TOPOLOGICAL TYPES OF POLYNOMIAL MAPPINGS.

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Submitted for the degree of Ph.D.
at the University of Warwick.

Mathematics Institute,
University of Warwick,
July, 1979.

SUMMARY.

We consider polynomial mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, and show that those of degree less than any integer k exhibit finitely many local topological types at 0, and that the proper polynomial mappings of degree less than k which are not singular everywhere exhibit finitely many global topological types. We explicitly construct the local types for the mappings which are not singular everywhere. Applications to jet spaces and mappings between compact 2-manifolds are outlined.

CONTENTS.

ACKNOWLEDGEMENTS.

I would like to thank everyone in the Maths Institute for all their help and encouragement during the years 1976 - 79 . A special debt of gratitude is owed to my supervisor, David Rand, without whose guidance this would not have been possible.

For the past three years, I have received money from the Science Research Council.

NOTATION.

Each part of Chapters 2 to 5 inclusive is labelled by three numbers (n.m.p), where n gives the chapter number, m gives the section in that chapter, and p the subsection. Chapter 1 has no subsections, so its parts are labelled (n.m). Whenever the thesis refers to another part of itself, these numbers are quoted in full. References to other sources are given in square brackets thus: [r] , where the number r indicates the item's position in the bibliography.

Our notation is mostly standard. The following list should prevent any misinterpretation.

Symbol.	Meaning and Remarks.
$\subset$	Inclusion: never strict inclusion.
$\amalg$	Disjoint union.
$\setminus$	Set-theoretic difference.
$\twoheadrightarrow$	We write $f : X \twoheadrightarrow Y$ if $fX = Y$.
$\mapsto$	We write $x \mapsto y$ if for some (contextually clear) map f , then $fx = y$. Note that $z \mapsto z^n$ has a special meaning: see (2.2.3).

Symbol.	Meaning and Remarks.
$\#A$	The cardinality of the set A. If A is uncountable, we write $\#A = \infty$.
$\mathbb{R}^n$	Real n-space, which is always assumed to have the normed-space topology.
0	The origin of $\mathbb{R}^n$.
$\mathbb{P}^n$	Real or complex projective n-space.
bM , $\mathrm{Int}\, M$	If M is a manifold, the bM denotes its boundary, and $\mathrm{Int}\, M$ its interior. We occasionally use b to denote the frontier of an open ball, but we always make this clear.
$\overline{X}$	The closure of X in some larger space, which is explicitly mentioned if confusion would otherwise result.
$B(0,r)$	The open r-ball about 0 in $\mathbb{R}^n$.
$\widehat{ab}$	The closed line-segment joining points a and b.
I	The interval $[0,1]$.
$f_\#$	If $f : X \to Y$ is a continuous map between topological spaces, then $f_\# : \pi_1(X) \to \pi_1(Y)$ is the induced map on the fundamental groups.
$f^{-1}A$	Inverse image of A under f. If $A = \{a\}$, we usually write $f^{-1}a$ for $f^{-1}\{a\}$.
$f\|A$	Restriction of f to A.

The words <u>map</u> and <u>mapping</u> are used synonymously to mean a continuous function between topological spaces. Some symbols appear at different meanings (e.g. S is used in (2.2) to denote the set of 2-dimensional semi-algebraic sets, and in Chapters 3 and 4 to denote the singular set of a polynomial mapping). We have endeavoured to indicate the scope of such symbols every time they are introduced.

CHAPTER 1 - INTRODUCTION

(1.1) In this thesis we shall be concerned with topological types of polynomial mappings, concentrating upon those from $\mathbb{R}^2$ to $\mathbb{R}^2$. Although we may use particular definitions in special cases, the general definition of <u>topological type</u> is as follows: Let X, Y be topological spaces, and $f, g : X \to Y$ be continuous maps. We say that f and g are <u>topologically equivalent</u>, written $f \sim g$, if there exist homeomorphisms $h : X \twoheadrightarrow X$, $k : Y \twoheadrightarrow Y$, such that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

commutes. Note that this definition does depend upon Y: we are <u>not</u> saying that there exist homeomorphisms $h : X \twoheadrightarrow X$, $k : \operatorname{Im} f \twoheadrightarrow \operatorname{Im} g$ such that the diagram commutes. Equivalence classes under $\sim$ are called <u>topological types</u>.

(1.2) Now suppose that $f, g : X \to Y$ are continuous maps satisfying $fx_0 = y_0 = gx_0$ for some $x_0 \in X$. We say that f and g are <u>locally topologically equivalent at</u> x_0, written $f \overset{x_0}{\sim} g$, if there exist neighbourhoods

U and V of x_0 in X, neighbourhoods U' and V' of y_0 in Y, with $fU \subset U'$, $gV \subset V'$; and homeomorphisms $h : U \twoheadrightarrow V$, $k : U' \twoheadrightarrow V'$, such that the diagram

$$\begin{tikzcd} U \arrow[r, "f|U"] \arrow[d, "h"'] & U' \arrow[d, "k"] \\ V \arrow[r, "g|V"'] & V' \end{tikzcd}$$

commutes. Equivalence classes under $\overset{x_0}{\sim}$ are called <u>local topological types at</u> x_0.

(1.3) Let $K = \mathbb{R}$ or $\mathbb{C}$. We define $P(n,m,k,K)$ to be the set of polynomial mappings $f : K^n \to K^m$ with $f0 = 0$ and $\deg f < k$, where, if $f(x) = (f_1x,\ldots,f_mx)$, $\deg f = \max \deg f_i$ for $i = 1,\ldots,m$. (Note that this latter definition is independent of linear changes of coordinates, but not of polynomial homeomorphisms, since e.g. $x \longmapsto x^n$ for n odd is a homeomorphism of $\mathbb{R}$.) We consider two questions about these maps.

<u>Global Question</u> $G(n,m,K)$. <u>For given values of</u> n,m,K, <u>is</u> $P(n,m,k,K)/\sim$ <u>a finite set for each</u> k ?

<u>Local Question</u> $L(n,m,K)$. <u>For given values of</u> n,m,K <u>is</u> $P(n,m,k,K)/\overset{0}{\sim}$ <u>a finite set for each</u> k ?

It is obvious that $G(n,m,K)$ implies $L(n,m,K)$.

(1.4) In [5] Fukuda has given an affirmative answer to $G(n,1,K)$ for all n, and $K = \mathbb{R}$ or $\mathbb{C}$. Thom has conjectured that $G(n,2,K)$ is true for all n and $K = \mathbb{R}$ or $\mathbb{C}$. Both questions have negative answers for $m \geqslant 3$.

<u>Example</u>.

Let $F_t : \mathbb{R}^3 \to \mathbb{R}^3 : (x,y,z) \longmapsto (X,Y,Z)$ where

$X = (x(x^2+y^2-a^2) - 2ayz)^2((ty+x)(x^2+y^2-a^2) - 2az(y-tz))^2$,

$Y = x^2+y^2-a^2$,

$Z = z$.

In [20] Thom showed that for $s \neq t$, F_s and F_t have different topological types. As this example cannot be briefly covered, the reader is referred to Thom's own excellent exposition.

(1.5) In this paper we shall consider the case in which $K = \mathbb{R}$, and $m = 2 = n$. We give a complete solution of the local problem, and solve the global problem for proper maps (see (2.2.11) for the precise definition) which are not everywhere singular, i.e. we prove:

THEOREM 1. $P(2,2,k,\mathbb{R})/\overset{0}{\sim}$ <u>is a finite set for each</u> k. (See (3.3.5) for the proof of this theorem).

THEOREM 2. $\{f \in P(2,2,k,\mathbb{R}) \mid f$ <u>is proper and</u> $\det Df_x \not\equiv 0\}/\sim$ <u>is a finite set for each</u> k. (See (4.2.24)).

I believe that $G(2,2,\mathbb{R})$ has an affirmative answer, but I am unable to prove it. A better understanding of this would shed some light on Hilbert's 17th Problem (see [13]) and the Jacobian Problem (see [1]). There is a discussion of the difficulties involved, and some partial results, in Chapter 4.

(1.6) The reason for studying topological types of polynomial mappings (other than for its own sake) is to enable one to deduce results about jet spaces, which, after choosing coordinates, can be identified with polynomial mappings of less than a certain degree. With the methods of this thesis, we actually construct the types which may occur, giving a clearer insight into the subsequent stratifications of the jet spaces than methods which merely prove the existence of such stratifications (see [23] and [24]). Results about jet spaces are proved in Chapter 5, where we also give the local structure of all C^∞ maps between compact 2-manifolds, except for a set of infinite codimension.

(1.7) The techniques of this thesis are mainly elementary. We use a little plane algebraic geometry, but nothing more difficult than Bezout's Theorem. The Tarski - Seidenberg Theorem (2.1.2) is used extensively, as most of the sets involved are semialgebraic. For the local result we require the Spherical Cone Lemma (2.1.5),

which I cannot claim to be original, although I can find no reference to it. The results about covering maps in section (2.3) are standard. In the field of differential topology we require Sard's Theorem (2.2.6), and use Church's results on light open maps (see [4]). We cannot use as many results on transversality as might at first be imagined. This is because we only get problems with non-generic properties, and if we have a proof in these cases, it is usually quite easy to extend it to all maps without specifically invoking transversality.

(1.8) The organization of this thesis is as follows:

Chapter 2 consists of mathematical preliminaries.

Chapter 3 contains the proof of the local result (Theorem 1).

Chapter 4 proves the global result (Theorem 2) and ends with a discussion of the difficulties involved if the 'proper' and 'not everywhere singular' hypotheses are removed.

Chapter 5 proves results about jet spaces and compact 2-manifolds.

CHAPTER 2 - MATHEMATICAL PRELIMINARIES

(2.0) Introduction.

This chapter contains most of the results we quote. In most cases proofs are not given, but there is a list of references at the head of each section.

(2.1) Semialgebraic sets (see [3,9,10,16,19]).

(2.1.1) Definition. Let $P_n = \mathbb{R}[X_1,\ldots,X_n]$. Then S_n is the smallest family of subsets of $\mathbb{R}^n$ satisfying the following conditions:

(a) If $f \in P_n$, then $f^{-1}0 \in S_n$.

(b) If $A,B \in S_n$, then $A \cap B$, $A \cup B$, $A \setminus B \in S_n$.

Elements of S_n are called semialgebraic sets.

(2.1.2) THEOREM (Tarski-Seidenberg).

If $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a polynomial mapping and* $A \in S_n$, *then* $fA \in S_n$.

(2.1.3) Definition. If $A \in S_n$ then $a \in A$ is a *regular point of* A *of dimension* k if a has a neighbourhood U in $\mathbb{R}^n$ such that $U \cap A$ is a smooth analytic variety of dimension k . The set of regular points of A of dimension k is an element of S_n .

The *dimension* of an element of S_n is the maximum of the dimensions of its regular points.

We shall henceforth be concerned with S_2 , which for the remainder of this section we shall write as S .

(2.1.4) <u>Definition</u>. Let $A \in S$, with $0 \in A$, and let U be an open ball about 0 in $\mathbb{R}^2$. We say that A is <u>conelike</u> in U if either $A \cap U = \{0\}$ or there exists $K \subset S^1$ (the topological circle) such that $(U, A \cap U, 0)$ is homeomorphic to

$$(S^1 \times [0,1) / (s,0) \sim (t,0), K \times [0,1) / \sim , 0).$$

If $U = B(0,r)$, then A is <u>spherically conelike</u> in U if for all r' with $0 < r' \leq r$, A is conelike in $B(0,r')$.

(2.1.5) LEMMA (<u>Spherical Cone Lemma</u>).
<u>Let</u> $A_1, \ldots, A_n$ <u>be a finite collection of sets in</u> S , <u>with</u> $0 \in A_i$ <u>for each</u> i . <u>Then there exists an open ball</u> $U = B(0,r)$ <u>such that for each</u> i , A_i <u>is spherically conelike in</u> U .

<u>Proof</u>. It suffices to prove the result for a single set A . If $\dim A \leq 0$, the result is obvious. If $\dim A = 1$, let $A = A_1 \sqcup Q$, where A_1 is the set of regular points of A of dimension 1 , and $\dim Q \leq 0$. Let $B_1 = \bar{A}_1 \setminus A_1$, and let V be an open ball about 0 such that $B_1 \cup Q$ is conelike in V . Let $M = V \cap A_1 \setminus \{0\}$, a smooth 1-manifold without boundary, and let

$$P_1 = \{x \in M \mid T_x M \text{ is perpendicular to } \widehat{0x}\}.$$

Choose a ball $U \subset V$ such that P_1 is conelike in U and so that $U \cap A_1$ is equal to the intersection of U with the connected components of $M \setminus P_1$ containing 0. Then A_1 is spherically conelike in U, and so is A. If $\dim A = 2$, let $A = A_1 \amalg A_2 \amalg Q$, where A_i is the set of regular points of A of dimension i, and let $B_i = \overline{A}_i \setminus A_i$, for $i = 1,2$. Choose a ball $U = B(0,r)$ such that A_1, B_1, B_2, Q and all intersections of these sets are spherically conelike in U. Then A_2 is spherically conelike in U, and so is A.

(2.1.6) <u>Definition</u>. Let A be a 1-dimensional semi-algebraic subset of $\mathbb{R}^2$, and let $a \in A$. Let U be an open ball about a in $\mathbb{R}^2$ such that A is conelike in U on a. The <u>branches</u> of A at a are the germs at A of the sets of the form $\overline{K}$, where K is a connected component of $(A \setminus \{a\}) \cap U$. We shall frequently refer to representatives of these germs as <u>branches</u>. If B is a branch of A at a, the <u>tangent</u> to B at a is the limiting direction of a chord $\widehat{ax}$, as x tends to a in B.

(2.1.7) LEMMA. <u>A semialgebraic subset of $\mathbb{R}^2$ is locally path connected, so its components and path components coincide</u>.

(2.1.7) Remark. As a consequence of Lemma (2.1.7), we shall henceforth fail to distinguish between path components and components of semialgebraic sets.

(2.2) Differential topology (see [4]).

(2.2.1) Definition. If M^n, N^n are n-manifolds, and $f : M \to N$ is continuous, then f is open if fU is open in N for each open $U \subset M$; and f is light if for every $y \in N$, $\dim f^{-1}y \leq 0$.

The description C^r, applied to maps, manifolds, etc., means r times continuously differentiable.

(2.2.2) Except when otherwise stated, we shall use the definition of topological equivalence as stated in (1.1). We do, however, extend the definition of local topological equivalence given in (1.2) as follows:

Let M_i, N_i $(i = 1,2)$ be manifolds, and $f_i : M_i \to N_i$ be continuous mappings. Then f_1 and f_2 are locally topologically equivalent at $m_1 \in M_1$, $m_2 \in M_2$, if there exist, for $i = 1,2$, neighbourhoods U_i of m_i in M_i, neighbourhoods V_i of $f_i m_i$ in N_i with $f_i U_i \subset V_i$, and homeomorphisms $h : U_1 \twoheadrightarrow U_2$, $k : V_1 \twoheadrightarrow V_2$

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such that the diagram

$$\begin{tikzcd} U_1 \arrow[r, "f/U_1"] \arrow[d, "h"'] & V_1 \arrow[d, "k"] \\ U_2 \arrow[r, "f/U_2"'] & V_2 \end{tikzcd}$$

commutes.

(2.2.3) <u>Definition</u>. By " $z \longmapsto z^n$ " we shall mean the polynomial mapping $f_n : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (u,v)$, where $u + iv = (x + iy)^n$.

(2.2.4) THEOREM (<u>Stoilow</u> [18]).
<u>If</u> $f : M^2 \to N^2$ <u>is light and open, then at each point of</u> M , f <u>is locally topologically equivalent to</u> $z \longmapsto z^n$ <u>at</u> 0 , <u>for some integer</u> n .

This theorem, and similar results, are considered by Whyburn in [27]. Church [4, Theorem (2.1)] has extended the above Theorem to maps between n-manifolds, showing that at points not in a set of codimension at least 3, a continuous light open map is locally topologically equivalent (with suitably chosen coordinates) to a map of the form $F_{n,d}: (x_1,\ldots,x_n) \longmapsto (y_1,\ldots,y_n)$, where $y_1 + iy_2 = (x_1 + ix_2)^d$ and $y_j = x_j$ for $j \geqslant 3$. We do not require this extension for the proof of our results.

(2.2.5) <u>Definition</u>. If $f : M^n \to N^p$ is C^1, let

$$R_q = \{x \in M \mid \operatorname{rank} Df_x \leq q\}.$$

(2.2.6) THEOREM (<u>Sard's Theorem</u>, see [4:(1.3)]).
<u>If</u> $f : M^n \to N^p$ <u>and</u> f, M, N <u>are</u> C^n, <u>then</u> $\dim fR_q \leq q$ <u>and</u> $\dim fM \leq n$. <u>If</u> f <u>is also light, then</u> $\dim R_q \leq q$

(2.2.7) <u>Definition</u>. Let $f : M^n \to N^n$ be C^1. Define $B_f = \{x \in M \mid f$ is not a local homeomorphism at $x\}$.

(2.2.8) THEOREM [4:(1.4)]. <u>Let</u> $f : M^n \to N^n$ <u>be open and</u> C^1. <u>If for</u> $x \in M$, $\operatorname{rank} Df_x \geq n - 1$, <u>then</u> f <u>is a local homeomorphism at</u> x, <u>i.e.</u> $B_f \subset R_{n-2}$.

The above is an extension of the Inverse Function Theorem to open maps. It is proved by a decomposition argument, and uses results on upper semicontinuity.

(2.2.9) COROLLARY [4:(1.6)]. <u>If</u> $f : M^n \to N^n$ <u>is light and</u> C^n, <u>then</u> f <u>is open if and only if</u> $B_f \subset R_{n-2}$.

(2.2.10) COROLLARY [4:(1.7)]. <u>Let</u> $f : \mathbb{R}^n \to \mathbb{R}^n$ <u>be</u> C^n <u>and light. Then</u> f <u>is open if and only if</u> $\det Df_x$ <u>is non-negative or nonpositive everywhere</u>.

(2.2.11) <u>Definition</u>. A continuous map $f : X \to Y$ between topological spaces is called <u>proper</u> if for every compact subset K of Y, $f^{-1}K$ is compact in X. The map f is <u>locally proper</u> (some writers say <u>proper onto its image</u>) if $f : X \to fX$ is proper. To appreciate the difference between these definitions, note that $\tan^{-1} : \mathbb{R} \to \mathbb{R} : x \longmapsto \tan^{-1}x$ is locally proper but not proper.

(2.3) <u>Covering spaces</u> (see [17 Ch 2]).

(2.3.1) <u>Definition</u>. Let X be a topological space. The <u>category of covering spaces of</u> X, Cov X, has as objects covering maps of the form $p : \tilde{X} \to X$, where X is a connected topological space, and as morphisms commutative triangles of the form

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd} .$$

(2.3.2) THEOREM. <u>Two covering maps</u> $p_i : \tilde{X}_i \to X$, $i = 1,2$, <u>are equivalent in</u> Cov X <u>if and only if</u>

$$p_{i\#}\pi_1(X_i, x_i), \quad i = 1,2,$$

<u>are conjugate subgroups of</u> $\pi_1(X,x)$, <u>where</u> $p_i x_i = x$.

(2.3.2) COROLLARY. <u>Let</u> $p_i : \tilde{X}_i \to X$, $i = 1,2$, <u>be objects of</u> Cov X . <u>Equivalence in this category implies right topological equivalence, i.e. there exists a homeomorphism</u> $h : X_1 \twoheadrightarrow X_2$ <u>such that</u> $p_2 \circ h = p_1$.

(2.4) <u>Stratifications</u>.

(2.4.1) <u>Definition</u>. Let M be a manifold without boundary and let S be a subset of M . A <u>stratification</u> $\mathcal{S}$ of S is a cover $\mathcal{S} = \{X_i\}$ by submanifolds of M lying in S (called <u>strata</u>) satisfying the following conditions:

(a) If $i \neq j$ then $X_i \cap X_j = \emptyset$.

(b) <u>Local finiteness</u>: Each point of M has a neighbourhood meeting only finitely many strata.

(c) <u>Condition of the frontier</u>: If $\overline{X}_i \cap X_j \neq \emptyset$, then

$$X_j \subset \overline{X}_i \setminus X_i .$$

If these conditions are satisfied, then S is called a <u>stratified set</u>.

(2.4.2) <u>Definition</u>. Let S , S' be stratified sets. A <u>stratified map</u> $f : S \to S'$ is a smooth map sending strata into strata such that if X is a stratum of S , then $f|X$ is a submersion. The stratified map f is <u>exact</u> if it maps strata onto strata.

Page 17 and section (2.4.3) do not exist.

(2.4.4) Definition. The set of i - strata of a stratification $\mathcal{S} = \{X_j\}$ of a set S is the set $\{X_j \in \mathcal{S} \mid \dim X_j = i\}$.

The i - skeleton of a stratification is the point-set union of its j - strata, for $j \leq i$.

(2.4.5) Remark. In this thesis we are not much concerned with stratifications themselves, and so do not introduce refinements like Whitney's conditions (a) and (b) (see [27]). We use the terminology of stratifications as a shorthand. Occasionally, and when no confusion can arise, we shall abuse the notation by using the same symbol for the stratified set and its stratification, i.e. identifying $\mathcal{S}$ and S (see for example (3.2.2(iii))). When this is done, it is obvious from the context whether we are considering the object to have as elements points or strata.

(2.5) Algebraic geometry.

(2.5.1) Any study of real algebraic geometry involves difficulties, the main one being the non-uniqueness of representations of sets, for example the point $(0,0) \in \mathbb{R}^2$ is the zero-set of both $x^2 + y^2$ and $x^2 + y^4$, which are a pair of coprime elements of $\mathbb{R}[x,y]$. (For a discussion of some of the difficulties involved, see [15]). In an attempt to circumvent this problem, we shall always consider

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the point-set determined by a polynomial, without regard to the multiplicity of components, nor irreducibility, nor existence of a relatively prime polynomial of lower degree determining the same point-set. This is made clearer in the results below.

(2.5.2) THEOREM (Bezout's Theorem).
Let V, W be hypersurfaces in $\mathbb{R}^2$ determined by $f, g \in \mathbb{R}[x,y]$, where $\deg f = n$, $\deg g = m$, having no common components. Then $\# V \cap W \leq nm$.

Proof. This follows from Bezout's Theorem for projective plane curves over an algebraically closed field, as given in, e.g. [6, p.112]. At each intersection point, the intersection number of f and g is at least 1, and the curves can only intersect in a maximum of nm points when the plane is complexified and extended to $\mathbb{P}^2$. The presence of multiple components does not affect the result unless the two polynomials have a common factor over $\mathbb{R}$ which determines an algebraic set of dimension at least 1.

(2.5.3) COROLLARY. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping of degree less than k. Then for each $y \in \mathbb{R}^2$, $\# f^{-1}y = \infty$ or is less than k^2.

Proof. With respect to some bases, let f_1, f_2 be the

components of f , and let $y = (y_1, y_2)$. If $f_1 - y_1$ and $f_2 - y_2$ have a common component determining a set of dimension $\geqslant 1$, then $\# f^{-1} y = \infty$. Otherwise, we can apply (2.5.2), and the result follows because $\deg (f_i - y_i) < k$ for $i = 1,2$.

(2.5.4) LEMMA. <u>Let</u> C <u>be a proper algebraic subset of</u> $\mathbb{R}^2$ <u>determined by a polynomial</u> f <u>of degree</u> d . <u>Then if</u> $x \in C$, C <u>has at most</u> $2d$ <u>branches at</u> x .

<u>Proof</u>. Consider a small circle about x , and apply Bezout's Theorem.

CHAPTER 3 - LOCAL TOPOLOGICAL TYPES OF POLYNOMIAL MAPPINGS OF THE PLANE.

(3.0) Introduction. The main result of this chapter is:

THEOREM 1. For each k, $P(2,2,k,\mathbb{R})/\overset{0}{\sim}$ is a finite set.

In the course of the proof we construct all the types which may occur for the maps which are not everywhere singular, and we end the chapter with some examples.

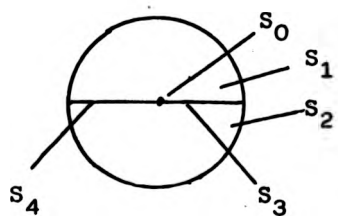
(3.1) Maps which are not everywhere singular.

(3.1.1) Throughout this section let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping with $f0 = 0$. Let $Q_0 = \{x \in \mathbb{R}^2 \mid \det Df_x = 0\}$, and let $Q' = fQ_0$, $Q = f^{-1}Q'$. By (2.1.2), all of these sets are semialgebraic. We further suppose that $Q \neq \mathbb{R}^2$ (dealing with the case in which $Q = \mathbb{R}^2$ in section (3.3)), so that $\dim Q \leq 1$. In this section we show that f is locally topologically equivalent at 0 to a map of standard type.

The idea of the proof is to choose neighbourhoods of the origin in which Q, $\operatorname{Im} f$, etc., are conelike. These neighbourhoods are then stratified, and counting arguments give a bounded number of strata for maps of degree less

than k. We then show that such maps can exhibit only finitely many local topological types.

As an example, consider the fold map $g : (x,y) \longmapsto (x,y^2)$. We stratify $B(0,1)$ as in the picture below:



With respect to diffeomorphic embeddings of this stratified set into the plane, we can make the restriction of g an exact stratified map, sending S_i onto S_i diffeomorphically for $i \neq 2$, and sending S_2 onto S_1 via an orientation reversing diffeomorphism.

In the proof, we show that knowledge of this type of information is sufficient to describe the map's local topological type.

(3.1.2) LEMMA. <u>There is an open ball</u> U_1 <u>about</u> 0 <u>in</u> $\mathbb{R}^2$ <u>such that</u> Q <u>and</u> $f^{-1}0$ <u>are spherically conelike in</u> U_1.

<u>Proof</u>. Since Q and $f^{-1}0$ are semialgebraic sets, the result follows from (2.1.2) and (2.1.5).

(3.1.3) <u>Definition</u>. Let $W_1 = fU_1$.

(3.1.4) LEMMA. <u>There is an open ball</u> V_1 <u>about</u> 0 <u>such that</u> $V_1 \cap \operatorname{Im} f \subset W_1$, <u>and</u> Q, W_1, $Q' \cap W_1$ <u>are spherically</u> conelike in V_1 .

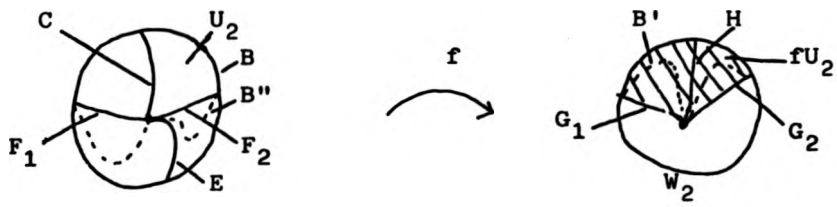
<u>Proof</u>. Again, everything is semialgebraic, so the Lemma follows from (2.1.2) and (2.1.5).

(3.1.5) LEMMA. <u>There is an open ball</u> U_2 <u>about</u> 0 <u>such that</u> $U_2 \subset U_1$, $W_2 = fU_2 \subset W_1$, <u>and the following is true</u>: <u>If</u> L <u>is a connected component of</u> $(Q \cap U_2) \setminus \{0\}$, <u>then either</u> fL <u>is a connected component</u> M <u>of</u> $(Q' \cap W_2) \setminus \{0\}$ <u>and</u> $f|L : L \to M$ <u>is a local homeomorphism of</u> C^0 1-<u>manifolds, or</u> $fL = \{0\}$.

<u>Proof</u>. Note first that, by construction, each connected component of $Q \cap U_1 \setminus \{0\}$ is a C^0 1-manifold. If L is such a component, and does not map onto 0 , the set of points where $f|L$ fails to be a homeomorphism of 1-manifolds is a 0-dimensional semialgebraic set (since it is contained in the set where $f|L$ fails to be a diffeomorphism, and so may be excluded by shrinking of U_1 to some U_2 . Since by (3.1.2), $f^{-1}0$ is spherically conelike in U_1 , either $f(L \cap U_2) = 0$, or $f(L \cap U_2)$ is contained in a connected component of $Q' \cap W_2 \setminus \{0\}$.

(3.1.6) <u>Definition</u>. Let $B = \overline{U}_2 \setminus U_2$, $B' = fB$, $B'' = (f|U_2)^{-1}B'$.

Picture:



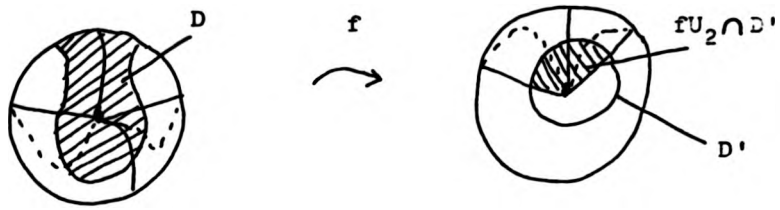
In this picture Q has branches C, E, F_1, F_2 at O , and Q' has branches H, G_1, G_2 . The branch C maps onto the origin, $fF_i = G_i$ $(i = 1,2)$, $fE = H$, and F_1 and F_2 are fold lines.

(3.1.7) LEMMA. <u>There is an open ball</u> D' <u>about</u> O <u>such that</u> $D' \subset V_1$, <u>and</u> fU_2, Q', B' <u>with all intersections of these sets, are spherically conelike in</u> D'.

<u>Proof</u>. All the sets mentioned are semialgebraic, so the result follows from (2.1.5).

(3.1.8) <u>Definition</u>. Let $D = (f|U_2)^{-1}D'$.

Picture:



The sets involved are the same as in the picture after Definition (3.1.6).

(3.1.9) LEMMA. <u>Let</u> Y <u>be a connected component of</u> $D' \setminus (B' \cup Q')$. <u>Let</u> $\gamma : I \to \overline{Y}$ <u>be a</u> C^1 <u>path such that</u> $\gamma[0,1) \in Y$ <u>and</u> $\gamma(1) = 0$. <u>Let</u> $x \in D$ <u>be such that</u> $f(x) = \gamma(0)$. <u>Then there is a</u> C^1 <u>path</u> $c : I \to D$ <u>with</u> $c(0) = x$ and $f \circ c = \gamma$.

<u>Proof</u>. Let X be the connected component of $D \setminus (B'' \cup Q)$ containing x. At x, f is a local diffeomorphism, so γ certainly lifts locally. Suppose that the maximal connected subset of I containing 0 upon which c may be defined is $[0,d)$, and let $\gamma(d) = z' \in Y$. Let $\lim_{t \to d} c(t) = z \in \overline{X}$. Then $fz = z'$ by continuity, and $z \in X$, so c may be defined on a larger set. This is a contradiction unless c is defined on all of I.

(3.1.10) LEMMA. D <u>is the connected component of</u> $U_2 \setminus (f|U_2)^{-1} bD'$ <u>containing</u> 0, <u>where</u> $bD' = \overline{D'} \setminus D'$.

<u>Proof</u>. Let K be the connected component in the statement of the Lemma. We show first that $D \subset K$. If $x \in U_2$ and $fx = 0$, then $x \in D$, and since $f^{-1}0$ is conelike in U_2, $x \in K$. If $x \in D \cap Q \setminus f^{-1}0$, then $fx = x' \in Q' \setminus \{0\}$. By (3.1.5), the connected component of $Q' \setminus \{0\}$ containing x' lifts via a local homeomorphism, so $x \in K$. If $x \in D \setminus Q$, $fx \in D' \setminus Q'$, and since Q' and fU_2 are conelike in D', there exists a path joining x' to 0 which misses Q except for 0. By (3.1.9) this path lifts along its length, showing that $x \in K$. Conversely, if $x \in K$, then the path from 0 to x in K maps down to a path which never crosses bD', and so $x \in D$.

(3.1.11) LEMMA D <u>is a topological open disc such that</u> Q, B'' <u>and</u> $Q \cap B''$ <u>are conelike in</u> D.

<u>Proof</u>. The first part follows from the proof of (3.1.10). The singular set Q is conelike in D because it is conelike in U_2. Each connected component of $B'' \setminus \{0\}$ is the lift via a local diffeomorphism of a connected component of $B' \setminus \{0\}$, and the latter is conelike in D' by (3.1.7). Since $Q \cap B'' \subset \{0\}$, the result follows.

<u>Remark</u>. We do not have that the semialgebraic sets are sperically conelike in D, because the latter is not an open ball.

(3.1.12) We now stratify D and D' as follows:
Strata for D are $\{0\}$ and the connected components of $(Q \cup B'') \cap D \setminus \{0\}$ and $D \setminus (Q \cup B'')$.
Strata for D' are $\{0\}$ and the connected components of $(Q' \cup B') \cap D' \setminus \{0\}$ and $D' \setminus (Q' \cup B')$.

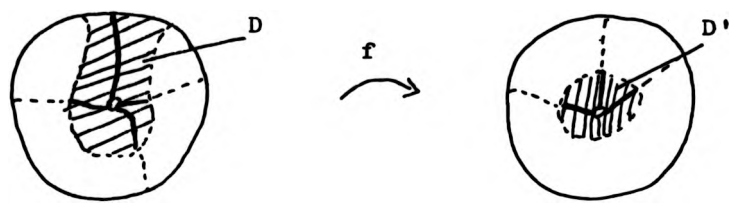
We shall henceforth assume that $\deg f < k$.

(3.1.13) LEMMA. <u>The partitions introduced in</u> (3.1.12) <u>do in fact give stratifications of</u> D <u>and</u> D'.
<u>The number of</u> 1-<u>strata in</u> D <u>is less than</u> $2(k-1)^2(1+k^2)$.
<u>The number of</u> 1-<u>strata in</u> D' <u>is less than</u> $4(k-1)^2$.
<u>With respect to these stratifications</u>, $f|D$ <u>is an exact stratified map</u>.

<u>Proof</u>. By (2.5.4), Q and Q' have less than $2(k-1)^2$ branches at 0 . The set B' has less than $2(k-1)^2$ branches, since we introduce at most 2 new branches for each connected component of $D \setminus Q$. Hence D' has less than $4(k-1)^2$ 1-strata. Each branch of B' (by (2.5.3) and the conic structure) has at most k^2 preimages in D , and so D has less than $2(k-1)^2(1+k^2)$ 1-strata. The conic structure implies that the partitions are in fact stratifications, and with respect to these, the map $f|D : D \to D'$ is obviously stratified. We already have exactness on 0- and 1-strata by (3.1.9) and (3.1.10). On 2-strata we can apply the path-lifting argument of

(3.1.9) and the fact that the strata are simply-connected to deduce that each 2-stratum of D maps onto a 2-stratum of D'.

Picture (the sets involved are the same as those in the picture following Definition (3.1.8)):



The 1-strata are marked with bold lines.

(3.2) <u>Wheel maps</u>.

(3.2.1) The n - <u>wheel</u> W^n is the stratification of the open unit ball about 0 in $\mathbb{R}^2$ with strata as follows:

$$W^n = \{0\} \cup \bigcup_{i=0}^{n-1} L_i \cup \bigcup_{j=0}^{n-1} M_j$$

defined in polar coordinates by

$$L_i = \{(r, 2\pi i/n) \mid 0 < r < 1 \quad ,$$

$$M_j = \{(r,\theta) \mid 0 < r < 1 \ , \quad 2\pi j/n < \theta < 2\pi(j+1)/n\} \ .$$

We now consider a map $g : W^n \to W^p$, and, when no confusion can arise, we shall denote the set of i - strata of W^n by S_i , the set of i - strata of W^p by S'_i , write $Y_i = \bigcup_{S \in S_i} S$, $Z_j = \bigcup_{i \leq j} Y_i$, with Z'_i and Y'_j defined analogously.

The set $W(n,p)$ is defined to be the set of exact stratified mappings $g : W^n \to W^p$ satisfying the following conditions:

(i) g is C^∞, with its singular set contained in Z_1 ;

(ii) $fY_0 = Y'_0$, $fZ_1 \subset Z'_1$, $fY_2 \subset Y'_2$;

(iii) if $Y \in S_1$ and $y \in Y$, then $(f|Y)^{-1}y$ is connected;

(iv) there exists an integer N (depending upon g) such that for all $y \in W^p$, $\# f^{-1}y$ is either infinite or less than N .

We call $W(n,p)$ the set of <u>wheel maps</u>.

(3.2.2) PROPOSITION. <u>Let</u> $f \in P(2,2,k,\mathbb{R})$, <u>and suppose that its singular set is not the whole of</u> $\mathbb{R}^2$. <u>Then</u> f <u>is locally topologically equivalent at</u> 0 <u>to a mapping</u> $g \in W(n,p)$ <u>for some</u> $n < 2(k-1)^2(k^2+1)$, $p < 4(k-1)^2$.

<u>Proof</u>. The mapping f satisfies all the hypotheses of section (3.1), and so the construction applies. Let n and p be the numbers of 1-strata of D and D' obtained in (3.1.13). If $n,p \geq 1$, let $h : D \twoheadrightarrow W^n$ and $h' : D' \twoheadrightarrow W^p$ be exact stratified homeomorphisms so that $h/(D \setminus Q)$ and

$h'|(D' \setminus Q')$ are local diffeomorphisms. Define $g = h' \circ f \circ h^{-1} : W^n \to W^p$. The construction guarantees that $g \in W(n,p)$.

If $n = 0$, then f is light on a neighbourhood of the origin, since Q is locally at most a single point. Hence $\det Df_x$ is non-negative or nonpositive on a neighbourhood of the origin, and so by (2.2.10), f is open near 0 . By (2.2.4), f is thus locally topologically equivalent to $z \mapsto z^d$ at 0 , where $d \leq k^2$ by (2.5.3), and so is locally topologically equivalent to an element of $W(d,1)$.

The case in which $p = 0$ and $n \geq 1$ does not occur, since $Q \cap D$ would have to map onto 0 , and we would have introduced more 1-strata into D' when we considered B'

(3.2.3) We have shown that f is locally topologically equivalent to a wheel map, and we now show that the elements of $W(n,p)$ display only finitely many local types. In order to do this, we define standard maps on <u>sectors</u> of W^n (i.e. closures in W^n of elements in S_2) and show that every element of W^n has the same local topological type as a map whose restriction to each sector is a standard map.

For the rest of this section we shall suppose that $g \in W(n,p)$.

(3.2.4) Let $M_i \in S_2$, $M'_j \in S'_2$ be defined as in (3.2.1). We define the <u>standard sector map</u>

$$\pi(n,p,i,j,s) : \overline{M}_i \longrightarrow \overline{M'_j}$$

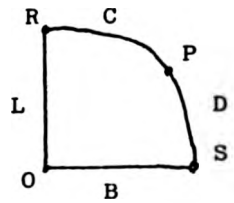
(where $^{-}$ denotes closure in $B(0,1)$) by

$$(r,\theta) \longmapsto (r, \frac{sn\theta}{p} + \frac{2\pi n}{p}(j - si + \frac{s+1}{2})) ,$$

where $0 \leq i < n$, $0 \leq j < p$, $s \in \{-1,1\}$. This is the obvious orientation preserving or reversing diffeomorphism according as $s = \pm 1$.

(3.2.5) Let $X = \{(r,\theta) \mid r \in I , 0 \leq \theta \leq \pi/2\}$. We define a map $F : X \to X$ sending the line segment $B = \{(r,0) \mid r \in I\}$ onto O .

Picture:



Let $J = \text{Int } Q$, $L = \{(r,\pi/2) \mid r \in I\}$, $C = \{(1,\theta) \mid \pi/4 \leq \theta \leq \pi/2\}$, $D = \{(1,\theta) \mid 0 \leq \theta \leq \pi/4\}$, $P = (1,\pi/4)$, $R = (1,\pi/2)$, $S = (1,0)$.

Choose a C^∞ mapping $F : X \to X$ satisfying

(i) $F|J : J \twoheadrightarrow J$ is an orientation preserving diffeomorphism;

(ii) $FB = O$;

(iii) $FR = R$, $FP = S$;

(iv) $F|C : C \twoheadrightarrow C \cup D$ and $F|D : D \twoheadrightarrow B$ are diffeomorphisms of 1-manifolds, and $F|L = id_L$.

The map F is called the <u>standard single collapse</u> of X .

(3.2.6) We now define $G : X \to X$, the <u>standard double collapse</u> of X , by the obvious analogy. Let

$$X_0 = \{(r,\theta) \in X \mid 0 \leq \theta \leq \pi/4\} ,$$

$$X_1 = \{(r,\theta) \in X \mid \pi/4 \leq \theta \leq \pi/2\} .$$

Define $G|X_0 = \Pi(4,8,0,0,1) \circ F \circ \Pi(8,4,0,0,1) : X_0 \to X_0$ and $G|X_1 = \Pi(4,8,0,0,1) \circ F \circ \Pi(8,4,1,0,-1) : X_1 \to X_1$.

(3.2.7) We say that $\tilde{g} : W^n \to W^p$ is a <u>standard exact wheel</u> map if $\tilde{g} \in W(n,p)$ and if $\tilde{g}(M_i) = M_j$ for some $M_i \in S_2$, $M_j' \in S_2'$, then

$$\tilde{g}|M_i = \Pi(4,p,1,j,s) \circ K \circ \Pi(n,4,i,1,t)$$

for some $s,t \in \{-1,1\}$ and $K \in \{id_X, F, G\}$.

(3.2.8) There are many choices for our standard maps F,G, but all these choices are topologically equivalent on $\operatorname{Int} X$. We need only choose maps which can be extended continuously to Z_1 , and this is guaranteed by the definitions.

(iii) $FR = R$, $FP = S$;

(iv) $F|C : C \twoheadrightarrow C \cup D$ and $F|D : D \twoheadrightarrow B$ are diffeomorphisms of 1-manifolds, and $F|L = id_L$.

The map F is called the <u>standard single collapse</u> of X .

(3.2.6) We now define $G : X \to X$, the <u>standard double collapse</u> of X , by the obvious analogy. Let

$$X_0 = \{(r,\theta) \in X \mid 0 \leq \theta \leq \pi/4\} ,$$

$$X_1 = \{(r,\theta) \in X \mid \pi/4 \leq \theta \leq \pi/2\} .$$

Define $G|X_0 = \Pi(4,8,0,0,1) \circ F \circ \Pi(8,4,0,0,1) : X_0 \to X_0$ and $G|X_1 = \Pi(4,8,0,0,1) \circ F \circ \Pi(8,4,1,0,-1) : X_1 \to X_1$.

(3.2.7) We say that $\tilde{g} : W^n \to W^p$ is a <u>standard exact wheel</u> map if $\tilde{g} \in W(n,p)$ and if $\tilde{g}(M_i) = M_j$ for some $M_i \in S_2$, $M_j' \in S_2'$, then

$$\tilde{g}|M_i = \Pi(4,p,1,j,s) \circ K \circ \Pi(n,4,i,1,t)$$

for some $s,t \in \{-1,1\}$ and $K \in \{id_X, F, G\}$.

(3.2.8) There are many choices for our standard maps F,G, but all these choices are topologically equivalent on $\operatorname{Int} X$. We need only choose maps which can be extended continuously to Z_1 , and this is guaranteed by the definitions.

(3.2.9) LEMMA. <u>If</u> $g \in W(n,p)$, <u>then there exists a standard exact wheel map</u> $\tilde{g} \in W(n,p)$ <u>and a homeomorphism</u> $h \in W(n,n)$ <u>such that</u> $\tilde{g} \circ h = g : W^n \to W^p$.

<u>Proof</u>. For each $M_i \in S_2$, we merely have to note the index j such that $gM_i = M_j' \in S_2'$, what collapses occur, the sign of $\det Dg_x$ for $x \in M_i$, and choose a standard map g such that $\tilde{g}_i = \tilde{g}|M_i$ has the same properties. Let $g_i = g|M_i$ and define $h_i = \tilde{g}_i^{-1} \circ g_i : M_i \to M_i$. As indicated in (3.2.8), the continuous extensions of h_i and h_{i+1} agree on $\overline{M}_i \cap \overline{M}_{i+1}$, so that $h \in W(n,n)$ can be defined.

(3.2.10) LEMMA. <u>The set</u> $\{g \in W(n,p) \mid n < N ,\ p < P\}/\overset{0}{\sim}$ <u>is finite for all integers</u> N , P .

<u>Proof</u>. Each $W(n,p)/\overset{0}{\sim}$ is a finite set by (3.2.9), so the result follows.

(3.1.11) PROPOSITION. <u>The set</u>

$$\{f \in P(2.2,k, \mathbb{R}) \mid f \text{ \underline{is not everywhere singular}}\}/\overset{0}{\sim}$$

<u>is finite for each</u> k .

<u>Proof</u>. In (3.2.2) we showed that f was equivalent to an element of $W(n,p)$, with n and p bounded, and in (3.2.10) we showed that the set of such maps exhibited only finitely many local topological types. The Proposition is proved.

(3.3) <u>Maps which are everywhere singular</u>.

(3.3.1) By Fukuda [5], $P(2,1,k,\mathbb{R})/\overset{0}{\sim}$ is a finite set. For fixed k, choose representatives $F_1,\ldots,F_M$ of $C^\infty(\mathbb{R}^2,\mathbb{R})/\overset{0}{\sim}$ having the same local topological types at 0 as elements of $P(2,1,k,\mathbb{R})/\overset{0}{\sim}$, such that F_i is a map from W^{n_i} to $(-1,1)$ and $F_i^{-1}0$ is the 1-skeleton of W^{n_i}. Note that for each i, we have that $n_i < 2k^2$ by (2.5.4). We call the F_i's the <u>standard</u> k - <u>maps</u>.

(3.3.2) Throughout this section let

$$X = \{f \in P(2,2,k,\mathbb{R}) \mid \det Df_x \equiv 0\},$$

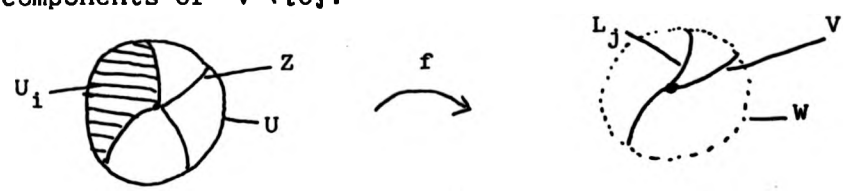
and suppose that $f \in X$.

This section proves that $X/\overset{0}{\sim}$ is a finite set. The idea of the proof is to project each branch of $\operatorname{Im} f$ (which is a 1-dimensional semialgebraic set) onto a line, so that composition with f gives a polynomial map from $\mathbb{R}^2$ to $\mathbb{R}$, and then deduce that the preimage of each branch can only map down in finitely many ways. We then show that matching-up on the preimages can only be carried out in finitely many ways.

(3.3.3) If $\operatorname{rank} Df_x$ is zero everywhere, then $f \equiv 0$. Otherwise $Z = f^{-1}0$ is an algebraic set of dimension ≤ 1. Since f is singular everywhere, its image is at most 1-dimensional by Sard's Theorem (2.2.6) and is semialgebraic by (2.1.2).

(3.3.4) PROPOSITION. $X/\overset{0}{\sim}$ <u>is a finite set</u>.

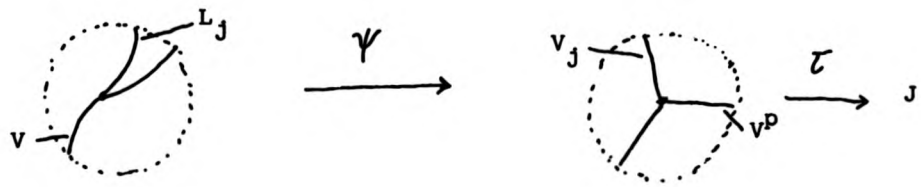
<u>Proof</u>. By (3.3.3) it suffices to prove the result for the maps f in X such that $Z = f^{-1}0$ is an algebraic set of dimension ≤ 1. Apply (2.1.5) to construct ball neighbourhoods U and W of 0 in $\mathbb{R}^2$ so that Z is spherically conelike in U and $\operatorname{Im} f$ is spherically conelike in W. Let $V = fU \cap W$. Now let $U_0,\ldots,U_{n-1}$ be the closures in U of the connected components of $U \setminus Z$ (labelled so that $U_i \cap U_{i-1} \neq \emptyset$)and let $L_0,\ldots,L_{p-1}$ be the closures in V of the connected components of $V \setminus \{0\}$.



Note that $n,p < 2k^2$. Since $f|U$ is continuous, we have that $fU_i \subset L_j$ for each i and some j. Choose a set $V^p = \{(r,\theta) \mid 0 \leq r < 1,\ \theta \in \{\theta_0,\ldots,\theta_{p-1}\}\}$ in polar coordinates with branches $V_j = \{(r,\theta_j) \mid 0 \leq r < 1\}$. Choose also a homeomorphism

$$\psi : (V;L_0,\ldots,L_{p-1}) \longrightarrow (B(0,1);V_0,\ldots,V_{p-1})$$

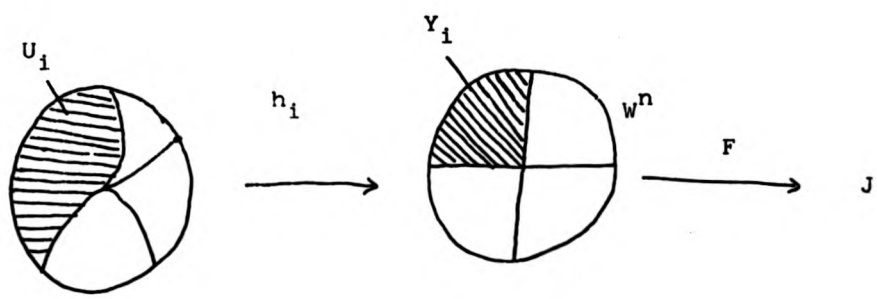
so that $\psi_j = \psi|L_j$ is the non-singular orthogonal projection of L_j onto V_j. Let $\tau : V^p \to J = [0,1)$ be the distance map $x \longmapsto |x|^2$ and let $\tau_j = \tau|V_j$.



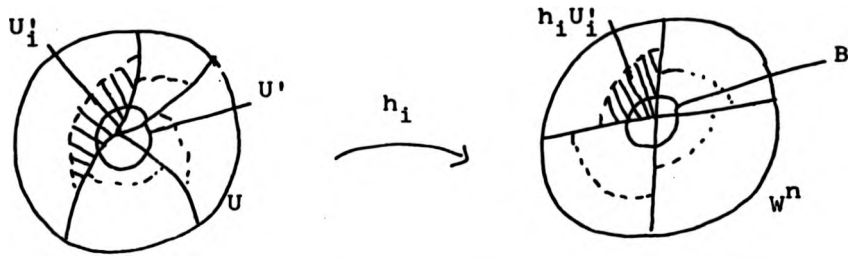
The map $\tau_j \psi_j f|U_i$ is the restriction to U_i of a polynomial mapping from $\mathbb{R}^2$ to $\mathbb{R}$, which is topologically equivalent on U_i to f. Hence for each i there is a set $U_i' \subset U_i$, a homeomorphism $h_i : U_i' \twoheadrightarrow Y_i$ (where Y_i is the closure in W^n of the ith 2-stratum of W^n), a homeomorphism $k_i : J \to J$, and a standard k-map $F ; W^n \to (-1,1)$, such that the diagram

$$\begin{tikzcd}
U_i' \arrow[r, "f/U_i'"] \arrow[d, "h_i"'] & L_j \arrow[r, "\psi_j"] & V_j \arrow[r, "\tau_j"] & J \arrow[d, "k_i"] \\
Y_i \arrow[rrr, "F/Y_i"'] & & & (-1,1)
\end{tikzcd}$$

commutes.



Now let B be an open ball about 0 in $\mathbb{R}^2$ contained in $\bigcup h_i U_i'$ and let U' be an open ball contained in $\bigcup h_i^{-1}(B \cap Y_i)$. We write U_i' for $h_i^{-1}(U_i' \cap B)$ and regard B as W^n. The other notation is retained with the obvious new meaning.



Define $\varphi_i : Y_i \to Y_i$, $i = 0,\ldots,n-1$, as follows: Let $\varphi_0 = id_{Y_0}$. For $i = 1,\ldots,n-1$, let φ_i be equal to $\varphi_{i-1} h_{i-1} h_i^{-1}$ on $Y_i \cap Y_{i-1}$, and let

$\varphi_{n-1} | Y_0 \cap Y_{n-1} = h_1 h_{n-1}^{-1} | Y_0 \cap Y_{n-1}$. Finally extend each φ_i to a homeomorphism of Y_i onto itself, and let $\Phi : W^n \to W^n$ be defined by $\Phi | Y_i = \varphi_i h_i | Y_i$. Then Φ is a homeomorphism of U' onto W^n and the diagram

$$\begin{tikzcd} U' \arrow[r, "f|U'"] \arrow[d, "\Phi"'] & fU' \arrow[d, "\eta"] \\ W^n \arrow[r, "\tilde{F}"'] & V^p \end{tikzcd}$$

commutes, where $\eta | L_i = \tau_i^{-1} k_i \tau_i \psi_i : L_i \to V_i$, and $\tau \tilde{F} | Y_i$ is a standard k-map. For elements of X, $n,p < 2k^2$, and there are only finitely many choices of standard $2k$-maps Hence the result.

(3.3.5) THEOREM 1. <u>For any</u> k , $P(2,2,k,\mathbb{R})/\overset{0}{\sim}$ <u>is a finite set</u>.

<u>Proof</u>. In (3.2.11) we proved the result for the maps which are not everywhere singular, and in (3.3.3) and (3.3.4) we gave the proof for the everywhere singular case.

(3.4) <u>Example</u>: $P(2,2,3,\mathbb{R})/\overset{0}{\sim}$ <u>is a finite set</u>.

(3.4.0) In this section we explicitly construct the topological types of polynomial mappings of degree < 3. We omit the case in which the maps are everywhere singular, as here our proof (relying upon [5]) is non-constructive. It was consideration of this type of example which led to the formulation of the proof given earlier in this chapter.

(3.4.1) Let $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map of degree less than 3, let $J(x) = \det Df_x$, let $S = J^{-1}0$, and suppose throughout that $S \neq \mathbb{R}^2$. If f and g have any linear terms, then either $J(0) \neq 0$, so that $F \overset{0}{\sim} id$, or $\operatorname{rank} Df_0 = 1$. We consider the latter case in (3.4.5).

(3.4.2) If S has an isolated point at 0, then F is light on a neighbourhood of 0, and is also open near 0 by (2.2.10), since J is non-negative or nonpositive there. Hence by (2.2.4) we have that $F \overset{0}{\sim} z \longmapsto z^n$, where $n \leqslant 4$. In this case F is locally topologically equivalent to the unique light orientation-preserving map in $W(n,1)$.

(3.4.3) Suppose that f and g (the components of F) are homogeneous of degree 2. After a change of coordinates, we can suppose that either

$$(1) \quad F = (x^2 + y^2,\ ax^2 + 2bxy + cy^2),$$

$$\text{or } (2) \quad F = (x^2 - y^2,\ ax^2 + 2bxy + cy^2).$$

This section concerns case (1).

Immediately we have that F is light near 0, since $f^{-1}0 = 0$.

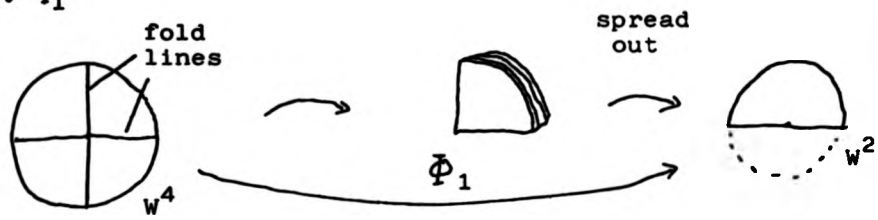
(i) Suppose that either a or c is non-zero. We can then assume that

$$F = (x^2 + y^2,\ x^2 + 2bxy + cy^2) \quad \text{and so}$$

$$J = 4(bx^2 + (c - 1)xy - by^2).$$

If $b \neq 0$, then S is either a point or two different lines. In the first case we have that $F \overset{0}{\sim} z \longmapsto z^n$ $(n \leqslant 4)$ by (2.2.4), and in the second F is equivalent to a collapse free standard map in $W(4,2)$ which has its

image contained in a single sector of W^2. We denote this map by Φ_1.



If $b = 0$ and $c = 1$, then F is everywhere singular. If $b = 0$ and $c \neq 1$, then S is given by $xy = 0$, and $F \overset{0}{\sim} \Phi_1$.

(ii) If $a = 0 = c$, we can perform the non-singular change of coordinates $(x,y) \longmapsto (x+y, x-y)$ to obtain a map of the type considered in case (2) below.

(3.4.4) This section deals with case (2), where we suppose that F has the form:

$$F = (x^2 - y^2,\ ax^2 + 2bxy + cy^2) .$$

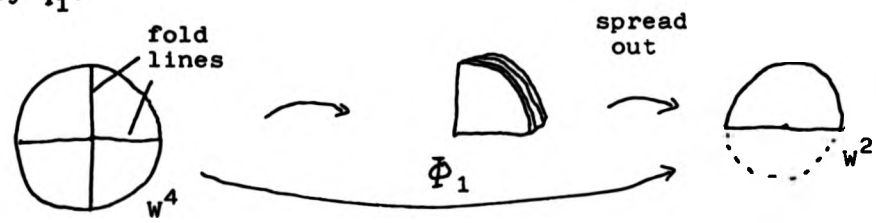
Note that collapses can occur in this case, since $f^{-1}0$ is no longer a point.

(i) Suppose that one of a and c is non-zero, so that we can assume

$$F = (x^2 - y^2,\ x^2 + 2bxy + cy^2) \quad \text{and}$$
$$J = 4(bx^2 + (c+1)xy + by^2) .$$

Here S is either a point or a line counted twice, and since J is always non-negative or nonpositive, if F is light then it is open, which implies that $F \overset{0}{\sim} z \longmapsto z^n$ $(n \leq 4)$. If S is a line and F is not light, we can

image contained in a single sector of W^2. We denote this map by Φ_1.



If $b = 0$ and $c = 1$, then F is everywhere singular. If $b = 0$ and $c \neq 1$, then S is given by $xy = 0$, and $F \overset{0}{\sim} \Phi_1$.

(ii) If $a = 0 = c$, we can perform the non-singular change of coordinates $(x,y) \longmapsto (x+y, x-y)$ to obtain a map of the type considered in case (2) below.

(3.4.4) This section deals with case (2), where we suppose that F has the form:

$$F = (x^2 - y^2,\ ax^2 + 2bxy + cy^2) .$$

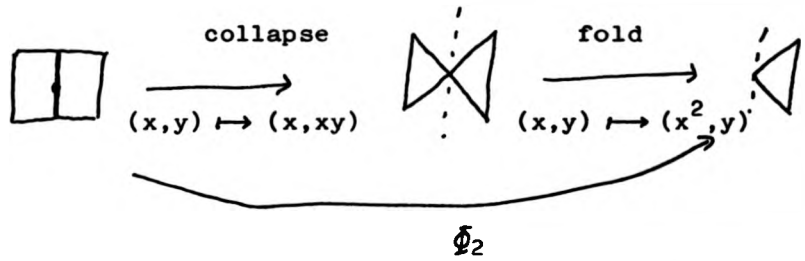
Note that collapses can occur in this case, since $f^{-1}0$ is no longer a point.

(i) Suppose that one of a and c is non-zero, so that we can assume

$$F = (x^2 - y^2,\ x^2 + 2bxy + cy^2) \quad \text{and}$$
$$J = 4(bx^2 + (c+1)xy + by^2) .$$

Here S is either a point or a line counted twice, and since J is always non-negative or nonpositive, if F is light then it is open, which implies that $F \overset{0}{\sim} z \longmapsto z^n$ $(n \leq 4)$. If S is a line and F is not light, we can

suppose that $F = (x^2 + y^2, (x + y)^2)$, or, after a change of coordinates, that $F = (x^2, xy)$. This map is a collapse along the y-axis followed by a fold about the y-axis. We call this map Φ_2 .



If $b = 0$, we have $F = (x^2 - y^2, x^2 + cy^2)$.

If $c = -1$, then F is singular everywhere.

If $c > 0$, we are in case (1).

If $c < 0$, $c \neq 1$, then $F \overset{0}{\sim} \Phi_1$.

(ii) If $a = 0 = c$ and $b \neq 0$, then S is just the origin, and F is light and open.

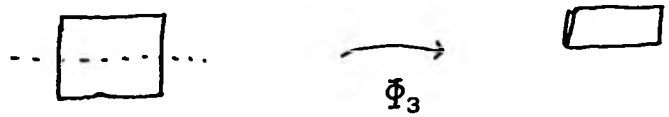
(3.4.5) We now consider the case in which F has rank 1 at 0 , i.e. we suppose that

$$F = (2x + ax^2 + 2bxy + cy^2, dx^2 + 2exy + hy^2) ,$$

$$J = 4(ex + hy + (ae-bd)x^2 + (ah-cd)xy + (bh-ce)y^2) .$$

(i) If $(e,h) \neq (0,0)$, then S is simple at 0 , and $g^{-1}0$ is either a point, two distinct lines, or a repeated

line. If S has no component in common with $g^{-1}0$, then F is light, and it is easy to check that either $F \overset{0}{\sim} \Phi_1$ or $F \overset{0}{\sim} \Phi_3$, the fold map $(x,y) \longmapsto (x,y^2)$.

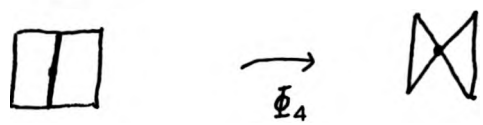


If F is not light, then $g^{-1}0 \cap f^{-1}0$ contains a line, so we can assume that $c = 0 = h$, and that

$$F = (x(2 + ax + 2by),\ x(dx + 2ey)),$$

$$J = 4x(e + 2by).$$

Since $e \neq 0$, S has only one component through 0, and $F \overset{0}{\sim} \Phi_4 = ((x.y) \longmapsto (x,xy))$.



(ii) If $(e,h) = (0,0)$, we can suppose that

$$F = (2x + 2bxy + cy^2,\ dx^2),$$

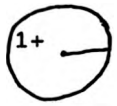
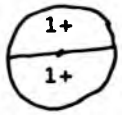
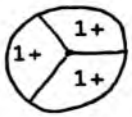
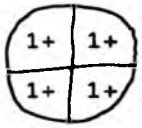
$$J = -4d(bx + cy)x.$$

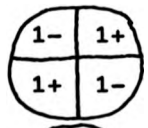
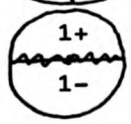
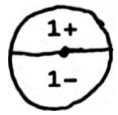
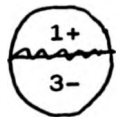
If $c \neq 0$, then $F \overset{0}{\sim} \Phi_1$.

If $c = 0$ and $b \neq 0$, then $F \overset{0}{\sim} \Phi_2$, the collapse-and-fold map.

(3.4.6) In order to describe the maps $\hat{\Phi}_i$, $i = 1,\ldots,4$ in terms of wheel maps, we introduce some new notation.

A standard map $g \in W(n,p)$ is represented by a single picture of the n-wheel, with a number and sign in each sector. The number in the ith sector indicates which sector of W^p it maps onto, and the sign is that of the determinant of the derivative. A 1-stratum of W^n has a wavy line drawn through it if it collapses. With this notation we have the following table:

Map	Equivalent wheel map	
$z \mapsto z$		$\in W(1,1)$
$z \mapsto z^2$		$\in W(2,1)$
$z \mapsto z^3$		$\in W(3,1)$
$z \mapsto z^4$		$\in W(4,1)$

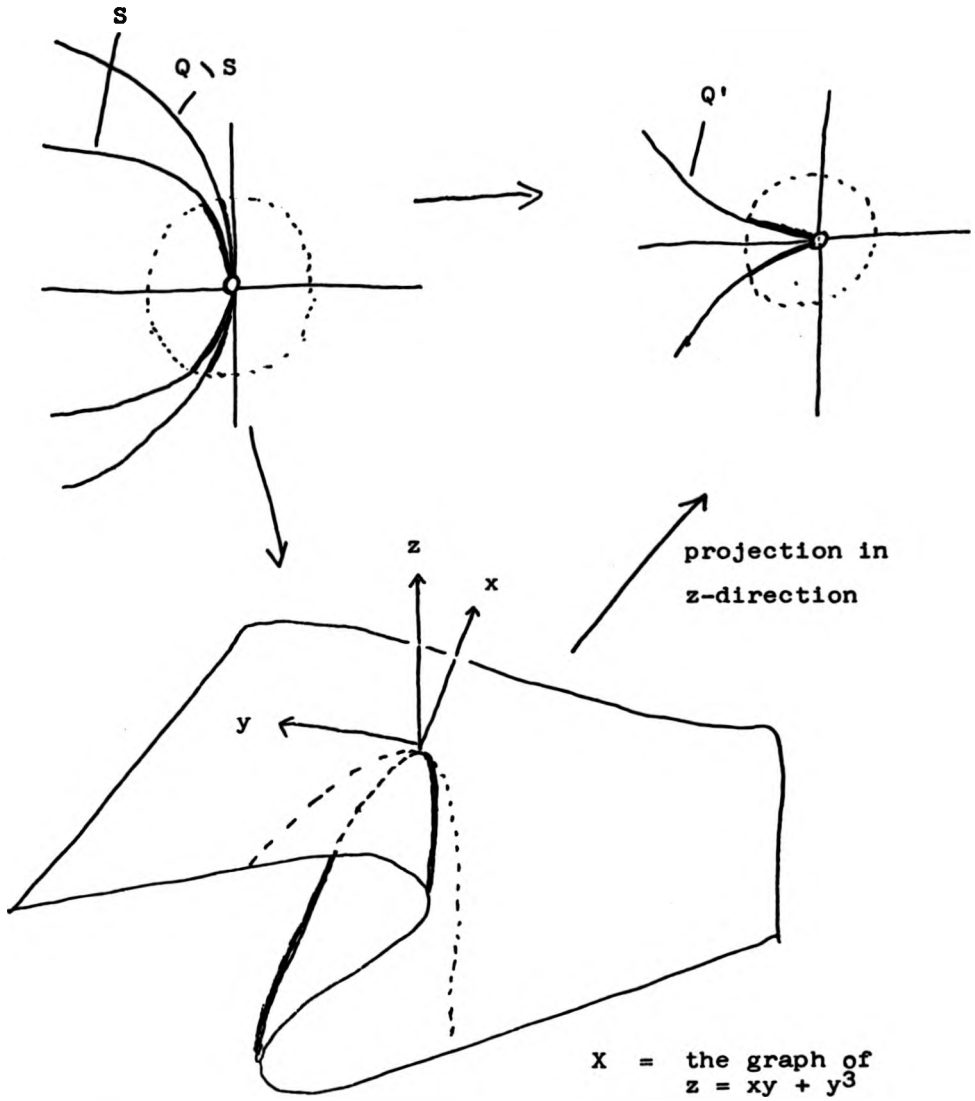
Map	Equivalent wheel map
Φ_1	 $\in W(4,2)$
Φ_2	 $\in W(2,2)$
Φ_3	 $\in W(2,2)$
Φ_4	 $\in W(2,4)$

(3.4.7) As a final example, consider the cusp map

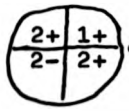
$$g : (x,y) \longmapsto (x, xy + y^3) .$$

(By the results of 26 , this map with the fold map Φ_2 gives the only local diffeomorphism types of 1-generic C^∞ mappings from the plane to the plane. The 1-generic maps are those maps f whose 1-jet is transverse to the set where f drops rank by r , for all r. See also [7, pp 145 - 149]).

In the picture below, we indicate the stratification of neighbourhoods of 0 (the 1-strata are marked with bold lines).



The singular set S maps onto the fold parabola in X. The other parabola lying over the cusp Q' is the image in X of $Q \setminus S$.

It is now easy to verify that $g \overset{0}{\sim}$ 

$\in W(4,2)$.

CHAPTER 4 - GLOBAL TOPOLOGICAL TYPES OF POLYNOMIAL MAPPINGS OF THE PLANE.

(4.0) Introduction.

The main result of this chapter is

THEOREM 2. The set

$\{f \in P(2,2,k,\mathbb{R}) \mid f$ is proper and $\det Df_x \neq 0\}/\sim$ is finite for each k.

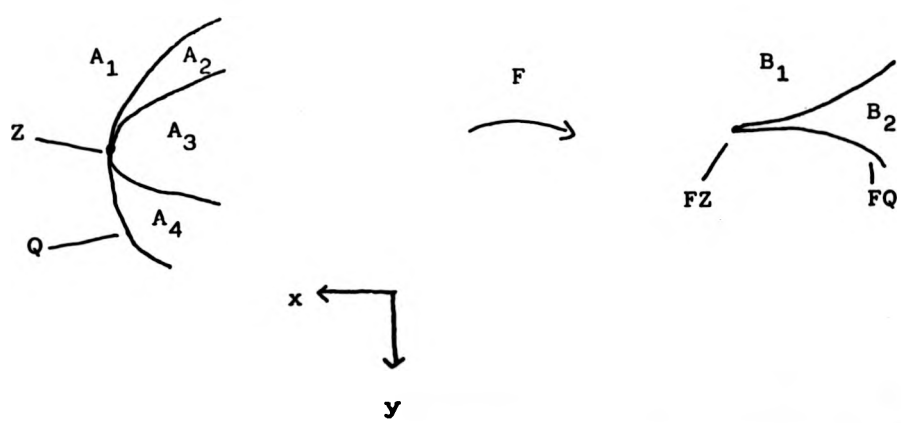
Section (4.1) contains an outline of the method of proof employed, and we also give a proof of the same result for $P(1,1,k,\mathbb{R})$, which is due to Fukuda [5]. The proof of Theorem 2 is followed by some examples, and a discussion of the hypotheses that f be proper and not singular everywhere, neither of which the author considers to be necessary. Relaxation of the 'proper' hypothesis means that the methods of this chapter are inapplicable, and has connections with the Jacobian Problem: see [1]. For the maps which are everywhere singular, but still proper, precise knowledge of the topology of the images, should, with the methods of this chapter, enable us to prove the result.

(4.1) Outline of the method of proof, and $P(1,1,k,\mathbb{R})/\sim$.

(4.1.1) The method of proof is essentially simple. Let $P(k) = \{f \in P(2,2,k,\mathbb{R}) \mid f$ is proper and $\det Df_x \neq 0\}$, and suppose that $f \in P(k)$. Let Q be the preimage of the image of the singular set. We show that if K is a connected component of $\mathbb{R}^2 \setminus Q$ and if L is a connected component of $\mathbb{R}^2 \setminus fQ$ with $fK \subset L$, then $f|K : K \twoheadrightarrow L$ is a covering map of degree less than k^2, and that there are only finitely many topological types for such maps. Turning to Q , we show that for a finite set $Z \subset Q$ (which is essentially the points where f has rank 0 and the points where Q is singular), such that $\#Z$ is bounded for all $f \in P(k)$, each connected component of $Q \setminus Z$ either maps down via a local homeomorphism, or collapses to a point. Counting the possible configurations of Q and fQ now enables us to complete the proof.

As an example, consider the cusp map discussed in (3.4.7):

$$F : (x,y) \longmapsto (x,\ xy + y^3).$$



The decomposition is made as indicated in the picture, with $Z = FZ = 0$. Note that A_1, A_2 and A_4 map diffeomorphically onto B_1, B_2 and B_2 respectively, with positive orientation, and that A_3 maps diffeomorphically onto B_2 with negative orientation. In this case the connected components of $Q \setminus Z$ map via diffeomorphisms of 1-manifolds onto connected components of $FQ \setminus FZ$: this is always the case if the latter components are simply-connected, and we only get non-trivial covering maps if some of these components are circles. There are more examples in section (4.3).

We now give an outline of the proof of Theorem 2 by sketching what happens in the 1-dimensional case.

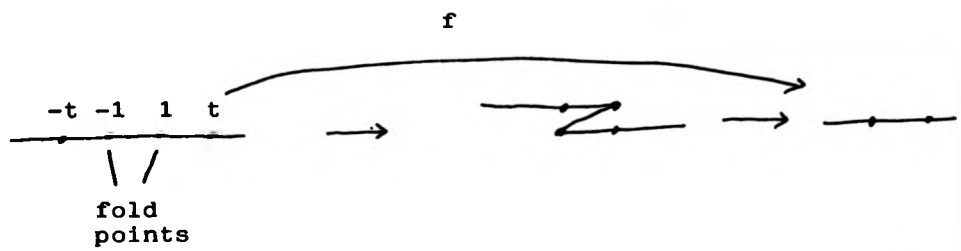
(4.1.2) THEOREM (<u>Fukuda</u> [5]).
<u>For each</u> k , <u>the set</u> $P(1,1,k,\mathbb{R})/\sim$ <u>is finite</u>.

<u>Sketch proof</u>. Let $\tilde{X}$ be the set of constant maps from $\mathbb{R}$ to $\mathbb{R}$, and let $X(k) = P(1,1,k,\mathbb{R}) \setminus \tilde{X}$. The constant maps contribute only one topological type, so it suffices to prove that $X(k)/\sim$ is finite. If $f \in X(k)$ then f is proper.(The 2-dimensional analogue of this fact is false). For $f \in X(k)$, let $S_f = \{x \in \mathbb{R} \mid f'(x) = 0\}$, $Q'_f = fS_f$, $Q_f = f^{-1}Q'_f$. Note that $\# S_f < k-1$, $\# Q_f < (k-1)k$ and that each connected component of $\mathbb{R} \setminus Q$ maps via a

diffeomorphism onto a connected component of $\mathbb{R} \setminus Q'$. Thus we need only know the order of the images of the points in Q to describe the topological type of the map, and the result follows.

<u>Example</u>.

Let $f(x) = x^3 - 3x$, and let $t > 0$ be such that $f(t) = 2$. Then $S = \{-1,1\}$, $Q = S \cup \{-t,t\}$ and the map looks like:



(4.2) <u>Proof of Theorem 2</u>.

(4.2.1) <u>Definition</u>. Throughout this section we shall suppose that

$$P(k) = \{f \in P(2,2,k,\mathbb{R}) \mid f \text{ is proper and } \det Df_x \not\equiv 0\}$$

and $$\tilde{P}(k) = \{f \in P(2,2,k,\mathbb{R}) \mid f \text{ is proper and } \det Df_x \equiv 0\}.$$

(4.2.2) LEMMA. <u>If</u> $f \in P(k)$ <u>and</u> $y \in \mathbb{R}^2$, <u>then</u> $\#f^{-1}y < k^2$ <u>or is infinite</u>.

<u>Proof</u>. This is the Corollary (2.5.3) to Bezout's Theorem.

(4.2.3) <u>Definition</u>. If $f \in P(k)$, let

$$S_f = \{x \in \mathbb{R}^2 \mid \det Df_x = 0\} ,$$

$Q'_f = fS_f$, $Q_f = f^{-1}Q'_f$. When no confusion can arise, we shall omit the subscripts from these and similar symbols.

(4.2.4) LEMMA. <u>If</u> $f \in P(k)$, <u>then</u> S_f, Q_f <u>and</u> Q'_f <u>are semialgebraic subsets of</u> $\mathbb{R}^2$ <u>of dimension at most</u> 1.

<u>Proof</u>. This Lemma follows from the Tarski-Seidenberg Theorem (2.1.2) and Sard's Theorem (2.2.6).

(4.2.5) <u>Definition</u>. If $f \in P(k)$, let

$$C'_f = \{y \in \mathbb{R}^2 \mid f^{-1}y \text{ is an infinite set}\} \subset Q'_f ,$$

and let $C_f = f^{-1}C'_f \subset Q_f$. We call C_f the <u>collapse set</u>.

Let Z'_f be the set of $y \in \mathbb{R}^2 \setminus C'_f$ such that

<u>either</u> y is a singular point of Q'

<u>or</u> $y = fx$ is simple on Q' and one of the following holds:

(i) $\operatorname{rank} Df_x = 0$;

(ii) x is a singular point of Q ;

(iii) x is a simple point of Q and $\ker Df_x \supset T_xS$.

(4.2.6) LEMMA. <u>If</u> $f \in P(k)$, <u>then</u> $\#C'_f < (k-1)^2$.

<u>Proof</u>. If $y \in C'$, then $f^{-1}y$ is an algebraic component of S , and these number less than $(k-1)^2$.

(4.2.7) LEMMA. <u>There exists an integer</u> $N = N(k)$ <u>such that if</u> $f \in P(k)$ <u>then</u> $\#Z_f$ <u>and</u> $\#Z_f'$ <u>are less than</u> N .

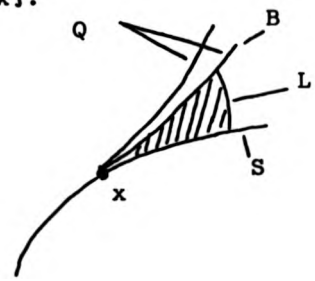
<u>Proof</u>. We merely have to check that the two sets are discrete and given by polynomials of bounded degree. Consequently we just give an outline of the proof.

Firstly note that S is an algebraic curve of degree less than $(k-1)^2$, so that there exists $N_1(k)$ such that if $f \in P(k)$ then S_f has less than N_1 singular points. If $x \in Q \setminus S$ is a singular point of Q, then fx is a singular point of Q' since f is a local diffeomorphism at x .

The rank 0 set in Z_f consists of isolated points because f is light on a neighbourhood of $Q \setminus C$, and so there exists an integer $N_2(k)$ such that for $f \in P(k)$, the rank 0 points in Z_f number less than N_2 .

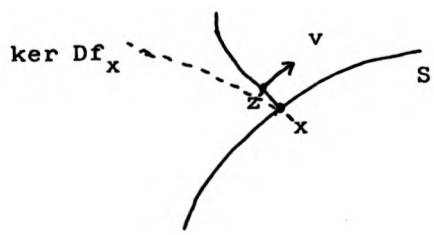
Suppose now that x is simple on S but singular on Q such that $y = fx$ is simple on Q', and $\ker Df_x \neq T_xS$. Then branches of Q near x must either have the same tangent at x as S , and map to branches of Q' at y , or have tangents in the direction of $\ker Df_x$. If a branch B of $Q \setminus S$ is the closest branch of Q to one of S , and $T_xB = T_xS$, let A be the closed region bounded by parts of B , S

and a line segment L as in the picture below, chosen so that $A \cap f^{-1}y = \{x\}$.



Then fA must be bounded by parts of Q' and fL, since at the preimage in A of any other boundary point, f is a local diffeomorphism. However, since fA contains points near fx other than points of Q', this is impossible.

Now suppose that B is a branch of $Q \setminus S$ at x such that $T_xB = \ker Df_x$. Choose a point z on B close to x and a vector v parallel to T_xS.



Then $(Df_z)(v)$ is close to $(Df_x)(v)$, i.e. close to the direction of T_yQ'. Remember that f is a local diffeomorphism at z, so that $(Df_z)(v)$ must be transverse to T_yQ'. This gives a contradiction.

We finally have to show that the set $\{x \in S \mid \ker Df_x = T_xS$, x simple on S , $fx = y$ simple on $Q'\}$ is discrete. This follows immediately from Sard's Theorem (2.2.6) applied to $f|S : S \to Q'$.

(4.2.8) <u>Definition</u>. If $f \in P(k)$, define

$K_f^2 = \{K \mid K$ is a connected component of $\mathbb{R}^2 \setminus Q_f\}$,

$L_f^2 = \{L \mid L$ is a connected component of $\mathbb{R}^2 \setminus Q_f'\}$,

$K_f^1 = \{K \mid K$ is a connected component of $Q_f \setminus (C_f \cup Z_f)\}$,

$L_f^1 = \{L \mid L$ is a connected component of $Q_f' \setminus (C_f' \cup Z_f')\}$.

(4.2.9) LEMMA. <u>If</u> f $P(k)$, <u>then</u> Z_f, Z_f' , Q_f,Q_f' <u>and the elements of</u> K_f^i, L_f^i <u>for</u> $i = 1,2$, <u>are semialgebraic sets</u>.

<u>Proof</u>. This Lemma follows from the Tarski-Seidenberg Theorem (2.1.2).

(4.2.10) LEMMA. <u>Let</u> $f \in P(k)$, <u>and let</u> $c : I \to \mathbb{R}^2 \setminus Q_f'$ <u>be a</u> C^1 <u>path such that</u> $c(0) = f(x_0)$ <u>for some</u> $x_0 \in \mathbb{R}^2$. <u>Then</u> c <u>can be lifted along its length from</u> x_0 , <u>that is, there exists a</u> C^1 <u>path</u> $\bar{c} : I \to \mathbb{R}^2 \setminus Q_f$ <u>with</u> $f \circ \bar{c} = c$ <u>and</u> $\bar{c}(0) = x_0$.

<u>Proof</u>. If $G' = cI$, then $G = f^{-1}G'$ is compact because f is proper. Let G_0 be the connected component of G

We finally have to show that the set $\{x \in S \mid \ker Df_x = T_xS\ ,\ \ x$ simple on $S\ ,\ fx = y$ simple on $Q'\}$ is discrete. This follows immediately from Sard's Theorem (2.2.6) applied to $f|S : S \to Q'$.

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<u>Proof</u>. If $G' = cI$, then $G = f^{-1}G'$ is compact because f is proper. Let G_0 be the connected component of G

containing x_0 , and suppose that $fG_0 = c[0,d]$, where $d < 1$. Since f is a local diffeomorphism at each point of $f^{-1}(c(d)) \cap G_0$, c can be lifted along its length.

(4.2.11) LEMMA. <u>If</u> $f \in P(k)$, <u>then</u> Im f <u>is a closed semialgebraic set with boundary contained in</u> Q_f' .

<u>Proof</u>. By the Tarski-Seidenberg Theorem (2.1.6), Im f is semialgebraic. If Im f is not closed, let J be a compact set crossing its frontier. Then $f^{-1}J$ is compact, contradicting the fact that Im $f \cap J$ is not closed.

(4.2.12) LEMMA. <u>If</u> $f \in P(k)$, $K \in K_f^2$, $L \in L_f^2$ <u>such that</u> $fK \subset L$, <u>then</u> $f|K : K \to L$ <u>is a covering map</u>.

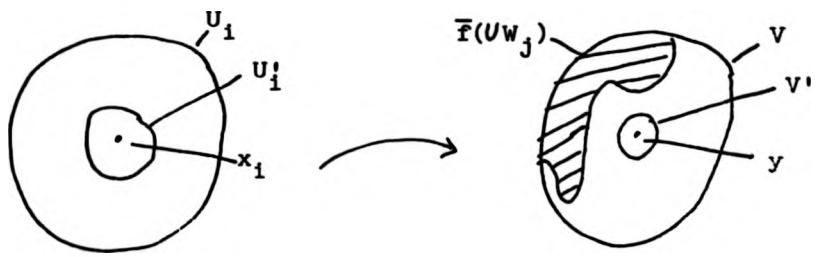
<u>Proof</u>. We have to show that each point of L has an open neighbourhood evenly covered by $\bar{f} = f|K$. Let $y \in L$, and let $\bar{f}^{-1}y = \{x_1,\ldots,x_n\}$, where $n < k^2$ by (4.2.2). Since $\bar{f}$ is a local diffeomorphism at each x_i , we can choose a closed disc neighbourhood V of y in L and closed neighbourhoods U_i of x_i in K (for $i = 1,..,n$) such that $\bar{f}$ maps each U_i diffeomorphically onto V . Each U_i is a connected component of $W = f^{-1}V$, for if not, let $x_i \in bU_i$ be such that every neighbourhood of x_i

meets $A_i' = A_i \setminus U_i$, where A_i is the connected component of W containing x_i . Because $\bar{f}$ is a local diffeomorphism at x_i , points of A_i' cannot be mapped into V. This is a contradiction.

Since W is semialgebraic, it has only finitely many connected components which are not equal to U_i for some $i \in \{1,\ldots,n\}$. Call these components $W_1,\ldots,W_r$. Since W is compact, each W_j $(j = 1,\ldots,r)$ is compact. Choose a closed disc $V' \subset V$ containing y such that

$$V' \cap \bar{f}\left(\bigcup_{j=1}^{r} W_j\right) = \emptyset .$$

Then $W' = \bar{f}^{-1}V'$ has connected components $U_1',\ldots,U_n'$, with $x_i \in U_i' \subset U_i$, such that $\bar{f}|U_i' \,;\, U_i' \twoheadrightarrow V'$ is a homeomorphism, and Int V' is evenly covered by $\bar{f}$.



(4.2.13) <u>Remark</u>. In the proof of (4.2.12) we rely heavily upon the fact that W has only finitely many components, and so those components which are not equal to some U_i map onto a compact set which misses y . If this does not happen, there is no guarantee that we can choose V' .

(4.2.14) LEMMA. <u>If</u> $f \in P(k)$, $K \in K_f^1$, $L \in L_f^1$ <u>are such that</u> $fK \subset L$, <u>then</u> $f|K : K \to L$ <u>is a covering map</u>.

<u>Proof</u>. Since K and L are C^1 1-manifolds, and for $x \in K$ we have that $\ker Df_x \neq T_xK$, we can deduce that $f|K : K \to L$ is a local diffeomorphism. The result is thus the 1-dimensional analogue of (4.2.12), and the proof is the same.

(4.2.15) LEMMA. <u>There exists an integer</u> $M = M(k)$ <u>such that if</u> $f \in P(k)$, <u>then the number of elements in each of the following sets is less than</u> M :

(i) <u>The set of connected components of</u> C_f.

(ii) C_f'.

(iii) Z_f, Z_f'.

(iv) K_f^i, L_f^i <u>for</u> $i = 1,2$.

<u>Proof (sketch)</u>. This result is a consequence of (4.2.6) and (4.2.7). The number of branches at each point of Q_f is bounded for all $f \in P(k)$, and so removal of Z_f leaves a bounded number of connected components. This proves that K_f^1 is bounded, and the other parts can be deduced similarly.

(4.2.16) <u>Definition</u>. Sets $A, B \subset \mathbb{R}^2$ are said to be <u>topologically equivalent</u>, written $A \overset{S}{\sim} B$ (S for set equivalence), if there exists a homeomorphism h of $\mathbb{R}^2$

with $hA = B$. Two pairs (A_i, B_i), $i = 1,2$, of sets are <u>topologically equivalent</u> if there exists a homeomorphism h of $\mathbb{R}^2$ with $hA_1 = A_2$, $hB_1 = B_2$.

This definition is stronger than that of homeomorphism with the subspace topology. For example, let

$$A = \{(x,y) \mid x^2 + y^2 < 1\} ,$$

$$B = A \setminus \{(x,y) \mid x = 0 , y \geqslant 0\} .$$

Then A and B are homeomorphic with the subspace topology, but not $\overset{S}{\sim}$ - equivalent.

(4.2.17) LEMMA. <u>The number of topological equivalence classes in each of the following sets (where</u> f <u>runs through</u> $P(k)$) <u>is finite</u>:

$\{(Q_f, Z_f)\}$, $\{(Q'_f, Z'_f)\}$, $\{(Q_f, C_f)\}$, $\{(Q'_f, C'_f)\}$, $\{(Q_f, K) \mid K \in K^1_f\}$, $\{(Q'_f, L) \mid L \in L^1_f\}$, $\{K \mid K \in K^2_f\}$, $\{L \mid L \in L^2_f\}$.

<u>Proof</u>. This Lemma follows from (4.2.15). For example, the set $\{(Q_f, K) \mid f \in P(k),\ K \in K^1_f\} / \overset{S}{\sim}$ is finite because Q is composed of a bounded number of 1-manifolds, joined at a bounded number of points.

(4.2.18) LEMMA. <u>If</u> $f \in P(k)$, <u>let</u> $K \in K_f^2$, $L \in L_f^2$ <u>be such that</u> $fK = L$. <u>Let</u> $\bar{f} = f|K$. <u>For any</u> $x \in K$, $y \in L$, <u>with</u> $fx = y$, <u>we have that</u> $\pi_1(K,x) = F_m$ (<u>the free group on</u> m <u>generators) and</u> $\pi_1(L,y) = F_n$, <u>where</u> $m,n < M(k)$ <u>as given in</u> (4.2.15). <u>Moreover we can choose generators of these groups such that if</u> a <u>is a generator of</u> $\pi_1(K,x)$, <u>then</u> $\bar{f}_* a = b^r$, <u>where</u> b <u>is a generator of</u> $\pi_1(L,y)$, <u>and</u> $0 \leq r < k^2$.

<u>Proof</u>. Let $\tilde{K} = \bar{K}/\sim$, where $\sim$ identifies each boundary component of $\bar{K}$ to a point, considered with the quotient topology, and let $K^* = \tilde{K} \setminus iK$, where $i : K \to \tilde{K}$ is the inclusion map. Let $\tilde{L}$ and L^* be defined analogously. If $K^* = m+1$, $L^* = n+1$, it is obvious that $\pi_1(K) = F_m$ and $\pi_1(L) = F_n$, and that $n,m < M$ by (4.2.15).

The map $\tilde{f} : \tilde{K} \to \tilde{L}$ induced by f is a well-defined, continuous, light open map between 2-manifolds, and so by (2.2.4) it is locally topologically equivalent at each point of K^* to a map of the form $z \mapsto z^d$, where $d < k^2$. Hence we can choose generators of $\pi_1(\tilde{K} \setminus K^*)$ which map via $\tilde{f}_*$ onto powers of generators of $\pi_1(\tilde{L} \setminus L^*)$, and the same is true of $\bar{f}_* : \pi_1(K) \to \pi_1(L)$.

(4.2.19) LEMMA. <u>Let</u> $F_m = \langle a_1,\ldots,a_m\rangle$, $F_n = \langle b_1,\ldots,b_n\rangle$ <u>be free groups, and let</u> Θ <u>be the set of homomorphisms</u> $\theta : F_m \to F_n$ <u>such that</u> $\theta(a_i) = b_{s(i)}^{r(i)}$, <u>where</u> $0 < r(i) < k^2$, $s(i) \in \{1,\ldots,n\}$, <u>for</u> $i = 1,\ldots,m$. <u>Then</u> $\{\theta F_m \mid \theta \in \Theta\}$ <u>is a finite set of subgroups of</u> F_n .

<u>Proof</u>. This is a simple counting argument.

(4.2.20) <u>Definition</u>. Let $g_i : A_i \twoheadrightarrow B_i$ be continuous maps, where $A_i, B_i \subset \mathbb{R}^2$, $i = 1,2$. We write $g_1 \overset{M}{\sim} g_2$ (M for map equivalence) if and only if there exist homeomorphisms $h : A_1 \twoheadrightarrow A_2$ and $k : (\mathbb{R}^2, B_1) \twoheadrightarrow (\mathbb{R}^2, B_2)$ such that $g_2 \circ h = k \circ g_1 : A_1 \to B_2$.

Note that if $g_1 \overset{M}{\sim} g_2$, then B_1 and B_2 are $\overset{S}{\sim}$ -equivalent, but A_1 and A_2 are just homeomorphic with the subspace topology from $\mathbb{R}^2$.

(4.2.21) LEMMA. <u>For each</u> k , <u>the set</u> $\{f|K : K \to L \mid f \in P(k),\ K \in K_f^2,\ L \in L_f^2,\ fK = L\} / \overset{M}{\sim}$ <u>is finite</u>.

<u>Proof</u>. By (4.2.17), K and L can exhibit only finitely many $\overset{S}{\sim}$ types as f runs through $P(k)$. For fixed topological type of L , by (4.2.18) and (4.2.19) we have that $(f|K)_\# \pi_1(K)$ is one of a finite number of subgroups

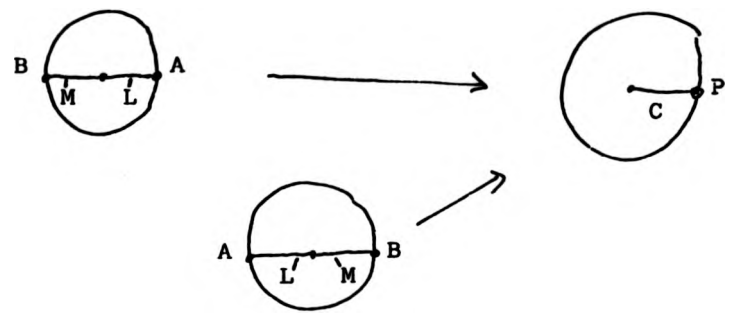
of $\pi_1(L)$. By (2.3.2) and (2.3.3), if two covering maps give rise to conjugate subgroups, then they are right topologically equivalent. This completes the proof.

(4.2.22) LEMMA. <u>For each</u> k , <u>the set</u> $\{f/K : K \to L \mid f \in P(k),\ K \in K^1_f,\ L \in L^1_f,\ fK = L\} / \overset{M}{\sim}$ <u>is finite</u>.

<u>Proof</u>. This Lemma is the 1-dimensional analogue of (4.2.21).

(4.2.23) <u>Remark</u>. We now have control of the types on the singular set, since connected components of $Q \setminus Z$ either behave as in(4.2.21) or collapse to points. We also know that on the connected components of the remainder, we just get covering maps. This final step checks that the maps on the components can be matched up in only finitely many ways. The rough idea of the proof is to take two maps which agree on everything checked so far, to carry out identifications via the corresponding homeomorphisms, and then to show that the matching-up introduces finitely many more types. We perform this process first on a single element of L^2, then use the fact that $\#L^2 < M$ to deduce the theorem.

<u>Example</u>. Suppose we know that a continuous map from the closed unit disc to itself behaves like $z \mapsto z^2$ on both the boundary and the interior of the disc. Then, having fixed the behaviour on the boundary, there are only two configurations which make sense: twisting the open disc through a half turn, or doing nothing.



In the picture, we know that the points A and B map onto P, and that the line segments M and L map onto C.

Think of the open disc 'clicking' into place.

(4.2.24) THEOREM 2. <u>For each</u> k, <u>the set</u> $\{f \in P(2,2,k,\mathbb{R}) \mid f$ <u>is proper and</u> $\det Df_x \neq 0\}/\sim$ <u>is finite</u>.

<u>Proof</u>. Suppose that $f_1, f_2 \in P(k)$ and there exist homeomorphisms α, β of $\mathbb{R}^2$ such that the diagram

$$\begin{tikzcd}
(Q_1, Z_1) \arrow[r, two heads, "f|Q_1"] \arrow[d, two heads, "\alpha|Q_1"'] & (Q_1', Z_1') \arrow[d, two heads, "\beta|Q_1'"] \\
(Q_2, Z_2) \arrow[r, two heads, "f|Q_2"'] & (Q_2', Z_2')
\end{tikzcd}$$

commutes, where $Q_i = Q_{f_i}$, etc.

Suppose that, for $i = 1,2$, we have that $K_i \in K_i^2$, $L_i \in L_i^2$, $f_i K_i = L_i$, and $\alpha K_1 = K_2$, $\beta L_1 = L_2$. Then $\beta \circ f_1 = f_2 \circ \alpha$ on $b\overline{K}_1$, so we can carry out identifications of $\overline{K}_1$ with $\overline{K}_2$ (which we call $\overline{K}$) and of $\overline{L}_1$ with $\overline{L}_2$ (which we call $\overline{L}$). We write $\overline{f}_1$, $\overline{f}_2$ for the induced maps, and continue to write K_i, L_i for the images of K_i, L_i under the identifications.

Suppose that $\overline{f}_{i\#}\pi_1(K)$ for $i = 1,2$ are conjugate subgroups of $\pi_i(L)$. Then by (2.3.2) there exists a homeomorphism $h : K \to K$ such that

$$\overline{f}_1|K = \overline{f}_2 \circ h : K \to L ,$$

where K and L are the interiors of $\overline{K}$ and $\overline{L}$.

Since $(\overline{K}_1, K_1)$ and $(\overline{K}_2, K_2)$ are topologically equivalent in the sense of (4.2.20), we can extend h by continuity to a homeomorphism $\overline{h} : \overline{K} \to \overline{K}$ such that

$$\overline{f}_1 = \overline{f}_2 \circ \overline{h} : \overline{K} \to \overline{L} .$$

(We need only check what happens on components of $b\overline{K}_i \setminus Z_i$ which are not in C_i : in the latter case any homeomorphism will do). By (4.2.22) there are only

finitely many such $\overset{M}{\sim}$ types, and so we can deduce the following:

If $f_1|K_1 \overset{M}{\sim} f_2|K_2$, then $\overline{f}_1|\overline{K}_1$ is in one of the equivalence classes in the set

$\{f|\overline{K} : \overline{K} \to \overline{L} \mid f \in P(k),\ K \in K_f^2,\ L \in L_f^2,\ fK = L\} / \overset{M}{\sim}$.

We now repeat this procedure on each element of $\{L \in L_f^2 \mid f \in P(k)\} / \overset{S}{\sim}$, and so deduce the Theorem.

(4.3) <u>Examples</u>.

(4.3.1) We now give a brief discussion of the global topological types of maps in $P(3)$. Since our proof does not explicitly construct the types exhibited, this section is necessarily less detailed than (3.4).

(4.3.2) Let $F = (f,g) : \mathbb{R}^2 \to \mathbb{R}^2$ be an element of $P(3)$. If F is homogeneous of degree 2, then the local types constructed in (3.4) are the global types. Note however that any such F cannot have a collapse, since its collapse set would be a union of lines, and so non-compact. Consequently the only global types for the homogeneous maps are:

(i) id ;

(ii) $z \mapsto z^n$ $(n \leq 4)$;

(iii) Φ_1 ;

where the latter two maps are considered to be from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(4.3.3) For the non-homogeneous maps in $P(3)$ the position is similar, since the zero-set of the Jacobian is a (possibly degenerate) conic. Considering F as given

in (3.2.5), it is clear that if $F \in P(3)$, then F cannot have any collapses, since $g^{-1}0$ is non-compact. In order to work out the global types here, it is necessary to follow through the proof given in (4.2), checking what can happen on Q, and calculating the type of covering map on each element of L_F^i, $i = 1,2$. While this process is not difficult, it is extremely tedious. Also, since no collapses can occur for maps in $P(3)$, it might appear that the proof provides for a bogus eventuality. To dispel this misapprehension, we now give an example of an element of $P(4)$ which has a genuine collapse.

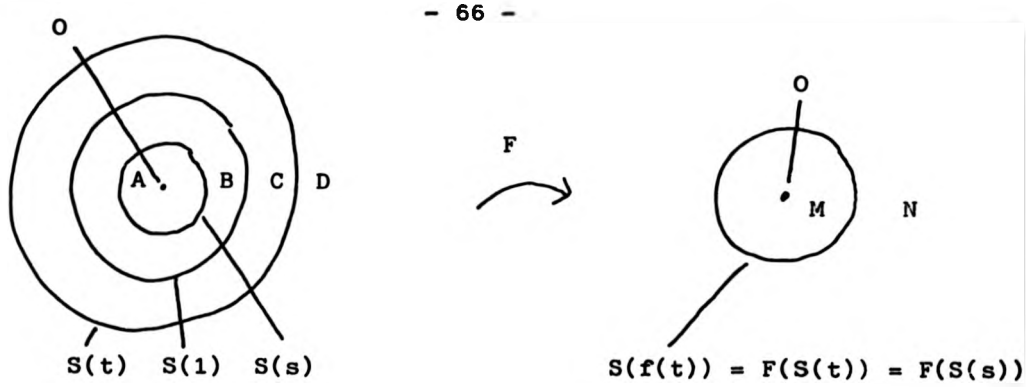
(4.3.4) Let $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (x(x^2+y^2-1), y(x^2+y^2-1))$, so that $J = (x^2 + y^2 - 1)(3(x^2 + y^2) - 1)$. It is clear that the circle determined by $x^2 + y^2 - 1$ collapses to $(0,0)$, but the map is proper by [12] because

$$\lim_{\|(x,y)\| \to \infty} \|F(x,y)\| = \infty .$$

The map is also rotationally symmetric about 0, so it suffices to look at its restriction to the x-axis.

Let $f : \mathbb{R} \to \mathbb{R} : x \longmapsto x^3 - x$. Then f has zeros at $0, \pm 1$, and turning points at $\pm s$, where $s = \sqrt{1/3}$. Let $t \neq s$ be such that $ft = fs$.

In the diagram below, $S(r)$ denotes the unit circle about 0 of radius r.



In the picture, $J^{-1}O = S(1)$ and $Q = S(s) \cup S(1) \cup S(t)$.

With the indicated stratifications, we can describe the topological type of F :

The collapse set is $S(1)$, which maps onto O .

The circles $S(s)$ and $S(t)$ map diffeomorphically onto $S(ft)$.

The regions A,C,D respectively map diffeomorphically onto M,M,N respectively with positive orientation.

The region B maps diffeomorphically onto M with negative orientation.

By composing the map F with $z \longmapsto z^n$ we can produce a similar map which is an n-fold cover on each of the strata except for the collapse set and O .

(4.4) <u>Remarks on the 'proper' hypothesis</u>.

(4.4.1) For the local result, considerations as to whether the maps are proper or not are irrelevant, since we are looking at small neighbourhoods. Consider, for example, the maps

$$f_1 : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (x^2 - x^2y^2,\ 2x^2y) \quad \text{and}$$

$$f_2 : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto ((x-1)((x-1)^2+y^2-1, y((x-1)^2+y^2-1)).$$

Here f_1 is the blowing-down map $(x,y) \longmapsto (x,xy)$ followed by $z \longmapsto z^2$, and f_2 is the example of (4.3.4) with co-ordinates centred at $(1,0)$. We have that $f_1 \overset{0}{\sim} f_2$, since both are locally topologically equivalent to

$\in W(2,2)$

in the notation of (3.4.6), but whereas f_2 is proper by (4.3.4), f_1 is not, because $f_1^{-1}0$ is the y-axis, a non-compact set. However, these apparent differences are not as important in the global case as might be expected. We consider this point next.

(4.4.2) As we saw in (4.1.2), every non-constant map in $P(1,1,k,\mathbb{R})$ is proper. While the corresponding statement for $P(2,2,k,\mathbb{R})$ is obviously false (as shown in (4.4.1)), several observations can be made.

In the proof of Theorem 2, the only places in

which we crucially use the fact that the maps in $P(k)$ are proper are those in which we deal with elements of K^i for $i = 1,2$; where we obtain a path-lifting result, and then deduce a covering map structure. Consequently, instead of proving Theorem 2 for $P(k)$, we could instead prove it for the set of $f \in P(2,2,k,\mathbb{R})$ with $\det Df_x \not\equiv 0$ and $f \mid \mathbb{R}^2 \setminus C_f : \mathbb{R}^2 \setminus C_f \longrightarrow \mathbb{R}^2 \setminus C_f'$ proper. Although this might appear to be a significant strengthening of the Theorem, it is not, since the hypotheses, which are difficult enough to verify for a particular map om the version given, now become extremely complicated to check. Ideally we would like to have simple algebraic hypotheses, but to illustrate the difficulties involved, we now turn our attention to the simplest case.

(4.4.3) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping such that for all $x \in \mathbb{R}^2$, $|\det Df_x| > 0$, i.e. suppose that f is a local diffeomorphism everywhere. One version of the Jacobian problem (see [1]) asks:

<u>Question 1. Under these hypotheses, is</u> f <u>a diffeomorphism of</u> $\mathbb{R}^2$ <u>onto</u> $\mathbb{R}^2$?

Now let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial mapping which is a local homeomorphism everywhere.

<u>Question 2. Under these hypotheses, is</u> g <u>a homeomorphism of</u> $\mathbb{R}^2$ <u>onto</u> $\mathbb{R}^2$?

(In both cases, the maps are onto by (4.2.11): there are no polynomial equivalents of the map which, in polar coordinates, sends (r,θ) onto $(\tan^{-1} r,\theta)$, or to put it another way, every locally proper polynomial local homeomorphism is proper).

We cannot give definitive answers to these questions, but we show they have affirmative answers on a large set of polynomial mappings.

(4.4.4) PROPOSITION. <u>Let</u> $F = (f,g) \in P(2,2,k,\mathbb{R})$ <u>be a polynomial mapping which is a local diffeomorphism everywhere. If the homogeneous terms of</u> f <u>and</u> g <u>of highest degree have no common non-trivial zeros, then</u> F <u>is proper and hence a diffeomorphism of</u> $\mathbb{R}^2$ <u>onto</u> $\mathbb{R}^2$.

<u>Proof</u>. Let $f = f_1 + \dots + f_n$, $g = g_1 + \dots + g_m$, where f_i and g_i are homogeneous of degree i, $f_n \neq 0 \neq g_m$, and n,m k. Consider the curve C in $\mathbb{R}^2$ determined by the polynomial $K = f^{2n} + g^{2m} - r^2$, where $r \neq 0$.

If C is compact for all r, then F is proper. Since by assumption f_n and g_m have no common non-trivial zeros, we can homogenize K as a map from $\mathbb{P}^2$ to $\mathbb{P}^1$, and the corresponding curve $\overline{C}$ in $\mathbb{P}^2$ has no points on the line at infinity. If C were unbounded in $\mathbb{R}^2$, $\overline{C}$ would have a non-isolated point on the line at infinity, so C is compact, F is proper, and the result follows.

(4.4.5) The same result, with the same proof, is true for maps which are local homeomorphisms everywhere. The Proposition can be improved with more careful analysis, but I have been unable to prove, nor find a counter-example to a version of (4.4.4) which has no conditions other than $|\det Df_x| > 0$.

(4.4.6) I still suspect that the set

$$\{f \in P(2,2,k,\mathbb{R}) \mid \det Df_x > 0\}/\sim$$

is finite for each k, although the methods of (4.2) are no longer applicable, since we can no longer ensure that paths lift along their lengths, or that the mappings behave like covering maps on each element of the stratification, i.e. in this case we can no longer prove the analogues of (4.2.10), (4.2.12) or (4.2.14).

(4.5) Remarks on the 'not everywhere singular' hypothesis.

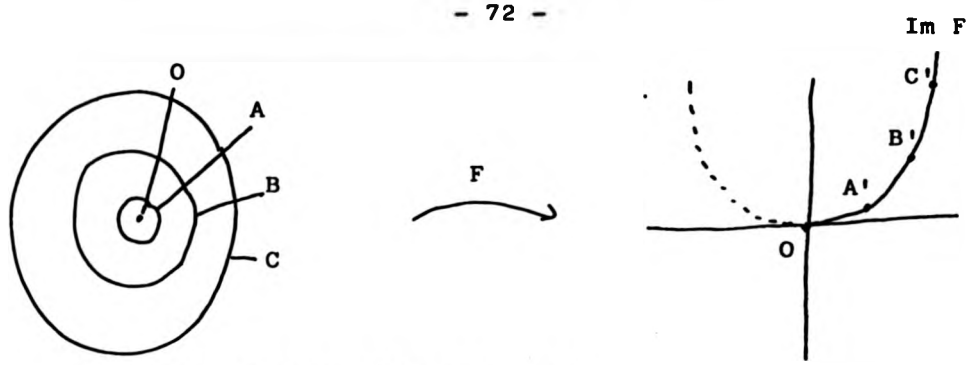
(4.5.1) In section (3.3) we proved the local result for the mappings in $P(2,2,k,\mathbb{R})$ which are everywhere singular by considering projections onto the inverse image of the origin, and it might be imagined that a similar method would yield a global result. Unfortunately, we do not have a firm enough hold on the topology of the image.

(4.5.2) Let $\tilde{P}(k) = \{f \in P(2,2,k,\mathbb{R}) \mid f$ is proper and everywhere singular$\}$. If $f \in \tilde{P}(k)$, then $\operatorname{Im} f$ is a semialgebraic set of dimension at most 1, by the theorems of Sard and Tarski-Seidenberg. In contrast to (4.2), we now have very little information, since the collapse set is the entire plane.

Before going any further, we show that $\tilde{P}(k)$ is non-empty.

(4.5.3) Examples.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (x^2 + y^2, (x^2 + y^2)^2)$, so that $\operatorname{rank} DF_x = 1$ except at 0. Think of F as the composition of $(x,y) \longmapsto (x^2 + y^2, 0)$ with the map $(x,y) \longmapsto (x,y^2)$, so that F maps each circle with centre 0 to a point on part of a parabola:



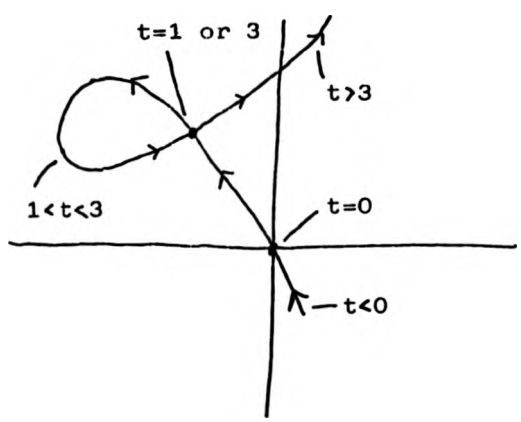
(Here A is mapped to A' , etc.)

It is clear that F is proper, and that its global topological type is the same as that of $(x,y) \longmapsto (x^2 + y^2, 0)$. In this example F has a particularly simple image, with no multiple points. We now give a more complicated example.

Consider the folium of Descartes determined by

$$(X - 3)^3 + (X - 3) - (Y + 6)^2 .$$

If we put $X = t^2 - 4t$ and $Y = t^3 - 6t^2 + 11t$, we parameterize the curve as shown in the picture below:



(the arrows on the curve are in the direction of increasing t)

The map

$G : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \longmapsto (g^2 - 4g,\ g^3 - 6g^2 + 11g)$,

where $g(x,y) = x^2 + y^2$, has as image that part of the curve parameterized by $t \geqslant 0$. The mappings F and G have the same local topological type at each point of $\mathbb{R}^2$, but obviously their global types differ.

(4.5.4) It is the presence of self-intersections of the image (i.e. points y at which $\operatorname{Im} f$ is not a C^0 1-manifold, but for all $x \in f^{-1}y$, the image of a sufficiently small neighbourhood of x in $\mathbb{R}^2$ is a 1-manifold, where $f \in \tilde{P}(k)$) which cause a difference in global types for maps which have the same local type everywhere, as we have just seen. If we could determine a bound on the number of these bad points for all $f \in \tilde{P}(k)$, then we ought to be able to show that $\tilde{P}(k)/\sim$ is a finite set for each k. A proof would go something like this:

Remove a set Z' from $\operatorname{Im} f$, where Z' consists of images of rank 0 points together with the singular points of the image, and let $Z = f^{-1}Z'$. The bound referred to above would limit the size of Z' and the number of

connected components of Z . The components of $\mathbb{R}^2 \setminus Z$ would map onto connected components of $\mathbb{R}^2 \setminus Z'$, and by choosing suitable projections, we should be able to relate the topological types on the components to elements of $P(2,1,k,\mathbb{R})$. An argument about matching-up should then show that $\tilde{P}(k)/\sim$ is a finite set.

CHAPTER 5 - APPLICATIONS.

(5.0) Introduction.

This final chapter contains applications of the results of Chapter 3 to jet spaces and compact 2-manifolds. Some of these results first appeared in the author's MSc dissertation [14]. Theorem 3 (5.1.9) was proved in greater generality by Varčenko [23], as given in (5.1.7), but our methods enable us to prove it very easily in this more limited case. Corresponding results are true, and the same proof is valid, for any $J(n,p)$ where $P(n,p,k,\mathbb{R})/\overset{0}{\sim}$ is finite for all k, but as mentioned in Chapter 1, the only case still undecided is the one in which n is arbitrary and $p = 2$.

Our notation for jet spaces, germs, etc., is more or less standard, see e.g. [11,25].

(5.1) Applications to jet spaces.

(5.1.1) Definition. Let $J(n,p)$ denote the set of germs at 0 of C^∞ mappings $(\mathbb{R}^n,0) \to (\mathbb{R}^p,0)$. We write $J^k(n,p)$ for the set of k - jets of these germs, and if $f \in J(n,p)$, we write $j^k f$ for its k - jet. Note that

with respect to a chosen basis, $J^k(n,p)$ can be identified with $P(n,p,k+1,\mathbb{R})$. Let

$$P_k : J(n,p) \to J^k(n,p) \quad \text{and}$$
$$P^r_s : J^r(n,p) \to J^s(n,p) \quad (r \geqslant s)$$

denote the obvious projections.

(5.1.2) <u>Definition</u>. Two germs $f, g \in J(n,p)$ are <u>topologically equivalent</u>, written $f \overset{\mathcal{T}}{\sim} g$, if there exist germs of homeomorphisms $h \in J(n,n)$, $h' \in J(p,p)$, such that $h' \circ f = g \circ h$. Note that if $f \overset{\mathcal{T}}{\sim} g$, then for any representatives $\hat{f}, \hat{g}$ of f, g, we have that $\hat{f} \overset{0}{\sim} \hat{g}$. Consequently we shall often write $f \overset{0}{\sim} g$, or even $f \overset{0}{\sim} \hat{g}$ when no confusion can arise.

Two k - jets $f, g \in J^k(n,p)$ are <u>topologically equivalent</u>, written $f \overset{\mathcal{T}}{\sim} g$, if there exist k - jets of homeomorphisms $h \in J^k(n,n)$, $h' \in J^k(p,p)$, such that $h' \circ f = g \circ h$. Again, if there can be confusion, and $f \overset{\mathcal{T}}{\sim} g$, we shall write $f \overset{0}{\sim} g$, and fail to distinguish between germs, jets and representatives.

(5.1.3) <u>Definition</u>. Germs $f, g \in J(n,p)$ are said to be $\mathcal{A}$ - <u>equivalent</u>, written $f \overset{\mathcal{A}}{\sim} g$, if there exist germs of diffeomorphisms $h \in J(n,n)$, $h' \in J(p,p)$ such that $h' \circ f = g \circ h$. We define $\mathcal{A}$ - equivalence of jets in a similar way. (The corresponding group $\mathcal{A}$, having as orbits the $\mathcal{A}$ - equivalence classes, is the right-left equivalence group considered by Mather in [11].

(5.1.4) <u>Definition</u>. A germ $f \in J(n,p)$ is said to be k-<u>determined relative to</u> $\mathcal{S}$ (written f is k-det rel $\mathcal{S}$), where $\mathcal{S} = \mathcal{A}$ or $\mathcal{J}$, if for any $g \in J(n,p)$ with $j^k f = j^k g$, we have that $f \overset{\mathcal{S}}{\sim} g$. A germ $f \in J(n,p)$ is <u>finitely determined relative to</u> $\mathcal{S}$ (written f is fd rel $\mathcal{S}$) if f is k-det rel $\mathcal{S}$ for some k.

It is clear that if f is k-det rel $\mathcal{A}$, then f is k-det rel $\mathcal{J}$.

(5.1.5) THEOREM (Mather [11]).

<u>For each positive integer</u> k <u>there is an algebraic set</u> $W^k \subset J^k(n,p)$ <u>with the following properties</u>:

(i) $(P_k^{k+1})^{-1} W^k \subset W^{k+1}$,

(ii) <u>the codimension of</u> W^k <u>in</u> $J^k(n,p)$ <u>tends to infinity as</u> k <u>tends to infinity</u>.

(iii) $f \in J(n,p)$ <u>is</u> fd rel $\mathcal{A}$ <u>if and only if</u> $j^k f \notin W^k$ <u>for some</u> k.

(5.1.6) COROLLARY. <u>If</u> $f \in J(n,p)$ <u>is</u> fd rel $\mathcal{A}$, <u>then there exists a neighbourhood</u> U <u>of</u> $j^k f$ <u>in</u> $J^k(n,p)$ <u>for some</u> k <u>such that if</u> $z \in U$ <u>then</u> z <u>is</u> fd rel $\mathcal{A}$.

<u>Proof</u>. Since f is fd rel $\mathcal{A}$, there exists k such that $j^k f \notin W^k$, by (5.1.5), and since W^k is algebraic, we can find a neighbourhood U of $j^k f$ which misses W^k.

(5.1.7) THEOREM (Varčenko [23, English version, p.1034]). <u>For</u> $k \geqslant 1$, <u>there exists a finite partition of</u> $J^k(n,p)$ <u>into disjoint semialgebraic sets</u>:

$$V^k_{-m(k)},\ldots,V^k_{-1},V^k_1,\ldots,V^k_{n(k)};$$

<u>with the following properties</u>:

(i) <u>if</u> $f_1,f_2 \in J(n,p)$ <u>are such that</u> $j^kf_1,\ j^kf_2 \in V^k_i$ <u>for some</u> $i > 0$, <u>then</u> $f_1 \overset{\mathcal{J}}{\sim} f_2$;

(ii) <u>for any</u> $f \in J(n,p)$ <u>such that</u> $j^kf \in V^k_i$ <u>for some</u> $i > 0$, <u>there exist triangulations of neighbourhoods of the origins in</u> $\mathbb{R}^n$ <u>and</u> $\mathbb{R}^p$ <u>with respect to which</u> f <u>is simplicial</u>;

(iii) <u>the codimension of</u> $V^k_0 = \bigcup_{i<0} V^k_i$ <u>in</u> $J^k(n,p)$ <u>tends to infinity as</u> k <u>tends to infinity</u>;

(iv) <u>for any</u> k <u>and</u> r <u>with</u> $k > r$, <u>and any</u> $i \neq 0$ <u>with</u> $-m(k) \leqslant i \leqslant n(k)$, $P^k_r(V^k_i)$ <u>is one of the sets</u> $V^r_{-m(r)},\ldots, V^r_{n(r)}$.

(5.1.8) <u>Remark</u>. Varčenko's Theorem does not imply anything about polynomial mappings exhibiting only finitely many topological types. The bad set, V^k_0 , is the set of k-jets which are not k-det rel $\mathcal{J}$, and although the codimension of this set tends to infinity as k tends to infinity, it is non-empty in each jet space. The proof of this theorem is very long and involved. We now state and prove a restriction of (5.1.7) to low dimensions, which with the results of the present work has a simple proof.

(5.1.9) THEOREM 3. *Let* $p = 1$ *or* $(n,p) = (2,2)$. *For* $k \geqslant 1$ *there exists a finite partition of* $J^k(n,p)$ *into disjoint sets* $W_0^k, \ldots, W_{q(k)}^k$ *with the following properties*:

(i) *if* $f_1, f_2 \in J(n,p)$ *are such that* $j^k f_1, j^k f_2 \in W_i^k$ *for some* $i > 0$, *then* $f_1 \overset{\mathcal{T}}{\sim} f_2$;

(ii) *if* $n = 2$ *and* $f \in J(n,p)$ *satisfies* $j^k f \in W_i^k$ *for some* $i > 0$, *then there exist triangulations of neighbourhoods of the origins in* $\mathbb{R}^n$ *and* $\mathbb{R}^p$ *with respect to which* f *is simplicial*;

(iii) *the codimension of* W_0^k *in* $J^k(n,p)$ *tends to infinity as* k *tends to infinity*.

(5.1.10) *Remark*. This theorem differs from (5.1.7) in several details, other than merely being a restriction. We do not prove that the partition is by semialgebraic sets. This would require showing that the orbits of the topological equivalence group on $J^k(n,p)$ are semi-algebraic. In our result, we lump together the k-jets which are not k -determined, because our proof uses the fact that if $j^k f$ is not fd rel $\mathcal{T}$, then f is not fd rel $\mathcal{A}$, with the infinite codimension result of (5.1.5). Hence we cannot deduce the finiteness of projection, part (iv) of (5.1.7).

(5.1.11) <u>Proof of Theorem 3</u>.

Let W_0^k be the W^k of (5.1.5). If $f \in J(n,p)$ is such that $j^k f \notin W_0^k$, then f is k-det rel $\mathcal{A}$, and hence k-det rel $\mathcal{J}$. Under an identification of $J^k(n,p)$ with $P(n,p,k+1,\mathbb{R})$ we have only finitely many local topological equivalence types by the results of Chapter 3, and so only finitely many $\mathcal{J}$- types. Hence the partition of $J^k(n,p) \setminus W_0^k$ by $\mathcal{J}$- equivalence types is finite, proving (i).

Part (ii) follows from the wheel map construction of (3.2), as this gives triangulations of the relevant neighbourhoods. We do not have this explicit construction for $p = 1$ and $n > 2$, and this is why the additional restriction is introduced for part (ii).

Part (iii) follows from (5.1.5(ii)).

(5.1.12) <u>Definition</u>. Let f $J(n,p)$ be fd rel $\mathcal{A}$. let V be a neighbourhood of $j^k f$ in $J^k(n,p)$, and let $r(V,k)$ be the number of $\mathcal{J}$- types of germs in V. Define $r(k) = \inf \{ r(V,k) \mid j^k f \in V \subset J^k(n,p) \}$. We say that f is <u>topologically simple</u> if $r(k)$ is bounded as k tends to infinity. (This definition is similar to that of <u>simplicity</u>, as applied to germs in $J(n,1)$. See, e.g. Arnol'd [2]).

(5.1.13) THEOREM 4. <u>If</u> $p = 1$, <u>or</u> $(n,p) = (2.2)$, <u>and</u> $f \in J(n,p)$ <u>is</u> fd rel $\mathcal{A}$, <u>then</u> f <u>is topologically simple</u>.

<u>Proof</u>. Let k and $U \subset J^k(n,p)$ be as in (5.1.6). Then if $s > k$, any neighbourhood U' of $j^s f$ in $J^s(n,p)$ satisfying $P_k^s U' = U$ contains no more $\mathcal{J}$ - types than U , since any element of U' is $\mathcal{A}$ - equivalent, and so $\mathcal{J}$ - equivalent, to an element of U . Hence $r(s) = r(k)$. By Theorem 3 (5.1.9), $r(k) < q(k)$, and the result is proved.

(5.3) Applications to compact 2-manifolds.

(5.3.1) In this section we state and prove a theorem about maps between compact 2-manifolds, showing that except for a set of infinite codimension, they behave like polynomial mappings. It was originally hoped that we could describe the global types in an analogous way to wheel maps, but this appears to be very difficult.

Our method is based upon Kurland and Robbin [8] and Tougeron [22, p 150] . Most of the proof of Theorem 5 (5.3.3) is due to Mather: we are merely extending it to obtain a finiteness result.

(5.3.2) Definition. We now define the notion of jet bundle. For further details see [7, p 39 et seq] .

Let X, Y be smooth (i.e. C^∞) manifolds with $p \in X$, and let $f, g \in C^\infty(X,Y)$ be such that $f(p) = g(p) = q$. We say that f and g have first order contact if

$$Df_p = Dg_p : T_pX \to T_qY .$$

We say that f and g have kth order contact at p , written $f \overset{k,p}{\sim} g$, if Df and Dg have $(k-1)$th order contact at every point of T_pX .

Let $J^k(X,Y)_{p,q} = \{f \in C^\infty(X,Y) \mid f(p) = q\} / \underset{\sim}{k,p}$,
then define

$$J^k(X,Y) = \coprod_{(p,q) \in X \times Y} J^k(X,Y)_{p,q} ,$$

the <u>kth jet bundle of</u> $C^\infty(X,Y)$.
(In general this is not a vector bundle, since there is no natural addition, unless for example $Y = \mathbb{R}^m$).

(5.3.3) THEOREM 5.

<u>Let</u> M <u>and</u> N <u>be</u> 2-<u>manifolds, with</u> M <u>compact, and let</u> P <u>be any manifold. Then there exists an open dense subset</u> G <u>of</u> $C^\infty(M \times P, N)$ <u>such that if</u> $F \in G$ <u>and</u> $p \in P$, <u>then</u>

$$F_p : M \to N : x \longmapsto F(x,p)$$

<u>has the following property</u>:

<u>There exists</u>

(i) <u>positive integers</u> n, k ;

(ii) points $x_1, \ldots, x_n \in M$;

(iii) <u>neighbourhoods</u> $U_1, \ldots, U_n$, <u>where</u> U_i <u>is a neighbourhood of</u> x_i <u>in</u> M , <u>which cover</u> M ;

(iv) <u>polynomial mappings</u> $g_1, \ldots, g_n \in P(2,2,k+1,\mathbb{R})$;

(v) <u>neighbourhoods</u> $V_1, \ldots, V_n$, <u>where</u> V_i <u>is a neighbourhood of</u> 0 <u>in</u> $\mathbb{R}^2$;

<u>such that</u> $F_p|U_i : U_i \longrightarrow F_p U_i$ <u>is topologically equivalent to</u> $g_i|V_i : V_i \longrightarrow g_i V_i$.

<u>Proof</u>. Choose an integer k so that the codimension of W^k in $J^k(2,2)$ is greater than $\dim P + 2$, where $W^k \subset J^k(2,2)$ is the set constructed in (5.1.5). Choose a finite cover of M (which we can do since M is compact) by coordinate charts (α, U') such that the open cover $\{U'\}$ contains a refinement $\{U\}$ (i.e. $U \subset U'$) which is a cover by relatively compact sets (i.e. $\overline{U}$ is compact). Then for each pair of charts (α, U), (β, V) we have a trivialization of the jet bundle $J^k(M,N) \to M \times N$:

$$J_{\alpha\beta} : U \times V \longrightarrow \alpha U \times \beta V \times J^k(2,2) .$$

Since $W^k \subset J^k(2,2)$, we can define

$$W^k = (J_{\alpha\beta})^{-1}(\alpha U \times \beta V \times W^k) ,$$

with $\overline{W}_{\alpha\beta}$, $W'_{\alpha\beta}$ defined similarly, by replacing U,V with $\overline{U},\overline{V}$ and U',V' respectively. Then define W (respectively $\overline{W}$, W') to be the union of the $W_{\alpha\beta}$ (respectively $\overline{W}_{\alpha\beta}$, $W'_{\alpha\beta}$).

For $F \in C^{\infty}(M \times P, N)$ define

$$F_k : M \times P \longrightarrow J^k(M,N) : (x,p) \longmapsto j^k F_p(x)$$

(i.e. the k-jet of F_p at x), and let G be the set of F with $F_k(M \times P) \cap \overline{W} = \emptyset$. The set G is open because $\overline{W}$ is closed, and the codimension of W' is greater than $\dim (M \times P)$, so that if F_k is transverse to W', it misses it. Hence $F_k(M \times P) \cap W' = \emptyset$ implies $F \in G$. The Thom Transversality Theorem (see [7, 2.4])

implies that G is dense. (This first part of the proof is essentially due to Mather).

Thus if $F \in G$, each F_p is k-det rel $\mathcal{A}$, and so k-det rel $\mathcal{J}$, at each point of M. This means that for $x \in M$ there exists an open neighbourhood of x in M such that $F_p|U : U \to F_pU$ is topologically equivalent to $g|V : V \to gV$, for some $g \in P(2,2,k+1,\mathbb{R})$ and some neighbourhood V of 0 in $\mathbb{R}^2$. Since M is compact we can refine this open cover $\{U\}$ of M to a finite cover $\{U_1,\ldots,U_n\}$, and so deduce the Theorem.

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SUMMARY.

We consider polynomial mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$, and show that those of degree less than any integer k exhibit finitely many local topological types at 0, and that the proper polynomial mappings of degree less than k which are not singular everywhere exhibit finitely many global topological types. We explicitly construct the local types for the mappings which are not singular everywhere. Applications to jet spaces and mappings between compact 2-manifolds are outlined.