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SOLUTION TO A PROBLEM OF NIRENBERG CONCERNING EXPANDING MAPS

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ABSTRACT. By constructing a map of a separable Hilbert space into itself that is continuous, expanding, non-surjective, and equal to the identity on the unit ball we answer a problem stated by Louis Nirenberg.

1. INTRODUCTION

We answer the following problem, which was stated (in an equivalent form) by Louis Nirenberg in [7, End of Section 5.2].

Problem N. *If the range of a continuous, expanding map T of a separable Hilbert space H into itself has non-empty interior, is T surjective?*

Recall that T is said to be expanding if $\|Tx - Ty\| \geq \|x - y\|$ for every $x, y \in H$. Of course, since T may be multiplied by an arbitrary constant, in this problem it is equivalent to require that it is c-expanding for some $c > 0$ (i.e., $\|Tx - Ty\| \geq c\|x - y\|$ for every $x, y \in H$).

Obviously, the above problem may be, and has been, asked for general Banach spaces and/or for maps between different spaces. On various levels of generality, a number of papers, including [2, 3, 10] provide additional conditions under which Nirenberg's problem is said to have positive answer. However, most of these results do not use the assumption that the range of T has non-empty interior, and so are in fact answers to a related but different question: Under what additional conditions is an expanding map $T : X \to Y$ surjective? As far as we see, all assumptions of Problem N are used only in the result of [1] that T is surjective provided that it is everywhere Fréchet differentiable and satisfies $\limsup_{y \to x} \|T'(y) - T'(x)\| < 1$ for every x.

In general Banach spaces the answer to Problem N is known to be negative: By a clever construction Jean-Michel Morel and Heinrich Steinlein [5] find a counterexample in the space ℓ_1 of absolutely convergent sequences. In a Hilbert space, Janusz Szczepański [8, 9] modified their construction to obtain several 'almost counterexamples'. For example, for any $\varepsilon > 0$, he constructed a continuous map $F_\varepsilon : \ell_2 \to \ell_2$ that is one-to one, non-surjective, satisfies $\|F_\varepsilon x - F_\varepsilon y\| \geq \|x - y\|$ whenever $\|y\| \notin (1, 1 + \varepsilon)$ and $\|F_\varepsilon x\| = c_\varepsilon \|x\|$ for every x, and its range contains the unit ball.

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Here we show that Problem N has negative answer even in Hilbert spaces. More generally, for any $1 \le p < \infty$ we will find a continuous, non-surjective, c-expanding (for a suitable $c > 0$, for example for $c = 1/10$) map $T : L_p(0, \infty) \to L_p(0, \infty)$, where $L_p(0, \infty)$ is equipped with the norm $\|x\| := (\int_0^\infty |x(t)|^p dt)^{1/p}$, which is equal to the identity on the unit ball (Theorem 4). Since $L_p(0, \infty)$ is isometric to $L_p(0, 1)$ (see, for example, [4, Theorem 2.7.3]), our examples immediately transfer to these more usually considered spaces. The Hilbert space example required for the Nirenberg's problem is, of course, obtained by specifying $p = 2$. For interest, we slightly strengthen it in Theorem 5: We modify T to get a continuous, non-surjective, expanding (so $c = 1$) map of a separable Hilbert space into itself, which is equal to the identity on the unit ball. We also notice that all our proofs apply, without any modification, to real as well as complex spaces.

Our arguments start from an interpretation of some of the ideas of Jean-Michel Morel and Heinrich Steinlein [5], which we briefly indicate as follows. For $x \in \ell_1$ (with the usual basis $e_1, e_2, \dots$) they join the identity $Tx := x$ when $\|x\| \le 1$ to the shift $Tx := \sum_{i=2}^\infty x_{i-1} e_i$ when $\|x\| \ge 2$, by choosing for every $1 < \|x\| < 2$ suitable $n = n_x \in \mathbb{N}$ and $\alpha_x, \beta_x \ge 0$ with $\alpha_x + \beta_x = 1$, and defining

$$Tx := \sum_{i=1}^{n-1} x_i e_i + \alpha_x x_n e_n + \beta_x x_n e_{n+1} + \sum_{i=n+2}^{\infty} x_{i-1} e_i. \tag{1}$$

Although this is not always true, one could imagine that $n_x \to \infty$ when $\|x\| \to 1$ and $n_x \to 0$ when $\|x\| \to 2$, and that the splitting of the n-th coordinate using α_x and β_x was done in such a way that T becomes continuous. (As pointed out in [6], T is even Lipschitz.) The proof that T is expanding is rather delicate: in particular, when trying to estimate the distance between Tx and Ty when n_x is much smaller than n_y, the values $(Tx)_i - (Ty)_i = x_{i-1} - y_i$ for $n_x + 2 \le i \le n_y - 1$ look like they cannot be estimated in any way that could contribute to the lower estimate by $c\|x - y\|$. For this, the choice of n_x, α_x, β_x is subject to the requirement

$$\Big\| \sum_{i=1}^{n} (Tx)_i e_i \Big\| = 2 - \|x\|. \tag{2}$$

Thanks to working in ℓ_1, this equality allows an estimate of the 'bad term' by $3\big|\|x\| - \|y\|\big| + \sum_{i \notin [n_x, n_y]} |x_i - y_i|$ (see [5, inequality (4)]). If $\big|\|x\| - \|y\|\big|$ is small compared to $\|x - y\|$, this leads to a lower estimate of $\|Tx - Ty\|$, and if it is not, it suffices to use that T preserves the norm. In a modified form, we will reuse this argument in the proof of our main estimate in Theorem 4 (iv).

The main difference between our approach and [5] is that for $\|x\| > 1$ we wish to replace (1) (ignoring for a moment the splitting of x_n) by

$$Tx := \sum_{i=1}^{n} x_i e_i + \sum_{i=n+1}^{\infty} x_{i-1} e_{\varphi_n(i)}, \tag{3}$$

where $\varphi_n : \mathbb{N} \to \mathbb{N}$ are strictly increasing and have disjoint ranges. The disjointness of the ranges of φ_n should allow an easier estimate of the corresponding part of $\|Tx - Ty\|$, although in the case when n_x is much smaller that n_y we still have some differences $|x_i - x_j|$ where $i \ne j$. However, it seems rather difficult to make the map T defined by any formula similar to (3) continuous. One reason for this is the discontinuity of n_x, which we solve by working in $L_p(0, \infty)$ and replacing the first sum in (3) by $x\mathbf{1}_{(0,r(x))}$ where r is continuous. (By $\mathbf{1}_E$ we denote the indicator function of the set E.) The second sum would be naturally replaced by distributing the values of $x\mathbf{1}_{(r(x),\infty)}$, which leads to

$$Tx := x\mathbf{1}_{(0,r(x))} + (x\mathbf{1}_{(r(x),\infty)}) \circ \psi_n, \tag{4}$$

where ψ_n are suitable maps defined on mutually disjoint sets G_n (so in some sense are inverses to the maps φ_n from (3)). However, (4) still contains the discontinuous quantity $n = n_x$. We will solve this difficulty by smoothing, and so get

$$Tx := x\mathbf{1}_{(0,r(x))} + \sum_{n\in\mathbb{N}} \gamma(n - r(x))(x\mathbf{1}_{(r(x),\infty)}) \circ \psi_n, \tag{5}$$

where γ is a suitable continuous function with compact support.

It remains to explain how to choose $r(x)$ so that for some $c > 0$ the map T defined by (5) satisfies $\|Tx - Ty\| \geq c\|x - y\|$. For that, we return to (2), which in our space would be $\|x\mathbf{1}_{(0,r(x))}\| = 2 - \|x\|$, and write it as

$$\|x\mathbf{1}_{(r(x),\infty)}\| = \varepsilon(x). \tag{6}$$

We are almost free to choose $\varepsilon(x)$. A rather minor restriction is the need for continuity of $r(x)$, which we solve by a simple averaging trick in (8). More importantly, this freedom together with the choice of γ is used to obtain a lower estimate of $\|Tx - Ty\|$. More precisely, the choice of γ is made so that for any x, y with $r(x) \leq r(y)$ either there is n such that $\gamma(n - r(x)) = \gamma(n - r(y)) = 1$ or there is n such that $\gamma(n-r(x)) = 0$ and $\gamma(n-r(y)) = 1$. (See Lemma 1.) With one exception, this allows us to obtain, for Cases 1 and 2 in the proof of Theorem 4 (iv), a lower estimate of the norm of $Tx - Ty$ by restricting it to a set in which $(Tx - Ty)(t)$ is not appearing as the difficult term $x(u) - y(v)$ where $u \neq v$. In the exceptional case the choice of ε leads to showing that $\|x - y\|$ is much bigger than $\varepsilon(x) + \varepsilon(y)$ (Lemma 2 (v)), whilst (6) implies that $\|(Tx - Ty) - (x - y)\|$ is bounded by a fixed multiple of $\varepsilon(x) + \varepsilon(y)$, which easily gives the required lower estimate of $\|Tx - Ty\|$ for Case 3 in the proof of Theorem 4 (iv).

2. Proofs and results

From now on we will fix $1 \leq p < \infty$ and denote $X := L_p(0,\infty)$. As usual, we will treat elements $x \in X$ as functions and, for example, write $x(t) = 0$ for $t \in E$ instead of $x(t) = 0$ for almost all $t \in E$. When $G \subset (0,\infty)$ is a measurable set, we denote by $|G|$ its Lebesgue measure, and by P_G the standard projection of X onto $\{x : x(t) = 0 \text{ for } t \notin G\}$ defined by changing to zero the values $x(t)$ for $t \notin G$. When $0 \leq r \leq s \leq \infty$, we simplify the notation by letting $P_{r,s} := P_{(r,s)}$; when $r \geq s$ this means that $P_{r,s}$ is identically zero.

When $G \subset (0,\infty)$ is a measurable set of infinite measure, we define the function $\psi_G : (0,\infty) \to (0,\infty)$ by $\psi_G(t) = |(0,t) \cap G|$. Then $|G \cap \psi_G^{-1}(E)| = |E|$ for every measurable set $E \subset (0,\infty)$, and so

$$\int_E g(t)\,dt = \int_{G\cap\psi_G^{-1}(E)} g(\psi_G(t))\,dt \tag{7}$$

whenever $E \subset (0,\infty)$ and $g : (0,\infty) \to [0,\infty)$ are measurable.

Finally, to avoid possible unpleasant surprises, we agree that 0 belongs to the set $\mathbb{N}$ of natural numbers.

Fix a continuous function $\gamma : [-\infty, \infty] \to [0, 1]$ with support in $(-3/2, 3/2)$ such that $\gamma(t) = 1$ if and only if $t \in [-1, 1]$. We list the properties of γ that will be used in what follows.

Lemma 1. *For any $r \in [0, \infty)$,*

(i) *if $\gamma(n - r) = 1$, then $|n - r| \le 1$, in particular $r \le n + 1$;*
(ii) *$\gamma(n - r) = 0$, when $|n - r| \ge 3/2$;*
(iii) *there are at most three values of $n \in \mathbb{N}$ such that $\gamma(n - r) \neq 0$;*
(iv) *for any $s \in [r, \infty)$ at least one of the following statements holds:*
  (a) *there is $k \in \mathbb{N}$ such that $\gamma(k - r) = \gamma(k - s) = 1$;*
  (b) *there is $k \in \mathbb{N}$ such that $\gamma(k - r) = 0$ and $\gamma(k - s) = 1$.*

Proof. The statements (i)–(iii) are obvious. For (iv), let $m \in \mathbb{N}$ be such that $m \le s < m + 1$; hence $\gamma(m - s) = \gamma(m + 1 - s) = 1$. So, if $\gamma(m - r) = 1$, (a) holds with $k = m$, and if $\gamma(m - r) < 1$, we use $r \le s$ to get $m - r \ge m - s \ge -1$ and infer from $\gamma(m - r) < 1$ that $m - r > 1$. Hence $m + 1 - r > 2$, which implies that $\gamma(m + 1 - r) = 0$ and (b) holds with $k = m + 1$. □

In the following Lemma we define the key function $r(x)$. It will give the position up to which the values of Tx are the same as those of x.

Lemma 2. *Given any $0 < \alpha < 1$ and $\beta > 0$, there is a function $r : X \to [0, \infty]$ such that*

(i) *r is finite and continuous on $\{x \in X : \|x\| > 1\}$;*
(ii) *$r(x) = \infty$ when $\|x\| \le 1$;*
(iii) *the map $S : X \to X$ defined by $Sx := P_{r(x),\infty}\, x$ is continuous;*
(iv) *$\varepsilon(x) := \|Sx\|$ is continuous and satisfies $0 \le \varepsilon(x) \le \max(0, \|x\| - 1)$;*
(v) *if $\varepsilon(y) < \alpha\varepsilon(x)$ then $\|x - y\| \ge \beta\varepsilon(x)$.*

Proof. For any choice of parameters $0 < a < b < 1$ we define a function r satisfying (i)–(iv), and at the end of the proof explain how to choose a, b so that (v) holds as well.

Given $0 < a < b < 1$ define $\eta : \mathbb{R}^2 \to [0, \infty)$ by

$$\eta(s, t) = \max\bigl(0, \min(t - a(s - 1), b(s - 1) - t)\bigr).$$

Observe that $\eta : \mathbb{R}^2 \to [0, \infty)$ is continuous and $\eta(s, t) > 0$ if and only if $a(s - 1) < t < b(s - 1)$; in fact, these are the only properties of η that we will use. We also remark that $a(s - 1) < t < b(s - 1)$ cannot hold when $s \le 1$, and so $\eta(s, t) = 0$ for $s \le 1$.

For $x \in X$ put $\xi_x(t) := \|P_{t,\infty} x\|$ and for $\|x\| > 1$ define

$$r(x) := \frac{\int_0^\infty t\eta(\|x\|, \xi_x(t))\, dt}{\int_0^\infty \eta(\|x\|, \xi_x(t))\, dt} \tag{8}$$

To show (i), notice first that ξ_x is continuous, non-increasing, $\xi_x(0) = \|x\|$ and $\lim_{t\to\infty} \xi_x(t) = 0$. When $\|x\| > 1$, the functions

$$u_x(t) := \eta(\|x\|, \xi_x(t)) \text{ and } v_x(t) := t\eta(\|x\|, \xi_x(t))$$

are non-negative and continuous on $[0, \infty)$ and non-zero precisely on the interval $(A_x, B_x) := \xi_x^{-1}\bigl(a(\|x\| - 1), b(\|x\| - 1)\bigr) \neq \emptyset$. Hence their integrals are finite and strictly positive, and so $r(x) = \int_0^\infty v_x(t)\, dt / \int_0^\infty u_x(t)\, dt$ is well defined. For the sake of future reference, we also notice that $A_x u_x(t) \le v_x(t) \le B_x u_x(t)$ for $t \in (A_x, B_x)$, hence $A_x \le r(x) \le B_x$ and

$$a(\|x\| - 1) \le \xi_x(r(x)) \le b(\|x\| - 1). \tag{9}$$

To prove continuity of r, consider a sequence $x^k \to x$ where $\|x\| > 1$ and choose $t_0 \in (0, \infty)$ such that $\xi_x(t_0) < a(\|x\| - 1)$. Since ξ_{x_k} converge uniformly to ξ_x, $x^k(t_0) < a(\|x\| - 1)$ for sufficiently large k. For these k the continuous functions u_{x^k} and v_{x^k} have support in $[0, t_0]$. Since they converge uniformly to u_x and v_x, respectively, $r(x^k)$ converge to $r(x)$.

To satisfy (ii), it suffices to define $r(x) = \infty$ for $\|x\| \le 1$.

For (iii) and (iv) we first observe that the inequality in (iv) is obvious when $\|x\| \le 1$ since then $r(x) = \infty$ and so $\varepsilon(x) = 0$, and follows from (9) when $\|x\| > 1$ since then $\varepsilon(x) = \xi_x(r(x))$ and $b < 1$. This inequality immediately implies that S and ε are continuous at every x with $\|x\| \le 1$. On the set $Y := \{x \in X : \|x\| > 1\}$, S is a composition of continuous maps $x \in Y \to (r(x), x) \in [0, \infty) \times X$ and $(t, x) \in [0, \infty) \times X \to P_{t,\infty} x \in X$, and so S and ε are continuous also on Y.

To prove (v), we notice that $\varepsilon(x) > 0$, so $\|x\| > 1$ and $\varepsilon(x) = \xi_x(r(x))$. Hence by (9), $\varepsilon(x) \le b(\|x\| - 1)$. If $\|y\| \le 1$, we use this to get

$$\|x - y\| \ge \|x\| - \|y\| \ge \|x\| - 1 \ge \varepsilon(x)/b. \tag{10}$$

If $\|y\| > 1$, we infer from (9) that

$$a(\|y\| - 1) \le \varepsilon(y) < \alpha\varepsilon(x) \le \alpha b(\|x\| - 1).$$

Dividing this inequality by a and rearranging leads to

$$\|x\| - \|y\| \ge (a - \alpha b)(\|x\| - 1)/a$$

and using $\|x\| - 1 \ge \varepsilon(x)/b$ once more gives

$$\|x - y\| \ge \|x\| - \|y\| \ge (a - \alpha b)(\|x\| - 1)/a \ge (a - \alpha b)\varepsilon(x)/(ab). \tag{11}$$

We are now ready to explain the choice of a, b for which (v) holds: First choose $0 < b < \min(1, (1 - \alpha)/\beta)$, and then use $\lim_{a \nearrow b} (a - \alpha b)/(ab) = (1 - \alpha)/b > \beta$ to choose $0 < a < b$ so that $(a - \alpha b)/(ab) > \beta$. Then the right side of (10) as well as the right side of (11) is $\ge \beta\varepsilon(x)$, as required. $\square$

For $n \in \mathbb{N}$ choose mutually disjoint measurable sets G_n such that $|G_n| = \infty$ and

$$G_n \subset (n + 4, \infty) \cap \bigcup_{k=0}^{\infty} (2k, 2k + 1).$$

For example, we may take $G_n := \bigcup_{k=n+2}^{\infty} \big(2k + 2^{-n-1}, 2k + 2^{-n}\big)$. Then, simplifying the notation by letting $\psi_n := \psi_{G_n}$ and defining $T_n : X \to X$ by

$$(T_n x)(t) := \begin{cases} x(\psi_n(t)) & \text{when } t \in G_n \\ 0 & \text{when } t \in (0, \infty) \setminus G_n, \end{cases}$$

the equation (7) with $g(t) := |x(t)|^p$ gives

$$\int_E |x(t)|^p dt = \int_{G_n \cap \psi_n^{-1}(E)} |x(\psi_n(t))|^p dt = \int_{\psi_n^{-1}(E)} |(T_n x)(t)|^p dt \tag{$\star$}$$

We now fix $0 < \alpha < 1$ and $\beta > 0$, and use the function r from Lemma 2 to define $T : X \to X$ by

$$Tx := P_{0,r(x)}x + \sum_{n=0}^{\infty} \gamma(n-r(x))T_n(P_{r(x),\infty}x) = x - Sx + \sum_{n=0}^{\infty} \gamma(n-r(x))T_n(Sx) \tag{12}$$

The following Lemma lists some easy to prove properties of T that will be used in our arguments.

Lemma 3. *For any $x \in X$,*

(i) $(Tx)(t) = 0$ *for* $t \in (r(x), \infty) \cap \bigcup_{n=1}^{\infty}(2n-1, 2n)$*;*
(ii) $(Tx)(t) = 0$ *when* $r(x) < t < r(x) + 2$*;*
(iii) $(Tx)(t) = 0$ *when* $t \in G_n$ *for some* $n > r(x) - 1$ *such that* $\gamma(n - r(x)) = 0$*;*
(iv) $(Tx)(t) = 0$ *if there is* $t \in G_n$ *such that* $\gamma(n - r(x)) \neq 0$ *and* $\psi_n(t) < r(x)$*;*
(v) $(Tx)(t) = x(t)$ *for* $t < r(x)$*;*
(vi) $(Tx)(t) = (T_n x)(t)$ *when* $\gamma(n - r(x)) = 1$ *and* $t \in G_n$*.*

Proof. If $(Tx)(t) \neq 0$, either $t < r(x)$ or there is m for which $t \in G_m$. Moreover, when $t > r(x)$, then the latter case has to occur, m is unique and so (12) gives $0 \neq (Tx)(t) = \gamma(m - r(x))(P_{r(x),\infty}x)(\psi_m(t))$. It follows that $\psi_m(t) > r(x)$ and $\gamma(m - r(x)) \neq 0$, implying $m > r(x) - 3/2$ and $G_m \subset (m + 4, \infty) \subset (r(x) + 2, \infty)$. These facts and (12) easily imply all statements of the Lemma. □

We are now ready to prove the main result of this note.

Theorem 4. *There are* $0 < \alpha < 1$ *and* $\beta > 0$ *such that the map* T *has the following properties.*

(i) $Tx = x$ *for* $\|x\| \le 1$*.*
(ii) *It is a well defined continuous map of* X *to* X*.*
(iii) $\|Tx - x\| \le 4\varepsilon(x)$ *for every* $x \in X$*.*
(iv) *There is* $c > 0$ *such that* $\|Tx - Ty\| \ge c\|x - y\|$ *for any* $x, y \in X$*.*
(v) *The* T *image of* $\{x \in X : \|x\| \ge 1\}$ *is nowhere dense in* X*.*
(vi) T *is not surjective.*

Proof of (i). When $\|x\| \le 1$ then $Sx = 0$ and so the formula (12) defining T gives $Tx = x$. □

Proof of (ii). On the ball $\{x \in X : \|x\| \le 1\}$, T is well defined and continuous by (i).

When $x \in X$ and $\|x\| > 1$, we use continuity of r established in Lemma 2 (i) to find $\delta > 0$ such that $|r(y) - r(x)| < 3/2$ for $\|y - x\| < \delta$ and infer from Lemma 1 (ii) that $\gamma(n - r(y)) = 0$ whenever $\|y - x\| < \delta$ and $|n - r(x)| \ge 3$. Hence T is well defined and, by Lemma 2 (iii), is continuous on the ball $\{y : \|y - x\| < \delta\}$. Since this holds for every x with $\|x\| > 1$, T is well defined and continuous on $\{x : \|x\| > 1\}$.

It remains to show that $x^k \to x$, $\|x^k\| > 1$ and $\|x\| = 1$ implies $Tx^k \to x$. This will follow once we prove (iii) since it and Lemma 2 (iv) will show that $\|Tx^k - x^k\| \le 4\varepsilon(x^k) \to 0$. □

Proof of (iii). By Lemma 1 (iii) there are at most three values of $n \in \mathbb{N}$ for which $\gamma(n - r(x)) \neq 0$. Hence by the triangle inequality, (12) and ($\star$),

$$\|Tx - x\| \le \|Sx\| + \sum_{n=0}^{\infty} \gamma(n - r(x))\|T_n(Sx)\| \le 4\varepsilon(x).$$ □

Proof of (iv). We fix $c \in (0,1)$, suppose there are $x, y \in X$ such that $r(x) \le r(y)$ and $\|Tx - Ty\| < c\|x - y\|$, and obtain various estimates of $\|Tx - Ty\|$. At the end these estimates will allow us to pick particular values of α, β and c for which (iv) holds.

Notice that $r(x) < \infty$ (which is equivalent to $\|x\| > 1$), since $r(x) = \infty$ implies $\|Tx - Ty\| = \|x - y\| \ge c\|x - y\|$.

We will consider three cases.

Case 1. There is $k \in \mathbb{N}$ such that $\gamma(k - r(x)) = \gamma(k - r(y)) = 1$. In particular, Lemma 1 (i) implies

$$r(x) \le r(y) \le k + 1 \le r(x) + 2. \tag{13}$$

Let $U_1 := (0, r(x))$, $U_2 := (r(x), r(y))$, $U_3 := (r(y), \infty)$, $E_1 := U_1$, $E_2 := U_2$, $E_3 := G_k \cap \psi_k^{-1}(U_2)$ and $E_4 := G_k \cap \psi_k^{-1}(U_3)$. Since $G_k \subset (k+1, \infty) \subset (r(y), \infty)$ by the middle inequality in (13), the sets E_j, $1 \le j \le 4$, are mutually disjoint. So we may estimate

$$\|Tx - Ty\|^p \ge \sum_{j=1}^{4} I_j \quad \text{where} \quad I_j := \int_{E_j} |(Tx - Ty)(t)|^p dt. \tag{14}$$

We express each of the integrals I_j with the help of integrals involving x and y.

(1) For $t \in E_1$, Lemma 3 (v) gives $(Tx - Ty)(t) = (x - y)(t)$, hence

$$I_1 = \int_0^{r(x)} |(x - y)(t)|^p dt.$$

(2) For $t \in E_2$, Lemma 3 (v) gives $(Ty)(t) = y(t)$ and (13) together with Lemma 3 (ii) imply $(Tx)(t) = 0$. Hence

$$I_2 = \int_{r(x)}^{r(y)} |y(t)|^p dt.$$

(3) For $t \in E_3$ we have $r(x) < \psi_k(t) < r(y)$, hence Lemma 3 (iv) and (vi) imply $(Ty)(t) = 0$ and $(Tx)(t) = (T_k x)(t)$, respectively. Hence $(\star)$ gives

$$I_3 = \int_{E_3} |(Tx)(t)|^p dt = \int_{G_k \cap \psi_k^{-1}(U_2)} |(T_k x)(t)|^p dt = \int_{r(x)}^{r(y)} |x(t)|^p dt.$$

(4) For $t \in E_4$ we have $r(x) \le r(y) < \psi_k(t)$, hence Lemma 3 (vi) implies $(Tx - Ty)(t) = (T_k x - T_k y)(t) = (T_k(x - y))(t)$. So $(\star)$ gives

$$I_4 = \int_{E_4} |(Tx - Ty)(t)|^p dt = \int_{\psi_k^{-1}(U_3)} |(T_k(x - y)(t)|^p dt = \int_{r(y)}^{\infty} |(x - y)(t)|^p dt.$$

Adding these estimates, we get from (14) that

$$\begin{aligned}\|Tx-Ty\|^p &\geq \int_0^{r(x)} |(x-y)(t)|^p\,dt + \int_{r(x)}^{r(y)} |(y(t)|^p\,dt + \int_{r(x)}^{r(y)} |(x(t)|^p\,dt \\ &\quad + \int_{r(y)}^{\infty} |(x-y)(t)|^p\,dt \\ &\geq \int_0^{r(x)} |(x-y)(t)|^p\,dt + 2^{1-p}\int_{r(x)}^{r(y)} |(x-y)(t)|^p\,dt \\ &\quad + \int_{r(y)}^{\infty} |(x-y)(t)|^p\,dt \\ &\geq 2^{1-p}\|x-y\|^p.\end{aligned}$$

Hence

$$\|Tx-Ty\| \geq 2^{(1-p)/p}\|x-y\| \geq \|x-y\|/2. \tag{15}$$

By Lemma 1 (iv), in the remaining cases we have $k \in \mathbb{N}$ such that $\gamma(k-r(x)) = 0$ and $\gamma(k-r(y)) = 1$.

Case 2. If $\varepsilon(x) \leq \varepsilon(y)/\alpha$, we use $k \geq r(y)-1 \geq r(x)-1$, $\gamma(k-r(x)) = 0$ and Lemma 3 (iii) to infer that $(Tx)(t) = 0$ for $t \in G_k$. Since $(Ty)(t) = (T_k y)(t)$ for $t \in G_k$ by Lemma 3 (vi), $(\star)$ implies

$$\|Tx-Ty\|^p \geq \int_{G_k} |(T_k y)(t)|^p dt = \int_{r(y)}^{\infty} |y(t)|^p dt = \varepsilon(y)^p.$$

To get from this a more usable estimate, we observe that by (iii),

$$\|x-y\| \leq \|Tx-Ty\| + 4(\varepsilon(x)+\varepsilon(y)) \leq c\|x-y\| + 4(\varepsilon(x)+\varepsilon(y)).$$

Hence $(1-c)\|x-y\| \leq 4(\varepsilon(x)+\varepsilon(y)) \leq 4(1+\alpha)\varepsilon(y)/\alpha$, and we conclude that

$$\|Tx-Ty\| \geq \varepsilon(y) \geq (1-c)\alpha\|x-y\|/(4(1+\alpha)). \tag{16}$$

Case 3. If $\varepsilon(y) < \alpha\varepsilon(x)$, Lemma 2 (v) implies

$$\|x-y\| \geq \beta\varepsilon(x) \geq \beta(\varepsilon(x)+\varepsilon(y))/(1+\alpha).$$

This and (iii) imply

$$\|Tx-Ty\| \geq \|x-y\| - 4(\varepsilon(x)+\varepsilon(y)) \geq \|x-y\| - 4(1+\alpha)\|x-y\|/\beta. \tag{17}$$

It follows from (15), (16) and (17) that (iv) holds provided α, β and c were chosen so that

$$0 < c \leq \min(1/2, (1-c)\alpha/(4(1+\alpha)), 1-4(1+\alpha)/\beta).$$

To see that such a choice is possible, we may first pick any $0 < \alpha < 1$, then find $0 < c \leq 1/2$ small enough to satisfy $c \leq (1-c)\alpha/(4(1+\alpha))$ and finally choose $\beta > 0$ so that $c \leq 1-4(1+\alpha)/\beta$. A particular choice coming from this is $\alpha = 4/5$, $\beta = 8$ and $c = 1/10$. $\square$

Proof of (v). Let $Y_m := \{y \in X : y(t) = 0 \text{ for } t \in (m,\infty)\cap\bigcup_{n=1}^{\infty}(2n-1,2n)\}$. Then Y_m is nowhere dense in X and by Lemma 3 (i), $\{Tx : x \in X,\ r(x) \leq m\} \subset Y_m$ for every m. Hence

$$M := \{Tx : \|x\| \geq 1\} \subset \{z \in X : \|z\| = 1\} \cup \bigcup_{m=1}^{\infty} Y_m$$

is a first category subset of X. Finally, using that (iv) implies that M is closed, we infer from the Baire Category Theorem that it is a nowhere dense subset of X. $\square$

Proof of (vi). By (v) there are points y with $\|y\| > 1$ that do not belong to the T-image of $\{x : \|x\| \geq 1\}$. Since by (i) such points cannot belong to the T-image of $\{x : \|x\| \leq 1\}$, they do not belong to the range of T. $\square$

Theorem 4 says that T multiplied by $1/c$ provides a counterexample to Problem N in any $L_p(0,\infty)$, $1 \leq p < \infty$, and so, as already pointed out, also in $L_p(0,1)$. In the Hilbert space case, which is our main interest, we strengthen this example by modifying T in such a way that the resulting map is expanding and equal to the identity on the unit ball.

Theorem 5. *There is a map of a separable Hilbert space into itself that is continuous, expanding, non-surjective, and equal to the identity on the unit ball.*

Proof. We will work in the space $H := L_2(0,\infty)$ and modify the map T from Theorem 4 to get a map with the required properties.

Let $C = 4/c$ where c is the constant from Theorem 4, define $a : H \to [0,\infty)$ by

$$a(x) := \begin{cases} C & \text{when } \|x\| \geq c \\ 1 & \text{when } \|x\| \leq c/2 \\ 1 + (C-1)(2\|x\|/c - 1) & \text{when } c/2 \leq \|x\| \leq c \end{cases}$$

and $T_0 : H \to H$ by $T_0 x = a(x)Tx$.

We suppose $\|x\| \leq \|y\|$ and, considering several cases, estimate $\|T_0 x - T_0 y\|$. For that we will use the simple inequality

$$\|au - bv\| \geq \max\bigl(a\|u - v\|, (b-a)\|v\|\bigr)$$

when $\|u\| \leq \|v\|$ and $0 \leq a \leq b$, which follows by expanding $\|a(u-v) - (b-a)v\|^2$ and using that the scalar product of $u - v$ and v is negative. We also notice that $c \leq 1$, $C \geq 4$ and $a(d) = C/2$ for $d := (3C-4)c/(4(C-1))$.

Case $\|y\| \leq 1$**:** Then $Tx = x$, $Ty = y$ and $1 \leq a(x) \leq a(y)$, so

$$\|T_0 x - T_0 y\| = \|a(x)x - a(y)y\| \geq a(x)\|x - y\| \geq \|x - y\|.$$

Case $\|x\| \geq c$**:** Then $a(x) = a(y) = C$, so

$$\|T_0 x - T_0 y\| = C\|Tx - Ty\| \geq Cc\|x - y\| \geq \|x - y\|.$$

Case $d \leq \|x\| < c$ **and** $\|y\| > 1$**:** Then $a(x) \geq C/2$ and $\|Ty\| \geq 1 \geq \|Tx\|$, so

$$\|T_0 x - T_0 y\| \geq a(x)\|Tx - Ty\| \geq Cc\|x - y\|/2 \geq \|x - y\|.$$

Case $\|x\| < d$ **and** $\|y\| > 1$**:** Then $a(y) - a(x) \geq C/2$ and $\|Ty\| \geq \|Tx\|$, so

$$\|T_0 x - T_0 y\| \geq C\|Ty\|/2 \geq Cc\|y\|/2 \geq \|y\| + 1 \geq \|x - y\|.$$

Since this covers all possible cases for x and y, T_0 is expansive. Also, it is continuous and satisfies $T_0 x = x$ when $\|x\| \leq c/2$. To see that it is not surjective, we use Theorem 4 (v) to find y with $\|y\| > 1$ that does not belong to the T-image of $\{x : \|x\| > 1\}$. We show that Cy does not belong to the range of T_0. Suppose for a contradiction that $Cy = T_0 x$. Then $\|x\| > 1$ since otherwise $\|T_0 x\| \leq C\|x\| \leq C < \|Cy\|$. Hence $Tx = T_0 x/C = y$, contradicting the choice of y.

To finish the proof, it suffices to infer from the above that the map $U : H \to H$ defined by $Ux := 2T_0(cx/2)/c$ is non-surjective, continuous, expansive and equal to the identity on the unit ball. $\square$

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