Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies*

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Abstract

We settle a problem of Havel by showing that there exists an absolute constant d such that if G is a planar graph in which every two distinct triangles are at distance at least d, then G is 3-colorable. In fact, we prove a more general theorem. Let G be a planar graph, and let $\mathcal{H}$ be a set of connected subgraphs of G, each of bounded size, such that every two distinct members of $\mathcal{H}$ are at least a specified distance apart and all triangles of G are contained in $\bigcup \mathcal{H}$. We give a sufficient condition for the existence of a 3-coloring ϕ of G such that for every $H \in \mathcal{H}$ the restriction of ϕ to H is constrained in a specified way.

1 Introduction

This paper is a part of a series aimed at studying the 3-colorability of graphs on a fixed surface that are either triangle-free, or have their triangles restricted in some way. Here, we are concerned with 3-coloring planar graphs. All *graphs* in this paper are finite and simple; that is, have no loops or multiple edges. All *colorings* that we consider are proper, assigning different colors to adjacent vertices. The following is a classical theorem of Grötzsch [18].

Theorem 1.1. *Every triangle-free planar graph is* 3*-colorable.*

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There is a long history of generalizations that extend the theorem to classes of graphs that include triangles. An easy modification of Grötzsch' proof shows that every planar graph with at most one triangle is 3-colorable. Even more is true—every planar graph with at most three triangles is 3-colorable. This was first claimed by Grünbaum [19], however his proof contains an error. This error was fixed by Aksionov [1] and later Borodin [5] gave another proof. There are infinitely many 4-critical planar graphs with four triangles, but they were recently completely characterized by Borodin et al. [6].

As another direction of research, Grünbaum [19] conjectured that every planar graph with no intersecting triangles is 3-colorable. This was disproved by Havel [20], who formulated a more cautious question whether there exists a constant d such that every planar graph such that the distance between every two triangles is at least d is 3-colorable. In [21], Havel shows that if such a constant d exists, then $d \geq 3$, and Aksionov and Mel'nikov [2] improved this bound to $d \geq 4$. Borodin [4] constructed a family of graphs that suggests that it may not be possible to obtain a positive answer to Havel's question using local reductions only.

The answer to Havel's question is known to be positive under various additional conditions (e.g., no 5-cycles [8], no 5-cycles adjacent to triangles [7], a distance constraint on 4-cycles [9]), see the on-line survey of Montassier [22] for a more complete list. The purpose of this paper is to describe a solution to Havel's problem.

Theorem 1.2. *There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d, then G is 3-colorable.*

Let us remark that our proof gives an explicit upper bound on the constant d of Theorem 1.2, which however is quite large (roughly 10^{100}), especially compared to the aforementioned lower bounds.

A natural extension of Havel's question is whether instead of triangles, we could allow other kinds of distant anomalies, such as 3-colorable subgraphs containing several triangles (the simplest one being a diamond, that is, K_4 without an edge) or even more strongly, prescribing specific colorings of some distant subgraphs. Similar questions have been studied for other graph classes. For example, Albertson [3] proved that if S is a set of vertices in a planar graph G that are precolored with colors $1, \ldots, 5$ and are at distance at least 4 from each other, then the precoloring of S can be extended to a 5-coloring of G. Furthermore, using the results of the third paper of this series [12], it is easy to see that any precoloring of sufficiently distant vertices of a planar graph G of girth at least 5 can be extended to a 3-coloring of G. We can even precolor larger connected subgraphs, as long as these precolorings can be extended locally to the vertices of G at some bounded distance from the precolored subgraphs. Both for 5-coloring planar graphs and 3-coloring planar graphs of girth at least five this follows from the fact that the corresponding critical graphs satisfy a certain isoperimetric inequality [23].

The situation is somewhat more complicated for graphs of girth four. Firstly, as we will discuss in Section 4, there is a global constraint on 3-colorings of plane graphs based on winding number, which implies that in graphs with almost all faces of length four, precoloring a subgraph may give restrictions on possible colorings of distant parts of the graph. For example, if we prescribed specific colorings of the triangles in Theorem 1.2, the resulting claim would be false, even though such precolorings extend locally. Secondly, non-facial (separating) 4-cycles are problematic as well and they need to be treated with care in many of the results of this series, see e.g. Theorem 2.2 below. Specifically, we cannot replace triangles in Theorem 1.2 by diamonds, even though this seems viable when considering only the winding number argument, as shown by the class of graphs (with many separating 4-cycles) constructed by Thomas and Walls [24].

Thus, in our second result, we only deal with graphs without separating 4-cycles, and we need to allow certain flexibility in the prescribed colorings of distant subgraphs. The exact formulation of the result (Theorem 5.1) is somewhat technical, and we postpone it till Section 5. Here, let us give just a special case covering several interesting kinds of anomalies. The *pattern* of a 3-coloring ψ is the set $\{\psi^{-1}(1), \psi^{-1}(2), \psi^{-1}(3)\}$. That is, two 3-colorings have the same pattern if they only differ by a permutation of colors.

Theorem 1.3. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane graph without separating 4-cycles. Let S_1 be a set of vertices of G. Let S_2 be a set of (≤ 5)-cycles of G. Let S_3 be a set of vertices of G of degree at most 4. For each $v \in S_1 \cup S_3$, let $c_v \in \{1, 2, 3\}$ be a color. For each $K \in S_2$, let ψ_K be a 3-coloring of K. Suppose that the distance between any two vertices or subgraphs belonging to $S_1 \cup S_2 \cup S_3$ is at least d. If all triangles in G belong to S_2, then G has a 3-coloring φ such that*

- *$\varphi(v) = c_v$ for every $v \in S_1$,*
- *φ has the same pattern on K as ψ_K for every $K \in S_2$, and*
- *$\varphi(u) = c_v$ for every neighbor u of a vertex $v \in S_3$.*

Let us remark that forbidding separating 4-cycles is necessary when the anomalies S_2 (except for triangles) and S_3 are considered, as shown by simple variations of the construction of Thomas and Walls [24]. On the other hand, there does not appear to be any principal reason to exclude 4-cycles when only precolored single vertices are allowed.

Conjecture 1.4. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane triangle-free graph, let S be a set of vertices of G and let $\psi : S \to \{1, 2, 3\}$ be an arbitrary function. If the distance between every two vertices of S is at least d, then ψ extends to a 3-coloring of G.*

In Theorem 5.1, we show that Conjecture 1.4 is implied by the following seemingly simpler statement.

Conjecture 1.5. *There exists an absolute constant $d \geq 2$ with the following property. Let G be a plane triangle-free graph, let C be a 4-cycle bounding a face of G and let v be a vertex of G. Let ψ be a 3-coloring of $C + v$. If the distance between C and v is at least d, then ψ extends to a 3-coloring of G.*

If an n-vertex planar triangle-free graph G has bounded maximum degree, then we can select a subset S_1 of its vertices of size $\Omega(n)$ such that the distance between any two of vertices of S_1 is at least d. If G does not contain separating 4-cycles, then by Theorem 1.3, we can 3-color G so that all vertices of S_1 have prescribed colors. By choosing the colors of vertices in S_1, we obtain exponentially many 3-colorings of G. This solves a special case of a conjecture of Thomassen [25] that all triangle-free planar graphs have exponentially many 3-colorings.

Corollary 1.6. *For every $k \geq 0$, there exists $c > 1$ such that every planar triangle-free graph G of maximum degree at most k and without separating 4-cycles has at least $c^{|V(G)|}$ 3-colorings.*

While the current paper was undergoing review and revisions, Conjecture 1.5 was confirmed to be true by Dvořák and Lidický [16]. Consequently, Conjecture 1.4 is true as well, and in Corollary 1.6, the assumption that there are no separating 4-cycles can be dropped.

The rest of the paper is structured as follows. In the next section, we state several previous results which we need in the proofs. In Section 3, we study the structure of graphs where no 4-faces can be collapsed without decreasing distances between anomalies, showing that they contain long cylindrical quadrangulated subgraphs. In Section 4, we study the colorings of such cylindrical subgraphs. Finally, in Section 5, we prove a statement generalizing Theorems 1.2 and 1.3.

Proof outline

Let us finish the introduction by describing the main ideas of the proof of Theorem 1.2.

To deal with the aforementioned problems with separating 4-cycles, as well as with other technicalities arising in the argument, we are actually going to prove a stronger result: In the situation of Theorem 1.2, if either C is a 4-cycle in G, or a 5-cycle in G disjoint from all triangles, and ψ is a 3-coloring of C, then ψ extends to a 3-coloring of G. Then we can without loss of generality assume G has no separating 4-cycles: Otherwise, $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 intersecting in a 4-cycle K, with $C \subset G_1$, and we can use induction to first extend ψ to a 3-coloring of G_1, then extend the resulting coloring of K to G_2.

Suppose now for a contradiction G is a counterexample with $|V(G)|+|E(G)|$ minimum; clearly, the graph G is connected. Let t denote the number of triangles in G. We have $t \geq 2$, as otherwise ψ extends to a 3-coloring of G by a result of Aksionov [1], see Lemma 2.1. By the main result of the previous paper in this series [13], see Theorem 2.2 below, the minimality of G and the fact that G does not contain separating 4-cycles implies that the total length of (≥ 5)-faces of G is at most ηt, for a constant $\eta \ll d$. Since G is connected, $t \geq 2$, and every two triangles in G are at distance at least d from each other, observe that for some triangle $T \subset G$, there exist integers $a \leq b < d/2$ such that $b - a = \Omega(d/\eta)$, all faces of G whose distance from T is between a and b have length 4, the total length of (≥ 5)-faces of G at distance less than a from T is at most 2η, and C is at distance more than b from T.

Let R denote the part of G at distance between a and b from T, and let f be a 4-face in R. Let G' be the graph obtained from G by identifying two vertices v_1 and v_2 that are opposite on f to a single vertex v. If G' satisfies the assumptions of the theorem, then ψ extends to a 3-coloring of G' by the minimality of G, and giving v_1 and v_2 the color of v, we obtain a 3-coloring of G extending ψ. This is a contradiction, and thus the described identification either creates a triangle, or decreases the distance between two triangles of G (one of these triangles necessarily has to be T, since f is at distance less than $d/2$ from T). This has to be the case for every 4-face in R, and as we show in Section 3, this is basically only possible if R contains a regular cylindrical grid R' whose length is significantly larger than its circumference.

Let C_1 and C_2 be the boundary cycles of this long cylindrical grid. In Section 4, we use the connection between 3-colorings and nowhere-zero 3-flows to show that any precoloring of $C_1 \cup C_2$ satisfying a certain simple constrain (winding numbers on C_1 and C_2 match) extends to a 3-coloring of R'. This enables us to finish the argument: We cut G in the middle of R', obtaining two subgraphs H_1 and H_2 with $C \subseteq H_1$. For $i \in \{1, 2\}$, we fill in the newly created face of H_i by a subgraph with a face bounded by a cycle C'_i of length at most five and all other faces of length four, obtaining a plane graph H'_i. By the minimality of G, we can extend ψ to a 3-coloring φ_1 of H'_1, color C'_2 the same way as φ_1 colors C'_1, and extend this coloring to a 3-coloring φ_2 of H'_2. This is easily seen to ensure that the winding numbers on C_1 and C_2 in these colorings match. Hence, the coloring of $C_1 \cup C_2$ given by φ_1 and φ_2 extends to a 3-coloring φ_3 of R'. We can now combine the restrictions of φ_1 and φ_2 to $H_1 - V(R')$ and $H_2 - V(R')$ with φ_3 to obtain a 3-coloring of G extending ψ.

In the more general setting of Theorem 1.3, there are further complications arising from the fact that we need to avoid creating separating 4-cycles (or at least, creating separating 4-cycles too close to the anomalies) and that we need to handle the case there is only one anomaly, essentially proving the analogue of Lemma 2.1 for a graph with one anomaly sufficiently far away from a precolored (≤ 5)-cycle.

2 Previous results

We use the following lemma of Aksionov [1].

Lemma 2.1. *Let G be a plane graph with at most one triangle, and let C be either the null graph or a facial cycle of G of length at most five. If C has length* *five and G contains a triangle T, also assume that C and T are edge-disjoint. Then every 3-coloring of C extends to a 3-coloring of G.*

We also need several results from previous papers of this series. Let G be a graph and C its subgraph. We say that G is *C-critical* if $G \neq C$ and for every proper subgraph G' of G that includes C, there exists a 3-coloring of C that extends to a 3-coloring of G', but does not extend to a 3-coloring of G. The following claim is a special case of the general form of the main result of [13] (Theorem 4.1).

Theorem 2.2. *There exists an absolute constant η with the following property. Let G be a plane graph and Z a (not necessarily connected) subgraph of G such that all triangles and all separating 4-cycles in G are contained in Z. If G is Z-critical, then $\sum |f| \leq \eta |V(Z)|$, where the summation is over all faces f of G of length at least five.*

The following is a simple consequence of Corollary 5.3 of [13].

Lemma 2.3. *Let G be a triangle-free plane graph with the outer face f_0 bounded by a cycle and with another face f bounded by a cycle of length at least $|f_0|-1$. If every cycle separating f_0 from f in G has length at least $|f_0|-1$, then every 3-coloring of the cycle bounding f_0 extends to a 3-coloring of G.*

Finally, let us state a basic property of critical graphs.

Proposition 2.4. *Let G be a graph and C its subgraph such that G is C-critical. If $G = G_1 \cup G_2$, $C \subseteq G_1$ and $G_2 \not\subseteq G_1$, then G_2 is $(G_1 \cap G_2)$-critical.*

3 Structure of graphs without collapsible 4-faces

Essentially all papers dealing with 3-colorability of triangle-free planar graphs first eliminate 4-faces by identifying their opposite vertices, thus reducing the problem to graphs of girth 5. However, this reduction might decrease distances in the resulting graph, which constrains its applicability for the problems we consider. In this section, we give a structural result on graphs in that no 4-face can be reduced.

Let F be a cycle in a graph G, and let $S \subseteq V(G)$. We say that the cycle F is *S-tight* if F has length four and the vertices of F can be numbered v_1, v_2, v_3, v_4 in order such that for some integer $t \geq 0$ the vertices v_1, v_2 are at distance exactly t from S, and the vertices v_3, v_4 are at distance exactly $t+1$ from S. We say that a face is S-tight if it is bounded by an S-tight cycle.

A triple $(G, \mathcal{S}, C)$ is a *scene* if G is a connected plane graph, $\mathcal{S}$ is a family of non-empty subsets of $V(G)$ each of which induces a connected subgraph of G, and C is either the null graph $\varnothing$ or a cycle of length at most five bounding the outer face of G. For a positive integer d, the scene is *d-distant* if for all distinct $S, S' \in \mathcal{S}$, the distance between S and S' in G is at least d.

Lemma 3.1. *Let $d \geq 1$ be an integer and let $(G, \mathcal{S}, C)$ be a $2d$-distant scene. Let F be a cycle in G of length four and assume that for each pair u, v of diagonally opposite vertices of F, two distinct sets in $\mathcal{S}$ are at distance at most $2d-1$ in the graph obtained from G by identifying u and v. Then there exists a unique set $S_0 \in \mathcal{S}$ at distance at most $d-1$ from F. Furthermore, F is S_0-tight.*

Proof. Let the vertices of F be v_1, v_2, v_3, v_4 in order. By hypothesis there exist sets $S_1, S_2, S_3, S_4 \in \mathcal{S}$, where S_i is at distance d_i from v_i, such that $S_1 \neq S_3$, $S_2 \neq S_4$, $d_1 + d_3 \leq 2d - 1$, and $d_2 + d_4 \leq 2d - 1$. From the symmetry we may assume that $d_1 \leq d - 1$ and $d_2 \leq d - 1$. The distance between S_1 and S_2 is at most $d_1 + d_2 + 1 \leq 2d - 1$, and thus $S_1 = S_2$. Let us set $S_0 = S_1$. If any $S \in \mathcal{S}$ is at distance at most $d - 1$ from F, then the distance between S and S_0 is at most $2(d-1) + 1 < 2d$, and thus $S = S_0$. It follows that S_0 is the unique element of $\mathcal{S}$ at distance at most $d - 1$ from F.

Note that $S_4 \neq S_2 = S_1$, and hence $d_1 + d_4 + 1 \geq 2d$, because S_1 and S_4 are at distance at least $2d$. This and the inequality $d_2 + d_4 \leq 2d - 1$ imply that $d_1 \geq d_2$. But there is a symmetry between d_1 and d_2, and hence an analogous argument shows that $d_1 \leq d_2$. Thus for $t := d_1 = d_2$ the vertices v_1, v_2 are both at distance t from $S_0 = S_1 = S_2$. If v_4 were at distance t or less from S_0, then S_0 and S_4 would be at distance at most $t + d_4 = d_2 + d_4 \leq 2d - 1$, a contradiction. The same holds for v_3 by symmetry, and hence v_3 and v_4 are at distance $t + 1$ from S_0; hence, F is S_0-tight. □

We often use the following observation on vertices only incident with tight faces.

Observation 3.2. *Let $(G, \mathcal{S}, C)$ be a distant scene and let $v \in V(G)$ be a vertex such that for some $S \in \mathcal{S}$, every face incident with v is S-tight. Let t be the distance between v and S. Then v has even degree, and in the clockwise ordering of the neighbors of v in the drawing of G, every second neighbor is at distance exactly t from S, while every other neighbor is at distance $t-1$ or $t+1$ from S.*

Let G be a graph, let $S \subseteq V(G)$ and let K be a cycle in G. We say that K is *equidistant* from S if for some integer $t \geq 0$, every vertex of K is at distance exactly t from S. We will also say that K is equidistant from S at distance t.

We say that a plane graph H is a *cylindrical quadrangulation with boundary faces f_1 and f_2* if the distinct faces f_1 and f_2 of H are bounded by cycles and all other faces of H have length four. The union of the cycles bounding f_1 and f_2 is called the *boundary* of H. The cylindrical quadrangulation H is a *joint* if $|f_1| = |f_2|$, every cycle of H separating f_1 from f_2 has length at least $|f_1|$ and the distance between f_1 and f_2 in H is at least $4|f_1|$. If H appears as a subgraph of another plane graph G, we say that the appearance is *clean* if every face of H except for f_1 and f_2 is also a face of G. An $r \times s$ *cylindrical grid* is the Cartesian product of a path with r vertices and a cycle of length s.

Let $(G, \mathcal{S}, C)$ be a scene, R a cycle in G, and $S \in \mathcal{S}$ a set disjoint from R. Removing R splits the plane into two open sets, and since $G[S]$ is connected, S is contained in one of them; let $\Omega_S(R)$ denote the other one. We say S is *tightly isolated by* R if R is an equidistant cycle of length $s \geq 3$ at some distance $d_0 \geq 1$ from S, and for $d_1 = d_0 + 2(s-2)(s+3)$, letting $V_G(S, R)$ be the set of vertices of G at distance at most d_1 from S that are drawn in the closure of $\Omega_S(R)$, every face of G drawn in $\Omega_S(R)$ and incident with a vertex of $V_G(S, R)$ is S-tight.

Lemma 3.3. *Let $(G, \{S\}, \varnothing)$ be a scene. If S is tightly isolated by a cycle R_0 in G and every vertex of $V_G(S, R_0)$ has degree at least three, then G contains a clean joint H such that $V(H) \subseteq V_G(S, R_0)$.*

Proof. Let $s = |R_0|$ and let d_0 be the distance between S and R_0 in G. For an integer j, let $d(j) = d_0 + 2(s-j)(s+j+1)$. Note that $d(j) + 4j = d(j-1)$ for every j, $d_0 = d(s)$, and every vertex of $V_G(S, R_0)$ is at distance at most $d_1 = d_0 + 2(s-2)(s+3) = d(2)$ from S. Choose the smallest integer $j \in \{3, \ldots, s\}$ for that there exists an equidistant cycle R of length j at distance t from S such that $d_0 \leq t \leq d(j)$ and R is drawn in the closure of $\Omega_S(R_0)$; note this implies $V(R) \subseteq V_G(S, R_0)$. Such an integer j exists, since R_0 satisfies the requirements for $j = s$. Let $p \leq 4j$ be the maximum integer such that G contains a clean $(p+1) \times |R|$ cylindrical grid H with boundary faces f_1 and f_2 as a subgraph such that f_1 is bounded by R and f_2 is bounded by an equidistant cycle K at distance $t + p$ from S, and f_2 is drawn in $\Omega_S(R)$; note this implies $V(H) \subseteq V_G(S, R_0)$. Such an integer p exists, since R (treated as a $1 \times |R|$ cylindrical grid) satisfies the requirements for $p = 0$.

We claim that $p = 4j$, and thus H satisfies the conclusion of the theorem. Suppose that $p \leq 4j - 1$. Note that every vertex of G drawn in $\Omega_S(K)$ is at distance at least $t + p + 1$ from S. Observe that K has no chord contained in $\Omega_S(K)$, as otherwise there exists an equidistant cycle of length less than j at distance $t + p \leq t + 4j - 1 < d(j-1)$ from S contradicting the minimality of j. Hence, Observation 3.2 implies that every vertex $v \in V(K)$ has exactly one neighbor v' drawn in $\Omega_S(K)$.

Let Z be the subgraph of G induced by $\{v' : v \in V(K)\}$; note that $V(Z)$ consists exactly of all vertices drawn in $\Omega_S(K)$ at distance $t + p + 1 \leq t + 4j \leq d(j-1)$ from S, and in particular $V(Z) \subset V_G(S, R_0)$. By the assumptions of this lemma, all vertices in $V(Z)$ have degree at least three in G, and thus Observation 3.2 implies Z has minimum degree at least two. Consequently, Z contains a cycle Z'. Note that Z' is equidistant at distance at most $d(j-1)$ from S and $|Z'| \leq |V(Z)| \leq |K| = j$. By the minimality of j, it follows that $|Z'| = j$, and thus $|V(Z)| = |K|$. Therefore, $v_1' \neq v_2'$ for distinct vertices $v_1, v_2 \in V(K)$. We conclude that we can extend H to a clean $(p+2) \times |R|$ cylindrical grid by adding Z' and the edges vv' for $v \in V(K)$, contradicting the maximality of p. This finishes the proof. □

Next, we consider the case that some of the relevant faces are not tight, but instead are near to a short separating cycle. A 4-face f is *attached* to a cycle R if the boundary cycle of f and R intersect in a path of length two. Let $d_2 < d_3$ and s be positive integers and let $(G, \mathcal{S}, C)$ be a scene. For $S \in \mathcal{S}$, we say that a cycle R *separates* S *from* C if C is not the null graph, $R \neq C$, and S is drawn in the open disk bounded by R (recall that C bounds the outer face of G). We say that the scene is (d_2, d_3)-*tight* if for every $S \in \mathcal{S}$, every 4-face of G at distance at least d_2 and at most d_3 from S is bounded by C, or S-tight, or attached to a (≤ 6)-cycle separating S from C. An (S, d_2, d_3)-*slice* is a subset L of vertices of G such that

- each vertex $v \in L$ is at distance at least d_2 and at most d_3 from S,
- if $v \in L$ has a neighbor in G not belonging to L, then the distance between S and v is either exactly d_2 or exactly d_3, and
- L contains a vertex at distance exactly $d_3 - 1$ from S.

Note that the last two conditions imply that L contains vertices at any distance d from S such that $d_2 \leq d \leq d_3 - 1$. The *interior* L° of L is the set of vertices at distance at least $d_2 + 1$ and at most $d_3 - 1$ from S. When the parameters are clear from the context, we call L just a slice. For a positive integer s, we say that a set $S \in \mathcal{S}$ is (d_2, d_3, s)-*isolated* by an (S, d_2, d_3)-slice L if

- $L \cap V(C) = \emptyset$ and every vertex of L has degree at least three,
- every face of G incident with a vertex of L has length four, and
- every cycle $K \subseteq G[L]$ equidistant from S has length at most s.

Lemma 3.4. *Let $d_2 \geq 4$ and $s \geq 3$ be integers, let $d_3 = d_2 + 34(s-2)(s+3) + 474$, and let $(G, \{S\}, C)$ be a (d_2, d_3)-tight scene. If S is (d_2, d_3, s)-isolated by a slice L, then G contains a clean joint H with $V(H) \subseteq L^\circ$.*

Proof. Let $\mathcal{K}$ be the set of all (≤ 6)-cycles $K \subset G[L^\circ]$ that separate S from C in G. For an integer t such that $d_2 \leq t \leq d_3$, let G_t denote the subgraph of $G[L]$ induced by vertices at distance exactly t from S. By assumptions, every cycle in G_t has length at most s.

If $d_2 + 4 \leq t \leq d_3 - 4$ and $v \in V(G_t)$ is at distance at least two from every element of $\mathcal{K}$, then all faces incident with v are S-tight and $\deg_{G_t}(v) \geq 2$. (1)

Subproof. Since $v \in L$, any face f of G incident with v is a 4-face not bounded by C. Since $d_2 + 4 \leq t \leq d_3 - 4$, if f were attached to a (≤ 6)-cycle K separating S from C, then we would have $K \subset G[L^\circ]$, and thus K would be an element of $\mathcal{K}$ at distance at most one from v, contradicting the assumptions. Since the scene is (d_2, d_3)-tight, we conclude every face incident with v is S-tight. Since $\deg_G(v) \geq 3$, Observation 3.2 implies $\deg_{G_t}(v) \geq 2$. ■

For a cycle $K \in \mathcal{K}$, let Δ_K be the closed disk bounded by K. For distinct $K_1, K_2 \in \mathcal{K}$, we write $K_1 \prec K_2$ if K_1 is drawn in Δ_{K_2}, and we write G_{K_1,K_2} for the subgraph of G drawn in $\Delta_{K_2} \setminus \Delta^\circ_{K_1}$.

Consider cycles $K_1, K_2 \in \mathcal{K}$ of the same length r such that $K_1 \prec K_2$ and no cycle $K \in \mathcal{K}$ of length less than r satisfies $K_1 \prec K \prec K_2$. For $i \in \{1, 2\}$, let k_i denote the distance between S and K_i. If $k_1 + 4r + 3 \le k_2 \le d_3 - 2(s-2)(s+3) - 12$, then G contains a clean joint H such that $V(H) \subseteq L^\circ$.

(2)

Subproof. Note that by the assumptions of the claim, no cycle in G_{K_1,K_2} that separates K_1 from K_2 has length less than r and the distance between K_1 and K_2 is at least $4r$. If $V(G_{K_1,K_2}) \subseteq L^\circ$, then since S is (d_2, d_3, s)-isolated by L, all faces of G_{K_1,K_2} not bounded by K_1 or K_2 have length four, and thus we can set $H = G_{K_1,K_2}$.

Therefore, assume that G_{K_1,K_2} contains a vertex not in L°; since L is a slice and G is connected, we conclude $G_{K_1,K_2} \cap G[L]$ contains vertices at any distance between k_1 and d_3 from S. Let $t = k_2 + 8$ and let Q be a connected component of G_t contained in G_{K_1,K_2}. Observe that every cycle $K \in \mathcal{K}$ which intersects G_{K_1,K_2} is at distance at most k_2 from S if $K \prec K_2$, and at most $k_2 + 3$ if K intersects K_2, and thus its distance from Q is at least two. By (1), Q has minimum degree at least two, and thus Q contains a cycle R, necessarily of length at most s. Furthermore, (1) implies every face f incident with a vertex $v \in V_G(S, R)$ is S-tight. By Lemma 3.3, G contains a clean joint H with $V(H) \subseteq V_G(S, R) \subseteq L^\circ$, as required. ■

Let $b_2 = d_2 - 1$ and $e_2 = d_3 - 2(s-2)(s+3) - 11$. For $3 \le r \le 6$, let b_r and e_r be chosen so that $b_{r-1} \le b_r \le e_r \le e_{r-1}$, every cycle in $\mathcal{K}$ of length r is at distance either at most b_r or at least e_r from S, and subject to these conditions, $e_r - b_r$ is as large as possible.

Consider a fixed $r \in \{3, 4, 5, 6\}$. If no cycle in $\mathcal{K}$ has length r and is at distance more than b_{r-1} and less than e_{r-1} from S, then we have $b_r = b_{r-1}$ and $e_r = e_{r-1}$. Otherwise, let $K_1 \in \mathcal{K}$ be a cycle of length r whose distance k_1 from S satisfies $b_{r-1} < k_1 < e_{r-1}$ and subject to that, k_1 is as small as possible; and, let $K_2 \in \mathcal{K}$ be a cycle of length r whose distance k_2 from S satisfies $b_{r-1} < k_2 < e_{r-1}$ and subject to that, k_2 is as large as possible. If $k_2 \ge k_1 + 4r + 3$, then (2) implies that the conclusion of this lemma holds, and thus we can assume that $k_2 \le k_1 + 4r + 2$. Note that the distance of every cycle in $\mathcal{K}$ of length r from S is at most b_{r-1}, or between k_1 and k_2 (inclusive), or at least e_{r-1}. Furthermore, $(k_1 - b_{r-1}) + (e_{r-1} - k_2) = (e_{r-1} - b_{r-1}) - (k_2 - k_1) \ge (e_{r-1} - b_{r-1}) - 4r - 2$, and thus, considering (b_{r-1}, k_1) and (k_2, e_{r-1}) as possible choices for (b_r, e_r), we have $e_r - b_r \ge \max(k_1 - b_{r-1}, e_{r-1} - k_2) \ge \frac{e_{r-1} - b_{r-1}}{2} - 2r - 1$.

It follows that $e_6 - b_6 > \frac{e_2 - b_2}{16} - 22 = \frac{d_3 - d_2 - 2(s-2)(s+3) - 362}{16} = 2(s-2)(s+3) + 7$. Let $t = b_6 + 5$ and let Q be a connected component of G_t (note that G_t is non-empty, since L is a slice). Observe the distance between Q and every element of $\mathcal{K}$ is at least two, and thus by (1), Q has minimum degree at least two. Consequently, Q contains a cycle R, necessarily of length at most s. Since $t + 2(s-2)(s+3) \le e_6 - 2$, every vertex $v \in V_G(S, R)$ is at distance at least $b_6 + 5$ and at most $e_6 - 2$ from S, and thus every cycle $K \in \mathcal{K}$ is at distance at least two from v. Consequently, (1) implies all faces incident with v are S-tight. By Lemma 3.3, G contains a clean joint H such that $V(H) \subseteq V_G(S,R) \subseteq L^\circ$, as required. $\square$

Let G be a plane graph. For a set $S \subseteq V(G)$, a path P from a vertex v to S is *S-geodesic* if P is a shortest path from v to S. Let B be an odd cycle in G, let Λ be one of the two connected open subsets of the plane bounded by B, let uv be an edge of B, let w be the vertex of B that is farthest (as measured in B) from uv and let z be a vertex of G such that either $z = w$, or z does not belong to the closure of Λ. Let P_u and P_v be the paths in $B - uv$ joining u and v, respectively, with w. We say that Λ is a *z-petal with top uv* if there exists a path Q in G between w and z such that the paths $Q \cup P_u$ and $Q \cup P_v$ are $\{z\}$-geodesic.

Let S be a set of vertices inducing a connected subgraph of G and consider a cycle K which is equidistant at some distance $t \geq 1$ from S. The removal of K splits the plane into two open sets, let Δ be the one containing S. For each $v \in V(K)$, choose an S-geodesic path P_v. We can choose the paths so that for every $u, v \in V(K)$, the paths P_u and P_v are either disjoint or intersect in a path ending in S. Removing $G[S]$ and the paths P_v for $v \in V(K)$ splits Δ to several parts; for each $e \in E(K)$, let Δ_e be the one whose boundary contains e. Clearly, Δ_e and $\Delta_{e'}$ are disjoint for distinct $e, e' \in E(K)$. We call the collection $\{\Delta_e : e \in E(K)\}$ a *flower of K with respect to S*. Let us remark that not all elements of a flower are necessarily petals: Δ_e is a z-petal with top e for some $z \in S$ if and only of the boundary of Δ_e does not contain any edge of $G[S]$.

Since a petal is bounded by an odd cycle, it contains an odd face of G. However, this face could in general be arbitrarily far from S. In the next lemma, we exploit the presence of S-tight faces to find a face of length other than four close to S.

Lemma 3.5. *Let d_4 be a positive integer and let $(G, \{S\}, \varnothing)$ be a scene such that every vertex v at distance exactly d_4 from S has degree at least three and all 4-faces incident with v are S-tight. For some $d \leq d_4$, let uv be an edge of G such that both u and v are at distance exactly d from S, and suppose $z \in S$ is at distance exactly d from both u and v. For every z-petal Δ with top uv, there exists a face $f \subseteq \Delta$ of G of length other than four at distance at most d_4 from S.*

Proof. We can assume that Δ is minimal, i.e., there is no $\Delta' \subsetneq \Delta$ such that Δ' is a z-petal satisfying the assumptions of the lemma. Since Δ is bounded by an odd cycle, there exists an odd face f contained in Δ. It suffices to consider the case that the distance between f and S is at least d_4+1. Let Q be the subgraph of G induced by vertices at distance exactly d_4 from S that are contained in the closure of Δ. Note that Q is non-empty since G is connected, and can intersect the boundary of Δ only in the edge uv.

If $Q = uv$, then $\{u, v\}$ forms a cut in G that separates the rest of the boundary of Δ from the vertices incident with f. Observe that this implies that there exists a face f' contained in Δ in whose boundary u and v appear non-consecutively. This implies f' is not S-tight, and thus it is not a 4-face. Hence, the conclusion of this lemma is satisfied.

Therefore, we can assume that $Q \neq uv$. By Observation 3.2, all vertices of Q other than u and v have degree at least two in Q. Since $uv \in E(Q)$, it follows that Q contains a cycle K, which is equidistant at distance d_4 from S. Let $F = \{\Delta_e : e \in E(K)\}$ be a flower of K with respect to S and let e_0 be the unique edge of K such that the closure of Δ_{e_0} contains the edge uv. Note that since every vertex in the boundary of Δ is contained in an S-geodesic path ending in z, every vertex of K is at distance exactly d_4 from z, and thus we can choose F so that $\Delta_e \subset \Delta$ and Δ_e is a z-petal for every $e \in E(K) \setminus \{e_0\}$. Since $|F| = |K| \geq 3$, it follows that each such z-petal Δ_e is a proper subset of Δ. This contradicts the minimality of Δ. □

Next, we apply Theorem 2.2 to prove existence of sufficiently isolated anomalies in hypothetical minimal counterexamples to Theorem 1.3. To this end, we need a few more definitions. For $p \geq 1$, we say that a scene $(G, \mathcal{S}, C)$ is *p-small* if every set in $\mathcal{S}$ has size at most p. The scene is *internally triangle-free* if for every triangle $T \neq C$ in G, there exists $S \in \mathcal{S}$ such that $T \subseteq G[S]$. For $S \in \mathcal{S}$, a cycle $F \neq C$ in G is *S-private* if the open disk bounded by F contains a vertex of S, but not of any other set from $\mathcal{S}$. For an integer $d \geq 1$, we say the scene has *no d-distant private 4-cycles* if for every $S \in \mathcal{S}$, every S-private 4-cycle in G is at distance less than d from S. We say that a 4-cycle is *$\mathcal{S}$-private* if it is S-private for some $S \in \mathcal{S}$.

Consider a face f of G, bounded by a closed walk $v_1 v_2 \ldots v_m$ going clockwise around f. A pair $(v_{i-1}v_i v_{i+1}, f)$ for $1 \leq i \leq m$ (where $v_0 = v_m$ and $v_{m+1} = v_1$) is called an *angle* in G, and v_i is its *tip*.

Lemma 3.6. *For all integers $D_1 \geq 2$ and $p \geq 1$ and for every function $h : \mathbb{N} \to \mathbb{N}$, there exist integers $s \geq 1$ and $D_2 > D_1$ with the following property. Let $(G, \mathcal{S}, C)$ be a (D_1, D_2)-tight $2D_2$-distant p-small internally triangle-free scene with no D_1-distant private 4-cycles. If $|\mathcal{S}| = 1$, assume furthermore that C is not the null graph and the distance between C and the unique element of $\mathcal{S}$ is at least $D_2 - 1$.*

Let $Z = C \cup \bigcup_{S \in \mathcal{S}} G[S]$. If G is Z-critical, then there exists an integer $d \geq D_1$ such that $d + h(s) \leq D_2$ and some element of $\mathcal{S}$ is $(d, d + h(s), s)$-isolated.

Proof. Let $\mu = 2\eta(3p+5)$, where η is the constant from Theorem 2.2, $s = \mu + 6p$, and $D_2 = D_1 + 3 + (\mu + 1)(h(s) + 1)$.

By removing some of the edges of $E(Z) \setminus E(C)$ from G if necessary, we can assume G contains no triangle other than C. Since G is Z-critical, note that Lemma 2.1 implies $\mathcal{S} \neq \emptyset$ and the open disk bounded by any separating 4-cycle in G contains a vertex of $\bigcup \mathcal{S}$. If G contains a non-$\mathcal{S}$-private separating 4-cycle, then let C_0 be such a 4-cycle with the closed disk Δ_0 bounded by C_0 minimal. Otherwise, let Δ_0 be the whole plane and $C_0 = C$.

For each $S \in \mathcal{S}$, let $\mathcal{F}_S$ denote the set of S-private 4-cycles F in G such that the open disk Λ_F bounded by F is contained in Δ_0 and is inclusionwise-maximal among all 4-cycles with this property. We claim that for distinct $F, F' \in \mathcal{F}_S$, the disks Λ_F and $\Lambda_{F'}$ are disjoint. Indeed, since G contains at most one triangle, the cycles F and F' are induced, and thus if $\Lambda_F \cap \Lambda_{F'} \neq \emptyset$, then the open disk $\Lambda_F \cup \Lambda_{F'}$ is also bounded by an S-private 4-cycle, contradicting the maximality of Λ_F or $\Lambda_{F'}$. Since each of the disks contains a vertex of S, we conclude $|\mathcal{F}_S| \leq |S|$.

Furthermore, for distinct $S, S' \in \mathcal{S}$ and any $F \in \mathcal{F}_S$ and $F' \in \mathcal{F}_{S'}$, the disks Λ_F and $\Lambda_{F'}$ are disjoint. Indeed, since the scene has no D_1-distant private 4-cycles, the distance between S and F, and between S' and F', is less than D_1, and since the scene is $2D_2$-distant, the cycles F and F' are vertex-disjoint. Futhermore $\Lambda_F \not\subseteq \Lambda_{F'}$ since Λ_F contains a vertex of S and F' is S'-private, and symmetrically $\Lambda_{F'} \not\subseteq \Lambda_F$. This implies $\Lambda_F \cap \Lambda_{F'} = \emptyset$.

Let $\mathcal{S}_1 \subseteq \mathcal{S}$ consist of the sets $S \in \mathcal{S}$ intersecting Δ_0; note that $\mathcal{S}_1 \neq \emptyset$. For $S \in \mathcal{S}_1$, let Δ_S be the complement of $\bigcup_{F \in \mathcal{F}_S} \Lambda_S$ and let B_S be the subgraph of $G[S] \cup \bigcup \mathcal{F}_S$ drawn in $\Delta_S \cap \Delta_0$. Let G_1 be the subgraph of G drawn in the subset $\Delta_1 = \Delta_0 \cap \bigcap_{S \in \mathcal{S}_1} \Delta_S$ of the plane. Let $Z_1 = C_0 \cup \bigcup_{S \in \mathcal{S}_1} B_S$; Proposition 2.4 implies that G_1 is Z_1-critical.

If some cycle $F \in \mathcal{F}_S$ is vertex-disjoint from S, then since $G[S]$ is connected, we conclude $\mathcal{F}_S = \{F\}$ and $B_S = F$. Otherwise, every cycle $F \in \mathcal{F}_S$ intersects S, and thus $G[S] \cup \bigcup \mathcal{F}_S$ is connected, and either B_S is connected or every component of B_S intersects C_0; and furthermore, $|V(B_S)| \leq 3|S| \leq 3p$. Since the scene has no D_1-distant private 4-cycles, every vertex of B_S is at distance at most $D_1 + 1$ from S.

Since the scene is $2D_2$-distant, at most one set in $\mathcal{S}_1$ is at distance at most $D_2 - 2$ from C_0. Moreover, if $|\mathcal{S}_1| = 1$, then we could not have chosen C_0 as a non-$\mathcal{S}$-private separating 4-cycle, and thus $C_0 = C$ is at distance at least $D_2 - 1$ from the unique element of $\mathcal{S}_1 = \mathcal{S}$ by the assumptions of this lemma. Therefore, letting $\mathcal{S}'_1$ consist of the sets $S \in \mathcal{S}_1$ at distance at least $D_2 - 1$ from C_0, we have $|\mathcal{S}'_1| \geq |\mathcal{S}_1|/2$.

A face f of G is *poisonous* if $f \subseteq \Delta_1$ and f has length at least 5. The construction of G_1 ensures that it has no separating 4-cycles, and thus Theorem 2.2 implies

$$\sum |f| \leq \eta|V(Z_1)| \leq \eta(3p|\mathcal{S}_1| + 5) \leq \mu|\mathcal{S}'_1|, \tag{3}$$

where the summation is over all poisonous faces of G. Consider $S \in \mathcal{S}'_1$. We say that an angle (xyz, f) in G is *S-contaminated* if f is poisonous and the distance between S and y in G is at most $D_2 - 1$. Since every S-contaminated angle contributes at least one toward the sum in (3), we deduce that there exists $S \in \mathcal{S}'_1$ such that there are at most μ angles that are S-contaminated. Let us fix such a set S.

By the choice of D_2, there exists an integer $d \geq D_1 + 2$ such that $d + h(s) \leq D_2 - 2$ and no angle with tip at distance at least d and at most $d + h(s)$ is S-contaminated. Let L consist of the vertices of G drawn in Δ_1 at distance at least d and at most $d + h(s)$ from S. Observe that L is an $(S, d, d + h(s))$-slice and every vertex of L is contained in the interior of Δ_1, since C_1 is at distance at least $D_2 - 1$ from S, every vertex of B_S is at distance at most $D_1 + 1$ from S, and for $S' \in \mathcal{S}_1$, the subgraph $B_{S'}$ is at distance at least $2D_2-(D_1+1) > D_2-1$ from S. In particular, $L \cap V(C) = \emptyset$. Since G_1 is Z_1-critical, every vertex of L has degree at least three. The choice of d implies that every face of G incident with a vertex of L has length four.

Hence, it remains to argue that every cycle $K \subseteq G[L]$ equidistant from S has length at most s. First, observe the argument from the previous paragraph also implies that every face f of G contained in Δ_S and at distance less than D_2-1 from S is contained in Δ_1. Let f_S be the face of B_S containing K, and let $W = \{\Delta_e : e \in E(K)\}$ be a flower of K in G with respect to S. Observe that if the closure of Δ_e does not contain any edge of the boundary of f_S, then Δ_e is a z-petal for some $z \in S$ and $\Delta_e \subset \Delta_S$. Lemma 3.5 applied with $d_4 = D_2 - 2$ to the scene $(G, \{S\}, \varnothing)$ implies that there exists a face $f \subseteq \Delta_e$ of G of length other than four at distance less than $D_2 - 1$ from S, and as we observed, this implies that f is contained in Δ; hence, f is poisonous and contributes an S-contaminated angle. Consequently, all but at most μ elements of W contain an edge of the boundary of f_S in their closure. Since $|V(B_S)| \leq 3p$, f_S has length at most $6p$, and thus $|K| = |W| \leq \mu + 6p = s$, as required. □

We can now combine the lemmas to obtain the main structural result of this section.

Lemma 3.7. *There exists a function $f_{3.7} : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let $D_1 \geq 2$ and $p \geq 1$ be integers and let $D_2 = f_{3.7}(D_1, p)$. Let $(G, \mathcal{S}, C)$ be a (D_1, D_2)-tight $2D_2$-distant p-small internally triangle-free scene with no D_1-distant private 4-cycles. If $|\mathcal{S}| = 1$, assume furthermore that C is not the null graph and the distance between C and the unique element of $\mathcal{S}$ is at least D_2-1. Let $Z = C \cup \bigcup_{S\in\mathcal{S}} G[S]$. If G is Z-critical, then G contains a clean joint H whose vertices are at distance at least D_1 and at most D_2-1 from some element of $\mathcal{S}$. Furthermore, H is vertex-disjoint from C.*

Proof. We choose D_2 and s according to Lemma 3.6 for the function $h(s) = 34(s-2)(s+3)+474$. By Lemma 3.6, there exists an integer $d_2 \geq D_1$ such that $d_3 = d_2 + h(s) \leq D_2$ and some $S \in \mathcal{S}$ is (d_2, d_3, s)-isolated by some slice L. By Lemma 3.4 applied to $(G, \{S\}, C)$, G contains a clean joint H with $V(H) \subseteq L^\circ$. Consequently, H is vertex-disjoint from C and at distance at least $d_2 + 1 > D_1$ and at most $d_3 - 1 \leq D_2 - 1$ from S. □

4 Colorings of quadrangulations of a cylinder

In this section, we give a lemma on extending a precoloring of boundaries of a quadrangulated cylinder. This is a special case of a more general theory which we develop in the following paper of the series [14].

Let C be a cycle drawn in plane, let $v_1, v_2, \ldots, v_k$ be the vertices of C listed in the clockwise order of their appearance on C, and let $\varphi : V(C) \to \{1, 2, 3\}$ be a 3-coloring of C. We can view φ as a mapping of $V(C)$ to the vertices of a triangle, and speak of the *winding number of φ on C*, defined as the number of indices $i \in \{1, 2, \ldots, k\}$ such that $\varphi(v_i) = 1$ and $\varphi(v_{i+1}) = 2$ minus the number of indices i such that $\varphi(v_i) = 2$ and $\varphi(v_{i+1}) = 1$, where v_{k+1} means v_1. We denote the winding number of φ on C by $W_\varphi(C)$.

Consider a plane graph G and its 3-coloring φ. For a face f of G bounded by a cycle C, we define the *winding number of φ on f*, which is denoted by $w_\varphi(f)$, as $-W_\varphi(C)$ if f is the outer face of G and as $W_\varphi(C)$ otherwise. The following two propositions are easy to prove.

Proposition 4.1. *Let G be a plane graph such that every face of G is bounded by a cycle, and let $\varphi : V(G) \to \{1, 2, 3\}$ be a 3-coloring of G. Then the sum of the winding numbers of all the faces of G is zero.*

Proposition 4.2. *The winding number of every 3-coloring on a cycle of length four is zero.*

Let G be a cylindrical quadrangulation with boundary faces f_1 and f_2. We say that the cylindrical quadrangulation is *boundary-linked* if every cycle K in G separating f_1 from f_2 and not bounding either of these faces has length at least $\max(|f_1|, |f_2|)$, and if $|K| = |f_i| = \max(|f_1|, |f_2|)$ for some $i \in \{1, 2\}$, then $V(K) \cap V(f_{3-i}) \neq \emptyset$. The cylindrical quadrangulation is *long* if the distance between f_1 and f_2 is at least $|f_1| + |f_2|$.

Lemma 4.3. *Let G be a long boundary-linked cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. Suppose that $|f_1| \geq \max(5, |f_2|)$ and let $v_1v_2v_3$ be a subpath of the cycle bounding f_1, where $\psi(v_1) = \psi(v_3)$. Then, there exists a long boundary-linked cylindrical quadrangulation G' with boundary faces f'_1 and f'_2 such that $|f'_1| = |f_1| - 2$ and $|f'_2| = |f_2|$ together with a 3-coloring ψ' of the boundary of G' such that $w_{\psi'}(f'_1) = w_\psi(f_1)$, $w_{\psi'}(f'_2) = w_\psi(f_2)$, and if ψ' extends to a 3-coloring of G', then ψ extends to a 3-coloring of G.*

Proof. Note that since $\max(|f_1|, |f_2|) \geq 5$ and G is boundary-linked, it follows that G contains no triangle other than possibly the cycle bounding f_2, and thus the neighbors of v_2 form an independent set in G_2. Furthermore, f_1 is an induced cycle. Let G' be the cylindrical quadrangulation obtained from $G - v_2$ by contracting all neighbors of v_2 (including v_1 and v_3) to a single vertex w and by suppressing the arising 2-faces. Let f'_1 and f'_2 be the faces of G' corresponding to f_1 and f_2, respectively. Clearly, G' is long.

Let ψ' be the coloring of the boundary of G' such that $\psi'(w) = \psi(v_1)$ and $\psi'(z) = \psi(z)$ for all vertices $z \neq w$ in the boundary. If ψ' extends to a 3-coloring φ of G', then we can turn φ into a 3-coloring of G extending ψ by setting $\varphi(z) = \psi(v_1)$ for every neighbor z of v_2 and $\varphi(v_2) = \psi(v_2)$.

Consider a cycle K' separating f'_1 from f'_2 in G' and not bounding either of these faces. Let K be the corresponding cycle in G (equal to K', or obtained from K' by replacing w by a neighbor of v_2, or obtained from K' by replacing w by a path xv_2y for some neighbors x and y of v_2).

Let us first consider the case that $|f_1| > |f_2|$. Note that $|f_1|$ and $|f_2|$ have the same parity, and thus $|f_1| \geq |f_2| + 2$ and $|f'_1| \geq |f_1| - 2 \geq |f_2|$. Consequently,

$|K'| \geq |K| - 2 \geq |f_1| - 2 = \max(|f_1'|, |f_2'|)$. Furthermore, the equality only holds if $v_2 \in V(K)$ and $|K| = |f_1|$. Since G is boundary-linked, the latter implies that K also contains a vertex incident with f_2. However, this contradicts the assumption that G is long. Therefore, we have $|K'| > \max(|f_1'|, |f_2'|)$.

Next, we consider the case that $|f_1| = |f_2|$, and thus $\max(|f_1'|, |f_2'|) = |f_2| > |f_1'|$. If $|K| = |f_2|$, then since G is boundary-linked, it would follow that K intersects both f_1 and f_2, contrary to the assumption that G is long. Therefore, $|K| > |f_2|$, and by parity, $|K| \geq |f_2| + 2$. Consequently, $|K'| \geq |K| - 2 \geq |f_2|$. The equality can only hold when K contains v_2, and thus K' contains the vertex w incident with f_1'. We conclude that G' is boundary-linked. $\square$

Lemma 4.4. *Let G be a long cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. If $|f_1| = |f_2| = 4$, then ψ extends to a 3-coloring of G.*

Proof. Let $v_1v_2v_3v_4$ be the cycle bounding f_1. Since ψ uses only three colors, we can without loss of generality assume $\psi(v_1) = \psi(v_3)$. Note that G is bipartite, and thus the vertices at distance exactly three from $\{v_2, v_4\}$ form an independent set. Let G' be the quadrangulation of the plane obtained from G by removing all vertices at distance at most two from $\{v_2, v_4\}$, identifying all vertices at distance exactly three from $\{v_2, v_4\}$ to a single (non-boundary) vertex w and by suppressing the arising 2-faces.

Let ψ' be a restriction of ψ to the 4-cycle bounding the face of G' corresponding to f_2. By Lemma 2.3, ψ' extends to a 3-coloring φ of G'. We can extend φ to a 3-coloring of G as follows. Give all vertices at distance exactly 1 from $\{v_2, v_4\}$ the color $\psi(v_1) = \psi(v_3)$, all vertices at distance exactly 3 from $\{v_2, v_4\}$ the color $\varphi(w)$ and all vertices at distance exactly 2 from $\{v_2, v_4\}$ an arbitrary color different from $\psi(v_1)$ and $\varphi(w)$. The resulting assignment is a 3-coloring of G extending ψ. $\square$

Next, we aim to use the connection between colorings and nowhere-zero flows first noticed by Tutte [26]. We only need the following implication from flows to colorings. A *nowhere-zero $\mathbb{Z}_3$-flow* in a graph G is an orientation of G such that the difference between the indegree and the outdegree of each vertex is divisible by 3. Given an orientation $\vec{G}^\star$ of the dual $G^\star$ of a connected plane graph G and a directed edge $e \in E(\vec{G}^\star)$, we define $l(e) = u$ and $r(e) = v$, where uv is the edge of G crossing e and u is to the left of e.

Proposition 4.5. *Let G be a connected plane graph and let $G^\star$ be its dual. If $\vec{G}^\star$ is a nowhere-zero $\mathbb{Z}_3$-flow, then G has a 3-coloring φ such that $\varphi(r(e)) - \varphi(l(e)) \equiv 1 \pmod 3$ for every $e \in E(\vec{G}^\star)$.*

We say that a 3-coloring ψ of a cycle $C = v_1 \ldots v_k$ is *rotating* if $3|k$, $\psi(v_1) = \psi(v_4) = \ldots = \psi(v_{3k-2})$, $\psi(v_2) = \psi(v_5) = \ldots = \psi(v_{3k-1})$, and $\psi(v_3) = \psi(v_6) = \ldots = \psi(v_{3k})$. Note that for any 3-coloring ψ of C, we have $W_\psi(C) \leq |C|/3$, with equality if and only if ψ is rotating.

Lemma 4.6. *Let G be a long boundary-linked cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G. The coloring ψ extends to a 3-coloring of G if and only if $w_\psi(f_1) + w_\psi(f_2) = 0$.*

Proof. If ψ extends to a 3-coloring of G, then $w_\psi(f_1) + w_\psi(f_2) = 0$ by Propositions 4.1 and 4.2.

Let us now show the converse implication. We proceed by induction on $|f_1| + |f_2|$, and thus we can assume that the claim holds for all graphs whose boundary has less than $|f_1| + |f_2|$ vertices. By symmetry, we can assume that $|f_1| \geq |f_2|$.

If $|f_1| = 4$, then since $|f_1|$ and $|f_2|$ have the same parity, we have $|f_2| = 4$, and ψ extends to a 3-coloring of G by Lemma 4.4. Thus, assume $|f_1| \geq 5$.

If the cycle bounding f_1 contains a path $v_1v_2v_3$ with $\psi(v_1) = \psi(v_3)$, then ψ extends to a 3-coloring of G by Lemma 4.3 and the induction hypothesis. Therefore, we can assume that the boundary cycle of f_1 contains no such path, and thus ψ is rotating on this cycle. It follows that $|f_1|$ is a multiple of 3 and $|w_\psi(f_1)| = |f_1|/3$. Since $w_\psi(f_1) + w_\psi(f_2) = 0$, we have $|w_\psi(f_2)| = |f_1|/3$, and since $|f_2| \leq |f_1|$, we conclude that ψ is also rotating on the boundary of f_2 and $|f_2| = |f_1|$. Since G is long and boundary-linked, every cycle in G that separates f_1 from f_2 and does not bound either of the faces has length at least $|f_1| + 2$.

Let $G^\star$ be the dual of G. Let K_i be the edge-cut in G consisting of the edges incident with $V(f_i)$ that do not belong to $E(f_i)$. Note that the dual $K_i^\star$ of K_i is a cycle in $G^\star$. Let $H = G^\star - (E(K_1^\star) \cup E(K_2^\star))$. Let $f_1^\star$ and $f_2^\star$ be the vertices of the dual corresponding to f_1 and f_2, respectively. Suppose that H contains an edge-cut of size less than $|f_1|$ separating $f_1^\star$ from $f_2^\star$, and thus $G^\star$ contains an edge cut $K^\star$ separating $f_1^\star$ from $f_2^\star$ with less than $|f_1|$ edges belonging to $E(K_1^\star) \cup E(K_2^\star)$. Choose $K^\star$ as a minimal edge-cut with this property; then the dual K to $K^\star$ is a cycle in G separating f_1 from f_2 such that $|E(K) \setminus (E(K_1) \cup E(K_2))| < |f_1|$. In particular, this implies K bounds neither f_1 nor f_2. Since G is long, K does not intersect both K_1 and K_2. As we observed before, $|K| \geq |f_1| + 2$, and thus we can by symmetry assume that K intersects K_1 in at least three edges. Let us choose such a cycle K that shares as many edges with the cycle bounding f_1 as possible. Let P be a subpath of K with both endpoints incident with f_1, but no other vertex or edge incident with f_1. Let Q_1 and Q_2 be the two subpaths of the cycle bounding f_1 joining the endpoints of P labelled so that $P \cup Q_2$ is a cycle separating f_1 from f_2. Consider the cycle $K' = (K - P) \cup Q_1$. Since K intersects K_1 in at least three edges, K' is not the cycle bounding f_1. Since K' shares more edges with the cycle bounding f_1 than K, the choice of K implies that

$$|E(K') \setminus (E(K_1) \cup E(K_2))| \geq |f_1| > |E(K) \setminus (E(K_1) \cup E(K_2))|, \text{ and thus}$$
$$|E(Q_1) \setminus (E(K_1) \cup E(K_2))| > |E(P) \setminus (E(K_1) \cup E(K_2))|.$$

Since $|E(Q_1) \cap (E(K_1) \cup E(K_2))| = 0$ and $|E(P) \cap (E(K_1) \cup E(K_2))| = 2$, we conclude that $|Q_1| > |P| - 2$. However, then the cycle $P \cup Q_2$ has length less than $|f_1| + 2$, contradicting the assumption that G is boundary-linked.

Therefore, H does not contain any edge-cut of size less than $|f_1|$ separating $f_1^\star$ from $f_2^\star$, and by Menger's theorem, H contains pairwise edge-disjoint paths $P_1, \ldots, P_{|f_1|}$ joining $f_1^\star$ with $f_2^\star$. Note that all vertices of $H' = H - E(P_1 \cup P_2 \cup \ldots \cup P_{|f_1|})$ have even degree, and thus H' is a union of pairwise edge-disjoint cycles $C_1, \ldots, C_m$. For $1 \le i \le m$, direct the edges of C_i so that all vertices of C_i have outdegree 1. For $1 \le i \le |f_1|$, direct the edges of P_i so that all its vertices except for $f_1^\star$ have outdegree 1. This gives an orientation $\vec{H}$ of H such that the indegree of every vertex of $V(H) \setminus \{f_1^\star, f_2^\star\}$ equals its outdegree, $f_1^\star$ has outdegree 0 and $f_2^\star$ has indegree 0. Let $\vec{G}_1^\star$ be the orientation of $G^\star$ obtained from $\vec{H}$ by orienting all edges of $K_1^\star$ and $K_2^\star$ in the clockwise direction along the cycles. Let $\vec{G}_2^\star$ be the orientation of $G^\star$ obtained from $\vec{G}_1^\star$ by reversing the orientation of the edges of $K_1^\star$, and let $\vec{G}_3^\star$ be the orientation of $G^\star$ obtained from $\vec{G}_2^\star$ by reversing the orientation of the edges of $K_2^\star$.

Since $|f_1| = |f_2|$ is a multiple of 3, it follows that the orientations $\vec{G}_1^\star$, $\vec{G}_2^\star$ and $\vec{G}_3^\star$ define nowhere-zero $\mathbb{Z}_3$-flows in $G^\star$. Let φ_1, φ_2 and φ_3 be the corresponding 3-colorings of G arising from Proposition 4.5. Since $f_1^\star$ has outdegree 0 in all three orientations, these 3-colorings are rotating on the boundary of f_1, and thus we can permute the colors so that the restrictions of φ_1, φ_2, and φ_3 to the cycle bounding f_1 match ψ. Similarly, for $i \in \{1, 2, 3\}$, the coloring φ_i is rotating on the boundary of f_2. Propositions 4.1 and 4.2 imply $w_{\varphi_i}(f_1) + w_{\varphi_i}(f_2) = 0$, and since $w_\psi(f_1) + w_\psi(f_2) = 0$ and $w_\psi(f_1) = w_{\varphi_i}(f_1)$, we conclude $w_{\varphi_i}(f_2) = w_\psi(f_2)$. Consequently, the restrictions of φ_1, φ_2 and φ_3 to the boundary of f_2 differ from ψ only by a cyclic permutation of colors. Observe that the colors $\varphi_1(v)$, $\varphi_2(v)$ and $\varphi_3(v)$ are pairwise distinct for every $v \in V(f_2)$, since the reversals of the orientations of $K_1^\star$ and $K_2^\star$ cyclically permute the colors on the boundary of f_2. Consequently, one of these colorings matches ψ on the boundary of f_2, and thus there exists $i \in \{1, 2, 3\}$ such that φ_i is a 3-coloring of G extending ψ. $\square$

The inspection of the proofs of Lemmas 4.3, 4.4, and 4.6 shows that they are constructive and can be implemented as linear-time algorithms to find the described 3-colorings (Lemma 2.3 is only used in the proof of Lemma 4.4 to extend the precoloring of a 4-cycle, and a linear-time algorithm for this special case appears in [10]). Hence, we obtain the following corollary which we use in the next paper of the series [14].

Corollary 4.7. *For all positive integers d_1 and d_2, there exists a linear-time algorithm as follows. Let G be a cylindrical quadrangulation with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G such that $w_\psi(f_1) + w_\psi(f_2) = 0$. Suppose that $|f_1| = d_1$, $|f_2| = d_2$, every cycle in G separating f_1 from f_2 and not bounding either of these faces has length greater than $\max(d_1, d_2)$, and the distance between f_1 and f_2 is at least $d_1 + d_2$. Then the algorithm returns a 3-coloring of G that extends ψ.*

We also need another result similar to Lemma 4.6.

Corollary 4.8. *Let G be a joint with boundary faces f_1 and f_2 and let ψ be a 3-coloring of the boundary of G such that $w_\psi(f_1) + w_\psi(f_2) = 0$. If $|w_\psi(f_1)| < |f_1|/3$, then ψ extends to a 3-coloring of G.*

Proof. Since $|w_\psi(f_1)| < |f_1|/3$, we have $|f_1| \neq 3$. If $|f_1| = 4$, then ψ extends to a 3-coloring of G by Lemma 4.4. Therefore, assume $|f_1| \geq 5$. Since $|w_\psi(f_1)| < |f_1|/3$ and $|w_\psi(f_2)| < |f_2|/3$, the coloring ψ is not rotating on the boundaries of f_1 and f_2, and thus there exist paths $u_1u_2u_3$ and $v_1v_2v_3$ in the cycles bounding f_1 and f_2, respectively, such that $\psi(u_1) = \psi(u_3)$ and $\psi(v_1) = \psi(v_3)$. Let G' be the cylindrical quadrangulation obtained from $G - u_2 - v_2$ by identifying all neighbors of u_2 to a single vertex w_1 and all neighbors of v_2 to a single vertex w_2. Let ψ' be the coloring of the boundary of G' such that $\psi'(w_1) = \psi(u_1)$, $\psi'(w_2) = \psi(v_1)$ and $\psi'(z) = \psi(z)$ for any other boundary vertex of G'. Clearly, it suffices to show that ψ' extends to a 3-coloring of G'.

Let f_1' and f_2' be the boundary faces of G' corresponding to f_1 and f_2, respectively. Note that every cycle in G' separating f_1' from f_2' has length at least $|f_1'|$, and each such cycle of length $|f_1'|$ contains either w_1 or w_2. We can assume that G' is drawn so that f_1' is its outer face. Let A be a subset of the plane homeomorphic to the closed annulus such that the boundary of A is formed by cycles in G' of length $|f_1'|$ separating f_1' from f_2', one of them containing w_1, the other one containing w_2, such that no other cycle separating f_1' from f_2' is contained in A. Let G_0 be the subgraph of G' drawn in A. Removing A splits the plane into two connected open sets B_1 and B_2, where $f_1' \subset B_1$. For $i \in \{1, 2\}$, let G_i be the subgraph of G' drawn in the closure of B_i. Note that G_0 is a long boundary-linked cylindrical quadrangulation. By Lemma 2.3, ψ' extends to a 3-coloring of $G_1 \cup G_2$, and by Lemma 4.6, the resulting coloring of the boundary of G_0 extends to a 3-coloring of G_0. This gives a 3-coloring of G' extending ψ'. $\square$

To use the results of this section, we need means to constrain the winding number of a coloring on a boundary of a face. We achieve this by filling the face by a carefully chosen cylindrical quadrangulation. An *s-cap* is a cylindrical quadrangulation G with boundary faces f_1 and f_2, such that G does not contain triangles and separating 4-cycles, $|f_1| = s$, $|f_2| = 4 + (s \bmod 2)$ and for every $u, v \in V(f_1)$, the distance between u and v in G is the same as their distance in the cycle bounding f_1. We call f_2 the *special face* of the s-cap.

Lemma 4.9. *For every $s \geq 4$, there exists an s-cap G that has fewer vertices than every joint with boundary faces of length s.*

Proof. Let G be an s-cap obtained from the $s \times s$ cylindrical quadrangulation by adding chords to one of its boundary faces. We have $|V(G)| = s^2$.

Consider any joint H with boundary faces f_1 and f_2 of length s. For $1 \leq i \leq 4s-1$, let V_i denote the set of vertices of H at distance exactly i from f_1. Observe that since all faces of H other than f_1 and f_2 have length 4, $H[V_i \cup V_{i+1}]$ contains a cycle separating f_1 from f_2 for $1 \leq i \leq 4s - 2$, and thus $|V_i| + |V_{i+1}| \geq s$. Therefore, $|V(H)| \geq |f_1| + |f_2| + (2s - 1)s = (2s + 1)s > |V(G)|$. $\square$

5 3-coloring with distant anomalies

An *anomaly* is a triple $T = (H_T, B_T, \Phi_T)$, where H_T is a connected plane graph, $B_T \subseteq V(H_T)$ and Φ_T is a set of 3-colorings of H_T such that for every $\psi \in \Phi_T$, there exist distinct colors a and b such that the 3-coloring obtained from ψ by swapping the colors a and b also belongs to Φ_T. An anomaly T *appears* in a plane graph G if H_T is an induced subgraph of G (where the plane embedding of H_T is induced by the embedding of G) and every $v \in B_T$ satisfies $\deg_G(v) = \deg_{H_T}(v)$. Given a 3-coloring φ of a plane graph G and an anomaly T appearing in G, we say that φ is *compatible with* T if $\varphi \restriction V(H_T) \in \Phi_T$.

An anomaly T is *locally extendable* if the following holds for every plane graph G: if T appears in G and all triangles in G are contained in H_T, then there exists a 3-coloring of G compatible with T. For an integer $r \geq 0$, an anomaly T is *strongly locally extendable with margin* r if for every plane graph G in that T appears so that all triangles of G are contained in H_T, and for every 4-face f of G at distance at least r from H_T, every 3-coloring ψ of the boundary of f extends to a 3-coloring of G compatible with T.

The following anomalies are of interest for Theorems 1.2 and 1.3. Recall that the pattern of a 3-coloring ψ is the set $\{\psi^{-1}(1), \psi^{-1}(2), \psi^{-1}(3)\}$.

- A single precolored vertex (H_T is a single vertex, B_T is empty and Φ_T consists of a coloring assigning to the vertex of H_T the prescribed color). This anomaly is locally extendable by Grötzsch' theorem. It is also strongly locally extendable with some margin, as we hypothesized in Conjecture 1.5 and was later proved in [16].

- A cycle of length at most 5 with a prescribed pattern of coloring (H_T is a (≤ 5)-cycle, B_T is empty and Φ_T consists of all 3-colorings of H_T with the prescribed pattern). This anomaly is locally extendable by Lemma 2.1. Furthermore, the same lemma implies that if the cycle has length 3, then the anomaly is strongly locally extendable with margin 0.

- A vertex of degree at most 4 with neighborhood precolored by one color (H_T is a star with at most 4 rays, B_T contains the center of the star and Φ_T consists of all 3-colorings of H_T which assign the prescribed color to the rays). This anomaly is locally extendable by the results of Gimbel and Thomassen [17] for degree at most 3 and Dvořák and Lidický [15] for degree 4 (given a vertex v of degree $k \leq 4$ with precolored neighborhood, split v into k vertices of degree two colored arbitrarily and extend the coloring of the resulting $2k$-cycle).

Thus, both Theorem 1.2 and Theorem 1.3 are implied by the following general statement (which also shows that Conjecture 1.5 implies Conjecture 1.4), by letting C be the null graph, $p = 5$ and $r = 0$.

Theorem 5.1. *For all integers $p \geq 1$ and $r \geq 0$, there exist constants $0 < d_0 < d_1$ with the following property. Let G be a plane graph and let $\mathcal{T} = \{T_i : 1 \leq i \leq n\}$ be a set of locally extendable anomalies appearing in G, such that*

$|V(H_{T_i})| \leq p$ for $1 \leq i \leq n$. Let C be either the null graph or a facial cycle of G of length at most five, at distance at least $2d_0$ from H_T for each $T \in \mathcal{T}$. Suppose that

- *for $1 \leq i < j \leq n$, the distance between H_{T_i} and H_{T_j} in G is at least $2d_1$,*
- *every triangle in G distinct from C is contained in H_T for some $T \in \mathcal{T}$, and*
- *if a separating 4-cycle K is at distance less than $2d_0$ from H_T for some $T \in \mathcal{T}$, then either K is contained in H_T, or T is strongly locally extendable with margin r.*

Then, every 3-coloring of C extends to a 3-coloring of G compatible with all elements of $\mathcal{T}$.

Proof. For the function $f_{3.7} : \mathbb{N}^2 \to \mathbb{N}$ from Lemma 3.7, let $d_0 = \max(r, f_{3.7}(r+4, p)) + 1$ and $d_1 = \max(2d_0, f_{3.7}(2d_0 + 3, p))$. We will prove by induction on $|V(G)|$ that d_0 and d_1 satisfy the conclusion of the theorem.

Let G be as stated, let ψ be a 3-coloring of C, and assume for a contradiction that ψ does not extend to a 3-coloring of G compatible with all elements of $\mathcal{T}$. Let $\mathcal{S} = \{V(H_T) : T \in \mathcal{T}\}$, $Z_0 = \bigcup_{S \in \mathcal{S}} G[S]$ and $Z = C \cup Z_0$. For a set $X \subseteq V(G)$, let $\mathcal{T}[X] = \{T \in \mathcal{T} : V(H_T) \subseteq X\}$. Note that G is connected, as otherwise we can color each component of G separately by the induction hypothesis. Without loss of generality, we can assume that if C is not null, then it bounds the outer face of G. Hence, $(G, \mathcal{S}, C)$ is a $2d_1$-distant p-small internally triangle-free scene. Note also that if C is not null then C is an induced cycle, since otherwise a triangle containing a chord of C would be contained in H_T for some $T \in \mathcal{T}$ and the distance between H_T and C would be zero, contradicting the assumptions.

Suppose H is a clean joint in G vertex-disjoint from Z, with boundary faces f_1 and f_2 labelled so that the face of G bounded by C (if any) is contained in f_1. For $i \in \{1, 2\}$, let G'_i be the subgraph of G drawn in the closure of f_i. Then $|\mathcal{T}[V(G'_2)]| \geq 2$ and H is at distance less than $2d_0$ from H_T in G for some $T \in \mathcal{T}[V(G'_2)]$. (4)

Subproof. Suppose for a contradiction that either $|\mathcal{T}[V(G'_2)]| \leq 1$ or H is at distance at least $2d_0$ from every subgraph H_T with $T \in \mathcal{T}[V(G'_2)]$.

For $i \in \{1, 2\}$, let H_i be an $|f_i|$-cap with its non-special boundary cycle equal to the boundary of f_i, but otherwise disjoint from G'_i, such that $|V(H_i)| < |V(H)|$, which exists by Lemma 4.9. Let h_i be the special face of H_i. Let $G_i = G'_i + H_i$. Note that the distance between any two elements of $\mathcal{S} \cup \{C\}$ in G_i is the same as the distance between them in G'_i, which is greater or equal to their distance in G. By the induction hypothesis, ψ extends to a 3-coloring φ_1 of G_1 compatible with all the elements of $\mathcal{T}[V(G'_1)]$. Consider the restriction of φ_1 to H_1. Propositions 4.1 and 4.2 imply that $w_{\varphi_1}(f_1) + w_{\varphi_1}(h_1) = 0$. Furthermore, since h_1 has length at most 5, we have $w_{\varphi_1}(h_1) = 0$ if $|h_1| = 4$ (f_1 has even length) and $|w_{\varphi_1}(h_1)| = 1$ if $|h_1| = 5$ (f_1 has odd length).

We now obtain a 3-coloring φ_2 of G_2 compatible with all the elements of $\mathcal{T}[V(G'_2)]$ such that $w_{\varphi_2}(h_2) = w_{\varphi_1}(f_1)$. Let C_2 be the cycle bounding h_2.

- Suppose $\mathcal{T}[V(G'_2)] = \emptyset$. Since h_2, h_1, and f_1 have the same parity and $|w_{\varphi_1}(f_1)| \leq 1$, there exists a 3-coloring ψ_2 of C_2 such that $w_{\psi_2}(h_2) = w_{\varphi_1}(f_1)$. Since G_2 is planar and triangle-free, ψ_2 extends to a 3-coloring φ_2 of G_2 by Lemma 2.1.

- Suppose $|\mathcal{T}[V(G'_2)]| = 1$. Then there exists a 3-coloring φ'_2 of G_2 compatible with T by the local extendability of T. Let a and b be distinct colors such that the 3-coloring φ''_2 obtained from φ'_2 by swapping the colors a and b is also compatible with T. Note that $w_{\varphi'_2}(h_2) = -w_{\varphi''_2}(h_2)$, $|w_{\varphi'_2}(h_2)| \leq 1$ and $w_{\varphi'_2}(h_2)$ and $w_{\varphi_1}(f_1)$ have the same parity, and thus we can choose φ_2 as one of φ'_2 and φ''_2.

- Suppose $|\mathcal{T}[V(G'_2)]| \geq 2$, and thus H is at distance at least $2d_0$ from every subgraph H_T with $T \in \mathcal{T}[V(G'_2)]$. Choose ψ_2 be an arbitrary 3-coloring of C_2 such that $w_{\psi_2}(h_2) = w_{\varphi_1}(f_1)$. The distance from C_2 to any subgraph H_T with $T \in \mathcal{T}[V(G'_2)]$ is also at least $2d_0$, and thus by the induction hypothesis, ψ_2 extends to a 3-coloring φ_2 of G_2 compatible with all elements of $\mathcal{T}[V(G'_2)]$.

By Propositions 4.1 and 4.2 for H_2, we have $w_{\varphi_2}(f_2) = -w_{\varphi_2}(h_2) = -w_{\varphi_1}(f_1)$. By Corollary 4.8, the restriction of $\varphi_1 \cup \varphi_2$ to the boundary cycles of f_1 and f_2 extends to a 3-coloring φ_3 of H. Consequently, the restriction of φ_1 to G'_1, the restriction of φ_2 to G'_2, and φ_3 together give a 3-coloring of G extending ψ and compatible with all the elements of $\mathcal{T}$. This is a contradiction. ■

We may assume, by taking a subgraph of G, that ψ extends to a 3-coloring compatible with all elements of $\mathcal{T}$ for every proper subgraph of G that includes Z. Using the fact that G is connected, we have $G \neq Z$, as otherwise either $\mathcal{T} = \emptyset$, $G = C$, and the claim is trivial, or C is the null graph and $|\mathcal{T}| = 1$ and the claim follows by the local extendability of the anomaly in $\mathcal{T}$. Consequently, G is Z-critical.

If K is a separating (≤ 5)-cycle and Δ_K is the open disk in the plane bounded by K, then at least one vertex or edge of Z is drawn in Δ_K, since G is Z-critical and every 3-coloring of a (≤ 5)-cycle extends to a 3-coloring of a triangle-free planar graph by Lemma 2.1. We claim that

if K is a separating cycle of length at most five in G, then K is at distance less than $2d_0$ from Z_0. Furthermore, if $|K| \leq 4$ and K is S-private for some $S \in \mathcal{S}$, then the distance between K and S is less than r. (5)

Subproof. Without loss of generality, we can assume that K does not have a chord e drawn in Δ_K; otherwise, e is contained in a triangle, and thus K

intersects Z_0, and moreover, if $|K| = 4$ and K is S-private, then one of the triangles in $K + e$ is S-private and we can consider it instead of K.

Suppose that for some anomaly $T \in \mathcal{T}$, H_T intersects Δ_K but is not contained in Δ_K. Since K does not have a chord drawn in Δ_K, a vertex of H_T is drawn in Δ_K, and thus if K is S-private, then $S = V(H_T)$. Since H_T is not contained in Δ_K, it follows that K is at distance 0 from H_T, and the claim follows.

Let G_1 be the subgraph of G drawn in the complement of Δ_K and G_2 the subgraph drawn in the closure of Δ_K. By the previous paragraph, we can assume the sets $\mathcal{T}_1 = \mathcal{T}[V(G_1)]$ and $\mathcal{T}_2 = \mathcal{T}[V(G_2) \setminus V(K)]$ partition $\mathcal{T}$. By the induction hypothesis, G_1 has a 3-coloring φ_1 extending ψ and compatible with all elements of $\mathcal{T}_2$. Since ψ does not extend to a 3-coloring of G compatible with all elements of $\mathcal{T}$, it follows the restriction of φ_1 to K does not extend to a 3-coloring of G_2 compatible with all elements of $\mathcal{T}_2$. By the induction hypothesis, we conclude that K is at distance less than $2d_0$ from H_T for some element $T \in \mathcal{T}_1$.

Furthermore, if K is S-private for some $S \in \mathcal{S}$, then $\mathcal{T}_1 = \{T\}$ and $S = V(H_T)$. If K is a triangle, then since K is at distance less then $2d_0$ from H_T, the assumptions of this lemma imply $K \subseteq H_T$. If K is a 4-cycle not contained in H_T, then the assumptions of this lemma imply H_T is strongly locally extendable with margin r, and thus the distance between K and S is at most r since the restriction of φ_1 to K does not extend to a 3-coloring of G_2 compatible with T. ■

In particular, the scene $(G, \mathcal{S}, C)$ contains no r-distant private 4-cycles. We now consider 4-faces of G.

Let f be a 4-face of G at distance at least $2d_0 + 3$ from Z_0. If f is not bounded by C, then f is S-tight for a unique set $S \in \mathcal{S}$ at distance at most $d_1 - 1$ from f. (6)

Subproof. Let the vertices of f be numbered u_1, u_2, u_3, u_4 in order. By (5), no vertex of f is contained in a separating 4-cycle. Since additionally C is an induced cycle if it is not null, the intersection of the boundary of f with C is a path of length at most two.

If the intersection contains three vertices, say u_1, u_2 and u_3, then note that u_2 has degree two. Consider the graph $G - u_2$ and color u_4 by $\psi(u_2)$. By the induction hypothesis, this coloring extends to a 3-coloring of $G - u_2$ compatible with all elements of $\mathcal{T}$, which also gives a 3-coloring of G extending ψ and compatible with all elements of $\mathcal{T}$, a contradiction.

Therefore, we can assume that $u_3, u_4 \notin V(C)$. Note that $u_1u_2u_3$ and $u_1u_4u_3$ are the only paths of length at most three joining u_1 with u_3, as otherwise, since f is at distance at least $2d_0 + 3$ from Z_0, G would contain a separating (≤ 5)-cycle contradicting (5). Let G_{13} be the graph obtained from G by identifying u_1 and u_3 and suppressing parallel edges, and observe that G_{13} contains no new triangles. Furthermore, C as well as every new separating 4-cycle in G_{13} is at distance at least $2d_0$ from Z_0. Let G_{24} be defined analogously.

If G_{13} or G_{24} satisfies the assumptions of Theorem 5.1, then it has a 3-coloring extending ψ and compatible with all elements of $\mathcal{T}$ by induction, which would give such a 3-coloring of G. Otherwise, both G_{13} and G_{24} contain a pair of anomalies at distance at most $2d_1 - 1$ from each other, and thus f is S-tight for a unique $S \in \mathcal{S}$ at distance at most $d_1 - 1$ from f by Lemma 3.1. ■

Therefore, the scene $(G, \mathcal{S}, C)$ is $(2d_0 + 3, d_1)$-tight. If $|\mathcal{S}| \geq 2$, then the choice of d_1 and Lemma 3.7 implies G contains a clean joint vertex-disjoint from C whose vertices are at distance at least $2d_0 + 3$ and at most $d_1 - 1$ from some element $S \in \mathcal{S}$. By (4), H is at distance less than $2d_0$ from some element $S' \in \mathcal{S}$, necessarily distinct from S. But then the distance between S and S' is less than $d_1 + 2d_0 - 1 \leq 2d_1$, contradicting the assumptions of this lemma.

Therefore, $|\mathcal{S}| \leq 1$. If $\mathcal{S} = \emptyset$, then ψ extends to a 3-coloring of G by Lemma 2.1. Therefore, we can assume that $|\mathcal{S}| = 1$; let $\mathcal{S} = \{S\}$ and $\mathcal{T} = \{T\}$. If C is the null graph, then G has a 3-coloring compatible with T, since T is locally extendable. Hence, suppose that C is a (≤ 5)-cycle. By (5) and the assumptions of this theorem, if T is not strongly locally extendable with margin r, then all separating 4-cycles of G are contained in H_T.

Let f be a 4-face of G at distance at least $r + 4$ and at most $d_0 - 1$ from S. If f is not S-tight, then f is attached to a (≤ 6)-cycle separating S from C. (7)

Subproof. Let the vertices of f be numbered u_1, u_2, u_3, u_4 in order. For $i \in \{1, 2\}$, let $G_{i(i+2)}$ the graph obtained from G by identifying u_i with u_{i+2} to a new vertex z_i and suppressing parallel edges. If the distance between S and C in both G_{13} and G_{24} is less than $2d_0$, then Lemma 3.1 applied to $(G, \{S, C\}, \varnothing)$ implies f is S-tight. Hence, we can assume that the distance between S and C in G_{13} is at least $2d_0$.

Suppose there exists a triangle in G_{13} not contained in H_T, which was necessarily created by identification of u_1 with u_3. Then G contains a 5-cycle $K = u_1u_2u_3xy$. Since G is Z-critical, u_2 has degree at least three, and thus K does not bound a face. Lemma 2.1 implies that K separates S from C, and thus the conclusion of the claim holds since f is attached to K. Therefore, we can assume every triangle in G_{13} is contained in H_T.

Since ψ does not extend to a 3-coloring of G compatible with T, ψ also does not extend to a 3-coloring of G_{13} compatible with T. Let G'_{13} be a minimal subgraph of G_{13} containing C and H_T such that ψ does not extend to a 3-coloring of G'_{13} compatible with T. It follows that the induction hypothesis cannot apply to G'_{13}, and thus T is not strongly locally extendable with margin r and there exists a separating 4-cycle K' in G'_{13} not contained in H_T, which was necessarily created by the identification of u_1 with u_3. The minimality of G'_{13} and Lemma 2.1 imply that K' separates S from C. Let K be the cycle in

G obtained from K' by replacing z_1 by the path $u_1u_2u_3$. Then f is attached to the 6-cycle K separating S from C. ■

Therefore, the scene $(G, \mathcal{S}, C)$ is $(r+4, d_0-1)$-tight. Since the distance between S and C is at least $2d_0 > d_0 - 2$, Lemma 3.7 and the choice of d_0 implies H contains a clean joint vertex-disjoint from Z. Since $|\mathcal{T}| = 1$, this contradicts (4) and finishes the proof. □

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