The Classification of Bifurcations in Maps with Symmetry

by

Anthony Graham Brown

Submitted for the degree of PhD.

The University of Warwick,

Coventry. CV4 7AL.

Department of Mathematics

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Summary.

The aim of this work is to classify the generic codimension 1 bifurcations of a map with symmetry.

Many of the results proved here are analogous to results already proved for vector fields, in fact many of the proofs given are straightforward adaptations. However the nature of mapping allows for more complicated dynamics to be observed, dynamics which have no direct counterparts in the world of continuous flows.

We start the thesis with a review of result for non symmetric maps on $\mathbb{R}$ and $\mathbb{R}^2$. We then motivate the introduction of symmetry with a system of identical coupled oscillators (simple predator prey models) which exhibit D_n symmetry, D_n is the symmetry group of an n-gon. In the next section we make the concept of symmetry more rigorous and introduce the language of groups which is a natural way to talk about symmetric systems. We also explain why symmetry effects can both complicate problems and the methods used to overcome these difficulties.

In the next chapter we prove some useful results about normal forms of symmetric mappings, these help in later chapters in the consideration of stability and bifurcation directions.

In the last two chapters we describe symmetric Hopf bifurcations, that is bifurcations to invariant circles and symmetric subharmonic bifurcations, i.e. bifurcations to periodic orbits. Using the spatial symmetries of the group action as well as the temporal symmetries which are introduce by the existence of periodic orbits and invariant circles we can predict generically the existence of solutions.

We finish with an example of how these solutions can be calculated and interpreted for a physical system.

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Declaration

As far as I am aware this thesis is the original work of the author, unless explicitly stated to the contrary in the course of the text.

Chapter 0

Introduction.

In recent years much work has been devoted to the study of dynamical systems, both continuous and discrete (an in depth introduction to this field of study can be found in Arrowsmith and Place [1990]). The main motivation for this work is the attempt to model physical systems with equations which capture the basic structure of the systems, whilst remaining simple enough to be tractable. One important area in this field is the study of stability of solutions and the types of behaviour expected when these solutions become unstable, this is Bifurcation Theory.

In this thesis we extend the ideas of bifurcation theory to the study of symmetric maps. There are many physical systems which exhibit almost exact symmetry which we would hope to be able to capture in our mathematical model. Furthermore the conditions put on certain solutions may produce symmetries not present in the original system.

Many of the results presented here already have analogues for continuous systems, i.e. systems governed by differential equations:-

$$dx/dt=f(x,\lambda)$$

where

$$f:\mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n$$

is a smoothly parameterised family of maps (Golubitsky et al [1988] gives a good discussion of these results and was used as a basis for this thesis). However for discrete systems, i.e. systems governed by the difference equation

$$x_{n+1}=f(x_n,\lambda),$$

there is no way to directly apply these results. Moreover the dynamics of discrete system are often much richer than for those of a corresponding continuous system, for example there is no corresponding continuous concept to that of periodic orbit for a discrete dynamical system.

Although discrete dynamical systems are not directly applicable to the study of most physical systems, there are ways to reduce a continuous system of equations to give an equivalent discrete system of equations. In addition, as computers become a more and more powerful tool in the study of dynamical systems, a better understanding of discrete dynamical systems should aid our understanding of how numerical simulations of physical systems relate to the actual observed types of behaviour.

This thesis attempts to classify the possible types of bifurcation expected when a fixed point of f loses stability. Before continuing we give a brief review of the contents of the thesis, indicating the sources use and how the thesis ties in with other related work in the field.

Chapter one uses as its main source Whitley [1983] and introduces some useful results for maps on $\mathbb{R}$ and $\mathbb{R}^2$ which in subsequent chapter we are able to use for the prediction of particular types of solution.

In the second chapter we motivate the introduction of symmetry with a simple symmetric system (which has a limited physical interpretation). The system is based on a set of n identical discrete predator prey environments (first introduced by Maynard Smith [1968]) which are coupled to create a D_n symmetric system of equations. This coupling could represent some migratory effect between consecutive environments. We also give some numerical simulations to indicate different possible modes of behaviour.

The third chapter introduces rigorously the idea of symmetric maps and the language of groups. It shows how symmetry effects can force the eigenvalues of the linearisation of f to have high multiplicity and also how these same symmetries can be utilised to reduce the complexity of the problem, in particular the idea of reduction to fixed point spaces in which we look for solutions with a predetermined symmetry and the use of invariant theory to restrict the form which f can take (these results are taken from Golubitsky *et al* [1987]). We end this chapter with the statement of the first bifurcation results for fixed point bifurcations (Chossat and Golubitsky [1988]) and period doubling bifurcations.

Chapter four adapts results of Elphick *et al* [1987] and Vanderbauwhede [1989] about normal forms of vector fields with symmetry and develops the work of Iooss [1987] to give analogous results about normal forms of maps with symmetry to show how they can be chosen to make symmetric problems more tractable.

Chapter five is concerned with Hopf type bifurcations, in symmetric maps, that is bifurcations from a fixed point of f to an invariant circle of solutions of f. It uses results from $\mathbb{R}^2$ and reduction to fixed point spaces to predict invariant circles with certain symmetries. The extra symmetry introduced by the presence of an invariant circle can be used to find further solutions. A result was given by Chossat and Golubitsky [1988], however their proof is unclear and seems to make an assumption which is not necessarily true, so here we have tried to fill in the detail of the proof and show how their assumption can possibly not hold, in doing so we prove a slightly weaker result.

The final chapter is entirely the author's own work unless specifically stated. The main results are concerned with finding periodic orbits of f. The first shows the existence of a map g whose zeros are in one to one correspondence with period q points of f. It uses a Liapunov Schmidt reduction of a mapping F whose zeros correspond with period q points of f to give a mapping g with the desired properties. This approach was first suggested by Andre Vanderbauwhede and is adapted here to look at maps with symmetry. We then use this result to show under what conditions we can expect to see periodic orbits bifurcating from a fixed point. We go on to show how f and g are related and discuss the concept of stability of a periodic point of x. We end the chapter with the discussion of a particular example, when f commutes with the action of D_n, looking at types of solutions expected and their stabilities.

Chapter 1

Review of diffeomorphisms in $\mathbb{R}^2$.

1.0 Introduction.

Before considering symmetric systems it will be useful to look at the results already known concerning bifurcations of maps in $\mathbb{R}$ and $\mathbb{R}^2$ and then see how we can generalise them for the study of bifurcations in maps with symmetry. Interested readers can find a more thorough discussion of these results in Whitley [1983]. Here we give details of the main results and outline some of their proofs, specifically those which can be modified to prove corresponding symmetric results.

Let f be a parameterised family of smooth mappings from $\mathbb{R}^n$ onto itself

$$f: \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n.$$

We denote points in $\mathbb{R}^n$ by x and those in Λ by λ and write $f_\lambda(x)$ for $f(x,\lambda)$.

Definition 1.0.1.

A point x_0 is a hyperbolic fixed point of f_λ if

$$f_\lambda(x_0)=x_0$$

and $(df_\lambda)_{x_0}$ has no eigenvalues whose modulus is unity.

If this is the case then $\mathbb{R}^n$ can be split into the direct sum of two distinct subspaces E_s and E_u, the stable and unstable generalised eigenspaces of $(df_\lambda)_{x_0}$, where E_s is the eigenspace corresponding to eigenvalues of modulus less than 1 and E_u that of the eigenvalues whose modulus is greater than 1.

Furthermore, in a sufficiently small neighbourhood of x_0, there exist two distinct submanifolds of $\mathbb{R}^n$, W_s and W_u the stable and unstable manifolds, where $x \in W_s$ if

$\text{Lim}_{n\to\infty} f^n(x,\lambda) = x_0$ and $x \in W_u$ if $\text{Lim}_{n\to\infty} f^{-n}(x,\lambda) = x_0$. These manifolds have the same dimension as E_s and E_u respectively and at x_0 W_s is tangent to E_s and W_u is tangent to E_u.

As we alter the parameter λ we change the position of the eigenvalues, hence it is possible for f_λ to become non hyperbolic. At such values f_λ will be structurally unstable and we should expect to see a bifurcation occurring. To simplify matters we can use a centre manifold reduction of f which gives a map equivalent to f on the centre manifold of f at x_0 in $\mathbb{R}^n$.

We now restrict our attention to a two parameter family Λ. Generically a two parameter family of maps can be non hyperbolic in a variety of ways;

(a) an eigenvalue equals 1,

(b) an eigenvalue equals -1,

(c) a complex conjugate pair of eigenvalues equal $e^{\pm 2\pi i\theta}$,

(d) an eigenvalue 1 has algebraic multiplicity two,

(e) an eigenvalue -1 has algebraic multiplicity two,

(f) an eigenvalue equals 1 and an eigenvalue equals -1,

(g) two complex conjugate pairs of eigenvalues lie on the unit circle at $e^{\pm 2\pi i\theta}$ and $e^{\pm 2\pi i\varphi}$,

(h) an eigenvalue equals 1 and a complex conjugate pair of eigenvalues equal $e^{\pm 2\pi i\theta}$,

(i) an eigenvalue equals -1 and a complex conjugate pair of eigenvalues equal $e^{\pm 2\pi i\theta}$.

This thesis will concentrate mainly on case (c) but we will also give in this chapter an idea of what kind of bifurcations would be expected in the cases (a) and (b), these are strictly codimension one problems but we need the second parameter in order that all of the dynamics of the solutions in case (c) can be observed. In cases (d)-(i) we see complex behaviour due to the interaction between the different eigenvalue modes which are beyond the scope of this thesis.

1.1. Maps of the line.

In cases (a) and (b) a centre manifold reduction implies we can see the full range of dynamics by considering f as simply a one parameter family of maps from $\mathbb{R}$ to $\mathbb{R}$ in which $(df_{\lambda_0})_{x_0}$ has eigenvalue equal to either 1 or -1. We can further assume without loss of generality that $x_0 = 0$ and that $\lambda_0 = 0$.

Under these assumptions we can describe the type of behaviour expected in the form of the following propositions.

Proposition 1.1.1. (the fold bifurcation)

Let f be a one parameter family of C^2 maps satisfying

(a) $f(0,0)=0$, (b) $(\partial f/\partial x)(0,0)=1$, (c) $(\partial^2 f/\partial x^2)(0,0)>0$,

(d) $(\partial f/\partial \lambda)(0,0)>0$.

Then there are intervals $(\lambda_1,0)$ and $(0,\lambda_2)$ such that

(i) If $\lambda \in (\lambda_1,0)$ then f_λ has two fixed points in the neighbourhood of 0, one stable the other unstable.

(ii) If $\lambda \in (0,\lambda_2)$ then f_λ has no fixed points in the neighbourhood of 0.

Note

Reversing inequalities (c) or (d) simply reverses the occurrence of (i) and (ii).

Proposition 1.1.2. (the flip bifurcation)

Let f be a one parameter family of C^3 maps satisfying

(a) $f(0,0)=0$, (b) $(\partial f/\partial x)(0,0)=-1$.

Then there is a unique branch of fixed points $x(\lambda)$ for λ small. If in addition

(c) $(\partial^2 f/\partial x \partial \lambda)(0,0)>0$ and (d) $(\partial^3 f/\partial x^3)(0,0)<0$.

Then there are intervals $(\lambda_1,0)$ and $(0,\lambda_2)$ such that

(i) If $\lambda \in (\lambda_1,0)$ then f_λ has one unstable fixed point and one stable period two orbit in the neighbourhood of 0.

(ii) If $\lambda \in (0,\lambda_2)$ then f_λ has a single stable fixed point in the neighbourhood of 0 and no period two orbits.

Note

Changing inequality (c) changes the stability of the fixed point solutions, while changing (d) effects the stability of the period two orbit, either change will reverse the interval in which the period two orbit lies.

Often there are restrictions put on the class of maps which may effect the genericity conditions of the previous propositions. Most commonly the restriction seen is that the origin is fixed for all parameter values (a common assumption in many physical phenomena) giving, rather than the fold bifurcation, the transcritical bifurcation described in the proposition below.

Proposition 1.1.3.(the transcritical bifurcation)

Let f be a one parameter family of C^2 maps satisfying

(a) $f(0,\lambda)=0$, (b) $\partial f/\partial x(0,0)=1$, (c) $\partial^2 f/\partial x^2(0,0)>0$,

(d) $\partial^2 f/\partial x\partial\lambda(0,0)>0$.

Then f has a unique branch of fixed point solutions $x(\lambda)$, for λ small, bifurcating from 0. The origin is stable for $\lambda<0$ and unstable for $\lambda>0$ while the solutions on the bifurcating branch have the opposite stability.

Again reversing inequality (d) reverses the stabilities of the solutions.

Another commonly occurring restriction is that of symmetry, such as when f is an odd function. Here we must have a trivial solution $x=0$ for all λ, but we cannot apply the previous proposition since the symmetry forces $\partial^2 f/\partial x^2(0,0)=0$. Instead we arrive at the followingstatement.

Proposition 1.1.4.(the pitchfork bifurcation)

Let f be a one parameter family of C^3 maps satisfying

(a) $f(0,\lambda)=0$, (b) $\partial f/\partial x(0,0)=1$, (c) $\partial^3 f/\partial x^3(0,0)<0$,

(d) $\partial^2 f/\partial x\partial\lambda(0,0)>0$.

Then there are intervals $(\lambda_1,0)$ and $(0,\lambda_2)$ such that

(i) If $\lambda\in(\lambda_1,0)$ then f_λ has only one stable fixed point at the origin, in the neighbourhood of the origin

(ii) If $\lambda\in(0,\lambda_2)$ then f_λ has three fixed points in the neighbourhood of the origin, the origin which is unstable and a branching pair of solutions, $x(\lambda)$ and $-x(\lambda)$, which are stable.

This proposition gives some idea of the problems associated with the introduction of symmetry. For example,generically the stability determining terms of the solution are forced by the symmetry to be of higher order than they otherwise would be. We shall come across similar difficulties throughout our discussion of symmetric bifurcations.

1.2. Maps of the plane.

When a complex conjugate pair of eigenvalues crosses the unit circle at $e^{\pm 2\pi i\theta}$ we see much richer behaviour than that in the previous section. The centre manifold of f has dimension at least two and so we must consider the reduced form of f to be a map on the plane. Furthermore the dynamics of the map are very delicate and require the two parameters to capture it fully, whereas the bifurcations in section one really only require one parameter to see the full dynamics.

In general if $\theta \neq {}^1/_4, {}^1/_3, {}^1/_2, {}^2/_3, {}^3/_4$ we will see a branch of invariant circles bifurcating from the fixed point, a Hopf bifurcation, and if in addition $\theta = p/q$ $p,q\in\mathbb{Z}$ we expect, in some specific region of parameter space, to see on this invariant circle two orbits of period q points.

We start this section with the statement of the Hopf bifurcation theorem.

Theorem 1.2.1. (Hopf Bifurcation theorem)

Let f_λ be a smooth one parameter family of maps of $\mathbb{R}^2$ with a fixed point at zero for all λ satisfying

(a) $(df_\lambda)_0$ has eigenvalues $s(\lambda)e^{\pm 2\pi i\theta(\lambda)}$ with $s(0)=1$,

(b) $ds/d\lambda|_{\lambda=0}>0$ (this says the eigenvalues cross the unit circle with non zero speed),

(c) $e^{\pm 2\pi i\theta(0)}$ is not an m-th root of unity for m=1, 2, 3, 4.

Then there is a smooth λ-dependant change of coordinates taking f_λ into the form

$$f_\lambda(X)=g_\lambda(X)+O(|X|^5)$$

and there are smooth functions a and b so that in polar coordinates

$$g_\lambda(r,\varphi) = \left(s(\lambda)r-a(\lambda)r^3, \varphi+\theta(\lambda)+b(\lambda)r^2\right).$$

Moreover if $a(0)>0$ ($a(0)<0$) for sufficiently small positive (negative) λ, f_λ has an attracting (repelling)invariantcircle.

We say the bifurcation is supercritical if the circle exists for positive λ and subcritical if it exists for negative λ.

Sketch of the proof (taken from Whitley [1983])

We derive a proof for the supercritical case. In the subcritical case the proof follows along the same lines. Condition (b) means we can use a smooth change of coordinates in parameter space to express the eigenvalues as $(1+\lambda)e^{\pm 2\pi i\theta(\lambda)}$.

Next by a successive polynomial change in coordinates we can bring f_λ into the form

$$f_\lambda(z) = (1+\lambda)e^{2\pi i\theta(\lambda)}z+\alpha(\lambda)z^2\bar{z}+\beta(\lambda)\bar{z}^4+O(|z|^5)$$

where $\beta(\lambda)=0$ if $5\theta(0)$ is not an integer (this is the normal form of f_λ see chapter four for a more detailed description of normal forms).

Letting $z=re^{i\varphi}$ and $f_\lambda(z)=Re^{i\Phi}$ we get

$$R = (1+\lambda)r+a_3(\lambda)r^3+a_4(\lambda)r^4+O(r^5)$$

and

$$\Phi = \varphi+\theta(\lambda)+b_2(\lambda)r^2+b_3(\lambda,\varphi)r^3+O(r^4).$$

Truncating R at order 3 we see there is an invariant circle given by

$$r_0 = \sqrt{\frac{\lambda}{-a_3(\lambda)}}$$

and since the untruncated R is only a slight perturbation of this we would expect that it also has an invariant circle. To show this we make a further change in coordinates

$$x = \frac{1}{\sqrt{\lambda}}\left(\sqrt{\frac{a(\lambda)}{\lambda}}r - 1\right) \qquad \text{if } 5\theta(0) \neq 1$$

or

$$x = \frac{1}{\sqrt{\lambda}}\left(\sqrt{\frac{a(\lambda)}{\lambda}}r - 1\right) + g(\varphi) \qquad \text{if } 5\theta(0) = 1$$

where $a=-a_3$ and g is $2\pi/5$ periodic and solves

$$\frac{b_2(0)}{a_3(0)}\frac{dg}{d\varphi}(\varphi) + 2g(\varphi) + \frac{a_4(0,\varphi)}{a_3(0)^{3/2}} = 0$$

This means that f_λ is now in the form

$$X = (1-2\lambda)x + \lambda^{3/2}X_1(x,\varphi,\lambda)$$
$$\Phi = \varphi + \theta(\lambda) + \lambda^{3/2}\Phi_1(x,\varphi,\lambda)$$

where X_1 and Φ_1 are smooth in x, φ and λ for $|x| \leq 1$, $\varphi\in[0, 2\pi]$ and λ small.

Next, for each λ small, we look for an f_λ invariant manifold of the form $M = \{ x = u(\varphi) \}$ where

(i) u is 2π periodic,

(ii) $|u| \leq 1$

(iii) For all $\varphi_1, \varphi_2 \in \mathbb{R}$

$$| u(\varphi_1)-u(\varphi_2) | \leq | \varphi_1-\varphi_2 | .$$

The set U of all u satisfying these properties is a complete metric space with metric

$$d(u_1, u_2) = \sup | u_1(\varphi)-u_2(\varphi) | .$$

It can be shown that f_λ maps U onto itself and further that the map restricted to U is in fact a contraction and thus by the contraction mapping theorem there is a unique fixed point u^* in U which is precisely the invariant circle required in the proof. Stability follows immediately since f_λ is a contraction onto this invariant circle.

Remark

The majority of the proof lies within this last paragraph and has only been outlined since it is somewhat technical.

Now we have an invariant circle we would like to look at the dynamics of f upon this circle. It has been shown (see Whitley[1983]) that generically we should expect to see periodic orbits upon the circle.

We let $\lambda = (\mu,\nu)$ be the two dimensional parameter space and let the eigenvalues of $(df_\lambda)_0$ be $e^{\pm 2\pi ip/q}$, two parameters are required in order that we can discuss fully the dynamics of the invariant circle. To simplify matters we can assume the map from parameter space to this pair of eigenvalues is non singular and thus we can reparameterise so the eigenvalues are of the form

$$(1+\mu)e^{\pm i((2\pi p/q)+\nu)}.$$

Thus in the previous Hopf theorem, bifurcation would occur as μ passed through the ν axis in parameter space. The following theorem in addition asserts that upon the invariant circle there exist a pair of periodic orbits for parameter values lying within a narrow cusped region.

Theorem 1.2.2. (Subharmonic Bifurcation Theorem)

Let f_λ be a smooth two parameter family of maps of $\mathbb{R}^2$ such that for all $\lambda=(\mu,\nu)$ in a neighbourhood U of 0:

(i) $f_{(\mu,\nu)}(0)=0$ for all $(\mu,\nu)\in U$,

(ii) $(df_{(\mu,\nu)})_0$ has a complex conjugate pair of eigenvalues $s(\mu,\nu)e^{\pm i\theta(\mu,\nu)}$ with $s(0,0)=1$ and $\theta(0,0)=2\pi p/q$, with $hcf(p,q)=1$ and $q\geq 5$,

(iii) the map $(\mu,\nu)\longrightarrow s(\mu,\nu)e^{i\theta(\mu,\nu)}$ is non singular for $\mu=\nu=0$.

Then there is a smooth reparameterisation so that near $\mu=\nu=0$

$$s(\mu,\nu)=1+\mu$$

and
$$\theta(\mu,\nu)=(2\pi p/q)+\nu$$

and a smooth change in coordinates taking $f_{(\mu,\nu)}$ into the form

$$f_{(\mu,\nu)}(z) = (1+\mu)e^{i((2\pi p/q)+\nu)} + \sum_{m=1}^{[(q-2)/2]} \alpha_{2m+1}(\mu,\nu)z^{m+1}\bar{z}^{m} + \beta(\mu,\nu)\bar{z}^{q-1} + O(|z|^q) \qquad (2.1)$$

which in polar coordinates, $z=re^{i\varphi}$, $f(\mu,\nu)=Re^{i\Phi}$ gives

$$R = (1+\mu)r + \sum_{m=1}^{[(q-2)/2]} a_{2m+1}(\mu,\nu)r^{2m+1} + b(\mu,\nu,\varphi)r^{q-1} + O(r^q)$$

and

$$\Phi = \varphi + \frac{2\pi p}{q} + \nu + \sum_{m=1}^{[(q-2)/2]} c_{2m}(\mu,\nu)r^{2m} + d(\mu,\nu,\varphi)r^{q-2} + O(r^{q-1}).$$

where [. . .] represents the integer part of . . . , the functions β, b, d and all α's, a's and c's are smooth in μ and ν for all $(\mu,\nu)\in U$ and b and d are of the form

$$A(\mu,\nu)\cos(q\varphi+\psi(\mu,\nu)).$$

For all $(\mu,\nu)\in U$ with $\mu>0$ (or $\mu<0$ respectively) $f_{(\mu,\nu)}$ has an attracting (repelling) invariant circle if $a_3 = a_3(0,0)<0$ ($a_3>0$).

Furthermore, if $\beta=\beta(0,0)\neq 0$ and $c_2=c_2(0,0)\neq 0$ then the map has two orbits of period q points alternately around the circle, one stable the other unstable, provided the parameter values μ and ν lie within a narrow tongue in the parameter plane whose boundaries are given by

$$\nu \simeq \frac{b_2}{a_3}\mu \pm \frac{|\beta|^2}{|a_3|^{(q-2)/2}}\mu^{(q-2)/2}.$$

Proof (Whitley [1983])

The existence and stability of the invariant circle comes via the direct application of the Hopf theorem from earlier in the chapter, however to establish the dynamics on this invariant circle it is convenient to adapt this proof slightly. Again in complex notation and successive polynomial change in coordinates we can put f_λ into the form (2.1), the normal form of f_λ, which has the desired polar form. We restrict our attention, as in the proof of the Hopf theorem, to the supercritical case (when $a_3<0$). Truncating R at order $q-1$ look for an invariant circle $R=r$ that is

$$\mu + \sum_{m=1}^{[(q-2)/2]} a_{2m+1} r^{2m} = 0$$

which has a unique positive solution

$$r_0^2(\mu,\nu) = \frac{-\mu}{a_3(\mu,\nu)} + O(\mu^2).$$

We now make a coordinate change for the full map, defined by

$$r = r_0(\mu,\nu)(1+\mu^{(q-4)/2}x)$$

giving the new map

$$X = (1-2\mu)x + \mu\frac{b(\mu,\nu,\varphi)}{a_3(\mu,\nu)^{(q-2)/2}} + O(\mu^{3/2})$$

$$\Phi = \varphi + \frac{2\pi p}{q} + \nu + \sum_{m=1}^{[(q-2)/2]} h_m\mu^m + h(\mu,\nu,\varphi)\mu^{(q-2)/2} + O(\mu^{(q-1)/2})$$

where $h_1 = c_2(\mu,\nu)/a_3(\mu,\nu)$ and $h(\mu,\nu,\varphi) = d(\mu,\nu,\varphi)/a_3(\mu,\nu)^{(q-2)/2}$.

Next we make a coordinate change similar to that for the Hopf theorem when $5\theta(0)=1$. Defining the new coordinate $\bar{x}$ by $x = g(\mu,\nu,\varphi)+\mu^\gamma\bar{x}$ where $0<\gamma<1/2$ and

$g(\mu,\nu,\varphi)=O(\mu^{1/2})$ solves the differential equation

$$2g(\mu,\nu,\varphi)+\frac{1}{\mu}(\nu+\sum h_m\mu^m)\frac{\partial g}{\partial\varphi}(\mu,\nu,\varphi)=\frac{b(\mu,\nu,\varphi)}{a_3(\mu,\nu)^{(q-2)/2}}$$

So now the map becomes (removing the hat from x)

$$X=(1-2\mu)x+O(\mu^{3/2-\gamma})$$

$$\Phi=\varphi+\frac{2\pi p}{q}+\nu+\sum_{m=1}^{[(q-2)/2]}h_m\mu^m+h(\mu,\nu,\varphi)\mu^{(q-2)/2}+O(\mu^{q/2+\gamma-1}).$$

Now we may apply the proof of the Hopf theorem to show the existence of a stable invariantcircle.

The map on the invariant circle can be written

$$f_{(\mu,\nu)}(\varphi)=\varphi+\frac{2\pi p}{q}+\nu+\sum_{m=1}^{[(q-2)/2]}h_m\mu^m+h_{(\mu,\nu)}(\varphi)\mu^{(q-2)/2}+O(\mu^{q/2+\gamma-1}).$$

This map truncated at order $\mu^{q/2+\gamma-1}$ has an orbit of period q solutions provided

$$\nu+\sum_{m=1}^{[(q-2)/2]}h_m\mu^m+h_{(\mu,\nu)}(\varphi)\mu^{(q-2)/2}=0$$

and since $h_{(\mu,\nu)}(\varphi)$ is of the form $A(\mu,\nu)\cos(q\varphi+\psi(\mu,\nu))$ then we have two solutions $\varphi_1{}^0$, $\varphi_2{}^0$ within the interval $[0,2\pi/q)$ provided that

$$|\nu+\sum_{m=1}^{[(q-2)/2]}h_m\mu^m|<\sup_{\varphi\in[0,2\pi]}h(\mu,\nu,\varphi)\mu^{(q-2)/2}$$

see Fig 1.1. This defines the region of existence whose boundaries are given in the statement of the proof.

From the implicit function theorem it follows that f has two orbits of period q points for (μ,ν) within this region of parameter space given by $f^j_{(\mu,\nu)}(\varphi_1(\mu,\nu))$ and $f^j_{(\mu,\nu)}(\varphi_2(\mu,\nu))$ for $j=0,\ldots,q-1$ where

$$\varphi_i=\varphi_i{}^0+O(\mu^\gamma).$$

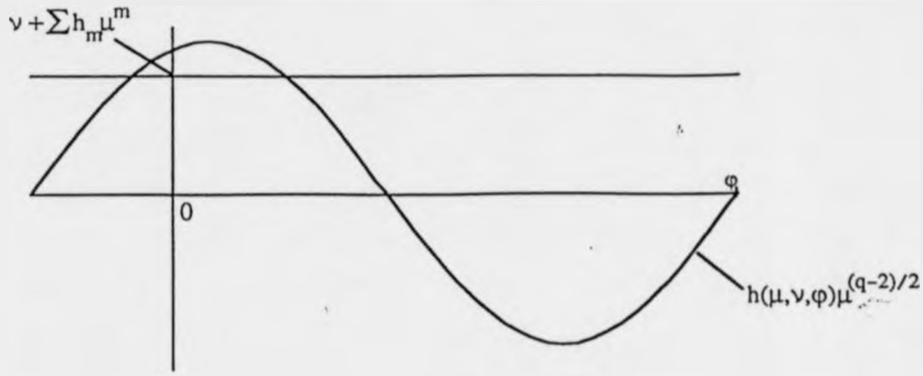


Fig 1.1 Diagram showing solutions of $\nu+\sum h_m \mu m = h(\mu,\nu,\varphi)\mu^{(q-2)/2}$.

For the stability of these two orbits note that the derivative of $f_{(\mu,\nu)}$ on the invariant circle with respect to φ is

$$f'_{(\mu,\nu)}(\varphi_i(\mu,\nu)) = 1 + h'_{(\mu,\nu)}(\varphi_i{}^0)\mu^{(q-2)/2} + O(\mu^{q/2+\gamma-1})$$

and from the diagram it is clear to see that the derivative of $h_{(\mu,\nu)}$ has opposite signs for $\varphi_1{}^0$ and $\varphi_2{}^0$ and thus these point and those sufficiently close to them will have opposite stabilities and our proof is complete.

Remark

If this two parameter family has a one parameter family embedded in it. Changing the parameter moves us along a path in the (μ,ν) plane which will typically intersect infinitely may of these tongues close to the bifurcation point. Therefore we should expect to see many large period periodic orbits on the invariant circle soon after the Hopf bifurcation.

We end this chapter with a useful result that can be used to compute where the bifurcations above will occur and also the regions in which maps will remain stable. It uses a coordinate independent system to determine this stability, namely the trace and determinant of the linearisation of the map.

Lemma 1.2.3.

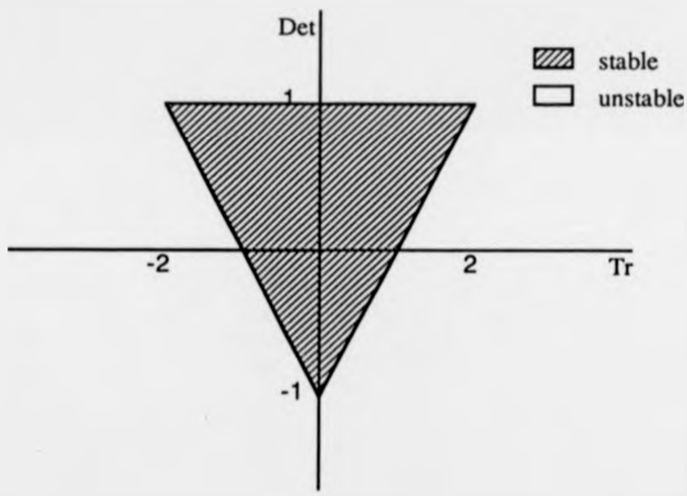
A fixed point x of a map f from $\mathbb{R}^2$ to $\mathbb{R}^2$ is asymptotically stable if and only if the trace Tr and determinant Det of $(df)_x$ satisfy the following inequalities:

i) $-2 < Tr < 2$,

ii) $-(1+Det) < Tr < 1+Det$,

iii) $Det < 1$.

That is in the (Tr , Det) plane (Tr , Det) lie within the shaded triangle in the diagram below.



Proof

A fixed point x of f is asymptotically stable if the eigenvalues of the linearisation of f around x lie within the unit circle (see Arrowsmith and Place Theorem 2.2.1 for an exact statement of the theorem).

Let

$$(df)_x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be the linearisation of f around x.

The eigenvalues of this map are the solutions of

$$\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = 0$$

i.e.
$$ad-(a+d)\lambda+\lambda^2-bc=0$$

or
$$\lambda^2-\mathrm{Tr}\,\lambda+\mathrm{Det}=0$$

where Tr = a+d is the trace of $(df)_x$ and Det = ad-bc is the determinant of $(df)_x$ and its solutions are given by

$$\lambda = \frac{\mathrm{Tr}\pm\sqrt{\mathrm{Tr}^2-4\mathrm{Det}}}{2}$$

If x is an asymptotically stable fixed point then the eigenvalues of $(df)_x$ have modulus less than unit and we split the proof into two part

a) $\lambda_1,\lambda_2\in\mathbb{R}$ and $1>\lambda_1\geq\lambda_2>-1$.

Since the eigenvalues are real then

$$\mathrm{Tr}^2-4\mathrm{Det}\geq 0$$

or
$$\mathrm{Tr}^2\geq 4\mathrm{Det}.$$

Furthermore we have the inequalities

$$\mathrm{Tr} + \sqrt{(\mathrm{Tr}^2-4\mathrm{Det})} < 2$$

and
$$\mathrm{Tr} - \sqrt{(\mathrm{Tr}^2-4\mathrm{Det})} > -2$$

i.e.
$$\sqrt{(\mathrm{Tr}^2-4\mathrm{Det})} < 2-\mathrm{Tr}$$

and
$$\sqrt{(\mathrm{Tr}^2-4\mathrm{Det})} < 2+\mathrm{Tr}.$$

Since we are considering the positive root we immediately get inequality (i) $-2 < \mathrm{Tr} < 2$ and since $\mathrm{Tr}^2 \geq 4\mathrm{Det}$ we have (iii) $\mathrm{Det} < 1$. Squaring both sides of the above inequalities gives

$$\mathrm{Tr}^2-4\mathrm{Det} < 4-4\mathrm{Tr}+\mathrm{Tr}^2$$

and
$$\mathrm{Tr}^2-4\mathrm{Det} < 4+4\mathrm{Tr}+\mathrm{Tr}^2$$

or $$\mathrm{Tr} < 1+\mathrm{Det}$$

and $$\mathrm{Tr} > -(1+\mathrm{Det})$$

so we have inequality (ii).

b) $\lambda_1, \lambda_2 \in \mathbb{C}$ and λ_1, λ_2 are conjugate.

Firstly note for non real eigenvalues

$$\mathrm{Tr}^2 < 4\mathrm{Det} \qquad \text{(and hence Det > 0).}$$

Observe also that $|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1\lambda_2 = \mathrm{Det}$, hence we need Det < 1

i.e. $$\mathrm{Tr}^2 < 4$$

or $$-2 < \mathrm{Tr} < 2.$$

Furthermore

$$2\mathrm{Det} < 1+\mathrm{Det}^2$$

since Det < 1 and Det < Det^2. Hence

$$\mathrm{Tr}^2 < 4\mathrm{Det} < 1+2\mathrm{Det}+\mathrm{Det}^2 = (1+\mathrm{Det})^2$$

or $$-(1+\mathrm{Det}) < \mathrm{Tr} < 1+\mathrm{Det}$$

and the forward proof is complete.

In the other direction we start by assuming the inequalities

i) $-2 < \mathrm{Tr} < 2.$

ii) $-(1+\mathrm{Det}) < \mathrm{Tr} < 1+\mathrm{Det} < 2.$

iii) $\mathrm{Det} < 1$

Again we consider the two cases of complex and real eigenvalues.

a) The eigenvalues are real (i.e. $\mathrm{Tr}^2 \geq 4\mathrm{Det}$).

From inequality (ii) we have

$$-(4+\mathrm{Det}) < 4\mathrm{Tr} < 4+4\mathrm{Det}$$

or $$\mathrm{Tr}^2-4\mathrm{Det} < \mathrm{Tr}^2+4\mathrm{Tr}+4 = (\mathrm{Tr}+2)^2$$

and $$\mathrm{Tr}^2-4\mathrm{Det} < \mathrm{Tr}^2-4\mathrm{Tr}+4 = (\mathrm{Tr}-2)^2$$

which gives

$$\sqrt{(\mathrm{Tr}^2-4\mathrm{Det})} < |\mathrm{Tr}+2|$$

and $$\sqrt{(Tr^2-4Det)} < | Tr-2 | \qquad \text{since } Tr^2-4Det \geq 0.$$

Now inequality (i) says $-2 < Tr < 2$ i.e.

$$Tr+2 > 0$$

and $$Tr-2 < 0$$

thus

$$\sqrt{(Tr^2-4Det)} < Tr+2$$

and $$\sqrt{(Tr^2-4Det)} < 2-Tr.$$

Hence $$Tr+\sqrt{(Tr^2-4Det)} < 2$$

and $$Tr-\sqrt{(Tr^2-4Det)} < -2.$$

b) The eigenvalues are complex conjugate (i.e. $Tr^2 < 4Det$).

Again $|\lambda_1| = |\lambda_2| = \lambda_1\lambda_2 = Det$, but $0 < Tr^2 < 4Det$ and from inequality (iii)

$$Det < 1$$

and so we are done.

From a simple consideration of this proof we see that the three sides to this triangle represent the three possible types of bifurcation. If $Tr=1+Det$ then we have an eigenvalue at 1 which corresponds to a fold bifurcation, if $Tr=-(1+Det)$ we have an eigenvalue at -1 corresponding to a flip bifurcation and finally if $Det=1$ then we have a complex conjugate pair of eigenvalues $e^{\pm i\theta}$ on the unit circle which will give a Hopf bifurcation. Moving along the line $Det = 1$ from $Tr = -2$ to $Tr = 2$ we will expect to see all possible values of θ and thus solutions exhibiting all possible periods.

This ends our discussion of diffeomorphisms in $\mathbb{R}$ and $\mathbb{R}^2$. We shall see in later chapters how some of the ideas from this chapter may be adapted when symmetry is introduced into the system, but first we motivate the development of symmetry via a simple example.

Chapter 2

A Motivating Example With Symmetry.

2.0 Introduction.

Before going on to discuss the effects of symmetry we give an example of how symmetries can arise in naturally occurring phenomenon. We introduce a discrete predator prey system of equations (which has been discussed in detail in Maynard Smith [1968]) and show by adding a simple linear coupling between identical sets of these systems we generate a system which exhibits D_n symmetry and further we show how steady state and periodic solutions of this set of equations can have less symmetry than the original problem.

2.1 A discrete predator prey system without symmetry.

Consider a model for a single species in an environment which can support a population X_E. We have

$$X_{n+1} - X_n = r\frac{(X_E - X_n)}{X_E}X_n$$

where r is the rate of unrestrained population growth (i.e. rate of growth if only a small number of the species existed or the environment could support an infinitely large population). This is multiplied by $(X_E - X_n)/X_E$ which reflects the effect of the environmental limiting factors on growth (that is the growth rate is positive if $X_n < X_E$ and negative if $X_n > X_E$).

Hence

$$X_{n+1} = (1 + r)X_n - \frac{rX_n^2}{X_E}$$

We now introduce a predator. Each predator can consume an amount of the above species proportional to the amount which exists (the more prey the greater the likelihood of encounter between predator and prey, hence the more prey each predator consumes). So the model must be adjusted to take this into account and we have

$$X_{n+1} = (1+r)X_n - \frac{rX_n^2}{X_E} - aX_nY_n \qquad r, X_E, a > 0$$

where a is a constant of proportionality which indicates the proficiency of the predator at hunting.

Assume also that the number of predators which exist in the next generation is proportional to the amount of prey consumed in the previous one (i.e. the predator turns food into offspring) we then get

$$Y_{n+1} = AX_nY_n \quad A > 0.$$

We now need to find equilibria of this system, that is (X_s, Y_s) where

$$X_s = (1+r)X_s - \frac{rX_s^2}{X_E} - aX_sY_s \qquad (1)$$

and

$$Y_s = AX_sY_s \qquad (2).$$

Giving three fixed points

$$(0,0), \ (X_E, 0) \text{ and } \left(\frac{1}{A}, \frac{r}{a}\left(1 - \frac{1}{AX_E}\right)\right).$$

For the linearisation of the equations around (0,0) we get

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1+r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \text{h.o.t}$$

where

$$x_n = X_n$$
$$y_n = Y_n$$

which is unstable for all $r > 0$.

The linearisation around the fixed point $(X_E,0)$ gives

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1-r & -aX_E \\ 0 & AX_E \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \text{h.o.t}$$

where $X_n = X_E + x_n$, $Y_n = y_n$

This is stable for $0 < r < 2$ and $AX_E < 1$.

Finally for the linearisation around

$$\left(\frac{1}{A}, \frac{r}{a}(1 - \frac{1}{AX_E}\right)$$

we get

$$x_{n+1} = \left(1 + r - \frac{2r}{AX_E} - \frac{ar}{a}\left(1 - \frac{1}{AX_E}\right)\right) x_n - \frac{a}{A} y_n - a x_n y_n - \frac{r x_n^2}{X_E}$$

$$y_{n+1} = \frac{Ar}{a}\left(1 - \frac{1}{AX_E}\right) x_n + A\frac{1}{A} y_n + A x_n y_n$$

where

$$X_n = X_s + x_n, \quad Y_n = Y_s + y_n \quad \text{and} \quad (X_s, Y_s) = \left(\frac{1}{A}, \frac{r}{a}\left(1 - \frac{1}{AX_E}\right)\right)$$

or

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{r}{AX_E} & -\frac{a}{A} \\ \frac{r}{a}\left(A - \frac{1}{X_E}\right) & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \text{h.o.t.}$$

We would now like to consider the stability of the fixed point and to do that we use Lemma 1.2.3 which relates the stability of a fixed point to the trace and determinant of df. It state that a fixed point is stable if (Tr, Det) the trace and determinant of the linearisation of the map around the fixed point lie within the triangle in (Tr, Det) plane determined by the three equations Det = 1, Tr = Det+1 and Tr = -(Det+1). Two straightforward calculations yield

$$Tr = 2-r/AX_E$$

and $$Det = 1+r-2r/AX_E.$$

Hence for the fixed point to be stable we need:

(i) $Det < 1$

or $$r < 2r/AX_E$$

and since $r > 0$ we must have $AX_E < 2$.

(ii) $Tr < Det+1$

that is $$-r/AX_E < r-2r/AX_E$$

or $$r/AX_E < r$$

and again since $r > 0$ we need $AX_E > 1$.

(iii) $Tr > -(Det+1)$

that is $$4 > 3r/AX_E-r$$

or $$4 > r(3-AX_E)/AX_E$$

and since in the interval $1 < AX_E < 2$

$$(3-AX_E)/AX_E > 0$$

we need

$$r < 4AX_E/(3-AX_E).$$

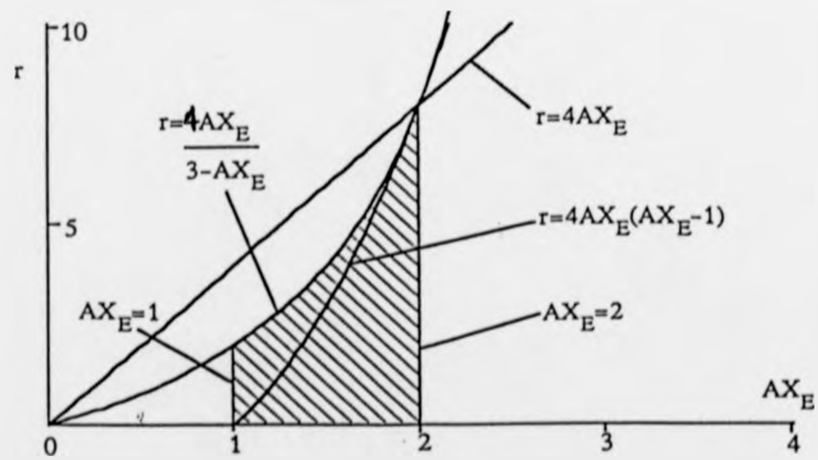


Fig 2.1 The shaded area is the region of stability of the fixed point.

From the discussion at the end of chapter 1 we see that there are three ways to lose stability corresponding to fold, flip and Hopf bifurcations. If $AX_E = 1$ we expect a transcritical bifurcation and to see a new branch of fixed points close to our original fixed point. If $r = 4AX_E/(3-AX_E)$ we expect a branch of period two points bifurcating from our original fixed point. Finally if $AX_E = 2$ we will see a Hopf type bifurcation. Furthermore if $Tr = 2\cos 2\pi p/q$ also then the complex conjugate pair of eigenvalues will be $e^{\pm 2\pi i p/q}$ so on the invariant circle we will see two branches of period q orbits alternately stable and unstable provided the eigenvalues exit the unit circle in the region defined in the statement of Theorem 1.2.2.

2.2 Coupled symmetric predator prey systems.

The next thing we must do is to introduce some symmetries into our system and we do this by simply coupling identical oscillators with a linear coupling thus.

Let (X^1,Y^1), (X^2,Y^2), . . . (X^n,Y^n) be n pairs of predator and prey in identical environments ('islands') each governed by the equations discussed before

$$X^i_{n+1} = (r+1)X^i_n - \frac{r\,(X^i_n)^2}{X_E} - aX^i_nY^i_n$$

$$Y^i_{n+1} = AX^i_nY^i_n$$

Add identical linear coupling (modelling some form of migration) between adjacent 'islands', i.e.

$$X^i_{n+1} = (r+1)X^i_n - \frac{r\,(X^i_n)^2}{X_E} - aX^i_nY^i_n + s(X^{i-1}_n - X^i_n) + s(X^{i+1}_n - X^i_n)$$

$$Y^i_{n+1} = AX^i_nY^i_n + s(Y^{i-1}_n - Y^i_n) + s(Y^{i+1}_n - Y^i_n)$$

We consider the equilibrium at

$$X_s^i = \frac{1}{A},\ Y_s^i = \frac{r}{a}(1 - \frac{1}{AX_E})$$

for i = 1, . . . ,n. Linearising around (X_s^i, Y_s^i) gives us the following equations

$$x_{n+1}^i = (1 - \frac{r}{AX_E} - 2s)\, x_n^i - \frac{a}{A} y_n^i - \frac{r(x_n^i)^2}{X_E} - a x_n^i y_n^i + s(x_n^{i-1} + x_n^{i+1})$$

$$y_{n+1}^i = \frac{Ar}{a}(1 - \frac{1}{AX_E})\, x_n^i + (1 - 2s)y_n^i + A x_n^i y_n^i + s(y_n^{i-1} + y_n^{i+1})$$

So for m identical oscillators the map we consider is

$$\begin{bmatrix} x_{n+1}^1 \\ y_{n+1}^1 \\ \cdot \\ \cdot \\ x_{n+1}^m \\ y_{n+1}^m \end{bmatrix} = M \begin{bmatrix} x_n^1 \\ y_n^1 \\ \cdot \\ \cdot \\ x_n^m \\ y_n^m \end{bmatrix} + \text{h.o.t.}$$

where

$$M = \begin{bmatrix} P\ Q\ 0\ .\ .\ Q \\ Q\ P\ Q\ .\ .\ 0 \\ \cdot \\ Q\ 0\ .\ .\ Q\ P \end{bmatrix}$$

with

$$P = \begin{bmatrix} 1 - \frac{r}{AX_E} - 2s & -\frac{a}{A} \\ \frac{Ar}{a}(1 - \frac{1}{AX_E}) & 1-2s \end{bmatrix}$$

and

$$Q = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}.$$

We will consider here the case where $m = 3$. So we have 3 identical coupled environments, E_1, E_2 and E_3, clearly these exhibit D_3 symmetry where the flip takes $E_2 \leftrightarrow E_3$ and E_1 to itself and the rotation takes $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow E_1$. So we have

$$M = \begin{bmatrix} P & Q & Q \\ Q & P & Q \\ Q & Q & P \end{bmatrix}.$$

Given a matrix of this form we can calculate its eigenvalues, in terms of P and Q. This idea is used by Golubitsky *et al* [1987] in the discussion of coupled non linear oscillators and can be generalised easily to $m > 3$.

The two dimensional vector space of $\mathbb{R}^6$ consisting of vectors of the form $[v,v,v]^T$, $v \in \mathbb{R}^2$ is invariant under M thus

$$\begin{bmatrix} P & Q & Q \\ Q & P & Q \\ Q & Q & P \end{bmatrix} \begin{bmatrix} v \\ v \\ v \end{bmatrix} = \begin{bmatrix} (P+2Q)v \\ (P+2Q)v \\ (P+2Q)v \end{bmatrix} = \begin{bmatrix} w \\ w \\ w \end{bmatrix},$$

so if v is an eigenvector of P+2Q with eigenvalue λ, that is $(P+2Q)v = \lambda v$, then $[v,v,v]^T$ is an eigenvector of M with eigenvalue λ.

Complexifying from $\mathbb{R}^6$ to $\mathbb{C}^6$ (or more generally from $\mathbb{R}^{km}$ to $\mathbb{C}^{km}$) we consider the subspaces $V_1 = \{[v, e^{2\pi i/3}v, e^{4\pi i/3}v]^T : v \in \mathbb{R}^2\}$ and $V_2 = \{[v, e^{4\pi i/3}v, e^{2\pi i/3}v]^T : v \in \mathbb{R}^2\}$ and show that these too are invariant under the action of M. Using similar calculations to those above we have

$$M\begin{bmatrix} v \\ e^{2\pi i/3}v \\ e^{4\pi i/3}v \end{bmatrix} = \begin{bmatrix} (P-Q)v \\ (P-Q)e^{2\pi i/3}v \\ (P-Q)e^{4\pi i/3}v \end{bmatrix}$$

and

$$M\begin{bmatrix} v \\ e^{4\pi i/3}v \\ e^{2\pi i/3}v \end{bmatrix} = \begin{bmatrix} (P-Q)v \\ (P-Q)e^{4\pi i/3}v \\ (P-Q)e^{2\pi i/3}v \end{bmatrix}$$

So if λ is an eigenvalue of P-Q with eigenspace v then

$$M\begin{bmatrix} v \\ e^{2\pi i/3}v \\ e^{4\pi i/3}v \end{bmatrix} = \begin{bmatrix} (P-Q)v \\ (P-Q)e^{2\pi i/3}v \\ (P-Q)e^{4\pi i/3}v \end{bmatrix} = \begin{bmatrix} \lambda v \\ \lambda v e^{2\pi i/3} \\ \lambda v e^{4\pi i/3} \end{bmatrix}$$

and

$$M\begin{bmatrix} v \\ e^{4\pi i/3}v \\ e^{2\pi i/3}v \end{bmatrix} = \begin{bmatrix} (P-Q)v \\ (P-Q)e^{4\pi i/3}v \\ (P-Q)e^{2\pi i/3}v \end{bmatrix} = \begin{bmatrix} \lambda v \\ \lambda v e^{4\pi i/3} \\ \lambda v e^{2\pi i/3} \end{bmatrix}.$$

So the eigenvalues of P+2Q are eigenvalues of M with multiplicity one and the eigenvalues of P-Q are eigenvalues of M with multiplicity two (the high multiplicity of this eigenvalue is a result of the symmetry in the system as was discussed in the introduction).

In general $\mathbb{C}^{km}$ decomposes into m invariant subspaces $V_0, \ldots, V_{m-1}$ where

$$V_j = \left\{[v, e^{2\pi ij/m}v, e^{4\pi ij/m}v, \ldots, e^{2(m-1)\pi j/m}v]^T;\ v \in \mathbb{R}^k\right\}$$

and for $v_j \in V_j$ a calculation similar to those above shows that

$$Mv_j = (P+2\cos(2\pi j/m)Q)v_j$$

thus if λ is an eigenvalue of $(P+2\cos(2\pi j/m)Q)$ then it is an eigenvalue of M also and since cosine is an even function these eigenvalues come in pairs unless $j = 0$ or $m/2$ ($j = m/2$ only if m is even).

Returning to the case where $m = 3$ we need to find the eigenvalues of P+2Q and P-Q and thus determine the regions of stability via Lemma 1.2.3. First we see that

$$P+2Q = \begin{bmatrix} 1-\dfrac{r}{AX_E} & -\dfrac{a}{A} \\ \dfrac{Ar}{a}(1-\dfrac{1}{AX_E}) & 1 \end{bmatrix}$$

which is the linearisation of the single predator prey system with no symmetry. Thus we get the same stability results as before, namely stability in the region indicated in fig 2.1.

Next we compute P-Q

$$P-Q = \begin{bmatrix} 1-\dfrac{r}{AX_E} -3s & -\dfrac{a}{A} \\ \dfrac{Ar}{a}(1-\dfrac{1}{AX_E}) & 1-3s \end{bmatrix}.$$

In order to find the regions of stability and use Lemma 1.2.3. we must calculate the trace and determinant of this matrix. We find

$$Tr = 2-6s-r/AX_E$$

and

$$Det = 1-6s+9s^2+r+(3s-2)r/AX_E.$$

First we note that for stability we need $Tr > -2$ and hence unless $s < 2/3$ the system will always be unstable for all values of r and AX_E. So throughout we shall assume $s < 2/3$.

For the fixed point to be stable we need:

(i) $Det < 1$

or

$$r(2-3s-AX_E)/AX_E > 9s^2-6s.$$

For $0 < AX_E < 2-3s$ we have

$$r > (9s^2-6s)AX_E/(2-3s-AX_E)$$

but this is trivially true since r is positive and the right hand side of the inequality is negative.

For $AX_E > 2-3s$ we have

$$r < (9s^2-6s)AX_E/(2-3s-AX_E).$$

(ii) $Tr < Det+1$

that is $$(1-3s-AX_E)r/AX_E < 9s^2.$$

Again there are two regions of interest, for $0 < AX_E < 1-3s$

$$r < 9s^2AX_E/(1-3s-AX_E)$$

and $AX_E > 1-3s$ in which case

$$r > 9s^2AX_E/(1-3s-AX_E)$$

which is trivially always true.

(iii) $Tr > -(Det+1)$

or $$(3-3s-AX_E)r/AX_E < (3s-2)^2$$

when $0 < AX_E < 3-3s$ this needs

$$r < (3s-2)^2AX_E/(3-3s-AX_E)$$

and when $AX_E > 3-3s$

$$r > (3s-2)^2AX_E/(3-3s-AX_E)$$

which for $r > 0$ is always true.

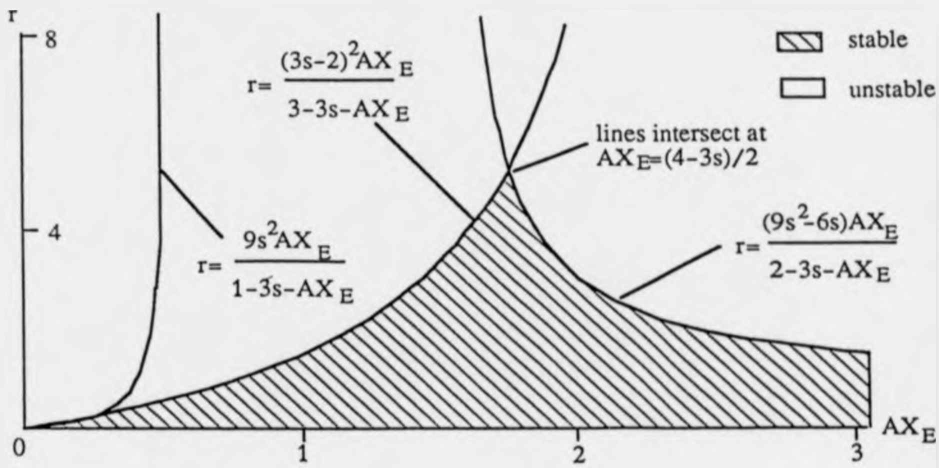
Thus the fixed point is stable for values of AX_E and r within the intersection of the following sets

$$\left\{0 < AX_E < 1-3s,\ 0 < r < 9s^2AX_E/(1-3s-AX_E)\right\} \cup \left\{AX_E > 1-3s,\ r > 0\right\},$$
$$\left\{0 < AX_E < 2-3s,\ r > 0\right\} \cup \left\{AX_E > 2-3s,\ 0 < r < (9s^2-6s)AX_E/(2-3s-AX_E)\right\},$$
$$\left\{0 < AX_E < 3-3s,\ 0 < r < (3s-2)^2AX_E/(3-3s-AX_E)\right\} \cup \left\{AX_E > 3-3s,\ r > 0\right\}.$$

We get a clearer picture of where stability occurs by graphing these curves and seeing what the stable area looks like.

Plotting for $s = 1/6$ and $s = 1/12$ in Fig 2.2, we get some idea of what will happen as $s \to 0$ in parameter space (r, AX_E). By looking at these examples it is clear to see that as s gets smaller the region of stability begins to resemble that of the non-coupled simple oscillators we had originally.

s=1/6:-



s=1/12:-

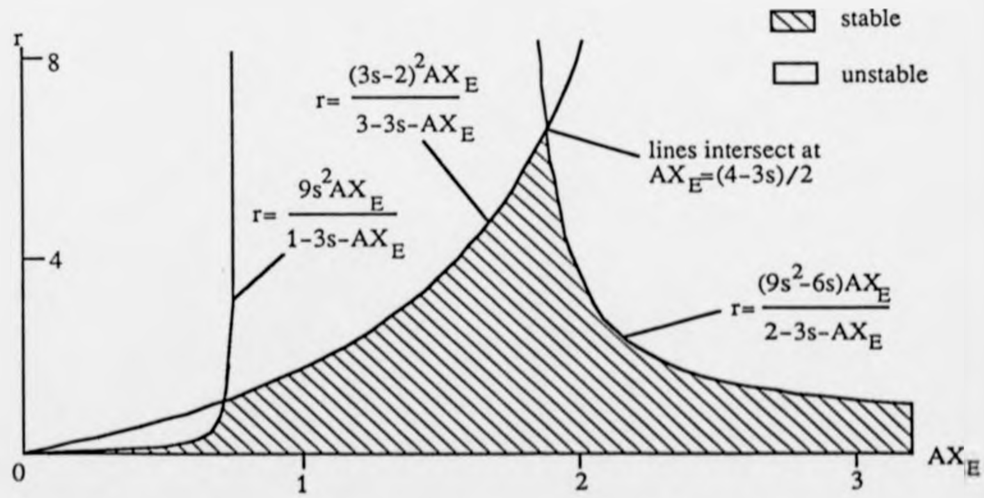


Fig 2.2 Stability diagrams coming from the eigenvalues of P-Q, comparing coupling strengths.

As we saw before there are three possible ways to lose stability:

i) some λ becomes greater than 1,

ii) some λ becomes less than -1,

iii) a pair of eigenvalues which are complex conjugate exit the unit circle.

On the diagrams on the previous page these three refer respectively to leaving the stable region through:

i) The line

$$r = 9s^2AX_E/(1-3s-AX_E),$$

ii) The line

$$r = (3s-2)^2AX_E/(3-3s-AX_E),$$

iii) The line

$$r = (9s^2-6s)AX_E/(2-3s-AX_E).$$

From the first of these we might expect to see bifurcating branches of fixed point solutions of the original equation, from the second branches of period two orbits and from the third Hopf bifurcation to branches of invariant circles (the symmetry will imply that more than one solution bifurcates in any particular case). However it is possible for the fixed point to lose stability via eigenvalues of P+2Q or P-Q passing through the unit circle. In the next section we discuss how the eigenvalues from the two different cases P+2Q and P-Q correspond to different actions of D_3 and hence give different types of solution.

2.3 Modes of behaviour for different solutions.

We first look in which regions of (AX_E, r) parameter space which eigenvalues become unstable. In Fig 2.3 we superimpose upon the stability diagram of the P-Q eigenvalues (for $s=1/6$) the diagram for the eigenvalues of P+2Q.

As we see from the diagram the fixed point bifurcation arises only from eigenvalues of P+2Q crossing the unit circle (as AX_E crosses the line $AX_E = 1$). The period 2 instability, however arises only from the eigenvalues of P-Q crossing the unit circle (or r crossing the line $(3s-2)^2AX_E/(3-3s-AX_E)$).

The invariant circle instability can arise from either set of eigenvalues. If $AX_E < 2$

otherwise it is from the eigenvalues of P+2Q (as AX_E crosses the line $AX_E = 2$).

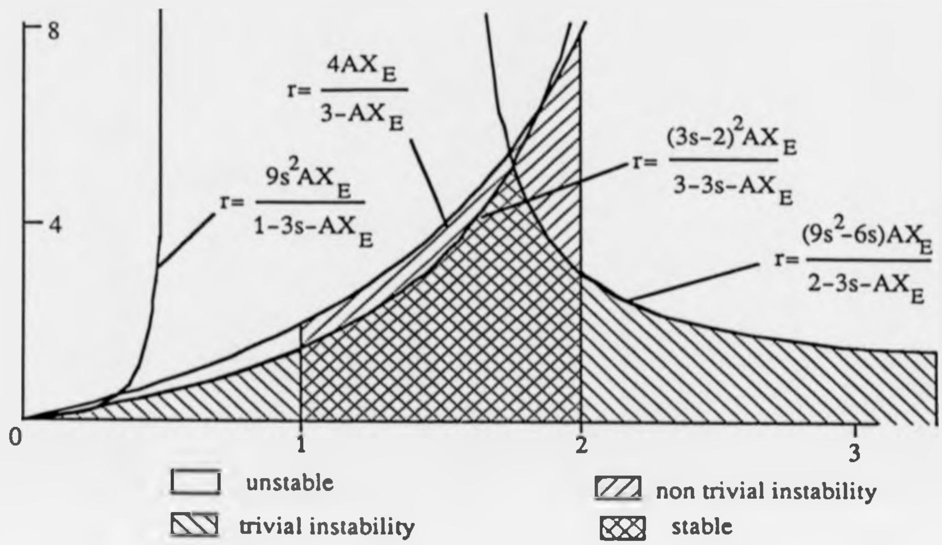


Fig 2.3 Regions of stability of the fixed point showing the possible ways of losing stability.

Given that we have a complex conjugate pair of eigenvalues we would like to know at what value of (AX_E, r) will we see the eigenvalues cross the unit circle at $e^{\pm i\theta}$.

For this to be the case we must be on the line

$$r = \frac{(9s^2-6s)AX_E}{2-3s-AX_E} \qquad \text{for} \qquad \frac{4-3s}{2} < AX_E < 2$$

or on the line

$$AX_E = 2 \qquad\qquad \text{for} \qquad 0 < r \le 4-6s.$$

Along the line $AX_E = 2$ the eigenvalues which cause the loss of stability are those that come from P+2Q so are equal to

$$\lambda = 1 - \frac{r}{4} \pm \frac{1}{2}\sqrt{\frac{r^2}{4} + 2r - 4r} \qquad \text{(i.e. } \lambda \text{ with } AX_E = 2\text{)}$$

or
$$\lambda = 1-\frac{r}{4} \pm \frac{1}{4}\sqrt{r^2 - 8r}$$

but since $0 < r < 4$

$$r^2-8r < 0$$

thus
$$\lambda = 1-\frac{r}{4} \pm \frac{i}{4}\sqrt{8r-r^2}$$

and so if $\lambda = \cos\theta + i\sin\theta$

then $\cos\theta = 1-r/4.$

Hence if $AX_E = 2$ and $r = 4(1-\cos\theta)$ there will be a complex conjugate pair of eigenvalues of P+2Q on the unit circle at $e^{\pm i\theta}$. Since for $AX_E = 2$ the possible values of r are $0 < r < 4-6s$, we can leave the unit circle for any value of θ from 0 to $\cos^{-1}3s/2$.

Along the line

$$r = \frac{(9s^2-6s)AX_E}{2-3s-AX_E}$$

the eigenvalues come from the P-Q component and are equal to

$$\lambda = \frac{(3s-2)^2+(3s-1)2AX_E \pm \sqrt{(9s^2-6s)(2AX_E+3s-4)(2AX_E+3s-2)}}{4-6s-2AX_E}$$

Since the root is again negative in the area of consideration, we have $\lambda = \cos\theta \pm i\sin\theta$ where

$$\cos\theta = \frac{(3s-2)^2+2(3s-1)AX_E}{4-6s-AX_E}.$$

So rearranging gives

$$AX_E = \frac{4\cos\theta-6s\cos\theta-(3s-2)^2}{6s-2+2\cos\theta}$$

and

$$r = (3s-2)^2 + 6s\cos\theta - 4\cos\theta.$$

Thus for $(4-3s)/2 < AX_E < 2$ and

$$r = \frac{(9s^2-6s)AX_E}{2-3s-AX_E}$$

a complex conjugate pair of eigenvalues of P-Q are equal to $e^{\pm i\theta}$ where θ is such that $\cos\theta$ solves

$$\cos\theta = \frac{(3s-2)^2+2(3s-1)AX_E}{4-6s-AX_E}.$$

Thus we can choose $\cos\theta$ in the range from -1 (if $AX_E = (4-3s)/2$) to $-3s/2$ (if $AX_E = 2$) and hence θ in the range $\cos^{-1}(-3s/2) < \theta < \pi$. So in the complex eigenvalue plane we have

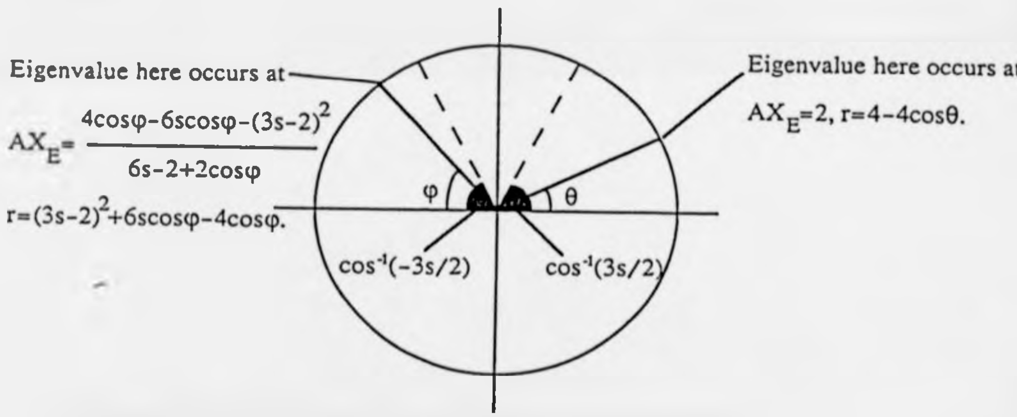


Fig 2.4 Diagram showing how and where eigenvalues can become unstable.

We notice immediately that for $s \neq 0$ we will not get eigenvalues crossing the unit circle at $\pm i$ from either P+2Q or P-Q, thus we will not see a branch of period 4 bifurcations emanating from the fixed point.

We now go on to discuss the difference between a loss of stability via the P+2Q eigenvalues and those of the P-Q eigenvalues. To do this we must return to where the eigenvalues actually came from.

We recall that the vector $[v,v,v]^T$ is invariant under the action of M and the eigenvalues from P+2Q have eigenvectors of this form. Further we note that the action of D_3 leaves this vector subspace fixed so we would expect the symmetry of the coupled system to remain unchanged, that is all three environments would behave in an identical fashion.

However the eigenvalues from P-Q with eigenvector v have corresponding eigenvectors for M of the form

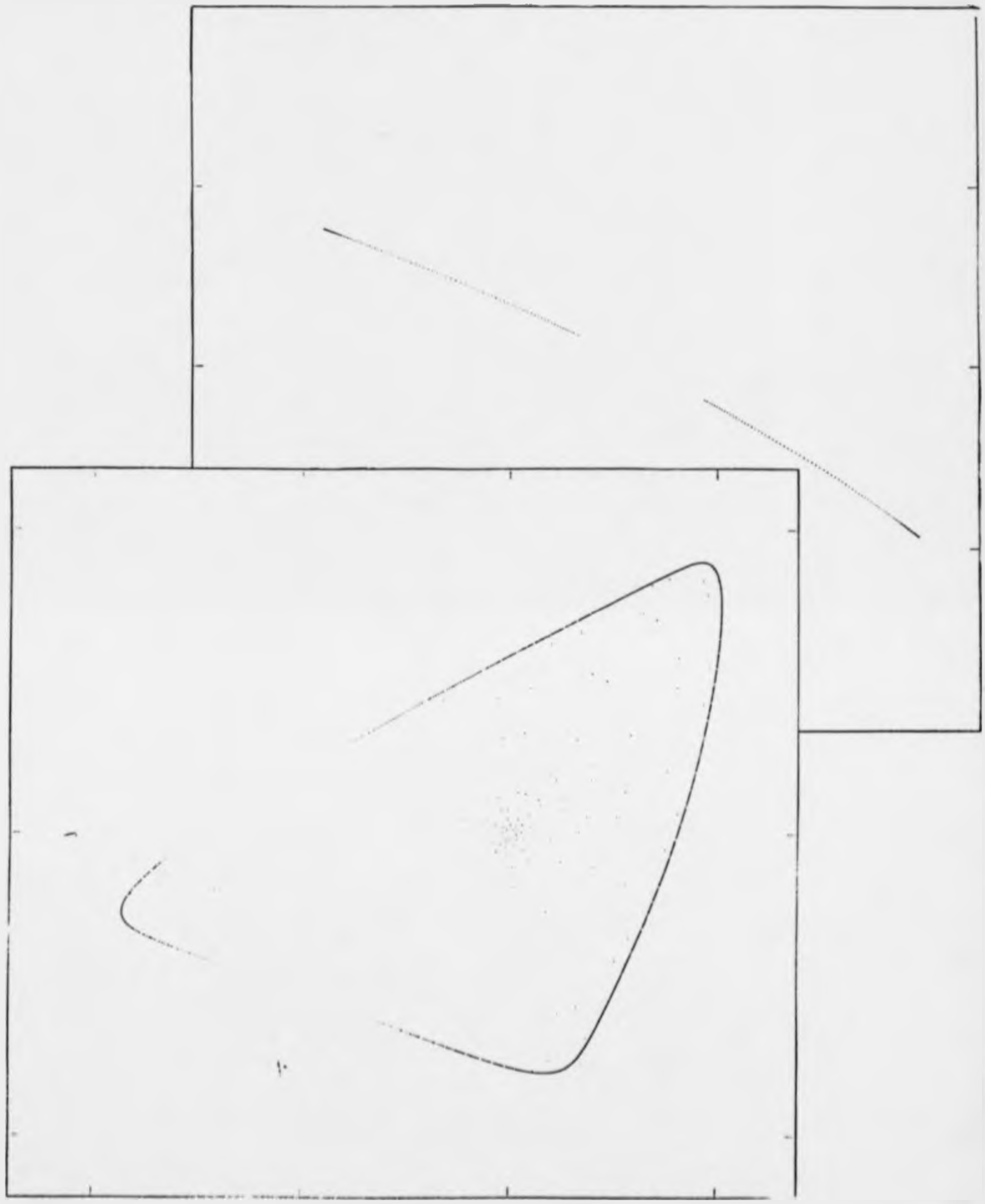
$$\begin{bmatrix} v \\ e^{\pm 2\pi i/3} v \\ e^{\pm 4\pi i/3} v \end{bmatrix}$$

Clearly D_3 acts upon this vector subspace in a non trivial way and thus we would expect the bifurcating solution not to be identical in all three environments. In the forthcoming chapters we show that for the flip bifurcation (an eigenvalue at -1) solutions will exist in which two environments exhibit the same period two behaviour whilst the other environment will be different and there will also be a solution in which two environment exhibit the same periodic behaviour but out of phase and the third will again be different. The Hopf bifurcation has solutions of a similar type and in addition a further solution in which the three environments are out of phase by $2\pi/3$.

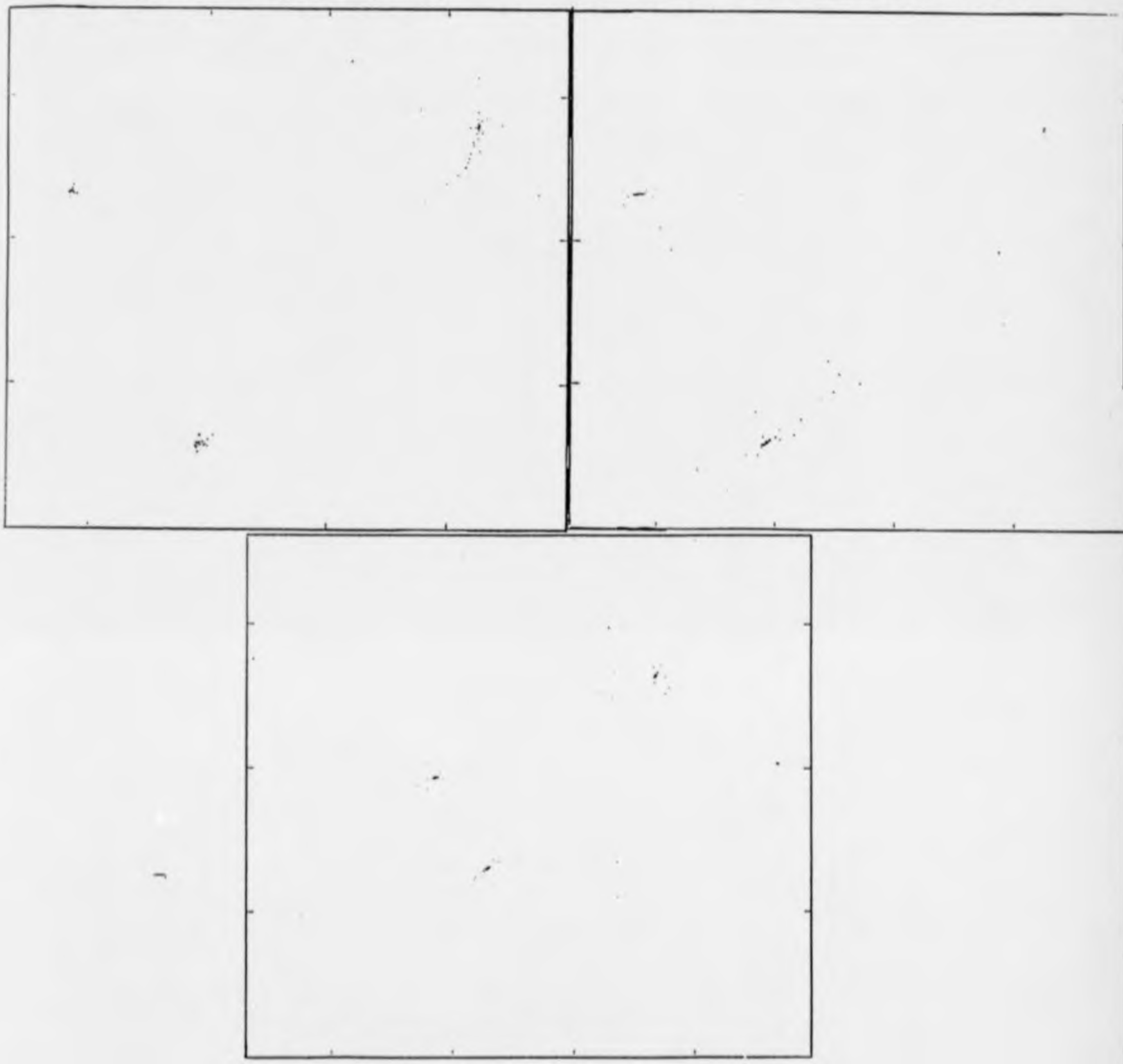
We end this chapter with a few of numerical simulation of this system to show the different types of solution which we hope to see.

2.4 Some Numerical Simulations.

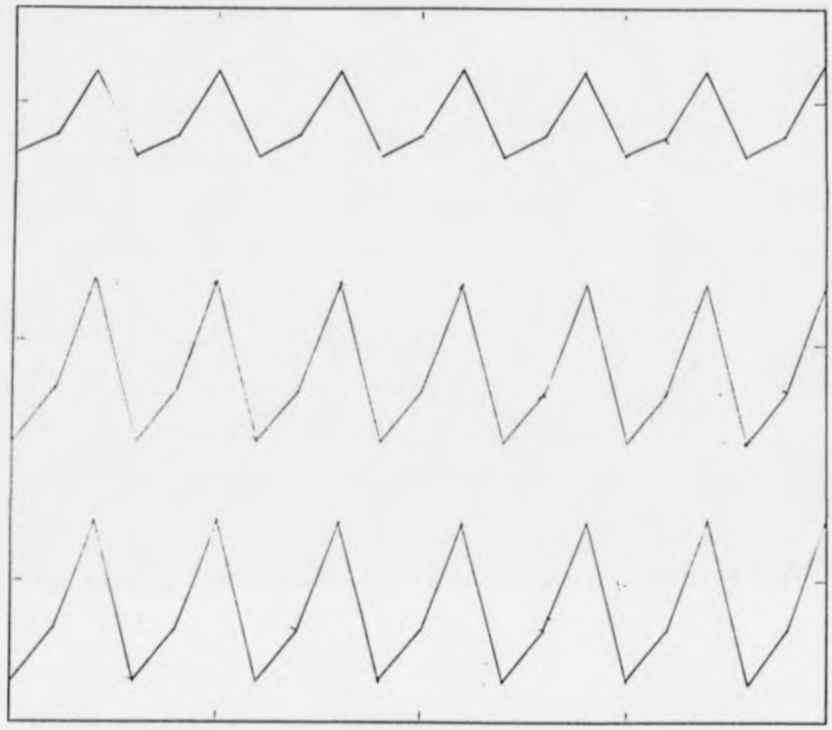
In this section we have used kaos, a computer package for running dynamical systems, to implement the system of three coupled oscillators. We try to give a feel of the types of behaviour we would expect with the diagrams shown on the following pages but we cannot be expected to give the whole picture due to the complexity of the system.



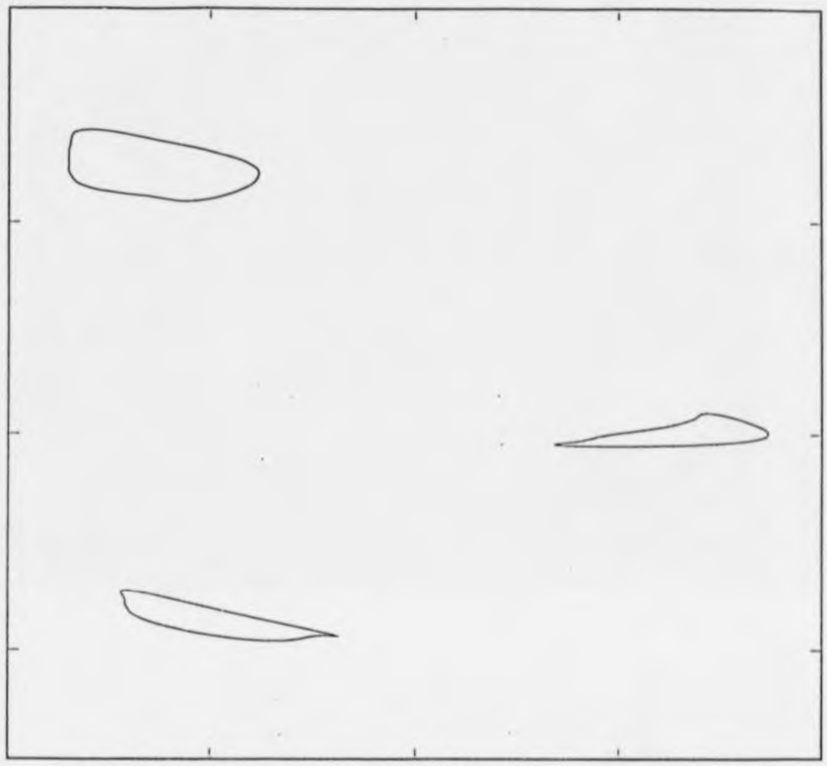
These picture come from the simple system with no coupling and are a typical stable period two orbit on either side of an unstable origin and a stable limit cycle around an unstable origin. Clearly here we would get the full D_3 symmetry in the solutions.



When coupling is introduced into the system, the fully Dn symmetric solutions are no longer seen. We will usually observe a reduced symmetry as above where two systems have identical period three orbits which are stable. There is no way to see from these diagrams if any temporal symmetry exists, we need to look at a time series of the points. We do this on the next page and see that in fact the two oscillators are in phase.



We end this section with an interesting picture which is half of a complete cycle. In the whole picture there are six of these limit cycles, the three we see and three more a relatively large distance away. This has occurred via three bifurcations, the first a period doubled bifurcation, the second is a period three bifurcation off this period two orbit to give a period 6 orbit. The final bifurcation is a Hopf bifurcation to the six limit cycles seen.



Chapter 3

An introduction to symmetric dynamical systems.

3.0 Introduction.

There are numerous physical systems which show symmetric effects both in their form and in the solutions they exhibit. In order to make the analysis of such systems easier we need to formulate these ideas in a mathematical context, for example how does a function exhibit symmetry and how do we say this rigorously. Furthermore we need some results about symmetric maps and symmetric dynamical systems which will make the discussion in subsequent chapters more straightforward.

3.1 Maps with symmetry.

Let f be a map

$$f:\mathbb{R}^n \longrightarrow \mathbb{R}^n$$

and let γ be an invertible $n \times n$ matrix. We say γ is a symmetry of f if

$$f(\gamma x) = \gamma f(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

If x_0 is a fixed point solution of f then clearly so is γx_0. There are then two possibilities either $\gamma x_0 \neq x_0$ in which case we have found a new steady state or $\gamma x_0 = x_0$ when we say that γ is a symmetry of the solution x_0.

If x_0 is a periodic point of f (that is $f^q(x_0) = x_0$ for some $q \in \mathbb{N}$) then again clearly so is γx_0. Here there are also two possibilities either $x_0 \neq \gamma f^c(x_0)$ for all $0 < c < q-1$ in which case we have found a new periodic point or $x_0 = \gamma f^c(x_0)$ for some c, in which case we say that the pair (γ, c) is a symmetry of the period q point solution x_0.

Because symmetry and groups are so closely linked it is often convenient to use the language of group theory in the discussion of symmetric systems.

If we let Γ be the set of all matrices γ which are symmetries of f then it is easy to show that Γ is a group. Since Γ is a subset of GL(n) then to show it is a group it is sufficient to show that it is closed under multiplication and every element has an inverse within Γ. Clearly if $\gamma_1, \gamma_2 \in \Gamma$ then

$$f(\gamma_1\gamma_2 x) = \gamma_1 f(\gamma_2 x) = \gamma_1\gamma_2 f(x)$$

so $\gamma_1\gamma_2 \in \Gamma$ and Γ is closed. To show the existence of an inverse for $\gamma \in \Gamma$ we let $y = \gamma x$ then we have

$$f(y)=f(\gamma x)=\gamma f(x)=\gamma f(\gamma^{-1}y)$$

multiplying through by γ^{-1} we get $\gamma^{-1}f(y) = f(\gamma^{-1}y)$ and hence $\gamma^{-1} \in \Gamma$.

Γ is a Lie group (since it is a topologically closed subgroup of GL(n)) and is called the group of symmetries of f. For a fixed point, x_0, of f we can define the isotropy subgroup of x_0

$$\Sigma_{x_0} = \{ \gamma \in \Gamma : \gamma x_0 = x_0 \}$$

this is a subgroup of Γ and is the group of symmetries of x_0 as defined previously. This is not always the whole of Γ and so the solutions of a certain problem with symmetry often has less symmetry than the original problem, we call this effect spontaneous symmetry breaking.

Given any fixed point solution x_0 of f we define the group orbit through x_0 as

$$\Gamma x_0 = \{ \gamma x_0 : \gamma \in \Gamma \}.$$

Every element in this orbit is a fixed point solution as well and can either be distinct from x_0 or equal to x_0. The relationship $\gamma x_0 = x_0$ is an equivalence relationship with the number of distinct solutions being equal to the number of equivalence classes. Furthermore for $\gamma_1 x_0$ and $\gamma_2 x_0$ to be in the same equivalence classes we need $\gamma_1\gamma_2^{-1}x_0 = x_0$, that is $\gamma_1\gamma_2^{-1} \in \Sigma_{x_0}$. Hence if Γ and Σ_{x_0} are finite each equivalence classes contains $|\Sigma_{x_0}|$ elements so the number of distinct solutions guaranteed by the existence of one solution is

$$|\Gamma|/|\Sigma_{x_0}|.$$

We can use similar language to describe the periodic solutions of f. However we must include the fact that the symmetries of the periodic point can contain 'temporal' as well as spatial components and hence we enlarge the group of symmetry operations from Γ to $\Gamma \times Z_q$ analogous results to those above hold replacing Γ with $\Gamma \times Z_q$.

Definition 3.1.1.

The spatio-temporal symmetry group of a period q point x_1 of f is

$$\Sigma_{x_1} = \{(\gamma,c) \in \Gamma \times Z_q : f^c(\gamma x_1) = x_1\}.$$

Consider the example from chapter two. If we have a period q orbit X, we would see this spatio-temporal symmetry exhibited as a subgroup of $D_3 \times Z_q$. For example a period 3 orbit may exist which is identical on all three environments, here the symmetry group will be $D_3 \subset D_3 \times Z_3$ since for $\gamma \in D_3$ $\gamma X = X$. If the periodic orbit has a phase lag in the three environments, that is they have the same periodic orbit but are at different points on that orbit, then the symmetry group is $Z_3 \times Z_3 \subset D_3 \times Z_3$.

With these ideas we now go on to discuss why symmetry effects can increase the complexity of a problem and how they can also be used to analyse this complexity.

3.2 The restrictions of symmetry.

Consider a bifurcation problem

$$f: \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n$$

where f has symmetry group Γ and in which f has a fixed point at $0 \in \mathbb{R}^n$ for all $\lambda \in \Lambda$.

When symmetry is introduced into the problem its eigenvalues are often forced to have high multiplicity (chapter 2 gives an example in which this is observed), this means that we cannot reduce the bifurcation problem via a centre manifold or Liapunov Schmidt reduction to be of a form in which we can apply the results of Chapter 1. To show why these high multiplicity eigenvalues are seen we use the following.

Lemma 3.2.1.

If f has a symmetry γ then the linearisation of f around 0, $(df)_0$, commutes with γ, that is

$$(df)_0\gamma = \gamma(df)_0.$$

Proof

Applying the chain rule to the identity $f(\gamma x) = \gamma f(x)$ we have

$$(df)_{\gamma x}\gamma = \gamma(df)_x.$$

Thus for x=0

$$(df)_0\gamma = \gamma(df)_0.$$

Now if λ is an eigenvalue of $(df)_0$ with an eigenvector v then γv is also an eigenvector of $(df)_0$ with the same eigenvalue thus

$$(df)_0\gamma v = \gamma(df)_0 v = \gamma\lambda v = \lambda\gamma v.$$

In general v is not always a scalar multiple of γv in which case the eigenspace related to λ is forced to be of dimension greater than 1 and so λ has multiplicity greater than 1.

Fortunately there are ways to reduce the complexity of the bifurcation problems in the presence of symmetry which rely on the very same symmetries that give rise to the complications. We use two approaches here, the first looks for solutions which have particular types of symmetry and the second considers the Taylor expansion of f and uses the symmetry to remove many of the terms in the expansion. We first assume that we have reduced the problem as far as possible via a Liapunov Schmidt or a centre manifold reduction and hence the only eigenvalues are on the unit circle. Let g be the germ on $\mathbb{R}^c$, equivalent to f in the reduced system then g is a bifurcation problem whose solutions are equivalent to those of f. Now we use the two ideas above to see how the complexity of the problem can be reduced still further.

(a) Restriction to fixed point subspaces

It is possible to prescribe, in advance, the symmetry type of the fixed point solution of g we are looking for, that is to look for solutions satisfying $g(x,\lambda) = x$ in which x has symmetry group $\Sigma \subset \Gamma$. The type of solution we are interested in must lie in

$$\mathrm{Fix}(\Sigma) = \left\{ x \in \mathbb{R}^c : \sigma x = x \text{ for all } \sigma \in \Sigma \right\}$$

the fixed point subspace of Σ. Thus we need only solve the equation

$$g|_{\mathrm{Fix}(\Sigma)}(x,\lambda) = x.$$

This task is simplified further by noting the following lemma

Lemma 3.2.2.

If g has symmetry group Γ and $\Sigma \subset \Gamma$ is the isotropy subgroup of some point x then g maps $\mathrm{Fix}\Sigma$ onto itself.

Proof

We need to show that $g(\mathrm{Fix}\Sigma) \subset \mathrm{Fix}\Sigma$

Take $y \in \mathrm{Fix}\Sigma$, for all $\sigma \in \Sigma$ we have

$$g(y,\lambda) = g(\sigma y,\lambda) \qquad \text{as } \sigma y = y$$

$$= \sigma g(y,\lambda) \qquad \text{as } \Sigma \subset \Gamma \text{ and g has symmetry group } \Gamma.$$

so $$g(y,\lambda) \in \mathrm{Fix}\Sigma \text{ for all } y \in \mathrm{Fix}\Sigma$$

that is

$$g(\mathrm{Fix}\Sigma) \subset \mathrm{Fix}\Sigma$$

so $$g : \mathrm{Fix}\Sigma \longrightarrow \mathrm{Fix}\Sigma.$$

Hence the dimension of the problem is reduced to $m = \dim \mathrm{Fix}(\Sigma)$ which can be much smaller than c the dimension of the original problem. For example if

$$\dim \mathrm{Fix}(\Sigma) = 1$$

we can apply proposition 1.1.4. to predict a branch of fixed point solutions, this result is stated more formally at the end of the chapter. Such a technique gives us information about what types of bifurcation to expect but tells us little of the dynamics of the solution, that is its stability and branching direction, to determine these we look to invariant theory.

(b) Invariant theory.

We would like to develop, in a coherent manner, a way of describing maps that commute with certain group actions. We begin with a discussion of maps invariant under the group action and then go on to look at how maps with this symmetry group are related to them. We start with a definition.

Definition 3.2.3.

Given a compact Lie group Γ and a function $f: V \longrightarrow \mathbb{R}$ we say that f is Γ invariant if

$$f(\gamma x) = f(x)$$

for all $\gamma \in \Gamma$, $x \in V$. An invariant polynomial is simply a polynomial which is an invariant function.

The most straightforward example is when $V = \mathbb{R}$ and the symmetry group is the non trivial action of $\mathbb{Z}_2$ (multiplication by -1). Here invariant functions are functions in which $f(-x) = f(x)$, that is even functions. It is clear to see that if f is an invariant polynomial it will be of the form

$$h(x^2)$$

(since $x^{2m} = (-x)^{2m}$ but $x^{2m+1} = -(-x)^{2m+1}$). This result gives some indication of how to approach the classification of invariant polynomials, the idea is generalised in the following theorem.

Theorem 3.2.4. (Hilbert-Weyl Theorem)

Let Γ be a compact Lie group acting on V. Then there is a finite subset of Γ invariant polynomials $u_1, \ldots, u_s$, for which given any Γ invariant polynomial f we can express f as

$$f(x) = p(u_1, \ldots, u_s)$$

where p is some polynomial from $\mathbb{R}^s \rightarrow \mathbb{R}$.

Next we make more rigorous the idea of the symmetry group of a function.

Definition 3.2.5.

Given a compact Lie group Γ and a function $f:V \longrightarrow V$ we say that f commutes with Γ or is Γ equivariant if

$$f(\gamma x) = \gamma f(x)$$

for all $\gamma \in \Gamma$, $x \in V$.

Again under the action of $\mathbb{Z}_2$ when $V = \mathbb{R}$, we have $f(-x) = -f(x)$ which are odd functions. In general these will be of the form

$$f(x) = h(x^2)x.$$

The Hilbert Weyl theorem can now be extended to give a corresponding result for Γ equivariant maps.

Theorem 3.2.6.

Let Γ be a compact Lie group acting on V. Then there exists a finite set of Γ equivariant polynomials $X_1, \ldots, X_r$ such that any Γ equivariant polynomial is of the form

$$g = f_1X_1 + \ldots + f_rX_r$$

where the f_i are Γ invariant polynomials.

Both of these theorems can be generalised from polynomials to any smooth function using the same set of generating functions $u_1, \ldots, u_s$ and $X_1, \ldots, X_r$.

For a general group Γ these invariant and equivariant generating functions are not easy to compute but it is often possible to look at the dynamics for a particular solution of a Γ equivariant bifurcation problem by considering the taylor expansion up to certain finite order, an approach like this is used at the end of chapter 6 to analyse the solutions of a $D_n \times Z_q$ equivariant map.

3.3 Some Bifurcations with Symmetry.

We end this chapter with two theorems the proofs of which rely on the techniques outlined above. But before this we introduce a definition.

Definition 3.3.1.

Let Γ be a compact Lie group acting on V. We say the action is absolutely irreducible if the only linear mappings on V that commute with Γ are scalar multiples of the identity.

Absolute irreducibility is a stronger concept than irreducibility (V is irreducible if the only Γ invariant subspaces are $\{0\}$ and V), since absolute irreducibility implies irreducibility.

This definition is not as restrictive as it first appears since there are many groups which exhibit this particular property, for example on $\mathbb{R}^2$ the standard action of D_n, $n \geq 3$, is absolutely irreducible as is that of O(2), the group of orthogonal 2×2 matrices.

Proposition 3.3.2.

Let f be a one parameter family of Γ-equivariant mappings

$$f: \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n.$$

If the action of Γ on $\mathbb{R}^n$ is non trivial and absolutely irreducibly then there is a trivial fixed point solution at zero for all $\lambda \in \Lambda$ and $(df)_{0,\lambda} = c(\lambda) I_n$.

If in addition there is a subgroup $\Sigma \subset \Gamma$ with dim $\mathrm{Fix}(\Sigma) = 1$, $c(0) = 1$ and $c'(0) \neq 0$ then generically $f(x,\lambda)$ has a unique branch of non trivial fixed points in $\mathrm{Fix}(\Sigma)$. Thus there is a branch of solutions with symmetry group precisely Σ.

Proof

As Γ acts absolutely irreducibly on $\mathbb{R}^n$ then it follows that it acts irreducibly.

Hence either $\mathrm{Fix}(\Gamma) = \{0\}$ or $\mathbb{R}^n$ (since it is a subspace of $\mathbb{R}^n$) but if $\mathrm{Fix}(\Gamma) = \mathbb{R}^n$ then the action of Γ is trivial. Applying lemma 3.2.2 to $\mathrm{Fix}(\Gamma)$ we have

$$f| \mathrm{Fix}(\Gamma) \times \Lambda \to \mathrm{Fix}(\Gamma)$$

i.e.

$$f(0,\lambda) = 0.$$

From lemma 3.2.1 $df_{(0,\lambda)}$ commutes with the action of Γ and thus by absolute irreducibility must be of the form $c(\lambda)I_n$.

For the remaining part of the proof we again apply lemma 3.2.2, this time to Σ, to give

$$f| \mathrm{Fix}(\Sigma) \times \Lambda \to \mathrm{Fix}(\Sigma).$$

This is a map from $\mathbb{R}\times\Lambda$ to $\mathbb{R}$ and since generically we can assume that at 0 $d^2f \neq 0$ then the hypotheses of proposition 1.1.3 which are necessary for the existence of a unique branch of fixed point solutions are satisfied and since we are restricted to $\mathrm{Fix}(\Sigma)$ this solution branch is the one required to complete the proof.

We now consider the case of period doubling, i.e. when $c(0) = -1$. First note that a period two point of f is a fixed point of f^2. So applying proposition 3.3.2 to f^2 we get the following result.

Proposition 3.3.3.

Let f be a one parameter family of Γ-equivariant mappings

$$f:\mathbb{R}^n \times \Lambda \longrightarrow \mathbb{R}^n$$

If the action of Γ on $\mathbb{R}^n$ is non trivial and absolutely irreducibly then there is a trivial fixed point solution at zero for all $\lambda \in \Lambda$ and $(df)_{0,\lambda} = c(\lambda)I_n$.

If in addition there is a subgroup $\Sigma \subset \Gamma$ with $\dim \mathrm{Fix}(\Sigma) = 1$, $c(0) = -1$ and $c'(0) \neq 0$ then $f(x,\lambda)$ has a unique branch of period two orbits with symmetry group precisely Σ.

Chapter 4

Normal form and maps with symmetry.

4.0 Introduction.

We consider the Taylor expansion of a map f around zero,that is

$$f(x)=Ax+f_2(x)+\ldots+f_k(x)+O(|x|^{k+1}),$$

where A is the linearisation of f at zero and f_i are homogeneous polynomials of degree i. By successive transformations of x we may hope to be able to remove some of these non linear terms and so simplify the map.

A theorem of Iooss [1987] shows that this can be done up to any finite order so that what remains commutes with the transpose of the linearisation of f.

Furthermore in this chapter we adapt ideas of Elphick *et al* [1987] and Vanderbauwhede [1987] from vector fields, to show that if f is already equivariant under the action of some compact Lie group Γ then the change of coordinates can be chosen so that this initial symmetry is preserved.

4.1 Normal form without symmetry

Before going any further we will give a definition of what it means for f to be in normal form. Given a function f and for each $k\in\mathbb{Z}$, then by a near identity polynomial change in coordinates (i.e. $x=x'+P_k(x')$, where P_k is a polynomial of degree k with $P_k(0)=0$ and $(dP_k)_0=0$) it is possible to write

$$f(x')=Ax'+N(x')+\ldots$$

where $A=(df)_0$, $N(0)=0$, $(dN)_0=0$, N is a polynomial of degree k which can be chosen to have a particular form (to be described later in the chapter) and ... indicates higher order terms, we say that f is in normal form. If these higher order terms vanish we say that f is in exact normal form.

To find out the form of what terms cannot be removed we first look at those terms which can be removed under the transformation

$$x = x' + P_i(x')$$

where P_i is a homogeneous polynomial of degree i.

Lemma 4.1.2

If

$$f(x) = Ax + f_2(x) + \ldots + f_i(x) + O(|x|^{i+1})$$

the terms which can be eliminated from f_i are any homogeneous terms of degree i of the form

$$\mathcal{L}_A(P_i(y)) = AP_i(y) - P_i(Ay).$$

Furthermore this change can be accomplished without effecting any of the terms of f_j for $j < i$.

Remark

If we let $\mathcal{P}_i$ be the space of homogeneous polynomial mappings of degree i then $\mathcal{L}_A$ is a linear map from $\mathcal{P}_i$ to $\mathcal{P}_i$.

Proof

Let $P(x') = x' + P_i(x') = x$ and let f transform to f' then

$$P(f'(x')) = f(P(x'))$$

or

$$f'(x') = P^{-1}f(P(x')).$$

Now $P^{-1}(y)$ is such that $P(P^{-1}(y)) = y$ that is

$$P^{-1}(y) + P_i(P^{-1}(y)) = y$$

or

$$P^{-1}(y) = y - P_i(P^{-1}(y)).$$

This implies

$$P^{-1}(y)=y+O(|y|^2)$$

thus $$P^{-1}(y)=y-P_i(y)+O(|y|^{i+1}).$$

So

$$f'(x')=f(P(x'))-P_i(f(P(x')))+O(|x'|^{i+1})$$

i.e. $$f'(x')=A(x'+P_i(x'))+f_2(x')+\ldots+f_i(x')-P_i(Ax')+O(|x'|^{i+1})$$

or $$f'(x')=Ax'+f_2(x')+\ldots+f_i(x')+AP_i(x')-P_i(Ax')+O(|x'|^{i+1})$$

hence using the transformation $x'\rightarrow x$ we can remove the term $AP_i(x')-P_i(Ax')$ without affecting terms of order less than i and the proof of lemma 4.1.2. is complete.

The fact that this transformation leaves unaltered terms of order less than i suggests that f can, by an iterative approach, be put into normal form up to any finite order.

This lemma indicates also that we can put f into a form in which the f_i lie in some complement of the image of $\mathcal{L}_A$ in $\mathcal{P}_i$

A result of Iooss [1987] gives a particularly useful way to represent this complimentarysubspace.

Theorem 4.1.3.

Let f be a map with $(df)_0 = A$ then up to any finite order k there exists a polynomial change of coordinates of order k taking f to the form

$$f(x)=Ax+N(x)+\ldots$$

where $N(0)=0$, $(dN)_0=0$ and N is a polynomial of degree k which commutes with A^t.

Proof

The idea behind the proof is a recursive one, we assume

$$f(x)=Ax+h_i(x)+f_i(x)+\ldots$$

where h_i is a polynomial of degree < i which commutes with A^t, f_i is a homogeneous polynomial of degree i and ... indicates higher order terms.

Lemma 4.1.2. shows that by a degree i polynomial change in coordinates (i.e. $x=x'+P_i(x')$) we can change f_i without altering h_i and further, that any such transformation will remove terms of the form $AP_i(x')-P_i(Ax')$ from f_i.

Hence f_i can be chosen in such a way as to lie in a complementary subspace of Im $\mathfrak{L}_A$ in $\mathcal{P}_i$. So we need only show that

$$\mathcal{P}_i=\mathrm{Im}\ \mathfrak{L}_A \oplus \mathcal{P}_i(A^t)$$

where $\mathcal{P}_i(A^t)=\{p\in\mathcal{P}_i : p \text{ commutes with } A^t\}$.

To do this we require two lemmas adapted from Golubitsky *et al* [1987] XVI 5.4 and 5.5.

Lemma 4.1.4

$\ker(\mathfrak{L}_{A^t}) = \mathcal{P}_i(A^t)$

Proof

If $(\mathfrak{L}_{A^t})p=0$ then by definition

$$A^t p(x)-p(A^t x)=0$$

or

$$A^t p(x)=p(A^t x)$$

that is p commutes with A^t.

Likewise if p commutes with A^t then

$$A^t p(x)-p(A^t x)=0$$

that is $(\mathfrak{L}_{A^t})p=0$.

Lemma 4.1.5

There exists an inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ on $\mathcal{P}_i$ such that

$$\langle\!\langle \mathfrak{L}_A P,Q\rangle\!\rangle=\langle\!\langle P,\mathfrak{L}_{A^t} Q\rangle\!\rangle$$

for all $P,Q\in\mathcal{P}_i$.

Proof

Let α,β be multi indices and x^α, x^β the associated monomials which are a basis of the space of real valued polynomials. A scalar product on these is given by

$$< x^\alpha , x^\beta > = \frac{\partial^{|\alpha|}}{\partial x^\alpha} x^\beta |_{x=0}.$$

Thus for any two real valued polynomials

$$<p(x),q(x)> = \left(p(\partial/\partial x_1, \ldots, \partial/\partial x_n)q\right)(0).$$

Finally if $P=(p_1, \ldots, p_n)$ and $Q=(q_1, \ldots, q_n)$ are in $\mathcal{P}_i$ we define the inner product

$$\langle\langle P, Q \rangle\rangle = \sum_{j=1}^{n} <p_j, q_j> \qquad (*)$$

Using this inner product we claim that for any linear map $A: \mathbb{R}^n \to \mathbb{R}^n$

(a) $$\langle\langle AP(x), Q(x) \rangle\rangle = \langle\langle P(x), A^t Q(x) \rangle\rangle$$

(b) $$\langle\langle P(Ax), Q(x) \rangle\rangle = \langle\langle P(x), Q(A^t x) \rangle\rangle$$

With these two and linearity we have

$$\begin{aligned} \langle\langle \mathcal{L}_A P, Q \rangle\rangle &= \langle\langle AP(x), Q(x) \rangle\rangle - \langle\langle P(Ax), Q(x) \rangle\rangle \\ &= \langle\langle P(x), A^t Q(x) \rangle\rangle - \langle\langle P(x), Q(A^t x) \rangle\rangle \\ &= \langle\langle P, \mathcal{L}_{A^t} Q \rangle\rangle \end{aligned}$$

and the lemma is proved.

Thus we need only prove the claims (a) and (b). For the first we use the linearity of the inner product to assume that $P(x)=x^\alpha e_j$ and $Q(x)=x^\beta e_k$, where $|\alpha|=|\beta|=i$ and e_m is the unit vector in the mth direction. Letting $A=(a_{ij})$ gives

$$\begin{aligned} \langle\langle A(x^\alpha e_j), x^\beta e_k \rangle\rangle &= \langle\langle x^\alpha \sum_l a_{lj} e_l , x^\beta e_k \rangle\rangle \\ &= \sum_l a_{lj} < x^\alpha , x^\beta > \delta_{lk} \\ &= a_{kj} < x^\alpha , x^\beta > \end{aligned}$$

Likewise

$$\begin{aligned} \ll x^{\alpha}e_j \, , A^t(x^{\beta}e_k) \gg &= \ll x^{\alpha}e_j \, , x^{\beta}\sum_l a^t_{lk}e_l \gg \\ &= \sum_l a^t_{lk} < x^{\alpha} \, , x^{\beta} > \delta_{jl} \\ &= a^t_{jk} < x^{\alpha} \, , x^{\beta} > \\ &= a_{kj} < x^{\alpha} \, , x^{\beta} > \end{aligned}$$

so part (a) of our claim is proved.

For part (b) we need only show that

$$< p(Ax) \, , q(x) > = < p(x) \, , q(A^t x) >$$

where p and q are homogeneous polynomials of degree k and then linearity provides the result.

To see this we consider the change of coordinates $x = A^t y$. This gives

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{\partial}{\partial x}\frac{\partial x}{\partial y} \\ &= \frac{\partial}{\partial x} A^t \\ &= A\frac{\partial}{\partial x} \end{aligned}$$

Hence using our definition of the scalar product $< \, , >$ we have

$$\begin{aligned} < p(Ax) \, , q(x) > &= p\left(A\frac{\partial}{\partial x} \right) q(A^t y)|_{y=0} \\ &= p\left(\frac{\partial}{\partial y} \right) q(A^t y)|_{y=0} \\ &= p\left(\frac{\partial}{\partial x} \right) q(A^t x)|_{x=0} \\ &= < p(x) \, , q(A^t x) > \end{aligned}$$

and our proof is complete.

We are now able to return to the proof of the main result

Proof of theorem 4.1.3.

By the Fredholm alternative

$$\operatorname{Im} \mathcal{L}_A = \left(\ker (\mathcal{L}_A)^t \right)^{\perp}.$$

Now from lemma 4.1.5.

$$\langle\langle \mathcal{L}_A P, Q \rangle\rangle = \langle\langle P, \mathcal{L}_{A^t} Q \rangle\rangle$$

we have $$(\mathcal{L}_A)^t = \mathcal{L}_{A^t}.$$

Thus

$$\operatorname{Im} (\mathcal{L}_A) = \left(\ker (\mathcal{L}_{A^t}) \right)^{\perp}$$

That is

$$\mathcal{P}_i = \operatorname{Im}(\mathcal{L}_A) \oplus \ker (\mathcal{L}_{A^t}).$$

Applying to this lemma 4.1.4 we have shown that f_i can be chosen so that it commutes with A^t and by letting $N = f_2 + \ldots + f_k$ we have the desired result for N.

Remark

When A is diagonalisable we can choose a coordinate system in which $A = A^t$ and thus in the statement of the theorem N will commute with A so the whole of f truncated at order k will commute with A. For example if A were $-I$ then f truncated to any order would commute with $-I$. Furthermore if A is orthogonal then $A^{-1} = A^t$ and again N can be chosen to commute with A and thus f will commute with A truncated at any order.

We end this section with a few examples of how this normal form reduction is performed.

(i) One of the simplest examples of normal form reduction is if $f:\mathbb{R} \to \mathbb{R}$ and df at 0 is -1. Normal form theory then says that up to any order k there is a change of coordinates in which f truncated at order k is an odd function and so will be of the form

$$f(x) = -x + \alpha_3 x^3 + \alpha_5 x^5 + \ldots + \alpha_{2j+1} x^{2j+1}.$$

(ii) f is a map on $\mathbb{R}^2$ and

$$A = \begin{bmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{bmatrix}$$

Thus we can apply the normal form theory of Iooss to conclude that f commutes with A since A is orthogonal. To make things easy we identify $\mathbb{R}^2$ with $\mathbb{C}$ so the action of A is multiplication by $e^{2\pi i\theta}$. In general f will be of the form

$$f(z) = \sum_{j,k} \alpha_{jk} z^j \bar{z}^k$$

and normal form theory says that to any order we can choose a coordinate system in which f commutes with this multiplication, so $\alpha_{jk} = 0$ unless

$$e^{2\pi i\theta} z^j \bar{z}^k = (e^{2\pi i\theta} z)^j (e^{-2\pi i\theta} \bar{z})^k$$

that is unless

$$\theta = j\theta - k\theta \quad (\text{mod } 1).$$

If θ is irrational this can only be solved by $j = k+1$ and hence

$$f(z) = \sum_j \alpha_j |z|^{2j} z.$$

However if $\theta = p/q$, $(p,q)=1$, we also have solutions satisfying

$$j-k = 1 \quad (\text{mod } q)$$

that is $j = k+1+rq$ or $k = j-1+rq$. These terms will be of the form

$$|z|^{2k} z^{rq+1} \text{ and } |z|^{2j} \bar{z}^{rq-1}$$

and hence

$$f(z) = \sum_{j,r} (\alpha_{jr} |z|^{2j} z^{rq+1} + \beta_{jr} |z|^{2j} \bar{z}^{rq-1})$$

We used this form in theorem 1.2.1 for the discussion of the Hopf bifurcation.

4.2 Normal forms with symmetry.

Our next step is obvious,we ask if f is a Γ-equivariant map, can N be chosen in such a way that it commutes with Γ as well as with A^t,that is, is there a Γ-equivariant theorem equivalent to theorem 4.1.3.

We must first give the Γ-equivariant version of lemma 4.1.2

Lemma 4.2.1.

Let f be a Γ-equivariant function, let $A=(df)_0$. Then for any k there exists a Γ-equivariant change of coordinates of degree k such that in the new coordinate system f is of the form

$$f(x)=Ax+g_2(x)+...+g_k(x)+O(|x|^{k+1})$$

where if

$$\mathcal{P}_i(\Gamma) = \{p\in\mathcal{P}_i : p \text{ is } \Gamma\text{-equivariant}\}$$

and

$$\mathcal{P}_i(\Gamma) = \mathcal{G}_i \oplus \mathcal{L}_A(\mathcal{P}_i(\Gamma))$$

then $g_i\in\mathcal{G}_i$.

This says that the terms which can be eliminated from f are any homogeneous terms of degree i of the form

$$\mathcal{L}_A(P_i(y))=AP_i(y)-P_i(Ay)$$

where $P_i\in\mathcal{P}_i(\Gamma)$.

Proof

We first note that in the Taylor expansion of f

$$f(x)=Ax+f_2(x)+...+f_k(x)+O(|x|^{k+1})$$

each f_i is Γ-equivariant, that is it lies in $\mathcal{P}_i(\Gamma)$. Furthermore if we choose P_i to be Γ-equivariant then our near identity change in coordinates $x=x'+P_i(x')$ will be also and thus f will remain Γ-equivariant. This change in coordinates will remove terms of the form $\mathfrak{L}_A(P_i(x))=AP_i(x)-P_i(Ax)$ for each i which are clearly Γ-equivariant themselves. Thus for each i the terms of order i remaining must lie in the complimentary subspace of $\mathfrak{L}_A(\mathcal{P}_i(\Gamma))$.

Next we state and prove the Γ-equivariant version of Theorem 4.1.3

Theorem 4.2.2

Let f be Γ-equivariant with $A=(df)_0$ and let the actions of Γ and A^t commute. Then there exists a Γ-equivariant polynomial change of coordinates of degree k, for any k, such that in the new coordinate system

$$f(y)=Ay+N(y)+\ldots$$

where N(y) is a polynomial,of degree k,which commutes with $\Gamma\times A^t$.

The proof of this theorem relies on our definition of a linear projection ρ with the following properties.

Lemma 4.2.3

If Γ is a compact Lie group which commutes with A^t then there exists a linear projection ρ which;

i) projects $\mathcal{P}_i$ onto $\mathcal{P}_i(\Gamma)$,

ii) projects $\mathcal{P}_i(A^t)$ onto $\mathcal{P}_i(\Gamma\times A^t)$ and

iii) projects $\mathfrak{L}_A(\mathcal{P}_i)$ onto $\mathfrak{L}_A(\mathcal{P}_i(\Gamma))$.

The proof of this lemma can be found in Golubitsky et al [87] XVI 5.10 we give an outline below.

Proof

For a compact lie group Γ we define, using the normalised invariant Haar measure a map

$$\rho(Y)(x) = \int_{\Gamma} \gamma^{-1} Y(\gamma x)\, d\gamma.$$

Clearly ρ is linear and maps $\mathcal{P}_i$ and $\mathcal{P}_i(A^t)$ to themselves. Furthermore a calculation shows that for $\tau \in \Gamma$

$$\tau^{-1}\rho(Y)(\tau x) = \rho(Y(x)),$$

that is ρ maps $\mathcal{P}_i$ onto $\mathcal{P}_i(\Gamma)$ and $\mathcal{P}_i(A^t)$ onto $\mathcal{P}_i(\Gamma \times A^t)$.

To show that ρ maps $\mathcal{L}_A(\mathcal{P}_i)$ onto $\mathcal{L}_A(\mathcal{P}_i(\Gamma))$ it is sufficient to show that for $P_i \in \mathcal{P}_i$ and $\gamma \in \Gamma$ then

$$\gamma^{-1}\mathcal{L}_A(P_i(\gamma x)) = \mathcal{L}_A(\gamma^{-1}P_i(\gamma x))$$

but

$$\begin{aligned}\gamma^{-1}\mathcal{L}_A(P_i(\gamma x)) &= \gamma^{-1}(A(P_i(\gamma x)) - P_i(A\gamma x)) \\ &= A\gamma^{-1}(P_i(\gamma x)) - \gamma^{-1}P_i(A\gamma x) = \mathcal{L}_A(\gamma^{-1}P_i(\gamma x))\end{aligned}$$

since A commutes with Γ.

We now return back to the proof of the main theorem.

Proof of theorem 4.2.2

From theorem 4.1.3 we have

$$\mathcal{P}_i = \mathcal{P}_i(A^t) \oplus \operatorname{Im} \mathcal{L}_A$$

thus

$$\rho(\mathcal{P}_i) = \rho(\mathcal{P}_i(A^t)) \oplus \rho(\operatorname{Im} \mathcal{L}_A)$$

and applying the results of the lemma we get

$$\mathcal{P}_i(\Gamma) = \mathcal{P}_i(\Gamma \times A^t) \oplus \mathcal{L}_A(\mathcal{P}_i(\Gamma)).$$

Hence for $f(x) = Ax + f_2(x) + \ldots + f_k(x) + O(|x|^{k+1})$ we can choose f_i to commute with $\Gamma \times A^t$ letting $N(x) = f_2(x) + \ldots + f_k(x)$ we are done.

4.3 Normal Forms of Parameterised Maps.

When f depends on a parameter there is a neighbourhood of 0 in the parameter space in which these normal form theorems still apply. Adapting a theorem of Vanderbauwhede [1989] we have

Theorem 4.3.1.

Let f be a smooth parameterised mapping

$$f:\mathbb{R}^n \times \Lambda \to \mathbb{R}^n$$

such that $f(0,\lambda)=0$ for all $\lambda\in\Lambda$ and $(df)_{0,0}=A$. Then up to any order $k\geq 2$ there is a neighbourhood Ω_k of the origin in Λ and a mapping

$$\Phi:\mathbb{R}^n \times \Omega_k \to \mathbb{R}^n$$

such that for all $\lambda\in\Omega_k$ Φ_λ is a diffeomorphism of $\mathbb{R}^n$, with $\Phi_\lambda(0)=0$ and

$$g(x,\lambda) = \Phi_\lambda^{-1}(f(\Phi_\lambda(x),\lambda)) = A(\lambda)x+f_2(x,\lambda)+\ldots+f_k(x,\lambda)+O(|x|^{k+1})$$

where $f_i\in W_i$ for $2 \leq i \leq k$, $(A(\lambda)-A)\in W_1$ and W_i is a complimentary subspace of $\mathcal{L}_A(\mathcal{P}_i)$) in $\mathcal{P}_i$ where $\mathcal{P}_1$ is $\mathcal{L}(\mathbb{R}^n)$, the space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. In fact W_i can be chosen to be the space of polynomial maps which commute with A^t.

Sketch of the Proof

The proof of this theorem is complicated since $(df)_{0,\lambda}$ the linearisation of f is no longer constant and must be dealt with separately. Let π_i be the projection of $\mathcal{P}_i$ onto $\mathcal{P}_i(A^t)$ whose kernel is $\mathcal{L}_A(\mathcal{P}_i)$ and let $B\in\mathcal{L}(\mathbb{R}^n)$ be invertible. Making a linear change in coordinates $x=By$ takes f into the form

$$f'(x,\lambda) = B^{-1}f(Bx,\lambda)$$

and for $(df')_{0,\lambda}$ to lie in W_1 we need to satisfy the equation

$$F(B,\lambda) = (I-\pi_1)(B^{-1}(df)_{0,\lambda}B-A) = 0.$$

However a calculation shows that

$$F(I,0) = 0$$

and $(d_BF)_{I,0}C = \mathcal{L}_A(C)$ for all $C \in \mathcal{L}(\mathbb{R}^n)$.

Hence since $(d_BF)_{I,0}$ is surjective on $\mathcal{L}_A(\mathcal{L}(\mathbb{R}^n))$ we can apply the implicit function theorem to show the existence of a neighbourhood Ω_1 of 0 in Λ and a smooth mapping $B^*:\Omega_1 \to GL(n)$ such that $B^*(0)=I$ and $F(B^*(\lambda),\lambda)=0$ for $\lambda \in \Omega_1$. We let

$$f'(x,\lambda) = (B*(\lambda))^{-1}f(B*(\lambda)x,\lambda)$$

and have $((df')_{0,\lambda}-A) \in W_1$ and $(df')_{0,0}=A$.

We continue as in the standard normal form theorem by induction. Assume that

$$f'(x,\lambda) = (df')_{0,\lambda}x+f'_2(x,\lambda)+\ldots+f'_j(x,\lambda)+O(|x|^{j+1})$$

where $f'_i \in W_i$ for $2 \leq i \leq j-1$ in a neighbourhood $\Omega_i \subset \Omega_{i-1}$ of $\lambda=0$.

Consider the linear mapping

$$(w_j,p_j) \mapsto w_j - \mathcal{L}_{A_\lambda}(p_j)$$

from $W_j \times \mathcal{P}_j$ into $\mathcal{P}_j$, which is smooth on $\lambda \in \Omega_1$ and surjective for $\lambda=0$. Hence we can find a neighbourhood $\Omega_j \subset \Omega_{j-1}$ of $\lambda=0$ and smooth mappings $w_j:\Omega_j \to W_j$ and $p_j:\Omega_j \to \mathcal{P}_j$ such that

$$w_j(\lambda)-\mathcal{L}_{A_\lambda}(p_j(\lambda))=f'_j(\,.\,,\lambda).$$

Thus using the change of coordinates $x=x'+p_j(\lambda)(x')$ and noting this polynomial change in coordinates will not affect lower order terms, we transform f' into

$$f'(x,\lambda) = (df')_{0,\lambda}x+f'_2(x,\lambda)+\ldots+w_j(\lambda)(x)+O(|x|^{j+1})$$

where all terms of order i lie in W_i for $2 \leq i \leq j$ and the induction is complete.

It remains only to construct the diffeomorphism Φ. If Φ' is the transformation equivalent to the successive coordinate transformations $x \mapsto x+p_i(x)$ for $2 \leq i \leq k$ then the map

$$\Phi(x,\lambda) = (B^*(\lambda))^{-1}\Phi'(B^*(\lambda)x,\lambda)$$

satisfies the requirements of the theorem.

As W_i is a complimentary subspace of $\mathcal{L}_A(\mathcal{P}_i)$ we can use the earlier results from this chapter so that W_i can be chosen to equal $\mathcal{P}_i(A^t)$.

As we saw before if $(df)_{0,0}$ is orthogonal or diagonalisable then there is a coordinate system in which A^t can be chosen to be A^{-1} or A respectively hence there is a coordinate system in which f commutes with A on some neighbourhood Ω of $\lambda=0$.

We finish with a theorem applying the ideas from this chapter which gives us a spatio-temporal version of theorem 3.3.3.

Proposition 4.3.2.

Let f be a one parameter family of Γ-equivariant mappings

$$f:\mathbb{R}^n \times \Lambda \longrightarrow \mathbb{R}^n$$

If the action of Γ on $\mathbb{R}^n$ is non trivial and absolutely irreducibly then there is a trivial fixed point solution at zero for all $\lambda\in\Lambda$ and $(df)_{0,\lambda} = c(\lambda)I_n$.

Assume that $c(0)=-1$ and $c'(0)\neq 0$ and f can be put into exact normal form. Then if there is a subgroup $\Sigma\subset\Gamma\times\mathbb{Z}_2$ with dim $\mathrm{Fix}(\Sigma)=1$ then $f(x,\lambda)$ has a unique branch of period two orbits with symmetry group precisely Σ.

The action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ is multiplication by $-I$.

Proof.

Since f is in exact normal form and the action of $(df)_0$ is diagonal then f is not simply Γ equivariant, it is $\Gamma\times\mathbb{Z}_2$ equivariant also. Hence applying theorem 3.3.3 to f we have the desired result.

We can strengthen this result still further and drop the requirement for f to be in exact normal form applying a result similar to that in chapter 6 which has been proved in the authors MSc. dissertation, Brown [1989].

Chapter 5

Hopf Bifurcations with Symmetry.

5.0 Introduction.

Consider a one parameter family of maps f from $\mathbb{R}^2$ to $\mathbb{R}^2$ with a fixed point at x_0. From chapter 1 we see that when a complex conjugate pair of eigenvalues of $(df)_{x_0,\lambda}$ cross the unit circle at $e^{\pm 2\pi i\theta}$ with non zero speed we expect a Hopf bifurcation,that is a branch of invariant circles, provided that $\theta \neq 0, {}^1/_4, {}^1/_3, {}^1/_2, {}^2/_3, {}^3/_4$. The stability and direction of branching of this invariant circle is determined, generically, up to third order.

When symmetry is introduced the group action often forces the eigenvalues of $(df)_{x_0,\lambda}$ to be real, for example when the group acts absolutely irreducibly and $(df)_{x_0,\lambda}$ must be a scalar multiple of the identity, thus the eigenvalues must be real. So we start our discussion with conditions for the existence of complex eigenvalues. We go on to give a simple generalisation of the Hopf theorem from $\mathbb{R}^2$ whose proof relies on the restriction to a two dimensional fixed point space. Finally we give a stronger result which derives from the extra 'temporal' symmetry of the normal form. The results of this chapter can be found in Chossat and Golubitsky [1987] although an assumption made by them may not be generically true as much of the detailed proof included here is omitted.

5.1 Requirements for the existence of complex eigenvalues.

A detailed discussion about and solution to a similar question can be found in Golubitsky *et al* [1987] XVI §1 here we will attempt to simply outline their basic ideas.

We first introduce a definition:

Definition 5.1.1

A representation W of Γ is Γ-simple if either

(a) $W \cong V \oplus V$ where V is absolutely irreducible, or

(b) W is irreducible but not absolutely (we say non absolutely irreducible).

This definition affords us a useful result.

If we have f as above and assume $(df)_{0,0}$ has eigenvalues $\mu = e^{\pm 2\pi i\theta}$. Let G_μ be the corresponding real generalised eigenspace of these eigenvalues of $(df)_{0,0}$. Then generically G_μ is Γ-simple.

Further we can also assume, generically that these are the <u>only</u> eigenvalues of $(df)_{0,0}$ (if not we can reduce f, via a centre manifold, so that it is the case).

With these assumptions we end up with the following result on the form of $(df)_{0,0}$.

Lemma 5.1.2.

Assume that $\mathbb{R}^n$ is Γ-simple, f is Γ-equivariant and $(df)_{0,0}$ has $e^{\pm 2\pi i\theta}$ as its only eigenvalues. Then:

(a) The eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $r(\lambda)e^{\pm 2\pi i\theta(\lambda)}$ each of multiplicity $m = n/2$.Moreover r and θ are smooth in λ.

(b) There exists an invertible linear map $S:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ commuting with Γ such that

$$(df)_{0,0} = S\begin{bmatrix} \cos(2\pi\theta)I_m & -\sin(2\pi\theta)I_m \\ \sin(2\pi\theta)I_m & \cos(2\pi\theta)I_m \end{bmatrix}S^{-1}$$

The proof follows that of lemma 1.5 ch.XVI §1 in Golubitsky *et al* [1987] and will be omittedhere.

Generically we may now assume that the f is a map from $\mathbb{R}^{2m} \times \Lambda \rightarrow \mathbb{R}^{2m}$ and that the eigenvalues of $(df)_{0,0}$ are all $e^{\pm 2\pi i\theta}$ each with multiplicity m.

5.2 A Simple Hopf Theorem with Symmetry.

In this section we show that by applying results from $\mathbb{R}^2$ and with very little extra work, we can predict the existence of invariant circles exhibiting particular symmetries. We use ideas similar to those used in proposition 3.2.8. by applying a reduction to certain fixed point spaces of dimension two we can use results from the standard Hopf theorem.

Theorem 5.2.1.

Let $\mathbb{R}^{2m}$ be Γ simple and let f be a Γ equivariant mapping

$$f:\mathbb{R}^{2m} \times \Lambda \to \mathbb{R}^{2m}$$

with a fixed point at $0 \in \mathbb{R}^{2m}$ for all $\lambda \in \Lambda$. Assume $(df)_{0,0}$ has as it only eigenvalues a complex conjugate pair $e^{\pm 2\pi i \theta}$ with $\theta \neq 0, {}^1/_4, {}^1/_3, {}^1/_2, {}^2/_3, {}^3/_4$. If $\Sigma \subset \Gamma$ is a subgroup with dim $\mathrm{Fix}(\Sigma) = 2$ then there exists a branch of invariant circles with symmetry group Σ.

Proof

We restrict our attention to solutions lying in $\mathrm{Fix}(\Sigma)$ and hence need only consider f restricted to $\mathrm{Fix}(\Sigma)$. Applying lemma 3.2.2 we are thus looking for invariant circles of the map

$$f|_{\mathrm{Fix}(\Sigma)}\mathrm{Fix}(\Sigma) \times \Lambda \to \mathrm{Fix}(\Sigma).$$

By applying lemma 5.1.2 and since dim $\mathrm{Fix}(\Sigma) = 2$ there will be a single complex conjugate pair of eigenvalues for $d(f|_{\mathrm{Fix}(\Sigma)})_{0,\lambda}$ what is more they will satisfy the requirements for the existence of an f invariant circle in the standard Hopf theorem (theorem 1.2.1). Therefore we expect to see an f invariant circle of solutions in $\mathrm{Fix}(\Sigma)$, that is an f invariant circle with symmetry group Σ as required.

5.3 The Hopf Theorem with Symmetry.

We can improve on this simple result by looking for subgroups of $\Gamma \times \mathbf{S}^1$ with two dimensional fixed point space. This additional $\mathbf{S}^1$ symmetry can be attributed to the existence of the invariant circle although there is no direct analogy to the idea of added temporal symmetry seen for a periodic orbit as described in chapter 3. Before the statement of the theorem we must make clear how to define the action of $\mathbf{S}^1$ on $\mathbb{R}^{2m}$. As $\mathbb{R}^{2m}$ is Γ simple we can find a coordinate system in which the linearisation of f is of the form

$$(df)_{0,0} = \begin{bmatrix} \cos(2\pi\theta)I_m & -\sin(2\pi\theta)I_m \\ \sin(2\pi\theta)I_m & \cos(2\pi\theta)I_m \end{bmatrix}$$

and in this coordinate system the action of $\varphi \in \mathbf{S}^1$ is simply

$$\varphi.x = \begin{bmatrix} \cos(2\pi\varphi)I_m & -\sin(2\pi\varphi)I_m \\ \sin(2\pi\varphi)I_m & \cos(2\pi\varphi)I_m \end{bmatrix} x$$

Since $(df)_{0,0}$ commutes with the group action (see lemma 3.2.1) then so must the action of $\mathbf{S}^1$ thus we have defined the action of $\Gamma \times \mathbf{S}^1$ on $\mathbb{R}^{2m}$.

Theorem 5.3.1.

Let f be defined as in theorem 5.2.1. If $\Sigma \subset \Gamma \times \mathbf{S}^1$ is a subgroup with dim Fix(Σ)=2 and dim(Γ)=0 then generically there exists a unique (upto conjugacy) branch of f invariant circles emanating from the trivial fixed point at 0 tangent to fix(Σ).

Proof

If we identify $\mathbb{R}^{2m}$ with $\mathbb{C}^m$ f becomes a map

$$f : \mathbb{C}^m \times \Lambda \to \mathbb{C}^m$$

in which $(df)_{0,\lambda} = e^{2\pi i\theta(\lambda)} I_m$ (since $\mathbb{R}^{2m}$ is Γ simple).

We now state a lemma which brings f into an appropriate normal form.

Lemma 5.3.2.

If $\theta(0) = \theta \neq 0, {}^1/_4, {}^1/_3, {}^1/_2, {}^2/_3, {}^3/_4$ then there is a smooth change of coordinates which takes f into the form

$$f(z,\lambda) = (df)_{0,\lambda}z + a_3(z,\lambda) + a_4(z,\lambda) + O(|z|^5)$$

where a_3 is a homogeneous polynomial of degree two in z and one in $\bar{z}$ and a_4 is a homogeneous polynomial of degree 4 in $\bar{z}$ which equals zero unless 5θ is an integer. Both a_3 and a_4 are smooth in λ.

Proof

By theorem 4.2.2 we can, to any order, put f into a normal form which commutes with both Γ and $(df)_{0,0}{}^t$ (this is in fact equal to $(df)_{0,0}{}^{-1}$ and thus f commutes with $(df)_{0,0}$). Thus we can put f into the form

$$f(z,\lambda) = (df)_{0,\lambda}z + \sum_{l=2,3,4} \sum_{|\alpha|+|\beta|=l} a_{\alpha\beta}(\lambda) z^{\alpha} \bar{z}^{\beta} + O(|z|^5)$$

in which each term in the polynomial commutes with $(df)_{0,0}$, that is $a_{\alpha\beta}(\lambda) = 0$ unless

$$e^{2\pi i\theta}(z^{\alpha}\bar{z}^{\beta}) = (e^{2\pi i\theta}z)^{\alpha}(e^{-2\pi i\theta}\bar{z})^{\beta}$$

orequivalently

$$(|\alpha| - |\beta| - 1)\theta = 0 \pmod 1.$$

If θ is irrational we are thus required to solve

$$|\alpha| - |\beta| = 1$$

and $$|\alpha| + |\beta| = k \qquad \text{for } k = 2, 3, 4.$$

These have only one solution at $k = 3$, $|\alpha| = 2$ and $|\beta| = 1$.

If $\theta = p/q$, $q \geq 5$ we need

$$|\alpha| = 1 + |\beta| + rq$$

and $$|\alpha| + |\beta| = k \qquad \text{for } k = 2, 3, 4$$

as $|\alpha|$ and $|\beta|$ are both positive this has only one pair of solution for $q = 5$ at $|\alpha| = 0$ and $|\beta| = 4$. Thus we have the desired normal form.

Let g be f truncated to third order, this has symmetry group $\Gamma \times S^1$ and hence as in theorem 5.2.1 we can apply the standard Hopf theorem to the restricted map

$$g:\mathrm{Fix}(\Sigma) \times \Lambda \to \mathrm{Fix}(\Sigma)$$

to find g invariant circles with the required symmetry. However we cannot conclude directly that these invariant circles persist when higher order terms are reintroduced since f does not commute with S^1 and hence will not necessarily leave $\mathrm{Fix}(\Sigma)$ invariant. To overcome this problem Chossat and Golubitsky [1987] use a theorem of Ruelle [1973], Theorem 3.1) which asserts the existence of invariant circles under certain conditions. Below we summarize these results and give the conditions which are needed in order that f will have invariant circles.

Theorem.(Ruelle[1983])

Let $\Phi_\lambda:\mathbb{C}^n \to \mathbb{C}^n$ be a C^1 diffeomorphism depending on a real parameter λ varying in an interval around 0:

$$\Phi_\lambda(z) = A_\lambda z+P_\lambda(z)+Q_\lambda(z)$$

where $\lambda \mapsto A_\lambda$ is a continuous complex function, and P_λ is a homogeneous polynomial of degree 2 in z and 1 in $\bar{z}$ with coefficients continuous in λ. We assume that there exists a function c(.) independent of λ such that c(.) ≥ 0, $\lim_{u\to 0} c(u)=0$ and

$$|Q_\lambda(z)| \leq c(|z|)|z|^3, \qquad |(dQ_\lambda)z| \leq c(|z|)|z|^2.$$

We also assume that $|A_0| = 1$ and $|A_\lambda| > 1$ for $\lambda > 0$. Let the vector field

$$z \to z+A_0^{-1}P_0(z)$$

be normally hyperbolic to the compact invariant manifold S. Suppose also that S is invariant under the transformation $z \to ze^{i\sigma}$ (for all real σ).

Then for small $\lambda > 0$, there exist $\Theta_\lambda \in C^1(S,\mathbb{C}^n)$ and $S_\lambda \subset \mathbb{C}^n$ such that

(a) Θ_λ is a diffeomorphism of S onto S_λ.

(b) S_λ is invariant under Φ_λ and Φ_λ is normally hyperbolic to S_λ.

(c) $\lim_{\lambda\to 0} S_\lambda = \{0\}$.

(d) If $\{\Gamma\}$ is a group of unitary transformations of $\mathbb{C}^n$ such that $\Gamma\Phi_\lambda = \Phi_\lambda\Gamma$ and $\Gamma S = S$, then $\Theta_\lambda\Gamma = \Gamma\Theta_\lambda$.

The proof of this theorem uses a vector field approximation to Φ_λ and results from Hirsch *et al* [1977] on invariant manifolds to give an S_λ which has the desired properties.

We now apply this theorem to f. Clearly from lemma 5.3.2 f has the desired form for the theorem to hold. The existence of the invariant manifold follows from the fact that up to third order f, which we have called g, is $\mathbf{S}^1$ equivariant and so provided that the third order term a_3 of g is non zero the vector field will have an invariant circle S which is invariant under the transformation $z \to e^{i\sigma}z$. However the equivariance of Γ means that this invariant circle must be considered as a submanifold of the invariant manifold ΓS. We say g_λ is normally hyperbolic to the invariant manifold ΓS in the sense that on the invariant manifold the only eigenvalues of (dg_λ) with modulus equal to one are those which are constrained to do so by the action of $\Gamma \times \mathbf{S}^1$, this is a generic condition. Thus the theorem of Ruelle predicts the existence of a diffeomorphism Θ_λ which maps ΓS onto S'_λ an invariant manifold of f.

It remains only to show that on the manifold S'_λ there is an invariant circle of f. We first note that since $\Gamma(\Gamma S)=\Gamma S$ and f is Γ equivariant we can apply part (d) from the theorem to say that

$$\Theta_\lambda(\Gamma S) = \Gamma\Theta_\lambda(S)$$

i.e.

$$S'_\lambda = \Gamma\Theta_\lambda(S)$$

but since Θ_λ is a diffeomorphism and S is an invariant circle then $\Theta_\lambda(S)$ is an invariant circle also. We now use the fact that $\dim(\Gamma)=0$, i.e. Γ is finite to say that $\Gamma\Theta_\lambda(S)$ is a finite set of invariant circles and thus on the manifold S'_λ there is an invariant circle of f.

Before carrying on to discuss the symmetry groups of the invariant circles we briefly outline the reason why we make the restriction that Γ is finite and why the proof used by Chossat and Golubitsky does not quite work. Consider a mapping f which is $\mathbf{O}(2)$ equivariant and satisfies the conditions of the theorem of Ruelle. So there exists an invariant torus of g, the $\mathbf{O}(2)\times\mathbf{S}^1$ equivariant truncated normal form of f, to which generically g is normally hyperbolic. However it does not follow that generically g is normally hyperbolic to its invariant circles, this is because the eigenvalues of dg which are forced to be on the unit circle may have generalised eigenspaces which are not tangent to the invariant circles. Thus when we introduce a perturbation to get f, these invariant circles may break up and although the perturbed invariant torus remains we will no longer see invariant circles but instead we see a rotating wave that proceeds around the torus.

If however g is normally hyperbolic to its invariant circles then we can apply Ruelle's theorem to these invariant manifolds an so f will have invariant circles also.

It is further conjectured that the theorem is true for any Γ although the proof has not been completed. The idea is to look at f restricted to the invariant manifold, quotient Σ, i.e. some Γ equivariant mapping h

$$h:(\Gamma\times S^1)/\Sigma \rightarrow (\Gamma\times S^1)/\Sigma$$

and show that when $\dim(\mathrm{Fix}(\Sigma))=2$ these are forced by the Γ equivariance to map invariant circles to invariant circles.

We now go on to discuss precisely what form the symmetry groups of each invariant circle predicted by theorem 5.3.1 takes. Let the action of $S^1=\mathbb{R}/\mathbb{Z}$ act on $\mathbb{C}^n$ by multiplication by $e^{2\pi i\theta}$, for $\theta\in S^1$. Let π be the natural projection of $\Gamma\times S^1$ onto Γ, i.e.

$$\pi:\Gamma\times S^1 \rightarrow \Gamma$$

$$\pi(\gamma,\theta) = \gamma.$$

If Σ is an isotropy subgroup of some point under the $\Gamma\times S^1$ action, let $H=\pi(\Sigma)$ and $K=\Sigma\cap\Gamma\times\{0\}$ then H is the ('spatio-temporal') symmetry group of the invariant circle and K is the purely spatial symmetry group of the invariant circle, that is H maps the invariant circle onto itself whilst K fixes pointwise the invariant circle.

To show this we return to the proof of the Hopf theorem, g the truncated normal form of f has a branch of invariant circles S with symmetry group precisely Σ, that is if $(\gamma,\theta)\in\Sigma$ then

$$\gamma e^{2\pi i\theta}z=z \qquad \text{for all } z\in S.$$

We have

$$f(\Theta(\gamma e^{2\pi i\theta}z))=f(\Theta(z))$$

i.e. $$f(\Theta(\gamma z))=f(\Theta(\gamma e^{-2\pi i\theta}z))$$

but since S is invariant under the action of $\mathbf{S}^1$ then $e^{-2\pi i\theta}z$ is on S also hence $\Theta(e^{-2\pi i\theta}z)$ is on the invariant circle of f. Since f and Θ are Γ equivariant we have

$$\gamma f(\Theta(z))=f(\Theta(e^{-2\pi i\theta}z))$$

and so γ maps the invariant circle of f onto itself. Since this is the projection of (γ,θ) onto Γ we are done.

To show K fixes pointwise the invariant circle we note that for $k\in K$ $k=(\gamma,0)$ and thus

$$\gamma z=z$$

so $$f(\Theta(z))=f(\Theta(\gamma z))=\gamma f(\Theta(z))$$

and hence K fixes the invariant circle pointwise.

From the way H and K are defined it is clear to see that $H/K\subset\mathbf{S}^1$ and what is more the restriction of f to their corresponding invariant circle must commute with the action of H/K.

We end this chapter with the discussion of an example relating to the motivating example of chapter 2. We take a few group theoretic results from Golubitsky *et al*[1987]:-

(i) The map

$$f:\mathbb{R}^4\times\Lambda\to\mathbb{R}^4$$

where f is D_n equivariant. We identify $\mathbb{R}^4$ with $\mathbb{C}^2$, it has been shown that all possible actions of D_n are equivalent to a standard action given below, by relabelling the group elements and dividing through by the kernel of the action. The action is given by the generators ζ and κ where

$$\zeta(z_1,z_2)=(e^{2\pi i/n}z_1,e^{-2\pi i/n}z_2)$$

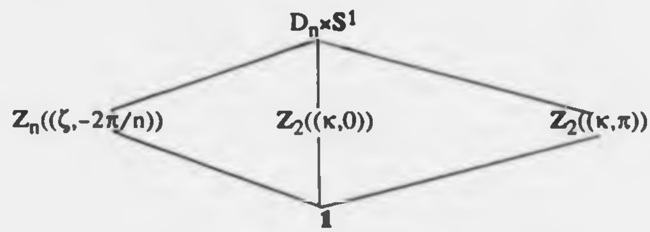
and $$\kappa(z_1,z_2)=(z_2,z_1).$$

The action of $\mathbf{S}^1$ is the multiplication described above, i.e. for $\theta \in \mathbf{S}^1$

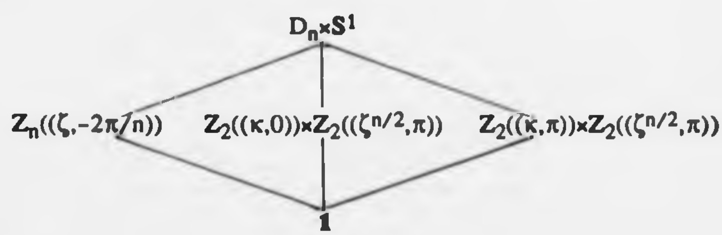
$$\theta(z_1,z_2)=(e^{2\pi i\theta}z_1,e^{2\pi i\theta}z_2).$$

We give the isotropy subgroup lattices below in fig 5.1 but do not attempt to show how to calculate the isotropy subgroups since this has been done by several others, see Golubitsky *et al* [1987].

n odd



n = 2 (mod 4)



n = 0 (mod 4)

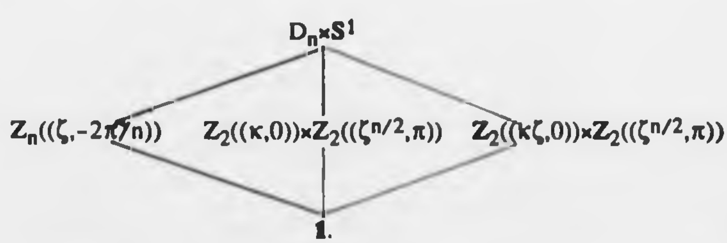


fig 5.1 Isotropy subgroup lattices for $D_n \times \mathbf{S}^1$ (the element in brackets is the generator of the subgroup).

We complete the example with a description of H and K and how these relate to the example from chapter 2. For n odd there are three isotropy subgroups as seen in fig 5.1 and for f the symmetry groups of the three invariant circles are $\mathbf{Z}_n(\zeta)$, $\mathbf{Z}_2(\kappa)$ and $\mathbf{Z}_2(\kappa)$

respectively hence in the motivating example when n=3 we would expect three types of invariant behavior two with the flip symmetry, i.e. on two islands we see the same cycle of solutions (possible not in phase) and one with rotational symmetry, i.e. all three islands have the same cycle of solutions (possibly not in phase). The respective K's tell us which solutions will be in phase, infact the only case in which K is non trivial is from the isotropy subgroup given by $\mathbf{Z}_2((\kappa,0))$ and hence this is the only case in which the respective solutions will be in phase.

We can give similar descriptions of the symmetry group when n is even also. For n=2 (mod 4) the H's will be $\mathbf{Z}_n(\zeta)$, $\mathbf{Z}_2(\kappa)\times\mathbf{Z}_2(\zeta^{n/2})$ and $\mathbf{Z}_2(\kappa)\times\mathbf{Z}_2(\zeta^{n/2})$ respectively. In this case K is trivial for $\mathbf{Z}_n((\zeta,-2\pi/n))$, equal to $\mathbf{Z}_2((\kappa,0))$ for $\mathbf{Z}_2((\kappa,0))\times\mathbf{Z}_2((\zeta^{n/2},\pi))$ and equal to $\mathbf{Z}_2((\kappa\zeta^{n/2},0))$ for $\mathbf{Z}_2((\kappa,\pi))\times\mathbf{Z}_2((\zeta^{n/2},\pi))$. When n=0 (mod 4) we find the H's to be $\mathbf{Z}_n(\zeta)$, $\mathbf{Z}_2(\kappa)\times\mathbf{Z}_2(\zeta^{n/2})$ and $\mathbf{Z}_2(\kappa\zeta)\times\mathbf{Z}_2(\zeta^{n/2})$ respectively and the K's to be $\mathbf{1}$, $\mathbf{Z}_2((\kappa,0))$ and $\mathbf{Z}_2((\kappa\zeta,0))$ respectively. These all have analogous physical interpretations similar to those given above, for example when n=4, the isotropy subgroup $\mathbf{Z}_2((\kappa\zeta,0))\times\mathbf{Z}_2((\zeta^2,\pi))$ gives rise to a solution which is left invariant by the action of $\mathbf{Z}_2(\kappa\zeta)\times\mathbf{Z}_2(\zeta^2)$, the invariance implies that diagonally opposite oscillators exhibit the same behavior and since K is $\mathbf{Z}_2((\kappa\zeta,0))$ then two of the oscillators will be in phase.

Chapter 6

Subharmonic Bifurcations and symmetry.

6.0 Introduction.

As seen in chapter 1, given a parameterised map f on $\mathbb{R}^2$ with a fixed point at 0 with an eigenvalue of $(df)_{0,\lambda}$ crossing the unit circle at the complex conjugate pair $e^{\pm 2\pi i p/q}$ not only would we expect an invariant circle type bifurcation, we would expect also to see two bifurcating branches of period q points emanating from the fixed point lying on this circle. These would be alternately stable and unstable (provided of course the eigenvalues of df lay within some region of parameter space defined in theorem 1.2.2).

If we now take f to be a map from $\mathbb{R}^{2m}$ to $\mathbb{R}^{2m}$ which is Γ-equivariant under the action of some compact Lie group Γ, what can we say about the occurence of bifurcating periodic solutions. The cases in which q = 1 and 2 (that is fixed point and period doubling bifurcations) have been considered in chapter 3, propositions 3.3.2 and 3.3.4. In the case of period doubling bifurcations the proof was delayed because it is a special case of the result from this chapter.

For the rest of this chapter we assume that $q \geq 3$, that is the eigenvalues are complex, but we will also discuss how we can apply similar ideas to the period doubling bifurcation. In chapter 5 we saw that generically we can expect Hopf bifurcations to occur in a one parameter family of maps, however as seen in the case for maps on $\mathbb{R}^2$, to see the full dynamics of a period q point we need to consider a two parameter family of mappings. We saw also that to ensure the existence of complex eigenvalues then Γ can be assumed to act simply on $\mathbb{R}^{2m}$ (refer to definition 5.1.1 for Γ-simple).

Thus throughout this chapter we assume that f is a two or more parameter family of mappings

$$f:\mathbb{R}^{2m} \times \Lambda \to \mathbb{R}^{2m}$$

which is Γ-equivariant and on which $\mathbb{R}^{2m}$ is Γ-simple. We first give a result which is a direct application of the period q theorem from $\mathbb{R}^2$ and then extend this result to take account of the extra 'temporal' symmetry of a period q orbit.

6.1 A simple period q theorem with symmetry.

In this section we use the method of restriction to fixed point spaces to reduce the problem as we did in the simple Hopf theorem to give.

Theorem 6.1.1.

Let $\mathbb{R}^{2m}$ be Γ simple and let f be a two parameter Γ-equivariant family of mappings

$$f:\mathbb{R}^{2m} \times \Lambda \to \mathbb{R}^{2m}$$

with a fixed point at $0 \in \mathbb{R}^{2m}$ for all $\lambda = (\lambda_1, \lambda_2) \in \Lambda$. Assume $(df)_{0,0,0}$ has as its only eigenvalues a complex conjugate pair $e^{\pm 2\pi i p/q}$ with $q \neq 1, 2, 3, 4$ which passes throughthe unit circle with non zero speed. If $\Sigma \subset \Gamma$ is a subgroup with dim $\mathrm{Fix}(\Sigma) = 2$ then there exists a branch of invariant circles with symmetry group Σ. What is more, within some well defined region of parameter space, the map restricted to the invariant circle has two period q orbits.

Proof

The proof is almost identical to that of the simple Hopf theorem 5.2.1. We first note that solutions with symmetry group Σ are precisely those solutions which lie in $\mathrm{Fix}(\Sigma)$. Thus we are looking for solutions of the map $f|_{\mathrm{Fix}(\Sigma)}$ and since $\mathrm{Fix}(\Sigma)$ has dimension 2 and is mapped onto itself by f we have reduced the problem to an equivalent one in $\mathbb{R}^2$. Furthermore the dimension of Fix(Σ) means that $d(f|_{Fix(\Sigma)})$ has a single complex complex conjugate pair of eigenvalues at $e^{\pm 2\pi i p/q}$. Hence we can apply theorem 1.2.2 and provided that the eigenvalues of d(f| Fix(Σ)) lie within a cusped region whose boundaries are tangent at $e^{2\pi i p/q}$ then we should expect to see two period q orbits on the predicted invariant circle. This region in the complex plane relates to a corresponding region in parameter space within which we see period q solutions of f lying in Fix(Σ) and the proof is complete.

6.2 Symmetry groups and isotropy subgroups.

We next discuss the idea of the symmetry group of a period q point. Now normally under the action of Γ the symmetry group of a point x is defined as that subgroup of Γ whose action fixes that point, i.e.

$$\Sigma_x = \left\{ \gamma \in \Gamma : \gamma x = x \right\}.$$

However a period q point has further symmetry due to its being a period-q-point. More precisely the action of $\gamma \in \Gamma$ on x may be the same as that of some iterate of f

$$f^c(\gamma x, \lambda) = x.$$

Hence we have:-

Definition 6.2.1.

The symmetry group of a period-q-point x(or more precisely (x,λ)), $\Sigma_x \subset \Gamma \times Z_q$, is

$$\Sigma_x = \left\{ (\gamma, c) \in \Gamma \times Z_q : f^c(\gamma x, \lambda) = x \right\}.$$

6.3 Period q theorem with symmetry.

We now state the main results of this chapter

Theorem 6.3.1

Let $\mathbb{R}^n$ be Γ-simple and f be a two parameter Γ-equivariant map

$$f:\mathbb{R}^n\times\Lambda\to\mathbb{R}^n$$

with a fixed point at 0 for all $\lambda=(\lambda_1,\lambda_2)\in\Lambda$.

Assume that $(df)_{0,\lambda_1,\lambda_2}$ has eigenvalues $r(\lambda_1,\lambda_2)e^{\pm i\theta(\lambda_1,\lambda_2)}$ (each of multiplicity $m=n/2$) where $r(0,0)=1$, $\theta(0,0)=2\pi p/q$.

Then there exists a $\Gamma\times\mathbf{Z}_q$-equivariant germ

$$g:\mathbb{R}^n\times\Lambda\to\mathbb{R}^n,$$

where the action of $(\gamma,c)\in\Gamma\times\mathbf{Z}_q$ is given by

$$(\gamma,c)x=\begin{bmatrix}\cos(2\pi p/q)I_m & -\sin(2\pi p/q)I_m\\ \sin(2\pi p/q)I_m & \cos(2\pi p/q)I_m\end{bmatrix}\gamma x$$

and for which $(dg)_{0,0,0}$ has as its only eigenvalue 0 with multiplicity n. There is also a Γ-equivariant map

$$\Phi:\mathbb{R}^n\times\Lambda\to\mathbb{R}^n$$

such that

(i) $\Phi(x,\lambda)$ is a diffeomorphism, onto its image, for all $\lambda\in\Lambda$,

(ii) $\Phi(x,\lambda)$ is a period q point of f if and only if $g(x,\lambda)=0$ (i.e. $f^q(\Phi(x,\lambda),\lambda)=\Phi(x,\lambda)$),

(iii) The symmetry group of $\Phi(x,\lambda)$ as a period q point of f is isomorphic to the isotropy subgroup of x in g, under the $\Gamma\times\mathbf{Z}_q$ action.

<u>6.3.2 Remark</u>

The assumptions of the theorem are not as restrictive as they at first may seem because if we assume the only eigenvalues of df are $e^{\pm 2i\pi p/q}$ at $\lambda=0$ via a centre manifold or Liapunov Schmidt reduction then these are the only eigenvalues and hence generically that $\mathbb{R}^n$ will be Γ-simple.

Proof

The idea of the proof will be to define a function $F:\mathbb{R}^{nq}\times\Lambda\to\mathbb{R}^{nq}$ which is $\Gamma\times\mathbf{Z}_q$-equivariant and whose zeroes correspond to the period q points of f and whose symmetry groups correspond to those of f. Next via a Liapunov Schmidt reduction of F onto $\mathbb{R}^n$ we get a map g with the desired properties.

We define

$$F:\mathbb{R}^n\times\mathbb{R}^n\times\ldots\times\mathbb{R}^n\times\Lambda\to\mathbb{R}^n\times\ldots\times\mathbb{R}^n$$

by $$(x_1,x_2,\ldots,x_q,\lambda)\mapsto\big(f(x_q,\lambda)-x_1,\ldots,f(x_{q-1},\lambda)-x_q\big)$$

(to reduce overly complicating notation, we will in future omit λ from f i.e. $f(x,\lambda)$ will be written as $f(x)$).

We start by examining some of the properties of F.

Lemma 6.3.3.

(a) The point $(x_1,x_2,\ldots,x_q,\lambda)$ is a zero of F if and only if the set $x_1,\ldots,x_q$ is a period q orbit of $f(\,.\,,\lambda)$.

(b) F is $\Gamma\times\mathbf{Z}_q$-equivariant where Γ acts on each block of n variables as it does on f ('the diagonal action') and $\mathbf{Z}_q$ acts by the cyclic permutation of the q blocks of n variables, i.e.

$$(\gamma,c).(x_1,\ldots,x_q)=(\gamma.x_{c+1},\gamma.x_{c+2},\ldots,\gamma.x_c).$$

Proof

(a) If $$F(x_1,x_2,\ldots,x_q,\lambda)=(0,\ldots,0)$$

then $$f(x_q)-x_1=0\Leftrightarrow f(x_q)=x_1$$

$$\vdots$$

$$f(x_{q-1})-x_q=0\Leftrightarrow f(x_{q-1})=x_q,$$

so $$f^q(x_i,\lambda)=f(\ldots f(f(x_i,\lambda),\lambda)\ldots,\lambda)=f(\ldots f(x_{i+1},\lambda)\ldots,\lambda)$$

$$= \ldots = f(x_{i+q-1},\lambda) = f(x_{i-1},\lambda) = x_i.$$

Conversely if $f^q(x_i,\lambda) = x_i$ and $f(x_i,\lambda) = x_{i+1}$ then

$$\begin{aligned} F(x_1,x_2,\ldots,x_q,\lambda) &= \big(f(x_q)-x_1,\ldots,f(x_{q-1})-x_q\big) \\ &= \big(x_1-x_1,\ldots,x_q-x_q\big) \\ &= \big(0,\ldots,0\big). \end{aligned}$$

(b)
$$\begin{aligned} F\big((\gamma,c).(x_1,x_2,\ldots,x_q),\lambda\big) &= F\big(\gamma.x_{c+1},\ldots,\gamma.x_c,\lambda) \\ &= \big(f(\gamma.x_c)-\gamma.x_{c+1},\ldots,f(\gamma.x_{c-1})-\gamma.x_c,\lambda\big) \\ &= \big(\gamma.f(x_c)-\gamma.x_{c+1},\ldots,\gamma.f(x_{c-1})-\gamma.x_c\big) \\ &= \big(\gamma.(f(x_c)-x_{c+1}),\ldots,\gamma.(f(x_{c-1})-x_c)\big) \\ &= (\gamma,c).\big(f(x_q)-x_1,\ldots,f(x_{q-1})-x_q\big) \\ &= (\gamma,c).F(x_1,x_2,\ldots,x_q,\lambda). \end{aligned}$$

We now perform a Liapunov Scmidt reduction on F to give a map g whose zeroes are in direct correspondence to those of F and has the desired properties from the statement of the theorem.

There are five steps involved in the reduction to arrive at g:-

(1) We first decompose $\mathbb{R}^{nq}$ in two ways. On the domain of F we decompose as in (a) and on the range of F as in (b)

$$\begin{aligned} &(a)\ \mathbb{R}^{nq} = \ker \mathfrak{L} \oplus M \\ &(b)\ \mathbb{R}^{nq} = \operatorname{coker} \mathfrak{L} \oplus \operatorname{range} \mathfrak{L} \end{aligned} \qquad (*)$$

where $\mathfrak{L}$ is $dF(0,0)$.

(2) The equation $F(y,\lambda) = 0$ can then be split into two separate equations which carry the same equilibria information as F, viz

$$\begin{aligned} &(a)\ EF(y,\lambda) = 0 \\ &(b)\ (I-E)F(y,\lambda) = 0 \end{aligned} \qquad (**)$$

where $E:\mathbb{R}^{nq} \longrightarrow \operatorname{range} \mathfrak{L}$ is the unique projection of $\operatorname{coker} \mathfrak{L} \oplus \operatorname{range} \mathfrak{L}$ onto $\operatorname{range} \mathfrak{L}$.

(3) Using (*) part (a) we write $y = v+w$ where $v \in \ker \mathcal{L}$ and $w \in M$. We can then use the implicit function theorem to solve (**) part (a) for w as a function of v and λ giving

$$W : \ker \mathcal{L} \times \Lambda \rightarrow M$$

where

$$EF(v+W(v,\lambda),\lambda) = 0.$$

(4) Next we define φ

$$\varphi : \ker \mathcal{L} \times \Lambda \rightarrow \operatorname{coker} \mathcal{L}$$

by

$$\varphi(v,\lambda) = (I-E)F(v+W(v,\lambda),\lambda).$$

(5) Finally we define g as

$$g : \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n$$

by

$$g(x,\lambda) = p\Big(\varphi(x, Ax, \ldots, A^{q-1}x, \lambda)\Big)$$

where p is the projection of the first n co-ordinates.

Thus in order to carry out a Liapunov Schmidt reduction on F at (0,0) we first need to know the form of $\ker (dF)_{0,0}$ and $\operatorname{coker} (dF)_{0,0}$.

We have

$$(dF)_{x_1,\ldots,x_q,\lambda} = \begin{bmatrix} -I_n & 0 & \cdot & \cdot & \cdot & (df)_{x_q} \\ (df)_{x_1} & -I_n & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & (df)_{x_{q-1}} & -I_n \end{bmatrix}$$

so

$$(dF)_{0,\ldots,0,\lambda} = \begin{bmatrix} -I_n & 0 & \cdot & \cdot & \cdot & (df)_0 \\ (df)_0 & -I_n & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & (df)_0 & -I_n \end{bmatrix}$$

As $\mathbb{R}^n$ is Γ-simple by Lemma 5.1.2 there exist a basis for $\mathbb{R}^n$ in which

$$(df)_{0,0} = \begin{bmatrix} \cos(2\pi p/q)I_m & -\sin(2\pi p/q)I_m \\ \sin(2\pi p/q)I_m & \cos(2\pi p/q)I_m \end{bmatrix}$$

where $m = n/2$.

If we call this matrix A and $(dF)_{0,0} = \mathcal{L}$ we get finally

$$(dF)_{0,0} = \begin{bmatrix} -I_n & 0 & . & . & . & A \\ A & -I_n & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & A & -I_n \end{bmatrix}$$

We can now calculate ker $\mathcal{L}$ and coker $\mathcal{L}$.

For $y \in \mathbb{R}^{nq}$ to be in ker $\mathcal{L}$ we need

$$\mathcal{L}y = 0$$

where $y = (x_1, x_2, \ldots, x_q)$ and $x_i \in \mathbb{R}^n$, i.e.

$$\begin{bmatrix} -I_n & 0 & . & . & . & A \\ A & -I_n & . & . & . & 0 \\ . & & & & & . \\ 0 & . & . & . & A & -I_n \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ x_q \end{bmatrix} = 0$$

or

$$Ax_q - x_1 = 0 \iff Ax_q = x_1$$

$$.$$

$$Ax_{q-1} - x_q = 0 \iff Ax_{q-1} = x_q$$

so if $y \in \ker \mathcal{L}$ then

$$y = \begin{bmatrix} x \\ Ax \\ A^2x \\ \cdot \\ A^{q-1}x \end{bmatrix}$$

To calculate coker $\mathcal{L}$ (i.e. $(\mathrm{Im}\,\mathcal{L})^\perp$) we use the Fredholm alternative, that is

$$\operatorname{coker} \mathcal{L} = (\mathrm{Im}\,\mathcal{L})^\perp = \ker(\mathcal{L}^t)$$

for some inner product defined on $\mathbb{R}^{nq}$.

We shall use the standard inner product on $\mathbb{R}^{nq}$, i.e.

$$\langle\langle x , y \rangle\rangle = \sum_{i=1}^{n} x_i y_i$$

with this inner product $\mathcal{L}^t$ is simply the transpose of the matrix $\mathcal{L}$ above, what is more, since A is orthogonal $A^t = A^{-1}$ and we have

$$\mathcal{L}^t = \begin{bmatrix} -I_n & A^{-1} & \cdot & \cdot & \cdot & 0 \\ 0 & -I_n & A^{-1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A^{-1} & \cdot & \cdot & \cdot & 0 & -I_n \end{bmatrix}$$

using a similar method to that above we calculate $\ker \mathcal{L}^t = \operatorname{coker} \mathcal{L}$

$$\operatorname{coker} \mathcal{L} = \begin{bmatrix} v \\ Av \\ A^2v \\ \cdot \\ A^{q-1}v \end{bmatrix} \quad \text{where } v \in \mathbb{R}^n$$

So by the generalities of the Liapunov Schmidt reduction there exists a function φ

$$\varphi: \ker \mathfrak{L} \times \Lambda \longrightarrow \operatorname{coker} \mathfrak{L}$$

whose zeroes are in one to one correspondence with those of F and thus the period q points of f. Moreover φ possesses the same $\Gamma \times \mathbf{Z}_q$ equivariance as F.

Now as both ker $\mathfrak{L}$ and coker $\mathfrak{L}$ are isomorphic to $\mathbb{R}^n$ we can define a function g, the projection of φ onto $\mathbb{R}^n$

$$g: \mathbb{R}^n \times \Lambda \longrightarrow \mathbb{R}^n$$

given by $$g(x,\lambda) = p(\varphi(x,Ax,\ldots,A^{q-1}x,\lambda))$$

where

$$p: \operatorname{coker} \mathfrak{L} \longrightarrow \mathbb{R}^n$$

$$p(y,Ay,\ldots,A^{q-1}y) = y.$$

The first thing we note is that if g has a zero then so do φ and F, hence f has a period q orbit. Likewise if there is a period q orbit of f then there is a zero of F and also of φ, which in turn corresponds to a zero of g. So the zeros of g are in one to one correspondence with period q points of f.

The next question we must tackle is how does the $\Gamma \times \mathbf{Z}_q$ equivariance on φ pass on to g. The answer comes via the next lemma.

Lemma 6.3.4.

With the action of $\Gamma \times \mathbf{Z}_q$ on $\mathbb{R}^n$ given by

$$(\gamma,c).x = \gamma.A^c x$$

where $A=(df)_{0,0}$ then g is $\Gamma \times \mathbf{Z}_q$ equivariant.

Proof

We need to show that

$$g\Big((\gamma,c).x,\lambda\Big) = (\gamma,c).g(x,\lambda)$$

for all $(\gamma,c) \in \Gamma \times \mathbf{Z}_q$.

By definition

$$g\big((\gamma,c).x,\lambda\big) = g(\gamma.A^c x,\lambda)$$

$$= p\big(\varphi(\gamma.A^c x, A\gamma.A^c x, \ldots, A^{q-1}\gamma.A^c x,\lambda)\big).$$

Since the action of Γ commutes with A

$$g\big((\gamma,c).x,\lambda\big) = p\big(\varphi(\gamma.(A^c x, \ldots, A^{c-1}x),\lambda)\big).$$

On $\mathbb{R}^{nq}$ the action of $c \in \mathbf{Z}_q$ is precisely this shift of coordinates, thus

$$g\big((\gamma,c).x,\lambda\big) = p(\varphi\big((\gamma,c).(x, Ax, \ldots, A^{q-1}),\lambda\big))$$

which by the $\Gamma \times \mathbf{Z}_q$ equivariance of φ

$$= p\big((\gamma,c).\varphi(x, Ax, \ldots, A^{q-1},\lambda)\big).$$

Assume that

$$\varphi(x, Ax, \ldots, A^{q-1},\lambda) = (y, Ay, \ldots, A^{q-1}y)$$

(this is the form which an element of coker $\mathcal{L}$ must take).

So $$p\big((\gamma,c).(y, Ay, \ldots, A^{q-1}y)\big) = p(\gamma.A^c y, \gamma.A^{c+1}y, \ldots, \gamma.A^{c-1}y)$$

and since A commutes with γ

$$p\big((\gamma,c).(y, Ay, \ldots, A^{q-1}y)\big) = p(\gamma.A^c y, A\gamma.A^c y, \ldots, A^{q-1}\gamma.A^c y)$$

$$= \gamma A^c y$$

$$= (\gamma,c).y$$

where now the action of $\Gamma \times \mathbf{Z}_q$ is that defined in the lemma

$$= (\gamma,c).p\big(\varphi(x, Ax, \ldots, A^{q-1}x,\lambda)\big)$$

$$= (\gamma,c).g(x,\lambda)$$

so g is $\Gamma \times \mathbf{Z}_q$ equivariant.

We next need to show that g has the desired properties given in the statement of the proposition, that is the zeros of g are in one to one correspondence with the period q points of f, the isotropy subgroups of the zeros of g are isomorphic to the symmetry group of the period q points of f and $(dg)_{0,0}$ has a single eigenvalue at 0 with multiplicity n.

The first part is easy to see from the generalities of the reduction but it is enlightening to see how these points correspond to complete the rest. For these we need to look more closely at the mechanics of the reduction which were discussed earlier.

Given x a zero of g such that

$$g(x,\lambda)=0$$

we need a bijection which will map it to a period q point of f (or equivalently a zero of F).

Now a zero of F is a solution to

$$EF(y,\lambda)=0$$

$$(I-E)F(y,\lambda)=0.$$

These solutions are given by $v+W(v,\lambda)$ where

$$W: \ker \mathcal{L}\times\Lambda\longrightarrow M.$$

So if x is a zero of $g(\,.\,,\lambda)$ then $(x,Ax,\ldots,A^{q-1}x) + W(x,Ax,\ldots,A^{q-1}x,\lambda)$ is a zero for $F\big(\,.\,,\lambda\big)$ thus by the earlier properties of F any of these q sets of n co-ordinates will give a period q point of f.

So we have a map

$$\Phi:\mathbb{R}^n\times\Lambda\longrightarrow\mathbb{R}^n$$

where $$\Phi(x,\lambda) = p\Big((x,Ax,\ldots,A^{q-1}x)+W(x,Ax,\ldots,A^{q-1}x,\lambda)\Big)$$

and p is the projection of the first set of n co-ordinates

$$p(x_1,\ldots,x_q) = x_1$$

such that if $$g(x,\lambda) = 0$$

then $$f^q\Big(\Phi(x,\lambda),\lambda\Big) = \Phi(x,\lambda).$$

To show that the isotropy subgroup $\Sigma_x\subset\Gamma\times Z_q$ of x a zero of g is isomorphic to the symmetry group of the corresponding period q point of f we proceed as follows.

If $(\gamma,c)\in\Sigma_x$, i.e.

$$\gamma.A^c x = x.$$

we need to show that $(\gamma,c)\in\Sigma_{\Phi(x,\lambda)}$ where $\Sigma_{\Phi(x,\lambda)}$ is the symmetry group of the period q point $\Phi(x,\lambda)$ of f, i.e.

$$f^c\Big(\gamma.\Phi(x,\lambda),\lambda\Big) = \Phi(x,\lambda).$$

As f is Γ equivariant

$$f^c\left(\gamma.\Phi(x,\lambda),\lambda\right) = \gamma.f^c\left(\Phi(x,\lambda),\lambda\right).$$

Furthermore due to the $\mathbf{Z}_q$ equivariance of F and W and the forms of ker $\mathcal{L}$ and coker $\mathcal{L}$ we have

$$f\left(\Phi(x,\lambda),\lambda\right) = \Phi(Ax,\lambda)$$

or more generally

$$f^c\left(\Phi(x,\lambda),\lambda\right) = \Phi(A^c x,\lambda).$$

Hence

$$\begin{aligned} f^c\left(\gamma.\Phi(x,\lambda),\lambda\right) &= \gamma.f^c\left(\Phi(x,\lambda),\lambda\right) \\ &= \gamma.\Phi(A^c x,\lambda). \end{aligned}$$

Thus we need to show is that Φ is Γ-equivariant.

By definition

$$\Phi(y,\lambda) = p\left((y,Ay,\ldots,A^{q-1}y)+W(y,Ay,\ldots,A^{q-1}y,\lambda)\right)$$

and since the action of γ on $\mathbb{R}^{nq}$ is diagonal and W is Γ equivariant then

$$\gamma.\Phi(y,\lambda) = p\left((\gamma.y,\gamma.Ay,\ldots,\gamma.A^{q-1}y)+W(\gamma.y,\gamma.Ay,\ldots,\gamma.A^{q-1}y,\lambda)\right).$$

As we saw earlier the action of Γ commutes with that of A, hence

$$\begin{aligned} \gamma.\Phi(y,\lambda) &= p\left((\gamma.y,A\gamma.y,\ldots,A^{q-1}\gamma.y)+W(\gamma.y,A\gamma.y,\ldots,A^{q-1}\gamma.y,\lambda)\right) \\ &= \Phi(\gamma.y,\lambda). \end{aligned}$$

Thus

$$f^c\left(\gamma.\Phi(x,\lambda),\lambda\right) = \Phi(\gamma.A^c x,\lambda)$$

but (γ,c) is in the isotropy subgroup of x, that is

$$\gamma.A^c x = x$$

and hence

$$f^c\left(\gamma.\Phi(x,\lambda),\lambda\right) = \Phi(x,\lambda).$$

So if (γ,c) is in the isotropy subgroup of x then (γ,c) is in the symmetry group of $\Phi(x,\lambda)$ the corresponding period q point of f.

Thus it remains only to show that the only eigenvalue of $(dg)_{0,0}$ is at 0. We know that for a solution x of $g(x,\lambda)=0$ there is a corresponding period q point $\Phi(x,\lambda)$ of f, that is

$$f^q(\Phi(x,\lambda),\lambda) = \Phi(x,\lambda).$$

We use the chain rule on the identity $g(x,\lambda)=f^q(\Phi(x,\lambda),\lambda)-\Phi(x,\lambda)$ to find

$$(dg)_{x,\lambda} = (df)_{f^{q-1}(\Phi(x,\lambda),\lambda),\lambda}\circ\cdots\circ(df)_{\Phi(x,\lambda),\lambda}\circ(d\Phi)_{x,\lambda} - (d\Phi)_{x,\lambda}$$

hence at 0,0 we have

$$(dg)_{0,0} = (df)_{f^{q-1}(\Phi(0,0),0),0}\circ\cdots\circ(df)_{\Phi(0,0),0}\circ(d\Phi)_{0,0} - (d\Phi)_{0,0}$$

but by the generalities of the reduction $\Phi(0,0)=0$ and hence

$$(dg)_{0,0} = ((df)_{0,0})^q(d\Phi)_{0,0} - (d\Phi)_{0,0}.$$

However we have already seen that in some coordinate system

$$(df)_{0,0} = \begin{bmatrix} \cos(2\pi p/q)I_m & -\sin(2\pi p/q)I_m \\ \sin(2\pi p/q)I_m & \cos(2\pi p/q)I_m \end{bmatrix}$$

so $((df)_{0,0})^q=I_n$ and $(dg)_{0,0}=0$ thus all the eigenvalues of $(dg)_{0,0}$ are 0.

This theorem means that if we have a map f with symmetry group Γ for which we are looking for period q points then we can equally look at the bifurcation problem g which is $\Gamma \times \mathbf{Z}_q$ equivariant. Hence we can get a stronger result than that of theorem 6.1.1 by looking at the bifurcation problem g.

Corollary 6.3.5.

Let $\mathbb{R}^n$ be Γ-simple, f be a two parameter Γ-equivariant map

$$f:\mathbb{R}^n\times\Lambda\rightarrow\mathbb{R}^n$$

with a fixed point at 0 for all $\lambda = (\lambda_1,\lambda_2)\in\Lambda$.

We assume that $(df)_{0,\lambda_1,\lambda_2}$ has eigenvalues $r(\lambda_1,\lambda_2)e^{\pm i\theta(\lambda_1,\lambda_2)}$ (each of multiplicity $m=n/2$) where $r(0,0)=1$, $\theta(0,0)=2\pi p/q$ and $q \neq 1, 2, 3, 4$. Let Σ be an isotropy subgroup of $\Gamma\times\mathbf{Z}_q$ with dim fix$\Sigma=2$, then in parameter space Λ there exists some region near to $\lambda = (0,0)$ within which there exists two branchs of period q orbits of f whose symmetry group is precisely Σ.

Remark

(i) The cases of $q = 1$ and 2 have been discussed in chapter 3, however when $q = 3$ or 4 the proof is greatly complicated by the appearance of low order terms in the normal form of g and these will not be discussed.

(ii) Since $\Gamma\times Z_q \subset \Gamma\times S^1$ and as the conditions above satisfy the requirements of the Hopf theorem with symmetry (theorem 5.3.1) we also expect an invariant circle of solutions which are tangent to $\mathrm{Fix}(\Sigma)$ at $x=0$ and since this invariant circle is unique (at least up to conjugacy) then the periodic solutions must lie on the unit circle.

Proof

To prove the corollary we need only show there exists a region of parameter space near to $\lambda=0$ within which two solution branches for $g(x,\lambda)=0$ bifurcating from the fixed point at zero, which have isotropy subgroup Σ. Then applying theorem 6.3.1 this implies that there are also two branches of period q points of f with symmetry group Σ.

Since we are looking for solutions with isotropy subgroup Σ we need only consider $g|\,\mathrm{Fix}(\Sigma)$ the map g restricted to the fixed point subspace of Σ. Furthermore by lemma 3.2.2 $\mathrm{Fix}(\Sigma)$ is mapped by g onto itself, thus we need only consider the map

$$g:\mathrm{Fix}(\Sigma) \to \mathrm{Fix}(\Sigma).$$

As $\mathrm{Fix}\Sigma$ is 2-dimensional and g is $\mathbf{Z}_q$-equivariant we can choose a co-ordinate sysytem (α,β) (with basis (v_1,v_2) in $\mathrm{Fix}\Sigma$) such that the $\mathbf{Z}_q$ action on (α,β) is given by the multiplication of the vector (α,β) by

$$R_{2\pi p/q} = \begin{bmatrix} \cos 2\pi p/q & -\sin 2\pi p/q \\ \sin 2\pi p/q & \cos 2\pi p/q \end{bmatrix}$$

we associate this with a mapping of the complex plane

$$\tilde{g}:\mathbb{C}\times\Lambda \to \mathbb{C}$$

where $z=\alpha+i\beta$and the rotation is now via multiplicatioon by $e^{2\pi i p/q}$.

The general form of a Z_q equivariant map from $\mathbb{C}$ to $\mathbb{C}$ is

$$f(z,\lambda)=P(z\bar{z}, z^q,\bar{z}^q,\lambda)z+iQ(z\bar{z}, z^q,\bar{z}^q,\lambda)z+R(z\bar{z}, z^q,\bar{z}^q,\lambda)\bar{z}^{q-1}+iS(z\bar{z}, z^q,\bar{z}^q,\lambda)\bar{z}^{q-1}.$$

where P,Q,R,S are real polynomials.

Expanding $\bar{g}$ around $(0,\lambda)$ we have

$$\bar{g}(z,\lambda)=P(0,0,0,\lambda)z+iQ(0,0,0,\lambda)z+P_{z\bar{z}}(0,0,0,\lambda)z^2\bar{z}+iQ_{z\bar{z}}(0,0,0,\lambda)z^2\bar{z}$$
$$+R(0,0,0,\lambda)\bar{z}^{q-1}+iS(0,0,0,\lambda)\bar{z}^{q-1}+O(|z|^5).$$

We need to include these two higher order terms in our expansion in order to observe the occurence of the periodic solutions.

We are looking for solutions of $\bar{g}(z,\lambda)=0$ which correspond to solutions of $g(\alpha v_1+\beta v_2,\lambda) = 0$ where $\{v_1,v_2\}$ is the suitable basis choice for $\mathrm{Fix}\Sigma$.

Close to the bifurcation point these can be approximated by solutions of the truncated expansion

$$\bar{g}(z,\lambda) = (\tau+i\sigma)z+(P_N+iQ_N)z^2\bar{z}+(R+iS)\bar{z}^{q-1} = 0$$

where $\tau=P(0,0,0,\lambda)$, $\sigma=Q(0,0,0,\lambda)$, $P_N=P_{z\bar{z}}(0,0,0,\lambda)$, $Q_N=Q_{z\bar{z}}(0,0,0,\lambda)$, $R=R(0,0,0,\lambda)$ and $S=S(0,0,0,\lambda)$ (the $\bar{z}^{q-1}$ term is required in this truncation in order to reflect the Z_q equivariance of g).

Letting $z=re^{i\theta}$ and multiplying through by $\bar{z}$ gives us

$$(\tau+i\sigma)r^2+(P_N+iQ_N)r^4+(R+iS)r^qe^{-iq\theta} = 0$$

further if $R+iS=se^{i\varphi}$ then we end up looking for solutions to

$$(\tau+i\sigma)r^2+(P_N+iQ_N)r^4+sr^qe^{i(\varphi-q\theta)} = 0$$

which is equivalent to solving the following real pair of simultaneous equations

$$\tau r^2+P_Nr^4+sr^q\cos(\varphi-q\theta) = 0$$

and $$\sigma r^2+Q_Nr^4+sr^q\sin(\varphi-q\theta) = 0.$$

Clearly one solution is given by $r=0$ which corresponds to the fixed point of f at 0. Therefore the solution corresponding to the period q points we are looking for is the solution to the following

$$\text{(a) } \tau+P_Nr^2+sr^{q-2}\cos(\varphi-q\theta) = 0$$

and $$\text{(b) } \sigma+Q_Nr^2+sr^{q-2}\sin(\varphi-q\theta) = 0.$$

To solve this pair of simultaneous equations we first note that, since $(dg)_{0,0}$ has all its eigenvalues equal to zero then for λ small τ and σ and will also be small. From equation (a) we see that any solution must be of the form

$$r = (-\tau/P_N)^{1/2}+O(\tau^2),$$

truncating at $O(\tau^2)$ and putting it into (b) we need to satisfy

$$\sigma-Q_N\tau/P_N+s(-\tau/P_N)^{(q-2)/2}\sin(\varphi-q\theta) = 0$$

but since $\sin(\varphi-q\theta)\in[-1,1]$ this can only have a solution if

$$|\sigma-Q_N\tau/P_N| \leq s(-\tau/P_N)^{(q-2)/2}$$

hence solutions exist in the region defined approximately by the boundary lines

$$\sigma \simeq Q_N\tau/P_N \pm s(-\tau/P_N)^{(q-2)/2}.$$

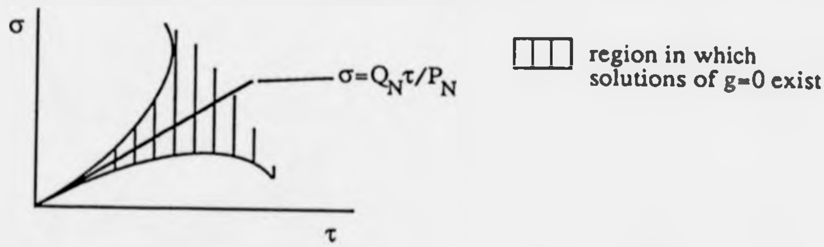


fig 6.1 the region of existence to solutions of g=0

Within this region of (τ,σ) space there are two distinct solutions for $\theta\in[0,2\pi/q)$ (if θ is a solution then so is $((2\varphi-\pi)/q)-\theta$). By applying the implicit function theorem when higher order terms are re-introduced we expect to see solutions which are close to those above within a region which is approximately the same.

So we have a diagram within whose outer boundaries solutions to $\bar{g}(z,\lambda)=0$ exist and these correspond to the period q points of f required. Since $\tau=P(0,\lambda)$ and $\sigma=Q(0,\lambda)$ we can calculate λ in terms of τ and σ (provided $(d_\lambda P)$ and $(d_\lambda Q)$ are non singular). Thus if we let $\lambda(\tau,\sigma)$ solve $P(0,\lambda(\tau,\sigma))=\tau$ and $Q(0,\lambda(\tau,\sigma))=\sigma$ then the region in parameter space for which periodic solutions exist is determined by the boundary

$$\lambda(\tau, Q_N\tau/P_N \pm s(-\tau/P_N)^{(q-2)/2})$$

6.4 The relationship between f and g when f is in normal form.

For the rest of this section we will assume f has been reduced to the centre manifold and that the eigenspace corresponding to the eigenvalue $e^{2\pi i p/q}$ is Γ-simple, hence there exists a coordinate system in which f commutes with A as defined above. We have seen in chapter 4 that there is a normal form for f whose non-linear part truncated to any order can be chosen to commute with A^t. However A is orthogonal thus $A^t = A^{-1}$ and hence the whole of f will commute with A.

In this section we will show that when f is in normal form, to degree k, then there is a simple formula connecting f and g, the Liapunov Schmidt reduced form of F from section 3. This will be a useful aid when we come to study the stability of f in section 5 since the stability of the zeros of g are the same generically as the corresponding period q points of f.

Theorem 6.4.1.

If f is in normal form to degree k then

$$j_k g(x) = A^{q-1}(j_k f(x) - Ax)$$

where $j_k f$ is the k-jet of f, the Taylor series expansion of f truncated at the kth order. Furthermore if x is a zero of g then it is a period q point of f.

Proof

In normal form

$$j_k f(x) = Ax + h_2(x) + \ldots + h_k(x)$$

where h_i is a homogeneous polynomial of degree i which commutes with A^t (hence also with A as $A^{-1}=A^t$). In this proof we shall refer to j_kf simply as f. We wish to find g, the reduced form of F, where

$$F(x_1,\ldots,x_q,\lambda) = \left(f(x_q)-x_1,\ldots,f(x_{q-1})-x_q\right)$$

and $(x_1,\ldots,x_q) = v+W(v,\lambda)$ where $v\in\ker\mathcal{L}$ and $W\in\operatorname{corange}\mathcal{L}$.

To help us we first state and prove a small lemma

Lemma 6.4.2.

Let $W(v,\lambda)$ be the function as defined in the proof of theorem 6.3.2 on f, the normal form truncated at order k, then $W(v,\lambda)=0$.

Proof

We quickly remind ourselves of where W comes from. It is found by applying the implicit function theorem to solve $EF(v+w,\lambda)=0$ for w in terrms of v and λ. So we need to show that $w=0$ is a solution (and hence the only one), i.e.

$$EF(v,\lambda) = 0.$$

This means $F(v,\lambda)\in\ker E = \operatorname{coker}\mathcal{L}$ by the definition of E. Now $v\in\ker\mathcal{L}$ and hence will be of the form

$$v = (x,Ax,\ldots,A^{q-1}x)$$

for some $x\in\mathbb{R}^n$ and so

$$\begin{aligned} F(v,\lambda) &= \left(f(A^{q-1}x)-x,\ldots,f(A^{q-2}x)-A^{q-1}x\right) \\ &= \left(A^{q-1}(f(x)-Ax),\ldots,A^{q-2}(f(x)-Ax)\right) \end{aligned}$$

by the A equivariance of f. Clearly this lies in $\operatorname{coker}\mathcal{L}$ and we have proved the lemma.

Returning to the proof of the theorem. We now have

$$\varphi:\ker\mathcal{L}\times\Lambda\to\operatorname{coker}\mathcal{L}$$

defined by $$\varphi(v,\lambda) = (I_{\mathbb{R}^{nq}}-E)F(v,\lambda)$$

where E is the projection which has as it's kernal coker$\mathcal{L}$ and

$$\begin{aligned} F(v,\lambda) &= F(x,Ax,\ldots,A^{q-1}x,\lambda) \\ &= \left(f(A^{q-1}x)-x,\ldots,f(A^{q-2}x)-A^{q-1}x\right) \\ &= \left(A^{q-1}(f(x)-Ax),\ldots,A^{q-2}(f(x)-Ax)\right). \end{aligned}$$

Thus $F(v,\lambda)\in$ coker L so $EF(v,\lambda)=0$ and hence

$$\varphi(v,\lambda)=F(v,\lambda)$$

Now

$$g(x,\lambda)=p\left(\varphi(x,Ax,\ldots,A^{q-1}x,\lambda)\right)$$

where p is the projection of the first set of n co-ordinates of $\mathbb{R}^{nq}$

$$p:\mathbb{R}^{nq}\longrightarrow\mathbb{R}^{n}$$

$$p(x_1,\ldots,x_q,\lambda)=x_1$$

and so along the period q branches of f we have

$$g(x,\lambda)=A^{q-1}(f(x)-Ax).$$

We finally note from theorem 6.3.1 that there is a map

$$\Phi:\mathbb{R}^n\times\Lambda\longrightarrow\mathbb{R}^n$$

where $\Phi(x,\lambda) = p\left((x,Ax,\ldots,A^{q-1}x)+W(x,Ax,\ldots,A^{q-1}x,\lambda)\right)$

which maps the zeros of g onto period q points of f and since W=0 along the solution branches then

$$\begin{aligned} \Phi(x,\lambda) &= p(x,Ax,\ldots,A^{q-1}x) \\ &= x \end{aligned}$$

so the zeros of g are precisely the period q points of f.

This means that if f is in normal form then for a period q point x of f

$$f(x) = A(g(x)+x)$$

and since x is a zero of g then

$$f(x) = Ax.$$

6.5 Asymptotic Orbital Stability of Period q points of f.

Next we will be discussing the stability of period q points of a function f which is Γ-equivariant.Firstly we will explain why the normal definition of asymptotic stability of a fixed point fails for a fixed point of a map which is possessed of symmetry,thus leading us to the definition of orbital asymptotic stability. Then we will explain how to adapt this definition to describe the stability of a period q point of the map.

Finally we show how the stability of a period q orbit of f is related to that of the corresponding equilibrium solution of

$$dx/dt=g(x,\lambda)$$

where g is the Liapunov Schmidt reduced function.

Definition6.5.1

A fixed point x_0 of a map f is asymptotically stable if for every point x near to x_0 $f^n(x,\lambda)$ stays near to x_0 for all n, and also $\lim_{n\to\infty} f^n(x,\lambda)=x_0$. It is neutrally stable if $f^n(x,\lambda)$ stays near to x_0 for all n and it is unstable if there are points arbitrarily close to x_0 whose trajectories $f^n(x,\lambda)$ do not stay near x_0 for all n.

We also have another set of stability criteria known as linear stability conditions, which are equivalent to those above. If all the eigenvalues of $(df)_{x_0,\lambda}$ lie within the unit circle centre the origin in $\mathbb{C}$ then x_0 is asymptotically stable. Moreover, if any one lies outside this circle then x_0 is unstable, it is however not true to say that if an eigenvalue lies on the unit circle then the fixed point is neutrally stable.

However when we introduce symmetry into the system these conditions are not adequate. Consider a Γ equivariant system in which a fixed point x_0 has as its isotropy subgroup Σ of lower dimension than Γ. If this is the case then there are points in the orbit Γx_0 arbitrarily close to x_0 (which by the Γ-equivariance of f are also fixed points). Thus the trajectory of these points in the limit do not tend to x_0 and hence x_0 cannot be asymptotically stable, however it can be neutrally stable which leads us to make the following definition.

Definition 6.5.2.

A fixed point x_0 is asymptotically orbitally stable if it is neutrally stable and for x starting near to x_0 then $\lim_{n\to\infty} f^n(x,\lambda)$ exists and lies in Γx_0.

The linear criterion for orbital stability is not the same as for normal asymptotic stability as the symmetry often forces eigenvalues to lie on the unit circle. For example consider

$$y(t) = \gamma(t)x_0$$

where $\gamma(t)$ is a smooth curve in Γ and $\gamma(0)=I_n$, now as x_0 is a fixed point then so is $y(t)$ because

$$\begin{aligned} f(y(t),\lambda) &= f(\gamma(t)x_0,\lambda) \\ &= \gamma(t)f(x_0,\lambda) \\ &= \gamma(t)x_0 \\ &= y(t). \end{aligned}$$

We have

$$\frac{d}{dt}\Big(f(y(t),\lambda)\Big)\Big|_{t=0} = (df)_{x_0}\Big(\frac{d}{dt}(\gamma(t))\Big|_{t=0} x_0\Big)$$

but also, since $f(y(t),\lambda)=y(t)$

$$\frac{d}{dt}\Big(f(y(t),\lambda)\Big)\Big|_{t=0} = \frac{d}{dt}(\gamma(t))\Big|_{t=0} x_0$$

which gives

$$(df)_{x_0}\Big(\frac{d}{dt}\gamma(t)\Big|_{t=0} x_0\Big) = \frac{d}{dt}\gamma(t)\Big|_{t=0} x_0$$

hence $d\gamma/dt(0)x_0$ is an eigenvector of $(df)_{x_0}$ with eigenvalue 1 and x_0 cannot be linearly asymptotically stable.

To overcome this problem we introduce the following definition.

Definition 6.5.3.

If x_0 is a fixed point of a Γ-equivariant map f then it is linearly orbitally stable if those eigenvalues of $(df)_{x_0,\lambda}$ other than those arising from $T_{x_0}\Gamma x_0$ (i.e. the eigenvectors of $(df)_{x_0}$ which lie on the unit circle) lie within the unit circle.

There is an equivalent result to that for non symmetric stability relating orbital stability and linear orbital stability which we state and prove below.

Theorem 6.5.4.

Let x_0 be a fixed point of a function f. If x_0 is linearly orbitally stable then it is also asymptotically orbitally stable.

Proof.

We first consider the solution to the linear difference equation

$$g(x,\lambda)=df_{(x_0,\lambda)}\,x.$$

Given $x\in\mathbb{R}^n$ it can be written in the form

$$x=y+w$$

where $y\in T_{x_0}\Gamma x_0$ and $w\in W$, the complimentary subspace of $T_{x_0}\Gamma x_0$ in $\mathbb{R}^n$, which is in fact the sum of the real parts of the generalised eigenspaces associated to the remaining eigenvalues. From the assumption of orbital linear stability these all lie within the unit circle in $\mathbb{C}$.

We now consider the map

$$h:T_{x_0}\Gamma x_0\times W\rightarrow T_{x_0}\Gamma x_0\times W$$

given by

$$h\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} y \\ Bw \end{bmatrix}$$

where B is a matrix with eigenvalues whose moduli are strictly less than 1 and which is equivalent to g (in the sense that if x=y+w then g(x)=h(y+w)). So

$$h^n\begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} y \\ B^n w \end{bmatrix}$$

but

$$| B^n w| \le | \lambda_{max}|^n | w|$$

and as the eigenvalues of B have modulus strictly less than 1 then $| \lambda_{max}| < 1$. Hence in the limit as $n \longrightarrow \infty$ $| \lambda_{max} |^n \longrightarrow 0$ and $B^n w \longrightarrow 0$. Thus in the limit $h^n(y,w)$ and consequently $g^n(y+w)$ tend to $y \in T_{x_0}\Gamma x_0$.

We have hence shown the theorem to be true for any linear function. Next we consider how this result is effected by the re-introduction of the non linear part of f. We do this using a result of Hirsch *et al* [1977].

Theorem

Let $V \subset M$ be a compact C^1 submanifold and $f:(M,V) \rightarrow (M,V)$ a diffeomorphism which is normally hyperbolic along V. Then there exists a neighbourhood A of V on which $W^s{}_A(f)$ and $W^u{}_A(f)$, the stable and unstable manifolds of f, intersect in V, tangent along V to $N_- \oplus TV$ and $N_+ \oplus TV$ respectively (N_- and N_+ are the stable and unstable real generalised eigenspaces of (df) along V). If $x \in W^s{}_A(f)$, there exists $y \in V$ such that $\lim_{n \to \infty} d(f^n x, f^n y) = 0$. If $x \in W^u{}_A(f)$, there exists $z \in V$ such that $\lim_{n \to \infty} d(f^{-n}x, f^{-n}z) = 0$.

The normal hyperbolicity of Γx_0 is ensured from the earlier part of the proof since g, the linear part of f contracts W to zero whilst leaving $T_{x_0}\Gamma x_0$ fixed. Furthermore since non of the eigenvalues are greater than 1 then $N_+ = \{0\}$ and so $W^u{}_A(f) = \{0\}$. Thus given $x \in A$ then $x \in W^s{}_A(f)$ and therefore there exists $y \in \Gamma x_0$ such that $\lim_{n \to \infty} d(f^n x, f^n y) = 0$ but as $f(y) = y$ we get $\lim_{n \to \infty} f^n(x) = y$ and x_0 is asymptotically orbitally stable.

We extend the definition of asymptoptic orbital stability for period q points as follows.

Definition 6.5.5.

A period q point x_0 of a function f is asymptotically orbitally stable if x_0 is an asymptotically orbitally stable fixed point of f^q in the sense of the previous definition.

We now state a lemma which gives the linear conditions required for a period q point of f to be asymptotically orbitally stable.

Lemma 6.5.6.

Let f be in exactnormal form and assume that it commutes with A as defined in the previous section, if x_0 is a period q point of f then x_0 is an orbitally stable period q point of f if the eigenvalues of $(df)_{x_0,\lambda}$ which are not forced be the symmetry to equal one have modulus stricly less than 1.

Proof.

From above x_0 is an orbitally stable period q point of f if it is an orbitally stable fixed point of $f^q(x,\lambda)$.

Applying the chain rule repeatedly we have

$$(df^q)_{x_0,\lambda} = (df)_{f^{q-1}(x_0,\lambda),\lambda} \cdots (df)_{x_0,\lambda}$$

but since f is in normal form and using theorem 6.4.1, $f(x_0) = Ax_0$ thus

$$(df^q)_{x_0,\lambda} = (df)_{A^{q-1}x_0,\lambda} \cdots (df)_{x_0,\lambda}.$$

Since f commutes with A we have $f(Ax,\lambda)=Af(x,\lambda)$, differentiating this gives

$$(df)_{Ax,\lambda}A = A(df)_{x,\lambda}$$

or

$$(df)_{Ax,\lambda} = A(df)_{x,\lambda}A^{-1}$$

and since A commutes with f and thus (df) we get

$$(df)_{Ax,\lambda} = (df)_{x,\lambda}$$

so

$$(df^q)_{x_0,\lambda} = [\,(df)_{x_0,\lambda}\,]^q.$$

Hence the stability problem for x_0 has been reduced to looking at the eigenvalues of $[\,(df)_{x_0,\lambda}\,]^q$. If $(df)_{x_0,\lambda}$ has eigenvalues $\sigma_1,\ldots,\sigma_s$ within the unit circle and $\chi_1,\ldots,\chi_c$ on the unit circle then $[\,(df)_{x_0,\lambda}\,]^q$ has eigenvalues $\sigma_1{}^q,\ldots,\sigma_s{}^q$ which must be within the unit circle and $\chi_1{}^q,\ldots,\chi_c{}^q$ which will lie on the unit circle (these are in fact the eigenvalues corresponding to the tangent space $T_{x_0}(\Gamma\times\mathbf{Z}_q)x_0)$. Thus applying theorem 6.5.4 we have that x_0 is an asymptotically orbitally stable fixed point of f^q and thus an asymptotically orbitally stable period q point of f.

In normal form we are able to write g explicitly in terms of f. So the next obvious step is to ask how the stability of f relates to that of g.

We have seen already that generically the period q bifurcations of f have corresponding steady state solutions of the system

$$dx/dt=g(x,\lambda) \qquad\qquad (*).$$

However it is not true that the stability of period q points of f correspond to the stability of steady states of (*). We start by showing the relationship between the eigenvalues of f and g and then go on to discuss the consequences of this relationship to the stabilities of f and g.

Lemma 6.5.7.

Let f be in normal form and Ω_g be the spectrum of $(dg)_{x,\lambda}$ i.e.

$$\Omega_g=(\omega_i\in\mathbb{C}:dg_{(x,\lambda)}v=\omega_i v \text{ where } v\in V \text{ is non zero})$$

then the spectrum of $(df)_{x,\lambda}$ is given by

$$\Omega_f=(e^{2\pi ip/q}(\omega_i+1)\mid\omega_i\in\Omega_g).$$

Proof

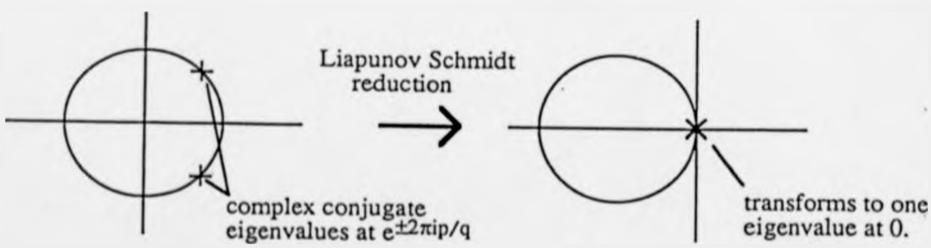
The proof of this lemma is straightforward if we use theorem 6.4.1 to note that in normal form

$$g = A^{q-1}f - I$$

or

$$f = A(g+I).$$

Consider the two differing stability crierion for the map f and for the differential equation $dx/dt=g$. For a map a solution becomes unstable when one of the eigenvalues of its linear part passes out of the unit circle, for a differential equation however instability occurs when the eigenvalue goes from having negative to positive real parts. The lemma shows that an eigenvalue of f passing through the unit circle (at $e^{2\pi ip/q}$) transforms to one passing through the origin for g, but the problem occurs because although at the origin the sphere is transformed by the reduction to be tangent to the imaginary axis thus



the eigenvalue may emerge in such a way as to lie inbetween the outside of the circle (unstable period q point) and the imaginary axis (stable equilibrium of g).

What we can say then is that if x_0 is an unstable equilibrium of g then it is also an unstable period q point of f and if x_0 is a stable period q point of f then it is also a stable equilibrium of g.

6.6 EXAMPLE: $\Gamma = D_m$, $n=4$.

Here we will be discussing what can be said about the system in just a group theoretic way we also will look at stability in the general case.

To start with let us remember how we define the action of D_m on $\mathbb{R}^4$, which we identify with $\mathbb{C}^2$. The group D_m has 2 generators, κ and ζ, where

$$\kappa^2 = 1,$$

$$\zeta^m = 1$$

and

$$\kappa\zeta = \zeta^{m-1}\kappa.$$

There is a standard action of D_m which is equivalent to every action on $\mathbb{C}^2$ by relabelling group elements and dividing through by the kernel of the action, it is given by the following:-

$$\kappa(z_1,z_2) = (z_2,z_1)$$

and

$$\zeta(z_1,z_2) = (e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2).$$

The action of $\mathbf{Z}_q$ is given by the action of τ the generator of the group, where

$$\tau^q = 1$$

and

$$\tau(z_1,z_2) = (e^{2\pi i p/q}z_1, e^{2\pi i p/q}z_2).$$

6.6.1 Isotropy Subgroups.

We wish to find the lattice of isotropy subgroups of $D_m \times \mathbf{Z}_q$. Upto conjugacy the isotropy subgroups of D_m are:-

<u>m even</u>

$$\begin{tikzcd}[column sep=small] & D_m \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbf{Z}_2(\kappa) \arrow[dr, dash] & & \mathbf{Z}_2(\zeta\kappa) \arrow[dl, dash] \\ & \mathbb{1} & \end{tikzcd}$$

<u>m odd</u>

$$\begin{tikzcd} D_m \arrow[d, dash] \\ \mathbf{Z}_2(\kappa) \arrow[d, dash] \\ \mathbb{1} \end{tikzcd}$$

where the element in the bracket is the generator of the $\mathbf{Z}_2$ isotropy subgroup. Note that

$Z_2(\kappa)$ and $Z_2(\zeta\kappa)$ are not conjugate in D_m when m is even but they are when m is odd.

For all m $Z_2(\kappa)$ is the isotropy subgroup of the fixed point space in which $z_1=z_2$ and for m even $Z_2(\zeta\kappa)$ is the isotropy subgroup of the fixed point space in which $z_1=e^{2\pi i/m}z_2$.

However when we look at the isotropy lattice for $D_m\times Z_q$ we get a somewhat different picture. Below we give a table of the isotropy subgroups of $D_m\times Z_q$ (upto conjugacy) including the corresponding fixed point spaces.

Table 6.1 The isotropy subgroups of the standard $D_m\times Z_q$ action.

	Isotropy subgroup	Fixed point space	dim Fix(Σ)
q odd, m odd			
	$Z_2((\kappa,1))$	$z_1=z_2=z$	2
	$\mathbf{1}$	$\mathbb{C}^2$	4
q even, m odd			
	$Z_2((\kappa,1))$	$z_1=z_2=z$	2
	$Z_2((\kappa,\tau^{q/2}))$	$z_1=-z_2=z$	2
	$\mathbf{1}$	$\mathbb{C}^2$	4
q odd,m even			
	$Z_2((\kappa,1))$	$z_1=z_2=z$	2
	$Z_2((\kappa\zeta,1))$	$z_1=e^{-2\pi i/m}z_2=z$	2
	$\mathbf{1}$	$\mathbb{C}^2$	4
m even, q even			
	$Z_2((\kappa,1))\times Z_2((\zeta^{m/2},\tau^{q/2}))$	$z_1=z_2=z$	2
	$Z_2((\kappa\zeta,1))\times Z_2((\zeta^{m/2},\tau^{q/2}))$	$z_1=e^{-2\pi i/m}z_2=z$	2
	$Z_2((\zeta^{m/2},\tau^{q/2}))$	$\mathbb{C}^2$	4

where the element in brackets is the generator of the group.

If the highest common factor of q and m $(q,m)\neq 1,2$ we get another conjugacy class of isotropy subgroups aswell. If $(m,q)=c$ it is given by

$$\mathbf{Z}_c((\zeta^{m/c},\tau^{-q/c}))$$

which has fixed point space $z_1=z$, $z_2=0$ of dimension 2.

Hence in the cases where the dimension of the fixed point space of the isotropy subgroup Σ is two we expect to see period q point bifurcations having precisely symmetry group Σ. Before going on to show how these reults are obtained we will give a brief description of the different modes of behaviour for the solutions in each isotropy subgroup.

Remark

If we look at the results of chapter 5 we see that for $\Sigma\subset D_n\times S^1$ with dim $\mathrm{Fix}(\Sigma)=2$ there exists under certain genericity conditions invariant circles of f which are tangent to $\mathrm{Fix}(\Sigma)$ at $x=0$. Hence we would expect to see periodic or quasiperiodic orbits of f in addition to those predicted above. So in the D_m example we have seen that there are always three isotropy subgroups, however the number of branches predicted by the period q theorem dependson m and q. For example if m is three and we are looking for a period five orbit then there are three branches of invariant circles on which we should see expect to see periodic or quasiperiodic solutions, yet the subharmonic theory predicts only one branch of periodic slolutions.

If $\Sigma=\mathbf{Z}_2((\kappa,1))$ or $\Sigma=\mathbf{Z}_2((\kappa\zeta,1))$ then solutions have no temporal symmetry and are simply of the form (z,z) or $(z,e^{2\pi i/m}z)$. For $\Sigma=\mathbf{Z}_2((\kappa,\tau^{q/2}))$ then the solutions will be of the form $(z,-z)$, with the additional temporal symmetry that $f^{q/2}(z,-z)=(-z,z)$. Solutions with isotropy subgroup $\Sigma=\mathbf{Z}_2((\kappa,1))\times\mathbf{Z}_2((\zeta^{m/2},\tau^{q/2}))$ will be of the form (z,z) with the temporal symmetry that $f^{q/2}(z,z)=(-z,-z)$. If $\Sigma=\mathbf{Z}_2((\kappa\zeta,1))\times\mathbf{Z}_2((\zeta^{m/2},\tau^{q/2}))$ solutions will take the form $(z,e^{2\pi i/m}z)$ they also have the additional temporal symmetry that $f^{q/2}(z,e^{2\pi i/m}z)=(-z,-e^{2\pi i/m}z)$. Finally for $\Sigma=\mathbf{Z}_c((\zeta^{m/c},\tau^{-q/c}))$ solutions are of the form $(z,0)$ which satisfy $f^{q/c}(z,0)=(e^{2\pi i/c}z,0)$. In all cases the temporal symmetry can be thought of as a discrete phase shift in the solution orbits.

To show these results we must consider how in general an element of $D_m \times \mathbb{Z}_q$ acts on $\mathbb{C}^2$, basically there are two different types of element to consider, namely

(i) (ζ^k, τ^j), $0 \leq k < m$ and $0 \leq j < q$which acts on (z_1, z_2) to give

$$(\zeta^k, \tau^j).(z_1, z_2) = (e^{2\pi i(pj/q+k/m)} z_1, e^{2\pi i(pj/q-k/m)} z_2)$$

and

(ii) $(\kappa\zeta^k, \tau^j)$, $0 \leq k < m$ and $0 \leq j < q$ which acts on (z_1, z_2) to give

$$(\kappa\zeta^k, \tau^j).(z_1, z_2) = (e^{2\pi i(pj/q-k/m)} z_2, e^{2\pi i(pj/q+k/m)} z_1).$$

For any of these elements to lie in an isotropy subgroup of $\mathbb{C}^2$ we need

$$(\gamma, r).(z_1, z_2) = (z_1, z_2)$$

cases (i) and (ii) are considered seperately.

(i) We must satisfy the two equations

$$z_1 = e^{2\pi i(pj/q+k/m)} z_1$$

and $$z_2 = e^{2\pi i(pj/q-k/m)} z_2$$

in the region $0 \leq j < q$, $0 \leq k < m$. Upto conjugacy there are three cases to consider:

(a) $z_1 \neq 0$, $z_2 \neq 0$,

(b) $z_1 \neq 0$, $z_2 = 0$ and

(c) $z_1 = z_2 = 0$.

For (a) we need

$$e^{2\pi i(pj/q+k/m)} = e^{2\pi i(pj/q-k/m)} = 1$$

that is $$pj/q+k/m = s_1$$

and $$pj/q-k/m = s_2$$

for $s_1, s_2 \in \mathbb{Z}$.

In order that we are not simply considering the trivial solutions (when $j=k=0$) and to lie in the region of interest we can assume without loss of generality that $s_1=1$, $s_2=0$ and $p=1$ (the last assumption derives from the fact that p and q are coprime so there exist

j' with $pj'=1$). So we are looking for the solutions of the following:

$$j/q+k/m = 1$$

and $$j/q-k/m = 0.$$

These together imply that $j=q/2$ and $k=m/2$, so for q and m both even we have the subgroup $\mathbf{Z}_2((\zeta^{m/2},\tau^{q/2}))$ (this may not be the whole of the isotropy subgroup) and for any (z_1,z_2)

$$(\zeta^{m/2},\tau^{q/2}).(z_1,z_2) = (e^{2\pi i}z_1,e^0z_2) = (z_1,z_2)$$

so its fixed point space is the whole of $\mathbb{C}^2$ and has dimension 4.

For (b) we must have

$$e^{2\pi i(pj/q+k/m)} = 1$$

that is $$pj/q+k/m = s_1$$

for $s_1 \in \mathbf{Z}$.

Again in order that we are not simply considering the trivial solutions (that is when $j=k=0$) and to lie in the region of interest we can assume without loss of generality that $s_1=0$ and $p=1$ so the solutions we are looking for are the solutions of the following:

$$j/q+k/m = 0$$

i.e. $$jm = -kq.$$

Assuming that the highest common factor of m and q, $hcf(m,q)=c\neq 1$ (if $c=1$ the solution is trivial) then we have an isotropy subgroup $\mathbf{Z}_c((\zeta^{m/c},\tau^{-q/c}))$ and for any element of $\mathbb{C}^2$ of the form $(z_1,0)$

$$(\zeta^{m/c},\tau^{-q/c})^j.(z_1,0) = (e^{2\pi i(-qj/cq+mj/cm)}z_1,0) = (z_1,0)$$

so the fixed point space is by $(z_1,0)$, $z_1 \in \mathbb{C}$, which has dimension 2.

For (c) we see that every element of this form fixes (0,0) so (0,0) has as part of its isotropy subgroup $\mathbf{Z}_m \times \mathbf{Z}_q$ (clearly it can be seen that the whole group fixes (0,0) so this point has as its isotropy subgroup $D_m \times \mathbf{Z}_q$).

We have now completed type (i) elements. For (ii), elements of the form $(\kappa\zeta^k,\tau^j)$, $0 \leq k < m$ and $0 \leq j < q$, we need to satisfy

$$z_1=e^{2\pi i(pj/q-k/m)}z_2$$

and $$z_2=e^{2\pi i(pj/q+k/m)}z_1$$

in the region $0\le j<q$, $0\le k<m$. Again, upto conjugacy, there are three cases to consider:

(a) $z_1\neq 0\neq z_2$,

(b) $z_2\neq 0$, $z_1=0$ and

(c) $z_1=z_2=0$.

For (a) we need to satisfy

$$e^{2\pi i(pj/q+k/m)}e^{2\pi i(pj/q-k/m)}=1$$

that is $$pj/q+k/m+pj/q-k/m = 2pj/q = s_1$$

for $s_1\in\mathbf{Z}$.

Again we can assume that $p=1$ and that s_1 is either 0 or 1. If $s_1=0$ then j must be zero. Now for any value of k, $0\le k<m$, consider how elements of the form $(\kappa\zeta^k,1)$ act

$$(\kappa\zeta^k,1).(z_1,z_2)=(e^{-2\pi ik/m}z_2,e^{2\pi ik/m}z_1).$$

Clearly for each k this fixes points of the space $(z,e^{2\pi ik/m}z)$, $z\in\mathbb{C}$ so the cyclic group of order 2 generated by $(\kappa\zeta^k,1)$, i.e.$\mathbf{Z}_2((\kappa\zeta^k,1))$ is a subgroup of, and possibly the whole of, the isotropy subgroup which fixes $(z,e^{2\pi ik/m}z)$, $z\in\mathbb{C}$ which has dimension 2.

We do not require however to look at all of these because each of these subgroups is conjugate to either $\mathbf{Z}_2((\kappa,1))$ or $\mathbf{Z}_2((\kappa\zeta,1))$ (and if m is odd $\mathbf{Z}_2((\kappa\zeta,1))$ is conjugate to $\mathbf{Z}_2((\kappa,1))$ and hence they all are).

When $s_1=1$ then j must equal q/2, so if q is odd we will get no extra information. If q is even however we will see other fixed point spaces.

For any value of k, $0\le k<m$, consider how elements of the form $(\kappa\zeta^k,\tau^{q/2})$ act

$$(\kappa\zeta^k,\tau^{q/2}).(z_1,z_2) = (e^{2\pi i(1/2-k/m)}z_2,e^{2\pi i(1/2+k/m)}z_1)$$
$$= (-e^{-2\pi ik/m}z_2,-e^{2\pi ik/m}z_1).$$

For each k this fixes points of the space $(z,-e^{2\pi ik/m}z)$ $z\in\mathbb{C}$ so the cyclic group of order 2 generated by $(\kappa\zeta^k,\tau^{q/2})$, i.e.$\mathbf{Z}_2((\kappa\zeta^k,\tau^{q/2}))$ is a subgroup (possibly the whole) of the isotropy subgroup which fixes $(z,-e^{2\pi ik/m}z)$, $z\in\mathbb{C}$ and hence has fixed point space of dimension 2.

Again we do not require however to look at all of these because each of these subgroups is conjugate to either $\mathbb{Z}_2((\kappa,\tau^{q/2}))$ or $\mathbb{Z}_2((\kappa\zeta\ ,\tau^{q/2}))$ (and if m is odd $\mathbb{Z}_2((\kappa\zeta,\tau^{q/2}))$ is conjugate to $\mathbb{Z}_2((\kappa,\tau^{q/2}))$ and hence they all are).

For case (b) when $z_1\neq 0$ and $z_2=0$ we have a siuation in which we are looking for solutions of

$$(e^{2\pi i(pj/q-k/m)}0, e^{2\pi i(pj/q+k/m)}z_1)=(z_1,0).$$

This implies that $z_1=0$ also which means we are just looking at case (c).

Finally for case (c) $z_1=z_2=0$ this point is fixed by all elements of $D_m\times Z_q$ as mentioned before and so we have completed the classification of isotropy subgroups of $D_m\times Z_q$ (upto conjugacy).

We now wish to look at the bifurcation directions and stability of each of the possible solutions which are derived from our result. This is done by looking at the eigenvalues of the linearisation of f^q along the solution branches and observing whether they lie inside or outside the unit circle in $\mathbb{C}$.

6.6.2 Equivariant theory for $D_m\times Z_q$.

As seen in chapter 3 section 2(a), symmetry puts restriction on the form of any map which commutes with the action of some group. Thus we can simplify the problem by first looking at the general form of a $D_m\times Z_q$ equivariant map. We have the following result.

Proposition 6.6.2.1.

Let n,q≥5. Then every smooth $D_m \times \mathbf{Z}_q$ equivariant map $f:\mathbb{C}^2 \rightarrow \mathbb{C}^2$ has the form:-

$$F(z_1, z_2) = \begin{bmatrix} h(|z_1|^2, |z_2|^2)z_1 + \alpha_{kln} z_2^k \bar{z}_1^l \bar{z}_2^n + R(z_1, z_2) \\ h(|z_2|^2, |z_1|^2)z_2 + \alpha_{kln} z_1^k \bar{z}_2^l \bar{z}_1^n + R(z_2, z_1) \end{bmatrix}$$

where $R(z_1,z_2)$ includes terms of order k+l+n+2 or more and k,l,n are given as follows:-

<u>(i) m,q both odd</u>

m>2q	k=0, l=q-1, n=q,	
2q>m≥q+2	k=(m-q)/2, l=(m+q)/2-1, n=0,	k=(m+q)/2, l=(m-q)/2-1, n=0,
m=q	k=n=0, l=q-1,	
2m>q≥m+2	k=0, l=(q+m)/2-1, n=(q-m)/2,	k=0, l=(q-m)/2-1, n=(q+m)/2,
q>2m	k=m, l=m-1, n=0.	

<u>(ii) at least one of m,q even</u>

m'>q'	k=0, l=q'/2-1, n=q'/2,	
m'=q'	k=0, l=q'/2-1, n=q'/2k=n=0, l=q'-1	k=q'/2, l=q'/2-1, n=0,
m'<q'	k=m'/2, l=m'/2-1, n=0	

where

$$m' = \begin{cases} m/2 & m \text{ even} \\ m & m \text{ odd} \end{cases}$$

and

$$q' = \begin{cases} q/2 & q \text{ even} \\ q & q \text{ odd.} \end{cases}$$

In addition the lowest order term, when hcf(m,q)=c≠1,2, containing just z_1 terms is $\beta\bar{z}_1^{qm/c-1}$ (this is important in the case when fixΣ is of the form (z,0)).

To help us to prove this theorem we state a lemma which will reduce the complexity of the problem.

Lemma 6.6.2.2.

If $f:\mathbb{C}^2 \to \mathbb{C}^2$ is $D_m \times Z_q$ equivariant under the action described above then

$$f(z_1,z_2) = \big(g(z_1,z_2), g(z_2,z_1)\big) \qquad (*)$$

where $g:\mathbb{C}^2 \to \mathbb{C}$ is $Z_m \times Z_q$ equivariant and where $Z_m \times Z_q$ acts on $\mathbb{C}^2$ as $Z_m \times Z_q \subset D_m \times Z_q$ and on $\mathbb{C}$, Z_m acts by multiplication by $e^{2\pi i/m}$ and Z_q acts by multiplication by $e^{2\pi ip/q}$.

<u>Proof</u>

Firstly note that for f to be equiariant under the action of the flip, $\kappa \subset D_m$, then

$$f(\kappa(z_1,z_2)) = \kappa f(z_1,z_2) = \kappa(f_1(z_1,z_2), f_2(z_1,z_2))$$

that is $$f(z_2,z_1) = (f_1(z_2,z_1), f_2(z_2,z_1)) = (f_2(z_1,z_2), f_1(z_1,z_2))$$

or $$f_1(z_1,z_2) = f_2(z_2,z_1)$$

and vica versa. Hence f must be of the form $(*)$.

Next we look at how the actions of $\zeta \subset D_m$ and $\tau \subset Z_q$ will reduce from $\mathbb{C}^2$ to $\mathbb{C}$. The ζ equivariance implies that

$$f(\zeta(z_1,z_2)) = \zeta f(z_1,z_2) = \zeta(f_1(z_1,z_2), f_2(z_1,z_2))$$

that is $$\begin{aligned} f(e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2) &= (f_1(e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2), f_2(e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2)) \\ &= (e^{2\pi i/m}f_1(z_1,z_2), e^{-2\pi i/m}f_2(z_1,z_2)) \end{aligned}$$

or $$f_1(e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2) = e^{2\pi i/m}f_1(z_1,z_2)$$

thus ζ must act as multiplication by $e^{2\pi i/m}$ on $\mathbb{C}$.

Likewise τ equivariance implies that

$$f(\tau(z_1,z_2)) = \tau f(z_1,z_2) = \tau(f_1(z_1,z_2), f_2(z_1,z_2))$$

that is $$\begin{aligned} f(e^{2\pi ip/q}z_1, e^{2\pi ip/q}z_2) &= (f_1(e^{2\pi ip/q}z_1, e^{2\pi ip/q}z_2), f_2(e^{2\pi ip/q}z_1, e^{2\pi ip/q}z_2)) \\ &= (e^{2\pi ip/q}f_1(z_1,z_2), e^{2\pi ip/q}f_2(z_1,z_2)) \end{aligned}$$

or $$f_1(e^{2\pi ip/q}z_1, e^{2\pi ip/q}z_2) = e^{2\pi ip/q}f_1(z_1,z_2)$$

thus τ must act as multiplication by $e^{2\pi ip/q}$ on $\mathbb{C}$.

Hence it only remains to show that if f is defined by (*) and g is $\mathbf{Z}_m \times \mathbf{Z}_q$ equivariant then f is $D_n \times \mathbf{Z}_q$ equivariant:

i)
$$\begin{aligned} f(\kappa(z_1,z_2)) &= f(z_2,z_1) \\ &= (g(z_2,z_1),g(z_1,z_2)) \\ &= \kappa(g(z_1,z_2),g(z_2,z_1)) \\ &= \kappa f((z_1,z_2)). \end{aligned}$$

ii)
$$\begin{aligned} f(\zeta(z_1,z_2)) &= f(e^{2\pi i/m}z_1,e^{-2\pi i/m}z_2) \\ &= (g(e^{2\pi i/m}z_1,e^{-2\pi i/m}z_2),g(e^{-2\pi i/m}z_2,e^{2\pi i/m}z_1)) \\ &= (e^{2\pi i/m}g(z_1,z_2),e^{-2\pi i/m}g(z_2,z_1)) \\ &= \zeta(g(z_1,z_2),g(z_2,z_1)) \\ &= \zeta f((z_1,z_2)). \end{aligned}$$

iii)
$$\begin{aligned} f(\tau(z_1,z_2)) &= f(e^{2\pi ip/q}z_1,e^{2\pi ip/q}z_2) \\ &= (g(e^{2\pi ip/q}z_1,e^{2\pi ip/q}z_2),g(e^{2\pi ip/q}z_2,e^{2\pi ip/q}z_1)) \\ &= (e^{2\pi ip/q}g(z_1,z_2),e^{2\pi ip/q}g(z_2,z_1)) \\ &= \tau(g(z_1,z_2),g(z_2,z_1)) \\ &= \tau f((z_1,z_2)). \end{aligned}$$

We now return to the proof of proposition 6.6.2.1

Proof

From the above lemma we have shown that to find a $D_m \times \mathbf{Z}_q$ equivariant function f from $\mathbb{C}^2$ to $\mathbb{C}^2$ we need only look for a $\mathbf{Z}_m \times \mathbf{Z}_q$ equivariant function g from $\mathbb{C}^2$ to $\mathbb{C}$.

Assuming g to be analytic it is possible to express it as a taylor expansion around (0,0) thus

$$g(z_1, z_2) = \sum_{j,k,l,n \in \mathbb{N}} \alpha_{jkln}\, z_1^j\, z_2^k\, \bar{z}_1^l\, \bar{z}_2^n.$$

For g to be $\mathbf{Z}_m \times \mathbf{Z}_q$ equivariant we need it to satisfy the following two conditions:

(a) $\quad g(\zeta(z_1, z_2)) = \zeta(g(z_1, z_2)),$

$$\text{(b)} \qquad g(\tau(z_1, z_2)) = \tau(g(z_1, z_2)).$$

These can be equivalently written as

$$g(e^{2\pi i/m}z_1, e^{-2\pi i/m}z_2)) = e^{2\pi i/m}(g(z_1, z_2))$$

and

$$g(e^{2\pi ip/q}z_1, e^{2\pi ip/q}z_2)) = e^{2\pi ip/q}(g(z_1, z_2))$$

respectively, which as taylor series say

$$\sum_{j,k,l,n} \alpha_{jkln}\, e^{2\pi ij/m} z_1^j\, e^{-2\pi ik/m} z_2^k\, e^{-2\pi il/m} \bar{z}_1^l\, e^{2\pi in/m} \bar{z}_2^n = e^{2\pi i/m} \sum_{j,k,l,n} \alpha_{jkln}\, z_1^j z_2^k \bar{z}_1^l \bar{z}_2^n.$$

and

$$\sum_{j,k,l,n} \alpha_{jkln}\, e^{2\pi ipj/q} z_1^j\, e^{2\pi ikp/q} z_2^k\, e^{-2\pi ilp/q} \bar{z}_1^l\, e^{-2\pi inp/q} \bar{z}_2^n = e^{2\pi ip/q} \sum_{j,k,l,n} \alpha_{jkln}\, z_1^j z_2^k \bar{z}_1^l \bar{z}_2^n.$$

Thus $\alpha_{jkln}=0$ unless

$$e^{2\pi i(j-k-l+n)/m} = e^{2\pi i/m}$$

and

$$e^{2\pi i(j+k-l-n)p/q} = e^{2\pi ip/q}$$

or

$$j-k-l+n = 1 \bmod m$$

and

$$j+k-l-n = 1 \bmod q.$$

So to find non zero terms in the taylor expansion of g we need to solve the pair of simultaneous equations

$$j-k-l+n = 1+rm$$

and

$$j+k-l-n = 1+sq$$

for $s,r \in \mathbb{Z}$, over the natural numbers.

This implies that

$$j=l+1+(rm+sq)/2 \qquad \text{or} \qquad l=j-1-(rm+sq)/2$$

and

$$k=n+(sq-rm)/2 \qquad \text{or} \qquad n=k+(rm-sq)/2$$

depending whether $1+(rm+sq)/2$ and $(sq-rm)/2$ are greater than or less than 0.

Firstly we note that when $r=s=0$ $j=l+1$ and $k=n$ thus terms are of the form

$$\beta_{jk} z_1^{j+1} z_2^k \bar{z}_1^j \bar{z}_2^k,$$

that is $$\beta_{jk}|z_1|^{2j}|z_2|^{2k}z_1$$

these are the terms that appear in h as defined in the proposition.

Next we look for the lowest order terms of g which are not of this form. The order of any term is given by $j+k+l+n$ and there are 4 possibilities, that is:

a) $2l+2n+1+sq$ (if $1+(rm+sq)/2 \geq 0$ and $(sq-rm)/2 \geq 0$).

b) $2l+2k+1+rm$ (if $1+(rm+sq)/2 \geq 0$ and $(sq-rm)/2 < 0$).

c) $2j+2n-1-rm$ (if $1+(rm+sq)/2 < 0$ and $(sq-rm)/2 \geq 0$).

d) $2j+2k-1-sq$ (if $1+(rm+sq)/2 < 0$ and $(sq-rm)/2 < 0$).

A little thought indicates that the lowest order terms of g should come from cases (c) or (d), depending on which of $-rm$ or $-sq$ is smaller and the parity of $sq+rm$. This is in fact the case but the proof although needing little in the way of complex calculations requires us to consider all possible parity conditions (i.e. one of, both of or neither of q and m being odd) and within each, whether sq or rm is greatest.

Hence we shall only give a small part of the complete proof, to give the reader an idea of the calculations involved and the regions of particular difficulty.

The simplest case to consider is when both m and q are even because whatever value s and r take $sq+rm$ will always be even. Thus the lowest possible orders will come when either $s=-1$ or when $r=-1$ (giving orders of $q-1$ and $m-1$ respectively).

We now split this case into 3 subcases; $m>q$, $m=q$ and $m<q$.

If $m>q$ we would like the lowest order elements to have order $q-1$, that is $s=-1$. So we consider all possible permutations in which $s=-1$,

$$j=l+1+(rm-q)/2$$

and $$k=n-(q+rm)/2.$$

If $r>0$ then the order is given by

$$2l+2k+1+rm$$

since $rm\geq m>q>0$, which is greater than $q-1$ for all $l,k,r>0$. If $r<0$ then the order is given by

$$2j+2n-1+rm$$

since $rm<-m<-q<0$, which is greater than $q-1$ for all $j,n,-r>0$. However if $r=0$ then the order is given by

$$2j+2k-1+q$$

giving the lowest order element $\bar{z}_1^{q/2-1}\bar{z}_2^{q/2}$ and we have the first result from the proposition.

When $m=q$, the cases of interest are when $r=-1$ and $s=-1$. When $s=-1$ the results obtained above hold (noting also that $r=-1$ from the second case gives another low order term $\bar{z}_1^{q-1}$). A similar arguement in which $r=-1$ gives the third and final low order term $\bar{z}_1^{q/2-1}z_2^{q/2}$.

When one of m and q is odd and the other is even, we get results similar to those above just doubling the odd one. This is because $rm+sq$ must be even, so the odd one of m and q must always have an even coefficient.

If both m and q are odd the proof becomes more complex since the coefficients r and s must both be odd or both be even and thus we do not always expect to see the lowest order term we would hope to find.

In order to tackle these problems we divide the proof into finer subcases, whose boundaries are the solutions of $sq-rm=0$ and $rm+sq=-2$. Fortunately we need only consider r and s between -2 and 2, since there is always an element with order $2q-1$ or $2m-1$. Thus the critical boundary values are $2q=m$, $q=m$, $q=2m$, $q=m-2$, $q=m+2$ (there are others but crossing them does not effect the form of the lowest order element). Simple calculations within each particular region, for the range of values of r and s between -2 and 2, gives a set of terms from which it is easy to pick the ones with lowest order and these are the last results left in the table.

All that is left to consider now is the lowest order term which contains only one of z_1 and z_2, this is important in studying the stability of the solution branch whose symmetry group is $\mathbf{Z}_c(\zeta^{m/c},\tau^{-q/c})$ which fixes $z_1=z$, $z_2=0$.

Clearly the low order terms which have already been derived will not be seen since these are forced by the symmetry group to be zero, so we need to find higher order terms which do not vanish.

Returning to the equivariant calulations from the start of the proof we need to solve the simultaneous equations, in which $k=n=0$, i.e.

$$j-l=1 \bmod m$$

and

$$j-l=1 \bmod q.$$

or

$$1+rm=1+sq.$$

Thus we need to find an r and s for which

$$rm=sq.$$

Since we know that the highest common factor of m and q is c then we have that r and s must be of the form

$$r=tq/c, \quad s=tm/c.$$

These give terms like $z_1^j\bar{z}_1^{j-1-tqm/c}$ and $z_1^{l+1+tqm/c}\bar{z}_1^l$, the lowest order of which is when $j=0$, $t=-1$ and is $\bar{z}_1^{qm/c-1}$.

6.6.3 Stability and branching information.

Now that we have seen what the general form of a $D_m\times\mathbf{Z}_q$ equivariant map is we can go on to investigate the branching directions and stability of the periodic solutions which exhibit the various symmeties shown to exist. From propsition 6.6.2.1 f has the form:-

$$F(z_1,z_2)=\begin{bmatrix} h(|z_1|^2,|z_2|^2)z_1+\alpha_{kln}z_2^k\bar{z}_1^l\bar{z}_2^n+R(z_1,z_2) \\ h(|z_2|^2,|z_1|^2)z_2+\alpha_{kln}z_1^k\bar{z}_2^l\bar{z}_1^n+R(z_2,z_1) \end{bmatrix}$$

where k, l and n are described in the proposition and R is a function with lowest order terms k+l+n+1. Since the stability of a period q orbit of f is that of a fixed point of f^q we define $G(z_1,z_2)=F^q(z_1,z_2)$, G being $D_n \times Z_q$ equivariant itself will be of the same form as F i.e.

$$G(z_1, z_2) = \begin{bmatrix} g(|z_1|^2, |z_2|^2)\, z_1 + K(z_1, z_2) \\ g(|z_2|^2, |z_1|^2)\, z_2 + K(z_2, z_1) \end{bmatrix}$$

where

$$K(z_1,z_2) = \beta_{kln} z_2^{k} \bar{z}_1^{l} \bar{z}_2^{n} + O(z_1,z_2)$$

and β_{kln} can be expressed as a function of α_{kln}.

We state now and prove later the stability criteria for the period q branches predicted by the corollary 6.3.5.

Lemma 6.6.3.1.

Assume K has a Taylor series of the form

$$K(z_1, z_2) = \sum_{j,k,l,n} \beta_{jkln}\, z_1^{j} z_2^{k} \bar{z}_1^{l} \bar{z}_2^{n}.$$

Let $\rho = \beta_{j'k'l'n'}$ be the coefficient of the lowest order term of K in which

$$j'-k'-l'+n' = a_1 \neq 1,$$

let $\sigma = \beta_{j''k''l''n''}$ be the coefficient of the lowest order term of K in which

$$j''+k''-l''-n'' = a_2 \neq 1$$

and let $\mu = \beta_{j'''0l'''0}$ be the coefficient of the lowest order term of K in which

$$j'''-l''' = a_3 \neq 1.$$

Under these assumptions the stability of the periodic branches is determined by the signs of the terms in the table below:

Table 6.2 Showing the branching and stability determining terms for the isotropy subgroups of $D_n \times Z_q$

Isotropy subgroup	Orbit representative	Branching equation	Stability determining terms
$Z_2((\kappa,1))$	(z,z)	$g(\lvert z\rvert^2,\lvert z\rvert^2)z+K(z,z)=z$	$\mathrm{Re}(g_{N_1}+g_{N_2})$
$Z_2((\kappa,1))\times Z_2^*$			$\mathrm{Re}\Big((g_{N_1}+g_{N_2})\bar{\sigma}a_2 z^{l''+n''+1}\bar{z}^{j''+k''}\Big)$
			$\mathrm{Re}(g_{N_1}-g_{N_2})$
			$\mathrm{Re}\Big((g_{N_1}-g_{N_2})\bar{\rho}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}\Big)$
$Z_2((\kappa,\tau^{q/2}))$	$(z,-z)$	$g(\lvert z\rvert^2,\lvert z\rvert^2)z+K(z,-z)=z$	$\mathrm{Re}(g_{N_1}+g_{N_2})$
			$(-1)^{k''+n''}\mathrm{Re}\Big((g_{N_1}+g_{N_2})\bar{\sigma}a_2 z^{l''+n''+1}\bar{z}^{j''+k''}\Big)$
			$\mathrm{Re}(g_{N_1}-g_{N_2})$
			$(-1)^{k'+n'}\mathrm{Re}\Big((g_{N_1}-g_{N_2})\bar{\rho}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}\Big)$
$Z_2((\kappa\zeta,1))$	$(z,e^{2\pi i/m}z)$	$g(\lvert z\rvert^2,\lvert z\rvert^2)z+K(z,e^{2\pi i/m}z)=z$	$\mathrm{Re}(g_{N_1}+g_{N_2})$
$Z_2((\kappa\zeta,1))\times Z_2^*$			$\mathrm{Re}(g_{N_1}+g_{N_2})\omega^{n''-k''}\bar{\sigma}a_2 z^{l''+n''+1}\bar{z}^{j''+k''}$
			$\mathrm{Re}(g_{N_1}-g_{N_2})$
			$\mathrm{Re}(g_{N_1}-g_{N_2})\omega^{n'-k'}\bar{\rho}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}$
$Z_c((\zeta^{m/c},\tau^{-q/c})$	$(z,0)$	$g(\lvert z\rvert^2,0)z+K(z,0)=z$	$\mathrm{Re}(g_{N_1})$
			$\mathrm{Re}(g_{N_1})\bar{\mu}a_3 z^{l'''+1}\bar{z}^{j'''}$

where $Z_2^*=Z_2((\zeta^{m/2},\tau^{q/2})$.

Proof

We first recall from theorem 6.5.4 that a fixed point x_0 of a map G is asymptotically orbitally stable if the eigenvalues of $(dG)_{x_0}$ not forced by the symmetry lie within the unit circle. The branches of solutions predicted by corollary 6.3.5 all lie within a two dimensional fixed point space and as such the map dG is restricted to having two, two dimensional subspaces invariant subspaces, namely $\mathrm{Fix}(\Sigma)$ on which Σ acts trivially and an invariant complimentary subspace W. Thus of the four eigenvalues of dG, two come from $(dG)_{x_0}|\,\mathrm{Fix}(\Sigma)$ and the other two come from $(dG)_{x_0}|\,W$.

To go further it is convenient to write G in coordinates

(a) $$Z_1 = g(|z_1|^2, |z_2|^2)z_1 + K(z_1, z_2)$$

(b) $$Z_2 = g(|z_2|^2, |z_1|^2)z_2 + K(z_2, z_1).$$

In these coordinates dG takes the form

$$dG\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Z_{1,z_1}w_1 + Z_{1\bar{z}_1}\bar{w}_1 + Z_{1,z_2}w_2 + Z_{1\bar{z}_2}\bar{w}_2 \\ Z_{2,z_1}w_1 + Z_{2\bar{z}_1}\bar{w}_1 + Z_{2,z_2}w_2 + Z_{2\bar{z}_2}\bar{w}_2 \end{bmatrix}$$

Since the invariant subspaces are both of dimension two we use the fact that an $\mathbb{R}$ linear map from $\mathbb{C} = \mathbb{R}^2$ has the form

$$w \to \alpha w + \beta\bar{w}$$

for $\alpha, \beta \in \mathbb{C}$. Furthermore if M is a matrix equivalent ot this map then

$$\mathrm{Tr} = \mathrm{Trace}(M) = 2\mathrm{Re}(\alpha), \qquad \mathrm{Det} = \mathrm{Det}(M) = |\alpha|^2 - |\beta|^2$$

and so using the results of lemma 1.2.3 we can calculate the stability determining terms of G for the predicted periodic orbit branches. We consider each isotropy subgroup seperately.

(i) Isotropy subgroup $\mathbf{Z}_2((\kappa,1))$ or $\mathbf{Z}_2((\kappa,1)) \times \mathbf{Z}_2((\zeta^{m/2}, \tau^{q/2}))$.

Here the two invariant subspaces are $\mathrm{Fix}(\Sigma) = \{(w,w)\}$ on which Σ acts trivially and $W = \{(w,-w)\}$ on which Σ acts as $-I$. The map $(dg)\,|\,\mathrm{Fix}(\Sigma)$ is given by

$$w \mapsto (dg)(w,w) = (Z_{1,z_1}(z,z) + Z_{1,z_2}(z,z))w + (Z_{1\bar{z}_1}(z,z) + Z_{1\bar{z}_2}(z,z))\bar{w}.$$

We have

$$Z_{1,z_1}(z,z) = g_{z_1}(|z|^2, |z|^2)z + g(|z|^2, |z|^2) + K_{z_1}(z,z),$$
$$Z_{1,z_2}(z,z) = g_{z_2}(|z|^2, |z|^2)z + K_{z_2}(z,z),$$
$$Z_{1\bar{z}_1}(z,z) = g_{\bar{z}_1}(|z|^2, |z|^2)z + K_{\bar{z}_1}(z,z),$$
$$Z_{1\bar{z}_2}(z,z) = g_{\bar{z}_2}(|z|^2, |z|^2)z + K_{\bar{z}_2}(z,z).$$

Furthermore letting $|z_i|^2 = N_i$ we have

$$g_{z_i} = g_{N_1} N_{1,z_i} + g_{N_2} N_{2,z_i} = g_{N_i} \bar{z}_i$$

$$g_{\bar{z}_i} = g_{N_1} N_{1,\bar{z}_i} + g_{N_2} N_{2,\bar{z}_i} = g_{N_i} z_i$$

and using the fact that along the solution branches

$$g(|z|^2, |z|^2) = 1 - K(z,z)/z$$

$$\begin{aligned} Tr &= 2\mathrm{Re}(1+(g_{N_1}+g_{N_2})|z|^2 + (K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z)). \\ &= 2+2\mathrm{Re}(g_{N_1}+g_{N_2})|z|^2 + O(|z|^{k+l+n-1}) \end{aligned}$$

For stability we need $Tr < 2$, i.e. $\mathrm{Re}(g_{N_1}+g_{N_2}) < 0$ which gives the first entry from the table in 6.6.3.1. For the other stability determining term we need to calculate Det and we find

$$\begin{aligned} Det = {} & |1+(g_{N_1}+g_{N_2})|z|^2 + (K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z)|^2 \\ & - |(g_{N_1}+g_{N_2})z^2 + (K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z))|^2. \end{aligned}$$

$$\begin{aligned} = {} & 1 + 2\mathrm{Re}\Big((g_{N_1}+g_{N_2})|z|^2 + (K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z)\Big) + |g_{N_1}+g_{N_2}|^2|z|^4 \\ & + 2\mathrm{Re}(g_{N_1}+g_{N_2})|z|^2\Big(\overline{K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z}\Big) + |K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z|^2 \\ & - \Big(|g_{N_1}+g_{N_2}|^2|z|^4 + 2\mathrm{Re}(g_{N_1}+g_{N_2})z^2\Big(\overline{K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z)}\Big) + |K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z)|^2\Big). \end{aligned}$$

or

$$Det = (Tr-1) + 2\mathrm{Re}(g_{N_1}+g_{N_2})\Big((\overline{K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z})|z|^2 - (K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z))z^2\Big).$$

For stability we need $Tr < Det+1$, i.e.

$$0 < 2\mathrm{Re}(g_{N_1}+g_{N_2})\Big((\overline{K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z})|z|^2 - (\overline{K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z)})z^2\Big)$$

so to find the stability determining term we must calculate the lowest order term to appear in the right hand side of the inequality. Since

$$K(z_1, z_2) = \sum_{j,k,l,n} \beta_{jkln} z_1^j z_2^k \bar{z}_1^l \bar{z}_2^n .$$

we have

$$\left(\overline{K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z}\right)|z|^2-\left(\overline{K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z)}\right)z^2 =$$

$$\sum_{j,k,l,n}\bar{\beta}_{jkln}(j+k-l-n-1)\, z^{l+n+1}\bar{z}^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$2\mathrm{Re}(g_{N_1}+g_{N_2})\bar{\sigma}a_2 z^{l''+n''+1}\bar{z}^{j''+k''}$$

and we have the next entry in our table.

The map $(dg)\,|\,W$ is given by

$$w \mapsto (dg)(w,-w) = (Z_{1,z_1}(z,z)-Z_{1,z_2}(z,z))w + (Z_{1,\bar{z}_1}(z,z)-Z_{1,\bar{z}_2}(z,z))\bar{w}.$$

A calculation similar to that above yields

$$\begin{aligned} Tr &= 2\mathrm{Re}(1+(g_{N_1}-g_{N_2})|z|^2+(K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z)). \\ &= 2+2\mathrm{Re}(g_{N_1}-g_{N_2})|z|^2+O(|z|^{k+l+n-1}) \end{aligned}$$

For stability we need $Tr < 2$, i.e. $\mathrm{Re}(g_{N_1}-g_{N_2}) < 0$ which gives the next entry from the table. For the other stability determining term we need to calculate Det, we find

$$\begin{aligned} Det = 1 &+ 2\mathrm{Re}\Big((g_{N_1}-g_{N_2})|z|^2+(K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z)\Big) + |g_{N_1}-g_{N_2}|^2|z|^4 \\ &+ 2\mathrm{Re}(g_{N_1}-g_{N_2})|z|^2\Big(\overline{K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z}\Big) + |K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z|^2 \\ &- \Big(|g_{N_1}+g_{N_2}|^2|z|^4 + 2\mathrm{Re}(g_{N_1}-g_{N_2})z^2\Big(\overline{K_{\bar{z}_1}(z,z)-K_{\bar{z}_2}(z,z)}\Big) + |K_{\bar{z}_1}(z,z)-K_{\bar{z}_2}(z,z)|^2\Big). \end{aligned}$$

or

$$Det = (Tr-1) + 2\mathrm{Re}(g_{N_1}-g_{N_2})\Big(\left(\overline{K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z}\right)|z|^2-\left(\overline{K_{\bar{z}_1}(z,z)-K_{\bar{z}_2}(z,z)}\right)z^2\Big).$$

For stability we need $Tr < Det+1$, i.e.

$$0 < 2\mathrm{Re}(g_{N_1}-g_{N_2})\Big(\left(\overline{K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z}\right)|z|^2-\left(\overline{K_{\bar{z}_1}(z,z)-K_{\bar{z}_2}(z,z)}\right)z^2\Big).$$

We have

$$\left(\overline{K_{z_1}(z,z)-K_{z_2}(z,z)-K(z,z)/z}\right)|z|^2-\left(\overline{K_{\bar{z}_1}(z,z)-K_{\bar{z}_2}(z,z)}\right)z^2 =$$

$$\sum_{j,k,l,n}\bar{\beta}_{jkln}(j+k-l+n-1)\, z^{l+n+1}\bar{z}^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$2\mathrm{Re}(g_{N_1}-g_{N_2})\bar{\rho}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}$$

and we have the final entry in our table for this isotropy subgroup.

(ii) Isotropy subgroup $\mathbf{Z}_2((\kappa,\tau^{q/2}))$.

Here the two invariant subspaces are $\mathrm{Fix}(\Sigma)=\{(w,-w)\}$ on which Σ acts trivially and $W=\{(w,w)\}$ on which Σ acts as $-I$. The map $(dg)\,|\,\mathrm{Fix}(\Sigma)$ is given by

$$w \mapsto (dg)(w,-w) = (Z_{1,z_1}(z,-z)-Z_{1,z_2}(z,-z))w + (Z_{1,\bar z_1}(z,-z)-Z_{1,\bar z_2}(z,-z))\bar w.$$

We have

$$Z_{1,z_1}(z,-z) = g_{z_1}(|z|^2,|z|^2)z + g(|z|^2,|z|^2) + K_{z_1}(z,-z),$$
$$Z_{1,z_2}(z,-z) = g_{z_2}(|z|^2,|z|^2)z + K_{z_2}(z,-z),$$
$$Z_{1,\bar z_1}(z,-z) = g_{\bar z_1}(|z|^2,|z|^2)z + K_{\bar z_1}(z,-z),$$
$$Z_{1,\bar z_2}(z,-z) = g_{\bar z_2}(|z|^2,|z|^2)z + K_{\bar z_2}(z,-z).$$

Using the fact that along the solution branches

$$g(|z|^2,|z|^2)=1-K(z,-z)/z$$

$$\mathrm{Tr} = 2\mathrm{Re}(1+(g_{N_1}+g_{N_2})|z|^2+(K_{z_1}(z,-z)-K_{z_2}(z,-z)-K(z,-z)/z)).$$
$$= 2+2\mathrm{Re}(g_{N_1}+g_{N_2})|z|^2+O(|z|^{k+l+n-1})$$

For stability we need $\mathrm{Tr} < 2$, i.e. $\mathrm{Re}(g_{N_1}+g_{N_2}) < 0$ which gives the first entry for this isotropy subgroup. For the other stability determining term we need to calculate Det, we find it equals

$$(\mathrm{Tr}-1) + 2\mathrm{Re}(g_{N_1}+g_{N_2})\Big(\overline{(K_{z_1}(z,-z)-K_{z_2}(z,-z)-K(z,-z)/z)}|z|^2-(\overline{K_{\bar z_1}(z,-z)-K_{\bar z_2}(z,-z)})z^2\Big).$$

For stability we need $\mathrm{Tr} < \mathrm{Det}+1$, i.e.

$$0 < 2\mathrm{Re}(g_{N_1}+g_{N_2})\Big(\overline{(K_{z_1}(z,-z)-K_{z_2}(z,-z)-K(z,-z)/z)}|z|^2-(\overline{K_{\bar z_1}(z,-z)-K_{\bar z_2}(z,-z)})z^2\Big)$$

so to find the stability determining term we must calculate the lowest order term to appear in the right hand side of the inequality. We have

$$\overline{(K_{z_1}(z,-z)-K_{z_2}(z,-z)-K(z,-z)/z)}|z|^2-(\overline{K_{\bar z_1}(z,-z)-K_{\bar z_2}(z,-z)})z^2 =$$

$$\sum_{j,k,l,n}(-1)^{k+n}\beta_{jkln}(j+k-l-n-1)z^{l+n+1}\bar z^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$(-1)^{k''+n''}\mathrm{Re}(g_{N_1}+g_{N_2})\bar\sigma a_2 z^{l''+n''+1}\bar z^{j''+k''}$$

and we have the next entry in our table.

The map $(dg)|_W$ is given by

$$w \mapsto (dg)(w,w) = (Z_{1,z_1}(z,-z)+Z_{1,z_2}(z,-z))w + (Z_{1,\bar{z}_1}(z,-z)+Z_{1,\bar{z}_2}(z,-z))\bar{w}.$$

Again we find

$$\begin{aligned}\mathrm{Tr} &= 2\mathrm{Re}(1+(g_{N_1}-g_{N_2})|z|^2+(K_{z_1}(z,-z)+K_{z_2}(z,-z)-K(z,-z)/z)).\\ &= 2+2\mathrm{Re}(g_{N_1}-g_{N_2})|z|^2+O(|z|^{k+l+n-1})\end{aligned}$$

For stability we need $\mathrm{Tr} < 2$, i.e. $\mathrm{Re}(g_{N_1}-g_{N_2}) < 0$ giving a further entry in the table. We next calculate Det to find it equals

$$(\mathrm{Tr}-1) + 2\mathrm{Re}(g_{N_1}-g_{N_2})\Big(\overline{(K_{z_1}(z,-z)+K_{z_2}(z,-z)-K(z,-z)/z)}|z|^2-\overline{(K_{\bar{z}_1}(z,-z)+K_{\bar{z}_2}(z,-z))}z^2\Big).$$

For stability we need $\mathrm{Tr} < \mathrm{Det}+1$, i.e.

$$0 < 2\mathrm{Re}(g_{N_1}-g_{N_2})\Big(\overline{(K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z)}|z|^2-\overline{(K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z))}z^2\Big).$$

We have

$$\overline{(K_{z_1}(z,z)+K_{z_2}(z,z)-K(z,z)/z)}|z|^2-\overline{(K_{\bar{z}_1}(z,z)+K_{\bar{z}_2}(z,z))}z^2 =$$

$$\sum_{j,k,l,n}(-1)^{k+n}\bar{\beta}_{jkln}(jk-l+n-1)z^{l+n+1}\bar{z}^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$2(-1)^{k'+n'}\mathrm{Re}(g_{N_1}-g_{N_2})\bar{\beta}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}$$

and we have the last entry for this isotropy subgroup.

(iii) Isotropy subgroup $\mathbf{Z}_2((\kappa\zeta,1))$ or $\mathbf{Z}_2((\kappa\zeta,1))\times\mathbf{Z}_2((\zeta^{m/2},\tau^{q/2}))$.

Here the two invariant subspaces are $\mathrm{Fix}(\Sigma)=\{(w,e^{2\pi i/m}w)\}$ on which Σ acts trivially and $W=\{(w,-e^{2\pi i/m}w)\}$ on which Σ acts as $-I$. The map $(dg)\,|\,\mathrm{Fix}(\Sigma)$ is given by

$$w \mapsto (dg)(w,\omega w) = (Z_{1,z_1}(z,\omega z)+\omega Z_{1,z_2}(z,\omega z))w + (Z_{1,\bar{z}_1}(z,\omega z)+\bar{\omega}Z_{1,\bar{z}_2}(z,\omega z))\bar{w}$$

where $\omega=e^{2\pi i/m}$. We have

$$\begin{aligned}Z_{1,z_1}(z,\omega z) &= g_{z_1}(|z|^2,|z|^2)z + g(|z|^2,|z|^2) + K_{z_1}(z,\omega z),\\ Z_{1,z_2}(z,\omega z) &= g_{z_2}(|z|^2,|z|^2)z + K_{z_2}(z,\omega z),\\ Z_{1,\bar{z}_1}(z,\omega z) &= g_{\bar{z}_1}(|z|^2,|z|^2)z + K_{\bar{z}_1}(z,\omega z),\\ Z_{1,\bar{z}_2}(z,\omega z) &= g_{\bar{z}_2}(|z|^2,|z|^2)z + K_{\bar{z}_2}(z,\omega z).\end{aligned}$$

So

$$Tr = 2Re(1+(g_{N_1}+g_{N_2})|z|^2+(K_{z_1}(z,\omega z)+\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)).$$

and $Re(g_{N_1}+g_{N_2})$ is again a stability determining term. For the other term we need to calculate Det which we calculate to be

$$(Tr-1)+$$
$$2Re(g_{N_1}+g_{N_2})\Big(\overline{(K_{z_1}(z,\omega z)+\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)}|z|^2-(\overline{K_{\bar{z}_1}(z,\omega z)+\bar{\omega}K_{\bar{z}_2}(z,\omega z)})z^2\Big).$$

We need $Tr < Det+1$, i.e.

$$0 < 2Re(g_{N_1}+g_{N_2})\Big(\overline{(K_{z_1}(z,\omega z)+\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)}|z|^2-(\overline{K_{\bar{z}_1}(z,\omega z)+\bar{\omega}K_{\bar{z}_2}(z,\omega z)})z^2\Big)$$

A calculation gives

$$\overline{(K_{z_1}(z,\omega z)+\omega K_{z_2}(z,\omega z)-K(z\omega z)/z)}|z|^2-(\overline{K_{\bar{z}_1}(z,\omega z)+\bar{\omega}K_{\bar{z}_2}(z,\omega z)})z^2 =$$

$$\sum_{j,k,l,n} \omega^{n-k}\bar{\beta}_{jkln}(j+k-l-n-1)z^{l+n+1}\bar{z}^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$2Re(g_{N_1}+g_{N_2})\omega^{n''-k''}\bar{\sigma}a_2 z^{l''+n''+1}\bar{z}^{j''+k''}.$$

The map $(dg)\,|\,W$ is given by

$$w \mapsto (dg)(w,-\omega w) = (Z_{1,z_1}(z,\omega z)-\omega Z_{1,z_2}(z,\omega z))w + (Z_{1,\bar{z}_1}(z,\omega z)-\bar{\omega}Z_{1,\bar{z}_2}(z,\omega z))\bar{w}.$$

We find

$$Tr = 2Re(1+(g_{N_1}-g_{N_2})|z|^2+(K_{z_1}(z,\omega z)-\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)).$$

So for stability we need $Tr < 2$, i.e. $Re(g_{N_1}-g_{N_2}) < 0$. For the other stability determining term we need to calculate Det, we find

$$Det = (Tr-1) + 2Re(g_{N_1}-g_{N_2})\Big(\overline{(K_{z_1}(z,\omega z)-\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)}|z|^2$$
$$-(\overline{K_{\bar{z}_1}(z,\omega z)-\bar{\omega}K_{\bar{z}_2}(z,\omega z)})z^2\Big).$$

For stability we need

$$2Re(g_{N_1}-g_{N_2})\Big(\overline{(K_{z_1}(z,\omega z)-\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)}|z|^2-(\overline{K_{\bar{z}_1}(z,\omega z)-\bar{\omega}K_{\bar{z}_2}(z,\omega z)})z^2\Big).$$

to be positive.

$$(K_{z_1}(z,\omega z)-\omega K_{z_2}(z,\omega z)-K(z,\omega z)/z)|z|^2-(K_{\bar{z}_1}(z,\omega z)-\bar{\omega}K_{\bar{z}_2}(z,\omega z))z^2 =$$

$$\sum_{j,k,l,n} \omega^{n-k}\bar{\beta}_{jkln}(j+k-l+n-1)z^{l+n+1}\bar{z}^{j+k}$$

and hence the lowest order term on the right hand side of the inequality is

$$2\mathrm{Re}(g_{N_1}-g_{N_2})\omega^{n'-k'}\bar{\beta}a_1 z^{l'+n'+1}\bar{z}^{j'+k'}$$

and we have the final entry in our table for this case.

(iv) The isotropy subgroup $\mathbf{Z}_c((\zeta^{m/c},\tau^{-q/c})$.

Here the two invariant subspaces are $\mathrm{Fix}(\Sigma)=\{(w,0)\}$ on which Σ acts trivially and $W=\{(0,w)\}$ on which Σ acts as multiplication by $e^{-4\pi i/c}$. The map $(dg)\mid\mathrm{Fix}(\Sigma)$ is given by

$$w \mapsto (dg)(w,0) = Z_{1,z_1}(z,z)w+Z_{1,\bar{z}_1}(z,z)\bar{w}.$$

We have

$$Z_{1,z_1}(z,0) = g_{z_1}(|z|^2,0)z + g(|z|^2,0) + K_{z_1}(z,0),$$
$$Z_{1,\bar{z}_1}(z,0) = g_{\bar{z}_1}(|z|^2,0)z + K_{\bar{z}_1}(z,0),$$

hence

$$\mathrm{Tr} = 2\mathrm{Re}(1+g_{N_1}|z|^2+(K_{z_1}(z,0)-K(z,0)/z)).$$
$$= 2+2\mathrm{Re}(g_{N_1})|z|^2+O(|z|^{k+l+n-1})$$

For stability we need $\mathrm{Tr} < 2$, i.e. $\mathrm{Re}(g_{N_1}) < 0$. For the other stability determining term we need to calculate Det, we find

$$\mathrm{Det} = (\mathrm{Tr}-1) + 2\mathrm{Re}(g_{N_1})\Big((\overline{K_{z_1}(z,0)-K(z,0)/z})|z|^2-\overline{K_{\bar{z}_1}(z,0)}z^2\Big).$$

For stability we need $\mathrm{Tr} < \mathrm{Det}+1$, i.e.

$$0 < 2\mathrm{Re}(g_{N_1})\Big((\overline{K_{z_1}(z,0)-K(z,0)/z})|z|^2-\overline{K_{\bar{z}_1}(z,0)}z^2\Big)$$

so to find the stability determining term we must calculate the lowest order term to appear in the right hand side of the inequality, we find

$$(K_{z_1}(z,0)-K(z,0)/z)|z|^2-K_{\bar{z}_1}(z,0)z^2 =$$

$$\sum_{j,l}\bar{\beta}_{j0l0}(j-l-1)z^{l+1}\bar{z}^{j}$$

and hence the lowest order term on the right hand side of the inequality is

$$2\mathrm{Re}(g_{N_1})\bar{\mu}a_3 z^{l'''+1}\bar{z}^{j'''}$$

and we have the next entry in our table.

The map $(dg)\,|\,W$ is given by

$$w \mapsto (dg)(0,w) = (Z_{1,z_2}(z,0))w + Z_{1,\bar{z}_2}(z,0))\bar{w}.$$

A calculation shows

$$\mathrm{Tr} = 2\mathrm{Re}((g_{N_2})|z|^2+K_{z_2}(z,0))$$

so for $|z|$ small this always satisfies $-2 < \mathrm{Tr} < 2$. Furthermore find

$$\mathrm{Det} = 2\mathrm{Re}(g_{N_2})\left(\overline{K_{z_2}(z,0)}|z|^2-\overline{K_{\bar{z}_2}(z,0)})z^2\right).$$

For stability we need $-1-\mathrm{Det} < \mathrm{Tr} < \mathrm{Det}+1$ but for $|z|$ small this is always the case.

Hence it remains only to discuss what these stability determining terms actually are. It is not always true that they are the lowest order terms in K derived in proposition 6.6.2.1. However using similar calculations to those in the proof of the proposition and noting that $D_n \times Z_q$ equivariance means that

$$j+k-l-n = 1 \mod q$$

and

$$j-k-l+n = 1 \mod m$$

we can say something about their form.

Since $z^{l+n+1}\bar{z}^{j+k}$ equals $|z|^{2(j+k)}z^{l+n+1-j-k}$ or $|z|^{2(l+n+1)}\bar{z}^{j+k-l-n-1}$ then the stability determining terms will always be of the form αz^{sq} for some $\alpha \in \mathbb{C}$, $s \in \mathbb{Z}$. This implies that the solution space splits up into 2sq sectors in which solutions are alternately stable and unstable. In any specific case it will be easy to compute the precise form of these stability determining terms but in general this is as exact as it is possible to be.

To calculate the branching directions we note that along the solution branches

$$1-g(|z|^2,|z|^2,\lambda)+K(z,z,\lambda)/z = 0$$

$$1-g(|z|^2,|z|^2,\lambda)+K(z,-z,\lambda)/z = 0$$

$$1-g(|z|^2,|z|^2,\lambda)+K(z,\omega z,\lambda)/z = 0$$

$$1-g(|z|^2,0,\lambda)+K(z,0,\lambda)/z = 0$$

for the cases (i)-(iv) respectively. Furthermore, given the way g is defined it is clear to see $g(0,0,0)=(df^q)_{0,0,0}=e^{2\pi ipq/q}=1$ and so the lowest order terms in the Taylor expansion along the solution branches are

$$g_{N_1}(0,0,0)|z|^2+g_{N_2}(0,0,0)|z|^2+g_\lambda(0,0,0)\lambda = 0$$

for cases (i)-(iii) and

$$g_{N_1}(0,0,0)|z|^2+g_\lambda(0,0,0)\lambda = 0$$

for (iv).

Hence branching directions are given by

$$\lambda = -(g_{N_1}(0,0,0)|z|^2+g_{N_2}(0,0,0)|z|^2)/g_\lambda(0,0,0)$$

for case (i)-(iii) and

$$\lambda = -(g_{N_1}(0,0,0)|z|^2)/g_\lambda(0,0,0)$$

for (iv).

6.7 Further Areas of interest.

This is by no means a complete classification of bifurcations of maps with symmetry. We mentioned briefly in chapter one the problems associated with strong resonance and how some of the dynamics can be observed by using the technique of vector field approximation. Bridges [1991] has used a Liapunov Schmidt type reduction to study strong resonance in higher dimensions and this approach could be extended to the study of strong resonance in maps with symmetry. This thesis has concerntrated on bifurcations from fixed points and has not even touched upon the possibilities of further bifurcations from periodic orbits and invariant circles, Krupa and Roberts [1991] have looked at bifurcations from invariant circles and using their ideas in combination with the Liapunov Schmidt type reduction in theorem 6.3.1 I believe it would be possible to get some useful results about bifurcations from periodic orbits. I am sure this thesis has merely touched the surface of a very rich source of possible further research.

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