Dynamic Production-Pricing Strategies for Multi-generation Products under Uncertainty

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Abstract

Due to rapid advances in technology and design, firms periodically release new generations of electronic products such as mobile phones and computers. In order to increase product variability, firms may wish to develop a multiple-generation product line rather than replace the older versions with new ones. However, when multiple generations are available in the market, different generations compete with each other as well as other products in the market. Firms need to take joint decisions for the purpose of inventory management and dynamic pricing of multiple generations to tackle impact of uncertain demand and market competition. In this paper, we present a dynamic joint production-pricing decision model to obtain optimal strategies for a firm selling multiple generations of a product. We account for the internal competition among multiple generations by evaluating customer choices. In order to tackle a curse of dimensionality, we introduce a forward dynamic programming approach for approximately solving the joint production-pricing problem. We also propose a two-stage heuristic algorithm as an alternative solution approach. Different pricing rules determined from the abridged model and a price list derived by theoretical bounds are integrated to improve further computational performance of the solution approaches. We design computational experiments to illustrate effectiveness and efficiency of these approximate methods and show the benefit of joint decision-making process in multi-generation product line.

Keywords: Multi-generation products, production-pricing, dynamic programming, uncertainty

Declarations of interest: none.

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Highlights:

- We study a dynamic joint production-pricing decision model.
- An approximation method and a heuristic approach are introduced to solve the problem.
- Different pricing rules are derived by theoretical bounds.
- Numerical experiments are designed to illustrate performance of solution approaches.

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Abstract

Due to rapid advances in technology and design, firms periodically release new generations of electronic products such as mobile phones and computers. In order to increase product variability, firms may wish to develop a multiple-generation product line rather than replace the older versions with new ones. However, when multiple generations are available in the market, different generations compete with each other as well as other products in the market. Firms need to take joint decisions for the purpose of inventory management and dynamic pricing of multiple generations to tackle impact of uncertain demand and market competition. In this paper, we present a dynamic joint production-pricing decision model to obtain optimal strategies for a firm selling multiple generations of a product. We account for the internal competition among multiple generations by evaluating customer choices. In order to tackle a curse of dimensionality, we introduce a forward dynamic programming approach for approximately solving the joint production-pricing problem. We also propose a two-stage heuristic algorithm as an alternative solution approach. Different pricing rules determined from the abridged model and a price list derived by theoretical bounds are integrated to improve further computational performance of the solution approaches. We design computational experiments to illustrate effectiveness and efficiency of these approximate methods and show the benefit of joint decision-making process in multi-generation product line.

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1 Introduction

In the technology industry, firms gain competitive advantage by periodically releasing a new-generation product every 12-15 months while keeping multiple generations in the market (Saxena 2018). Having a multi-generation product line is more profitable for a firm than selling a single model due to the increase in product variability (Kilicay-Ergin et al. 2015). Many leading firms have developed multi-generation product lines to target different customer segments. For instance, Apple has simultaneously offered four generations of iPhone (namely iPhone 8, XR, XS and 11) since 2017. Similarly, Samsung has had five generations of Galaxy S (Apple 2020, Samsung 2020).

Despite the profitability of multi-generation product line, it creates various challenges at strategic and operational levels. In comparison to the older generation products, the new-generation is always assumed to have innovative and improved features, which have never been exposed to the market yet. Therefore, there is a high uncertainty in customers' initial response towards a new product. Apart from the innovative features, price as another important factor affects customers' behaviour to distinguish among multiple generations. It is expected that price of the older generations drops with the introduction of a new product. Although a new release attracts customers, the older generation products' sales might still increase due to price drop. For instance, when iPhone 7's release date was announced, Apple cut the price of iPhone 6 by $100 and iPhone 6 attained the largest market share in the US (Smith 2016, Munbodh 2016). The surge in sales of old generations is another strong motivation behind the firm maintaining a multi-generation product line. On the other hand, customers' reaction to the older generations, particularly in the presence of new release is unpredictable, as well. In fact, customers may attempt to "get more for less" by delaying purchase in hope of price reductions of the older generations followed by a new release (Levin et al. 2010). Customer anticipation for prices may lead to *internal competition* across the multi-generation product line and internal competition cannibalises sales of existing products by the newest generation or vice versa. Thus, internal competition between the old- and new-generation products often creates conflicting benefits for the firm (Ferguson & Koenigsberg 2007, Li et al. 2010).

A forward-looking approach has been often used in practice to predict future business conditions and determine optimal strategies balancing supply and demand uncertainties (Li et al. 2010). This approach involves critical decision-making problems such as production planning and pricing that affect the success of a multi-generation product line. During a new release, a firm needs to determine the prices for both new and old-generations, as well as the production and inventory plan for all generations. The new-generation is usually priced higher than the previous generations based on the additional features and upgrades. On the other hand, the price reduction decision for old generations may be affected by unsold inventory of the current product line. Apart from pricing decisions, the unsold inventory levels also impact firms' production plan for multiple generations. Thus, production and pricing strategies cannot be developed in isolation. Joint decision-making models are necessary to successfully manage multi-generation product lines. Although many academics (Davis 1993, Chen & Simchi-Levi 2004, Talluri & van Ryzin 2004, Karaesmen et al. 2011) and practitioners (Webber et al. 2011) advocate the importance of joint pricing-inventory techniques as essential tools to mitigate demand uncertainty for fixed-age products, this research area has not yet received enough attention. Unpredictable customers' response during a new product release does not only cause demand uncertainty for all generations, but also creates internal competition

among available generations in the market.

In this paper, we are concerned with dynamic joint production-pricing strategies for a multi-generation product line by considering demand uncertainty and internal competition. Our contribution in this research paper is two-fold:

- We formulate the joint production-pricing decision making problem of a firm selling multiple generations of a product under demand uncertainty as a stochastic dynamic programming model. The existing literature on multi-product pricing and production problems, as we review in the next section, generally focuses on only two products and assumes a constant product value over time. However, during a new product release, the old generation becomes less attractive due to the new generation's technological improvements or additional features. In contrast to the existing literature, we account for the internal competition among multiple generations by evaluating the dynamic changes in customer choice model. The consideration of multiple generations in the joint-production and pricing model flares up the state space. Moreover, the customer choice probabilities depending on pricing decisions lead to a nonlinear (high degree polynomials) optimisation problem to be solved at each state of the system. Therefore, the underlying dynamic programming model is computationally intractable to solve by a traditional (backward) dynamic programming technique. This requires efficient approximation method.
- In order to tackle curse of dimensionality on the state space, we propose an approximation method and a heuristic approach. The first approach considers a forward dynamic programming algorithm for approximately solving the joint production-pricing problem. At each iteration of the algorithm, for a given customer arrival path, the decision-making model is solved to determine joint production-pricing strategy. The second approach applies for a two-stage heuristic algorithm that adopts the idea of partial planning introduced by Chan et al. (2006). The pricing decisions (determined in the first stage of the algorithm) are integrated into the optimisation model at the second stage to determine the optimal production level for each product. In order to improve further computational performance of the solution approaches, we investigate different pricing rules determined from the abridged model and a list of prices derived by theoretical bounds. We design numerical experiments to illustrate performance of solution approaches and derive some managerial insights. In our numerical experiments, we analyse the benefits of selling multiple generations of a product on firm profit. We also quantify the benefits of dynamic joint production-pricing decisions as opposed to using fixed policies based on either production or pricing. Our analysis indicates that joint production-pricing strategy performs significantly better than the fixed policies since it considers recent changes while matching demand with production. We also observe that the customer choices play an important role on the performance of the joint decision making process.

The remaining part of the paper is organized as follows. Section 2 focuses on the literature review by providing details of existing studies relevant to our research. The stochastic dynamic programming formulation of the joint inventory-pricing problem is presented in Section 3. The solution methodology and computational results are explained in Sections 4 and 5, respectively. The concluding remarks are provided in Section 6.

2 Literature Review

Research on dynamic inventory and pricing management problems has attracted significant attention over the years. The recent surveys for these problems are provided by Chen & Simchi-Levi (2012) and Janssen et al. (2016). In this study, we consider a firm producing a multi-generation product such as mobile phones and laptops. Although electronic products do not have a short shelf lifetime like perishable products have, older versions of a multi-generation product are generally discontinued from the market after some time due to technological developments. In this respect, a multi-generation product can be considered as a perishable product. Therefore, we focus our review on joint inventory-pricing management research for both non-perishable and perishable products.

Within the joint inventory-pricing management research on non-perishable products, initial studies consider sale of only one product and assume that the seller has convex production and holding costs, and unlimited production capacity (Thowsen 1975, Federgruen & Heching 1999). By using the properties of the model, they show that the base-stock policy (place an order when the inventory level drops below the base-stock level) is optimal for such problems and the optimal price is a decreasing function of the starting inventory. Similarly, Chen & Simchi-Levi (2004) extend the work of Federgruen & Heching (1999) by considering the fixed setup cost for ordering. The optimal policies of joint ordering and pricing problem for a multi-period problem are derived by assuming an additive demand model. Chao et al. (2012) focus on the model proposed by Chen & Simchi-Levi (2004) and investigate the pricing and inventory control policies under the limited production capacity. Chan et al. (2006) propose partial-planning strategies for pricing and inventory replenishment problem by considering capacity constraints. Under a partial planning strategy, the seller separates the pricing and the production decisions and decides either pricing or production schedule at the beginning of the planning horizon. The remaining decision (pricing or production schedule) is made by considering demand uncertainty. They proposed several heuristics based on the proposed dynamic programming model to solve these partial-planning problems.

A few researchers have addressed joint inventory and pricing management problem for multiple non-perishable products. Gilbert (2000) develops a solution method for the joint decision-making model with deterministic demand. The proposed demand model does not consider the cross-price effect between the non-perishable products. Zhu & Thonemann (2009) extend this case and focus on a two-product model in which demand for each non-perishable product depends linearly on the prices of both products. They show that the optimal inventory policy is similar to the base-stock policy for the one-product problem. Song & Xue (2007) consider a more general demand setting for substitutable multiple products. They formulate the problem as a dynamic program and develop a solution algorithm by exploiting the special problem structure. Yan et al. (2017) study joint production and pricing policies for a firm selling new and remanufactured products by considering possible product returns. They assume that the firm either adopts make-to-order or make-to-stock strategy for the new product and under this set-up, they show that the base-stock type production policy is optimal for the make-to-stock strategy for additive demand model. A complete literature review for joint inventory-pricing management of non-perishable product is provided by Elmaghraby & Keskinocak (2003) and Chen & Simchi-Levi (2012).

Products with short and fixed lifetimes are known as perishable products, such as vegetables, dairy

products and medicines. Perishability is also observed in many high-tech products such as laptops, mobile phones, digital cameras due to the rapid obsolescence in a fast moving market (Ferguson & Koenigsberg 2007). Within the research on joint inventory-pricing management for perishable products, there are various studies exclusively developed for food or healthcare products (Li et al. 2009, Chen & Sapra 2013, Chen et al. 2014, Chintapalli 2015, Herbon 2017). We only discuss the studies which are relevant to management of a multiple generation product line. Ferguson & Koenigsberg (2007) consider a firm selling a food product with exactly two-period life cycle. In the first time period, procurement and pricing decisions of a fresh product is made in the presence of demand uncertainty. At the beginning of second time period, the decisions are concerned with how much leftover inventory from first period (old inventory) to carry over, how much fresh products to procure (new inventory), and what prices to charge for new and old inventories. Uncertain demand in the first time period becomes deterministic in the second time period. They use a customer utility model to obtain demand functions of new and old products based on price and features of the product using a subgame perfect equilibrium in the second time period. A perfect equilibrium is achievable in a deterministic setting. Sainathan (2013) extend Ferguson & Koenigsberg (2007)'s joint inventory-pricing decision model to the case of a firm experiencing uncertain demand. He uses a linear utility customer choice model to derive optimal pricing and replenishment policies for a product with two-period shelf life. He assumes that old and new perishable products compete with each other in the market under demand uncertainty that is incorporated through the process of dynamic demand substitution; i.e. demand for an old food item is replaced by a new one. Chintapalli (2015) works on the joint inventory-pricing decision model for a firm experiencing substitutable demand for a n-period shelf-life food product. However, demand for multiple generation products, like electronics, is not substitutable since price difference among its various versions is higher than the old and new food items due to improved features.

In some high-tech industries, the firm's aim to release a new product is to eventually replace the older versions. The transition from the current product to a new one does not occur instantaneously but rather involves a period of time, referred as the product transition or product rollover (Li et al. 2010). A firm can either completely replace the old generation by the new one or continue to sell multiple generations until the sales of the old generations diminishes. These strategies are known as the single and the dual product rollover, respectively (Corey Billington & Tang 1998, Feryal Erhun & Hopman 2007). Several studies compare the benefits of both strategies by considering the internal competition between two generations (Lim & Tang 2006, Arslan et al. 2009, Zhou et al. 2015). Liang et al. (2014) extend the earlier work on rollover strategies by analysing the interaction between rollover strategy and strategic waiting behaviour. They formulate a two-period problem in which a firm releases a new generation of the product in each period. By analysing the customers' optimal purchase decisions, they conclude that optimal rollover strategy significantly depends on the new product's innovation and the number of strategic customers in the market. In a related work, Liu et al. (2018) compare product rollover strategies when customers are allowed to trade-in the older version of the product with the new one. They propose a two-period dynamic game model and analyse the value of trade-in policy for rollover strategies. Li et al. (2010) focus on an inventory management problem with no replenishment during the product transition from an old generation to a new one. They develop a dynamic model to find inventory levels for two generations of products where the release date of the new generation is assumed to be unknown. Li & Graves (2012) present a dynamic pricing model for the product transition stage in which two generations are sold simultaneously with no

product replenishment. All of the above models address the transition between exactly two generations of a product line. However, in various cases like mobile phones, computers and e-tablets, the motive of a new release is not to replace the older versions, rather develop a product line of multiple generations. In fact, in some cases, a new release boosts the sales of its predecessors, due to their reduced prices (Munbodh 2016). Moreover, simultaneously selling multiple generations is reported to be profitable for the firm (Kilicay-Ergin et al. 2015). Akçay et al. (2010) consider the case of a firm simultaneously selling multiple perishable products over a finite time. They derive optimal pricing policies by introducing a linear random utility framework to model consumer choices in a differentiated assortment of products. Thus, their focus is a pricing problem where no replenishment opportunities are present during the planning horizon.

In this paper, we develop a joint inventory-pricing model for a firm selling multiple generations simultaneously in the market using stochastic dynamic program. We primarily investigate the release of newer generations in the presence of older ones. This model differs from the joint production and pricing models introduced by Ferguson & Koenigsberg (2007) and Sainathan (2013) in terms of problem set-up and demand function. At every decision stage, orders for all generations in market are placed, in contrast to Ferguson & Koenigsberg (2007) and Sainathan (2013). Moreover, we develop a customer choice model based on the models proposed by Caplin et al. (1991) and Petrin & Train (2003) to account for both uncertain demand and internal competition among multiple generations.

The curse of dimensionality is often experienced while obtaining the solution for the dynamic joint decision-making problem of multiple products. Therefore, most of the literature discussed above limits the joint production-pricing model for a simplified two product case. The intertwined structure of the multiple inventory levels with their production and pricing decisions results in a complex and non-decomposable decision problem, which is computationally intractable by the classical dynamic programming solution method (backward dynamic programming). Thus, forward dynamic programming (FDP) has emerged as a successful methodology to tackle the curse of dimensionality by reducing the state space calculations. Although FDP has been widely used in various areas, the literature on FDP for solving the joint inventory-pricing problems is limited. For instance, Çimen & Kirkbride (2017) consider the FDP approach for the flexible production-inventory problem for multiple products at multiple locations. Coşgun et al. (2013) use FDP to solve the problem of markdown optimization between substitutable products in retail chains.

In this research, the curse of dimensionality is encountered for solving the underlying dynamic programming model for management of a multiple generation product line. Thus, we propose two approximation methods to solve the joint production-pricing problem. The first approximation method is a two-stage heuristic based on the idea of partial planning model introduced by Chan et al. (2006). In the first stage, we solve an abridged adaption of the original problem as a dynamic programming model to determine the pricing policy. In the second stage, this pricing information is then used to obtain the joint production pricing policies. Our second approximation method is based on the forward dynamic programming approach to approximately solve the joint production-pricing model. The numerical experiments (presented in Section 5) show that the two-stage heuristic in conjunction with different pricing rules produces policies with higher expected profits while FDP is accounted as more efficient than other approaches.

3 The Dynamic Production-Pricing Model

Consider a firm (such as Apple and IBM) that designs and develops various generations of innovative products (such as electronics, mobile phones and computers). The firm releases a new generation of the product whilst the older versions still continue to sell in the market. The new generation of the product uses innovative technologies and/or involves improved features in comparison to previous models. It is difficult to predict customers' reaction toward the latest technological developments of the new generation of the product as well as the older generations in presence of a new release. Thus, demand for all generations of the product is assumed to be uncertain. The firm tackles demand uncertainty and internal competition among multi-generations of the product by a forward-looking planning of joint decisions on production level and their prices. We therefore formulate the firm's joint production-pricing problem using a finite discrete-time stochastic dynamic programming model. In this section, we first present the underlying production-pricing problem and then formulate the problem as a dynamic program under a customer choice model. Before that, we introduce notation used for the problem formulation.

Notation: We use tilde ($\tilde{*}$) to denote randomness; e.g., $\tilde{y}$ denotes random variable y. Boldface is used to denote vectors; for example, $\mathbf{a} \in \mathbf{R}^n$ is a n-dimensional vector. In particular, we denote a vector of ones by $\mathbf{1} = [1, \cdots, 1]$ in appropriate dimension and "$\mathbf{x} \cdot \mathbf{y}$" displays a scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$. A description of the notation used in the paper is provided in Table 1.

Table 1: Description of notation

	Model parameters
T	planning horizon discretised by time periods $t = 0, 1, \cdots, T$
G_t	set of generations available in the market at time t
$\bar{m}$	maximum number of versions available in the market
$\alpha_{k,t}$	level of desirability for version $k \in G_t$ at time t
$\gamma_{k,t}$	choice probability of version $k \in G_t$ at time t
$\mathbf{c}_t$	vector of unit ordering costs $c_{k,t}$ of generation $k \in G_t$ at time t
h_t	unit inventory holding cost from time $t-1$ to t
κ	production capacity
δ	discount factor
$\tilde{\mathbf{d}}_t$	vector of uncertain demand $\tilde{d}_{k,t}$ for generation $k \in G_t$ at time t
	State variables and actions
$\mathbf{x}_t$	vector of inventory levels $x_{k,t}$ for generation $k \in G_t$ at time t
$\mathbf{p}_t$	vector of prices $p_{k,t}$ for generation $k \in G_t$ at time t
$\mathbf{q}_t$	vector of production levels $q_{k,t}$ for generation $k \in G_t$ at time t

We interchangeably use *models*, *versions* and *generations* of the underlying (electronics) product in the remaining of the paper. The maximum function $(a)^+ = \max\{a, 0\}$ takes value of a if and only if $a > 0$; otherwise, it is zero. On the other hand, the minimum function $\min\{b, c\}$ for $b, c \geq 0$ takes value of b if $b \leq c$; otherwise, it is equal to c. Both "innovation level" and "quality of product" that basically represent "level of desirability" for the product are interchangeably used in this paper. As the innovation level (quality) of a product increases, the level of desirability increases. Even though the level of desirability $\alpha_{k,t}$ for version k of a product is the same for a price-sensitive and quality-sensitive customers, their utilities at time t will be different because of customers quality preferences.

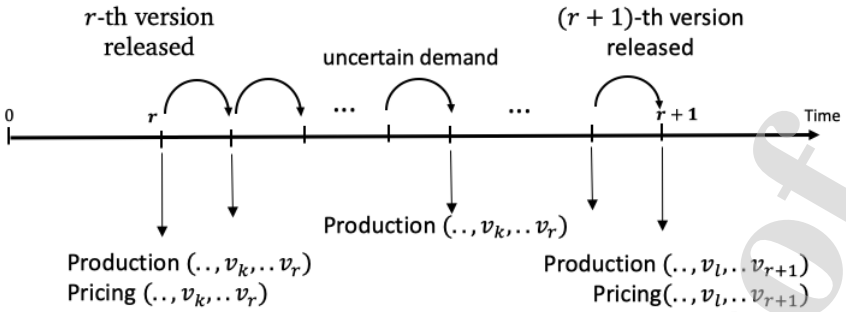

Figure 1: Decision-making process in a multi-generation product line

Dynamic Programming Model: Assume that the firm releases different generations (indexed by r) of the product over a finite planning horizon. The planning horizon is discretized into $t = 0, 1, \cdots, T$ time periods where the operational (such as production and pricing) decisions and tactical decisions (such as release of new generation) are made; in particular, $t = 0$ represents today. We assume that the firm regularly launches new generations of the product and the lead time for procurement is zero so that the products can be received instantaneously. The release time of generation r is denoted as t_r such that $0 \leq t_r \leq T$. The firm may sell at most $\bar{m} \geq 2$ generations at each time period. We denote a set of different generations of the product available in the market at time t by G_t. Figure 1 displays a graphical timeline of firm's decision-making process. When r-th version (v_r) is released, as displayed in Figure 1, the firm decides the production levels and prices for the available versions. The firm then observes the uncertain demand and focuses on the production of the set of available versions till the next generation (v_{r+1}) is released.

Let $\tilde{\mathbf{d}}_t = \{\tilde{d}_{r,t} : r \in G_t\}$ represent a vector of uncertain demand $\tilde{d}_{r,t}$ of generations $r \in G_t$ at time t. In addition, let $\mathbf{c}_t = \{c_{r,t} : r \in G_t\}$ show a vector of unit production cost $c_{r,t}$ of all generations $r \in G_t$ at time t. The carrying cost h_t for holding one unit of inventory from $t-1$ to t remains the same for all generations.

We assume that the inventory level $x_{r,t}$ of generations $r \in G_t$ (that are currently selling in the market) is reviewed at the beginning of time period t before the production process begins. We define vector $\mathbf{x}_t = \{x_{r,t} : r \in G_t\}$ of the inventory levels of all generations available in the market as a state of the dynamic system at time t. Given a state of the system, an action set consists of decisions made at each i) release time as *production and selling prices of the new generation as well as previously released generations of the product*, and ii) intermediate time period between two consecutive release times in terms of *production of all generations available in the market*. Let $p_{r,t}$ and $q_{r,t}$ denote unit selling price and amount of products to be produced for each generation $r \in G_t$ at time t, respectively. Similarly, we introduce vectors $\mathbf{p}_t = \{p_{r,t} |\ p_{r,t} \geq 0, r \in G_t\}$ and $\mathbf{q}_t = \{q_{r,t} |\ q_{r,t} \geq 0, r \in G_t\}$ corresponding to market prices and amount of production of generations at time t, respectively.

The system dynamics lead to state transition of inventory levels from t to $t+1$. The following balance equations

$$\mathbf{x}_{t+1} = \max\{\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t, 0\}, \text{ for } t = 0, 1, \cdots, T$$

imply that the inventory $\mathbf{x}_{t+1}$ to be carried over from t to $t+1$ is determined by the inventory level $\mathbf{x}_t$, amount of production $\mathbf{q}_t$ and customers' demand $\tilde{\mathbf{d}}_t$ at time t. Note that if $\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t > 0$, then the firm incurs an inventory holding cost h_t per unit to carry unsold inventory to the next period. On the other hand, if demand exceeds the current inventory level, i.e., $\tilde{\mathbf{d}}_t - \mathbf{x}_t + \mathbf{q}_t > 0$, then the firm is unable to fulfill the customers' demand. In this case, the unmet demand is assumed to be lost (not backlogged).

Let κ represent the production capacity of the firm. We ensure that total number of products to be produced at time t do not exceed the available production capacity: $\mathbf{1}(\mathbf{q}_t) \leq \kappa$.

The firm aims to maximize the expected profit over the planning horizon while maintaining a multiple-generation product line through a joint inventory-pricing decision framework. The expected profit is computed as the expected revenue minus the expected total cost of production and holding. Given the selling price of all generations, the revenue depends on future realisations of the customer demand. When the demand at time t is low (i.e., $\tilde{\mathbf{d}}_t \leq \mathbf{x}_t + \mathbf{q}_t$), the revenue becomes $\mathbf{p}_t \cdot \tilde{\mathbf{d}}_t$. In case of high demand (i.e., $\tilde{\mathbf{d}}_t > \mathbf{x}_t + \mathbf{q}_t$), we compute the revenue as $\mathbf{p}_t \cdot (\mathbf{x}_t + \mathbf{q}_t)$. Thus, we can state the revenue earned at time t as $\mathbf{p}_t \cdot \min\{\mathbf{x}_t + \mathbf{q}_t, \tilde{\mathbf{d}}_t\}$. The total ordering and holding costs are expressed as $\mathbf{c}_t \cdot \mathbf{q}_t$ and $h_t \mathbf{x}_{t+1}$, respectively. The single-period profit $\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t)$ at time t is obtained as

$$\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t) = \mathbf{p}_t \cdot \min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\} - h_t \mathbf{x}_{t+1} - \mathbf{c}_t \cdot \mathbf{q}_t.$$

We formulate the joint production-pricing decision making problem as a stochastic dynamic optimisation model that requires different action sets at the release time of a new generation. During the launch of a new generation, along with production decisions, the firm needs to determine price of the latest generation of the product, based on enhanced features and/or innovations, while adjusting prices of older generations by evaluating the market value of innovative evolution of technologies and predicting customers' willingness to pay for the new and old innovative improvements. Let us assume that a new generation r is to be released at time t (i.e., $t = t_r$). The firm needs to determine how many products to produce for each generation and what market price to assign for all generations $r \in G_t$ in the market. Let $V_t(\mathbf{x}_t)$ denote the value function at time t given a state $\mathbf{x}_t$ of the system. The value function for the dynamic joint production-pricing decision making problem can be written as follows;

$$\begin{aligned} V_t(\mathbf{x}_t) = \quad & \max_{\mathbf{p}_t, \mathbf{q}_t} \ \mathbb{E}\left[\pi_t(\mathbf{x}_t, \mathbf{q}_t, \mathbf{p}_t) + \delta V_{t+1}(\mathbf{x}_{t+1})\right] \\ & \text{s.t.} \\ & \quad \mathbf{x}_{t+1} = (\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \\ & \quad \mathbf{1} \cdot (\mathbf{q}_t) \leq \kappa, \quad \mathbf{q}_t \geq \mathbf{0} \\ & \quad \mathbf{p}_t \in \mathcal{F}_t \end{aligned} \tag{1}$$

where $\mathcal{F}_t$ is a set of feasible prices of all generations of the product. We will further refine construction of this set in the next section by introducing the customer choice model that takes into account internal competition of different generations of the product and also the customer preferences in terms of quality and price of the products selling in the market. The expectation operator in the value function is always taken over customer demand uncertainty. Note that in this formulation, the ending inventory for the last time period is assumed to be held till the end of the planning horizon and then it is discarded at zero cost. Finally, the boundary condition at the end of planning horizon is $V_{T+1}(\mathbf{x}_{T+1}) = 0$.

The firm can produce any version of the product at intermediate time periods between consecutive release times. The problem formulation (1) then becomes the following dynamic programming model where the production decisions are made at any intermediate time periods t' for $t_r < t' < t_{r+1}$ assuming that the unit prices of different versions (G_{t_r}) available in the market at time t' remain the same from t_r till the next launch at t_{r+1}.

$$\begin{aligned} V_{t'}(\mathbf{x}_{t'}|\mathbf{p}_{t_r}) = \quad & \max_{\mathbf{q}_{t'}} \ \mathbb{E}\Big[\mathbf{p}_{t_r} \cdot \min\{\tilde{\mathbf{d}}_{t'}, \mathbf{x}_{t'} + \mathbf{q}_{t'}\} - h_{t'}\mathbf{x}_{t'+1} + \delta V_{t'+1}(\mathbf{x}_{t'+1})\Big] - \mathbf{c}_{t'} \cdot \mathbf{q}_{t'} \\ & \text{s.t.} \\ & \quad \mathbf{x}_{t'+1} = (\mathbf{x}_{t'} + \mathbf{q}_{t'} - \tilde{\mathbf{d}}_{t'})^+ \\ & \quad \mathbf{1} \cdot (\mathbf{q}_{t'}) \leq \kappa, \quad \mathbf{q}_{t'} \geq \mathbf{0} \end{aligned} \tag{2}$$

Notice that when intermediate time periods between any two consecutive release times are ignored, the problem complexity of the dynamic program in (1) can be slightly reduced. Then the dynamic optimisation model (so-called an *'abridged model'*) provides an approximate solution to the initial joint production-pricing problem. The abridged model will be used to develop a two-stage heuristic and it is discussed in Section 4.2.

3.1 The Customer Choice Model

When the firm releases a new generation of the product, customers anticipate both price at which the new version is released and also potential change in selling prices of the older versions. As reported by Li et al. (2010), the customer's anticipation for prices of multiple versions may cannibalize sales of existing generations by the new one and vice versa. The cannibalization of sales occurs because new and existing generations internally compete to be a preferable choice of customers. A customer choice model examines various factors governing the customer's decision to buy a specific version of the product or leave without a purchase. In general, there exists different observable and unobservable factors that determine the customer's choice of buying a particular version (Train 2009). Price, innovation and technological levels of a generation, and the customer's sensitivity toward technology can be listed as examples of observable factors impacting their choices. On the other hand, unobservable factors influencing the customer's predilection toward buying a specific version are described as idiosyncratic and mostly depend on individual's preferences, like personal style and acceptance of innovative technology. In this paper, we consider only observable factors such as price and quality of generations to analyze the customer's choice. Next a description of our customer choice model follows.

We assume that a customer is rational while making a decision and aims to maximize his/her own utility. Moreover, the latest generation of the product is assumed to be more attractive than the previous models in terms of technological features of multiple versions. Let $U_{k,t}$ denote the utility that a customer achieves from purchasing generation $k \in G_t$ at time t. Following Train (2009), Akçay et al. (2010) and Sainathan (2013), the customer's utility can be expressed by a linear function of the innovation level $\alpha_{k,t}$ and price $p_{k,t}$ of generation k at time t as follows;

$$U_{k,t} = \theta\alpha_{k,t} - p_{k,t}, \quad k \in G_t, \ t = 0, 1, \cdots, T, \tag{3}$$

where θ represents the customer's quality sensitivity toward the technology. The innovation level $\alpha_{k,t}$

measures the attractiveness of a generation and is explicitly associated with technological features of multiple versions available in the market. Although the innovation level and the market price of each generation are the same for all customers, the quality sensitivities of customers toward innovative technology may vary. Note that one can also define the utility for an individual customer by using specific sensitivity parameter associated with the individual customer; the reader is referred to Akçay et al. (2010) and Sainathan (2013) for further information on the customer specific choice models.

We assume that parameter θ (representing customer's quality sensitivity) follows a uniform distribution over an interval of $[0, 1]$. The customer prefers the latest generation of the product possessing innovative technologies when $\theta = 1$ that indicates the highest quality sensitivity. On the other end, $\theta = 0$ reflects the least quality sensitivity where the customer prefers not to buy any version of the product. Let us now consider a set $G_t = \{k|\ k = 1, 2, \cdots, m\}$ of $m \leq \bar{m}$ generations available in the market at time t where $k = 1$ represents the earliest generation and $k = m$ is the latest generation of the product. Further assumptions regarding the customer choice model are enlisted below.

A1: Since the recent generation of the product is always perceived to have a better quality than the older models, the corresponding innovation level of the newest generation is assumed to be higher than those of previous models. Thus, we can construct the following relationship $\alpha_{m,t} \geq \alpha_{m-1,t} \geq, \ldots, \geq \alpha_{1,t}$ among innovation levels of generations available in the market.

A2: The market prices $p_{m,t}, p_{m-1,t}, \cdots, p_{1,t}$ of generations should also reflect quality difference in terms of innovative technologies and/or improved features employed in development of generations. In order to balance between prices and features of generations, we impose the following conditions:

$$\frac{p_{m,t} - p_{m-1,t}}{\alpha_{m,t} - \alpha_{m-1,t}} \geq \frac{p_{m-1,t} - p_{m-2,t}}{\alpha_{m-1,t} - \alpha_{m-2,t}} \geq, \cdots, \geq \frac{p_{2,t} - p_{1,t}}{\alpha_{2,t} - \alpha_{1,t}} \geq \frac{p_{1,t}}{\alpha_{1,t}}. \tag{4}$$

This relationship is based on the quality-aligned prices condition provided by Akçay et al. (2010). This condition basically states that a recently released model of the product would be priced higher than the older models available in the market due to improved or additional features. This relationship allows the firm to charge a larger price for a higher quality model.

It is worthwhile to mention that a set of linear constraints in (4) must be included into the joint production-pricing model (1) as they construct the feasibility set $\mathcal{F}_t$ for the dynamic pricing problem.

Next, we will describe how to compute the customer choice probability in view of different features of the multi-generation product. As mentioned above, while purchasing a specific generation of the product, customers are assumed to compare its attributes in terms of price and innovation level with its predecessors as well as possible successive generations. Thus, due to internal competition among all versions $k \in G_t$ of the product available in the market, the choice probability $\gamma_{k,t}$ of generation k at time t depends on its own price and innovation level of successive generation $k + 1$ as well as predecessor generation $k - 1$ for $k = 2, 3, \cdots, m - 1$. Since the latest generation $k = m$ has no successor at time t and the oldest generation $k = 1$ has no predecessor, their choice probabilities need to be computed accordingly. The following proposition states the choice probabilities for all generations of the product available in the market.

Proposition 1 *For the given set G_t of currently available generations $k = 1, \cdots, m - 1, m$ (in order from the earliest to the latest released versions) at time t, the customer's choice probabilities $\gamma_{k,t}$ are determined*

as follows;

$$\gamma_{k,t} = \begin{cases} \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}}{\alpha_{k,t}}, & k = 1, \\ \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}}, & k = 2, \cdots, m-1, \\ 1 - \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}}, & k = m. \end{cases}$$

Proof: The proof is provided in Appendix. ■

As the latest generation possesses the highest innovative technology, which hasn't been exposed to the market before, the customer's response toward the newest version of the product is unpredictable. Moreover, the release of a new version reduces the prices of the older versions and may increase the customer's willingness to buy. Thus, during a new release, the customer's response toward the older generations is also difficult to predict.

In our model, customer demand for multiple generations is assumed to be uncertain and the underlying random variable follows a probability distribution. At time t, demand of generation k specifically depends on the customer choice probability for generation k and also the number of customer arrivals. We also suppose that customer arrival is uncertain and follows a discrete probability distribution. Let $\lambda_{j,t}$ denote the probability of j customer arrivals at time t such that $\sum_{j=0}^{M} \lambda_{j,t} = 1$ where M represents the maximum number of customers expected to arrive at any time period. The following proposition expresses the probability mass function of demand for multiple versions.

Proposition 2 *Let $f_{k,t}(.)$ denote a probability mass function for generation $k \in G_t$ at time t. Then the probability of having demand for j number of products from generation $k \in G_t$ at time t can be computed as follows:*

$$f_{k,t}(j) = \quad Pr(\tilde{d}_{k,t} = j) = \sum_{i=j}^{M} \binom{i}{j} (\gamma_{k,t})^j (1-\gamma_{k,t})^{i-j} \lambda_{i,t}, \; for \; j = 0, 1, \cdots, M.$$

Proof: The proof is provided in Appendix. ■

Note that Proposition 2 will be used to define the probability mass function of having demand for certain number of products within the stochastic dynamic programming model. Next, we will focus on approximation methods for solving the dynamic joint production-pricing model. In particular, we introduce a simulation based stochastic dynamic programming method (namely forward dynamic programming) and a two-stage heuristic method.

4 Approximation Methods for Joint Production-Pricing Problem

As in most real-life stochastic dynamic programming applications, finding an optimal policy for the joint production-pricing problem of multi-generation products under uncertainty is computationally expensive due to large number of states. The traditional dynamic programming algorithm uses the backward recursion

principle where the optimal decisions and value functions are calculated iteratively starting from the terminal time and stepping backwards in time. Although this procedure can produce an exact analytical solution, it is affected by the curse of dimensionality since the value function is computed at each state and all possible actions are evaluated and stored in look-up tables. Because, at each decision epoch, enumeration of the entire state and its feasible action spaces becomes computationally expensive.

For the joint production-pricing model, the action and state spaces magnify with increase in production capacity and the number of generations released in the market. Even solving the abridged optimisation model (described as in form of a two-stage heuristic in the next section) becomes computationally cumbersome for the realistic size of problems when the backward dynamic programming method is applied. For instance, for a firm selling at most m versions of the product at time t with fixed production capacity of κ, the state space comprises of $(\kappa+1)^m$ number of states. At any state, there exists in total $(\kappa+1)^m$ number of actions to take for the production quantity of all available versions. Moreover, we need to solve a non-linear optimisation problem for the optimal pricing decisions at any state of the system. Thus, the state and action spaces exponentially grow as more generations are released over time and/or the capacity is expanded. In order to tackle curse of dimensionality on the state space, we propose iterative approximation algorithms based on a forward dynamic programming and a two-stage heuristic.

4.1 Forward Dynamic Programming

Forward dynamic programming (FDP) is a simulation based algorithmic framework and solves the underlying dynamic programming problem using a strategy that steps forward through time starting from an initial state. As opposed to visiting the entire state (and action) space, FDP selects a sample path and moves forward iteratively. Each sample path is generated using a Monte Carlo simulation from the same initial state. The value functions are evaluated for all states (a look-up table) or updated at states on a random path (reached from the initial state) using aggregation of states or regression models. In this sense, the FDP algorithm differs from the backward dynamic programming algorithm that computes the value function at every state. The interested reader is referred to Powell (2011) and Topaloglu and Powell (2005) for further information. Next a brief description of the FDP algorithm for solving the joint production-pricing problem follows.

The forward dynamic programming algorithm is especially designed to reduce state space by adopting an approximation technique for the value function. The FDP algorithm starts with initialisation of value function. In our case, we set it to be zero. The performance of the FDP algorithm highly depends on how the value function is initialized; hence, the decisions can be suboptimal. The best initialization of the value function encourages FDP to explore different states. At the n-th iteration of the FDP algorithm, as presented in its pseudo code in Algorithm 1, the value function $\hat{V}_t^n(\mathbf{x}_t^n)$ is updated for state $\mathbf{x}_t^n$ to approximate the real value function $\hat{V}_t(\mathbf{x}_t)$. Let $\hat{V}_t^n(\mathbf{x}_t^n)$ and $\bar{V}_t^n(\mathbf{x}_t^n)$ denote the optimized and approximated value functions given state $\mathbf{x}_t^n$ of iteration n, respectively. At each iteration of the FDP algorithm, we simulate a path of customer arrivals for each time period. The customer requests are generated by using customer choice model presented in Section 3.1. For given customer arrivals at iteration n, the joint

production-pricing decisions at state $\mathbf{x}_t^n$ are obtained by solving the following maximisation problem

$$\hat{V}_t^n(\mathbf{x}_t^n) = \max_{\mathbf{p}_t, \mathbf{q}_t} \left\{ \mathbf{p}_t \min\{\mathbf{x}_t^n + \mathbf{q}_t, \tilde{\mathbf{d}}_t\} - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \bar{V}_{t+1}^{n-1}((\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+) \right\},$$

where $\bar{V}_{t+1}^{n-1}$ is an approximation of the value function at state $\mathbf{x}_{t+1}^n = (\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+$ at time $t+1$. For any feasible production level $\mathbf{q}_t$, this is a convex optimization problem that determines optimal pricing decisions for $\mathbf{p}_t$. Using the production level ($\mathbf{q}_t$) and pricing ($\mathbf{p}_t$) decisions, we can then update the look-up table as follows;

$$\bar{V}_t^n(\mathbf{x}_t) = \begin{cases} (1-\eta)\bar{V}_t^{n-1}(\mathbf{x}_t^n) + \eta\hat{V}_t^n(\mathbf{x}_t^n), & \mathbf{x}_t = \mathbf{x}_t^n \\ \bar{V}_t^{n-1}(\mathbf{x}_t), & \text{otherwise.} \end{cases}$$

where η is a step size between 0 and 1. We then calculate the next state on the basis of random demand outcomes. The algorithm terminates when it satisfies the stopping criteria. The final value function is an approximate solution to the problem since the FDP algorithm does not compute the value function at every state, but only those reached from the initial state.

Algorithm 1: Pseudo code of the FDP algorithm

1: **Initialization:** Initialize iteration number N and set step size η.
2: Set value function at $\mathbf{x}_t^0$ as $\bar{V}_t^0(\mathbf{x}_t^0) = 0$.
3: **for** $n = 1, \cdots, N$ **do**
4: Select an arrival path ω^n.
5: **for** $t = 0, 1, \cdots, T$ **do**
6: - Set initial state (inventory for version 1) at $t = 0$ as $\mathbf{x}_0^n = \{x_{1,0}^n\} = \{0\}$.
7: - Obtain $(\mathbf{p}_t, \mathbf{q}_t)$ by solving the following maximization problem;

$$\hat{V}_t^n(\mathbf{x}_t^n) = \max_{\mathbf{p}_t, \mathbf{q}_t} \left\{ \mathbf{p}_t \min\{\mathbf{x}_t^n + \mathbf{q}_t, \tilde{\mathbf{d}}_t\} - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \bar{V}_{t+1}^{n-1}((\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+) \right\}$$

8: - Update the value function at state $\mathbf{x}_t$ as follows;

$$\bar{V}_t^n(\mathbf{x}_t) := \begin{cases} (1-\eta)\bar{V}_t^{n-1}(\mathbf{x}_t^n) + \eta\hat{V}_t^n(\mathbf{x}_t^n) & \text{if } \mathbf{x}_t = \mathbf{x}_t^n \\ \bar{V}_t^{n-1}(\mathbf{x}_t) & \text{if } \mathbf{x}_t \neq \mathbf{x}_t^n \end{cases}$$

- Compute new states based on random outcome of demand: $\mathbf{x}_{t+1}^n = (\mathbf{x}_t^n + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+$.
9: - Set $t := t + 1$. If $t < T + 1$, then go to Step 6.
10: **end for**
11: Set $n := n + 1$. If $n < N$, then go to Step 4.
12: **end for**

4.2 A Two-stage Heuristic

In addition to the FDP algorithm we adopt an alternative approach based on the partial-planning strategy introduced by Chan et al. (2006). The pseudo code summarising the main steps of this approach is

presented in Algorithm 2. This heuristic consists of two stages. In the first stage, we determine a range of prices of the released products available in the market. Then, in the second stage, we obtain the optimal production level of each product for any price point determined in the first stage of the algorithm.

Stage 1: Derivation of Price Bounds: A range of prices of products available in the market can be specified by the lower and upper bounds determined by different ways. In particular, we consider pricing rules by solving the abridged model of the joint production-pricing problem and a list of prices derived by theoretical bounds as described below.

The Abridged Model: Since the joint dynamic production-pricing model is computationally intractable, one can consider the (reduced) abridged optimisation model where decisions related to production and pricing of available products are made at only release times. In other words, production does not take place between two subsequent release times. The elimination of the production decisions between release times can be interpreted as the firm placing cumulative production decisions between two release times in advance. The resulting (abridged) dynamic program can be solved for only certain small-size problems by the standard technique of backward dynamic programming algorithm. The average, minimum and maximum value of product prices derived from the policy tables can be then used at the second stage of the heuristic approach to define a range for prices of the products.

Theoretical Bounds: From the policy table obtained by solving the abridged model, we investigate patterns between the production and pricing decisions of multiple versions. We observe the minimum value of the pricing decision for a specific version is obtained when its inventories are at the maximum level. This observation is used to theoretically derive the lower and upper bounds for the pricing decisions for multiple versions. The following propositions state theoretical bounds in order to determine the pricing decision of a generation.

Proposition 3 *Assume that maximum number of demand for any generation of the product is M at the final time period. For $\mathbf{x}_T + \mathbf{q}_T = M$, the market price of generation $k \in G_T$ at time T can be computed in terms of innovation level $\alpha_{k,T}$ of generation $k \in G_T$ and holding cost h_T as $p^*_{k,T} = \frac{\alpha_{k,T} - h_T}{2}$.*

Proof: The proof is provided in Appendix. ■

Proposition 4 *The market prices of multiple generations $k \in G_t$ at time t of the product are bounded as follows;*

a) $\frac{\alpha_{k,t} - h_t}{2} \leq p_{k,t} \leq \alpha_{k,t}$ *for* $k = m$, *and*

b) $\frac{\alpha_{k,t} - h_t}{2} \leq p_{k,t} \leq \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}}$ *for* $k = 1, \cdots, m-2, m-1$.

Proof: The proof is provided in Appendix. ■

Stage 2: The Optimal Production Strategies: Given the pricing strategy (determined by certain rules or bounds at the first stage of the algorithm), we need to compute the optimal production strategy. Let $\hat{\mathbf{p}}_t = \{\hat{p}_{k,t}, k \in G_t\}$ represent the pre-determined price of all generations selling in the market at time

t. Given pre-determined price $\hat{\mathbf{p}}_t$, we can formulate the value function $\hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t)$ at state $\mathbf{x}_t$ as follows;

$$\begin{aligned} \hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t) = \max_{\mathbf{q}_t \geq 0} \quad & \mathbb{E}\left[\hat{\mathbf{p}}_t . \min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\} - h_t \mathbf{x}_{t+1} + \delta V_{t+1}(\mathbf{x}_{t+1})\right] - \mathbf{c}_t \mathbf{q}_t \\ \text{s.t.} \quad & \mathbf{x}_{t+1} = (\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \\ & \mathbf{1}(\mathbf{q}_t) \leq \kappa \end{aligned} \tag{5}$$

Assuming that no new generation is to be launched at the end of planning horizon $t = T$, the value function $\hat{V}_T(\mathbf{x}_T \mid \hat{\mathbf{p}}_T)$ states the boundary condition determining the optimal order quantity $\mathbf{q}_T$ at time period T as follows;

$$\begin{aligned} \hat{V}_T(\mathbf{x}_T \mid \hat{\mathbf{p}}_T) = \max_{\mathbf{q}_t \geq \mathbf{0}} \quad & \mathbb{E}\left[\hat{\mathbf{p}}_T . \min\{\tilde{\mathbf{d}}_T, \mathbf{q}_T + \mathbf{x}_T\} - h_T \mathbf{x}_{T+1}\right] - \mathbf{c}_T \mathbf{q}_T \\ \text{s.t.} \quad & \mathbf{x}_{T+1} = (\mathbf{x}_T + \mathbf{q}_T - \tilde{\mathbf{d}}_T)^+ \\ & \mathbf{1}(\mathbf{q}_T) \leq \kappa \end{aligned} \tag{6}$$

where prices $\hat{\mathbf{p}}_T$ of all available versions at time T are set during the last release time. Note that in this formulation, the ending inventory for the last time period is assumed to be held till the end of the planning horizon and then it is discarded at zero cost.

The following proposition establishes convexity of the optimal production model for given prices of products. Thus, the optimal order policy is obtained when the prices are known.

Proposition 5 *Given approximate prices $\hat{\mathbf{p}}_t$ of all generations in the market at time t, the value function $\hat{V}_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t)$ of the dynamic production planning model is concave in production decisions.*

Proof: The proof is provided in Appendix. ■

Notice that the original joint production-pricing problem is simplified to the dynamic production problem where the optimal ordering policy can be obtained by solving the dynamic optimisation model as follows,

$$\mathbf{q}_t^* = \arg\max \quad \hat{\mathbf{p}}_t \mathbb{E}\left[\min\{\tilde{\mathbf{d}}_t, \mathbf{x}_t + \mathbf{q}_t\}\right] - h_t \mathbb{E}\left[\mathbf{x}_{t+1}\right] + \delta \mathbb{E}\left[V_{t+1}(\mathbf{x}_{t+1})\right] - \mathbf{c}_t \mathbf{q}_t$$

where fixed prices $\hat{\mathbf{p}}_t$ of multiple generations are selected from a range of $\left[\hat{\mathbf{p}}_t^L\ ,\ \hat{\mathbf{p}}_t^U\right]$ that is obtained at the first stage of the two-stage algorithm.

Algorithm 2: Pseudo code of the two-stage algorithm

Stage 1: *Obtain interval of prices* $[\hat{p}^L_{k,t},\ \hat{p}^U_{k,t}]$ *for* $k \in G_t$ *at* $t = 1, 2, \cdots, T$ *by either solving the abridged model or applying Proposition 5.*

Stage 2: *Given a price range, determine the optimal production strategy*

- Initialize step size $\phi_{k,t}$ for each version $k \in G_t$ at time $t = 1, \cdots, T$.

- Set initial inventory level at time $t = 1$ as $\mathbf{x}_1 = \{x_{k,1} = 0, k \in G_1\}$

- Compute a feasible price set:

$$\mathcal{P} = \left\{ \hat{p}^L_{k,t} + e.\phi_{k,t} \mid \gamma_{k,t} \geq 0,\ e = 0, 1, \cdots, \frac{\hat{p}^U_{k,t} - \hat{p}^L_{k,t}}{\phi_{k,t}},\ \text{for } k \in G_t,\ t = 1, \cdots, T \right\}$$

for *each price point* i *in set* $\mathcal{P}$ **do**

  Set boundary condition: $V^i_{T+1}(\mathbf{x}_{T+1} \mid \hat{\mathbf{p}}_T) = 0$ for all possible states $\mathbf{x}_{T+1}$ at $t = T + 1$

  for $t = T, T-1, \cdots, 1$ **do**

    Solve the following maximisation model at each state $\mathbf{x}_t$

$$V^i_t(\mathbf{x}_t \mid \hat{\mathbf{p}}_t) = \max_{\mathbf{q}_t \geq 0} \left\{ \hat{\mathbf{p}}_t \min(\mathbf{x}_t + \mathbf{q}_t, \tilde{\mathbf{d}}_t) - \mathbf{c}\mathbf{q}_t - h_t(\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ + \delta V^i_{t+1}((\mathbf{x}_t + \mathbf{q}_t - \tilde{\mathbf{d}}_t)^+ \mid \hat{\mathbf{p}}_{t+1}) \right\},$$

  end for

  return $\hat{\mathbf{q}}_t = \arg \max_{i \in \mathcal{P}} V^i_1(\mathbf{x}_1 \mid \hat{\mathbf{p}}_t)$

end for

5 Computational Experiments

In this section, we first describe the design and data structure used for numerical experiments and then present the computational results of different approaches studied for solving the joint production-pricing problem of multi-generation product line. A brief description of these approaches (presented in Section 4) with different pricing rules adopted follows;

- *Forward Dynamic Programming (FDP)* is a simulation based stochastic dynamic programming method presented in Section 4.1. Different from the two-stage heuristic, this method determines the pricing and production decisions together. Three different versions of this approach are used to solve the joint production-pricing model (1).
  - *FDP-1* solves the joint production-pricing model (1) at each iteration in Algorithm 1.
  - *FDP-2* uses price sets that are determined from the maximum and minimum prices given by the abridged model to solve model (1).
  - *FDP-3* defines sets for pricing decisions determined from the theoretical price bounds using Proposition 4.
- *Two-stage heuristic (TSH)* applies pricing decisions made (in stage 1) for multiple generations of a product using the model (4.2)-(6) to determine the production policy (in stage 2). Different pricing strategies to find production quantities are abbreviated as follows;

– *ABridged Model (ABM)* assumes that there is no intermediate ordering between release times of the generations. The resulting model is solved by dynamic programming and the average, minimum and maximum value of the product prices are implemented at the second stage of the two-stage heuristic.

– *Bounds Algorithm 1 (BA-1)* requires the maximum and minimum prices obtained from the abridged model to form the price bounds. These price bounds are then used in Algorithm 2 to search for the price sets for each generation.

– *Bounds Algorithm 2 (BA-2)* employs the theoretical price bounds given by Proposition 4 to search for the price sets in Algorithm 2.

These algorithms were implemented in MATLAB and all computational experiments were run in a desktop computer with Intel Core i5-7500, 3.4GHz, 8GB RAM.

5.1 Design of Experiments and Data

We design a series of computational experiments in order to illustrate the performance of different algorithms developed for solving the dynamic programming models. Specifically, the numerical experiments aim to answer the following managerial questions:

- What is the impact of selling multiple generations of a product on firm profit?
- What is the added value of joint decision making while managing a multi-generation product line?
- How do varying characteristics of customer segments affect the management of multiple generations?

Our experimental design considers different parameter sets related to the number of generations in the market, the innovation level and the innovation sensitivity. We define three base cases with respect to the number of generations in the market, which are given in Table 2. In these base cases, we set the production capacity to $\kappa = 25$, the inventory holding cost to $h = 0.001$, and the discount factor to $\delta = 1$. Additional test instances (adopted from Sainathan (2013) and Akçay et al. (2010)) are also generated by varying some of the base parameters.

Table 2: A list of parameters set in the base case

Generations (n)	Innovation Level ($\alpha_1, ..., \alpha_n$)	Production Cost ($c_1, ..., c_n$)	Planning Horizon (T)
2	(3.5, 4)	(0.5 ,1)	4
3	(3.5, 4, 4.5)	(0.5, 1, 1.5)	6
4	(3.5, 4, 4.5, 5)	(0.5, 1, 1.5, 2)	8

We simulate the arrival of customer requests over a planning horizon with length T. Given the optimal pricing strategy obtained by different solution methodologies at each time period, we first make the production decision for each generation and then, generate the customer requests. Customer arrivals follow a

Poisson distribution with mean arrival rate of 25. The arriving customer either chooses one of the offered generations according to the choice probabilities described in Proposition 1 or leaves with no purchase.

We assume that customers are classified into two segments, namely price and innovation sensitive. Each customer is assigned a value of $\theta \in [0, 1]$, denoting the customer's sensitivity towards innovation. A customer with a higher θ value, closer to 1, is highly sensitive towards innovation and hence, s/he is classified as innovation sensitive. On the other hand, a customer with a lower θ value is price sensitive. In order to have an equal number of customers in each segment, we uniformly distribute the value of θ in the initial experiments. We also perform experiments where we vary the proportion of customer segments. We estimate the expected profits by simulating the arrivals of customer requests over 1000 sample paths.

5.2 Numerical Results and Analysis

In this section, we present results of the numerical experiments under three main categories: performance of different approximate approaches, impact of number of multi generations available in the market and effect of customer choices on the firm's profitability.

Performance Comparison of Different Approaches: We are first concerned with evaluating the performance of the proposed approximation methods with respect to varying production capacities. The performance of each algorithm is measured in terms of total expected profit achieved at the end of planning horizon and the CPU time taken to solve each problem instance. We estimate the expected profits by simulating the decisions made by different solution methods under multiple customer arrival trajectories.

We consider three experiment setups where a firm is selling two, three and four generations of a product, respectively. Since the production capacity has a significant effect on the computation time of the methods, we vary it over $\{5, 10, 15, 25\}$ to evaluate the performances of each solution strategy. Table 3 summarises our simulation results for two, three and four generation product lines. The first column in Table 3 shows the production capacity used in these tests whereas the next six columns present the expected profits and the related computation times obtained by solving the corresponding dynamic optimisation models using the two-stage heuristic and FDP strategies. In addition, the best performance of an approach (defined as the highest expected profit achieved and the lowest CPU time taken to solve the underlying problem by a method) is presented in **bold** and – highlights the specific cases with 'no solution obtained' by ABM, BA-1, FDP-1 and FDP-2 approaches for the production capacity higher than five within three and four-generation production lines due to the computational difficulty as seen from Table 3. Both forward and backward dynamic programming approaches require solving a nonlinear optimisation problem (with high-degree polynomial objective function) at each state of every time period. The non-linearity arises because of the definition of choice probabilities and the demand function. Moreover, the degree of the polynomial function expands as the capacity and number of generations increase. Therefore, it becomes computationally intractable to solve ABM, BA-1, FDP-1 and FDP-2 approaches when capacity is higher than five for three and four generation product line problems.

By comparing the expected profits in Table 3, we observe that BA-1 and BA-2 typically generate the highest profits followed by FDP-1, FDP-2, FDP-3 and ABM without a specific ordering between the latter

four solution methods. Expected profits obtained by BA-1 and BA-2 are significantly close. The small performance gaps between BA-1 and BA-2 show that the theoretical price bounds perform well compared to the price bounds obtained from ABM. The performance of the FDP-based methods in terms of expected profits significantly depends on the production capacity. Note that the range for randomness in the demand enhances as capacity increases. Therefore, as the production capacity increases, the FDP-based methods perform better since they can explore more states.

Table 3: Performance of different solution methods for different generation product lines

Production Capacity	Performance Metrics	Two-stage Heuristic			Forward Dynamic Programming		
		ABM	BA-1	BA-2	FDP-1	FDP-2	FDP-3
Two-generation Product Line							
5	Exp. Profit	12.69	**12.89**	12.47	12.21	12.56	11.94
	Time (s)	52.23	54.13	**1.83**	2739.21	74.23	38.35
10	Exp. Profit	23.96	24.42	**24.47**	23.75	23.67	23.83
	Time (s)	1126.32	1130.41	**9.12**	6888.62	1150.30	41.98
15	Exp. Profit	35.08	**37.13**	36.62	36.89	35.16	35.52
	Time (s)	17,800.88	1725.03	59.13	21425.36	17654.32	**58.91**
25	Exp. Profit	57.62	60.18	60.16	**62.38**	60.71	60.82
	Time (s)	456,809.12	564,809.34	374.62	131,367.64	458,342.35	**220.44**
Three-generation Product Line							
5	Exp. Profit	17.80	**18.57**	18.01	16.85	15.49	17.68
	Time (s)	1506.32	1507.34	**18.60**	17652.34	1886.46	50.43
10	Exp. Profit	–	–	**33.92**	–	–	31.43
	Time (s)	–	–	**49.86**	–	–	70.32
15	Exp. Profit	–	–	**51.24**	–	–	48.79
	Time (s)	–	–	102.92	–	–	**100.85**
25	Exp. Profit	–	–	**83.54**	–	–	79.85
	Time (s)	–	–	741.21	–	–	**557.6**
Four-generation Product Line							
5	Exp. Profit	–	–	**20.32**	–	–	18.07
	Time (s)	–	–	**33.47**	–	–	566.32
10	Exp. Profit	–	–	**39.99**	–	–	38.00
	Time (s)	–	–	**224.04**	–	–	983.82
15	Exp. Profit	–	–	**59.66**	–	–	56.04
	Time (s)	–	–	1464.83	–	–	**1234.51**
25	Exp. Profit	–	–	**104.67**	–	–	97.5
	Time (s)	–	–	75737.16	–	–	**6765.43**

The size of the state space expands exponentially with the number of generations. With the same number of sample paths, FDP is able to explore more number of states in two generation product line compared to the three and four generation lines. Since the performance of FDP depends on the number

of states it can explore, the FDP-based methods perform better when there are small number of versions available.

In terms of computation times, we observe that the most computational effort is invested in solving the abridged model (ABM). Since BA-1 and FDP-2 use the price bounds obtained from the abridged model, their computation times become high. On the other hand, BA-2 and FDP-3 apply for the theoretical price bounds, and therefore they are computationally much faster. We should also emphasize that FDP-1 is relatively slower since it doesn't use price sets unlike FDP-2 and FDP-3 approaches. Instead, at each iteration of FDP-1, the high-degree non-linear optimisation problem (see Section 4.1) is solved at each decision stage. In general, the CPU times increase as the problem size increases.

Impact of Multiple Generations on Profitability: We also design experiments to analyse the impact of strategic decisions regarding to the management of multiple generations on the company's profitability. The main question to answer is under what conditions offering multiple generations improves expected profit over a planning horizon. We quantify the benefits of offering multiple generations as opposed to selling only one generation in the market by comparing the expected profits obtained by different product line strategies. We assume that a firm producing four generations of a product over the planning horizon of eight time periods. At the release time of a new generation, the firm decides how many generations to keep in the market depending on the adopted product line strategy. In order to explore the impact of different product line strategies on the expected profit over a fixed planning horizon, we consider four settings: (a) one-generation product line where the firm sells only the latest generation, (b) two-generation product line where the latest two generations are sold, (c) three-generation product line where the latest three generations are available in the market, and (d) four-generation product line where all four generations are sold. We apply the FDP-3 strategy due to the computational efficiency of the forward dynamic programming approach. In particular, we vary the production cost and innovation level of the multi-generation product line to analyse the firm's production decisions and the related profitability. Table 4 presents our results. The initial production cost and innovation levels of the oldest generation of the product line are displayed in the first two columns of Table 4. When a new generation is released, its innovation level and production cost are increased by 0.5 units each in comparison with its previous generation. The expected profits are obtained by the FDP-3 approach using different generations of production line strategies at the end of the planning horizon.

Table 4: Expected profit obtained over a planning horizon

Initial	Initial	Product Lines			
Cost	Quality	1-generation	2-generation	3-generation	4-generation
0.5	3.5	88.07	91.25	95.87	97.50
0.5	4.5	131.16	131.88	135.86	141.74
1.5	3.5	29.54	31.09	32.46	41.55
1.5	4.5	61.46	65.35	67.50	79.34

Comparing the expected profits obtained from different product line strategies, we observe that profitability of the firm increases as the number of generations offered in the market increases. In particular, there is a significant gap between the expected profits obtained by single-generation and four-generation

strategies. This difference can be attributed to the impact of capturing different customer segments with wider choice of products. On the other hand, the difference between the expected profits obtained by three-generation and four-generation product lines are very close when the production cost and the innovation level are low for the oldest generation (the first test instance). In our simulation experiments, we observe that while operating a four-generation product line, the firm generally decides to sell three generations instead of four to yield the maximum profit. Due to low innovation level of the oldest generation, customer preferences shift to the other versions. On the other hand, when the innovation level for the oldest generation is high (at level of 4.5), selling it with the other versions in the market is more profitable. The introduction and discontinuation of multiple generations depend on their innovation levels. The firm is likely to sell different versions when the innovation levels are high. In practice, technology firms may upgrade the design and software features of the old products when they release a new one. In fact, a similar strategy was implemented by Apple in 2016. During the release of new mobile, iPhone 6, Apple also launched an upgraded version of iPhone 5S by increasing its capacity (Welch 2017).

Impact of Joint Production-Pricing Strategy: We are also concerned with an effectiveness of dynamic joint production-pricing decision making in the multi-generation product line problem. We therefore adopted planning strategies (so-called "partial planning") introduced in Chan et al. (2006) to compare with the proposed dynamic strategies.

In a partial planning strategy, we fixed one decision (production or pricing) at the beginning of the planning horizon while the other decision is dynamically determined by the optimisation model at each time period. In practice, we see that decisions related to pricing and production may be taken in advanced due the various limitations related to legal contracts experienced by firms (Rasmussen 2018). This kind of advance decisions is generally made on the basis of preset parameters (Chan et al. 2006) without taking into account uncertainty. Therefore, we determine the partial planning strategies based on the deterministic formulation of the joint decision-making model (1)-(2). In particular, we consider two partial planning strategies, abbreviated as F-price and F-prod. In the F-price strategy, prices of all generations are fixed at the beginning of the planning horizon. These prices are basically input to the FDP-3 algorithm to find the dynamic production policy. Similarly, in the F-prod strategy, we fix the production decision and find the dynamic pricing policy by using FDP-3. Note that the fixed pricing and production decisions in the F-price and F-prod strategies, respectively, are obtained by the deterministic formulation of the joint decision-making model. In this experiment, we consider a four-generation product line and compare the performance of F-price and F-prod with the joint production-pricing policy, abbreviated as JPP. We present our numerical results in Table 5. The first two columns in Table 5 show the production cost and innovation level of the oldest generation of the product line. The next three columns give the expected profits obtained by JPP, F-price and F-prod, respectively. The last two columns display the percentage gaps between JPP and the two partial planning strategies. Figure 2 illustrates the production and inventory levels at each time period by using the second test instance given in Table 5.

As seen from Table 5, JPP outperforms the partial planning strategies. While the average performance gap between JPP and F-price is 7.72%, it is 18.41% for F-prod. The performance gaps are more striking when the initial cost of the oldest generation is high. The poor performance of partial planning strategies is due to the fixed decision (production or pricing) made at the beginning of the planning horizon. Comparing

Table 5: Performance comparison of joint production-pricing and partial strategies

Initial Cost	Initial Quality	Strategies			% Gap with JPP	
		JPP	F-price	F-prod	F-price	F-prod
0.50	3.50	97.50	92.26	85.15	5.68	12.67
0.50	4.50	141.74	138.10	125.74	2.63	11.29
1.50	3.50	41.55	37.84	30.24	9.81	27.22
1.50	4.50	79.34	70.37	61.51	12.76	22.47

the top and middle panels of Figure 2, we note that the F-price strategy cannot manage multi-generation product line effectively as opposed to JPP. It generally offers two generations during the planning horizon. On the other hand, from the top and bottom panels of Figure 2, we observe that JPP and F-prod have a similar product line. In addition, the F-prod strategy behaves closely to the JPP policy in terms of production and inventory levels. However, there is a significant gap between the expected profits of F-prod and JPP in all test instances. Due to the fixed production decision set at the beginning of the planning horizon, the F-prod strategy cannot balance its inventory and production levels. As illustrated in Figure 2, the inventory level for version 1 at time 3 is approximately same in JPP and F-prod, respectively. However, the corresponding production decisions for version 1 are different for these two strategies. The JPP policy dynamically reacts to a high inventory level by producing less number of products. On the other hand, when we fix the production levels, the firm is unable to dynamically respond to the changes in inventory levels. Thus, the F-prod strategy suggests a higher production for version 1 in comparison to the JPP policy. The variation in inventory levels of multiple versions is primarily caused by uncertain demand. Thus, dynamically deciding the production levels for multiple versions is essential to mitigate demand uncertainty. In summary, fixing one decision related to price or production does not only decreases profits, but also has several other drawbacks. When the prices are fixed at the beginning of the planning horizon, the firm cannot react to the changes in customer demand and may fail to utilise the benefit of internal competition among multiple versions. On the other hand, when production decision is fixed, the firm cannot manage the demand uncertainty efficiently which may result in shortages or overstocking. Thus, the internal competition among multiple versions and uncertainty in their demand must be handled by a joint production-pricing strategies.

Impact of Customer Segments: We also investigate impact of customer preferences on the management of a multi-generation product line. In particular, we explore how product sales alternate between different generations when the proportion of each customer segment changes. Recall that the segment of a customer is determined with respect to innovation sensitivity parameter (i.e., θ). While a customer with a higher θ value (closer to 1) is highly sensitive towards innovation, a customer with a lower θ value is defined more price sensitive. We simulate customer requests by changing the value of θ and evaluate the product sales by using the optimal production plan of obtained by the BA-2 strategy. Since the FDP-based strategies do not take θ into account while determining the production and pricing policies, we use the BA-2 strategy for this experiment. Figure 3 illustrates the average number of sales (left panel) and the average production quantities (right panel) for three- (top panel) and four- (bottom panel) generation product lines at the last two time periods with respect to varying customer segments. In these figures, the horizontal axis displays

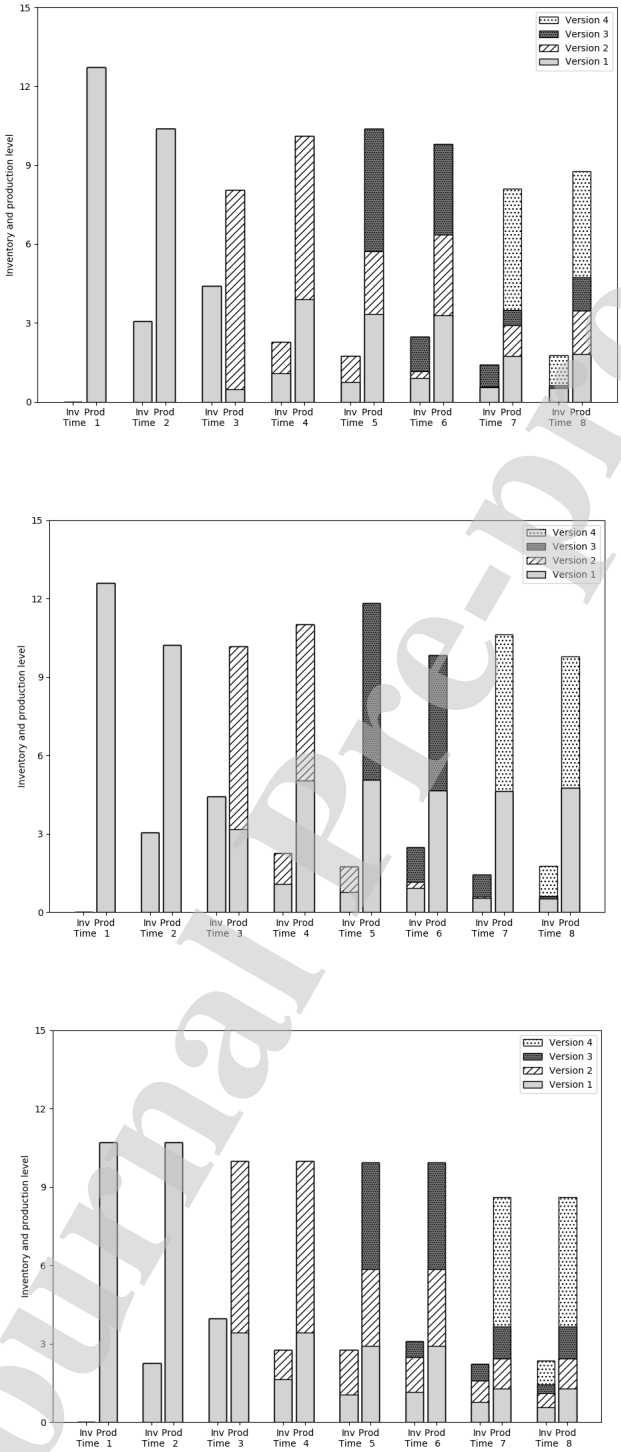

Figure 2: Inventory and production levels obtained by the joint production-pricing (top panel), F-price (middle panel) and F-prod (bottom panel) policies using four generation product line

market classification in terms of various percentage customer segments as starting from 90-10 up to 10-90. For instance, the case '70-30' represents 70% of customers in the market as being quality sensitive whereas the remaining 30% of customers is price sensitive.

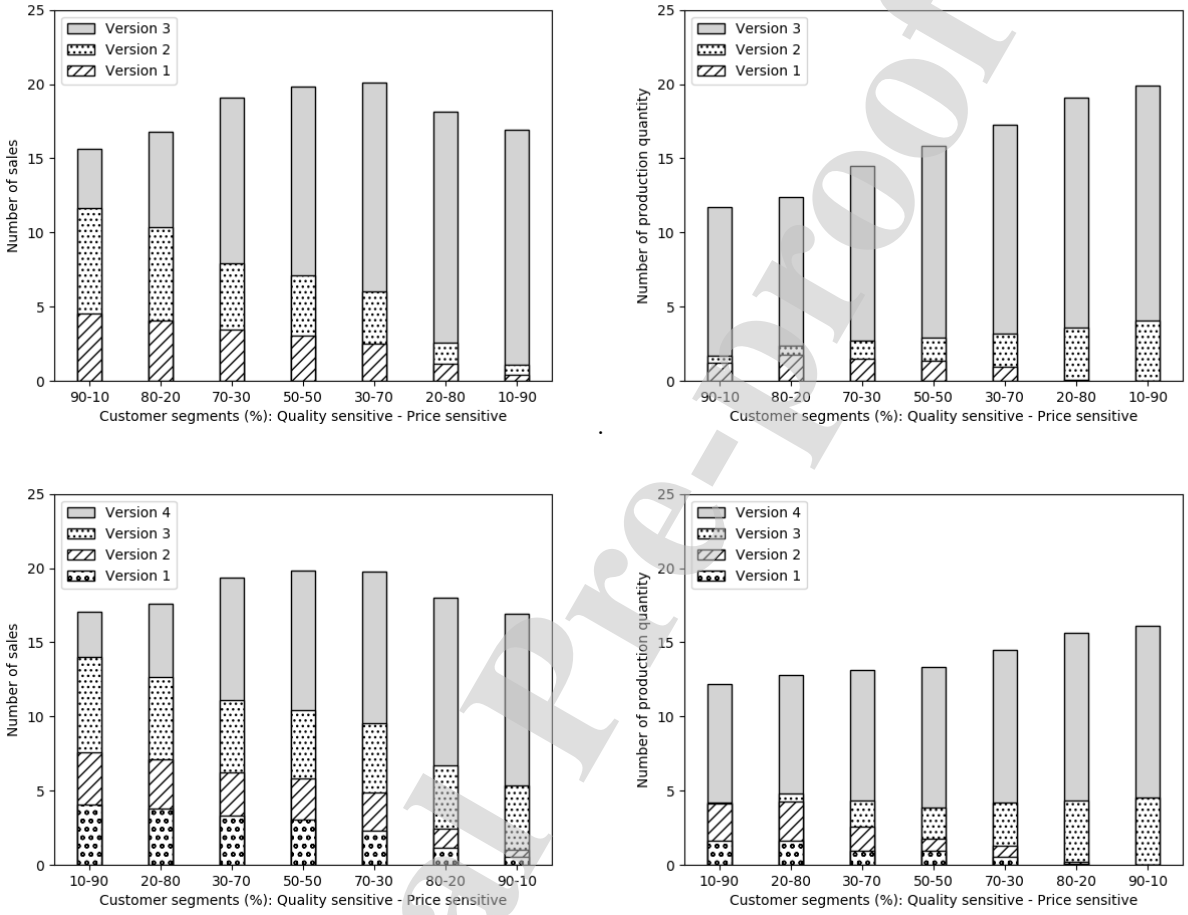

Figure 3: Average sales and production quantity in three-generation (at top panel) and four-generation (at bottom panel) product lines with respect to varying customer segments

Overall, we observe that results obtained for three- and four-generation production lines show similar performance patterns. By comparing the average number of sales and production quantities in Figure 3, one can see that when the proportion of price sensitive customers is high, demand for the oldest product remains high and the production of all generations becomes profitable. On the other hand, when there are more quality sensitive customers in the market, demand shifts towards the newer generations. An interesting result at this point is that the average production quantities increases as the proportion of quality sensitive customers increases. Because the production decisions at the last two time periods depend on the starting inventory, and a change in the market segments results in different starting inventories. Thus, our model is adaptive of the changing customer segments due to its dynamic nature. It is important to point out that the average production for the oldest generations decreases to zero when the market is dominated by mainly innovative sensitive customers. This shows that the optimal product line strategy significantly depends on the proportion of price and innovation sensitive customers in the market.

6 Conclusions

In this paper, we study the joint production-pricing decision-making process of a firm selling a multi-generation product line under demand uncertainty. We account for the internal competition between multiple generations by examining customer choices and derive a stochastic dynamic model for joint production-pricing problem for multi-generation product line. Finding the optimal production and pricing policies for this problem requires solving a stochastic dynamic program with a high-dimensional state vector. By analysing the structural properties of the problem, we present two approximations, based on FDP and heuristic in conjunction with pricing strategies obtained by different solution methods, for solving the dynamic joint production-pricing problem. A computational study is conducted to investigate the performance of the approximation methods and to derive managerial insights.

We first evaluate how selling multi-generation product line affects the firm profitability. Our numerical results show that the profitability of selling the oldest version increases with its innovation level. Therefore, by improving the innovation levels of the older generation during a new product release, the firm can increase the sales of older generations. Moreover, the computational results indicate that the gap between innovation levels of the oldest and the newest generations must be at minimum level to make the multi-generation production line more profitable. We then illustrate the benefit of joint decision-making process in multi-generation product line. The joint decision-making policy proposed in this paper is compared with partial decision-making policies. Our results indicate that the dynamic joint policy outperforms the fixed production and pricing policies since it takes the recent changes into account to match with demand by production.

We also analyse the effect of customer segments on the management of multi-generation product line. By varying the proportion of customer's price and quality sensitivity (toward the underlying technology) in the market, we investigate the production decisions for multi-generation products. The results indicate that when the proportion of innovation sensitive customers is high, the production and sale of older generations drop due to the high demand towards the new generations. Similarly, as the percentage of price sensitive customers increases, it becomes more profitable to sell older generations with new release. This shows that the number of generations to be kept in the market should be determined by considering varying customer segments.

Appendix: Proofs of Propositions

The proofs of all propositions are provide below.

Proof of Proposition 1

The proof follows from the quality aligned prices results presented by Akçay et al. (2010). We assume that the following relation between prices and features of generations holds at each time period in the planning horizon.

$$\frac{p_{m,t}-p_{m-1,t}}{\alpha_{m,t}-\alpha_{m-1,t}} \geq \frac{p_{m-1,t}-p_{m-2,t}}{\alpha_{m-1,t}-\alpha_{m-2,t}} \geq, \cdots, \geq \frac{p_{2,t}-p_{1,t}}{\alpha_{2,t}-\alpha_{1,t}} \geq \frac{p_{1,t}}{\alpha_{1,t}}.$$

By using this relation, we can formulate the choice probabilities. Let us consider the latest generation $m \in G_t$ at time t. A customer will choose generation m, if it provides the maximum utility, in other words, if $\theta\alpha_{mt} - p_{mt} \geq \theta\alpha_{jt} - p_{jt}$, for $j = 1, \ldots, m-1$. Equivalently, generation m would be chosen if $\frac{p_{m,t}-p_{j,t}}{\alpha_{m,t}-\alpha_{j,t}} \leq \theta$ for $j = 1, \ldots, m-1$ or $\max\limits_{j<m}\left\{\frac{p_{m,t}-p_{j,t}}{\alpha_{m,t}-\alpha_{j,t}}\right\} \leq \theta$. Based on the quality aligned prices condition, we have

$$\max_{j<k}\left\{\frac{p_{k,t}-p_{j,t}}{\alpha_{k,t}-\alpha_{j,t}}\right\} = \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}} \qquad \text{and} \qquad \min_{j>k}\left\{\frac{p_{j,t}-p_{k,t}}{\alpha_{j,t}-\alpha_{k,t}}\right\} = \frac{p_{k+1,t}-p_{k-1,t}}{\alpha_{k+1,t}-\alpha_{k,t}}.$$

Thus, the choice probability for generation m can be given as

$$\gamma_{m,t} = Pr\left(\frac{p_{m,t}-p_{m-1,t}}{\alpha_{m,t}-\alpha_{m-1,t}} < \theta \leq 1\right) = 1 - \frac{p_{m,t}-p_{m-1,t}}{\alpha_{m,t}-\alpha_{m-1,t}}.$$

We can extend this result and formulate the choice probability for any generation $k \in G_t$. A customer will choose generation k, for $1 < k < m$, if $\theta\alpha_{kt} - p_{kt} \geq \theta\alpha_{jt} - p_{jt}, \forall j \neq k, j = 1, \ldots, m$. In other words, generation k, for $1 < k < m$, would be chosen if $\max\limits_{j<k}\left\{\frac{p_{k,t}-p_{j,t}}{\alpha_{k,t}-\alpha_{j,t}}\right\} \leq \theta$ and $\min\limits_{j>k}\left\{\frac{p_{j,t}-p_{k,t}}{\alpha_{j,t}-\alpha_{k,t}}\right\} \geq \theta$. Based on the quality aligned prices condition, the choice probability for generation k can be given as

$$\gamma_{k,t} = Pr\left(\max_{j<k}\left\{\frac{p_{k,t}-p_{j,t}}{\alpha_{k,t}-\alpha_{j,t}}\right\} \leq \theta \leq \min_{j>k}\left\{\frac{p_{j,t}-p_{k,t}}{\alpha_{j,t}-\alpha_{k,t}}\right\}\right) = \frac{p_{k+1,t}-p_{k,t}}{\alpha_{k+1,t}-\alpha_{k,t}} - \frac{p_{k,t}-p_{k-1,t}}{\alpha_{k,t}-\alpha_{k-1,t}}.$$

By carrying out comparison of the current and previous versions available at time t in the same manner, we find that if $0 \leq \theta \leq \frac{p_{k,t}}{\alpha_{k,t}}$ for $k = 1$, then the customer prefers not to purchase. As a result the choice probabilities of multiple generations are obtained as stated above.

■

Proof of Proposition 2

In order to prove that $f_{k,t}(.)$ is the probability mass function for generation $k \in G_t$ at time t, we need to show that $\sum_{j=0}^{M} f_{k,t}(j) = 1$ holds.

$$\sum_{j=0}^{M} f_{k,t}(j) = \sum_{j=0}^{M}\sum_{i=j}^{M} \binom{i}{j} (\gamma_{k,t})^j (1-\gamma_{k,t})^{i-j} \lambda_{i,t}$$

We expand the first summation,

$$= \sum_{i=0}^{M} (1-\gamma_{k,t})^i \lambda_{i,t} + \sum_{i=1}^{M} \binom{i}{1} (\gamma_{k,t})(1-\gamma_{k,t})^{i-1} \lambda_{i,t} + \cdots + (\gamma_{k,t})^M \lambda_{M,t}$$

By rearranging it for $\lambda_{i,t}$ and applying the binomial theorem, we obtain

$$\begin{aligned} &= \lambda_{0,t} + \lambda_{1,t}\left(\binom{1}{0} (\gamma_{k,t})^0 (1-\gamma_{k,t})^1 + \binom{1}{1} (\gamma_{k,t})^1 (1-\gamma_{k,t})^0 \right) + \cdots + \\ &\quad \lambda_{M,t}\left(\binom{M}{0} (\gamma_{k,t})^0 (1-\gamma_{k,t})^M + \binom{M}{1} (\gamma_{k,t})^1 (1-\gamma_{k,t})^{M-1} + \cdots + \binom{M}{M} (\gamma_{k,t})^M (1-\gamma_{k,t})^0 \right) \\ &= \lambda_{0,t} + \lambda_{1,t}(\gamma_{k,1} + 1 - \gamma_{k,1})^1 + \cdots + \lambda_{M,t}(\gamma_{k,1} + 1 - \gamma_{k,1})^M = 1 \end{aligned}$$

■

Proof of Proposition 3

Consider the dynamic joint production-pricing optimization model. Given the boundary conditions at time $T+1$, the value function at time T is as follows;

$$V_T(\mathbf{x}_T) = \max_{0 \le \mathbf{q}_T \le \kappa,\ \mathbf{p}_T \in \mathcal{F}_T} \mathbb{E}\left[\sum_{k \in G_T} p_{k,T}.\min\{\tilde{d}_{k,T}, q_{k,T} + x_{k,T}\} - c_k q_{k,T} - h_T(x_{k,T} + q_{k,T} - \tilde{d}_{k,T})^+ \right]$$

By using $\min\{a,b\} = b - (b-a)^+$ for $a, b \in \mathbb{Z}^+$, we can rewrite the value function as

$$V_T(\mathbf{x}_T) = \max_{0 \le \mathbf{q}_T \le \kappa,\ \mathbf{p}_T \in \mathcal{F}_T} \sum_{k \in G_T} p_{k,T}\,(x_{k,T} + q_{k,T}) - c_k q_{k,T} - (h_T + p_{k,T})\,\mathbb{E}\left[\left(x_{k,T} + q_{k,T} - \tilde{d}_{k,T} \right)^+ \right] \quad (7)$$

Suppose that there are S number of different models available in the market (i.e., $|G_T| = S$) and total number of products available for each generation $k \in G_T$ is assumed to be $x_{k,T} + q_{k,T} = M$. In this case, we obtain

$$\mathbb{E}\left[\left(x_{k,T} + q_{k,T} - \tilde{d}_{k,T} \right)^+ \right] = \sum_{j=0}^{x_{k,T}+q_{k,T}} f_{k,T}(j)\,(x_{k,T} + q_{k,T} - j) \quad (8)$$

We know the expression $\sum_{j=0}^{M} j f_{k,T}(j) = \gamma_{k,T} \sum_{j=0}^{M} j\lambda_j$ holds true. This expression is obtained by plugging the value of $f_{k,T}(.)$ from proposition 2 and mathematically expanding the equation. The mathematical terms

are rearranged to apply algebraic tools and the binomial theorem which results in the simplified expression. By using $\sum_{j=0}^{M} j f_{k,T}(j) = \gamma_{k,T} \sum_{j=0}^{M} j\lambda_j$, and re-injecting (8) into (7) as

$$V_T(\mathbf{x}_T) \quad = \quad \max_{0\leq \mathbf{q}_T \leq \kappa,\ \mathbf{p}_T \in \mathcal{F}_T} \sum_{k\in G_T} p_{k,T}\left(x_{k,T}+q_{k,T}\right) - c_k q_{k,T} - (h_T + p_{k,T})\left(x_{k,T}+q_{k,T} - \gamma_{k,T}\sum_{j=0}^{M} j\lambda_j\right).$$

One can easily show that the hessian matrix H of the value function with respect to $p_{k,T}$ for $k \in G_T = \{1, \cdots, s-1, s\}$ (in order from the older to recent generations currently available at T) is a symmetric diagonally dominant with real non-positive diagonal entries, and also negative semi-definite matrix.

$$H = \sum_{j=0}^{M} j\lambda_j \begin{pmatrix} \frac{-2\alpha_2}{\alpha_1(\alpha_2-\alpha_1)} & \frac{2}{\alpha_2-\alpha_1} & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{2}{\alpha_2-\alpha_1} & \frac{-2(\alpha_3-\alpha_1)}{(\alpha_3-\alpha_2)(\alpha_2-\alpha_1)} & \frac{2}{\alpha_3-\alpha_2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{2}{\alpha_3-\alpha_2} & \frac{-2(\alpha_4-\alpha_2)}{(\alpha_4-\alpha_3)(\alpha_3-\alpha_2)} & \frac{2}{\alpha_4-\alpha_3} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{2}{\alpha_s-\alpha_{s-1}} & \frac{-2}{\alpha_s-\alpha_{s-1}} \end{pmatrix} \tag{9}$$

Let μ_k and ξ_k for each version $k \in G_T$ denote the Lagrangian multipliers with respect to linear constraints. Then the Lagrangian function can be written as

$$\begin{aligned} \mathcal{L}(\mathbf{p}, \mu, \xi) \quad = \quad & \sum_{k\in G_T} p_{k,T}(x_{k,T}+q_{k.T}) - c_k q_{k,T} - (h_T + p_{k,T})\left(x_{k,T}+q_{k,T} - \gamma_{k,T}\sum_{j=0}^{M} j\lambda_{j,T}\right) \\ & - \sum_{k\in G_T} \xi_k p_{k,T} - \sum_{i=1}^{S-1} \mu_i \left(\frac{p_i}{\alpha_i} - \frac{p_{i+1}}{\alpha_{i+1}}\right) \end{aligned}$$

Next, we will show that the first order optimality conditions $\frac{\partial \mathcal{L}}{\partial p_{i,T}} = 0$ for $i = 1, \cdots, S-1$ and $\frac{\partial \mathcal{L}}{\partial p_{S,T}} = 0$, as well as complimentarity conditions are satisfied at the optimal production strategy. In other words, we have for $i = 1, \cdots, S-1$

$$\left(\frac{2p_{i-1,T}}{\alpha_{i,T}-\alpha_{i-1,T}} - \frac{2(\alpha_{i+1,T}-\alpha_{i-1,T})p_{i,T}}{(\alpha_{i+1}-\alpha_{i,T})(\alpha_{i,T}-\alpha_{i-1,T})} + \frac{2p_{i+1,T}}{\alpha_{i+1,T}-\alpha_{i,T}} - \frac{h}{\alpha_{i,T}-\alpha_{i-1,T}}\right)\sum_{k=0}^{M} k\lambda_{k,T} + \mu_{i-1} - \mu_i - \xi_i = 0,$$

and

$$\begin{aligned} & \left(\frac{2p_{i-1,T}}{\alpha_{i,T}-\alpha_{i-1,T}} - \frac{2p_{i,T}}{\alpha_{i,T}-\alpha_{i-1,T}} + 1\right)\sum_{k=0}^{M} k\lambda_{k,T} + \mu_{i-1} - \mu_i - \xi_i = 0, \quad i = S, \\ & \mu_i\left(\frac{p_{i,T}}{\alpha_{i,T}} - \frac{p_{i+1,T}}{\alpha_{i+1,T}}\right) = 0,\ \xi_i p_{i,T} = 0,\ \mu_i \geq 0,\ \xi_i \geq 0, \quad i = 1, \cdots, S. \end{aligned}$$

Since $\frac{p_{i,T}}{\alpha_{i,T}} < \frac{p_{i+1,T}}{\alpha_{i+1,T}}$ and $p_{i,T} \neq 0$, we find $\mu_i = \xi_i = 0$ for $i = 1, \cdots, S$. In this case, for $\sum_{k=0}^{M} k\lambda_{k,T} \neq 0$, the first order conditions for sufficiency of the optimality become

$$\begin{aligned} & \frac{2p_{2,T}}{\alpha_{2,T}-\alpha_{1,T}} - \frac{2\alpha_{2,T}p_{2,T}}{\alpha_{1,T}(\alpha_{2,T}-\alpha_{1,T})} - \frac{h_T}{\alpha_{1,T}} = 0 \\ & \frac{2p_{i-1,T}}{\alpha_{i,T}-\alpha_{i-1,T}} - \frac{2(\alpha_{i+1,T}-\alpha_{i-1,T})p_{i,T}}{(\alpha_{i+1,T}-\alpha_{i,T})(\alpha_{i,T}-\alpha_{i-1,T})} + \frac{2p_{i+1,T}}{\alpha_{i+1,T}-\alpha_{i,T}} - \frac{h_T}{\alpha_{i,T}-\alpha_{i-1,T}} = 0, \quad i = 2, \dots, S-1 \\ & \frac{2p_{S-1,T}}{\alpha_{S,T}-\alpha_{S-1,T}} - \frac{2p_{S,T}}{\alpha_{S,T}-\alpha_{S-1,T}} + 1 = 0 \end{aligned} \tag{10}$$

This linear equation system for S unknowns provides the optimal price strategy as $p^*_{i,T} = \frac{\alpha_{i,T}-h_T}{2}$ for all generations $i = 1, \cdots, S$. Note that if $\frac{p_{i,T}}{\alpha_{i,T}} = \frac{p_{i+1,T}}{\alpha_{i+1,T}}$, then $\lambda_{i,T} > 0$, $\xi_i > 0$ for $i = 1, 2, \cdots, S$. The first order conditions for sufficiency of the optimality leads to a system of $2S-1$ linear equations with S variables. As the number of linear equations are greater than number of variables, the solutions for this system of linear equations will be inconsistent.

■

Proof of Proposition 4

We prove this proposition in two parts by considering the lower and upper bounds of prices.

a) We first prove by contradiction that the lower bound $p^L_{k,t}$ of market price can be obtained as $p^L_{k,t} = \frac{\alpha_{k,t}-h_t}{2}$ such that $p^L_{k,t} \leq p_{k,t}$ for the oldest generation $k \in G_t = \{1, 2, \cdots, m\}$ (where generations are represented in order from the oldest to the recent model) for any time period t. Assume the price of the oldest version $k = 1$ at time t is $p_{k,t} = \frac{\alpha_{k,t}-h_t}{2} - a$, where $a > 0$ and $a \in \mathbb{R}$. From Proposition 1, we can compute the total purchase probability over all available products at time t as $\sum_{k=1}^{m} \gamma_{k,t} = 1 - \frac{p_{k,t}}{\alpha_{k,t}}$. We then obtain $\sum_{k=1}^{m} \gamma_{k,t} = 1 - \frac{\alpha_{k,t} - h_t - 2a}{2\alpha_{k,t}} = \frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}}$.

   – Suppose that $h_t \leq \alpha_{k,t}$. In this case, we have $\frac{h_t}{\alpha_{k,t}} \leq 1$ that leads to $\frac{h_t}{2\alpha_{k,t}} \leq \frac{1}{2}$ and also $\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} \leq 1$. Similarly, we can write an equivalent form of the inequality

$$\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}} \leq 1 + \frac{a}{\alpha_{k,t}} \implies \sum_{k=1}^{m} \gamma_{k,t} \leq 1 + \frac{a}{\alpha_{k,t}}$$

   For $a > \alpha_{k,t}$ or $a \leq \alpha_{k,t}$, the equation $\sum_{k=1}^{m} \gamma_{k,t} \leq 1 + \frac{a}{\alpha_{k,t}}$ is no longer a binding restriction on choice probabilities to lie between 0 and 1 and we may obtain $\sum_{k=1}^{m} \gamma_{k,t} \geq 1$ for some parameter values which contradicts the assumption. For example, let $h_t = \alpha_{k,t} - a$ (since $h_t \leq \alpha_{k,t}$),

$$\begin{aligned} \frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}} &= \frac{1}{2} + \frac{\alpha_{k,t} - a}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}} \\ &= 1 + \frac{a}{2\alpha_{k,t}} \geq 1 \\ \implies \sum_{k=1}^{m} \gamma_{k,t} &\geq 1 \end{aligned}$$

   – Suppose that $h_t \geq \alpha_{k,t}$. In this case, $\frac{h_t}{\alpha_{k,t}} \geq 1$ that leads to $\frac{h_t}{2\alpha_{k,t}} \geq \frac{1}{2}$ and also $\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} \geq 1$. Similarly, we can write an equivalent form of the inequality

$$\frac{1}{2} + \frac{h_t}{2\alpha_{k,t}} + \frac{a}{\alpha_{k,t}} \geq 1 + \frac{a}{\alpha_{k,t}} \implies \sum_{k=1}^{m} \gamma_{k,t} \geq 1 + \frac{a}{\alpha_{k,t}}$$

   For $a > \alpha_{k,t}$ or $a \leq \alpha_{k,t}$, we obtain $\sum_{k=1}^{m} \gamma_{k,t} \geq 1$ that contradicts the assumption.

Therefore, we conclude that $p^L_{k,t} = \frac{\alpha_{k,t} - h_t}{2} \leq p_{k,t}$ for $k \in G_t = \{1, 2, \cdots, m\}$.

b) Next, we will show that $p_{k,t} \leq p^U_{k,t}$ where $p^U_{k,t} = \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}}$ for generation $k \in G_t$ at ant time t. Let's consider the choice probability $\gamma_{k,t} \geq 0$ for the oldest generation k. Then we find

$$\gamma_{k,t} = \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} - \frac{p_{k,t}}{\alpha_{k,t}} \geq 0 \implies \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}} \geq p_{k,t}$$

The choice probability for successive generation $k+1$ is $\gamma_{k+1,t} = \frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} - \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} \geq 0$.

$$\begin{aligned}
&\frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} - \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} \geq 0 \\
&\implies \frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} \geq \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} \\
&\implies \frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} \geq \frac{p_{k+1,t} - p_{k,t}}{\alpha_{k+1,t} - \alpha_{k,t}} \geq \frac{p_{k+1,t} - \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}}}{\alpha_{k+1,t} - \alpha_{k,t}} \\
&\implies \frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} \geq \frac{p_{k+1,t} - \frac{\alpha_{k,t} p_{k+1,t}}{\alpha_{k+1,t}}}{\alpha_{k+1,t} - \alpha_{k,t}} = \frac{p_{k+1,t}}{\alpha_{k+1,t}} \\
&\implies \frac{p_{k+2,t} - p_{k+1,t}}{\alpha_{k+2,t} - \alpha_{k+1,t}} \geq \frac{p_{k+1,t}}{\alpha_{k+1,t}} \\
&\implies \frac{\alpha_{k+1,t} p_{k+2,t}}{\alpha_{k+2,t}} \geq p_{k+1,t}
\end{aligned}$$

By carrying out in the same manner, we obtain the lower and upper bounds for market prices as stated in the proposition.

■

Proof of Proposition 5

We prove the concavity of $\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t})$ by mathematical induction. At the boundary condition $t = T$, we prove the concavity of the value function $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ in $\mathbf{q}_T$. The value function $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ at the boundary condition is rewritten as,

$$\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T}) = \max_{0 \leq \mathbf{q_T} \leq \kappa} \sum_{k \in G_T} \hat{p}_{kT}(x_{kT} + q_{kT}) - c_k q_{kT} - B_{kT}(\mathbf{x}_T, \mathbf{q}_T) \tag{11}$$

where $B_{kT}(\mathbf{x}_T, \mathbf{q}_T) = (h_T + \hat{p}_{kT}) \sum_{j=0}^{x_{kT}+q_{kT}} f_{kT}(j)(x_{kT} + q_{kT} - j)$ is a differentiable function with respect to q_{kT}. In addition, $B'_k(\mathbf{x}_T, \mathbf{q}_T)$ and $B''_k(\mathbf{x}_T, \mathbf{q}_T) \geq 0$ represent the first and second-order derivatives, respectively, with respect to q_{kT}. We establish the concavity w.r.t $\mathbf{q_t}$ by writing its Hessian matrix as follows,

$$\begin{pmatrix} -B''_1(\mathbf{x_T}, \mathbf{q_t}) & 0 & 0 & 0 & \dots & 0 \\ 0 & -B''_2(\mathbf{x_T}, \mathbf{q_T}) & 0 & 0 & \dots & 0 \\ \\ 0 & 0 & 0 & 0 & \dots & -B''_k(\mathbf{x_T}, \mathbf{q_T}) \end{pmatrix} \tag{12}$$

that is a symmetric diagonally dominant matrix with real non-positive diagonal entries. As the Hessian matrix is negative semi-definite, it directly follows that $\hat{V}_T(\mathbf{x}_T|\hat{\mathbf{p}}_\mathbf{T})$ is a concave function in $\mathbf{q}_T$. Let us assume $\hat{V}_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})$ is concave in production level $\mathbf{q}_{t+1}$. We rewrite $\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t})$ as follows,

$$\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t}) = \max_{0\leq \mathbf{q_t} \leq \kappa} \sum_{k\in G_t} \hat{p}_{kt}(x_{kt}+q_{kt}) - c_k q_{kt} - B_{kt}(\mathbf{x}_t, \mathbf{q}_t) + E[\hat{V}_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})] \tag{13}$$

We canestablish the concavity w.r.t $\mathbf{q_t}$ by writing its Hessian matrix as follows by introducing $\phi = E[\hat{V}''_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})]$.

$$\begin{pmatrix} -B''_{\gamma_t}(\mathbf{x_t}, \mathbf{q_t}) + \phi & 0 & 0 & 0 & \dots & 0 \\ 0 & -B''_{\gamma_t+1}(\mathbf{x_t}, \mathbf{q_t}) + \phi & 0 & 0 & \dots & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & \dots & -B''_t(\mathbf{x_t}, \mathbf{q_t}) + \phi \end{pmatrix} \tag{14}$$

By induction hypothesis, $E[\hat{V}''_{t+1}(\mathbf{x}_{t+1}|\hat{\mathbf{p}}_\mathbf{t+1})] \leq 0$. The given Hessian matrix is a symmetric diagonally dominant matrix with real non-positive diagonal entries. As the Hessian matrix is negative semi-definite, it directly follows that $\hat{V}_t(\mathbf{x}_t|\hat{\mathbf{p}}_\mathbf{t})$ is a concave function in $\mathbf{q}_t$.

■

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