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Abstract—This letter develops a one-dimensional (1D) diffusion-based molecular communication system to analyze channel responses between a single transmitter (TX) and two fully-absorbing receivers (RXs). Incorporating molecular degradation in the environment, rigorous analytical formulas for i) the fraction of molecules absorbed, ii) the corresponding hitting rate, and iii) the asymptotic fraction of absorbed molecules as time approaches infinity at each RX are derived when an impulse of molecules are released at the TX. By using particle-based simulations, the derived analytical expressions are validated. Simulations also present the distance ranges of two RXs that do not impact molecular absorption of each other, and demonstrate that the mutual influence of two active RXs reduces with the increase in the degradation rate.

I. INTRODUCTION

Molecular communication (MC) is one of the most promising solutions to nano-scale communications. In MC, information is encoded into small particles that are released by a transmitter (TX) into a fluid medium and propagate until they arrive at a receiver (RX). Moreover, MC can be biocompatible and consumes low energy. These characteristics make MC suitable for applications such as targeted drug delivery, pollution control, and environmental monitoring [1]. For each application, accurate channel modeling is essential for analysis and design of MC systems [2].

Most existing MC papers have focused on the modeling of a single-RX MC system [3]. Some papers, e.g., [4]–[7], have considered a multi-RX MC system. The majority of papers involving a multi-RX MC system have assumed transparent RXs for tractability, due to the independence among observations at multiple transparent RXs. However, many practical RX surfaces might interact with the molecules of interest, e.g., by providing binding sites for absorption or other reactions [8]. In an environment where multiple non-transparent RXs co-exist, one non-transparent RX would impact molecules received by other non-transparent RXs. Hence, an accurate characterization of such dependence makes the derivation of channel response (CR) cumbersome. Motivated by this, [5]–[7] have considered a multi-RX system with non-transparent RXs. In [5], the capture probability for each receiver was obtained via simulations. Considering a one-dimensional (1D) environment with two fully-absorbing RXs, [6] derived the sum of absorbed molecules by both RXs. Notably, [7] derived

Channel Characterization for 1D Molecular Communication with Two Absorbing Receivers

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the fraction of molecules absorbed at each RX in a three-dimensional (3D) environment with two fully-absorbing RXs. However, this derivation is not applicable in a 1D environment, as we will show in Section IV. Thus, an exact closed-form expression for the fraction of molecules absorbed over time at each RX (i.e., the CR) has not been derived yet for a 1D environment.

Despite the aforementioned challenges, we provide closed-form expressions for the fraction of molecules absorbed at RXs when multiple fully-absorbing RXs co-exist, by taking into account the mutual influence between RXs. Such expressions accurately characterize the CR at fully-absorbing RXs and lay the foundation for future performance evaluation, detection design, and diverse applications (e.g., target detection using two fully-absorbing RXs) of a realistic multi-RX system. In this letter, we consider a 1D environment where one TX communicates between two fully-absorbing RXs. The 1D environment is worthy of investigation since it is a good first approximation for regions between two close cells, such as chemical synapses in a human body [9]. To capture the effect of molecular chemical reaction on the received molecules at fully-absorbing RXs, we also consider molecular degradation in the environment.

Our major contributions are summarized as follows. We derive i) the exact closed-form expressions for the fraction of molecules absorbed, ii) the corresponding hitting rate, and iii) the asymptotic fraction of molecules absorbed as time approaches infinity at each RX with an impulse emission at the TX. Aided by a particle-based simulation method, we verify our analytical results. In addition, we present the distance ranges of two RXs that do not impact molecular absorption of each other. We also show that the mutual impact between two RXs reduces with the increase of degradation rate in the environment.

The rest of this paper is organized as follows. In Section II, we introduce the system model. In Section III, the closed-form expression for the fraction of absorbed molecules is derived. We also derive the corresponding hitting rate and the asymptotic fraction of absorbed molecules as time approaches infinity. In Section IV, we discuss the numerical results, and conclusion is presented in Section V.

II. SYSTEM MODEL

In this letter, we consider a 1D unbounded environment where a single TX is located between two fully-absorbing RXs, i.e., RX1 and RX2, with distance $d_1$ from the RX1 and distance $d_2$ from the RX2, as depicted in Fig. 1. We consider the TX as a point source that can release an impulse of particles. We assume that the TX transmission starts at
Fig. 1. Illustration of the system model, where one TX communicates with two fully-absorbing RXs in a one-dimensional environment.

t = 0 s. Once released, particles diffuse randomly with a constant diffusion coefficient $D$. We also consider the first-order chemical reaction (i.e., unimolecular degradation) in the environment, where type A molecules degrade into a new type of molecule $\phi$ that cannot be identified by the RXs, i.e., $A \xrightarrow{k} \phi$ [10, Ch. 9], where $k$ [s$^{-1}$] is the degradation rate constant. We model the two RXs as point fully-absorbing RXs, which means that information molecules $A$ are absorbed as soon as they hit the point RX$_1$ or RX$_2$.

III. DERIVATION OF CHANNEL IMPULSE RESPONSE

In this section, we derive closed-form expressions for the expected fraction of absorbed molecules at each RX for impulsive emission at the TX, and the asymptotic fraction of absorbed molecules at each RX as $t \rightarrow \infty$. We first derive the CR for impulsive emission when one RX exists, which builds the foundation for deriving the CR when two RXs exist. According to [11], the hitting rate for a single RX with molecular degradation can be obtained via the hitting rate without molecular degradation and multiplying by $\exp(-kt)$. Based on the existing expression for the hitting rate without molecular degradation in [6], the hitting rate at the RX at time $t$, denoted by $f(d,t)$, with molecular degradation is

$$f(d,t) = \frac{d}{\sqrt{4\pi Dt^3}} \exp \left( -\frac{d^2}{4Dt} - kt \right),$$

(1)

where $d$ is the distance between the TX and the RX. Using $\int_0^t f(d,u)du$, we obtain the fraction of molecules absorbed by time $t$, denoted by $F(d,t)$, as

$$F(d,t) = \frac{1}{2} \exp \left( -\sqrt{\frac{k}{D}} d \right) \operatorname{erfc} \left( \frac{d}{\sqrt{4Dt}} - \sqrt{kt} \right) + \frac{1}{2} \exp \left( \sqrt{\frac{k}{D}} d \right) \operatorname{erfc} \left( \frac{d}{\sqrt{4Dt}} + \sqrt{kt} \right).$$

(2)

As $t \rightarrow \infty$, we derive the asymptotic absorbed molecules $F(d,t \rightarrow \infty)$ as

$$F(d,t \rightarrow \infty) = \exp \left( -\sqrt{\frac{k}{D}} d \right).$$

(3)

In the following, we derive the CR when two RXs exist. We denote the fraction of absorbed molecules at RX$_1$ and RX$_2$ by time $t$ for impulsive emission by $P_1(t)$ and $P_2(t)$, respectively. We also denote the corresponding hitting rates at RX$_1$ and RX$_2$ by $p_1(t)$ and $p_2(t)$, respectively.

To derive $p_1(t)$ and $p_2(t)$, we first discuss the impact of the existence of RX$_1$ on $p_2(t)$, based on [7]. As shown in Fig. 1, we classify all possible diffusion paths of molecules by time $t$ in this environment into three paths, namely path 1, path 2, and path 3. Path 1 is for molecules diffusing in the environment, and path 2 and path 3 are for molecules that have hit RX$_2$ and RX$_1$, respectively. If only RX$_2$ exists, we can further classify path 3 into path 3a and path 3b. Path 3a represents molecules that do not hit RX$_2$ after firstly arriving at the location of RX$_1$ at time $\tau < t$, and path 3b represents molecules that hit RX$_2$ after firstly arriving at the location of RX$_1$ at time $\tau$. Given that $f(d_2,t)$ denotes the hitting rate from the TX to RX$_2$ when only RX$_2$ exists, we find that $p_2(t)$ is less than $f(d_2,t)$, due to the existence of RX$_1$. Accordingly, $p_2(t)$ is obtained as [7, eq. (12)]

$$p_2(t) = f(d_2,t) - \gamma(t),$$

(4)

where $\gamma(t)$ is the reduced hitting rate impacted by the existence of RX$_1$. Based on the division in Fig. 1, $\gamma(t)$ is the hitting rate of path 3b and derived as

$$\gamma(t) = \int_0^t p_1(\tau)f(d_1 + d_2, t - \tau)d\tau,$$

(5)

where $f(d_1 + d_2, t - \tau)$ is the hitting rate from RX$_1$ to RX$_2$ when RX$_1$ is regarded as the TX and RX$_2$ is the only RX. Combining (4) and (5), we derive $p_2(t)$ as [7, eq. (17)]

$$p_2(t) = f(d_2,t) - \int_0^t p_1(\tau)f(d_1 + d_2, t - \tau)d\tau.$$  

(6)

Similarly, we obtain $p_1(t)$ as

$$p_1(t) = f(d_1,t) - \int_0^t p_2(\tau)f(d_1 + d_2, t - \tau)d\tau.$$  

(7)

where $f(d_1,t)$ is the hitting rate from the TX toRX$_1$ when only RX$_1$ exists.

Based on (6) and (7), we solve the closed-form expressions for $P_2(t)$ and $p_2(t)$ in the following theorem:

**Theorem 1:** The fraction of absorbed molecules at RX$_2$ by time $t$ for an impulsive emission of molecules is given by

$$P_2(t) = \sum_{i=0}^\infty \left( R(2(i+1)d_1 + (2i+3)d_2, t, 2) - R(2(i+2)d_1 + (2i+3)d_2, t, 2) - R(2id_1 + (2i+1)d_2, t, 0) + R(2(i+1)d_1 + (2i+1)d_2, t, 0) \right),$$

(8)

where $R(x, t, a)$ is given by

$$R(x, t, a) = \frac{\theta}{2\sqrt{xD}} \left( \alpha(t) - \hat{\alpha}(t) \right) - \frac{i}{2} \hat{\beta}(t) + \alpha \hat{\beta}(t) - \frac{\theta}{\sqrt{\pi Dt}} \exp \left( -\frac{x^2}{4Dt} - kt \right) + \left( i + 1 \right).$$

(9)

In (9), $\theta = (d_1 + d_2)(i+1)(i+a)$, $\alpha = \exp \left( x\sqrt{\frac{k}{D}} \right)$, $\hat{\alpha} = \exp \left( -x\sqrt{\frac{k}{D}} \right)$, $\beta(t) = \operatorname{erfc} \left( \frac{x}{\sqrt{4Dt}} + \sqrt{kt} \right)$, and $\hat{\beta}(t) = \operatorname{erfc} \left( \frac{x}{\sqrt{4Dt}} - \sqrt{kt} \right).$ The corresponding hitting rate at RX$_2$ by time $t$, $p_2(t)$, is obtained by taking the derivative of (8) with
respect to \( t \). By doing so, the expression for \( p_2(t) \) is similar to (8), except for replacing \( R(x,t,a) \) with \( r(x,t,a) \), where 
\[
r(x,t,a) = \frac{dR(x,t,a)}{dt}
\]
and is given by 
\[
r(x,t,a) = \frac{i+1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt} - kt\right) \times \left(1 - \frac{x^2}{2Dt}\right)(d_1 + d_2)(i + a - x).
\]

**Proof:** Please see Appendix A.

We denote the asymptotic fraction of absorbed molecules as \( t \to \infty \) at RX1 and RX2 by \( P_{1,asy} \) and \( P_{2,asy} \), respectively. We derive and present \( P_{2,asy} \) in the following theorem:

**Corollary 1:** The asymptotic fraction of absorbed molecules at RX2 as \( t \to \infty \) is given by 
\[
P_{2,asy} = \left\{ \begin{array}{ll}
\frac{\exp\left(-d_2\sqrt{\frac{\Delta t}{D}}\right) - \exp\left(-\frac{(2d_1 + d_2)\sqrt{\Delta t}}{2}\right)}{1 - \exp\left(-\frac{(2d_1 + d_2)\sqrt{\Delta t}}{2}\right)}, & k \neq 0 \\
\frac{d_1}{d_1 + d_2}, & k = 0.
\end{array} \right.
\]

**Proof:** Please see Appendix B.

**Remark 1:** The closed-form expressions for \( p_1(t) \), \( p_1(t) \) and \( P_{1,asy} \) are obtained by exchanging \( d_1 \) and \( d_2 \) therein for \( P_{2,asy} \), \( p_2(t) \) and \( P_{2,asy} \), respectively.

**Remark 2:** By setting \( k = 0 \) in the expressions for \( p_2(t) \) and \( p_2(t) \), we can obtain the corresponding expressions without the occurrence of molecular degradation.

### IV. Numerical Results

In this section, we present numerical results to validate our theoretical analysis in Section III and provide insightful discussions. The simulation results are conducted using a particle-based simulation method [12], where all results are averaged over 2000 realizations and the simulation time step is \( \Delta t_{sim} = 0.001 \text{s} \). Throughout this section, we set the diffusion coefficient \( D = 79.4 \mu \text{m}^2/\text{s} \) [13], an impulse of emission \( N_{ex} = 5000 \) molecules, \( \Delta t = 0.5 \text{s} \), and \( k = 0.8 \text{s}^{-1} \), unless otherwise stated. In all figures, we observe precise agreement between our simulation results and the analytical curves generated from Section III, which demonstrate the validity of our analysis.

In Fig. 2, we investigate the number of summation terms that should be applied in (8). In Fig. 2(a), we apply three data sets to investigate the impact of \( d_1 \), \( d_2 \), and \( k \) on the number of summation terms. First, when only applying \( i = \{0,1\} \) in (8), we observe that the equation first reaches the highest point, i.e., \( P_{2,asy} \), and then drops. Applying a larger number of terms results in \( P_2(t) = P_{2,asy} \) for a longer time, but increases computational complexity. To reduce such complexity, we clarify that applying \( i = \{0,1\} \) is adequate since it enables (8) to reveal the absorbed molecules before becoming visually indistinguishable from the asymptotic value, and increasing terms in (8) does not change the absorbed molecules calculated before reaching the asymptotic value. After reaching the asymptotic value, we set \( P_2(t) = P_{asy} \). Second, comparing data set i) with data set ii) and data set iii), we observe that changing \( d_1 \), \( d_2 \), and \( k \) does not change the fact that applying \( i = \{0,1\} \) is adequate for (8). In Fig. 2(b), we apply two data sets to investigate the impact of \( D \) on the number of terms.

We still observe that changing \( D \) does not impact the fact that applying \( i = \{0,1\} \) is adequate for (8).

In Fig. 3, we plot molecules absorbed at RX2 by time \( t \) using (8) and [7, eq. (13)], respectively. We note that [7, eq. (13)] was initially applied to a 3D environment but can also be applied to a 1D environment\(^1\). In this figure, we first keep \( k = 0 \text{s}^{-1} \) and vary \( d_1 \), \( d_2 \), and \( D \) to investigate the accuracy of (8) and [7, eq. (15)]. We also investigate the molecular degradation by setting \( k = 0.8 \text{s}^{-1} \) and plot the absorbed

\(^1\)In the 3D environment, [7] assumed that molecules are absorbed at the same points before reaching another RX. The points on RX1 and RX2 are denoted by \( s'_1 \) and \( s'_2 \), where \( s'_1 \) and \( s'_2 \) are found numerically. As RX1 and RX2 are regarded as points in a 1D environment, \( s'_1 \) and \( s'_2 \) are points RX1 and RX2. Substituting \( s'_1 \) and \( s'_2 \) with RX1 and RX2 in [7, eq. (13)], we obtain the fraction of absorbed molecules at RX1 in a 1D environment, based on the method in [7].
molecules at RX2 with (8). First, we clearly observe that the simulation matches well with (8) and a gap exists between the simulation and [7, eq. (13)] for $3 \leq t \leq 30$ s, which demonstrates the accuracy advantage of (8) relative to [7, eq. (13)]. Second, we observe that the asymptotic value of (8) and [7, eq. (13)] converge as $t \to \infty$. Last, we observe that (11) matches with the simulation for both $k = 0$ and $k = 0.8$ s$^{-1}$ when $t$ is large, which demonstrates the correctness of (11).

In Fig. 4, we plot the minimum $d_1$ that does not impact molecular absorption at RX2 versus $d_2$, and the minimum $d_2$ that does not impact molecular absorption at RX1 versus $d_1$, for different $k$. We examine the impact of RX1 on RX2 based on the gap between the fraction of absorbed molecules at RX2 and the fraction of absorbed molecules for the single RX as $t \to \infty$, which is expressed as $F(d_2, t \to \infty) - P_{2,asy}$, where $F(d_2, t \to \infty)$ is given by (3) and $P_{2,asy}$ is given by (11). We calculate the minimum $d_1$ which satisfies the condition $(F(d_2, t \to \infty) - P_{2,asy}) / F(d_2, t \to \infty) < 0.01$ for given $d_2$. We also suppose that RX1 does not impact the molecular absorption at RX2 when this condition is satisfied. Similarly, we calculate the minimum $d_2$ for given $d_1$. From this figure, we first observe that minimum $d_1$ intersects minimum $d_2$ when $k \neq 0$. The upper right area of the intersection for each $k$ represents the range for $d_1$ and $d_2$ of two RXs that do not impact each other, because the condition for two RXs not impacting each other’s molecular absorption is that $d_1$ and $d_2$ are simultaneously larger than minimum $d_1$ and minimum $d_2$, respectively. For example, if $d_1$ and $d_2$ are in the grey area, then two RXs do not impact each other when $k = 0.3$ s$^{-1}$. When $k = 0$, there is no intersection such that two RXs always impact each other. Second, we observe that the range for two RXs not impacting each other decreases with decreasing $k$. When $k$ decreases, there is a larger number of molecules in the environment such that the mutual influence between two RXs is higher, which results in the less range for two RXs not impacting each other. Third, we observe that minimum $d_1$ and minimum $d_2$ firstly increase when $d_2$ and $d_1$ increase, respectively, and then become constant after the intersection. This is because increasing $d_2$ means that the molecular absorption at RX2 decreases. In this case, if $d_1$ keeps the same value, then the molecular absorption at RX1 relatively increases, which results in a larger impact on the molecular absorption at RX2. Therefore, the minimum $d_1$ increases to reduce the impact on RX2. Beyond the intersection, two RXs will not impact each other such that increasing $d_2$ will not lead to increase in minimum $d_1$.

V. Conclusion

In this letter, we focused on a 1D molecular communication system to investigate channel responses between a single TX and two fully-absorbing RXs. We derived new closed-form expressions for i) the fraction of absorbed molecules, ii) the corresponding hitting rate, and iii) the asymptotic fraction of absorbed molecules as time approaches infinity at each RX. Our results showed that our analytical expressions are accurate. We also investigated distance ranges for two RXs that do not impact molecular absorption of each other, which showed that the mutual influence between two RXs decreases with the increase in the degradation rate. Future work includes extending the 1D environment to 3D and deriving the CR between one TX and multiple RXs that partially absorb molecules.

APPENDIX A

**Proof of Theorem 1**

Taking the integral for both (6) and (7) over the interval $[0, t]$, we obtain

$$P_2(t) = F(d_2, t) - P_1(t) * f(d_1 + d_2, t),$$

$$P_1(t) = F(d_1, t) - P_2(t) * f(d_1 + d_2, t),$$

where $*$ is the convolution operator. Substituting $P_1(t)$ in (12) with (13) and performing the Laplace transform, we obtain

$$P_2(s) = \frac{\exp(-2d_1\sqrt{\frac{s+k}{D}}) - \exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}})}{s(1 - \exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}}))},$$

where $P_2(s)$ is the Laplace transform of $P_2(t)$. To obtain the inverse Laplace transform of (14), we define two new equations as

$$G(s) = \frac{\exp(-d_2\sqrt{\frac{s+k}{D}}) - \exp(-2d_1 + d_2)\sqrt{\frac{s+k}{D}})}{(s+k)(1 - \exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}}))},$$

and

$$H(s) = \frac{\exp(-d_2\sqrt{\frac{s+k}{D}})}{s^2(1 - \exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}}))} - \frac{\exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}})}{s^2(1 - \exp(-2(d_1 + d_2)\sqrt{\frac{s+k}{D}}))}.$$

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We note that \( P_2(s) = G(s) + \frac{1}{s} \mathcal{L}^{-1}(G(s)) \) and \( G(s) = H(s) - H(s) \). Thus, we first solve the inverse Laplace transform of \( H(s) \). From (10), we observe that \( H_1(s) \) and \( H_2(s) \) have similar forms. Thus, we only show the process of performing the inverse Laplace transform of \( H_1(s) \). We re-write \( H_1(s) \) as
\[
H_1(s) = \exp \left( \frac{-d_2}{\sqrt{D}} s \right) \times \frac{1}{s \left( 1 - \exp \left( \frac{-d_1 + d_2}{\sqrt{D}} s \right) \right)}.
\]

According to [14, eqs. (5.1), (5.34), (5.36), (1.18)], the inverse Laplace transform of \( H_1(s) \), denoted by \( h_1(t) \), is
\[
h_1(t) = \begin{cases} 
0, & 0 < t < \frac{d_2}{\sqrt{D}} \\
(i + 1) \left( t - \frac{d_2}{\sqrt{D}} - \frac{d_1 + d_2}{\sqrt{D}} i \right), & \frac{d_2}{\sqrt{D}} < t < \frac{d_1}{\sqrt{D}} + 2(i + 1) \frac{d_1 + d_2}{\sqrt{D}}, \\
\frac{d_2}{\sqrt{D}} + \frac{2(i + 1) d_1 + (i + 1) d_2}{\sqrt{D}} \exp \left( -\frac{u^2}{4t} \right), & i = 0, 1, 2, 3, \ldots 
\end{cases}
\]

According to [14, eq. (1.27)], the inverse Laplace transform of \( H_1(\sqrt{s}) \) is
\[
\mathcal{L}^{-1} \left\{ H_1(\sqrt{s}) \right\} = \frac{1}{\sqrt{4\pi t}} \int_0^\infty u \exp \left( -\frac{u^2}{4t} \right) h_1(u) du
\]
\[
= \frac{1}{\sqrt{4\pi t}} \sum_{i=0}^\infty \left[ 2 \exp \left( -\frac{u^2}{4t} \right) \left( i d_1 + (i + 1) d_2 \right) - u \right]
+ \frac{1}{\sqrt{4\pi t}} \exp \left( \frac{u}{\sqrt{4t}} \right),
\]
where \( F(x)_i^b = F(b) - F(a) \). As aforementioned, \( H_1(s) \) and \( H_2(s) \) have similar forms. Therefore, the inverse Laplace transform of \( H_2(\sqrt{s}) \), denoted as \( \mathcal{L}^{-1} \{ H_2(\sqrt{s}) \} \), can be derived analogously. Based on [14, eq. (1.3)], (19), and \( \mathcal{L}^{-1} \{ H_2(\sqrt{s}) \} \), the inverse Laplace transform of \( G(s) \), denoted by \( g(t) \), is derived as
\[
g(t) = \exp \left( -kt \right) \mathcal{L}^{-1} \left\{ H_1(\sqrt{s}) \right\} - \mathcal{L}^{-1} \left\{ H_2(\sqrt{s}) \right\} \quad \text{(20)}
\]

Given \( P_2(s) = G(s) + \frac{1}{s} \mathcal{L}^{-1}(G(s)) \), the inverse Laplace transform of \( P_2(s) \) is
\[
P_2(t) = g(t) + \int_0^t g(u) du \quad \text{(21)}
\]

Substituting (20) into (21), we obtain (8).

**APPENDIX B**

**PROOF OF THEOREM 1**

According to the final value theorem, if \( P_2(t) \) has a finite limit as \( t \to \infty \), we have
\[
\lim_{t \to \infty} P_2(t) = \lim_{s \to 0} sP_2(s). \quad \text{(22)}
\]

When \( k \neq 0 \), substituting (14) into (22), we obtain
\[
P_{2,asy} = \frac{\exp \left( -d_2 \sqrt{\frac{k}{D}} \right) - \exp \left( -2d_1 + d_2 \sqrt{\frac{k}{D}} \right)}{\exp \left( -2d_1 + d_2 \sqrt{\frac{k}{D}} \right)} \quad \text{(23)}
\]

When \( k = 0 \), we apply L'Hôpital's rule [15] to (23), and we have
\[
P_{2,asy} = \lim_{k \to 0} \frac{\exp \left( -d_2 \sqrt{\frac{k}{D}} \right) - \exp \left( -2d_1 + d_2 \sqrt{\frac{k}{D}} \right)}{\exp \left( -2d_1 + d_2 \sqrt{\frac{k}{D}} \right)} = \frac{d_1}{d_1 + d_2} \quad \text{(24)}
\]
Combining (23) and (24), we obtain (11).

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