+1)z.
\]

So \( e_2 = e_1 \). Similarly we can show that \( e_2 = e_2 \).

Now the simple components of \( H \) are clearly

\[
He_0, He_1, He_2, He_3, He_4, He_5, 1 \leq j \leq m-1.
\]

Each such component annihilates every other component different from itself, so the element \( e_0 \) is represented by the zero matrix on every irreducible module different from \( H_0 \) which affords the representation \( T_{s_1} = u, 1 = 1,2 \).

The argument is similar for the elements \( e_1, e_2, e_3 \). Our theorem is now proved.

We next give some examples to illustrate the situation.

1. \( W(A_2) = D_6, n = 3 = 2.1 + 1, m = 1, \rho_1 = 2 \cos \frac{2\pi}{3} = -1 \)

\[
e_0 = \frac{1}{(u+1)(u^2+u+1)} \sum_{T_w \in D_6} T_w, e_0 = \frac{1}{(u+1)(u^2+u+1)} \sum_{T_w \in D_6} (-1)^{T_w}u^{3-L(w)}T_w
\]

\[
e_1 = \frac{1}{u^2+u+1} \left[uT_1 + uT_2 - T_{s_1s_2} \right] = \frac{1}{u^2+u+1} (uT_1 + T_{s_1s_2})
\]

\[
e_2 = \frac{1}{u^2+u+1} \left[uT_1 - uT_2 + uT_1 - T_{s_1s_2} \right] = \frac{1}{u^2+u+1} (uT_1 + T_{s_1s_2})
\]

2. \( W(B_2) = D_8, n = 4 = 2.2, m = 2, \rho_1 = 2 \cos \frac{\pi}{2} = 0 \)

\[
e_0 = \frac{1}{(u+1)(u^3+u^2+u+1)} \sum_{T_w \in D_8} T_w, e_0 = \frac{1}{(u+1)(u^3+u^2+u+1)} \sum_{T_w \in D_8} (-1)^{T_w}u^{4-L(w)}T_w
\]

\[
e_1 = \frac{1}{u^2+u+1} \left[uT_1 - uT_2 - T_{s_1s_2} \right] = \frac{1}{u^2+u+1} (uT_1 + T_{s_1s_2})
\]

\[
e_2 = \frac{1}{u^2+u+1} \left[uT_1 - uT_2 - uT_1 - T_{s_1s_2} \right] = \frac{1}{u^2+u+1} (uT_1 + T_{s_1s_2})
\]
\[ e_{\sigma_{1}} = \frac{1}{2u(u+1)^{2}} [u^{2}T_{1} - uT_{s_{1}} + u^{2}T_{s_{2}} - uT_{s_{1}s_{2}} - uT_{s_{2}s_{1}} + T_{s_{1}}s_{2}s_{1} - uT_{s_{2}s_{1}s_{2}} + T_{s_{1}s_{2}s_{1}s_{2}} ] \]

\[ e_{\sigma_{2}} = \frac{1}{2u(u+1)^{2}} [u^{2}T_{1} - uT_{s_{2}} + u^{2}T_{s_{1}} - uT_{s_{1}s_{2}} + uT_{s_{2}s_{1}} + T_{s_{1}s_{2}s_{1}} - uT_{s_{1}}s_{2}s_{1} + T_{s_{1}}s_{2}s_{1}s_{2} ] \]

\[ e_{1}^{(1)} = \frac{1}{2u(u+1)^{2}} [u^{2}T_{1} - uT_{s_{1}} + u^{2}T_{s_{2}} - uT_{s_{1}s_{2}} + uT_{s_{2}s_{1}} - T_{s_{1}s_{2}s_{1}} + uT_{s_{2}s_{1}s_{2}} - T_{s_{1}s_{2}s_{1}s_{2}} ] \]

\[ = \frac{1}{2u(u+1)^{2}} (u^{2}T_{1} - uT_{s_{1}} + uT_{s_{2}s_{1}} - T_{s_{1}s_{2}s_{1}})(T_{1} + T_{s_{2}}) = \]

\[ = \frac{1}{2u(u+1)^{2}} (uT_{1} - T_{s_{1}})(uT_{1} + T_{s_{2}})(T_{1} + T_{s_{2}}) \]

\[ e_{2}^{(1)} = \frac{1}{2u(u+1)^{2}} [u^{2}T_{1} - uT_{s_{2}} + u^{2}T_{s_{1}} - uT_{s_{1}s_{2}} + uT_{s_{2}s_{1}} - T_{s_{2}s_{1}s_{2}} + uT_{s_{1}s_{2}s_{1}} - T_{s_{1}s_{2}s_{1}s_{2}} ] \]

\[ = \frac{1}{2u(u+1)^{2}} [u^{2}T_{1} - uT_{s_{2}} + uT_{s_{1}s_{2}} - T_{s_{2}s_{1}s_{2}}](T_{1} + T_{s_{1}}) = \]

\[ = \frac{1}{2u(u+1)^{2}} (uT_{1} - T_{s_{2}})(uT_{1} + T_{s_{1}})(T_{1} + T_{s_{1}}) \]

(3) \( W = D_{10}, n = 5 = 2.2 + 1, m = 2, \rho_{1} = 2 \cos \frac{2\pi}{5}, \rho_{2} = 2 \cos \frac{4\pi}{5} \).

\[ e_{0} = \frac{1}{(u+1)(u^{4}+3u^{2}+u+1)} \sum_{w \in D_{10}} T_{w}, \quad e_{\sigma} = \frac{1}{(u+1)(u^{4}+3u^{2}+u+1)} \sum_{w \in D_{10}} (-1)^{w} w^{5} \chi(w)T_{w} \]

\[ e_{1}^{(1)} = \frac{1}{u(\rho_{1}-\rho_{2})(u^{2}-u_{w_{0}}+1)} [u_{w_{0}}(uT_{1} - T_{s_{1}} + uT_{s_{2}} - T_{s_{2}s_{1}} + uT_{s_{2}s_{1}} - T_{s_{2}s_{1}s_{2}} + uT_{s_{1}s_{2}s_{1}} - T_{s_{1}s_{2}s_{1}s_{2}}) ] \]

\[ = \frac{1}{u(\rho_{1}-\rho_{2})(u^{2}-u_{w_{0}}+1)} (uT_{1} - T_{s_{1}})(u_{w_{0}}T_{1} + T_{s_{2}s_{1}})(T_{1} + T_{s_{2}}) \]

\[ e_{2}^{(1)} = \frac{1}{u(\rho_{1}-\rho_{2})(u^{2}-u_{w_{0}}+1)} [u_{w_{0}}(uT_{1} - T_{s_{2}} + uT_{s_{1}} - T_{s_{1}s_{2}} + uT_{s_{1}s_{2}} - T_{s_{1}s_{2}s_{1}} + uT_{s_{1}s_{2}s_{1}} - T_{s_{1}s_{2}s_{1}s_{2}}) ] \]

\[ = \frac{1}{u(\rho_{1}-\rho_{2})(u^{2}-u_{w_{0}}+1)} (uT_{1} - T_{s_{2}})(u_{w_{0}}T_{1} + T_{s_{1}s_{2}})(T_{1} + T_{s_{1}}) \]
\[ e_1^{(2)} = \frac{1}{u(p_2-p_1)(u^2-up_2+1)} \left[ u_0(p_2u_1-T_1 + uT_2 - T_1 s_2 s_1 + T_1 s_2 s_1 + uT_2 s_1 s_2 - T(s_1 s_2)^2) \right] \]
\[ = \frac{1}{u(p_2-p_1)(u^2-up_2+1)} \left[ u(p_2u_1 + T) s_2 (T_1 + T_2) \right] \]
\[ e_2^{(2)} = \frac{1}{u(p_2-p_1)(u^2-up_2+1)} \left[ u_0(p_2u_1-T_2 + uT_1 - T_2 s_1 s_2 + T_1 s_2 s_1 + uT_2 s_1 s_2 - T(s_1 s_2)^2) \right] \]
\[ = \frac{1}{u(p_2-p_1)(u^2-up_2+1)} \left[ u(p_2u_1 + T) s_2 (T_1 + T_2) \right] \]

(4) \( W(G_2) = D_{12}, n = 6, m = 3, \rho_1 = 2 \cos \frac{\pi}{3} = 1, \rho_2 = 2 \cos \frac{2\pi}{3} = -1. \)

\[ e_0 = \frac{1}{(u+1)(u^2+u^3+u+1)} \sum_{\omega \in D_{12}} \frac{1}{\omega} \]

\[ e_0^{(2)} = \frac{1}{(u+1)(u^2+u^3+u+1)} \sum_{\omega \in D_{12}} (-1)^{\bar{\omega}(w)} \omega \]

\[ e_0^{(2)} = \frac{1}{3u^2(u+1)^2} \left[ u^3 T_1 u^2 T_1 + u^3 T_2 u^2 T_2 + u^2 T_1 s_1 s_2 - u^2 T_2 s_1 s_2 - u^2 T_1 s_2 s_1 + u^2 T_2 s_2 s_1 - uT(s_1 s_2)^2 + uT(s_2 s_1)^2 - T(s_1 s_2)^2 \right] \]

\[ e_0^{(2)} = \frac{1}{3u^2(u+1)^2} \left[ u^3 T_1 u^2 T_2 + u^3 T_2 u^2 T_2 + u^2 T_1 s_1 s_2 - u^2 T_2 s_1 s_2 - u^2 T_1 s_2 s_1 + u^2 T_2 s_2 s_1 + uT(s_1 s_2)^2 + uT(s_2 s_1)^2 - T(s_1 s_2)^3 \right] \]
\[ e_2^{(1)} = \frac{1}{6u^2(u^2-u^1+1)} \left[ u^2(uT_1-T_{s_2}+uT_{s_1}-T_{s_1}s_2) + 2u(uT_{s_1}s_2-T_{s_2}s_1s_2+uT_{s_1}s_2s_1-T(s_2^2s_1)^2) + \\
\quad + T(t_1)^2(t_1s_2)s_1 - T(t_1s_2)s_2 - T(t_1s_2)^3 \right] \]

\[ = \frac{1}{6u^2(u^2-u^1+1)} (uT_1-T_{s_2})(u^2T_1+2uT_{s_1}s_2+T(t_1s_2)^2)(T_1 + T_{s_1}) \]

\[ e_1^{(2)} = \frac{1}{2u^2(u^2+u^1+1)} (u^2(uT_1-T_{s_1}+uT_{s_2}-T_{s_1}s_2) - uT_{s_2}s_1s_2 - T(t_1s_2)^2 + T(t_1s_2)^3) - \\
\quad - \frac{1}{2u^2(u^2+u^1+1)} (uT_1-T_{s_1})(u^2T_1-uT(s_2s_1)^2+T(t_1s_2)^2)(T_1 + T_{s_1}) \]

\[ e_2^{(2)} = \frac{1}{2u^2(u^2+u^1+1)} (u^2(uT_1-T_{s_2}+uT_{s_1}-T_{s_2}s_2) - uT_{s_1}s_2s_2 - T(t_1s_2)^2 + T(t_1s_2)^3) \]

\[ = \frac{1}{2u^2(u^2+u^1+1)} (uT_1-T_{s_2})(u^2T_1-uT(s_1s_2)^2+T(s_2s_1)^2)(T_1 + T_{s_1}). \]
OPEN QUESTIONS

We now mention two open questions which arise naturally from our work.

(A): In Chapter 3 we investigate the centre of $H$ following some ideas of R.W. Carter. These ideas give a natural basis for the centre of Hecke algebras of dihedral type, and under the specialization $u \rightarrow 1$ this basis specializes to the class sums.

These elements are parametrized by the conjugacy classes of the group and for each conjugacy class $C$ a typical element $z_C$ of this basis has the form: $z_C = \sum_{\ell(w) = \text{maximal inside } C} T_w + \text{linear combination of other } T_w's$, not involving any $T_w$ with $w$ of maximal length in any other conjugacy class different from $C$.

In the same chapter we determine a basis of the same form for the case $W = S_4$.

It is therefore natural to make the following conjecture:

Let $H$ be the Hecke algebra over the polynomial ring $\mathbb{Q}[u]$ associated to a finite indecomposable Coxeter group $W$. Then there exists a basis \{ $z_C | C$ runs over the conjugacy classes of $W$ \} of the centre of $H$, where each $z_C$ has the form $z_C = \sum_{\ell(w) = \text{maximal inside } C} T_w + \text{linear combination of other } T_w's$, not involving any $T_w$ with $w$ of maximal length in any other conjugacy class different from $C$.

Furthermore the coefficient of each $T_w$ involved in $z_C$ belongs to $\mathbb{Z}[u]$, and this basis specializes to the class sums under the specialization $u \rightarrow 1$. 
(B): In Chapter 6 we have introduced a maximal commutative subalgebra $M(u)$ inside the generic Hecke algebra. The definition of this subalgebra depends upon a chosen $W$-graph.

Nevertheless we have evidence that $M(u)$ does not in fact depend on the choice of the $W$-graph. To prove this it would be enough to show that $M(u) = L(u)$. In order to show this it would be sufficient to prove that the subalgebra $L(1)$ of $QW$ is commutative. We conjecture that this is true for all finite Coxeter groups $W$.

The validity of this conjecture together with the results of Proposition 6.1 and 6.2(11) would imply that $M(u) = L(u)$. This result might be of significant help in the effort to decompose the identity of $M(u)$ into a sum of orthogonal primitive idempotents.

We have checked the truth of this conjecture when $W$ is a dihedral group and when $W$ is the symmetric group $S_4$. 

REFERENCES


