KAC-MOODY GROUPS AND COMPLETIONS

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Abstract. In this paper we construct a new “pro-$p$-complete” topological Kac-Moody group and compare it to various known topological Kac-Moody groups. We come across this group by investigating the process of completion of groups with BN-pairs. We would like to know whether the completion of such a group admits a BN-pair. We give explicit criteria for this to happen.

In this paper we study Kac-Moody groups over finite fields. A Kac-Moody group is a generalisation of the notion of a reductive group to a more general Kac-Moody root datum $\mathcal{D}$ or a closely related group with a BN-pair. Connected reductive groups are classified via a one-to-one correspondence to root data of finite type. A root datum of finite type yields a group scheme, generalised by Tits to a construction of a functor $G_{\mathcal{D}}$ from the category of commutative rings to the category of groups [T2, T3]. For instance, the Steinberg central extension $\hat{\text{SL}}_n(\mathbb{F}_q[z, z^{-1}])$ is a Kac-Moody group $G_{\mathcal{D}}(\mathbb{F}_q)$ for the simply-connected root datum $\mathcal{D}$ of the affine type $\tilde{A}_{n-1}$ (see [CKRu, sec. 6] for further details). On the other hand, $\text{SL}_n(\mathbb{F}_q[z, z^{-1}])$ is not of the form $G_{\mathcal{D}}(\mathbb{F}_q)$, yet it is still called a Kac-Moody group.

A topological Kac-Moody group is a locally compact totally disconnected topological group that contains a Kac-Moody group. Often it is obtained by completion of a Kac-Moody group. In the examples of the previous paragraph, one arrives at topological Kac-Moody groups $\hat{\text{SL}}_n(\mathbb{F}_q((z)))$ and $\text{SL}_n(\mathbb{F}_q((z)))$.

There are several known topological Kac-Moody groups\footnote{Since this paper a new book by Marquis [M3] was published, which would be a comprehensive source for further reading on the subject.}. They are the Mathieu-Rousseau group $G^{ma+}$, the Carbone-Garland group $G^{ca}$ and the Caprace-Rémy-Ronan group $G^{cr}$. Each of them contains $G_{\mathcal{D}}(\mathbb{F}_q)$. In this paper we show the existence of a new topological Kac-Moody group $\hat{G}$.

Main Theorem. Let $A$ be an irreducible generalised Cartan matrix, $\mathcal{D}$ a simply connected root datum of type $A$ and $\mathbb{F} = \mathbb{F}_q$ a finite field of characteristic $p$ ($q = p^a$, $a \in \mathbb{N}$). Let $G := G_{\mathcal{D}}(\mathbb{F}_q)$ be the corresponding minimal Kac-Moody group. Recall that it has a BN-pair $(B, N)$ with $B = U \rtimes T$ where $U = \langle X_\alpha \mid \alpha \in \Delta^+_r \rangle$ (see Section 2, where the notations are introduced).
There exists a locally compact totally disconnected group $\hat{G}$ satisfying the following conditions:

1. $G$ is a dense subgroup of $\hat{G}$.
2. $\hat{G}$ has a BN-pair $(\hat{B}, N)$ where $\hat{B} = \hat{U} \times T$ and $\hat{U}$ is the full pro-$p$ completion of $U$.
3. If $G = G_{crr}$ or $C_{c\lambda}$, or in the case when $G$ is dense in $G_{ma} +$ and $G = G_{ma} +$, then there exists an open continuous surjective homomorphism $\hat{G} \rightarrow G$.
4. Let $Z'(\hat{G}) := Z(G) \times C(\hat{G})$ where $C(\hat{G}) = \bigcap_{g \in \hat{G}} \hat{U}^{g}$.
   a. $\hat{G} / Z'(\hat{G})$ is a topologically simple group.
   b. If $A$ is 2-spherical, then $\hat{G} / Z'(\hat{G})$ is an abstractly simple group.

Let us explain the content of the present paper. We investigate the process of completion of a group $G$ with a BN-pair in Chapter 1. The main result is Theorem 1.1 that contains a sufficient condition for the completion $\hat{G}$ to inherit a BN-pair from $G$. It relies on Tits’ description of groups with BN-pairs by generators and relations [T1]. The remainder of Chapter 1 contains several technical or user-friendly results about these completions. For instance, Theorem 1.2 conveniently constructs the completion $\hat{G}$ together with its BN-pair.

We put our completion results to good use in Chapter 2. After quickly recalling the definition of $G_D(\mathbb{F}_q)$ we construct the new group $\hat{G}_D(\mathbb{F}_q)$. We compare it to other known completions and address its topological simplicity in Theorem 2.2 and its algebraic simplicity in Theorem 2.4. Thus, the Main Theorem is a combination of results in Section 2.

There have been previous attempts to compare various topological Kac-Moody groups: Capdeboscq and Rémy [CR], Baumgartner and Rémy [CarE Ri, 2.6], Marquis [M2], and Rousseau [Rou] all discuss the maps between different completions at length. We address these questions in Section 2 only modulo “congruence kernel” $Z'(\hat{G})$ whose full computation remains mysterious. We devote the last chapter of the paper to several observations about $Z'(\hat{G})$. Our major insight into the nature of the congruence kernel is its parabolic decomposition in Theorem 3.5.

Hristova and Rumynin study representations of topological Kac-Moody groups [HrRu]. The groups $\hat{G}$ are new examples for their theory.

1. Completion Theorem

Let $(G, T)$ be a Hausdorff topological group ($G$ is a group, $T$ is a topology). The topology determines a right uniformity on $G$ that we now describe following Bourbaki [B1]. Pick a basis $\mathcal{J}$ of the topology at 1. The basis of uniformity $\mathcal{J}^{\hat{}}$ is $\mathcal{J}^{\hat{}} = \{ V^{\hat{}} | V \in \mathcal{J} \}$ where $V^{\hat{}} = \{(x, y) \in G^2 | xy^{-1} \in V \}$. The completion $\hat{G}$ is the set of all minimal Cauchy filters on $(G, T^{\hat{}})$. Recall that a filter is a non-empty collection $\mathcal{F}$ of open sets closed under intersections and oversets. A filter is Cauchy if it contains arbitrary “small” subsets, i.e., for each $V \in \mathcal{J}$ there exists $U \in \mathcal{F}$ such that $xy^{-1} \in V$ for all $x, y \in U$. We define the left uniformity $\mathcal{J}^{\hat{}} T$ in a similar way, using $\mathcal{J}^{\hat{}} V = \{(x, y) \in G^2 | x^{-1} y \in V \}$ instead. The inverse map is an isomorphism of uniform spaces $\text{Inv} : (G, T^{\hat{}}) \rightarrow (G, \mathcal{J}^{\hat{}} T)$ imposing an isomorphism between the right and left completions.
The completion $\hat{G}$ is always a monoid, although the multiplication $\text{Mult} : (G, T^\diamond) \times (G, T^\diamond) \to (G, T^\diamond)$ is not uniformly continuous in general. It is a monoid because $\text{Mult}(F, G)$ is a Cauchy filter for two Cauchy filters $F$, $G$ on $(G, T^\diamond)$ [B1 Prop. III.3.4(6)]. On the other hand, the completion is not necessarily a group: $\text{Inv} : (G, T^\diamond) \to (G, T^\diamond)$ is uniformly continuous but we have no information about uniform continuity of $\text{Inv} : (G, T^\diamond) \to (G, T^\diamond)$. The latter uniform continuity is a sufficient (but not necessary) condition for $\hat{G}$ to be a group. A necessary and sufficient condition is the following: if $\mathcal{F}$ is a Cauchy filter on $(G, T^\diamond)$, then $\text{Inv}(\mathcal{F})$ is a Cauchy filter on $(G, T^\diamond)$ [B1 Th. III.3.4(1)].

For reader’s convenience we sketch an example of a group $G$ with non-a-group $\hat{G}$ following a hint in Bourbaki [B1 Exercise X.3(16)]. Let $G$ be the group of auto-homeomorphisms of $[0, 1]$ with the topology of uniform convergence. It suffices to exhibit a uniformly convergent sequence $f_m$ of homeomorphisms such that the sequence of inverses $f_{m}^{-1}$ is not uniformly convergent. The following sequence fits the bill:

$$f_m(x) = \begin{cases} x^m & \text{if } x \leq \frac{1}{2} \\ (2 - 2^{1-m})x + (2^{1-m} - 1) & \text{if } x \geq \frac{1}{2} \end{cases}$$

Now suppose that $G$ admits a BN-pair $(B, N)$. The key question is whether the completion $\hat{G}$ admits a BN-pair. Let $\overline{B}$ be the closure of $B$ in $\hat{G}$. It is a moot point that $\overline{B}$ is isomorphic to the completion of $B$ in the restriction uniformity $T^\diamond |_B$ [B1 Cor. II.3.9(1)]. A candidate BN-pair on $\hat{G}$ is $(\overline{B}, N)$ but it does not work in general. Let $G$ be a simple split group scheme, $G = G(\mathbb{F}[z, z^{-1}])$ its points over Laurent polynomials over a finite field, $N \leq G$ the group of monomial matrices, $I_\leq = \{G(\mathbb{F}[z^{-1}]) \xrightarrow{z^{-1} \mapsto 0} G(\mathbb{F}) \}^{-1}(B)$ its negative Iwahori. The pair $(I_\leq, N)$ is a BN-pair on $G$ but $(\overline{I_\leq}, N) = (I_\leq, N)$ is not a BN-pair on the positive completion $\hat{G} = G(\mathbb{F}(z))$: the countable groups $I_\leq$ and $N$ cannot generate uncountable $\hat{G}$. A reader can see that the condition 3 of Theorem [14] fails for the negative Iwahori. On the other hand, if $\mathbb{F}$ is finite, all conditions of Theorem [14] holds for the positive Iwahori $I_\geq = \{G(\mathbb{F}[z]) \xrightarrow{z \mapsto 0} G(\mathbb{F}) \}^{-1}(B)$ so that the theorem yields the standard BN-pair on $\hat{G}$.

Nevertheless, we can prove the following partial affirmative answer, sufficiently general for the study of Kac-Moody groups. Let us explain some notations before stating the theorem. The notations $(B, N)$ and $W = (W, S)$ are standard for the groups with BN-pairs. The homomorphism $\pi : N \to W$ is the natural surjection. For elements $s \in S$, $w \in W$ we choose some liftings $\dot{s} \in \pi^{-1}(s)$, $\dot{w} \in \pi^{-1}(w)$. The minimal parabolic $P_s$ is the subgroup generated by $B$ and $\dot{s}$.

**Theorem 1.1.** Let $G$ be a Hausdorff topological group with a BN-pair $(B, N)$ with the Weyl group $(W, S)$ where $S$ is finite. If the following three conditions hold, then $(\overline{B}, N)$ is a BN-pair on the completed group $\hat{G}$.

1. The completion $\hat{G}$ is a group.
2. The index $|P_s : B|$ is finite for all $s \in S$.
3. $B$ is open in $G$.

**Proof.** We have systems of subgroups: $\mathfrak{A} = (B, N, P_s; s \in S)$ of $G$ and $\mathfrak{B} = (\overline{B}, N, \overline{P_s}; s \in S)$ of $\hat{G}$, where $\overline{X}$ is the closure of $X$. The system of groups $\mathfrak{A}$ satisfies all conditions of Tits’ Theorem as observed by Tits [11]. We claim that under the assumptions of this theorem the system $\mathfrak{B}$ also satisfies these conditions.
We will verify this claim at the end of the proof. For the reader’s convenience we restate Tits theorem (cf. [Ku, Th. 5.1.8]):

**Tits Theorem.** Suppose that the system $\mathcal{B}$ satisfies the following conditions:

1. If $s \neq t \in S$, then $F_s \cap F_t = F_{s \circ t}$.
2. The subgroup $B \cap N$ is normal in $N$.
3. Given $s \in S$, let $N_s := F_s \cap N$. Then $N_s/(B \cap N)$ is of order 2 for all $s \in S$.
4. $F_s = B \cup BsB$ for all $s \in S$.
5. The pair $(N/(B \cap N), S)$ is a Coxeter group.
6. Let $\pi : N \to W := N/(B \cap N)$ be the quotient map. For any $n = s_1s_2\cdots s_i \in N$ with $s_i \in S$ such that $s_1s_2\cdots s_i$ is a reduced word in $W$, the subgroup $B(s_1, \ldots, s_i)$ (see (1)) depends only on $w := \pi(n) = s_1\cdots s_i \in W$ and the homomorphism $\gamma(s_1, \ldots, s_i)$ (see (2)) depends only on $n$. (This justifies the notation $B_w$ and $F_n$ from now on.)
7. If $w \in W$, $s \in S$ satisfy $l(ws) = l(w) + 1$, then $B_wB_s = B_{ws}$.
8. If $w \in W$, $s, t \in S$ satisfy $tw^{-1} = s$ and $l(wt) = l(w) + 1$, then for all $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and $b \in B \setminus B_t$, there exist $y \in bB_t \cap B_n$ and $y', y'' \in B_n$ such that
   \begin{itemize}
   \item[(a)] $x'^{-1}yx' = y'y''$ in $F_t$
   \item[(b)] $x^{-1}y^t x^{-1}x = y^{-1}y''x^{-1}x^{-1}y''^{-1}$ in $F_s$.
   \end{itemize}
   where $x' := n^{-1}x^{-1}n \in \pi^{-1}(t)$.
9. $B$ is not normal in any $F_s$.

Then the canonical map

$$N \cup (\cup_{s \in S} F_s) \longrightarrow \tilde{G} := \ast H$$

to the amalgam $\tilde{G}$ is injective. The amalgam $\tilde{G}$ admits a $BN$-pair $(B, N)$ with a set of simple reflections $S$, where we identify the groups $B$ and $N$ with their images in $\tilde{G}$ under the canonical map.

Furthermore, consider a group $G'$ and an injective function

$$\varphi : N \cup (\cup_{s \in S} F_s) \longrightarrow G'$$

such that $\varphi|_N$ and all $\varphi|_{F_s}$ are group homomorphisms. If $G'$ is generated by the image of $\varphi$, then the canonical homomorphism $\varphi^* : \tilde{G} \to G'$ is an isomorphism.

Tits’ Theorem applies to $G' = \tilde{G}$, allowing us to conclude that the group $\tilde{G}$ admits a $BN$-pair $(B, N)$ and the natural group homomorphism $f : \tilde{G} \to \tilde{G}$ is injective.

It remains to check surjectivity of $f$. The image $f(\tilde{G})$ contains $N$ and $B$, which generate $G$. Hence, $f(\tilde{G})$ is dense in $\tilde{G}$. On the other hand, $f(\tilde{G})$ contains $\mathcal{B}$, which is open in $\tilde{G}$ because $B$ is open in $\tilde{G}$. Thus, $f(\tilde{G})$ is open in $\tilde{G}$ but it is a subgroup, hence, $f(\tilde{G})$ is also closed in $\tilde{G}$. Being closed and dense, $f(\tilde{G})$ must be equal to $\tilde{G}$.

It only remains to verify all nine conditions in Tits’ Theorem for $\mathcal{B}$. Our starting point is that these nine conditions hold in $G$ for $\mathfrak{A}$ as shown by Tits [11].

1. We know that $B \cap N$ is normal in $N$. Clearly, $B \cap N \subseteq B \cap N$. In the opposite direction, $B \cap N = (B \cap G) \cap N$. An element $x \in B \cap G$ is a limit of a net

$$x = \lim_{m \in \mathbb{M}} b_m, \quad b_m \in B, \quad \mathbb{M} \text{ is an ordinal.}$$
This limit works in $G$ as well where $B$ is open, hence, closed. Thus, $x \in B$ and $\overline{B} \cap G = B$. Therefore, $\overline{B} \cap N = B \cap N$ is normal in $N$.

(P1) Let $s \neq t \in S$. Since $P_s = B \dot{s} B \cup B$, we conclude that $P_{s \cdot t} = B \dot{s} B \cup B$. Let us prove now that $\overline{B \dot{s} B} \cap \overline{B \dot{t} B} = \emptyset$. An element $x \in \overline{B \dot{s} B} \cap \overline{B \dot{t} B}$ is a limit of two nets

$$x = \lim_{m \in M} a_m \bar{s} b_m = \lim_{m \in M} c_m \dot{l} d_m , \quad a_m, b_m, c_m, d_m \in B.$$ 

Since $\overline{B}$ is open there exists an ordinal $L < M$ such that $a_m \bar{s} b_m x^{-1} \in \overline{B} \ni x(c_m \dot{l} d_m)^{-1}$ and consequently $a_m \bar{s} b_m d_m^{-1} c_m^{-1} \in \overline{B}$ for all $m \geq L$. Clearly $a_m \bar{s} b_m d_m^{-1} c_m^{-1} \in G$. It is shown $(P_2)$ that $\overline{B} \cap G = B$, thus, $a_m \bar{s} b_m d_m^{-1} c_m^{-1} \in B$ for all $m \geq L$. On the other hand, these elements $a_m \bar{s} b_m d_m^{-1} c_m^{-1}$ lie in $B \dot{s} B \dot{t} B$ equal to the double coset $B \dot{s} t B$ since $l(\dot{s} t) = 2 = l(\dot{s}) + l(\dot{t})$. Since $B \dot{s} B \dot{t} B \cap B = \emptyset$, no such $x$ exists. Therefore, $P_{s \cdot t} \cap P_t = \overline{B}$.

(P3) The minimal parabolic $P_s$ is a union of cosets of $B$, hence, open in $G$. Similarly to the proof in $(P_2)$, $N_s = \overline{P_s} \cap N$ is equal to $(\overline{P_s} \cap G) \cap N = P_s \cap N$. Therefore, $N_s / (\overline{B} \cap N)$ is of order 2 for all $s \in S$.

(P4) By condition (2), $B$ has finite index in $P_s$. Hence $B \dot{s} B = X \dot{B}$ for some finite subset $X \subseteq P_s$. Observe now that

$$\overline{B \dot{s} B} \subseteq \overline{B \dot{s} B} = X \dot{B} \supseteq X \dot{B} \subseteq \overline{B \dot{s} B}.$$ 

Since $X$ is finite, $X \dot{B}$ is closed and the inclusion $f$ is an equality. Thus, $\overline{B \dot{s} B} = \overline{B \dot{s} B}$. Therefore, $\overline{P_s} = \overline{B} \cup \overline{B \dot{s} B} = \overline{B} \cup \overline{B \dot{s} B}$.

(P5) We have proved in $(P_2)$ that $\overline{B} \cap N = B \cap N$. Therefore, $(N / \overline{B} \cap N, S) = (W, S)$ is a Coxeter group.

(P6) Let us first observe that if $A, K \leq G$ are open subgroups, then $A \cap K = A \cap K \cap \overline{A \cap K}$ in $G$. Indeed, the inclusion $\supseteq$ is obvious. To prove the inclusion $\subseteq$ consider $x \in A \cap K$. It is a limit of two nets

$$x = \lim_{m \in M} a_m = \lim_{m \in M} k_m , \quad a_m \in A, \ k_m \in K.$$ 

Then the net $a_m k_m^{-1} = (a_m x)(k_m^{-1})$ converges to $1 \in G$. Since $A \cap K$ is open there exists an ordinal $L < M$ such that $a_m k_m^{-1} \in A \cap K$, and consequently $a_m, k_m \in A \cap K$ for all $m \geq L$. Thus, $x \in A \cap K$.

The following subgroups are defined recursively for a reduced word $s_1 s_2 \cdots s_t \in W$ and its fixed lift $n = s_1 \dot{s}_2 \cdots \dot{s}_t \in N$

$$B(s_1, \ldots, s_t) := B \cap \dot{s}_1^{-1} B(s_1, \ldots, s_{t-1}) \dot{s}_1,$$

(1) $$\overline{B}(s_1, \ldots, s_t) := \overline{B} \cap \dot{s}_1^{-1} \overline{B}(s_1, \ldots, s_{t-1}) \dot{s}_1.$$ 

The aforementioned observation implies that $\overline{B}(s_1, \ldots, s_t) = \overline{B}(s_1, \ldots, s_t)$. Consequently, the homomorphism

(2) $$\overline{\gamma}(\dot{s}_1, \ldots, \dot{s}_t) : \overline{\overline{B}}(s_1, \ldots, s_t) \to \overline{B}, \quad x \mapsto \dot{s}_1 \cdots \dot{s}_t x \dot{s}_t^{-1} \cdots \dot{s}_1^{-1}$$

is uniquely determined by its restriction $\gamma(\dot{s}_1, \ldots, \dot{s}_t) : B(s_1, \ldots, s_t) \to B$. Property (P6) hold for $\overline{A}$. This means that $B(s_1, \ldots, s_t)$ depends only on the element $w = s_1 \cdots s_t \in W$, not the word or the choice of the liftings $\dot{s}_i \in N$. It also means that $\gamma(\dot{s}_1, \ldots, \dot{s}_t)$ depends only on the element $n = \dot{s}_1 \cdots \dot{s}_t \in N$. Therefore, the subgroup $\overline{B}(s_1, \ldots, s_t)$ and the homomorphism $\overline{\gamma}(\dot{s}_1, \ldots, \dot{s}_t)$ depend only on $w \in W$ and $n \in N$ correspondingly. (We denote these $B_w, \overline{B}_w, \gamma_n, \overline{\gamma}_n$.)
(P$_7$) We begin by proving that all subgroups $B_w$, $w \in W$ are commensurable. We proceed by induction on the length $l(w)$ to show that $B_w$ has finite index in $B$.

If $l(w) = 1$, then $w = s$ for some $s \in S$. Since $xB_s \mapsto xsbB$ is an embedding of quotient sets $B/B_s \mapsto P_s/B$, we conclude that $|B : B_s| \leq |P_s : B| < \infty$ by assumption (2).

Suppose the case of $l(w) = m - 1$ is settled. Consider $w \in W$, $s \in S$ with $l(ws) = m$. Then $B_{ws} = B \cap s^{-1}Bs$. Hence,

$$|B : B_{ws}| = |B : B_s||B \cap s^{-1}Bs| \leq |B : B_s||B : B_w| < \infty$$

since by induction assumption $|B : B_w| < \infty$.

Property (P$_7$) for $\mathfrak{A}$ ensures that $B_wB_s = B$ if $l(ws) = l(w) + 1$ as before. Hence, $B = B_wB_s \geq B_wB_s = B_wB_s$.

because $XY \supseteq X Y$ for all subsets $X, Y$ and $B_w = B_w$ for all $w \in W$ as shown in (P$_6$). Since $B_w$ and $B_s$ are commensurable, $B_wB_s = XB_s$ for a finite subset $X \subseteq B_w$.

Then $B_wB_s = XB_s = X \bar{B}_s \subseteq B_wB_s$

since $X \bar{B}_s$ is closed as a finite union of closed cosets of $B_s$. Therefore, $B_wB_s = B$.

(P$_8$) Let $s, t \in S$, $w \in W$ such that $wtw^{-1} = s$ and $l(wt) = l(w) + 1$. Let us fix arbitrary $x \in \pi^{-1}(s)$, $n \in \pi^{-1}(w)$ and define $x' := n^{-1}x^{-1}n \in \pi^{-1}(t)$. Now pick any $b \in \bar{B} \setminus \bar{B}_t$. As shown in (P$_7$), $B = XB_t$ for a finite set $X$. Without loss of generality, $1 \in X$ and $B \setminus B_t = (X \setminus \{1\})B_t$. By the argument as above ($X \bar{B}_t$ is closed etc.), $\bar{B}/\bar{B}_t = (X \setminus \{1\})\bar{B}_t$. Hence, $b = b_1b_2$ for some $b_1 \in X \setminus \{1\}$ and $b_2 \in \bar{B}_t$. This brings property (P$_8$) down to the system $\mathfrak{A}$ where we know it (T1). Therefore, there exist elements $y \in b_1B_t \cap B_w \subseteq b\bar{B}_t \cap \bar{B}_w$ and $y', y^2 \in B_w \subseteq \bar{B}_w$ satisfying $x'y' = y'(y')^{-1}x' \in P_s \subseteq F_s$ and $x\gamma_n(y)y^{-1} = \gamma_n(y')x^{-1}\gamma_n(y^2) \in P_s \subseteq F_s$.

(P$_9$) Recall that $\bar{B} \cap G = B$ and $\bar{F} \cap G = P_s$ as shown in (P$_2$). If $\bar{B}$ were normal in $F_s$, then $B$ would be normal in $P_s$, contradicting (P$_9$) for $\mathfrak{A}$. Therefore, $\bar{B}$ is not normal in $F_s$. 

A shortcoming of Theorem 1.1 is that it requires the group $\hat{G}$ to exist first. It would be useful to tweak the theorem to enable construction of new groups. The next theorem addresses this issue. If $\mathcal{T}$ is a topology on a group $B$, we denote $T_1 := \{ A \in \mathcal{T} \mid 1 \in A \}$.

**Theorem 1.2.** Let $G$ be a group with a BN-pair $(B, N)$ with the Weyl group $(W, S)$ where $S$ is finite. Suppose further that a topology $\mathcal{T}$ on $B$ is given such that the four conditions (1)–(4) hold.

1. $(B, \mathcal{T})$ is a topological group.
2. The completion $\hat{B}$ is a group.
3. $T_1$ is a basis at 1 of topology on each minimal parabolic $P_s$, $s \in S$ that defines a structure of topological group on $P_s$.
4. The index $|P_s : B|$ is finite for each $s \in S$.

Under these conditions the following statements hold:

a. $T_1$ is a basis at 1 of topology on $G$ that defines a structure of topological group on $G$.

b. The completion $\hat{G}$ is a group and $\hat{B} = \bar{B} \subseteq \hat{G}$.  


(c) The completion $\hat{G}$ is isomorphic to the amalgam $\ast H$ where

$$B = \{ B, N, P_s ; s \in S \}.$$

(d) The pair $(\overline{B}, N)$ is a BN-pair on the completed group $\hat{G}$.

Proof. (a) We already know that $T_1$ is a filter of neighbourhoods of 1 in a topological group $B$. To verify (a) it suffices to show that for all $g \in G$, $A \in T_1$ it holds that $gAg^{-1} \in T_1$ [B1 Prop. III.1.2(1)]. By (3) we know this property for all $g \in P_s$. Since $G$ is generated by all $P_s$, we conclude the proof.

(b) Denote the aforementioned topology on $G$ by $T_G$. We need to show that the monoid $(G, T_G)$ is a group. Consider $x \in (G, T_G)$ and a convergent net $x_m \to x$, $m \in M$, $x_m \in G$. Since $B \in T_1$, $B$ is open in $(G, T_G)$. The net $x_m$ is Cauchy, so there exists an ordinal $L$ such that $x_m x^{-1}_L \in B$ for all $m \geq L$. Let

$$y_m = \begin{cases} 1 & \text{if } m < L, \\ x_m x^{-1}_L & \text{if } m \geq L. \end{cases}$$

Since $y_m y^{-1}_m = (x_m x^{-1}_L)(x_L x^{-1}_L)^{-1} = x_m x^{-1}_L$, the net $y_m$ is a Cauchy net in $B$. Let $y = \lim y_m \in B$. The inverse $y^{-1} \in \hat{B}$ exists because $B$ is a group. Then

$$(x_L y^{-1}) \cdot x = x_L^{-1} \cdot \lim m y_m^{-1} \cdot \lim x_m = x_L^{-1} \cdot \lim m y_m^{-1} x_m = x_L^{-1} \cdot \lim x_L = 1.$$ 

Similarly, $x \cdot (x_L y^{-1})$. Thus, we have found the inverse $x^{-1} = x_L^{-1} y^{-1}$) $\in \hat{G}$ so that $\hat{B}$ is a group. Coincidence of the completion and the closure is standard.

(c+d) These follow immediately from Theorem 1.1. □

Suppose that a group $G$ admits two topological group structures $(G, S)$ and $(G, T)$ such that $S \subseteq T$. Then the identity map $\text{Id} : (G, T) \to (G, S)$ is a homomorphism of topological groups that admits a unique extension $\hat{\text{Id}} : (G, \overline{T}) \to (G, \overline{S})$ [B1 Prop. III.3.4(8)]. This extension $\hat{\text{Id}}$ may or may not be injective in general. Ditto for surjective [B1 Exercise III.3.12]. However, we can give nice criteria for surjectivity and injectivity for the topologies we are interested in.

Corollary 1.3. Consider a group $G$ with a BN-pair $(B, N)$ that admits two topological group structures $(G, S)$ and $(G, T)$ such that $S \subseteq T$. Suppose $B \in S$. Then the kernel of $\hat{\text{Id}} : (G, \overline{T}) \to (G, \overline{S})$ is equal to the kernel of $\hat{\text{Id}}_B : (B, \overline{T}_B) \to (B, \overline{S}_B)$.

Proof. Clearly, $\ker(\hat{\text{Id}}) \supseteq \ker(\hat{\text{Id}}_B)$. In the opposite direction, consider $x \in \ker(\hat{\text{Id}})$. This element is a limit of a Cauchy net $x_m \in G$, $m \in M$ in $T$ such that $x_m \to 1$ in $S$. Since $B \in S$ there exists an ordinal $L < M$ such that $x_m \in B$ for all $m \geq L$. Thus, $x \in (B, \overline{T}_B)$ and $x \in \ker(\hat{\text{Id}}_B)$. □

Corollary 1.4. Consider a group $G$ with a BN-pair $(B, N)$ that admits two topological group structures $(G, S)$ and $(G, T)$ such that $S \subseteq T$. Suppose $(G, S)$ satisfies the conditions of Theorem 1.1 or Theorem 1.2. If $\hat{\text{Id}}_B : (B, \overline{T}_B) \to (B, \overline{S}_B)$ is surjective, then $\hat{\text{Id}} : (G, \overline{T}) \to (G, \overline{S})$ is surjective.

Proof. This holds because $(G, \overline{S})$ is generated by $N$ and $\overline{B}$. □

Surjectivity of $\hat{\text{Id}}$ has a very interesting consequence as pointed out to us by Guy Rousseau. Note that the map $\hat{\text{Id}} : (G, \overline{T}) \to (G, \overline{S})$ defines an injective map of Tits...
buildings $\mathcal{TB}((\widehat{G}, \mathcal{T})) \rightarrow \mathcal{TB}((\widehat{G}, \mathcal{S}))$. Surjectivity of $\widehat{\mathcal{I}}$ implies that this map of Tits buildings is bijective.

We finish this section with a convenient corollary of Theorem 1.2 whose proof is straightforward.

**Corollary 1.5.** Let $G$ be a group with a BN-pair $(B, N)$ with the Weyl group $(W, S)$ where $S$ is finite. Suppose further that a system $\mathcal{G}$ of subgroups of $B$ is given such that the following three conditions hold.

1. $\mathcal{G}$ forms a topology basis at $1$ of a topological group $(B, \mathcal{T})$.
2. Each minimal parabolic $P_s$ is split as a semidirect product $P_s = L_s \rtimes U_s$ where $L_s$ is a finite group and $U_s$ is a subgroup of $B$.
3. $L_s$ acts continuously on $(U_s, \mathcal{T}_{U_s})$.

Then the four conclusions of Theorem 1.2 hold.

2. **Completions of Kac-Moody Groups**

Let $A = (a_{ij})_{n \times n}$ be a generalised Cartan matrix, $\mathcal{D} = (I, A, \mathcal{X}, \mathcal{Y}, \Pi, \Pi')$ a root datum of type $A$. Recall that this means

- $I = \{1, 2, \ldots, n\}$,
- $\mathcal{Y}$ is a free finitely generated abelian group,
- $\mathcal{X} = \mathcal{Y}^\vee = \text{hom}(\mathcal{Y}, \mathbb{Z})$ is its dual group,
- $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a set of simple roots, where $\alpha_i \in \mathcal{X}$,
- $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$ is a set of simple coroots, where $\alpha_i^\vee \in \mathcal{Y}$,
- for all $i, j \in I$, $\alpha_i(\alpha_j^\vee) = a_{ij}$.

Recall that $\mathcal{D}$ is simply connected if $\Pi^\vee$ is a basis of $\mathcal{Y}$. We call $A$ 2-spherical if for each $J \subseteq I$ with $|J| = 2$, the submatrix $A_J := (a_{ij})_{i, j \in J}$ is a Cartan matrix of finite type. Let $\Delta_{re}$ be the set of real roots. Recall that $\Delta_{re} = W(\Pi)$ where $W$ is the Weyl group. Note that $\Delta_{re} = \Delta_{re}^+ \cup \Delta_{re}^-$. Let $\mathbb{F} = \mathbb{F}_q$ be a finite field of $q = p^n$ elements $(a \geq 1$ and $p$ a prime). Tits [12, 13] gives a definition of a Kac-Moody group $G := G_{\mathcal{D}}(\mathbb{F})$, which is generated by the torus $T = \text{hom}(\mathcal{X}, \mathbb{F}^\times)$ and root subgroups $X_\alpha \cong \mathbb{F}^+$, $\alpha \in \Delta_{re}$. For all $i \in I$, set

$$M_i := \langle X_{\alpha_i} \cup X_{-\alpha_i} \rangle.$$

The group $G$ admits a BN-pair $(B, N)$ where $B = U \rtimes T$ with $U := \langle X_\alpha \mid \alpha \in \Delta_{re}^+ \rangle$ and $N := N_G(T)$, the normaliser of $T$. If $\mathcal{D}$ is simply connected, $T \cong (\mathbb{F}^\times)^n$. Moreover, the Coxeter group $(N/T, S = \{s_i, i \in I\})$ and $(W, S)$ are isomorphic. We denote by $P_i := P_{a_i}$, $i \in I$, a minimal parabolic subgroup of $G$. It is known [CaR2, 6.2] that $P_i = U_i \rtimes L_i$ where $L_i := M_iT$ and $U_i = U \cap s_iUs_i^{-1}$. In particular, $|P_i : B|$ is finite for all $i \in I$.

Consider the set of subgroups $\mathcal{F}$ of $B$ where

$$\mathcal{F} = \{A \mid A \leq U \leq B, |U : A| = p^a \text{ for some } a \in \mathbb{N}\}.$$ 

Elements of $\mathcal{F}$ form a basis at $1$ of a topology on $B$. Then the completion $\widehat{B}$ of $B$ with respect to this topology is a group. Since $|P_i : U| < \infty$, conditions (1)–(4) of Theorem 1.2 are satisfied, thus its conclusions hold. In particular, $\widehat{G}$ is a topological group with an open subgroup $\widehat{B}$ and $(\widehat{B}, N)$ is a BN-pair of $\widehat{G}$.

Since $U$ is a residually finite-$p$ group [12] Remark after Th. 4.1] and $|B : U| = (q - 1)^n$, our completion $\widehat{B}$ is equal to $\widehat{U} \rtimes T$ where $\widehat{U}$ is the full pro-$p$ completion of $U$. 

Let us recall other known topological Kac-Moody groups. They are the Mathieu-Rousseau group $G_{ma}^+$, the Carbone-Garland group $G^c\lambda$ and the Caprace-Rény-Ronan group $G^{err}$. Each of them contains a quotient $G^\perp := G/Z$ by a central subgroup $Z$, which depends on the completion and could be trivial. In fact, $G^\perp$ is always dense in $G^{c\lambda}$ and $G^{err}$. Let $G^+ := G^\perp$ be the closure of $G^\perp$ in $G_{ma}^+$. Rousseau [Rou 6.10] investigates whether $G^+$ equals $G_{ma}^+$ and show that this happens when $p > \max\{|a_{ij}|, i \neq j\} \geq 1$ [Rou 6.11]. Rousseau and later Marquis give examples when it does not happen [Rou, M2].

There are two further known completions of $G$ where the closure $\overline{U}$ is compact totally disconnected [ReW]. The Belyaev group $G^b$ is the “largest” such completion. The Schlichting group $G^s$ is the “smallest” such completion. Our completion admits a characterisation similar to the Belyaev group: $\hat{G}$ is the “largest” completion where the closure $\overline{U}$ is a pro-$p$-group.

Let $\overline{U}$ and $\overline{B}$ be the closures of $U$ and $B$ correspondingly in either of the topological groups $G^+$, $G^{c\lambda}$ or $G^{err}$. The group homomorphism $\hat{U} \to \overline{U}$ extends to $\hat{B} \to \overline{B}$ and $G \to \overline{G}$ (cf. Section 6.3 of [Rou]). Using this and the universal properties of the Belyaev and Schlichting completions, we have open continuous homomorphisms $\overline{G} \to G^b \to G^{\perp} \to G^{c\lambda} \to G^{err} \to G^s$.

It is known that for $\overline{G} \in \{G^+, G^{c\lambda}, G^{err}\}$, $G/Z'(G)$ is topologically simple (where $Z'(G) = \bigcap_{g \in G} gBg^{-1}$). What about our new group $\hat{G}$?

Recall the following criterion of Bourbaki [B2].

**Proposition 2.1.** Let $(G, B, N, S)$ be a Tits system with Weyl group $W = N/(B \cap N)$. Let $U$ be a subgroup of $B$. We set $Z'(G) = \bigcap_{g \in G} gBg^{-1}$. Assume that $G$ is a topological group topologically generated by the conjugates of $U$ in $G$. Assume further $B$ a closed subgroup of $G$, and the following conditions hold:

1. We have $U \triangleleft B$ and $B = UT$ where $T = B \cap N$.
2. For any proper normal closed subgroup $V \triangleleft U$, we have $[U/V, U/V] \subsetneq U/V$.
4. The Coxeter system $(W, S)$ is irreducible.

Then for any normal closed subgroup $K$ in $G$, $K \leq Z'(G)$. In particular, $G/Z'(G)$ is topologically simple.

We now prove the following statement.

**Theorem 2.2.** If $A$ is an irreducible generalised Cartan matrix, then $\hat{G}/Z'(\hat{G})$ is topologically simple.

**Proof.** If $V$ is a closed normal subgroup of $\hat{U}$, then $[\hat{U}/V, \hat{U}/V] \neq \hat{U}/V$ as shown in [CarERi 4.4]. Now Proposition 2.1 finishes the proof.

There is a similar criterion for the abstract simplicity [B2].

**Proposition 2.3.** Let $(G, B, N, S)$ be a Tits system with Weyl group $W = N/(B \cap N)$. Let $U$ be a subgroup of $B$ such that $G$ is generated by the conjugates of $U$. Assume that the following holds:

1. We have $U \triangleleft B$ and $B = UT$ where $T = B \cap N$.
2. For any proper normal subgroup $V \triangleleft U$, we have $[U/V, U/V] \subsetneq U/V$.


We have $G = [G, G]$.

(4) The Coxeter system $(W, S)$ is irreducible.

Then for any normal subgroup $K$ in $G$, $K \leq Z'(G)$. In particular, $G/Z'(G)$ is abstractly simple.

This allows us to prove the following statement.

**Theorem 2.4.** Suppose $q \geq 4$. If $A$ is irreducible and 2-spherical, then $\hat{G}/Z'(\hat{G})$ is abstractly simple, and there are natural isomorphisms

$$\hat{G}/Z'(\hat{G}) \cong G^+/Z'(G^+) \cong G^{c\lambda}/Z'(G^{c\lambda}) \cong G^{crr}/Z'(G^{crr}).$$

**Proof.** Let us first show that $\hat{G}/Z'(\hat{G})$ is abstractly simple. To do that it suffices to check the conditions of Proposition 2.3 for the Tits system $(\hat{G}, \hat{B}, N, S)$.

By construction of $\hat{G}$, condition (1) holds because it holds in $(G, B, N, S)$.

Abramenko proves that for $q \geq 4$, $U$ is finitely generated if and only if $A$ is 2-spherical [A]. Thus, $\hat{U}$ is topologically finitely generated. By [CarERi] Lemma 4.4, condition (2) of Proposition 2.3 holds for $\hat{U}$ and any proper normal subgroup $V$ of $\hat{U}$.

Moreover, $[\hat{G}, \hat{G}] \geq [G, G][\hat{U}, \hat{U}]$. Since $q \geq 4$, for every $\alpha \in \Delta^{re}$, the subgroup $M_{\alpha} := (X_{\alpha}, X_{-\alpha})$ is perfect (in fact, it is $\text{PSL}_2(\mathbb{F})$ or $\text{SL}_2(\mathbb{F})$), and thus $[M_{\alpha}, M_{\alpha}] = M_{\alpha} \supseteq X_{\alpha}$. Hence, $G = [G, G]$. Now the argument of Carbone, Ershov and Ritter [CarERii] 4.3(b)] shows that $[\hat{U}, \hat{U}]$ is an open subgroup of $\hat{G}$, and so $\hat{G} = [\hat{G}, \hat{G}]$.

Finally, condition (4) holds since $A$ is irreducible. Therefore, $\hat{G}/Z'(\hat{G})$ is abstractly simple.

Observe that the homomorphisms (3) yield open surjective homomorphisms

$$\hat{G}/Z'(\hat{G}) \twoheadrightarrow G^+/Z'(G^+) \twoheadrightarrow G^{c\lambda}/Z'(G^{c\lambda}) \twoheadrightarrow G^{crr}/Z'(G^{crr})$$

that are isomorphisms of abstract groups due to simplicity of $\hat{G}/Z'(\hat{G})$. They are isomorphisms of topological groups because they are open. \qed

### 3. Congruence Kernel

We finish the paper with some observations on the structure of $Z'(\hat{G})$. To facilitate our discussion we use the following notation for arbitrary groups $K \leq H$:

- $\hat{H}$ – the completion of $H$ in the pro-$p$ topology on $H$ or its canonical (such as $U$) subgroup,
- $\breve{H}$ – the completion of $H$ in some other topology,
- $\bar{K}^{cH}$ (or simply $\bar{K}$) – the closure of $K$ in $H$,
- $C(H, K) := \cap_{g \in H} gKg^{-1}$ – the normal core of $K$ in $H$.

The group $Z'(\hat{G})$ contains two commuting subgroups: the centre (before completion) $Z(G)$ and the normal core $C(\hat{G}, \hat{U})$. In fact, as $\hat{U} \cong \hat{U}^{G(G)}$ is a Sylow pro-$p$ subgroup of $\hat{G}$, $C(\hat{G}, \hat{U}) = C(\hat{G}, V)$ for any Sylow pro-$p$ subgroup $V$ of $\hat{G}$. Therefore, we may use the notation $C(\hat{G})$ instead of $C(\hat{G}, \hat{U})$. Sometimes it is convenient to use the full notation $C(\hat{G}, \hat{U})$. We will use both notations depending on circumstances.

Following the argument of Rousseau [Rou, Prop. 6.4], we can prove that

$$Z'(\hat{G}) = Z(G) \times C(\hat{G}).$$
We can compute the centre $Z(G)$ from the Cartan matrix but we see no efficient way of computing the normal core $C(\hat{G})$. Observe that in the Caprace-Rémy-Ronan completion, $Z'(G^{crr}) = Z(G)$. Hence, Theorem 2.4 implies that $C(\hat{G}) = \ker(\phi)$ where $\phi : \hat{G} \to G^{crr}$ is the natural continuous open surjective homomorphism. The kernels of the natural maps between two different completions of the same groups are commonly known as congruence kernels, the term used later in the paper. Can we describe $C(\hat{G})$ explicitly?

Let $\mathcal{P}$ be the collection of all normal index $p^n$, $n \in \mathbb{N}$, subgroups of $U$ so that

$$\hat{U} \cong \{(x_HH) \mid x_H \in U, H \in \mathcal{P}, x_HH = x_H'H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} U/H.$$ Let us examine the action of $\hat{U}$ on the Tits building $\mathcal{T}\mathcal{B}(\hat{G})$. Let $U_n$ be the pointwise stabiliser of the ball of radius $n$ around the simplex $\tilde{B}$ in $\mathcal{T}\mathcal{B}(\hat{G})$. Then

$$\mathcal{P}^0 := \{H \in \mathcal{P} \mid \exists n : H \geq U_n\}$$

is a basis of topology on $U$. We can describe the completion of $U$ in this topology as

$$U^{crr} \cong \{(x_HH) \mid x_H \in U, H \in \mathcal{P}^0, x_HH = x_H'H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}^0} U/H.$$ Clearly, $C(G^{crr}, U^{crr}) = 1$ because it consists of those elements $(x_HH)$ that act trivially on $\mathcal{T}\mathcal{B}(\hat{G})$. This forces $x_H \in U_n$ for all $n$ and $(x_HH) = 1$. The natural map $\hat{U} \to U^{crr}$ is the projection whose kernel is exactly $C(\hat{G}, \hat{U})$ that we can describe now as

$$C(\hat{G}) = \{(x_HH) \mid x_H \in H^*, H \in \mathcal{P}, x_HH = x_H'H \text{ for } H \geq H'\} \leq \prod_{H \in \mathcal{P}} H^*/H$$

where $H^* := \cap_{H \leq K \in \mathcal{P}^0} K$. This description tells us that one of the three following statements holds:

1. $\mathcal{P} \setminus \mathcal{P}^0$ is finite. Then $C(\hat{G}, \hat{U})$ is a finite group.
2. $\mathcal{P} \setminus \mathcal{P}^0$ is infinite but $\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is finite. Then $C(\hat{G}, \hat{U})$ is a finitely generated pro-$p$ group.
3. $\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}$ is infinite. Then $C(\hat{G}, \hat{U})$ may be an infinitely generated pro-$p$ group.

A natural question to address is whether $C(\hat{G})$ is central. We can do it under some strong assumptions.

**Lemma 3.1.** If $A$ is irreducible of indefinite type and $q \geq n > 2$, then at least one of the following statements holds:

1. $C(\hat{G})$ is not a finitely generated pro-$p$ group,
2. $C(\hat{G}) \leq Z(\hat{G})$.

In particular, if $C(\hat{G})$ is finite, then $C(\hat{G}) \leq Z(\hat{G})$.

**Proof.** If $A$ is irreducible of indefinite type and $q \geq n > 2$, then $G/Z(G)$ is a simple non-linear group as shown by Caprace and Rémy [CaR].

Let us assume that $C := C(\hat{G})$ is a finitely generated pro-$p$ group. In this case the Frattini quotient $C/\Phi(C)$ is a finite elementary abelian $p$-group. Since $\Phi(C) \triangleleft \hat{G}$, it follows immediately that $\hat{G}/Z'(\hat{G})$ acts on $C/\Phi(C)$. Now $\hat{G}/Z'(\hat{G})$ contains a dense subgroup isomorphic to $G/Z(G)$. This subgroup is simple non-linear, hence,
it must act trivially on the finite group \(C/\Phi(C)\). Since the subgroup is dense, the whole \(\hat{G}/\Z'(\hat{G})\) acts trivially. Since the action is given by conjugation \(g \cdot (c\Phi(C)) = (gcq^{-1})\Phi(C)\), we can say that \(\hat{G}/\Z'(\hat{G})\) centralises \(C/\Phi(C)\).

Now let \(T\) be the torus of \(G\), defined at the start of Section 2. Then \([T, C/\Phi(C)] = 1\). Let \(C_i = \Phi_i(C)\), the \(i\)-th Frattini subgroup. Since \(C\) is finitely generated, \(C/C_i\) is a finite group, \(\Phi_i(C/C_i) = \Phi(C)/C_i\) and \(\{C_i, i \in \mathbb{N}\}\) is a fundamental system of open neighbourhoods of 1 in \(C\). \([\text{RibZ}\ 2.8.13]\). It follows that \(T\) acts on \(C/C_i\) and centralises \((C/C_i)/\Phi(C/C_i)\). A theorem of Burnside states that a \(p^r\)-automorphism of a \(p\)-group \(P\), inducing the identity automorphism on \(P/\Phi(P)\), is the identity itself \([\text{Gi} 5.1.4]\). It follows that \([T, C/C_i] = 1\). Since \(\{C_i, i \in \mathbb{N}\}\) is a fundamental system in \(C\), it follows that \([T, C] = 1\). Obviously \([T^g, C] = 1\) for all \(g \in \hat{G}\).

Therefore, \(([T^g, g \in \hat{G}], C) = 1\). Since \(Z(G)\) is a subgroup of \(T\) \([\text{CaRZ} Cor. 5.14]\), \(G = \langle T^g, g \in G \rangle \leq \langle T^g, g \in \hat{G} \rangle\). The result now follows.

In some cases we can describe \(C(\hat{G})\) fully.

**Proposition 3.2.** Suppose that the generalised Cartan matrix \(A = (a_{ij})_{n \times n}\) is irreducible, of untwisted affine type and \(n \geq 3\). Then \(C(\hat{G}) = 1\). In particular, \(\hat{G} \cong G^{ma+} \cong G^c \cong G^{crr}\) and \(G(\mathbb{F}_q[t, t^{-1}]) \cong G(\mathbb{F}_d((t)))\).

**Proof.** The root datum \(\mathcal{D}\) changes only Cartan subgroup and has no effect on \(U\) or \(C(\hat{G}) = C(\hat{G}, U)\). Thus we may choose \(\mathcal{D}\) so that \(G \cong G(\mathbb{F}_d[t, t^{-1}])\) for the corresponding Chevalley group scheme \(G\). Now Lemma 7 of \([\text{CLR}]\) gives us that \(U\) is a Sylow pro-\(p\) subgroup of \(G(\mathbb{F}_d((t)))\). This implies that \(\hat{G} \cong G(\mathbb{F}_d((t)))\) which gives the desired result.

We expect Proposition 3.2 to hold for a twisted affine \(\hat{A}\) as well. As pointed out by the referee, it would be interesting to establish whether \(\hat{G} \cong G^{crr}\) implies that \(G\) is of affine type. For instance, the isomorphism fails in rank 2 as shown in the next proposition.

**Proposition 3.3.** Suppose that the generalised Cartan matrix \(A = (a_{ij})_{2 \times 2}\) is not of finite type and \(p > \max\{|a_{12}|, |a_{21}|\}\). Then \(C(\hat{G})\) is an infinitely-generated pro-\(p\)-group and \(\{H^* \mid H \in \mathcal{P} \setminus \mathcal{P}^0\}\) is infinite.

**Proof.** For such \(A\), Morita \([\text{Mo} 3(6)]\) gives the description of \(U\) in \(G_{2D}(\mathbb{F})\) for a field \(\mathbb{F}\) of characteristic 0. His description extends to the case \(\mathbb{F} = \mathbb{F}_p\), as an interested reader can verify. If \(\min\{|a_{12}|, |a_{21}|\} \geq 2\), then \(U = U_1 \ast U_2\), where \(U_i \cong \mathbb{F}_p[t]\) for \(i = 1, 2\). If \(\min\{|a_{12}|, |a_{21}|\} = 1\), then \(U = U_1 \ast U_2\) and each \(U_i\) is a metabelian infinitely generated group.

Consider \(\hat{U}\). By \([\text{RibZ} 9.1.1]\), \(\hat{U} = \hat{U}_1 \amalg \hat{U}_2\), the free pro-\(p\) product of the pro-\(p\) groups \(\hat{U}_1\) and \(\hat{U}_2\). Since each \(\hat{U}_i\) is an infinitely-generated pro-\(p\) group, \([\text{RibZ} 9.1.15]\) implies that \(\hat{U}\) is infinitely-generated.

On the other hand, as \(p > \max\{|a_{12}|, |a_{21}|\}\), the results of \([\text{CR} 2.2\) and 2.4\] give us that \(U^{crr}\) is a finitely generated pro-\(p\) group. The proposition follows immediately.

**Corollary 3.4.** Let \(A = (a_{ij})_{n \times n}\) be an irreducible generalised Cartan matrix whose Dynkin diagram contains an infinite edge, i.e., there exists \(1 \leq i \neq j \leq n\)
with $a_{ij}a_{ji} \geq 4$. Suppose that $p > \max\{|a_{ij}|, i \neq j\}$. Then $C(\hat{G}, \hat{U})$ is an infinitely-generated pro-$p$-group and $\{H^* \mid H \in \mathcal{P}^0\}$ is infinite.

**Proof.** Let $P$ be a parabolic subgroup of $G$ whose Levi complement corresponds to the subdiagram of the Dynkin diagram $\Delta$ of $G$ based on $\alpha_i$ and $\alpha_j$. Then $P = U_P \rtimes L$ where

$$L = \langle X_\alpha, T \mid \alpha \in \Delta^{\text{re}} \cap \text{Span}_\mathbb{R}\{\alpha_i, \alpha_j\}\rangle$$

is a Levi complement of $P$ and $U_P = \cap_{g \in P}^g \mathcal{P}$ [R, 6.2.2]. Hence, $U = U_P \rtimes U_L$ where $U_L = U \cap L$. It follows that $\hat{U}_L \leq \hat{U}$. Moreover, the natural isomorphism $U/U_P \xrightarrow{\cong} U_L$ yields an exact sequence

$$(4) \quad 1 \to U_P \to U \to U_L \to 1.$$

Since pro-$p$-completion is a right exact functor, the sequence

$$(5) \quad \hat{U}_P \xrightarrow{\hat{a}} \hat{U} \xrightarrow{\hat{b}} \hat{U}_L \to 1$$

is exact as well. Therefore, $\hat{U}_L$ is a homomorphic image of $\hat{U}$. Since $\hat{U}_L$ is an infinitely-generated pro-$p$ group, so is $\hat{U}$.

As $p > \max\{|a_{ij}|, i \neq j\}$, the results of [CR] imply that $U^{\text{cr}}$ is a finitely generated pro-$p$ group. Note that $C(\hat{G}, \hat{U})$ is the kernel of the homomorphism $\hat{U} \to U^{\text{cr}}$.

This finishes the proof. □

It is possible to relate the calculations of the congruence kernel of a Levi factor and of the unipotent radical of a parabolic. Let $J \subseteq I$ and $P := P_J$ a parabolic in $G_D(F)$ with the unipotent radical $U_P$ and a Levi complement $L = \langle X_\alpha, T \mid \alpha \in \Delta^{\text{re}} \cap \text{Span}_\mathbb{R}(J)\rangle$. Then $P = U_P \rtimes L$ [R, 6.2.2] and we have a natural isomorphism $\hat{L} \simeq \mathcal{L}^{\mathbb{C}\hat{G}} = \mathcal{L}^{\mathbb{C}\hat{F}}$ where $\hat{F} := \mathcal{F}^{\mathbb{C}\hat{G}}$. Let $U_L := U \cap L$. Then $U_L$ is the unipotent radical of a Borel subgroup of $L$. Two “parabolic” congruence kernels “approximate” $C(\hat{G}, \hat{U})$.

**Theorem 3.5.** There exists an exact sequence of topological groups

$$(6) \quad 1 \to C(\hat{G}, \overline{U_P}) \to C(\hat{G}, \hat{U}) \to C(\hat{L}, \hat{U}_L) \to 1.$$

Moreover, $C(\hat{L}, \hat{U}_L)$ is a subgroup of $C(\hat{G}, \hat{U})$. In particular, if $C(\hat{L}) \neq 1$ then $C(\hat{G}) \neq 1$.

**Proof.** Notice that $\hat{L}$ is a topological Kac-Moody group on its own letting us talk about $C(\hat{L}) = C(\hat{L}, \hat{U}_L)$.

Let us examine the exact sequence [5] in the proof of Corollary [3,4]. The image $\tilde{a}(\hat{U}_P)$ is a closed subgroup containing $a(U_P)$. Since $U_P$ is dense in $\hat{U}_P$, $a(U_P)$ is dense in $\tilde{a}(\hat{U}_P)$. This yields another exact sequence

$$(7) \quad 1 \to \overline{U_P} \xrightarrow{\tilde{\pi}} \hat{U} \xrightarrow{\tilde{b}} \hat{U}_L \to 1.$$

The same argument applied to the semidirect decomposition $P = U_P \rtimes L$ gives an exact sequence with $\tilde{c}(\hat{L}) = \hat{b}$:

$$(8) \quad 1 \to \overline{U_P} \xrightarrow{\pi} \mathcal{P} \xrightarrow{\tilde{c}} \hat{L} \to 1$$

where the closure $\mathcal{P} = \mathcal{P}^{\mathbb{C}\hat{G}}$ is the completion of $P$ in the uniformity induced from $\hat{G}$. Loosely speaking, both $\mathcal{P}$ and $\hat{G}$ are obtained by pro-$p$-completion of $U$. Both $\overline{U_P}$
and $\hat{U}$ are subgroups of $\hat{\mathcal{P}}$. The map $\varpi$ is the inclusion of subgroups. Conjugating them by all $g \in \hat{\mathcal{P}}$ and then intersecting yields the inclusion $\varpi: C(\hat{\mathcal{P}}, \hat{U}) \hookrightarrow C(\mathcal{P}, \hat{U})$. Moreover, the sequence (7) restricts to a new sequence 

\[ 1 \to C(\mathcal{P}, \mathcal{U}) \xrightarrow{\varpi} C(\hat{\mathcal{P}}, \hat{U}) \to C(\hat{\mathcal{L}}, \hat{U}_L) \to 1 \] 

Observe that $p \cdot y = \hat{c}(p) y \hat{c}(p)^{-1}$, $p \in \hat{\mathcal{P}}$, $y \in \hat{\mathcal{L}}$, gives a $\hat{\mathcal{P}}$-action on $\hat{\mathcal{L}}$. It follows that $\hat{b}(C(\mathcal{P}, \hat{U})) \subseteq C(\hat{\mathcal{L}}, \hat{U}_L)$, so that the sequence (9) is well-defined.

We can conjugate $C(\mathcal{P}, \mathcal{U})$ and $C(\hat{\mathcal{P}}, \hat{U})$ by all $g \in \hat{G}$ and intersect further. This yields a subsequence of the sequence (9):

\[ 1 \to C(\hat{G}, \hat{U}) \to C(\hat{\mathcal{G}}, \hat{U}) \to C(\hat{\mathcal{L}}, \hat{U}_L) \to 1 \] 

This sequence is precisely the sequence (9) in the statement of the theorem. It remains to establish surjectivity of $\hat{b}$ in the sequence (9).

Consider the restriction of the continuous homomorphism $\phi: \hat{G} \to G^{\text{crr}}$ to $\hat{\mathcal{L}}$. Clearly, $\phi(\hat{\mathcal{L}}) = L^{\text{crr}}$. In particular, $\phi(C(\hat{\mathcal{L}}, \hat{U}_L)) \subseteq L^{\text{crr}}$. We have an $\hat{\mathcal{L}}$-equivariant map

\[ \eta: \mathcal{TB}(\hat{\mathcal{L}})_m = \hat{\mathcal{L}}/(\mathcal{B} \cap \hat{\mathcal{L}}) \to \mathcal{TB}(\hat{\mathcal{G}})_m = \hat{\mathcal{G}}/\bar{\mathcal{B}}, \quad g(\mathcal{B} \cap \hat{\mathcal{L}}) \mapsto g\bar{\mathcal{B}}, \]

where by $\mathcal{TB}(\hat{\mathcal{G}})_m$ and $\mathcal{TB}(\hat{\mathcal{L}})_m$ we denote the set of simplices of maximal dimension in the corresponding Tits buildings. As a subset of $\mathcal{TB}(\hat{\mathcal{G}}) = \mathcal{TB}(G^{\text{crr}})$, the image of $\eta$ consists of those simplices that have $\hat{P}$ as a face because, corestricted to its image, $\eta$ can be identified with the natural map $\hat{\mathcal{L}}/(\mathcal{B} \cap \hat{\mathcal{L}}) \to \hat{P}/\bar{\mathcal{B}}$.

Since $C(\hat{\mathcal{L}}, \hat{U}_L)$ acts trivially on $\mathcal{TB}(\hat{\mathcal{L}})$, it follows that $C(\hat{\mathcal{L}}, \hat{U}_L)$ fixes the image of $\eta$. Since the stabiliser of an individual simplex is a Borel subgroup, the fixator of all these simplices is $C(\mathcal{P}, \mathcal{B})$. It follows from \cite[Th 6.3]{CaR2} that $C(\mathcal{P}, \mathcal{B})$ is equal to $U^{\text{crr}}_p T'$, where $T'$ is a subgroup of a torus in $\mathcal{B}$. Therefore, $\phi(C(\hat{\mathcal{L}}, \hat{U}_L)) \subseteq U^{\text{crr}}_p T'$.

Now $C(\hat{\mathcal{L}}, \hat{U}_L)$ and $U^{\text{crr}}_p$ are pro-$p$-groups, while $T'$ is a finite $p'$-group, i.e., a group of order coprime to $p$. Thus, $\phi(C(\hat{\mathcal{L}}, \hat{U}_L)) \subseteq U^{\text{crr}}_p$. Furthermore, $\phi(C(\hat{\mathcal{L}}, \hat{U}_L)) \subseteq L^{\text{crr}} \cap U^{\text{crr}}_p = 1$. It follows that $C(\hat{\mathcal{L}}, \hat{U}_L)$ is contained in the kernel of $\phi$ that is equal to $Z'(\hat{\mathcal{G}}) = Z(\hat{G}) \times C(\hat{\mathcal{G}}, \hat{U})$. Since $Z(\hat{G})$ is a subgroup of the torus $\mathcal{B}$, Cor. 5.14, it is a finite $p'$-group, while $C(\hat{\mathcal{G}}, \hat{U})$ is a pro-$p$-group. It follows that $C(\hat{\mathcal{L}}, \hat{U}_L)$ is contained in $C(\hat{\mathcal{G}}, \hat{U})$. This inclusion splits the sequence (10) proving surjectivity of $\hat{b}$. \hfill \Box

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