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STRUCTURE OF MULTICORRELATION SEQUENCES WITH INTEGER PART POLYNOMIAL ITERATES ALONG PRIMES

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Abstract. Let T be a measure preserving $\mathbb{Z}^\ell$-action on the probability space $(X, \mathcal{B}, \mu)$, $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ vector polynomials, and $f_0, \ldots, f_m \in L^\infty(X)$. For any $\epsilon > 0$ and multicorrelation sequences of the form $\alpha(n) = \int_X f_0 \cdot T^{\lfloor q_1(n) \rfloor} f_1 \cdots T^{\lfloor q_m(n) \rfloor} f_m \, d\mu$ we show that there exists a nilsequence ψ for which $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \epsilon$ and $\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}\cap[1,N]} |\alpha(p) - \psi(p)| \leq \epsilon$. This result simultaneously generalizes previous results of Frantzikinakis [2] and the authors [11, 13].

1. Introduction and main result

Since Furstenberg's ergodic theoretic proof of Szemerédi's theorem [5], there has been much interest in understanding the structure of *multicorrelation sequences*, i.e., sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^n f_1 \cdots T^{kn} f_k \, d\mu, \quad n \in \mathbb{N}, \tag{1}$$

where $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $f_0, \ldots, f_k \in L^\infty(X)$. The first to provide deeper insight into the algebraic structure of such sequences were Bergelson, Host, and Kra, who showed in [1] that if the system (X, μ, T) is ergodic then for any multicorrelation sequence α as in (1) there exists a uniform limit of k-step nilsequences ϕ such that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \phi(n)\big| = 0. \tag{2}$$

Here, a *k-step nilsequence* is a sequence of the form $\psi(n) = F(g^n x)$, $n \in \mathbb{N}$, where F is a continuous function on a k-step nilmanifold $X = G/\Gamma$,[1] $g \in G$, $x \in X$. A *uniform limit of k-step nilsequences* is a sequence ϕ such that for every $\epsilon > 0$ there exists a k-step nilsequence ψ with $\sup_{n\in\mathbb{N}} |\phi(n) - \psi(n)| \leq \epsilon$.

Later, Leibman extended the result of Bergelson, Host and Kra to polynomial iterates in [14], and removed the ergodicity assumption in [15]. Another extension was obtained by the second author in [12], and independently by Tao and Teräväinen in [17], answering a question raised in [3]. There, it was shown that in addition to (2) one also has

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} \big|\alpha(p) - \phi(p)\big| = 0, \tag{3}$$

where $\mathbb{P}$ denotes the set of prime numbers, $[1, N] := \{1, \ldots, N\}$, and $\pi(N) := |\mathbb{P} \cap [1, N]|$.

The proofs of all the aforementioned results depend crucially on the structure theory of Host and Kra, who established in [7] that the building blocks of the factors that control

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[1] A *k-step nilmanifold* is a homogeneous space $X = G/\Gamma$, where G is a k-step nilpotent Lie group and Γ is a discrete and co-compact subgroup of G.

multiple ergodic averages are nilsystems. Since the analogous factors for $\mathbb{Z}^\ell$-actions are unknown, extending the results above from $\mathbb{Z}$-actions to $\mathbb{Z}^\ell$-actions proved to be a challenge. Nevertheless, in [2] Frantzikinakis concocted a different approach and gave a description of the structure of multicorrelation sequences of $\mathbb{Z}^\ell$-actions, which we now explain.

Henceforth, let $\ell \in \mathbb{N}$ and let T be a measure preserving $\mathbb{Z}^\ell$-action on a probability space $(X, \mathcal{B}, \mu)$. The system $(X, \mathcal{B}, \mu, T)$ gives rise to a more general class of multicorrelation sequences,

$$\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu, \quad n \in \mathbb{N}, \tag{4}$$

where $q_1, \ldots, q_m \colon \mathbb{Z} \to \mathbb{Z}^\ell$ are integer-valued vector polynomials and $f_0, \ldots, f_m \in L^\infty(X)$. Note that (1) corresponds to the special case of (4) when $\ell = 1$ and $q_i(n) = in$. Frantzikinakis showed in [2] that for every α as in (4) and every $\epsilon > 0$ there exists a k-step nilsequence ψ such that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \psi(n)\big| \le \epsilon, \tag{5}$$

where k only depends on ℓ, m, and the maximal degree among the polynomials $q_1, \ldots, q_m$. Moreover, in the special case where each polynomial iterate is linear, it was proved in [2] that one can take $k = m$. It is still an open question whether in (5) one can replace ϵ with 0 after replacing the nilsequence ψ with a uniform limit of such sequences (see Question 2 in Section 3).

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer which is smaller or equal to x, while for $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ we let $\lfloor x \rfloor \coloneqq (\lfloor x_1 \rfloor, \ldots, \lfloor x_\ell \rfloor)$. In [11], the first author extended Frantzikinakis' results to all multicorrelation sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^{\lfloor q_1(n) \rfloor} f_1 \cdots T^{\lfloor q_m(n) \rfloor} f_m \, d\mu, \quad n \in \mathbb{N}, \tag{6}$$

where $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ are real-valued vector polynomials.

More recently, the last three authors showed that the conclusion of Frantzikinakis' result also holds along the primes:

Theorem 1 ([13, Theorems A and B])**.** *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{Z} \to \mathbb{Z}^\ell$ with degree at most d, measure preserving $\mathbb{Z}^\ell$-action T on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in* (4)*, there exists a k-step nilsequence ψ satisfying* (5) *and*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(rp+s) - \psi(rp+s)| \le \varepsilon. \tag{7}$$

In the special case $d = 1$ one can choose $k = m$.

Our main theorem simultaneously generalizes the main results from [11] and [13].

Theorem A. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ with degree at most d, any measure preserving $\mathbb{Z}^\ell$-action T on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in* (6)*, there exists a k-step nilsequence ψ satisfying* (5) *and* (7)*. In the special case $d = 1$ one can choose $k = m$.*

The proof of Theorem A, presented in the next section, follows closely the strategy implemented in [11], but uses Theorem 1 instead of Walsh's theorem [18] as a blackbox.

Remark 2. Both Theorems 1 and A are equivalent to seemingly stronger versions involving commuting actions. We say that two actions T_1 and T_2 of a group G commute if for every $g, h \in G$ we have $T_1^g \circ T_2^h = T_2^h \circ T_1^g$. When G is an abelian group, a collection of m commuting G-actions $T_1, \ldots, T_m$ can be identified with a single G^m-action T via $T^{(g_1,\ldots,g_m)} = T_1^{g_1} \cdots T_m^{g_m}$. Using this observation, and the identification $(\mathbb{Z}^\ell)^m = \mathbb{Z}^{\ell m}$, one sees that, given commuting measure preserving $\mathbb{Z}^\ell$-actions $T_1, \ldots, T_m$ in a probability space (X, μ, T), Theorem 1 holds when (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{q_1(n)} f_1 \cdots T_m^{q_m(n)} f_m \, d\mu, \tag{8}$$

and Theorem A holds when (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{\lfloor q_1(n) \rfloor} f_1 \cdots T_m^{\lfloor q_m(n) \rfloor} f_m \, d\mu.$$

Remark 3. Let $\lceil x \rceil$ and $[x]$ denote the smallest integer $\geq x$ and the closest integer to x, respectively. Using the relations $\lceil x \rceil = -\lfloor -x \rfloor$ and $[x] = \lfloor x + 1/2 \rfloor$, we see that Theorem A remains true if (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T^{[q_1(n)]_1} f_1 \cdots T^{[q_m(n)]_m} f_m \, d\mu, \quad n \in \mathbb{N},$$

where $[x]_i = ([x_1]_{i,1}, \ldots, [x_\ell]_{i,\ell})$ and $[\cdot]_{i,1}, \ldots, [\cdot]_{i,\ell}$ are any of $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$, or $[\cdot]$.

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2. Proof of main result

We start by proving a theorem concerning flows, which stands halfway in between Theorems 1 and A. The idea behind this result is that for a real polynomial $q(x) = a_d x^d + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$ and a measure presenting flow $(S^t)_{t \in \mathbb{R}}$ we can write $S^{q(n)} = (S^{a_d})^{n^d} \cdots (S^{a_0})^1$, an expression which can be handled by Theorem 1.

Theorem 4. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ with degree at most d, commuting measure preserving $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $s \in \mathbb{Z}$, letting*

$$\alpha(n) = \int_X f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\mu, \tag{9}$$

there exists a k-step nilsequence ψ satisfying (5) *and* (7)*. In the special case $d = 1$ one can choose $k = m$.*

Proof. For each $i \in [1, m]$, let $q_i = (q_{i,1}, \ldots, q_{i,\ell})$ for some $q_{i,j} \in \mathbb{R}[x]$. Next, for each $j \in [1, \ell]$, write $q_{i,j}(x) = \sum_{h=0}^d a_{i,j,h} x^h$, where the $a_{i,j,h}$'s are real numbers. Also, for each $j \in [1, \ell]$, let e_j be the j-th vector of the canonical basis of $\mathbb{R}^\ell$ and let $T_{i,j,h}$ be the measure preserving transformation defined by $T_{i,j,h} = S_i^{a_{i,j,h} e_j}$. Next, let $T_{i,h}$ be the composition $T_{i,h} = T_{i,1,h} \cdots T_{i,\ell,h}$, let T_i be the $\mathbb{Z}^{d+1}$-action defined by $T_i^{(n_0,\ldots,n_d)} = T_{i,0}^{n_0} \cdots T_{i,d}^{n_d}$, and let $q \colon \mathbb{Z} \to \mathbb{Z}^{d+1}$ be the polynomial $q(n) = (1, n, \ldots, n^d)$.

With this setup, for each $i \in [1, m]$ and $n \in \mathbb{N}$, we have

$$S_i^{q_i(n)} = \prod_{j=1}^{\ell} S_i^{q_{i,j}(n) e_j} = \prod_{j=1}^{\ell} \prod_{h=0}^{d} T_{i,j,h}^{n^h} = \prod_{h=0}^{d} T_{i,h}^{n^h} = T_i^{q(n)}.$$

Since the $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ commute, so do the $\mathbb{Z}^{d+1}$-actions $T_1, \ldots, T_m$. This implies that the multicorrelation sequence α can be represented by an expression of the form (8). The conclusion now follows directly from Theorem 1 and Remark 2. $\square$

Next we need a result concerning the distribution of real polynomials.

Lemma 5. *Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. Then, denoting by $\{\cdot\}$ the fractional part, we have*

$$\lim_{\delta\to 0^+} \lim_{N-M\to\infty} \frac{1}{N-M} \left|\Big\{n \in [M,N) : \big\{q(n)\big\} \in [1-\delta, 1)\Big\}\right| = 0,$$

and

$$\lim_{\delta\to 0^+} \lim_{N\to\infty} \frac{1}{\pi(N)} \left|\Big\{p \in \mathbb{P} \cap [1,N] : \big\{q(rp+s)\big\} \in [1-\delta, 1)\Big\}\right| = 0.$$

Proof. Let

$$A(\delta) = \lim_{N-M\to\infty} \frac{1}{N-M} \left|\Big\{n \in [M,N) : \big\{q(n)\big\} \in [1-\delta, 1)\Big\}\right|,$$

and

$$B(\delta) = \lim_{N\to\infty} \frac{1}{\pi(N)} \left|\Big\{p \in \mathbb{P} \cap [1,N] : \big\{q(rp+s)\big\} \in [1-\delta, 1)\Big\}\right|.$$

If $q - q(0)$ has an irrational coefficient, then by Weyl's Uniform Distribution Theorem [19] and Rhin's Theorem [16] we have $A(\delta) = B(\delta) = \delta$ which approach 0 as $\delta \to 0^+$.

Assume otherwise that $q \in \mathbb{R}[x]$ satisfies $q - q(0) \in \mathbb{Q}[x]$, say $q(x) = q(0) + b^{-1} \sum_{j=1}^{\ell} a_j x^j$ where $b \in \mathbb{N}$, $a_j \in \mathbb{Z}$ for $1 \le j \le \ell$, and $q(0) \in \mathbb{R}$. It follows that for all $n \in \mathbb{N}$,

$$q(n) - q(0) \bmod 1 \in \left\{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b}\right\}.$$

In particular, the fractional part $\{q(n)\}$ takes only finitely many values. Therefore, if δ is small enough, for every $n \in \mathbb{N}$ we have $\{q(n)\} \notin [1-\delta, 1)$ and hence $A(\delta) = B(\delta) = 0$, which implies the desired conclusion. $\square$

For the proof of Theorem A we adapt arguments from [10, 11], i.e., we use a multidimensional suspension flow to approximate α by a multicorrelation sequence of the form (9). The arising error consists of terms of the form $1_{\{n\in\mathbb{N}\colon q(n)\in[1-\delta,1)\}}$ that can be controlled by Lemma 5.

Proof of Theorem A. Given $\ell, m, d \in \mathbb{N}$, let k be as guaranteed by Theorem 4. Let $q_1, \ldots, q_m$, T, $f_0, \ldots, f_m$, $\epsilon > 0$, $r \in \mathbb{N}$, $s \in \mathbb{Z}$ and α be as in the statement. By multiplying each function by a constant if needed, we can assume without loss of generality that $\|f_i\|_\infty \le 1$ for each $i \in [1, m]$.

We start by considering a multidimensional suspension flow with a constant 1 ceiling function. More precisely, let $Y := X \times [0,1)^{m\times\ell}$ and $\nu = \mu \otimes \lambda$, where λ denotes the Lebesgue measure on $[0,1)^{m\times\ell}$. For each $i \in [1,m]$ define the measure preserving $\mathbb{R}^\ell$-action S_i on (Y, ν) as follows: for any $t \in \mathbb{R}^\ell$ and $(x; b_1, \ldots, b_m) \in Y = X \times \big([0,1)^\ell\big)^m$, let

$$S_i^t(x; b_1, \ldots, b_m) := \big(T^{\lfloor b_i + t\rfloor} x; b_1, \ldots, b_{i-1}, \{b_i + t\}, b_{i+1}, \ldots, b_m\big),$$

where $\{u\} := u - \lfloor u \rfloor$ for any $u \in \mathbb{R}^\ell$. Observe that the actions $S_1, \ldots, S_m$ commute.

Let $\pi\colon Y \to X$ be the natural projection and $\delta > 0$ a small parameter to be determined later. For each $i \in [1,m]$ let $\hat{f}_i \in L^\infty(Y)$ be the composition $\hat{f}_i := f_i \circ \pi$, and $\hat{f}_0 := 1_{X\times[0,\delta]^{m\times\ell}} \cdot f_0 \circ \pi$. Define

$$\tilde{\alpha}(n) = \int_Y \hat{f}_0 \cdot S_1^{q_1(n)} \hat{f}_1 \cdots S_m^{q_m(n)} \hat{f}_m \, d\nu.$$

By Theorem 4 there exists a k-step nilsequence $\tilde{\psi}$ such that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\tilde{\alpha}(n) - \tilde{\psi}(n)| \leq \delta^{\ell m}\epsilon/2, \tag{10}$$

and

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} |\tilde{\alpha}(rp+s) - \tilde{\psi}(rp+s)| \leq \delta^{\ell m}\epsilon/2. \tag{11}$$

On the other hand,

$$\tilde{\alpha}(n) = \int_{[0,\delta]^{\ell m}} \int_X f_0(x) f_1\Big(T^{\lfloor q_1(n)+b_1\rfloor}x\Big) \cdots f_m\Big(T^{\lfloor q_m(n)+b_m\rfloor}x\Big)\, d\mu(x)\, d\lambda(b_1,\ldots,b_m),$$

which implies

$$\alpha(n) - \frac{\tilde{\alpha}(n)}{\delta^{\ell m}} = \frac{1}{\delta^{\ell m}} \int_{[0,\delta]^{\ell m}} \int_X f_0(x) \left(\prod_{i=1}^m f_i\Big(T^{\lfloor q_i(n)\rfloor}x\Big) - \prod_{i=1}^m f_i\Big(T^{\lfloor q_i(n)+b_i\rfloor}x\Big) \right) d\mu\, d\lambda. \tag{12}$$

In particular, it follows from (12) that $|\alpha(n) - \delta^{-\ell m}\tilde{\alpha}(n)| \leq 2$ for all $n \in \mathbb{N}$. If $b_i \in [0,\delta]^\ell$ and $\big\{q_i(n)\big\} \in [0, 1-\delta)^\ell$ then $\lfloor q_i(n) + b_i\rfloor = \lfloor q_i(n)\rfloor$. Therefore (12) also implies that $\alpha(n) = \delta^{-\ell m}\tilde{\alpha}(n)$ whenever

$$n \notin \Big\{ n \in \mathbb{N} : \big\{q_i(n)\big\} \in [1-\delta, 1)^\ell \text{ for some } i \in [1,m] \Big\}.$$

In view of Lemma 5, by choosing δ small enough, we have

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \Big|\alpha(n) - \delta^{-\ell m}\tilde{\alpha}(n)\Big| < \frac{\epsilon}{2} \tag{13}$$

and

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} \Big|\alpha(rp+s) - \delta^{-\ell m}\tilde{\alpha}(rp+s)\Big| < \frac{\epsilon}{2}. \tag{14}$$

Letting $\psi = \delta^{-m\ell}\tilde{\psi}$ and combining (10) with (13) and (11) with (14) we obtain the desired conclusion. □

Remark 6. As it was already mentioned in Section 1, it is an open problem whether one can improve upon the approximation in Frantzikinakis' main result in [2] and Theorem 1 and take $\epsilon = 0$ in (5) and (7) (see Question 2 below). However, as the following example shows, in the case of Theorem A it is not possible to improve upon the approximation in that manner.

Example 7. Take $X = \mathbb{T} := \mathbb{R}/\mathbb{Z}$, $T(x) = x + 1/\sqrt{2}$, $q(n) = \sqrt{2}n$, $f_0(x) = e(x)$ and $f_1(x) = e(-x)$, where $e(x) := e^{2\pi i x}$. Then we have

$$\begin{aligned}\alpha(n) &= \int f_0 \cdot T^{\lfloor q(n)\rfloor} f_1\, d\mu = \int e(x) e\left(-x - \frac{1}{\sqrt{2}}\lfloor\sqrt{2}n\rfloor\right) dx \\ &= e\left(-\frac{1}{\sqrt{2}}\lfloor\sqrt{2}n\rfloor\right) = e\left(\frac{1}{\sqrt{2}}\{\sqrt{2}n\}\right).\end{aligned}$$

In particular, we can write $\alpha(n)$ as $F(T^n x_0)$ with $x_0 = 0 \in \mathbb{T}$ and $F(x) = e(\{x\}/\sqrt{2})$ for $x \in \mathbb{T}$. Assume for the sake of a contradiction that there exists a uniform limit of nilsequences ϕ for which

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N |\alpha(n) - \phi(n)| = 0. \tag{15}$$

By [9, Lemma 18], ϕ can be written as $\phi(n) = G(S^n y_0)$ for all $n \in \mathbb{N}$, where G is a continuous function on an inverse limit of nilsystems (Y, S) and $y_0 \in Y$.

We claim that $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. If not, then there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\alpha(n_0) - \phi(n_0)| = |F(T^{n_0}x_0) - G(S^{n_0}y_0)| \geq \delta. \tag{16}$$

Since the system $(X \times Y, T \times S)$ is the product of two distal systems, is a distal system itself. This implies that the point $(T^{n_0}x_0, S^{n_0}y_0)$ is uniformly recurrent, i.e., the sequence $(T^n x_0, S^n y_0)$ visits any neighborhood of $(T^{n_0}x_0, S^{n_0}y_0)$ in a syndetic set. This fact together with (16) and the fact that both the real and imaginary parts of F are almost everywhere continuous and semicontinuous imply that the set

$$\{n \in \mathbb{N} : |F(T^n x_0) - G(S^n y_0)| \geq \delta/2\}$$

is syndetic, which contradicts (15). Hence $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. However, by [6, Proposition 4.2.5], the sequence α is not a distal sequence; in particular, it is not a uniform limit of nilsequences, contradicting our assumption.

3. Open questions

We close this article with three open questions. Theorem A provides an approximation result of multicorrelation sequences along an integer polynomial of degree one, evaluated at primes. We can ask whether a similar result is true along other classes of sequences.

Question 1. *Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $c > 0$, and p_n denote the n-th prime. Suppose $r_n = q(n), q(p_n), \lfloor n^c \rfloor$ or $\lfloor p_n^c \rfloor$ for $n \in \mathbb{N}$. Is it true that for any α as in* (6) *and $\epsilon > 0$, there exists a nilsequence ψ satisfying*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(r_n) - \psi(r_n)| \leq \epsilon?$$

Variants of the following question have appeared several times in the literature, e.g., [2, Remark after Theorem 1.1], [3, Problem 20], [4, Problem 1], and [8, Page 398].

Question 2. *Let α be as in* (4). *Does there exist a uniform limit of nilsequences ϕ such that*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \phi(n)\big| = 0?$$

As mentioned in Example 7, the answer to Question 2 is negative when α is a multicorrelation sequence as in (6). Nevertheless, it makes sense to ask for the following modification of it.

Question 3. *Let α be as in* (6). *Does there exist a uniform limit of* Riemann integrable nilsequences *ϕ satisfying*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \phi(n)\big| = 0?$$

Here we say that ϕ is a uniform limit of Riemann integrable nilsequences if for every $\epsilon > 0$ there exists a nilmanifold $X = G/\Gamma$, a point $x \in X$, $g \in G$ and a Riemann integrable function[2] *$F\colon X \to \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} |\phi(n) - F(g^n x)| < \epsilon$.*

[2] A function F is *Riemann integrable* on a nilmanifold if its points of discontinuity is a null set with respect to the Haar measure.

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