DISTAL TRANSFORMATION GROUPS

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**A**

**ON THE STRUCTURE OF MINIMAL DISTAL TRANSFORMATION GROUPS WITH TOPOLOGICAL MANIFOLDS AS PHASE SPACES**

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References

References are given separately for each of the parts A, B, C of the thesis. For each part, the references are almost, but not quite, in alphabetical order.

A  ON THE STRUCTURE OF MINIMAL DISTAL TRANSFORMATION GROUPS WITH TOPOLOGICAL MANIFOLDS AS PHASE SPACES

I. Namioka, Right topological groups, distal flows and a fixed point theorem, Math. Systems Theory 6 (1972), 193 - 209.


ON THE FIBRES OF A MINIMAL DISTAL EXTENSION OF A TRANSFORMATION GROUP


NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A $C^1$ SKEN-PRODUCT


NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A $C^1$ SKEW-PRODUCT


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I should like to confine myself here mainly to thanking my supervisor W. Parry, in particular for his considerable help during the period of my transfer to his supervision, without which I could never have continued my studies, and for far more assistance and moral support than could be reasonably expected, throughout my time at Warwick.

Many other acknowledgements are due, for help with different parts of the thesis; the acknowledgements will be made at the start of the relevant parts. I am also grateful to anyone I have omitted to thank for assistance rendered.

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The thesis consists of three parts A, B, C, part A being the longest part. The objects of interest throughout are minimal distal transformation groups, in particular those for which the phase space is a compact topological manifold. Although many of the results obtained are true for a transformation group in which the group acting is an arbitrary topological group, there is an emphasis, particularly in the latter part of part A, on the groups of integers and of reals.

Part A is concerned mainly with a classification of those minimal distal transformation groups \((X, T)\) for which \(X\) is a compact manifold. A refinement of the Furstenburg Structure Theorem, for such a phase space \(X\), is proved, to show that there exists a finite sequence \(\{(X_i, T)\}_{i=0}^r\) of transformation groups with \((X_0, T)\) trivial, \((X_r, T) = (X, T)\), and \((X_{i+1}, T)\) an almost periodic extension of \((X_i, T)\), where \(r \leq \dim X\). An important step in the proof is the Addition Theorem: if \((W, T)\) is any minimal distal transformation group with factor \((Z, T)\) and factor map \(W\), then the covering dimension of the fibres \(W^{-1}(z)\) is constant, and:

\[
\dim Z + \dim W^{-1}(z) = \dim W \quad \text{for all } z \in Z.
\]

It is also shown in the course of the proof that the compact group associated with the extension \((X_{i+1}, T)\) is Lie, so that \(X_{i+1}\) is the total space of a fibre bundle with base \(X_i\), homogeneous fibre, and Lie structure group. One can then use fibre bundle theory to obtain information about the structure of \((X, T)\) when \(\dim X\) is low (i.e. \(\leq 3\)). The details of this are worked out in the second half of part A.

Since fibre results are thus available for \((X, T)\) with \(X\) a compact manifold, one might expect such transformation groups to be less pathological than in general, and part B gives an example of this:
using the Homotopy Covering Theorem, it is shown that any two fibres of a minimal distal transformation group over a factor are homeomorphic if the corresponding two points in the factor can be connected by a path (which implies all fibres are the same up to homeomorphism if, for example, the phase space of the transformation group is a compact manifold). But it is shown that two fibres need not be homeomorphic in general. In the example given, the phase space of the transformation group is a connected metric 3-dimensional space resembling a nilmanifold, and the phase space of the factor is a solenoid. Points in the solenoid, over which the fibres belong to a fixed homeomorphism class, lie in a nowhere dense set consisting of at most countably many pathwise-connected components of the solenoid.

The "classification" of part A is a topological one in a topological category, since the Furstenburg Structure Theorem deals with topological structure. A differentiable analogue of the Furstenburg Structure Theorem might be the following: if \((X', T)\) is minimal distal with \(X'\) a \(C^r\) manifold and \(T\) a group of \(C^r\) diffeomorphisms, then \((X', T)\) is topologically conjugate to \((X, T)\) (where \(X\) is a \(C^r\) manifold and \(T\) again a group of \(C^r\) diffeomorphisms) such that there exists a finite sequence \(\left\{ (X_i, T) \right\}_{i=0}^{\infty} \) of transformation groups with \((X_0, T)\) trivial, \((X_r, T) = (X, T)\), \(X_i\) a \(C^r\) manifold, \((X_{i+1}, T)\) an almost periodic extension of \((X_i, T)\), the factor map being \(C^r\), and the associated fibre bundle being \(C^r\).

(One sees at once that one has to allow topological conjugacy between \(X\) and \(X'\) by considering minimal diffeomorphisms of the circle, which are always topologically, but not necessarily \(C^1\), conjugate to rotations). Part C shows that the differentiable analogue suggested here does not hold. By part A, a minimal distal positively-oriented non-almost-periodic homeomorphism of the torus is
topologically conjugate to a topological skew-product of the form:

\[(x,y) \mapsto (x+\xi, y+g(x)),\]

where \(\xi\) is uniquely determined up to its sign. In part C, a minimal distal positively oriented analytic diffeomorphism of the torus is constructed which is not topologically conjugate to any \(C^1\) skew product, thus providing a counterexample to the suggested differentiable structure theorem. However, the construction, (which is similar to Arnold's construction of an analytic diffeomorphism of the circle which is not \(C^1\)-conjugate to a rotation) depends on the associated irrational, which has, among other things, to be Liouville. It is not clear what happens if, for example, the irrational is of bounded density, in which case, as Arnold conjectured, and Herman has shown, an analytic diffeomorphism with the irrational as rotation number is analytically conjugate to a rotation, so that Arnold's construction most certainly does not work.
ON THE STRUCTURE OF MINIMAL DISTAL TRANSFORMATION

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M. REPS

§1 Introduction and statement of the two basic theorems

The first purpose of this paper is to show how the Furstenberg Structure Theorem for minimal distal transformation groups [2], [3] can be refined when applied to a minimal distal transformation group \((X, T)\) for which \(X\) is a compact topological manifold. The refinement is given by the Manifold Structure Theorem 1.2, for which we need a result concerning the dimension of a factor of a minimal distal transformation group, namely the Addition Theorem 1.1.

§§2 - 7 are devoted to proving these two basic theorems - the actual proofs are given in §§6 - 7. The rest of the paper is devoted to examining, in some detail, what the structure theorem tells us in the case of connected manifolds of dimension \(\leq 3\); an explanation of how the structure theorem gives us some sort of classification of the transformation groups is given in §9, and the results are summarized there in tabular form, using the notation in the index of §9, which is a constant reference for the rest of the paper. Details of the results are worked out in §§10 - 13.

There is some overlap in this work with that of Bronstein [1] which will be discussed where it seems appropriate to do so.

I should like to thank my supervisor, Professor W. Parry, for considerable help, particularly in the preparation of this paper. This paper will be part of my Ph.D. thesis, and I should like to thank the S.R.C. for financial support.

We now proceed to the two basic theorems:

1.1 The Addition Theorem

Let \((X, T)\) be a minimal distal transformation group (4.1) and let \((Y, T) \leq \pi(X, T)\) (4.2). Then if "\(\text{dim}\)" denotes covering dimension, \(\text{dim} \pi^{-1}(y)\) is constant for \(y \in Y\) and:

\[
\text{dim} Y + \text{dim} \pi^{-1}(y) = \text{dim} X \quad (y \in Y),
\]
with the convention that \( n + \infty = \infty \) (\( n = \infty \) or \( n \) an integer).

1.2. The Manifold Structure Theorem

Let \((X,T)\) be a minimal distal transformation group (4.1) and let \(X\) be finite-dimensional with finitely many arcwise-connected components. (These hypotheses are automatically satisfied if \(X\) is a topological manifold.) Then the following conclusions hold:

(i1) If \((Y,T) \prec (X,T)\) then \(Y\) is a topological manifold (and, in particular, \(X\) is a manifold).

(i1) \((X,T)\) has order \(r\), where \(r \leq \max(1, \dim X)\) (4.10).

(iii) Let \((X_0,T)\) denote the trivial transformation group and let \((X_{i+1},T)\) denote the (unique up to isomorphism) maximal almost periodic extension of \((X_i,T)\) in \((X,T)\) (4.9). Then there exists a minimal distal transformation group \((Y_{i+1},T)\), a compact Lie group \(G_i\) and a closed subgroup \(H_i\) such that \(G_i\) acts freely and jointly continuously on \(Y_{i+1}\),

\[
(g,y)t = \gamma(y) \quad \text{for all } g \in G_i, \ y \in Y_{i+1}, t \in T,
\]

\[
\bigcap_{g \in G_i} g^{-1} Y_{i+1} = \{0\},
\]

and the following diagram is commutative for \(1 \leq i \leq r_i\):

\[
\text{Diagram 1.2(a)}
\]

so that \(S_i = (Y_{i+1}, Y_{i+1}/W_i, Y_i, X_i)\) is a fibre bundle (3.1) for \(1 \leq i \leq r_i\) and the \(X_i\)'s and \(Y_i\)'s are manifolds.

\[
\dim X_{i+1} > \dim X_i \text{ unless } \dim X = 0 \text{ (in which case } X \text{ is finite).}
\]

If \(\dim G_i/H_i = r_i\) then \(\dim G_i \leq r_i(r_i+1)/2\) by a result of [10].

(iv) \(G_i/H_i\) is connected for \(i > 2\) and \(G_i/H_i\) is connected if and only if \(X\) is connected.
(v) (A uniqueness property.) Let \((X, T) \cong (X', T)\).

Let \((X', T)\) denote the trivial transformation group and let \((X'_{+1}, T)\) be a maximal almost periodic extension of \((X'_1, T)\) in \((X', T)\), and let \((X'_1, T) \cong (X'_{+1}, T)\) so that (by 4.9) there exist T-isomorphisms \(\psi_i (0 \leq i \leq r)\) such that the following diagram is commutative:

**Diagram 1.2(b):**

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi_1} & X'_1 \\
\downarrow & & \downarrow \\
X'_{1-1} & \xrightarrow{\psi_{1-1}} & X'_{1-1}
\end{array}
\]

Let \(Y_i, G'_1, H_{1-i}, \xi'_i, \eta_i, \phi'_i, (1 \leq i \leq r)\) bear the same relation to \(X'_1, H'_1\) as \(Y_i, G_i, H_{1-i}, \xi_i, \eta_i, \phi_i, (1 \leq i \leq r)\) bear to \(X_i, H_i\) in (iii).

Then there exist T-isomorphisms \(\tau_i: (Y_i, T) \rightarrow (Y'_i, T)\), and topological group isomorphisms \(\phi'_i: G'_1 \rightarrow G'_1\) carrying \(H'_1\) onto \(H'_1\) such that

\[
\phi'_i(e, y) = \phi_i(e, \eta_i(y)) \text{ for all } y \in Y_i, e \in G'_1,
\]

and such that the following diagram commutes:

**Diagram 1.2(c):**

\[
\begin{array}{ccc}
Y_i & \xrightarrow{\tau_i} & Y'_i \\
\downarrow & & \downarrow \\
X_i & \xrightarrow{\phi_i} & X'_i \\
\downarrow & & \downarrow \\
X'_{1-1} & \xrightarrow{\phi_{1-1}} & X'_{1-1}
\end{array}
\]

In [1] Bresstein proved, among other things, a slightly different formulation of theorem 1.2(i)-(iii) with the hypothesis that \(X\) have finitely many arcwise-connected components replaced by the hypothesis that \(X\) be locally connected; neither of these conditions on \(X\) implies the other.
Bronstein seems to use in the proof the following: if \((Y,T) < (X,T)\) for \((X,T)\) minimal distal, then \(\dim Y < \dim X\) (which, of course, follows from 1.1), but this result does not seem to be stated in [1] as either a theorem or an assumption, which is part of our justification for duplicating some of Bronstein's work.

1.4 A similar theorem to 1.2 holds if the hypothesis that \(X\) have finitely many arcwise-connected components is omitted, and the hypothesis "\(T \in \mathcal{J}\)" is added, where:

\[ T \in \mathcal{J} \text{ if and only if there exists a compact } K \subseteq T \text{ such that every neighbourhood of } K \text{ generates } T. \]

Roughly speaking, the second version of (1.2) is obtained by replacing the words "manifold" and "Lie group", wherever they occur, by "finite-dimensional space" and "finite-dimensional group" respectively, and omitting all reference to fibre bundles. This second version of (1.2) will not be proved here.

§2. Preliminaries on Dimension Theory

It seems helpful to list here various properties of covering dimension which will be used subsequently, particularly in the proof of the Addition Theorem 1.1 (see §6).

Covering dimension is defined on the category of compact Hausdorff spaces [11], [12].

2.1 Covering dimension is a topological invariant.

2.2 If \(Y\) is a closed subset of \(X\), \(\dim Y \leq \dim X\).

2.3 For \(x \in X\), let \(\dim_x(X) = \inf \{\dim U : U\text{ is a closed neighbourhood of } x\}\). Then \(\dim X = \sup_{x \in X} \dim_x(X)\) ([11] 11.6-11.11).

2.4 \(\max (\dim X, \dim Y) < \dim X \times Y < \dim X + \dim Y\) ([11] 26.4).

2.5 \(\dim [0,1]^n = n\) ([12] Chapter IV).

From 2.5, 2.3, it follows that the covering dimension of a manifold is the same as the usual dimension.
2.6 If \( D \) is a partially ordered net and \( \left\{ \left( X_i, V_i, \alpha_i, \beta_i \right) \right\} \) is an inverse system of compact Hausdorff spaces with inverse limit \( (X, \left\{ V_i \right\}_{\alpha, \beta}) \), then
\[
\dim X = \operatorname{lim} \sup \dim X_i.
\]

§ 3 Preliminaries on Fibre Bundles

The relevance of fibre bundles to the study of minimal distal transformation groups follows, of course, from the Furstenberg Structure Theorem (4.7). The definitions given here are considerably less general than the customary ones, but are used for simplicity.

3.1 Definition \( \mathcal{G} = (Y, W, X, G, H, \tau, \psi) \) is a fibre bundle (or bundle) if:

(i) \( Y, W, X \) are compact Hausdorff spaces and \( \tau, \psi \) are continuous surjective maps.

(ii) \( \mathcal{G} \) is compact Lie, \( H \leq G \) is closed and \( \bigcap_{g \in G} g^{-1}Hg = \{ e \} \).

(iii) \( \mathcal{G} \) acts freely on the left of \( Y \), the action \( (g, y) \mapsto gy \) being jointly continuous.

(iv) The following diagram commutes:

Diagram 3.1

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & X/G \\
\downarrow{\tau} & & \downarrow{\cong} \\
W & \xrightarrow{\phi} & Y/H
\end{array}
\]

\( X \) is called the base of the bundle, \( \mathcal{G} \) the group of the bundle and \( H \) the isotropy subgroup.

If \( H \) is trivial, \( \mathcal{G} \) is a principal bundle, and we write \( \mathcal{G} = (Y, X, G, W) \).

3.2 The above definition of fibre bundle is essentially the same as that of [17] Chapter 1, §2, because of the following, which will be used in the proof of the Addition Theorem 1.1 (see [9] Theorem 1 in §5.4).

(1) If \( Y, W, X, G, H, \tau, \psi \) satisfy (1)-(iv) of 3.1, then for each \( y \in Y \),
If \( V(y) = x \), there exists a compact neighbourhood \( U \) of \( x \) and a continuous one-to-one map \( \phi : U \rightarrow Y \) such that \( \psi \circ \phi = \text{identity on } U \).

(ii) Let \( V = V^-(U) \). If \( \lambda : V \rightarrow G \) is defined by \( \lambda(v)^{-1} = v \), then \( \lambda(g.v) = g \cdot \lambda(v) \) for all \( g \in G, v \in V \), \( \lambda \) is continuous, and

\[ \forall x \lambda : V \rightarrow U \times G \text{ is a homeomorphism of } V \text{ onto } U \times G. \]

(iii) \( \exists : f(V) \rightarrow G/H = \{gH : g \in G\} \) is well defined by:

\[ \lambda(f(v)) = \lambda(v) \text{ (v \in V)}, \text{ and is continuous, and} \]

\[ \forall x \times \exists : f(V) \rightarrow U \times G/H \text{ is a homeomorphism of the neighbourhood } f(V) \text{ of } f(y) \text{ onto } U \times G/H. \]

3.3 Lemma If \( G = (Y,X,G,H,y,v) \) is a fibre bundle, then

\[ \dim W \leq \dim X + \dim G/H. \]

Proof: By 2.2 and 2.3, it suffices to show that given \( w \in Y \), there exists a closed neighbourhood \( V \) of \( w \) such that:

\[ \dim V \leq \dim W(V) + \dim G/H. \]

By 3.2, \( w \) has a neighbourhood \( V \) homeomorphic to \( W(V) \times G/H \), so that

\[ \dim V = \dim(W(V) \times G/H) \quad (2.1) \]

\[ \leq \dim W(V) + \dim G/H \quad (2.4). \]

3.4 We now define three different types of isomorphisms of fibre bundles. This may seem cumbersome, but for the justification see §9. Isot-isomorphism, essentially the type generally used in fibre bundle theory, is essentially the same as equivalence of bundles as in §17. Roughly speaking, isot-isomorphism is necessary because we shall usually regard the base space of a bundle as the phase space of a transformation group, and shall want to consider certain transformation-group-isomorphisms of it.

Definitions Let \( G = (Y,X,G,H,n,v) \) and \( G' = (Y',X',G',H',n',v') \) be two fibre bundles.

a) \( G \) and \( G' \) are isot-isomorphic under \( (\phi, \xi) \) (write \( (\phi, \xi) : G \rightarrow G' \)) if \( \xi \) is a topological group isomorphism of \( G \) onto \( G' \) carrying \( n \) onto \( n' \), and

\[ \phi : Y \rightarrow Y' \text{ is a homeomorphism satisfying:} \]
\[ \mathcal{F}(c,y) = \mathcal{G}(c) \mathcal{F}(y) \text{ for all } y \in Y, c \in G. \]

Note that \( \mathcal{G} \) induces homeomorphisms of \( W \) onto \( W' \) and \( X \) onto \( X' \) (\( \mathcal{U}_1, \mathcal{U}_2, \mathcal{W} \) may) such that the following diagram commutes:

**Diagram 3.4**

\[
\begin{array}{ccc}
Y & \xrightarrow{\mathcal{G}} & Y' \\
\downarrow & & \downarrow \\
W & \xrightarrow{\mathcal{U}} & W' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\mathcal{W}} & X'
\end{array}
\]

b) \( \mathcal{G} \) and \( \mathcal{G}' \) are \( 2nd \)-isomorphic under \((\mathcal{G}', \alpha)\) if \( \mathcal{G} \xrightarrow{\alpha} \mathcal{G}' \) is a \( 3rd \)-isomorphism, \( X = X' \), and the map \( \mathcal{G}_2 \) in Diagram 3.4 is in the identity.

c) \( \mathcal{G} \) and \( \mathcal{G}' \) are \( 1st \)-isomorphic under \( \mathcal{G} \) if \( \mathcal{G} = \mathcal{G}' \), \( R = R' \), \( X = X' \) and \((\mathcal{G}, 1) : \mathcal{G} \rightarrow \mathcal{G}' \) is a \( 2nd \)-isomorphism, where 1 denotes the identity isomorphism.

**3.5 Definition** The product bundle with base \( X \), group \( G \) and isotropy subgroup \( H \) is the bundle \((X \times G/R, X, G, H, \mathcal{G}, \mathcal{F})\), where the action of \( G \) on \( X \times G \) is given by:

\[ g \cdot (x, g') = (x, gg') \text{ for all } x \in X, g, g' \in G. \]

\[ \mathcal{U}(x, Hg) = x, \quad \mathcal{V}(x, g) = x, \quad \mathcal{F}(x, g) = (x, Hg). \]

**3.6 Theorem** (See [17] 11.6) Any bundle with base \([0,1]^I\) (where \( I \) is any index set) is \( 1st \)-isomorphic to a product bundle.

**4.4 Preliminaries on Transformation Groups**

**4.1 Definition** Throughout this work, we shall be considering transformation groups \((t.g.'s)\) where the phase space \( X \) is compact Hausdorff and \( T \) is an arbitrary topological group acting on \( X \) (on the right) such that the map \((x,t) \rightarrow xt\) is jointly continuous.

**4.2 Definition** If \((X,T)\) is a factor of \((Y,T)\) and \( F : (Y,T) \rightarrow (X,T)\) is the factor homomorphism, write \((X,T) \triangleleft (Y,T)\). (The suffix \( \triangleright \) will frequently be omitted.)

**4.3 Definition** Given a \( t.g. \) \((X,T)\), write \( R(X) \) for the envelopping semigroup.
of $X$. $E(X)$ is a compact Hausdorff space when given the topology $\mathcal{J}_p$ of pointwise convergence. Write $(E(X),T)$ for the canonical t.g. with phase space $E(X)$ and group $T$ (§2 Chapter 3).

4.4 Let $(X,T)$ be a minimal distal t.g. A reference for the following is [13]. (Note that [13] deals with left, rather than right, t.g.’s.)
(a) For any $x \in X$, the map $\Pi_x: (E(X),T) \to (X,T)$ is a $T$-homomorphism onto $(X,T)$, where $\Pi_x(p) = xp$.
(b) $(E(X),\mathcal{J}_p)$ is a group in which the following maps are continuous:
\[ p \mapsto qp \quad (p, q \in E(X)) \]
\[ p \mapsto pt \quad (t \in \text{the image of } T \text{ in } E(X), p \in E(X)) \]
(c) Let $\sigma$ be the weakest topology on $E(X)$ making the map $\psi$ continuous, where $\psi: (E(X) \times E(X), \mathcal{J}_p \times \mathcal{J}_p) \to E(X)$ is given by $\psi(p,q) = pq^{-1}$. Then $\sigma \subset \mathcal{J}_p$.
(d) If $H$ is a subgroup of $E(X)$, $(E(X)/H, \mathcal{J}_p)$ is Hausdorff if and only if $H$ is $\sigma$-closed, where $E(X)/H = \{ \hat{p} : p \in E(X) \}$.

Define $(H \Pi)t = H(\Pi t)$ ($t \in T$).

Then $(E(X)/H, T) \prec (E(X), T)$, where $\Pi(p) = Hp$.
(e) If $(T,T) \prec (E(X), T)$, then if $e$ is the identity of $E(X)$, let $H = \sigma^{-1}(e)$. Then $H$ is a $\sigma$-closed subgroup of $E(X)$ and the following diagram commutes:

\[
\begin{array}{ccc}
(E(X),T) & \overset{p}{\longrightarrow} & (X,T) \\
\downarrow & & \downarrow \\
(E(X)/H,T) & \overset{\Pi}{\longrightarrow} & (E(X)/H,T)
\end{array}
\]

(*) $E(X)$ can be identified with the group of $T$-isomorphisms of $(E(X),T)$. For consider the map $p \mapsto L_p$ where $L_p(q) = pq$ ($q \in E(X)$).

(g) Similarly, the group of $T$-isomorphisms of $(E(X)/H,T)$ can be identified with $L/H$, where $L = \{ p \in E(X) : pH = Hp \}$ (so $L$ is $\sigma$-closed).

(h) For a $\sigma$-closed $H \leq E(X)$, define $\text{alg}(H) = \{ f \circ C(E(X)) : L_p f = f \}$ (see (f)) so that $\text{alg}(H)$ is a $T$-invariant (i.e. $tf \in \text{alg}(H)$ for all $f \in \text{alg}(H)$, $t \in T$, where $tf(p) = f(tp)$) $C^*$-subalgebra of $C(E(X))$. 

For a \( T \)-invariant \( C^* \)-subalgebra \( \mathcal{A} \) of \( C(E(X)) \), define
\[ \text{gp}(\mathcal{A}) = \{ p \in E(X) : L_p f = f \text{ for all } f \in \mathcal{A} \} \]. Then \( \text{gp}(\mathcal{A}) \) is a \( g \)-closed subgroup of \( E(X) \).

We have \( \text{alg}(\text{gp}(\mathcal{A})) = \mathcal{A} \) and \( \text{gp}(\text{alg}(H)) = H \) (use Urysohn's lemma and the Stone-Weierstrass theorem).

### 4.5 Definition
A minimal t.g. \( (\mathcal{Y}, T) \) is a quotient-group-extension of \( (X, T) \) if there exists a compact topological group \( G \) with closed subgroup \( H \) such that \( \bigcap_{g \in G} g^{-1} H = \{ e \} \), and a minimal t.g. \( (Y, T) \) such that \( G \) acts freely on the left of \( Y \), the action \( (g, y) \mapsto gy \) being jointly continuous, \( (gy)t = g(yt) \) for all \( g \in G, y \in Y, t \in T \), and such that the following diagram commutes:

\[ \begin{array}{ccc}
Y & \xrightarrow{\varphi} & X \ (\cong Y/H) \\
\downarrow f & & \downarrow \pi \\
W \ (\cong Y/H) & \xrightarrow{\varphi} & \end{array} \]

In this diagram and all subsequent diagrams, if the objects in the diagram are phase spaces of t.g.'s with respect to a group \( T \), and the arrows denote \( T \)-homomorphisms.

We also say \( (W, T) \) is a \( G/H \)-extension of \( (X, T) \). If \( G \) is Lie, finite etc., we say \( (W, T) \) is a quotient-Lie-group-extension etc. of \( (X, T) \). If \( H \) is trivial, we say \( (W, T) \) is a group-extension of \( (X, T) \)

Note that if \( G \) is Lie, \( (Y_0, W_0, X_0, G, E, \pi, \varphi, W_0) \) is a fibre bundle (3.1) for any cloned \( X_0 \subset X \) with \( Y_0 = \varphi^{-1}(X_0) \) and \( W_0 = \pi^{-1}(X_0) \).

### 4.6 Let \( (X, T) \prec (\mathcal{Y}, T) \) with \( (\mathcal{Y}, T) \) minimal. The following are equivalent conditions for \( (\mathcal{Y}, T) \) to be an almost periodic (a.p.) extension of \( (X, T) \) ([2],[3]):

1. Given an index \( \mathcal{E} \) on \( \mathcal{Y} \), there exists an index \( \mathcal{E} = \mathcal{E}(\mathcal{E}) \) on \( \mathcal{Y} \) such that \((m_{\mathcal{E}} \in \mathcal{E} \) and \( (m_{\mathcal{E}}(w_1) = m_\mathcal{E}(w_2)) \) imply \( ((w_1, t, w_2) \in \mathcal{E} \text{ for all } t \in T)\).
2. \( (\mathcal{Y}, T) \) is a quotient-group-extension of \( (X, T) \).
For (iii) and (iv), we make the additional assumption that \((\mathcal{W}, T)\) is distal, and choose \(\sigma\)-closed subgroups \(H, G\), of \(\Sigma(\mathcal{W})\) (4.3) such that the following diagram commutes (see 4.4(e)):

**Diagram 4.6**

\[
\begin{array}{ccc}
\mathcal{E}(\mathcal{W}), T & \xrightarrow{H_p} & (\mathcal{W}, T) \\
\downarrow & & \downarrow \\
\mathcal{E}(\mathcal{W})/H, T & \xleftarrow{G_p} & (\mathcal{W}/G, T) \\
\end{array}
\]

(iii) \(N(0) \subseteq H\), where \(N(0)\) is the intersection of the \(\sigma\)-closed \(\sigma\)-neighbourhood of the identity in \(G\) (5.3) is a group).

(iv) \((G/H, \sigma) = (G/H, \sigma')\).

### 4.7

We shall use the following formulation of the Furstenburg Structure Theorem (see [2] Chapter 15, and [3] for the elimination of the assumption of quasiparability):

**Theorem**

(a) Let \((X, T)\) be a minimal distal t.s. Let \((Y, T) \not\leq (X, T)\). Then there exists \((Z, T)\) with \((Y, T) \leq (Z, T) \leq (X, T)\) such that \((Z, T)\) is an a.p. extension of \((Y, T)\).

(b) If \((Y, T) \leq (X, T)\), then by transfinite induction on \(a\), there exists an ordinal \(\omega + \alpha\) satisfying:

1. \(\{ (X_\beta, T) : \alpha \leq \beta \leq \omega \} \) is a system satisfying:

   \[ (X_\beta, T) \leq (X_{\beta+1}, T) \]

(iii) \((X_\omega, T) = (Y, T), (X_\alpha, T) = (X_\beta, T)\), for all \(\alpha \leq \beta \leq \omega\).

(iv) \((X_{\beta+1}, T)\) is a proper a.p. extension of \((X_\beta, T)\) for \(\beta < \omega\).

(v) \(\beta\) is a limit ordinal, \((X_\beta, T)\) is the inverse limit of \(\{(X_\gamma, T)\}_{\gamma < \beta}\).

### 4.8

In 4.7(b), (iv) can be replaced by:

(iv') \((X_{\beta+1}, T)\) is a proper quotient-Lie-group-extension of \((X_\beta, T)\).
This will be proved in 5.1-5.3. It was shown by Brunstein in [1]. However, a slight error in the proof led to the conclusion that one could assume that $(X_{\beta+1}, T)$ was a $G_{\beta+1}/H_{\beta+1}$-extension of $(X_{\beta}, T)$, where $G_{\beta+1}$ was either a connected Lie group or finite. This is not true: for example, if $T$ is an arbitrary group, and $(X, T)$ is a minimal distal t.g., and $(Y, T)$ is the trivial t.g., then it is not possible to choose $(X_{\beta}, T)^{0}_{\beta+1}$ such that all the groups $G_{\beta+1}$ are connected Lie or finite. We omit the details.

4.9 Given a minimal t.g. $(X, T)$, there is a natural correspondence between factors of $(X, T)$ and $T$-invariant $C^*$-subalgebras of $C(X)$, and any two factors associated with the same subalgebra are isomorphic [2].

If $(X, T)$ is minimal and $(Y, T) < (X, T)$, then there exists $(Z, T)$ such that $(Y, T) < (Z, T) < (X, T)$ ($X = H$). $(Z, T)$ is an a.p. extension of $(Y, T)$, and the subalgebra of $C(X)$ corresponding to $(Z, T)$ is at least as large as that corresponding to any other a.p. extension of $(Y, T)$ in $C(X)$. $(Z, T)$ is called the maximal almost periodic extension of $(Y, T)$ in $(X, T)$ [2].

4.10 Definition Let $(X, T)$ be minimal distal, and $(Y, T)$ the trivial factor. If, in the transfinite induction procedure of 4.70, we take $(X_{\beta+1}, T)$ to be the maximal a.p. extension of $(X_{\beta}, T)$ in $(X, T)$, then we obtain the smallest ordinal $\alpha$ for which there exists a system $\{(X_{\beta}, T): C \subseteq \beta < \alpha: \eta_{\beta}: 0 \rightarrow \beta \rightarrow 0 \alpha\}$ satisfying (i)-(v) of 4.70. This $\alpha$ is called the order of $(X, T)$.

4.5 On Quotient-Group-Extensions

In this section, various results on quotient-group-extensions (see 4.5 for definition) are collected together. 5.1-5.3 contain the proof of the modified Furstenberg Structure Theorem (4.5). The main result is 5.5, which concerns the "uniqueness" of a group-extension associated with a given quotient-group-extension.

5.1 Lemma Let $G$ be a compact topological group, and $H$ a closed subgroup. Let $N_1 \triangleleft G$ with $G/N_1$ Lie and $HN_1 \neq H$. Then there exists $N \triangleleft G$ with $N \triangleleft N_1$,
Proof Choose $x \in (G/\mathcal{N}) \cap \mathcal{H}_1$, and let $\phi$ be a finite-dimensional representation of $G$ such that $\phi(x) \neq \phi(h)$ for any $h \in H$. ($\phi$ exists by Urysohn's lemma and the Peter-Weyl Theorem [15] Section 33). Put $H_2 = H_1 \cap \ker \phi$, and put $N = \bigcap_{g \in G} g^{-1} H N_2 g$.

5.2 Lemma Let $(X,T)$ be a minimal distal t.g. and let $(Y,T) \subsetneq (X,T)$. Then there exists $(Z,T)$ with $(Y,T) \subsetneq (Z,T) \subsetneq (X,T)$ and $(Z,T)$ a quotient-Lie-group-extension of $(Y,T)$.

Proof By 4.7(1) we can assume $(X,T)$ is a $G/\mathcal{N}$-extension of $(Y,T)$ for some compact topological group $G$. By 5.1 (with $N_1 = G$) we can find $N \subset G$ with $G/N$ Lie.

$G/\mathcal{N} \subsetneq N$ and $H \neq G$. Then $\bigcap_{g \in G/N} g^{-1} \mathcal{H} \mathcal{N} g^{-1} = \{1\}$, and we have the following commutative diagram:

Diagram 5.2

So $(Z/H \mathcal{N}, T)$ is a quotient-Lie-group-extension of $(Y,T)$, and $(Y,T) \subsetneq (Z/H \mathcal{N}, T) \subsetneq (X,T)$.

5.3 Let $(X,T)$ be a minimal distal t.g., and $(Y,T) \subsetneq (X,T)$. By using 5.2 to obtain a quotient-Lie-group-extension $(X_{n+1}, T)$ of $(Y,T)$, find by transfinite induction a system $\{(X_{\beta_n}, T)\}_{\beta \in \alpha}$ satisfying (i), (ii), (iii), (v) of 4.7(6) and (iv) of 4.8. Hence 4.8 is proved.

5.4 It follows from 5.1 that if $(X,T)$ is minimal, and a finite a.p. extension of $(Y,T)$, then $(X,T)$ is a quotient-finite-group-extension of $(Y,T)$, hence a covering of $(Y,T)$.

5.5 The following proposition holds without the assumption that the $(X_{\beta_n}, T)$ $(\beta = 1, 2)$ be distal, but the proof of this will not be given here.
Proposition: Let \((Z_i, τ_i)\) be minimal distal \(i = 1, 2\) and suppose we have the following commutative diagram:

Diagram 5.5a)

where \(G_1\) is (as usual) a compact topological group acting freely and continuously on \(Z_1\), and \(H_1\) is a closed subgroup with \(\bigcap_{g \in G_1} g^{-1}H_1g = \{e\}\).

Then there exists a \(T\)-isomorphism \(\Phi: (Z_1, τ_1) \rightarrow (Z_2, τ_2)\) and a topological group isomorphism \(\xi: \mathcal{G}_1 \rightarrow G_2\) carrying \(H_1\) onto \(H_2\) such that

\[\Phi(gz) = \xi(g) \Phi(z)\]

for all \(z \in Z_1\) and \(g \in G_1\), and such that diagram 5.5a) remains commutative when the arrow \(Z_1 \rightarrow Z_2\) is inserted.

Proof 1. Define \(\xi_1: E(Z_1) \rightarrow E(Y_1)\) as follows (see 4.3):

For \(p \in E(Z_1)\) and \(g \in G_1\), define \(\xi_1(p)(z) = \Phi(zp)\), whenever \(\Phi(z) = g\).

Then \(\xi_1\) is well-defined. To show \(\xi_1\) is one-to-one:

Let \(p, q \in E(Z_1)\) and suppose \(\xi_1(p) = \xi_1(q)\). Then \(\xi_1(p)(z) = \xi_1(q)(z)\) for all \(z \in Z_1\).

Fix \(z \in Z_1\). For each \(g \in G_1\), there exists \(h \in H_1\) such that \(g\xi_1(p) = h\xi_1(q)\) (because \(\Phi(gzp) = \Phi(gzq)\)).

i.e. \(zp = (g^{-1}h)zq\), i.e. \(zp = k\zeta\), where \(k \in \bigcap_{g \in G_1} g^{-1}H_1g = \{e\}\).

i.e. \(zp = zp\), and hence, since \(z\) is arbitrary, \(p = \zeta\).

\(\xi_1: E(Z_1) \rightarrow E(Y_1)\) is a \(T\)-isomorphism and (clearly) a group isomorphism.

2. Let \((Y_1, \mathcal{Y}_1, \mathcal{P})\) denote the semigroup of (not necessarily continuous) maps from \(Y_1\) to \(Y_1\), with the topology \(\mathcal{J}_P\) of pointwise convergence. Consider the map \(Y_1 \rightarrow Y_2 Y_2\) given by \(h \mapsto y hy^{-1}\). The restriction \(\Phi\) of this map to \(E(Y_1)\) is a \(T\)-isomorphism and group isomorphism onto \(E(Y_2)\). By 4.4(3), it is
possible to find $G_1$, $H_1$ of $E(Y_1)$, and $T$-homomorphisms $T_1$ such that the following diagram commutes:

\[ \begin{array}{ccc}
E(Y_1) & \xrightarrow{T_1} & E(Y_2) \\
\downarrow & & \downarrow \\
E(Y_1)/H_1 & \xrightarrow{T_1 \circ \pi} & E(Y_2)/H_2 \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\pi} & G_2
\end{array} \]

3. Let $T_1(e) = y_1$ (the identity of $E(Y_1)$) and choose $z_1 \in Z_1$ such that $f_1(z_1) = y_1$. Now define $\sigma_1 : E(Z_1) \to Z_1$ by $\sigma_1(p) = z_1 p$ ($p \in E(Z_1)$).

Then $\tilde{f}_1 \circ \sigma_1 = T_1 \circ \tilde{g}_1$.

Then 4 implies the existence of a $T$-isomorphism $\tilde{\varphi}_1$ and $K_1$ (a $G$-closed subgroup of $E(Y_1)$) such that the following diagram commutes:

\[ \begin{array}{ccc}
E(Y_1) & \xrightarrow{\text{identity}} & E(Y_1) \\
\downarrow & & \downarrow \\
E(Y_1)/K_1 & \xrightarrow{T_1 \circ \pi} & E(Y_2)/K_2 \\
\downarrow & & \downarrow \\
G_1 & \xrightarrow{\pi} & G_2
\end{array} \]

4. Let $L_1 = \{ p \in G_1 : pK_1 = K_1 p \}$. Then $(L_1/K_1, p)$ is a group and a compact
A Hausdorff space, and identifies with the group of T-isomorphisms of E(Y_1)/K_1 which transposes leave C(E(Y_1)/G_1) invariant, with the topology of pointwise convergence (4.46g) and (h)).

Minimality of (Z^,T) implies that G_1 identifies with the group of T-isomorphisms of Z_1 whose transposes leave C(Z_1) invariant. Hence there exists \( \beta_1 : (L_1/K_1, \lambda) \rightarrow G_1 \) (a group isomorphism and homeomorphism, so that \((L_1/K_1, \lambda)_p\) is, in fact, a topological group, and \(\beta_1\) is a topological group isomorphism) such that:

\[ \Psi_1(K_1 pq) = \beta_1(K_1 p) \Psi_1(K_1 q) \quad \text{for all } p \in L_1, \ q \in E(Y_1). \]

Since \( C(Z_1) \) is the fixed algebra of \( G_1 \), \( C(E(Y_1)/G_1) \) must be the fixed algebra of \( L_1/K_1 \), and hence \( L_1 = G_1 \) by 4.46g).

1. e. \( K_1 \subseteq G_1 \).

Since \( \beta_1(K_1/K_1^1) = L_1 \), we have \( K_1 = \bigcap_{g \in G_1} \beta^{-1}(g) \).

5. We have \( \Psi : (E(Y_1), \lambda_p) \rightarrow (E(Y_2), \lambda_p) \) is a group isomorphism and homeomorphism, where \( \Psi(G_1) = G_2 \), \( \Psi(K_1) = E_2 \).

Hence, since \( K_1 = \bigcap_{g \in G_1} \beta^{-1}(g) \), \( \Psi(K_1) = K_1 \).

Then \( \Psi \) induces a T-isomorphism \( \Phi' : (E(Y_1)/K_1', T) \rightarrow (E(Y_2)/K_2', T) \)

and a topological group isomorphism \( \gamma : (G_1/K_1', \lambda_p) \rightarrow (G_2/K_2^1, \lambda_p) \)

such that \( \Phi'(K_1' p q) = \gamma(K_1 p) \Phi'(K_1' q) \) for all \( p \in G_1, \ q \in E(Y_1) \).

Then define \( \gamma \) by:

\[ \gamma : G_1 \rightarrow G_2, \quad \gamma = \beta_2^{-1} \beta_1^{-1} \]

\( \Phi \) and \( \gamma \) have the required properties.

Corollary Let \((X,T) < (Z,T)\), where \((Y,T)\) is a \(G/E\)-extension of \((X,T)\)

and \((Y,T)\) in minimal distal. Let \((X,T) < (W,T) < (Y,T)\). Then there exists a closed subgroup \( L \) of \( G \), \( H \subseteq L \subseteq G \), such that the following diagram commutes:
Diagram 5.6

Proof. By Proposition 5.5 and 4.6, we can assume \((Y,T) = (E(Y)/H^*,T)\) and
\((X,T) = (E(Y)/G^*,T)\) where \(G^*, H^*\) are \(G\)-closed subgroups of \(E(T)\) with
\(H(G^*) \leq H^* \leq G^*\) (see 4.4 and 4.6), \(G = (G^*/K^*,\varphi) = (G^*/K^*,\varphi')\), and
\(R = (H^*/K^*,\varphi') = (H^*/N',\varphi')\), where \(N' = \bigcap_{g \in G^*} g^{-1} H^* g\).

In this case, \((W,T) = (E(Y)/L^*,T)\) for some \(L', H^* \leq L' \leq G^*\) (4.4), and we can
take \(L = (L'/N',\varphi) = (L'/M*,\varphi)\).

5.7 The following proposition will be needed in the proof of the Manifold
Structure Theorem 1.9:

Proposition Let \((W,T)\) be minimal distal and \((X,T) \rightarrow (Y,T) \rightarrow (W,T)\) (5.5) with
\((Y,T)\) is a quotient-Lie-group-extension of \((X,T)\) and \((W,Y)\) is an a.p.
\(N\)-extension of \((Y,T)\). (i.e., \(\Pi^{-1}_N(y)\) has \(N\) elements for one, hence all, \(y \in Y\)).

Then \((W,T)\) is a quotient-Lie-group-extension of \((X,T)\).

Proof. By 4.6, 5.5 and repeated application of 5.1, it suffices to prove \((W,T)\)
is an a.p. extension of \((X,T)\).

Use the following standard notation: for an index \(z\) on a uniform space
\(Z\), let \(B(z) = \{z^2 : z^2 \in \mathbb{Z}\} \cap [z]\).

The proof is analogous to that of theorem 3 in [16].

\(Y\) is a \(G/H\)-extension of \(X\), say, where \(G\) is compact Lie. Choose any open
\(W_0 \leq W\) such that \(\Pi_1|W_0\) is a homeomorphism (5.4) and such that there exists a
homeomorphism of the form:

\[\Pi_2 \times \lambda : \Pi_1(W_0) \rightarrow \Pi(W_0) \times U\] where \(U\) is open in \(G/H\) (3.2(iii)).

Write \(f = (\Pi_2 \times \lambda)^{-1} \mid (\Pi(W_0) \times \Omega)\) and \(g = ((\Pi_2 \times \lambda) \circ \Pi_1)^{-1} \mid (\Pi(W_0) \times \Omega)\).

Find open \(U_1 \leq U\) and an open neighbourhood \(U_2\) of \(e \in G\) such that \(U_1 U_2 \leq U\).
To complete the proof, it suffices to prove the following:
5.7.1 Suppose given an index \( \xi \) on \( W \) such that \( \xi_j \) is a homeomorphism onto \( \xi_1(B_j(w)) \) for all \( w \in W \). Then there exists an index \( \delta \) on \( G/H \) such that \( B_\delta(u) \subseteq u \xi_1(B_j(w)) \) for all \( u \in G/H \) and:

\[
(g(x) \times x_1 B_\delta(u)).t \subseteq B_\delta(g(x,u)).t
\]

for all \( x \in \xi_1(U_0) \), \( u \in U_1 \), \( t \in T \).

For if 5.7.1 holds, then it follows from the minimality of \( (W,T) \) that \( L \) is satisfied, i.e. \( (W,T) \) is an a.p. extension of \( (X,T) \).

Suppose given such an \( \xi \). Choose an index \( \xi \) on \( U \) such that \( B_\xi(\xi_1(w)) \subseteq \xi_1(B_j(w)) \) for all \( w \in W, \ t \in T \) (Lemma 2). Since \( (X,T) \) is an a.p. extension of \( (X,T) \), choose an index \( \xi \) on \( G/H \) such that:

5.7.2 \( f([x] \times x_1 B_\xi(u)).t \subseteq B_\delta(f(x,u)).t \) for all \( x \in \xi_1(U_0), \ t \in T \).

Now choose an index \( \delta \) on \( G/H \) and a connected neighbourhood \( V \subseteq U_2 \) of \( \xi \) such that \( B_\delta(u) \subseteq u \subseteq B_\delta(u) \) for all \( u \in G/H \).

Fix \( x \in \xi_1(U_0) \) and \( u \in U_1 \). Then \( u \subseteq V \subseteq U \) and:

\[
g([x] \times x_1 B_\delta(u)).t \subseteq g([x] \times x uV)).t \subseteq \xi_1^{-1} g([x] \times x uV)).t
\]

\[
\subseteq \xi_1^{-1} \xi_1 B_\delta(g(x,u)).t \subseteq \xi_1 B_\delta(g(x,u)).t \cup U_2 \cdots \cup U_n
\]

Since \( g(x,u).t \subseteq (g([x] \times x uV)).t \) and \( V \) is connected,
\[
g([x] \times x_1 B_\delta(u)).t \subseteq g([x] \times x uV)).t \subseteq B_\delta(g(x,u)).t
\]

as required.

\[\text{Proof of the Addition Theorem 1.1}\]

5.1 The hypothesis is that \( (X,T) \) is minimal distal, and \( (T,T') \subseteq (X,T) \).

Using 4.8 (twice), choose ordinals \( \omega_1, \omega_2 \) (\( \omega_1 \leq \omega_2 \)) and a system \( \{ (x_0, T_0), T_{i+1}, \hdots \} \) of factors of \( (X,T) \) such that:

(i) \( (X_0, T_0) \subseteq (X_1, T_1), \ 0 \leq \beta \leq \gamma \leq \omega_1 \).

(ii) \( \bigcap T_{\beta} = \emptyset \leq \gamma \leq \omega_1, \ T_{\beta} = T \).

(iii) \( (X_0, T) \) is the trivial t.g., \( (X_0, T) = (T,T), (X_0, T) = (X,T) \).

(iv) For \( \beta < \omega_1 \), \( (X_{\beta+1}, T) \) is a \( G_{\beta+1}/R_{\beta+1} \)-extension of \( (X_\beta, T) \), where \( G_{\beta+1} \) is compact Lie, and \( \bigcap G_{\beta+1}/R_{\beta+1} = \{ e \} \) (4.5).
(v) If $\beta$ is a limit ordinal, $(X_\beta, T)$ is the inverse limit of $\{ (X_{i+1}, T_{i+1}) \}_{i<\beta}$.

Write $n_\beta = \dim \frac{G_\beta}{H_\beta}$ for $\beta$ not a limit ordinal, $\beta > 0$, and $n_\beta = 0$ for $\beta$ a limit ordinal.

The Addition Theorem will be proved if it can be proved that:

a) $\dim \mathbb{1} = 2 \cdot n_\beta$

b) $\dim \varphi^{-1}(y) = \sum_{\alpha < \beta} n_\alpha$ for all $y \subseteq Y$

c) $\dim X = \sum_{\alpha \in \beta} n_\alpha$ (where these sums are interpreted as $\infty$ if they do not converge).

Only b) will be proved; c) is proved in the same way as b) with $Y$ replaced by the trivial factor and $\omega_1$ by $0$, and a) is proved in the same way as b) with $X$ replaced by the trivial factor, $\omega_1$ by $0$, and $\omega_1$ by $\omega_1$.

6.2 Proof of 6.1b) Fix $y \in Y$. For $\alpha_1 < \beta \leq \alpha_2$ we shall construct by transfinite induction closed sets $Q_{\beta}$ and homeomorphisms $\varphi_{\beta}$ such that:

(i) $Q_\beta \subseteq X_\beta$, $\prod_{\alpha < \beta} (Q_\alpha) = Q_\beta$, $\omega_1 \leq \beta \leq \alpha_2$.

(ii) $\varphi_{\beta} : Q_\beta \rightarrow \prod_{\alpha < \beta} [0,1]^{n_\alpha}$ is a homeomorphism (where $[0,1]^{n_\alpha} = \{0\}$ by definition) such that the following diagram commutes for $\beta < \beta$:

Diagram 6.2a)

\[ \begin{array}{ccc}
Q_\beta & \rightarrow & Q_\beta \\
\varphi_{\beta} \downarrow & & \downarrow \varphi_{\beta} \\
\prod_{\alpha < \beta} [0,1]^{n_\alpha} & \rightarrow & \prod_{\alpha < \beta} [0,1]^{n_\alpha} \\
\varphi_{\beta}^{-1} \downarrow & & \downarrow \varphi_{\beta}^{-1} \\
[0,1]^{n_\beta} & \rightarrow & [0,1]^{n_\beta}
\end{array} \]

where we regard $\prod_{\alpha < \beta} [0,1]^{n_\alpha}$ as $\prod_{\alpha < \beta} [0,1]^{n_\alpha} \times \prod_{\alpha < \beta} [0,1]^{n_\alpha}$, and $\varphi_{\beta}^{-1}$ is the natural projection.

(iii) $\dim Q_\beta = \dim \prod_{\alpha < \beta} [0,1]^{n_\alpha}$

Then, since it is clear that $\dim Q_\beta = \sum_{\alpha < \beta} n_\alpha$, (by (ii)), 6.1b) will be proved by putting $\beta = \alpha_2$ in (iii).

Case $\beta = \omega_1 + 1$ Clearly $\varphi_{\omega_1,1}(y)$ is homeomorphic to $\prod_{\alpha < \omega_1} [0,1]^{n_\alpha}$. Hence find a closed subset $Q_{\omega_1,1}$ of $\varphi_{\omega_1,1}(y)$ homeomorphic under some $\varphi_{\omega_1,1}$ to $[0,1]^{n_{\omega_1}}$.

Now suppose $Q_{\alpha,1}$ have been constructed for $\alpha < \omega_1$. 
Case $\beta = \gamma + 1$, some $\gamma \geq \kappa$

By 6.1(iv) there exists a minimal distal t.o. $(Z_\beta, T)$ and continuous surjective maps $\xi, \psi$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Z_\beta & \xrightarrow{\psi} & X_\beta \setminus \{e \neq Z_\beta / T_\beta\} \\
\downarrow & & \downarrow \\
\psi \circ \xi & \xrightarrow{\psi \circ \xi} & X_\beta \setminus \{e \neq Z_\beta / T_\beta\}
\end{array}
\]

Then $(\psi^{-1}(Q_\gamma), \pi^{-1}(Q_\gamma), Q_\gamma, T_\gamma, H_\gamma, H_\gamma, \pi, \psi)$ is a fibre bundle (4.5, 3.1) and by 3.6 there exists a homeomorphism

\[
(\pi_\gamma \times \lambda_\gamma) : \pi^{-1}(Q_\gamma) \longrightarrow Q_\gamma \times H_\gamma.
\]

Now let $N_\beta \subseteq Q_\gamma \times H_\gamma$ be a closed subset isomorphic to $[0,1]$ under $\lambda_\gamma$, and let $Q_\beta$ be the inverse image under $(\pi_\gamma \times \lambda_\gamma)$ of $(Q_\gamma \times N_\gamma)$.

Let $Q_\beta : Q_\beta \longrightarrow \prod_{\kappa < \beta} [0,1]_{\kappa}$ be defined by $Q_\beta = (Q_\gamma \times H_\gamma) \circ (\pi_\gamma \times \lambda_\gamma)$.

Clearly (i) and (ii) of 6.2 hold, and $\dim Q_\beta = \sum_{\kappa < \beta} n_\kappa$.

By considering the fibre bundle

\[
(\pi_\gamma \times \lambda_\gamma)^{-1}(y), \pi^{-1}(Q_\gamma), \pi^{-1}(y), Q_\gamma, H_\gamma, H_\gamma, \pi, \psi),
\]

we see that

\[
\dim \pi^{-1}(y) \leq \dim \pi^{-1}(y) + \dim Q_\gamma / H_\gamma \quad \text{(by 3.3)} \leq \sum_{\kappa < \beta} n_\kappa \quad \text{and hence:}
\]

\[
\dim Q_\beta = \sum_{\kappa < \beta} n_\kappa, \quad \text{since} \quad Q_\beta \subseteq \pi^{-1}(y) \quad \text{(2.4), and so (iii) is satisfied.}
\]

Case $\beta$ a limit ordinal, $\beta > \kappa$.

Define $Q_\beta = \bigcap_{\gamma < \beta} \pi^{-1}(Q_\gamma)$.

Define $Q_\beta : Q_\beta \longrightarrow \prod_{\kappa < \beta} [0,1]_{\kappa}$ by:

\[
P_\beta \cdot Q_\beta (z) = Q_\beta \cdot \pi_\beta (z) \quad (z \in Q_\beta) \quad \text{for all} \quad Y \subseteq \beta.
\]

Then $Q_\beta$ is well-defined and a homeomorphism. (i) and (ii) of 6.2 are clearly satisfied, and clearly:

\[
\sum_{\kappa < \beta} n_\kappa = \dim Q_\beta \leq \dim \pi^{-1}(y).
\]
But by 2.6, \( \dim \Pi_i^{-1}(y) \leq \lim \sup \dim \Pi_i^{-1}(y) = \sum_{i \geq 1} n_i \), and (iii) is satisfied.

6.3 Corollary to the Addition Theorem  Let \((X, T)\) be minimal distal. Let \((X^T)\) denote the maximal a.p. factor of \((X, T)\). Then \(X\) is connected if and only if \(X^T\) is connected.

**Proof**  If \(X\) is connected, clearly \(X^T\) is connected.

Conversely, suppose \(X\) is not connected. For \(x, y \in X\), define \(x \sim y\) if \(x\) and \(y\) lie in the same connected component of \(X\). Then \(\sim\) is a closed \(T\)-invariant equivalence relation on \(X\), hence induces a factor \((X/\sim, T)\) of \((X, T)\). By the Furstenburg Structure Theorem (4.7), \((X/\sim, T)\) has a non-trivial a.p. factor \((Y, T)\), which, by the Addition Theorem 1.1, must be 0-dimensional, hence totally disconnected. But \(Y\) is a continuous image of \(X^T\). Hence \(X^T\) is not connected.

6.7 Proof of the Manifold Structure Theorem 1.2

7.1 Throughout this section use the notation of 1.2.

First note that, since, in 1.2(i), \(Y\), like \(X\), has finitely many arcwise connected components and hence, like \(X\), satisfies the hypotheses of the theorem, it suffices to prove \(X\) is a manifold, which will follow from 1.2(ii) and (iii) (since each \(X_i\) is there proved to be a manifold).

(ii) will follow from (iii) \((\dim X_{i+1} > \dim X_i)\) and the Addition Theorem \((\dim X_i < \dim X \text{ for all } i)\).

In 1.2(iv), "\(G_i/H_i\) connected if and only if \(X\) is connected" is precisely 6.3. 1.2(v) follows from Proposition 5.5.

Thus we only need to prove 1.2(iii), and that \(G_i/H_i\) is connected for \(i > 2\), which we proceed to do.

7.2 Suppose (inductive hypothesis) that \(X_1, Y_1, G_1, H_1, G_1/H_1, \ldots, Y_1\) have been constructed for \(1 < s \leq \text{order}(\text{a}), \ldots, a\) satisfying all the conditions in 1.2(iii), and with \(G_i/H_i\) connected for \(2 \leq i < a\). Let \((X_\pi, T)\) be the maximal
a.p. extension of \((X_{n-1}, T)\) in \((X, T)\). Let \(Y_b, G_b, H_b, \mathcal{V}_b, \mathcal{V}_n\), \(V_b\) be such that
the following diagram commutes \(4.6(\mathrm{ii})\):

**Diagram 7.2**

\[
\begin{array}{ccc}
Y_b & \xrightarrow{\gamma} & X_{b-1} \ (\cong Y_b/G_b) \\
\downarrow & & \downarrow \\
X_b \ (\cong Y_b/H_b) & \xrightarrow{\gamma_b} & Y_b \\
\end{array}
\]

Suppose also that \(\bigcap_{g \in G_b} g^{-1}H_b = \{e\}\).

We know that \(G_b\) is non-trivial, and it is easily seen that \(X_1\) is not finite.
Hence, to complete the proof of \((\mathrm{iii})\) and \((\mathrm{iv})\) of 1.2 for \(X_b, Y_b, G_b, H_b, W_b, \mathcal{V}_b\), it will suffice to prove:

(a) \(G_b\) is a Lie group (for then \(Y_b\) and \(X_b\) will be manifolds by 3.2).
(b) \(G_b/H_b\) is connected if \(b \geq 2\).

7.3 **Proof of 7.2(a)** If \(G_b\) is not Lie then there exists a strictly decreasing sequence \(\{H_i\}_{i=1}^{\infty}\) of normal subgroups of \(G_b\) such that \(H_{i+1} < H_i \leq H_{i+1}\) and \(G_b/H_i\) is Lie \((5.1)\), where \(\dim G_b/H_{i+1} < \dim G_b/H_i < \infty\) by the
Addition Theorem 1.1. \(Y_b/H_b\) is a manifold for each \(i\) \(3.2\). We obtain the
required contradiction to \(G_b\) not being Lie from the following lemma:

**Lemma** Let \((W, T)\) be minimal distal with \(W\) having finitely many arcwise-
connected components, and let \((V,T) \leq (W,T)\), with \(V\) a manifold. Then it is
not possible to find a strictly increasing sequence \(\{(V_{n,T})\}_{n=1}^{\infty}\) of factors
of \((W, T)\) such that \((V_{0,T}) = (V, T)\) and \((V_n, T)\) is a finite a.p. extension of
\((V, T)\).

**Proof** Suppose for contradiction that such a sequence exists. Replacing \(T\) by
a syndetic subgroup, \(\{(V_{n,T})\}_{n=1}^{\infty}\) by a proper subsequence and \(W, V_b\) by one of
the connected components of \(W, V_b\) if necessary, we can assume that \(W\) is arcwise-
connected. We can also assume \((W, T)\) is the inverse limit of \(\{(V_{n,T})\}_{n=1}^{\infty}\). Then
\((W, T)\) is an a.p. extension of \((V, T)\), hence a \(G/H\)-extension of \((V, T)\) for some
compact topological group \(G\). Fix \(v_0 \in V\). Then \(\Pi v_0\) is infinite and totally
a.p. extension of \((X_{i-1}, T)\) in \((X, T)\). Let \(Y^a, G^a, H^a, W^a, S^a, V^a\) be such that the following diagram commutes \((4.6(\text{iii}))\):

**Diagram 7.2**

\[
\begin{array}{ccc}
Y^a & \rightarrow & X_{i-1} \ (\cong Y^a/G^a) \\
\downarrow S^a & & \downarrow V^a \\
X^a \ (\cong Y^a/H^a) & \rightarrow & \\
\end{array}
\]

Suppose also that \(\bigcap_{c \in G^a} e^{-1}H^a c = \{e\}\).

We Know that \(G^a\) is non-trivial, and it is easily seen that \(X^a\) is not finite. Hence, to complete the proof of \((\text{iii})\) and \((\text{iv})\) of 1.2 for \(X^a, Y^a, G^a, H^a, W^a, S^a, V^a\), it will suffice to prove:

(a) \(G^a\) is a Lie group (for then \(Y^a\) and \(X^a\) will be manifolds by 3.2).
(b) \(G^a/H^a\) is connected if \(i \geq 2\).

**7.3 Proof of 7.2(a)** If \(G^a\) is not Lie then there exists a strictly decreasing sequence \(\left\{H^a_n\right\}\) of normal subgroups of \(G^a\) such that \(H^a_{n+1} \leq H^a_n\) and \(G^a/H^a_n\) is Lie (5.1), where \(\dim G^a/H^a_n < \dim G^a/H^a_{n+1}\) \(\leq \dim G^a/H^a < \infty\) by the Addition Theorem 1.1. \(Y^a/H^a_n\) is a manifold for each \(i\) (3.2). We obtain the required contradiction to \(G^a\) not being Lie from the following lemma:

**Lemma** Let \((W, T)\) be minimal distal with \(W\) having finitely many arcwise-connected components, and let \((V, T) \subset (W, T)\), with \(V\) a manifold. Then it is not possible to find a strictly increasing sequence \(\left\{V^a_n,T\right\}_{n=0}^{\infty}\) of factors of \((W, T)\) such that \((V^a_0, T) = (V, T)\) and \((V^a_0, T)\) is a finite a.p. extension of \((V, T)\).

**Proof** Suppose for contradiction that such a sequence exists. Replacing \(T\) by a syndetic subgroup, \(\left\{V^a_n,T\right\}\) by a proper subsequence and \(W, V\) by one of the connected components of \(W\), \(V\) if necessary, we can assume that \(W\) is arcwise-connected. We can also assume \((W, T)\) is the inverse limit of \(\left\{V^a_n,T\right\}\). Then \((W, T)\) is an a.p. extension of \((V, T)\), hence a \(G/H\)-extension of \((V, T)\) for some compact topological group \(G\). Fix \(v_0 \in V\). Then \(H^{-1}(v_0)\) is infinite and totally
disconnected. Since $\pi^{-1}(v_0)$ is homeomorphic to $G/H$, it is also compact and perfect, hence uncountable. Since each $V_n$ is a finite cover of $V$ (5.4), a loop in $V$ based at $v_0$ will lift to a unique path in $\tilde{X}$ joining $w_0$ (a fixed point in $\pi^{-1}(v_0)$) to another point in $\pi^{-1}(v_0)$, and a homotopy between two loops lifts to a homotopy between the corresponding paths in $\tilde{X}$ ([9] Chapter 6 Theorem 4).

Since $\pi^{-1}(v_0)$ is totally disconnected, if $w_1, w_2 \in \pi^{-1}(v_0)$, a path in $\tilde{X}$ joining $w_0$ to $w_1$ cannot be homotopic to a path joining $w_0$ to $w_2$, if the endpoints are restricted to $\pi^{-1}(v_0)$ and $w_1 \neq w_2$. Hence the fundamental group of $V$ (based at $v_0$) is uncountable. But this is impossible since $V$ is a compact manifold.

**7.4 Proof of 7.2(b)** If $n \geq 2$ and $G/H$ is not connected, then define an equivalence relation $\sim$ on $X_\sim$ by:

$$(x \sim y) \text{ if and only if } (\pi_\sim(x) = \pi_\sim(y)) \text{ and } x \text{ and } y \text{ lie in the same connected component of } \pi^{-1}_\sim(\pi(x)).$$

Then $(X_\sim, T)$ is a proper finite extension of $(X_\sim, T)$, so that $(X_\sim, T)$ is (5.3) an a.p. extension of $(X_\sim, T)$ — which contradicts $(X_\sim, T)$ being the maximal a.p. extension of $(X_\sim, T)$ in $(X, T)$. Therefore $G/H$ must be connected.

**8 Index of Notation and List of Fundamental Groups**

In this section we give a list of the symbols used from now on to denote the indicated standard (topological) groups and topological spaces. There follows (8.3) a table of fundamental groups which is sufficient for proof that most of the topological spaces mention in 8.2 are of distinct topological types.

**8.1 Note** If $X$ is a topological space and $\sim$ is an equivalence relation on $X$, $X/\sim$ will denote the space of equivalence classes with the quotient topology. For $x \in X$, $[x]$ will denote the $\sim$-equivalence class of $x$; square brackets will be used without mention of the associated equivalence relation, if it is though that no confusion can arise. In particular, $[x]$ will often denote the orbit of $x$ under the action of some group on $X$. 
8.2 List of symbols

\( A_n \)
- Group of permutations of \( \{1 \ldots n\} \) which can be written as the product of an even number of 2-cycles (so \( |A_n| = (n!) / 2 \)).

\( \text{Aut}(G) \)
- The automorphism group of a topological group \( G \).

\( \mathbb{C} \)
- The field of complex numbers.

\( D_n \)
- \((n \geq 2)\): Dihedral group of order \( 2n \), \( \langle a, b : a^n = b^2 = 1, ab = ba^{-1} \rangle \).

As a subgroup of \( \text{SO}(3) \): the group generated by the set

\[
\left\{ \begin{pmatrix} \cos 2\pi r/n & \sin 2\pi r/n & 0 \\ -\sin 2\pi r/n & \cos 2\pi r/n & 0 \\ 0 & 0 & 1 \end{pmatrix} : r = 0 \ldots n - 1 \right\} \cup \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}
\]

\( \text{GL}(n, \mathbb{R}) \)
- Group of real \( n \times n \) invertible matrices.

\( \text{GL}(n, \mathbb{Z}) \)
- Group of matrices with integer coefficients and determinants \( \pm 1 \).

\( K \)
- Circle group \( \{ z \in \mathbb{C} : |z| = 1 \} \) (group operation being multiplication).

As a subgroup of \( \text{SO}(3) \):

\[
\left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\}
\]

As a subgroup of \( \text{SU}(2) \):

\[
\left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - x^2} \end{pmatrix} : x \in \mathbb{R} \right\}
\]

\( K^n \)
- \( K \times \ldots \times K \) \((n \text{ copies})\) \( n \)-dimensional torus.

\( K \times Z_2 \)
- \( \mathbb{Z}_2 \) is identified with \( \text{Aut}(K) = \{1, \xi\} \), where \( \xi(k) = k^{-1} \).

\( K \times Z_2 \)
- \( \{ (k, \sigma) : k \in K, \sigma = 1 \text{ or } -1 \} \).

Multiplication is defined by \( (k_1, \sigma_1)(k_2, \sigma_2) = (k_1 \xi(k_2), \sigma_1 \sigma_2) \).

As a subgroup of \( \text{SO}(3) \), \( K \times Z_2 \) is the group generated by the set:

\[
\left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}
\]

As a subgroup of \( \text{SO}(3) \) \( K \times Z_2 \) \((12.711i)\),

\( K \times Z_2' \)
- \( \{ (k, \sigma) : k \in K \text{ as a subgroup of } \text{SO}(3) \text{ and } \sigma \in \mathbb{Z}_2 \text{ as a } \} \text{ subgroup of } \text{Aut}(\text{SO}(3)) \).
K^3/σ

(σ is an automorphism of K^2 of order r.) This denotes the orbit space of K^3 under the free action of σ defined by:
σ·(k_1,k_2,k_3) = (e^{2πi/r}k_1, σ(k_2,k_3)).

K^3/(r_{11} r_{12} r_{21} r_{22})

This denotes K^3/σ where σ corresponds to \((r_{11} r_{12}) \in GL(2, Z)\) (see 10.8.).

(K x S^2)/~

~ denotes the equivalence relation \((k, x) \sim (-k, -x)\) for \(k \in K\) and \(x \in S^2\) (see 8.1).

Kß

This denotes the Klein bottle K^2/~ where \((k_1,k_2) \sim (-k_1,k_2)\).

K/Γ_n

(\(n > 1\)): Γ denotes the Lie group of matrices:
\[
\begin{pmatrix}
1 & x \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
y \\
z
\end{pmatrix}
\]

\(\Gamma_n\) denotes the subgroup
\[
\begin{pmatrix}
1 & m_1 z/n \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]

Where no confusion can arise, \([x,y,z]\) denotes the element:
\[
\begin{pmatrix}
1 & z \\
0 & 1
\end{pmatrix}
\]
of K/Γ_n.

O(n)

Group of real orthogonal n x n matrices.

P^n

(\(n > 2\)): n-dimensional real projective plane \(S^n/\sim\) where \(x \sim -x\).

R

The field of real numbers.

S^n

(\(n > 1\)): \(\{(x_1,...,x_n) \in \mathbb{R}^{n+1}; \sum x_i^2 = 1\} \sim\) S^n.

S_n

Group of permutations of \{1,...,n\} (so \(|S_n| = n!\).

SU(2)

\[
\begin{pmatrix}
λ & μ \\
-μ & λ
\end{pmatrix}
\] : \(λ, μ \in \mathbb{C}, |λ|^2 + |μ|^2 = 1\).

(S^2 x k)/~

~ denotes the equivalence relation \((x,k) \sim (-x,-k)\) for \(x \in S^2\), \(k \in K\). This space is homeomorphic to \((K x S^2)/\sim\).

(S^2 x K)/\sim

\(\sim\) denotes the equivalence relation \((x,k) \sim (x,k^{-1})\).
\[ W_0 \quad \mathbb{K}^3/\sim \quad \text{where } \sim \text{ is the equivalence relation:}
\]

\[(k_1, k_2, k_3) \sim (-k_1, k_2^{-1}, k_3) \sim (k_1, -k_2, k_3^{-1}) \sim (-k_1, -k_2^{-1}, k_3).
\]

\[ W_2 \quad \mathbb{K}^3/\sim \quad \text{where } \sim \text{ is the equivalence relation:}
\]

\[(k_1, k_2, k_3) \sim (-k_1, k_2^{-1}, k_3) \sim (k_1, -k_2, k_3) \sim (-k_1, -k_2^{-1}, k_3).
\]

\[ \mathbb{Z} \quad \text{Group of integers.}
\]

\[ \mathbb{Z}_n \quad \text{Cyclic group of order } n \langle a : a^n = 1 \rangle.
\]

As a subgroup of \( \text{SO}(3) \): the group generated by:

\[
\begin{pmatrix}
\cos \frac{2\pi}{n} & \sin \frac{2\pi}{n} & 0 \\
-\sin \frac{2\pi}{n} & \cos \frac{2\pi}{n} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

As a subgroup of \( \text{SU}(2) \): the group generated by:

\[
\begin{pmatrix}
e^{\frac{2\pi i}{n}} & 0 \\
0 & e^{-\frac{2\pi i}{n}}
\end{pmatrix}
\]
<table>
<thead>
<tr>
<th>Space</th>
<th>Fundamental Group</th>
<th>Number of homomorphisms of fundamental group into:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\mathbb{Z}^3$</td>
<td>$\mathbb{Z}^2$ $\mathbb{Z}^3$ $\mathbb{Z}^4$ $\mathbb{Z}^5$ $\mathbb{Z}^6$</td>
</tr>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\langle a, b, c: a^2 = b^{-1}a, ac = c^{-1}a, bc = cb \rangle$</td>
<td>$8$ $3$ $48$</td>
</tr>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\langle a, b, c: cb = ba, ac = c^{-1}a, bc = cb \rangle$</td>
<td>$8$ $9$ $36$</td>
</tr>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\langle a, b, c: ab = ba, ac = c^{-1}ab, bc = cb \rangle$</td>
<td>$4$ $9$ $24$</td>
</tr>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\langle a, b, c: bc = cb, ba = ac^{-1}, ca = ac^{-1}b \rangle$</td>
<td>$2$ $3$</td>
</tr>
<tr>
<td>$\mathbb{R}^3$</td>
<td>$\langle a, b, c: cb = cb, ba = ac, ca = ab^{-1}c \rangle$</td>
<td>$2$ $3$</td>
</tr>
<tr>
<td>$\mathbb{W}_{10}$</td>
<td>$\langle a, b, c: ab = b^{-1}a, bc = c^{-1}b, ac = ca \rangle$</td>
<td>$9$ $3$ $36$</td>
</tr>
<tr>
<td>$\mathbb{W}_{12}$</td>
<td>$\langle a, b, c: ab = cb^{-1}a, bc = c^{-1}b, ac = ca \rangle$</td>
<td>$4$ $3$ $16$</td>
</tr>
<tr>
<td>$\mathbb{N}/\mathbb{R}_1$</td>
<td>$\langle a, b, c: ab = ba, ac = cab, bc = cb \rangle$</td>
<td>$4$ $9$ $\mathbb{L}^2$ $18$</td>
</tr>
<tr>
<td>$\mathbb{N}/\mathbb{R}_2$</td>
<td>$\langle a, b, c: ab = ba, ac = cab^2, bc = cb \rangle$</td>
<td>$9$ $9$ $\mathbb{L}^3$ $36$</td>
</tr>
<tr>
<td>$\mathbb{N}/\mathbb{R}_3$</td>
<td>$\langle a, b, c: ab = ba, ac = cab^3, bc = cb \rangle$</td>
<td>$4$ $27$ $\mathbb{L}^2(3, r)$ $\mathbb{L}^2(n, r)$</td>
</tr>
<tr>
<td>$\mathbb{N}/\mathbb{R}_{(n, r)} (n &gt; 4)$</td>
<td>$\langle a, b, c: ab = ba, ac = cab^2, bc = cb \rangle$</td>
<td></td>
</tr>
<tr>
<td>$S^2 \times \mathbb{R}$</td>
<td>$\mathbb{Z}$</td>
<td></td>
</tr>
<tr>
<td>$(S^2 \times \mathbb{R})^*$</td>
<td>$\mathbb{Z}$</td>
<td></td>
</tr>
<tr>
<td>$P^2 \times \mathbb{R}$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>$(S^2 \times \mathbb{R})^*$</td>
<td>$\mathbb{Z} \times \mathbb{Z}_2$</td>
<td></td>
</tr>
</tbody>
</table>
Notes on table

(i) $(n, r)$ denotes the highest common factor of $n$ and $r$.

(ii) It can be shown that $S^2 \times K$ and $(S^2 \times K)/\sim$ are not homeomorphic, even though they have the same fundamental group, and similarly for $P^2 \times K$ and $(S^2 \times K)/\sim$.

§ 9 On the String of a Minimal Distal Transformation Group

First we need some definitions (9.1-9.2):

9.1 Definitions A string $(G_1 \ldots G_n)$ of bundles is a finite sequence of bundles $G_i = (Y_i, X_i, X_{i-1}, G_i, \pi_i, \pi_{i-1}, U_i) \ (1 \leq i \leq n)$ where $X_0$ is a one-point set. $n$ is called the length of the string, which is denoted by $\langle n \rangle$, say.

If $(G' = (G'_1 \ldots G'_n))$ is another string with $G'_i = (Y'_i, X'_i, X'_{i-1}, G'_i, \pi'_i, \pi'_{i-1}, U'_i)$, then $G$ and $G'$ are isomorphic under $(\xi, \ldots, \xi_n)$ if there exist 3rd-isomorphisms $(\xi') (G_i, \xi'_i) : G_i \longrightarrow G'_i$ such that the following diagram commutes:

Diagram 9.1

9.2 Definition Let $G = (G_1 \ldots G_r)$ be a string with:

$G'_i = (Y'_i, X'_i, X'_{i-1}, G'_i, \pi'_i, \pi'_{i-1}, U'_i) \ (1 \leq i \leq r).

$G$ is $n$-allowable if each $G'_i / \pi'_i$ is connected, $\dim G'_i / \pi'_i \geq 1$ and $n = \sum_{i=1}^{r} \dim (G'_i / \pi'_i) \ (\dim X'_i)$.

$G$ is allowable if $G$ is $n$-allowable for some $n$.

9.3 Use the notation of the Manifold Structure Theorem (1.2): this theorem shows that given a minimal distal t.g. $(X, T)$ where $X$ is a compact connected
n-dimensional manifold, we can associate with it an n-allowable string 
\((\sigma_1, \ldots, \sigma_r) = \sigma(X, T)\) where \(r\) is the order of \((X, T)\) (4.10). There is some 
choice in the strings which can be associated with \((X, T)\) in this way, but 
any two choices are isomorphic as strings. Moreover, if \((X, T) \cong (X', T)\), then 
\(\sigma(X, T) \cong \sigma(X', T)\). Therefore we have:

2.4 Definition Given a topological group \(T\), a string \(\sigma\) is admissible if \(\sigma\) is 
\(\sigma(X, T)\) (up to isomorphism) for some minimal distal t.g. \((X, T)\) where \(X\) is 
a compact connected topological manifold.

\(\sigma\) is admissible if \(\sigma\) is \(T\)-admissible for some \(T\). Clearly (9.3)
admissible strings are allowable.

9.5 Later (9.7) we give a complete list of \(Z\)-admissible and \(R\)-admissible 
n-allowable strings for \(n \leq 3\), and hence obtain a coarse classification of 
minimal distal \(Z\)- and \(R\)-actions on compact connected manifolds of dimension \(\leq 3\)
It is easy - but rather tedious, so we shall not do it - to give a complete 
list of the admissible n-allowable strings for \(n \leq 3\), by using the results 
of §§10-11 and analogues of the results of §12. However, we list (9.6) the 
compact connected manifolds of dimension \(\leq 3\) which can be phase spaces of 
minimal distal group actions for some group. It is clear that such a list is 
a "corollary" of a list of the isomorphism classes of admissible strings.

Clearly the problem of finding the isomorphism classes of strings 
\(\sigma = (\sigma_1, \ldots, \sigma_r)\) with \(\sigma_1 = (T_1X_1, X_{1-1}, G_1, H_1, \phi_1, \theta_1)\) (1 \(\leq i \leq r\) is inductive 
on the length of the string and related to the following two problems:

(i) Find the possibilities for \((G_1, H_1)\) up to isomorphism (§10).

(ii) Having found \(X_{1-1}, G_1, H_1\), find the 1st, 2nd and 3rd-isomorphism classes 
of bundles with base \(X_{1-1}\), group \(G_1\) and isotropy subgroup \(H_1\). 1st-isomorphism 
classes are given in §11. 2nd- and 3rd-isomorphism classes are easily deduced 
from these.

9.6 Manifolds of dimension \(\leq 3\) supporting minimal distal actions of some 
group (for notation see §§8, §10):

Actions of order 1 (almost periodic): \(K, K^2, S^1, P^2, K^3, SU(2)/Z_\infty (n \geq 1),\)
SO(3)/D_{2n} (n ≥ 2), SO(3)/A_4, SO(3)/S_4, SU(3)/A_5, S^2 × K, P^2 × K, (S^2 × K)/\sim

Actions of order 2: K^2, KB, K^2, K^3/(-1 0 0), K^3/(0 -1 1), K^3/(1 0 1), K^3/(1 0 1), K^3/(0 1 1)

K^3/(0 1 1), K^3/(0 -1 1), SU(2)/Z_2 (n ≥ 1), SO(3)/D_{2n} (n ≥ 2), N/\Gamma_n (n ≥ 1),

S^2 × K, P^2 × K, (S^2 × K)/\sim, (S^2 × K)/\sim

Actions of order 3: K^3, K^3/(1 0 0), K^3/(0 -1 1), K^3/(1 0 1), N/\Gamma_n (n ≥ 1), W_e, \mathbb{Z}

2.7 If \( G = (G_1, \ldots, G_n) \) is an n-allowable string for \( n ≤ 3 \) and
\( G_1 = (Y_1, X_1, X_{-1}, G_1^1, H_1, F_1, V_1, Y_1) \) and \( G = (G_2, T) \) for some \( T \), then \( T \) acts on \( G_1 \) by right multiplication on \( G_1 \) of a homomorphic image (in \( G_1 \)) of \( T \). So if \( T \) is abelian, \( H_1 \) is trivial, \( G_1 \) is abelian and \( X_1 = Y_1 \).

Therefore, in table A and B we list the n-allowable strings (\( n ≤ 3 \)) of length 2 and 3 for which \( G_1 \) is abelian and \( H_1 \) is trivial, stating which of them are \( \mathbb{Z} \)-admissible and which are \( \mathbb{R} \)-admissible. Each line in the tables except \( A_4 \) represents exactly one isomorphism class.

The tables are intended merely as a summary of information and can only be understood in conjunction with Section 10 and 11.
<table>
<thead>
<tr>
<th>dim $X_2$</th>
<th>$(G_2, H_2)$ (§10)</th>
<th>$\mathcal{B}_2$ (§11)</th>
<th>$X_2$</th>
<th>$\mathcal{L}$-admissible</th>
<th>$\mathcal{L}$-admissible</th>
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<tr>
<td>A1</td>
<td>2</td>
<td>$\mathbb{F} (\mathbb{F}, {1})$</td>
<td>$\mathcal{X}(\mathbb{F})$</td>
<td>11.4</td>
<td>$\mathbb{F}^2$</td>
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<tr>
<td>A2</td>
<td>$n$</td>
<td>$\mathbb{F} (\mathbb{F}, {1})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>11.6</td>
<td>$n$</td>
</tr>
<tr>
<td>A3</td>
<td>$n$</td>
<td>$\mathbb{F} (\mathbb{F}, {1})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>KB</td>
<td>$n$</td>
</tr>
<tr>
<td>A4</td>
<td>3</td>
<td>$\mathbb{F} (\mathbb{F}, {1})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>9.0</td>
</tr>
<tr>
<td>A5</td>
<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>8.2</td>
</tr>
<tr>
<td>A6</td>
<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>8.2</td>
</tr>
<tr>
<td>A7</td>
<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>8.2</td>
</tr>
<tr>
<td>A8</td>
<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>8.2</td>
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<tr>
<td>A9</td>
<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
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<td>8.2</td>
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<tr>
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<td>$n$</td>
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<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathbb{F}^2$</td>
<td>8.2</td>
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<tr>
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<td>$n$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
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<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
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<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
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<tr>
<td>A14</td>
<td>$n$</td>
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<td>$\mathcal{X}(\mathbb{F}, \mathbb{F})$</td>
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<td>( (B_1, B_2) \text{ as in:} )</td>
<td>( (B_3, K_3) )</td>
<td>( \Phi_3 )</td>
<td>( k_3 )</td>
<td>( Z-\text{numbr.} )</td>
<td></td>
</tr>
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<td>-----------------</td>
<td>-----------------</td>
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<td></td>
</tr>
<tr>
<td>B1 A1 ( (K_1, {13}) )</td>
<td>( \Phi_0 )</td>
<td>11.7</td>
<td>( K^3 )</td>
<td>8.2</td>
<td>Yes</td>
</tr>
<tr>
<td>B2 A2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B3 A1</td>
<td>( \Phi_0, \Phi_2, \Phi_1 )</td>
<td>11.7</td>
<td>( K^3 )</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>B4 A2</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>B5 A1 ( (Kx_z, z_2, z_2) )</td>
<td>( \Phi_0^\prime, \Phi_0 )</td>
<td>11.7</td>
<td>( K^3 )</td>
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<td></td>
</tr>
<tr>
<td>B6 A2</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B7 A1</td>
<td>( \Phi_0^\prime, \Phi_0 )</td>
<td>11.7</td>
<td>( K^3 )</td>
<td>No</td>
<td></td>
</tr>
<tr>
<td>B8 A2</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>B9 A1</td>
<td>( \Phi_0^\prime, \Phi_0 )</td>
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<td>( K^3 )</td>
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<td>B10 A2</td>
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</tr>
<tr>
<td>B11 A1</td>
<td>( \Phi_0 (k_1, -k_2, k_3^2) )</td>
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<td></td>
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</tr>
<tr>
<td>B12 A2</td>
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<td></td>
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</tr>
<tr>
<td>B13 A1</td>
<td>( \Phi_0 (k_1, -k_2, k_3^2) )</td>
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<td>B14 A2</td>
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</tr>
<tr>
<td>B15 A1</td>
<td>( \Phi_0 (k_1, -k_2, k_3^2) )</td>
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<tr>
<td>B16 A2</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>B17 A3 ( (K, {13}) )</td>
<td>( K \Phi_0 )</td>
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<td>( K^3 )</td>
<td>8.2</td>
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<td>B18</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>B19 ( (Kx_z, z_2, z_2) )</td>
<td>( K \Phi_0 \Phi_2, \Phi_0^\prime, \Phi_0 )</td>
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<td>( K^3 )</td>
<td>8.2</td>
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</tr>
<tr>
<td>B20</td>
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<td>11.9</td>
<td>( K^3 )</td>
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</tr>
<tr>
<td>B21</td>
<td>( K \Phi_0^\prime, \Phi_0 )</td>
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</tr>
<tr>
<td>B22</td>
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<tr>
<td>B23</td>
<td>( K \Phi_0^\prime, \Phi_0 )</td>
<td>11.9</td>
<td></td>
<td></td>
<td>8.2</td>
</tr>
</tbody>
</table>
9.8 Notes
(i) None of the strings of table 3 are $\mathcal{K}$-admissible ($\S$ 13).
(ii) In $M_0$, $A$ can be assumed to be one of the groups of $10.8$, and $\mathcal{F} \in A$.
For $G$, $T \in A$, $\mathcal{K}(K^2, A, T)$ and $\mathcal{K}(K^2, A, \mathcal{I})$ give rise to isomorphic strings if and only if there exists $\eta \in \text{Aut}(K^2)$ with $\eta A \eta^{-1} = A$ and $\eta \mathcal{I} \eta^{-1} = \mathcal{I}$.
(iii) In $M_0$, $\mathcal{K}(K^2, A, \mathcal{I})$ gives rise to a $\mathcal{Z}$-admissible string if and only if $\mathcal{F} \subseteq A$ and $\mathcal{F} / \mathcal{I}$ is cyclic ($\S$ 12).

4.10 On Connected Irreducible Pairs

In this section we give some isomorphism classes of irreducible pairs $(G, H)$ (definitions 10.1 - 10.2) - this information is needed for finding the isomorphism classes of strings (9.1) and can doubtless be found elsewhere, but is collected together for convenient reference.

10.1 Definition An irreducible pair $(G, H)$ consists of a compact Lie group $G$ and a closed subgroup $H$ such that $\bigcap_{g \in G} g^{-1}H g = \{ e \}$.

$(G, H)$ is connected if $G/H$ is connected.
$(G, H)$ is group-connected if $G$ is connected.

10.2 Definition Irreducible pairs $(G_1, H_1)$ and $(G_2, H_2)$ are isomorphic if there exists a topological group isomorphism of $G_1$ onto $G_2$ which maps $H_1$ onto $H_2$ (written $(G_1, H_1) \cong (G_2, H_2)$).

10.3 The following lemma gives the relationship between connected irreducible pairs and group-connected irreducible pairs. The proof is straightforward and will be omitted.

Lemma (1) If $(G, H)$ is a connected irreducible pair, then $(G_0, H \cap G_0)$ is a group-connected irreducible pair, where $G_0$ denotes the identity component of $G$.

$G_0 = H G_0 = G$, and the map $E G \mapsto (H \cap G_0) G$ ($G \in G_0$) defines a homeomorphism of $G/H$ onto $G / (G \cap G_0)$.

(ii) If, for $h \in H$, $g \in G_0$, $G_h(g) = h g h^{-1}$, the map $h \mapsto G_h$ is a topological group isomorphism of $H$ into the subgroup $S(G_0, H \cap G_0)$ of $\text{Aut}(G_0)$ of automorphisms leaving $H \cap G_0$ invariant, where $\text{Aut}(G_0)$ is given the topology...
of pointwise convergence. So identify \( H \) with \( \{ G^{\alpha} : \alpha \in H \} \subseteq S(G_0, H \cap G_0) \).

(iii) Suppose, further, there exists a subgroup \( S_1 = S_1(G_0, H \cap G_0) \) of \( S(G_0, H \cap G_0) \) such that each element of \( S(G_0, H \cap G_0) \) can be uniquely written in the form \( xy \), where \( x \in S_1 \) and \( y \in H \cap G_0 \subseteq S(G_0, H \cap G_0) \) (which clearly happens if \( H \cap G_0 \) is trivial). Then write \( A = S_1 \cap H \subseteq S(G_0, H \cap G_0) \). \( A \) is finite and \( (G, H) = (G_0 \times A, (H \cap G_0)x_A) \) where \( G_0 \times A = \{(g, a) : g \in G_0, a \in A \} \) and multiplication is defined by \( (g_1, \sigma_1)(g_2, \sigma_2) = (g_1g_2, \sigma_1\sigma_2) \).

(iv) If \( A_1 \) and \( A_2 \) are finite subgroups of \( \text{Aut}(G_0) \) then \( (G_0 \times A_1, A_1) \cong (G_0 \times A_2, A_2) \) if and only if \( A_1 \) and \( A_2 \) are conjugate in \( \text{Aut}(G_0) \).

10.4 Detailed proof that the list of 10.6 is exhaustive will not be given, but the following facts are used:

(i) If \((G, H)\) is an irreducible pair and \( \dim G/H \leq n \) then \( \dim G \leq n/((n+1)/2) \) \[16\]

(ii) A compact connected Lie group is isomorphic to one of the form \((S \times T)/Z\) where \( S \) is semisimple compact connected, \( T \) is a torus and \( Z \) is a finite central subgroup with \( S \cap Z \) and \( T \cap Z \) trivial \([10] \) Chapter XII.1 Theorem 1.

(iii) Given a compact semi-simple Lie algebra \( \mathfrak{g} \), there is a unique compact simply-connected connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \) (up to isomorphism), and if \( G \) is another connected Lie group with Lie algebra \( \mathfrak{g} \), then \( G \cong G/Z \) for some finite central subgroup \( Z \) of \( G \) \([15]\).

(iv) A compact semisimple Lie algebra is a direct sum of compact simple Lie algebras, which have been completely classified \([5]\).

(v) Any toral subgroup of a compact connected Lie group \( G \) is contained in a maximal torus, and any two maximal tori are conjugate \([3] \) Chapter XIII.4.

10.2 Definition If \( G \) is a compact Lie group, and \( H \) is a closed subgroup, and \( Z = \bigcap_{g \in G} g^{-1}Hg \), then \([G, H]\) will denote the irreducible pair \((G/Z, H/Z)\).
10.6 We now list the group-connected irreducible pairs \((G, H)\) with \(\dim G/H \leq 3\) using the notation of 8.2.

(i) \((K, \{1\})\)

(ii) \((\mathbb{K}, \{1\})\)

(iii) \((\text{SO}(3), K) \cong [\text{SU}(2), K]. \) \(\text{SO}(3)/K\) is homeomorphic to \(S^2.\)

(iv) \((\text{SO}(3), \mathbb{K}, \mathbb{Z}_2) \cong [\text{SU}(2), M]\) where \(M\) is the subgroup generated by the set \(\left\{(0, 0, 0, 0) : \lambda \in K \right\} \cup \left\{(0, 0, 0, 0) : \lambda \in \mathbb{K} \right\}\)

\(\text{SO}(3)/(K \times \mathbb{Z}_2)\) is homeomorphic to \(P^2.\)

(v) \((K^3, \{1\})\)

(vi) \([\text{SU}(2), \mathbb{Z}_2]\)

(vii) \((\text{SO}(3), D_{2n}) \cong [\text{SU}(2), H_{4n}]\) \((n \geq 2)\) where \(H_{4n}\) is the subgroup generated by the set \(\left\{\begin{pmatrix} e^{\pi i/2n} & 0 \\ 0 & e^{-\pi i/2n} \end{pmatrix}, \left\{0, 0, 0, 1 \right\}\right\}.\)

(viii) \((\text{SO}(3), A_4), (\text{SO}(3), S_4), (\text{SO}(3), A_5).\) All subgroups of \(\text{SO}(3)\) isomorphic to \(A_4, S_4, A_5\) respectively are conjugate (\(\S\) Chapter 2).

(ix) \([\text{SU}(2) \times K, \mathbb{Z}_n]\) where \(Z_n = \left\{\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \lambda^n \right) : \lambda \in K \right\}\) \((n \geq 0)\).

If \(n = 0, G/H\) is homeomorphic to \(S^2 \times K.\) If \(n > 0, G/H\) is homeomorphic to \(\text{SU}(2)/\mathbb{Z}_n.\)

(x) \([\text{SU}(2) \times K, \mathbb{Z} \times \{1\}]\) where \(M\) is as in (iv). \(G/H\) is homeomorphic to \(P^2 \times K.\)

(xi) \([\text{SU}(2) \times K, \mathbb{W}]\), where \(W = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \lambda^2 \right\} \cup \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \lambda \right\} : \lambda \in K \right\}\)

\(G/H\) is homeomorphic to \((S^2 \times K)/\sim.\)

(xii) \([\text{SU}(2) \times \text{SU}(2), V_1]\), where \(V_1 = \left\{u, v : u, v \in \text{SU}(2) \right\}. \) \(G/H\) is homeomorphic to \(\text{SU}(2),\) equivalently to \(S^3.\)

(xiii) \([\text{SU}(2) \times \text{SU}(2), V_2]\), where \(V_2 = V_1 \cup \left\{u, -u : u, v \in \text{SU}(2) \right\}. \) \(G/H\) is homeomorphic to \(\text{SU}(2)/\mathbb{Z}_2,\) equivalently to \(\text{SO}(3)\) and to \(P^3.\)

10.7 Let \((G', H')\) be one of the group-connected irreducible pairs of 10.6(1)–(v) and let \(S(G', H') = \{G \in \text{Aut}(G') : \theta(H') = H' \} \), so that \(H' \leq S(G', H') (10.5(11).\)
For each such \((G',H')\), we define a \(S_1(G',H') \subseteq S(G',H')\) such that each element of \(S(G',H')\) can be uniquely written in the form \(xy\) \((x \in H', y \in S_1(G',H')\) and hence show that if \((G,H)\) is a connected irreducible pair with \(\dim G/H = 2\), then \((G,H) = (G_0 x A, (H_0 x A))\), for a finite \(A \subseteq S_1(G_0,H_0)\) (10.3(iii)).

\((G_0,H_0,G_0)\) being (up to isomorphism) one of the pairs 10.6(1) - (iv).

(i) \((G_0,H_0,G_0) = (K^2,\{1\})\) \(S_1(K^2,\{1\}) = \text{Aut}(K) = \{1, k \in \mathbb{Z}_2\}\)

(ii) \((G_0,H_0,G_0) = (K^2,\{1\})\) \(S_1(K^2,\{1\}) = \text{Aut}(K) = GL(2,2)\) (8.2)

(iii) and (iv) \((G_0,H_0,G_0) = (F^2,\{1\})\) \(S_1(F^2,\{1\}) = \text{Aut}(F^2)\) where the isomorphism is given by:

\(\sigma \mapsto \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\) \(\text{where } \sigma(k_1,k_2) = (k_1^a k_2^b, k_1^c k_2^d)\).

All automorphisms of \(SO(3)\) are inner, so that \(\text{Aut}(SO(3)) = SO(3)\). Under this identification, \(S(SO(3),K) = S(SO(3),K x \mathbb{Z}_2) = K x \mathbb{Z}_2 \subseteq SO(3)\).

(iii) \((G_0,H_0,G_0) = (SO(3),K)\) \(S_1(SO(3),K) = \{1, k \in \mathbb{Z}_2\} = \mathbb{Z}_2\), where \(\xi(k) = k^{-1}(k \in K)\).

(iv) \((G_0,H_0,G_0) = (SO(3),K x \mathbb{Z}_2)\) \(S_1(SO(3),K x \mathbb{Z}_2)\) is trivial.

10.6 In order to completely classify the connected irreducible pairs \((G,H)\) for which \((G_0,H_0,G_0) = (K^2,\{1\})\), it remains to find the conjugacy classes of finite subgroups of \(\text{Aut}(K^2) \cong GL(2,2)\) (10.3(iv) and 10.7). I am indebted to my father, D. Rees, for finding the conjugacy classes, although he says the answer must be known. Note that:

(a) If \(u \in GL(2,\mathbb{Z})\) has finite order then \(u\) must have order 1, 2, 3, 4 or 6.

(b) A finite subgroup of \(GL(2,\mathbb{Z})\) is conjugate in \(GL(2,\mathbb{Z})\) to a subgroup of \(O(2)\), which is, of course, isomorphic to \(K x \mathbb{Z}_2\) (16.9.1).

(a) and (b) imply that a non-trivial finite subgroup of \(\text{Aut}(K^2)\) must be
isomorphic to \(Z_n\) or \(D_{2n}\) (\(n = 2, 3, 4\) or \(6\)). It can be shown, further, that the conjugacy classes of finite subgroups are as follows:

1. \(\langle 1 \rangle\) (trivial subgroup)
2. \(\langle -I \rangle\)
3. \(\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle\), \(\cong Z_2\)
4. \(\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle\)
5. \(\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rangle\), \(\cong Z_3\
6. \(\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle\)
7. \(\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rangle\), \(\cong Z_6\
8. \(\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle\)
9. \(\langle \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \rangle\)
10. \(\langle 0, 1 \rangle\)

All let-isomorphism Classes of Fibre bundles

In order to determine the isomorphism classes of strings, it is necessary (9.5(iii)) to find the list-isomorphism classes (3.4) of bundles:

(a) with base \(K\), group \(G\) and isotropy subgroup \(H\), where \((G, H)\) is a connected irreducible pair with \(\dim G/H < 2\). (For the possibilities for \((G, H)\), see §10.) See 11.6.

(b) with base \(K^2\), \(K\mathbb{R}\), \(S^2\) or \(P^2\), group \(G\) and isotropy subgroup \(H\) where \((G, H) = (K, \{1\})\) or \((K \times Z_2, Z_2^2)\). See 11.5 - 11.10.

This is just a matter of collecting together known results. List-isomorphism classes rather than set-isomorphism classes are given (the latter would in some ways be more convenient) mainly because list-isomorphism is the type of isomorphism usually used in fibre bundle theory.

The notation of §8 will be used throughout this section.

I should like to thank E. Cesar de Sa, M. Eastwood, J. Sells and D. Epstein for helpful suggestions and discussion.

11.1 We define a complete list-isomorphism invariant \(\chi: C_1 \to D_1\) for each of the following classes \(C_1\) of principal
bundles, where $\mathcal{P}_1$ is the given range space.

(1) ([17] 13.5) $\mathcal{C}_1(X, G)$ is the set of principal bundles with base $X$ (a compact connected manifold) and finite group $G$. Let $\mathcal{B} = (Y, X, G, \pi) \in \mathcal{C}_1(X, G)$. Fix $x_0 \in X$, $y_0 \in Y$ with $\pi(y_0) = x_0$. If $\pi$ is a finite covering map, hence determines (with $x_0, y_0$) a homomorphism $\varphi_x : \pi_1(X) \to G$, where $\pi_1(X)$ denotes the fundamental group of $X$. Two homomorphisms in $\operatorname{Hom}(\pi_1(X), G)$ are said to be equivalent if one is the composition of the other with an inner automorphism of $G$. $\mathcal{D}_1(X, G)$ is the set of equivalence classes in $\operatorname{Hom}(\pi_1(X), G)$.

$\chi(\mathcal{B})$ is defined to be the equivalence class of $\varphi_x$, and $\chi : \mathcal{C}_1(X, G) \to \mathcal{D}_1(X, G)$ is independent of the $x_0, y_0$ chosen for each $\mathcal{B}$.

(11) ([17] 18.5) $\mathcal{C}_2(G)$ is the set of principal bundles with base $K$ and group $G$. $\mathcal{D}_2(G)$ is the set of conjugacy classes in $G/G_0$, where $G_0$ is the component of the identity in $G$.

Let $\mathcal{B} = (Y, K, G, \pi) \in \mathcal{C}_2(G)$. For $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{B}$, let $\{e^{i\theta_1}, e^{i\theta_2}\}$ denote $\{e^{i\theta} : \mathcal{B}_1 \leq \mathcal{B}_2\}$. Define $V_1 = \{e^{i\theta/4}, e^{i\theta/4}\}$ and $V_2 = \{e^{i\theta/4}, e^{7i\theta/4}\}$.

Diagram 11.1(a)

Choose maps $\varphi_1 : V_1 \to Y$ with $\varphi_1(0) = \text{identity}$ (3-2) and $\varphi_1(1) = \varphi_1(1)$. Define $\varphi_1, \varphi_2 : V_1 \cap V_2 \to G$ by $\varphi_1(x) = \varphi_1(x) \varphi_2(x)$ for all $x \in V_1 \cap V_2$.

Define $\lambda(\mathcal{B})$ to be the conjugacy class in $G/G_0$ of $G_0 \varphi_1, \varphi_2(-1)$.

(a) The definitions of $\lambda$ on $\mathcal{C}_1(X, G)$ and on $\mathcal{C}_2(G)$, for finite $G$, do not quite coincide on $\mathcal{C}_1(X, G) \cap \mathcal{C}_2(G)$, but there is a natural correspondence between the two definitions, and in any case no confusion should arise.
(b) It follows from (ii) that a principal bundle with base $K$ and group $K$ must be a product bundle. Hence ([17] 11.4) a principal bundle with base $K \times [0,1]$ and group $K$ must be a product bundle.

(iii) and (iv)

$C_3$ is the set of principal bundles with base $k^2$ and group $K$.

$C_4$ is the set of principal bundles with base $KB$ and group $K$.

$0_3 = Z$, $0_4 = \{0, 1\}$.

Let $V_1 = \{\{e^{-i\pi/3}, e^{\pi i/4}\} \times K, \quad V_2 = \{\{e^{-3i\pi/4}, e^{9i\pi/4}\} \times K, $ $S = \begin{bmatrix} 1 & 1 \end{bmatrix} \times K, \quad T = \begin{bmatrix} -1 & 1 \end{bmatrix} \times K. \quad \text{So } k^2 = V_1 \cup V_2.$

Let $V_1 = \{[k_1, k_2] : k_1, k_2 \in k^2 \}$ (6.2).

Let $V_2 = \{[k_1, k_2] : k_1, k_2 \in k^3 \}$.

Let $S = \{[1, k] : k \in K \}$, $T = \{[k, 1] : k \in K \}$.

So $KB = V_1 \cup V_2.$

Diagram 11.1(b)

This diagram represents how $V_1$ and $V_2$ are related for both (iii) and (iv).

If $Z = (Y, X, e, s) \in C_3$ (i = 3 or 4, so that $X = K^2$ or $KB$), choose maps $q_j : V_j \rightarrow Y$ (j = 1, 2) with $q_1 = q_2 = 0_3$ identity (see (ii)(b)) and:

$q_1 | S = q_2 | S.$

Define $e_{1,2} : V_1 \cap V_2 \rightarrow K$ by $q_1(x) = q_{1,2}(x), \quad q_2(x), \quad x \in V_1 \cap V_2.$

Then $e_{1,2} | T$ is homotopic to:

(iii) $(-1, k) \rightarrow k^n$ for a unique $n \in Z.$

(iv) $[1, k] \rightarrow k^n$ for a unique $n \in Z.$
Define $\chi$ by:

(iii) $\chi(\mathcal{B}) = n$

(iv) $\chi(\mathcal{B}) = 0$ if $n$ is even and $1$ if $n$ is odd.

$\chi$ is independent of the choice of $\mathcal{B}$.

11.2 We "define" a lat-isomorphism invariant $\chi$ on the class $C_2(X)$ of principal bundles with group $K \times \mathbb{Z}_2$ and base $X$, for a fixed compact Hausdorff $X$.

Suppose $\mathfrak{B} = (Y, K \times \mathbb{Z}_2, Y)$. Let $Y/K$ denote the orbit space of $Y$ under $K \leq K \times \mathbb{Z}_2$, and $Y = Y/K$ the orbit map. Let $\mathcal{B}_1 = (Y/K \times \mathbb{Z}_2, \mathcal{V}_1)$ and $\mathcal{B}_2 = (Y, K \times \mathbb{Z}_2, \mathcal{V}_2)$, where $\mathcal{V}_1 = \mathcal{V}_2 = Y$.

Define $\chi(\mathcal{B}) = (\mathcal{B}_1, \mathcal{B}_2)$.

It is simple but tedious to give a more rigorous definition of $\chi$ making it a lat-isomorphism invariant on $C_2(X)$. But $\chi$ is not a complete invariant and does not map onto any simply defined domain. However, $\chi$ will be a help in determining the lat-isomorphism classes of bundles in $C_2(X)$.

"Non-surjectivity" For an example of how one determines when a couple $(\mathcal{B}_1, \mathcal{B}_2)$ of fibre bundles is not in the image of $\chi$, see 11.8.

"Non-completeness" For an example of how one determines how many lat-isomorphism classes in $C_2(X)$ have the same image under $\chi$, see 11.10.

11.3 Definition Given a principal bundle $\mathfrak{B} = (Y, K, \mathcal{V})$, define $\mathfrak{B}' = (Y \times \mathbb{Z}_2, \mathcal{V}, \mathfrak{F}, \mathcal{E}, \mathcal{Y})$, a bundle with group $K \times \mathbb{Z}_2$, as follows:

the action of $K \times \mathbb{Z}_2$ on $Y \times \mathbb{Z}_2$ is defined in terms of the action of $K$ on $Y$ for $\mathfrak{B}$ by $k.(y, 1) = (k.y, 1)$

$k.(y, \epsilon) = (k^{-1}.y, \epsilon)$ for all $y \in Y$, $k \in K$, where $\mathbb{Z}_2 = \{1, \epsilon\}$.

$\mathcal{E}$ acts on $Y \times \mathbb{Z}_2$ by $\mathcal{E}.(y, \epsilon) = (y, \epsilon)$ for all $y \in Y$, $\epsilon \in \mathbb{Z}_2$.

$\mathcal{F}(y, \epsilon) = \mathcal{V}(y)$.

$\mathfrak{F}(y, \epsilon) = \mathfrak{V}(y)$.

Note that if $\mathfrak{B}$ is a product bundle, so is $\mathfrak{B}' = \mathfrak{B} \times \mathbb{Z}_2$. 
We wish to find the lst-isomorphism classes of bundles \((Y, \pi X, G, H, s, V)\)
where \((G, H)\) is a connected irreducible pair with \(\dim G/H = 2\). So (10.6 - 10.6)
\((G, H) = (G_0 \times A, (H \cap G_0) \times A),\) where \(A\) is a finite subgroup of
\(\text{Aut}(G_0)\) and \((G_0, H \cap G_0) = (K_1, [1]), (K_2, [1]), (SO(3), K),\) or \((SO(3), K \times \mathbb{Z}_2)\).

Fix \((G, H) = (G_0 \times A, H' \times A)\) and \(\sigma \in A\). The lst-isomorphism classes are
given by the bundles \(\mathcal{K}(G_0, H', A, \sigma) = \{Y(\sigma), \pi_X, K, G_0, E, V, \psi\}\) \((\mathcal{K}(G_0, A, \sigma)
\text{or } \mathcal{K}(G_0, H', A)\text{ or } \mathcal{K}(G_0, A)\text{ or } \mathcal{K}(G_0),\) depending on which of \(H', A, \sigma\) are trivial) where \(\sigma\) runs through the \(A\)-conjugacy classes in \(A\), and
the principal \(G\)-bundle associated with \(\mathcal{K}(G_0, H', A, \sigma)\), which is an element
of \(\mathcal{C}_2(G)\) (11.1), is mapped to \(\sigma\) under \(\chi\).

Define \(Y(\sigma) = K \times (G_0 \times A/A')\), where \(A' = \langle \sigma \rangle \) and \(A/A' = \left\{ A' : \text{Tr} A' \right\}\).

There is a natural left \(G\)-action on \(G_0 \times A/A'\).

Let \(r = \text{order of } \sigma\). There is a natural left \(K \times \mathbb{Z}_2\)-action on \(K\). Define
left \(G\)-action on \(K\) by \((g_0 \in G_0, k \in K, \sigma \in K) \mapsto (g_0 \cdot \pi_X).k = F_\sigma(\pi_X)k\) for all \(g_0 \in G_0, \pi_X \in K, \sigma \in K,\) where \(F_\sigma : A \rightarrow K \times \mathbb{Z}_2\) is some chosen homomorphism for which
\(F_\sigma(\sigma) = (e^{2\pi i/4}, 1)\) (always possible for the \(A\)'s being considered).

Define action of \(G\) on \(Y(\sigma)\) by \(g \cdot (k, x) = (g, k, x)\) for all \(g \in G, k \in K, x \in G_0 \times A/A').\)

Define \(\nu : Y(\sigma) \rightarrow K\) by \(\nu((1, r), (k, g_0, A')) = k^r\) for all \(r \in A, k \in K, g_0 \in G_0, x \in A/A').\) (\(\nu\) is well-defined.)

Define \(\phi_1 : V_2 \rightarrow Y(\sigma)\) (see 11.1(ii)) by \(\phi_1(e^{i\theta}) = (e^{i\theta}, 1, A')\)
for \(5\pi/4 \leq \theta \leq 7\pi/4.

\(q_2 : V_2 \rightarrow Y(\sigma)\) by \(q_2(e^{i\theta}) = (e^{i\theta}, 1, A')\) for \(\pi/4 \leq \theta \leq 3\pi/4.

Then \(\delta_{1,2}(-1) = (1, e^{i\theta})\) as required (see 11.1(ii)).
For definition of $\mathcal{K}(\sigma)$ and $\mathcal{S}$ consider the different possibilities for $(G,H)$:

$(G,H) = (K \times \mathbb{A}, \mathbb{A})$ where $\mathbb{A} = \{1\}$ or $\mathbb{A} = \{2,3\}$ and $\mathbb{Z}_2$ where $\xi(k) = k^{-1}$ ($k \in K$).

$\mathcal{K}(K)$ and $\mathcal{K}(K, \mathbb{Z}_2)$ are the product bundles.

$\mathcal{K}(K, \mathbb{Z}_2, \xi) = (K^2, K, K, K, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2)$ where $\xi : K^2 \to KB$ is defined by

$\xi(k_1, k_2) = [k_1, k_2]$ (see 8.1, 8.2).

$(G,H) = (K^2 \times \mathbb{A}, \mathbb{A})$

$\mathcal{K}(K^2, \mathbb{A}, \sigma) = (K^2 \times \mathbb{A}, K^2, K^2, K^2, K^2, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$ where $\xi : K^3 \times \mathbb{A} / \mathbb{A} \to K^3 / \sigma$ is well-defined by $\xi((1, \tau), (k_1, k_2, k_3, \mathbb{A})) = [k_1, k_2, k_3]$ for all $k_1, k_2, k_3 \in K$ and $\tau \in \mathbb{A}$ (see 8.2 for definition of $K^3 / \sigma$ and 8.1).

$K^3 / \sigma$ is homeomorphic to the unique $K^3 / \gamma$ in the list of 8.3 for which $\sigma$ is conjugate in Aut($K^3$) to $\gamma$.

$(G,H) = (SO(3), K)$ or $(SO(3), K \times \mathbb{Z}_2)$

$\mathcal{K}(SO(3), K) = (K \times SO(3), K \times \mathbb{S}_2, K, SO(3), K, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$ and

$\mathcal{K}(SO(3), K \times \mathbb{Z}_2) = (K \times SO(3), K \times \mathbb{S}_2, K, SO(3), K \times \mathbb{Z}_2, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$

are product bundles where $\xi_1$ and $\xi_2$ are defined by

$\xi_1(k, (u_{11})) = (k, (u_{11} \cdot u_{12} \cdot u_{13}))$ $\xi_2 : K \times SO(3) \to K \times \mathbb{S}_2$

$\xi_2(k, (u_{11})) = (k, [(u_{11}, u_{12}, u_{13})])$ $\xi_2 : K \times SO(3) \to K \times \mathbb{S}_2$

See 8.2 for the definition of $\mathbb{S}_2$ and 8.1.)

$(G,H) = (SO(3), K \times \mathbb{Z}_2, \mathbb{Z}_2)$

$\mathcal{K}(SO(3), K \times \mathbb{Z}_2) = (K \times SO(3) \times \mathbb{Z}_2, K \times \mathbb{S}_2, SO(3) \times \mathbb{Z}_2, K \times \mathbb{Z}_2, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$ is

the product bundle.

$\mathcal{K}(SO(3), K \times \mathbb{Z}_2, \xi) = (K \times SO(3), \mathbb{Z}_2, K \times \mathbb{Z}_2, \mathbb{Z}_2, K \times \mathbb{Z}_2, \mathbb{A}, \mathbb{A}, \mathbb{A}, \mathbb{A})$

has $\mathcal{K}(K, SO(3), \mathbb{Z}_2)$ defined by:

$\xi(k, (u_{11})) = [k, (u_{11} \cdot u_{12} \cdot u_{13})]$ where, if $k = e^{i\theta}$, $\omega_k \in SO(3)$ is

defined by $\omega_k = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$
11.5 ist-isomorphism classes of bundles with base $S^2$

We state the results without proof. (See [17] 10.5.)

(i) Bundles with group $K$

The distinct ist-isomorphism classes are given by $K_n$ ($n \in \mathbb{Z}$).

$n = 0$ $\mathcal{E}_0$ is the product bundle.

Fix $n > 0$ $\mathcal{E}_n = (SU(2)/\mathbb{Z}_n, S^2, K, \pi_n)$.

The action of $K$ on $SU(2)/\mathbb{Z}_n$ is defined by:

$$k \cdot Z_n \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = Z_n \begin{pmatrix} k^{1/n} & 0 \\ 0 & k^{-1/n} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

Diagram 11.5

$S^2$ is given "cylindrical coordinates", so $S^2 = ((-1,1)x1) \cup \{p, p'\}$.

Using these coordinates, $\pi_n : SU(2)/\mathbb{Z}_n \rightarrow S^2$ is defined by:

$$\pi_n \left( Z_n \left( \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) \right) = \begin{cases} (1-2r, e^{i(\theta-\theta')}), & 0 < r < 1 \\ p, & r = 1 \\ p', & r = 0 \end{cases}$$

For $n > 0$, $\mathcal{E}_n = (SU(2)/\mathbb{Z}_n, S^2, K, \pi_n)$ where $\pi_n = \pi_n$ and the action of $K$ on $SU(2)/\mathbb{Z}_n$ is defined by:

$$k \cdot Z_n \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = Z_n \begin{pmatrix} k^{1/n} & 0 \\ 0 & k^{-1/n} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

(ii) Bundles with group $K \times \mathbb{Z}_2$

The distinct ist-isomorphism classes are given by $K \times \mathbb{Z}_2$, $n > 0$ (see 11.4).
We state the results without proof.

(1) Bundles with group $K$

There are two isomorphism classes of bundles, denoted by $\mathcal{F}_0$ and $\mathcal{F}_1$.

$\mathcal{F}_0$ is the product bundle.

\[ \mathcal{F}_1 = \{(S^2 \times K)/\sim, P^2, K, \pi \} \]
where $K$ acts on $(S^2 \times K)/\sim$ by

\[ k \cdot [x_1, k] = [x_1, k k_1] \text{ for all } x_1 \in S^2, k, k_1 \in K. \]

$\pi : (S^2 \times K)/\sim \rightarrow P^2$ is defined by $\pi([x, k]) = [x]$.

(II) Bundles with group $K \times \mathbb{Z}_2$

The isomorphism classes are given by bundles denoted by $\mathcal{F}_0 \times \mathbb{Z}_2$.

$\mathcal{F}'_0 = \{(S^2 \times \mathbb{Z}_2)/\sim, P^2, K \times \mathbb{Z}_2, z_2, \pi, \gamma \}$ (see 8.2 for definition of $(S^2 \times K)/\sim$).

Action of $K \subseteq K \times \mathbb{Z}_2$ on $S^2 \times K$ is defined by

\[ k \cdot (y_1, k_1) = (y_1, k k_1) \text{ for all } y_1 \in S^2, k, k_1 \in K. \]

Action of $\mathbb{Z}_2 \subseteq K \times \mathbb{Z}_2$ on $S^2 \times K$ is defined by

\[ \varepsilon \cdot (x, k) = (-x, k^{-1}). \]

$\gamma : S^2 \times K \rightarrow P^2$ is defined by $\gamma(x, k) = [x]$ ($x \in S^2, k \in K$).

$\mathcal{F}'_n = \{(SU(2)/\mathbb{Z}_2, SO(3)/\mathbb{Z}_2, P^2, K \times \mathbb{Z}_2, z_2, \pi_n, \gamma_n, \gamma'_n) \}
Action of $K$ on $SU(2)/\mathbb{Z}_2$ is as for $\mathcal{F}_2n$ (11.5), and $\gamma_n$ is as $\pi_2n$ for $\mathcal{F}_2n$.

Action of $\varepsilon \in \mathbb{Z}_2 \subseteq K \times \mathbb{Z}_2$ on $SU(2)/\mathbb{Z}_2$ is defined by

\[ \varepsilon \cdot z_2n \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = z_2n \begin{pmatrix} u_{21} & u_{22} \\ -u_{11} & -u_{12} \end{pmatrix} \]

11.7 Isomorphism classes of bundles with base $K^2$

(1) Bundles with group $K$ use $\chi : \mathbb{Z}_2 \rightarrow \Theta_2$ (11.1).

$\chi(0) = 0$ This gives the product bundle, denoted by $\mathcal{F}_0$.

$\chi(\mathbb{Z}_2) = n > 0$. A bundle with this characteristic is $\mathcal{F}_n$, defined as follows.

For $n > 0$, define $\mathcal{F}_n = \mathcal{F}_n$ (8.2).
and let \( J_n = \langle N/r_n, K^2, K_n \rangle \).

Action of \( K \) on \( N/r_n \) is given by \( e^{2\pi i t} [x, y, z] = [x, y + t/2, z - t/2] \).

\( \Pi_n : N/r_n \rightarrow K^2 \) is defined by \( \Pi_n ([x, y, z]) = (e^{2\pi i x}, e^{2\pi i z}) \).

Define \( \varphi_n : N/r_n \rightarrow N/r_n, n = 1, 2 \) (11.1(1.1)) by
\[
\varphi_n (e^{2\pi i x}, e^{2\pi i z}) = [x, 0, z], \quad 0 \leq x \leq 1
\]

(11) Bundles with group \( Z_2 \): use \( \chi_1 : C_1(k^2, z_2) \rightarrow \mathcal{D}_1(k^2, z_2) \) (11.1).

\( \mathcal{D}_1(k^2, z_2) = \text{Hom}(N_1(k^2), z_2) = \{ \eta_1, \eta_2, \eta_3, \eta_4 \} \).

\[ N_1(k^2) = \langle a, b : ab = ba \rangle. \]

\( a \) and \( b \) are the homotopy classes corresponding to the paths \( t \mapsto (e^{2\pi it}, 1) \)
and \( t \mapsto (1, e^{2\pi it}) \) respectively.

We define a bundle \( \mathcal{B}_1 = (X_1^1, k^2, z_2, \nu_1^1) \) with \( \chi(\mathcal{B}_1) = \eta_1 (i = 1, \ldots, n) \).

\( \eta_1(a) \cdot \eta_1(b) = 1 \) : \( \mathcal{B}_1 \) is the product bundle.

\( \eta_2(a) = \xi, \eta_2(b) = 1 : X_1^2 = K^2, \quad \chi_1(k_1, k_2) = (-k_1, k_2), \quad \nu_2(k_1, k_2) = (k_1^2, k_2) \)

\( \eta_3(a) = 1, \eta_3(b) = \xi : X_1^3 = K^2, \quad \chi_3(k_1, k_2) = (k_1, k_2), \quad \nu_3(k_1, k_2) = (k_1^2, k_2) \)

\( \eta_4(a) = \eta_4(b) = \xi : X_1^4 = K^2, \quad \chi_4(k_1, k_2) = (-k_1, k_2), \quad \nu_4(k_1, k_2) = (k_1^2, k_2) \).

(11.1) Bundles with group \( K \times Z_2 \): use \( \mathcal{I}_2 \) defined on \( C_2(k^2) \) (11.2).

i.e., we find isomorphism classes of bundles

\( \mathcal{B} = (Y, Y/Z_2, K^1, K \times Z_2, z_2, \nu, \nu_2) \) in terms of the bundles

\( \mathcal{B}_1 = (X_1, k^2, z_2, \nu_1) \) and \( \mathcal{B}_2 = (Y, X_1, k, \nu_2) \). Complete proofs will not be given.

\( \chi(\mathcal{B}_1) = \eta_2 \) : \( \mathcal{B}_2 \) is the product bundle.

\( \chi(\mathcal{B}_2) = \eta_1, \eta_3, \eta_4 \) : \( \mathcal{B}_2 \) is the product bundle.

It can be shown that the only possibilities up to isomorphism are \( \eta_2, x \neq 2 \) (11.3).

The action of \( Z_2 \) is determined by \( \mathcal{B}_2 \) up to isomorphism, and can be taken to be as follows:

\( \chi(\mathcal{B}_2) = \eta_2 \) : \( \mathcal{B}_2 = (Y, k_1, k_2, k_3) = (-k_1, k_2, k_3) \) : given bundle \( \eta_2 (-k_1, k_2, k_3) \).
or \( \xi(k_1, k_2, k_3) = (-k_1, k_2, k_2 k_1^{-1}) \) : gives bundle \( \mathcal{J}_0(-k_1, k_2, k_2 k_1^{-1}) \)

or \( \chi(\mathcal{B}_1) = \eta_1 \) : gives bundle \( \mathcal{J}_0(-k_1, -k_2, k_3) \)

or \( \chi(\mathcal{B}_1) = \eta_1 \) : gives bundle \( \mathcal{J}_0(-k_1, -k_2, k_3) \)

or \( \chi(\mathcal{B}_1) = \eta_1 \) : gives bundle \( \mathcal{J}_0(-k_1, -k_2, k_3) \)

In each case \( Y/Z_2 \) is homeomorphic to \( k^3/\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( k^3/\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) respectively.

11.8 As an example of the method used in the calculation of 11.7(iii), we show that if \( \chi(\mathcal{B}_1) = \eta_2 \) and \( \mathcal{B}_2 = \mathcal{J}_0 \), then up to list-isomorphism, the action of \( \xi \) on \( k^3 \) must be:

\[ \xi(k_1, k_2, k_3) = (-k_1, k_2, k_1 k_2^{-1}) \text{ or } (-k_1, k_2, k_1 k_2^{-1}). \]

It can be shown that the action of \( \xi \) must be of the form:

\[ \xi(e^{2\pi i} \theta_1, e^{2\pi i} \theta_2, e^{2\pi i} \theta_3) = (e^{2\pi i} \theta_1, e^{2\pi i} \theta_2, e^{2\pi i} \theta_3) \]

where \( f \in \mathbb{C}(\mathbb{R}^2, \mathbb{R}) \) has \( f(\theta_1 + \pi, \theta_2) = f(\theta_1, \theta_2) \pmod{2} \) and:

\[ e^{2\pi i} f_2(\theta_1, \theta_2) = e^{2\pi i} f_2(\theta_1, \theta_2) \]

i.e. \( f_2(\theta_1, \theta_2) = f_2(\theta_1, \theta_2) \pmod{2} \).

Given \( f_1 \), we can choose a suitable \( \mathcal{F}(\theta_1, \theta_2) = (f_2(\theta_1, \theta_2))/2 - \delta \theta_2^2 + \beta \theta_1 \)

\( (\theta = 0 \text{ or } 1, \beta = 0 \text{ or } 1) \) so that \( f_2(\theta_1, \theta_2) = 0 \) or \( \theta_2 \) as required.
11.0. Lat-isomorphic classes of bundles with base $K^2$

(i) Bundles with group $K$: use $\chi : C^4 \rightarrow C^4$ (1.1).
$\chi(0) = 0$ This gives the product bundle, denoted by $K \times K$.
$\chi(1) = 1$ This gives $K \times K = (K^2, K, K, \pi)$, where the action of $K
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
is defined by $k. [k_1, k_2, k_3] = [k_1, k_2, k_3] (k_1, k_2, k_3, k \in K)$.
$\Pi : K^2 \oplus \mathbb{Z} \rightarrow K^2$ is defined by $\Pi ([k_1, k_2, k_3]) = [k_1, k_2]$. 

(ii) Bundles with group $Z_2$: use $\chi : C^4 \rightarrow C^4$ (1.1).
$\chi_1(KB, Z_2) = H_1(\chi, Z_2) = \langle \eta_1, \eta_2, \eta_3, \eta_4 \rangle$.
$\Pi_1(KB) = \{a, b : ab = b^{-1}a \}.$
a and $b$ are the homotopy classes corresponding to the paths $t \mapsto [a^t b, 1]$ and $t \mapsto [1, a^t b]$ ($t \in [0, 1]$) respectively.

We define a bundle $\overline{\beta}_1 = (X_1, KB, Z_2, Y_1)$ with $\chi(\overline{\beta}_1) = \eta_1 (i = 1, \ldots, 4)$.

(iii) Bundles with group $X_1 \times Z_2$: use $\chi$ defined on $C^4(2, Z_2)$ (1.2).

i.e., we find $\chi$-isomorphic classes of bundles
$\beta = (\gamma, \gamma_2, KB, X_1, Z_2, Z_2, Y_1, Y_2)$ in terms of the bundles $\overline{\beta}_1 = (X_1, KB, Z_2, Y_1)$ and $\overline{\beta}_2 = (Y_1, KB, Y_2)$. 

$\chi(\overline{\beta}_1) = \eta_1$; It can be shown that up to isomorphism, the only possibilities for $\beta$ are $KB \times Z_2$ and $KB \times Z_2$ (1.9(1), 1.3).
$\chi(\overline{\beta}_1) = \eta_2$; It can be shown that $\beta_2 \ast \beta_2 \ast \beta_2$ must be $\sum_0, \sum_1$ (1.7), and then the only possibility for $\beta$ is $\sum_0 = (K^3, K^3 \oplus \mathbb{Z}, KB, KB, K, \pi, \tau, \nu)$. 

where the action of $K \times \mathbb{Z}_2$ on $\mathbb{R}^3$ is given by:

$$k_1(k_2, k_3) = (k_1k_2, k_1k_3), \quad \varepsilon(k_1k_2, k_3) = (-k_1, k_2^{-1}, k_3).$$

$\varepsilon$ is given by $\varepsilon(k_1, k_2, k_3) = [k_1, k_2].$

$\gamma(\mathfrak{A}_1) = \eta_3$: It can be shown (11.10) that $\mathfrak{A}_2$ must be $\mathfrak{A}_2$ - not $\mathfrak{A}_3$ - up to a left-isomorphism. Up to left-isomorphism, there are two possibilities for $\mathfrak{A}_2$, determined by two possible actions of $\mathfrak{A}$ on $Y = KB \times K$:

- $\varepsilon([k_1, k_2], k_3) = ([k_1, k_2], k_1k_3^{-1})$: gives bundle $\mathfrak{A}_2([k_1, k_2], k_1k_3^{-1})$
- $\varepsilon([k_1, k_2], k_3) = ([k_1, k_2], k_1k_3^2)$: gives bundle $\mathfrak{A}_2([k_1, k_2], k_1k_3^2).$

$Y/\mathbb{Z}_2$ is homeomorphic to $\mathfrak{A}_0, \mathfrak{A}_2$ respectively (see 6.2, 6.3).

$\gamma(\mathfrak{A}_1) = \eta_3$: There is a natural correspondence of the possible bundles with those for $\gamma_3$, therefore they will not be listed.

11.10 As an example of the method used in the calculation of 11.9(11), we sketch the proof that if $\gamma(\mathfrak{A}_1) = \gamma_3$ then $\mathfrak{A}_2$ must be $\mathfrak{A}_2$ up to left-isomorphism.

If $\mathfrak{A}_2$ is $\mathfrak{A}_2$, then there exists a homeomorphism $\varepsilon: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\varepsilon^2 = \text{identity}$ and the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\varepsilon} & \mathbb{R}^3 \\
(k_1, k_2, k_3) & \downarrow & (k_1, k_2, k_3) \\
\mathfrak{A}_2 & \xrightarrow{\gamma} & \mathfrak{A}_2 \\
[k_1, k_2] & \downarrow & [k_1, k_2] \\
\end{array}
\]

i.e. $\gamma$ is of the form:

$$\gamma(z, x, y) = (2z + x, e^{2\pi i(x+y)}, e^{2\pi i(\Psi(x, y) - 2)})$$

where $\Psi \in C(\mathbb{R}^2, \mathbb{R})$ and:

- $\Psi(x, y+1) = \Psi(x, y) \mod \mathbb{Z}$
- $-2y + \Psi(x+1, y) = \Psi(x, y) + \frac{1}{2} \mod \mathbb{Z}$
- $\Psi(x, y+2) = \Psi(x, y) \mod \mathbb{Z}$ (since $\varepsilon^2 = \text{identity}$).

By considering a suitable function $\gamma_3(x, y) = \gamma(x, y) + ax$, we can assume:
Let $l(x, 0) = t$. Evaluating $v_t(\frac{1}{2}, t)$ in two different ways, we see $v_t$ cannot exist.

4.12 $Z$-admissibility

12.1 We use the notation of 8.2 throughout this section. Denote a t.g. $(x, z)$ by $(x, t)$ where $t$ is the homeomorphism of $X$ corresponding to $t \in Z$. In this section we prove (without full details) that the n-allovable strings ($n \leq 3$) are $Z$-admissible or not as recorded in tables A and B of §9. 12.2 - 12.6 are devoted to showing that the strings for which non-$Z$-admissibility is claimed in tables A and B are indeed not $Z$-admissible. 12.7 - 12.16 are devoted to reducing the problems of $Z$-admissibility of the remaining strings to problems concerning the existence of minimal group extensions of certain t.g.'s, and 12.17 - 12.18 are devoted to solving these problems.

12.2 Definition For a homeomorphism $q$ of $K^2$, let $r(q)$ be the unique $(r_1, r_2)$ in $GL(2, Z)$ such that $r(q)$ is homotopic to:

$$(k_1, k_2) \longrightarrow (r_{11}, r_{12}, r_{21}, r_{22})$$

Then $\det r(q) = \pm 1$.

12.3 If $(K^2, t)$ is minimal almost periodic, it is immediate that $\det r(t) = 1$.

12.4 If $(x, t)$ is a minimal distal t.g. with $Q(x, t) \neq 1$ in $A_4$ of table A, it is clear that $t : X \longrightarrow X$ must be of the form:

1. $(k_1, k_2) \rightarrow (\sigma(k_1), \sigma(k_2))$

2. $(k_1, k_2) \rightarrow (\sigma(k_1), \sigma(k_2)^{-1})$

3. $(k_1, k_2) \rightarrow (\sigma(k_1), \sigma(k_1)k_2)$

for all $k_1, k_2 \in K$.

In each case, $\sigma = e^{2\pi i \beta}$, where $\beta$ is irrational, and $\sigma \in C(K, K)$.

For $A_3$, $\sigma(x^{-1}) = \sigma(k_1)$ for all $x_1 \in K$.

Therefore, if $Q(K^2, t)$ is as in $A_1$, set $r(t) = 1$, and if $Q(K^2, t)$ is as in $A_2$, set $r(t) = -1$.

12.5 The following lemma shows that the strings of $A_2$, $B_4$, $B_7$ are not
\[ Z \text{-admissable, and will help prove the } Z \text{-admissibility of the strings } A_0, B_3, B_5. \]

**Lemma** For \( n \geq 0 \), let \( J_n = (N/R_n, K^2, \pi_n, \eta_n) \) be as in 11.7. Let \( \phi: K^2 \to K^2 \) be a homeomorphism.

(i) det \( r(\phi) = 1 \) if and only if \( \phi \) exists as in the commutative diagram 12.5 with \( \Psi(x) = k \cdot \phi(x) \) for all \( x \in N/R_n, k \in K \) (action of \( K \) on \( N/R_n \) as for \( J_n \)).

(ii) det \( r(\phi) = -1 \) if and only if \( \phi \) exists as in diagram 12.5 with \( \Psi(x) = k^{-1} \cdot \phi(x) \) for all \( x \in N/R_n, k \in K \).

**Diagram 12.5**

\[
\begin{array}{ccc}
N/R_n & \xrightarrow{\phi} & N/R_n \\
\downarrow{\pi_n} & & \downarrow{\pi_n} \\
K^2 & \xrightarrow{\phi} & K^2
\end{array}
\]

**Proof** It suffices to show the bundle \((N/R_n, K^2, \pi_n, \eta_n, \phi)\) (where the action of \( K \) on \( N/R_n \) is as for \( J_n \)) is \( K \)-admissible to \( J_n \) if det \( r(\phi) = 1 \) and to \( J_{n\cdot} \) if det \( r(\phi) = -1 \). By the First Homotopy Covering Theorems (17 11.5) it suffices to prove this for \( \phi \) of the form:

\[
\phi(k_1, k_2) = (k_1^{r_{11}}, k_2^{r_{21}}, k_1^{r_{12}}, k_2^{r_{22}}), \quad (r_{ij}) \in GL(2, \mathbb{Z}).
\]

But this is a straightforward computation.

**12.6 Lemma** The string of \( A_\infty \) corresponding to \( K^2, A_\sigma \) is \( Z \)-admissable only if \( \Phi^\sigma \in A \) and \( N(\sigma) \) is cyclic (see table A).

**Proof** Suppose \((K^2/\sigma, t)\) is a minimal distal t.g. with \( \Phi(K^2/\sigma, t) \) the string of \( A_\infty \) corresponding to \( K^2, A_\sigma \). Then \( (K^2/\sigma, t) \preceq (K^2, A_\sigma, e) \) where \( e \) is a minimal distal homeomorphism commuting with the action of \( K^2, A \) (11.4).

Fix \( T \in A \). Write \( 1 = (1, 1, 1, 1, \eta) \). \( \lambda_0^e = K^2 \cdot \phi(\eta) \) for some \( \eta \), by the minimality of \( e \).

\[
\lambda_0^e = (1, 1, 1, \eta(\sigma))e^m \text{ for all } \eta \not\prec e\}
\]

\[
= ((1, 1)(F, \eta^{-1}), 1, 1, \eta(\sigma))e^m \in K^2 \cdot \eta \not\prec e\}
\]

Therefore \( T^{-1} \eta \not\prec e \) for all \( \eta \not\prec e^\sigma \), \( \tau \in A \).
Therefore \(\omega \neq 0\). Clearly \(A/\langle \alpha \rangle\) must be cyclic.

12.7 Definitions Let \((Y,X,G,V)\) be a principal bundle (3.1) and let \(t\) be a homeomorphism of \(X\). Let \(\text{Hom}(Y:X,G,V,t)\) denote the set of homeomorphisms of \(Y\) such that:

1. \((g \cdot y)a = g \cdot (ya)\) for all \(g \in G, y \in Y\).
2. \((X,t) \prec (Y,a)\)

Write \(\text{Hom}(Y,G,t)\) for \(\text{Hom}(Y:X,G,V,t)\) if the definition of \(X, V\) are clear from the context.

Let \(\text{Hom}(Y,G,t) \neq \emptyset\). Let \(Y\) be metric, and note that all metrics on \(Y, G\) respectively (giving rise to the right topologies) are equivalent. Let \(\text{Hom}(Y,G)\) and \(C(Y,G)\) be given supremum metrics (any two such metrics on \(\text{Hom}(Y,G,t)\), \(C(Y,G)\) respectively are equivalent). Then \(\text{Hom}(Y,G,t)\) is a complete metric space and is isomorphic (as a metric space) to:

\[\left\{ f \in C(Y,G) : f(g \cdot y) = g(f(y)) \text{ for all } y \in Y, g \in G \right\}\]

which is in turn isomorphic (as a metric space) to \(C(X,G)\) if \(G\) is abelian, or if \((Y,X,G,V)\) is a product bundle. In the latter case, for a homeomorphism \(t\) of \(X\), the element of \(\text{Hom}(X \times G,G,t)\) corresponding to \(f \in C(X,G)\) is denoted by \(s\) if this notation cannot give rise to confusion, where:

\[(x,g)f(x) = (gt(x),g(x)) \text{ for all } x \in X, g \in G.\]

If \(h \in G\) and \(t' \in \text{Hom}(Y:X,G,H,t)\) then define:

\[\text{Hom}(Y:G,H,t,t') = \text{Hom}(Y:H,t) \cap \text{Hom}(Y,H,t')\]

so that if \(s \in \text{Hom}(Y:H,t,t')\), in particular the following diagram commutes:

\[
\begin{array}{ccc}
Y/H & \xrightarrow{s} & Y/G \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
[y]_{H} & \xrightarrow{t} & [y]_{G} \\
\end{array}
\]

Diagram 12.7
If $t$ is minimal, let $D(Y; G, t)$ (or $D(Y; G, t')$) be:

\[ \{ a \in \text{Hom}(Y; G, t) : a \text{ is minimal} \} \]

If $t$, $t'$ are minimal, let $D(Y; G, t, t') = D(Y; G, t) \cap \text{Hom}(Y; G, t, t')$.

12.3 Definitions Let $O = (O_1 \ldots O_n)$ be a string (1.1) with $O_1 = (X_1, X_2, X_1, X_2, \ldots)$ (1 < 1 < 2). Let $t$ be a minimal distal homeomorphism of $X_t$ with $O(X_{t-1}, t) = (O_1 \ldots O_{t-1})$.

Define $D(Y; O, t)$ as follows: $a \in D(Y; O, t)$ is in $D(Y; O, t)$ if and only if $O(X_t, u) = O$ where $u$ is the unique homeomorphism making the following diagram commutative:

Diagram 12.8

Using induction, $Z$-admissability of strings of tables $A$ and $B$ is implied by the following proposition, which we shall spend the rest of the section in proving using the notation of 12.6 (and of 12.7) throughout.

Proposition Let $O, t, t'$ be as in 12.6, and $L_t = G_{G_t}$, the identity component of $G_t$, and suppose $O$ is one of the strings of tables $A$ and $B$ for which $Z$-admissability is claimed. Then:

\[ D(Y; O, t, t') \text{ is dense in } \text{Hom}(Y; G_t, t, t') \]

For all the $Z$-admissable strings of tables $A$ and $B$ except $A_{10}$, $A_{33}$, $A_{33}$, proof of $Z$-admissability is achieved by reducing the problem to a similar
problems concerning minimal extensions and strings in which the final bundle \( \mathcal{B}_r \) is a product bundle with connected group and (possibly non-connected) base (see 12.17 for statement of the reduced problem). First (12.10 - 12.11) we deal with the strings of \( A10, B3, B8 \).

12.10 Lemma. If \( \mathcal{S} \) is one of the strings \( A2, A3, A5 - A6, A10, A11, A13, A14, B3, B8, B22, B23 \), then \( M(Y_r; \mathcal{G}_r, t) = \mathcal{D}(Y_r; \mathcal{G}_r, t) \), so that \( M(Y_r; \mathcal{G}_r, \mathcal{G}_r', t', t') = \mathcal{D}(Y_r; \mathcal{G}_r, t', t') \).

Proof. Let \( s \in \mathcal{M}(Y_r; \mathcal{G}_r, t) \) and suppose \( s \notin \mathcal{D}(Y_r; \mathcal{G}_r, t) \).

We shall assume \( \mathcal{S} \) is one of the strings of table A (proof is similar for \( B3, B8, B22, B23 \)). So \( r = 2 \). If \( s \notin \mathcal{D}(Y_r; \mathcal{G}_r, t) \) then the phase space of the maximal almost periodic factor of \((X_2, \mu)\), where \((X_2, \mu) \sim \mathcal{L}(Y_r, \mathcal{G}_r, m) \), must be \( T_2/I_2 \) where \( L_2 \) and \( g_2 \) are \( L_2 \), with \( g_2 = L_2 \) if \( L_2/I_2 \) is finite (5.5, 5.6, 5.7). In the particular cases considered, this implies \( H_2 = L_2 \), hence \((X_2, \mu)\) is almost periodic, \( g_2 \) is trivial, \( X_2 \) is abelian and \( X_2 \) is a torus - which is not true for the strings of \( A2, A3, A5 - A6, A10, A11, A13, A14 \).

12.11 If \( \mathcal{S} \) is the string of \( A10, [2]6.19.2.6 \). implies \( \text{Hom}(Y_r; \mathcal{G}_r; \mathcal{G}_r', t, t') = \mathcal{M}(Y_r; \mathcal{G}_r; \mathcal{G}_r', t, t') \) and hence by a simple argument the same is true of the strings of \( B3, B8 \). By 12.10 this implies proposition 12.9 is proved for the strings of \( A10, B3, B8 \).

12.12 Now we need some definitions:

Definitions: (i) If \( f \in C(X, \mathcal{K}) \), \( f \) can be uniquely written in the form

\[
f(k_1) = P_{k_1} e^{i p(k_1)}
\]

where \( p \in \mathbb{Z} \), \( e \in X \) and \( \int \Re k_1 \text{dk}_1 = C_\mathcal{K} \), \( e \in C(\mathcal{K}, \mathbb{R}) \).

Define \( P : C(X, \mathcal{K}) \to C(X, \mathcal{K}) \) by \( Pf(k_1) = e^{i p(k_1)} \).

(ii) If \( f \in C(X, X) \), \( f \) can be uniquely written in the form:

\[
f(k_1, k_2) = P_{k_1} P_{k_2} e^{i \Re (k_1, k_2)}, \quad \text{where } \int \Re k_1 k_2 \text{dk}_2 = 0.
\]

Define \( P : C(X, X) \to C(X, X) \) by \( Pf(k_1, k_2) = e^{i \Re (k_1, k_2)} \).

(iii) For \( f \in C(X, \mathcal{K}^n) \) or \( C(X, \mathcal{K}^n) \), define \( Pf = (P_f, \ldots, P_f) \) if \( f = (f_1, \ldots, f_n) \).
(iv) If \( B \) is a finite set then \( C(K^s \times B, K^p) \) is isomorphic (as a group, with pointwise multiplication) to \( C(K^s, (K^p)^B) \) under the map:

\[
 f \mapsto \{f_b(x) = f(b, x) \mid b \in K^s, x \in \mathbb{B}\}.
\]

Using this isomorphism, define \( P : C(K^s \times B, K^p) \to C(K^s \times B, K^p) \) for \( s = 1, 2 \).

(v) In each case, \( P \) is a continuous group homomorphism with respect to the uniform topology, and \( P^2 = P \).

12.13 Definitions (1) For \( X \) a compact Hausdorff space and \( G \) a compact group, a group \( C \) is said to be an automorphism group of \((X, G)\) if \( C \) acts freely on \( X \) and acts as a group of automorphisms on \( G \), both actions being on the left.

(ii) If \( C \) is an automorphism group of \((X, G)\), let \( \mathcal{A}_C \) denote the closed subgroup of the group \( C(X, G) \) (pointwise multiplication) defined by:

\[
\mathcal{A}_C = \{f \in C(X, G) : f(c. x) = c. f(x) \quad \text{for all} \quad x \in X, c \in C \}.
\]

(iii) If \( C \) is an automorphism group of \((X, G)\), define \( \mathcal{R}_C : C(X, G) \to C(X, G) \)

by \( (\mathcal{R}_C f)(x) = f(c. x) \quad (c \in C) \).

If \( X = K^s \times B \ (s = 1, 2 \text{ and } B \text{ finite}) \) and \( G = K^p \), \( C \) is said to be a \( P \)-invariant automorphism group if \( \mathcal{R}_C P = P \mathcal{R}_C \) for all \( c \in C \). If this condition is satisfied, \( P(\mathcal{A}_C) \subseteq \mathcal{A}_C \).

12.14 Suppose \( G \) is one of the strings \( A_1 - A_9, A_{11}, A_{13}, A_{14}, B_1, B_2, B_3, B_6 \).

(i) The principal bundle defined by the action of \( \mathcal{A}_{X} \) on \( Y_x \) is a product bundle. Recall (410) that we can assume \( G_x = G_{X.0} \times H_x \) where \( H_x \) is a finite subgroup of \( \text{Aut}(G_{X.0}) \) in these particular cases. Thus \( H_x \) is canonically isomorphic to \( G_x / G_{X.0} \), which acts on \( Y_x / G_{X.0} \) (and commutes with \( t^* \)). Therefore \( H_x \) is a finite automorphism group of \( (Y_x / G_{X.0}, G_{X.0}) \).

(ii) There exists \( f_1 \in C(Y_x / G_{X.0}) \) such that \( \text{Hom}(Y_x : G_x, G_{X.0}, t, t') = \{f_1 : f \in \mathcal{A}_{Y_x}\} \)

where we can take \( f_1 \equiv 1 \) except in the case of the string of \( A_{13} \), when

\[
C(Y_x / G_{X.0}, G_{X.0}) = C(K^s, X)
\]

and we can assume \( f_1 \equiv 1 \). \( \mathcal{A}_{Y_x} f_1 = \{f_1 f_2 : f_2 \in \mathcal{A}_{Y_x}\} \).

(iii) If \( G \) is one of the strings of \( A_1 - A_4, A_{11}, A_{13}, A_{14}, B_1, B_2, B_3, B_6 \), then \( H_x \) is \( P \)-invariant.
12.15 If \( G' \) is one of the strings of \( 39 - 323 \) then the proof of 12.9 for \( \mathcal{B} \) reduces to proving:
\[
\mathcal{O}(Y; G, t, t') \cap \{ a_f : f \in \mathcal{O}_c \}
\]
is dense in \( \text{Hom}(Y; G, t, t') \cap \{ s_f : f \in \mathcal{O}_c \} \)
where:

(i) \( G \) is the string of \( B_1, B_2, B_5, B_6 \).

(ii) \( C \) is a finite \( P \)-invariant automorphism group of \( (Y; G, t, t') \) such that the actions of \( C \) on \( Y; G, t, t' \) with those of \( K_1 \) and \( \text{CH}_K \) acts freely on \( Y; G, t, t' \). Hence \( \text{CH}_K \) is a finite \( P \)-invariant automorphism group of \( (Y; G, t, t') \).

12.16 Lemma If \( G \) is one of the strings \( A_1, A_4, A_9, B_1, B_2, B_5, B_6 \) and \( a_f \in \text{Hom}(Y; G, t, t') \) then a sufficient condition for \( c_f \in \mathcal{O}(Y; G, t, t') \) is that \( a_f \) be minimal.

Proof We indicate the proof only when \( G \) is one of \( B_1, B_2, B_5, B_6 \). For the strings \( B_1, B_2, B_5, B_6 \), we can assume \( G \) is \( B_1 \). This follows from the fact that if \( G(K, u) \) is as in \( B_2, B_5, B_6 \), then \( G(K^2, u^2) \) is as in \( B_1 \).

Hence suppose \( G \) is the string of \( B_1 \), so that \( t = t' : K^2 \to K^2 \) is of the form:
\[
(k_1, k_2)t = \omega k_1 g(k_1)k_2), \quad g \in G(k, k).
\]
and \( (k_1, k_2, k_3)s_f = (k_1, k_2, f(k_1, k_2))k_3) \).

Suppose \( a_f \) is minimal. By \([14]\) Theorem 1.1, this is true if and only if there is no continuous solution \( \mathfrak{a} \) to the equation:
\[
(12.16.1) \quad \gamma ((k_1, k_2)t). (\mathfrak{a}(k_1, k_2)) = \gamma (k_1, k_2) \text{ for any } \mathfrak{a} \in \mathcal{O}_C.
\]

If this equation cannot hold for \( \mathfrak{a} \), it also cannot hold for \( f \), so that \( a_f \) is minimal.

Suppose \( a_f \in \mathcal{O}(Y; G, t) \). Then (5.5, 5.5, 5.7) \( (Y; G, t) = (Y; s_f) = (K^3, s_f) \)
must be an almost periodic extension of \((K, \gamma) \) where \( \gamma \) denotes the homeomorphism \( k_1 \to k_1 (k_1 \in K) \).

Hence \((K^3, s_f) \) is a \( G/H \)-extension of \((K, \gamma) \) where there exists a group \( L \) with \( H \in L \in G \) and \( L/H \cong G/H \cong K \) (5.5, 5.6). This forces \((G, H) = (K^2, \mathfrak{a}) \).

This means there is a minimal homeomorphism \( s' \) of \( K^3 \) of the form:
\[
(k_1, k_2, k_3)s' = (\omega k_1, l(k_1)k_2, m(k_1)k_3) \text{ with } \omega, l \in G(K, K), \text{ and a homeomorphism}
\]

\[ \text{Proof.} \]
\( \Phi \) of \( K^3 \) of the form:
\[ \Phi(k_1, k_2, k_3) = (\tilde{\Phi}_2(k_1)k_2, \tilde{\Phi}_3(k_1, k_2)k_3) \]
where \( \tilde{\Phi}_2 \in C(K, K), \tilde{\Phi}_3 \in C(K^2, K) \), such that \( \Phi : (K^3, \sigma_f) \rightarrow (K^3, \sigma_g) \) is an isomorphism.
This implies \( \tilde{\Phi}_3((k_1, k_2)t) = \tilde{\Phi}_3(k_1, k_2) \cdot m(k_t) \)
and hence \( Pr((k_1, k_2)t) = Pr(k_1, k_2) = Pr(k_1, k_2) \)
which contradicts (12.16.1) having no continuous solution.
So \( s_f \in \mathcal{O}(Y, \Phi) \). Q.E.D.

12.17 The previous paragraphs, in particular 12.14 - 12.16, indicate that the proof of 12.9, except for the strings A10, B3, B4 (see 12.11) is a consequence of the following proposition, which is analogous to a result of [4].

**Proposition** Let \( t \) be a minimal distal homeomorphism of \( X \). Let \( G \) be a compact connected Lie group and let \( G \) be a finite automorphism group of \( (X, G) \) (12.13) such that \( c(x_t) = c(x_t) \) for all \( c \in C, x \in X \). Let \( s \) denote the homeomorphism of \( Y = X \times G \) defined by:
\[ (x, g)s = (x, f(x)) \]
Then \( \mathcal{O} \cap \{ s : s \text{ is minimal} \} \) is dense in \( \mathcal{O} \) where \( \mathcal{O} \) is a closed subset of \( C(X, G) \) of one of the following forms (see 12.12, 12.13):
(i) \( \mathcal{O} = \mathcal{O}_G \)
(ii) \( X = X^a \times B \) for \( a = 1 \) or 2 and \( B \) a finite set, \( G = K^a \) and \( \mathcal{O} = \mathcal{O}_G \cap \{ f : f = Pf \} \) where \( G \) is \( P \)-invariant.

**Proof** The following are true:
(a) Given an open cover \( \{ V_1 \ldots V_n \} \) of \( G \) there exists an integer \( p \) such that if \( \bigcup_i V_i = \bigcup_i V_i \) \( (1 = 1 \ldots p) \) then \( G = \bigcup_1 \ldots \bigcup_p \).
(b) Given \( f \in \mathcal{O} \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property:
if \( x_0 \in X \) is fixed and \( u : F \rightarrow G \) satisfies \( d(u(y), f(y)) < \delta \) for all \( y \in F \)
where \( F \) is a finite set with \( c.F \cap F = \emptyset \) for all \( c \in C \), then there exists \( v \in \mathcal{O} \) with \( v|F = u \), and sup \( d(v(x), f(x)) < \varepsilon \), where \( d \) is a metric on \( G \).

(a) is proved in [4] Proposition 2. For the proof of (b) when \( \mathcal{O} \) is as in (i), see lemma 12.18. The proof of (b) when \( \mathcal{O} \) is as in (ii) is omitted.
Now fix $y_o = (x_0, 1) \in Y$. For $U$ open in $Y$, let

$$E(U) = \{ f \in Q : f(y_{o_1}) \in U \}.$$  

Using (i) and (ii), use an argument similar to that of Lemma 2 to show that $E(U)$ is dense in $C(X, G)$. Then note that $\cap_{U \cap \text{open} \subset Y} E(U)$, hence is dense in $Q$, since $Y$ has a countable basis of open sets.

**Lemma (b) of 12.17 is true for $Q$ as in (i) of 12.17.**

**Proof** Assume without loss of generality that the metric $d$ is $G$-invariant.

Choose $\epsilon > 0$ and $S(g) (g \in G)$ such that:

$$\{ \epsilon' : d(g, g') < \epsilon' \leq S(g) \leq \epsilon' : d(g, g') < \epsilon'/2 \}$$

where $S(g)$ is homeomorphic to $\mathbb{R}^n$ for some $n$.

For $y \in F$, choose $U_y$, an open neighbourhood of $y$, such that:

$$f(U_y) \subseteq \{ \epsilon' : d(g', f(y)) < \epsilon'/2 \}, \text{ and } U_y \cap c.U_y = \emptyset \text{ for } c \in C.$$

If $y = c.y_1 (y_1 \in F)$, define $U_y = c.U_y$.

For $y \in \bigcup_{c \in C} F$, define $v_y(z) = c.u(y_1)$ if $y = c.y_1$ for $c \in C, y_1 \in F$

$$v_y(x) = f(x), \text{ for } x \in \text{the boundary of } U_y,$$

and extend $v_y$ to a function $v_y : U_y \rightarrow S(f(y))$ such that $v_{c.y}(c.x) = c.v_y(z)$.

Then define $v = v_y$ on $U_y \cap F$.

$\text{ otherwise.}$

**R-admissibility**

12.1 The different types of minimal distal $\mathcal{W}$-actions on compact connected topological manifolds of dimension $\leq 3$ were obtained by Bronstein [1], though not quite in the form given here.

12.2 Clearly $\mathcal{W}$ (with the usual topology) can only act minimally on a connected space. Then the following lemma, quoted by Bronstein for roughly the same purpose, and easily verified, shows that all the strings in tables A and B except for A9, A10, A13, A14, are not $\mathcal{W}$-admissible.

**Lemma (5)** Let $(X, \mathcal{R})$ be a minimal periodic t.g. and $(Y, \mathcal{R})$ a minimal almost periodic extension of $(X, \mathcal{R})$. Then $(Y, \mathcal{R})$ is almost periodic.
(Note that an almost periodic action of $\mathbb{R}$ on $K$ must be periodic, hence the lemma implies a distal action of $\mathbb{R}$ on a 2-dimensional manifold must be almost periodic.)

13.13 The string of $\mathcal{A}^0$ is $\mathbb{R}$-admissable.

$\mathcal{O}(X,\mathbb{R})$ is the string of $\mathcal{A}^0$ if and only if $X = \mathbb{R}^2$ (8.2), and the action of $\mathbb{R}$ is given by:

$$[x,y,z] = [x+at,y+bt,ct^2/2+xt+gt(x,z),z+ct]$$

for all $x, y, z, t \in \mathbb{R}$, where $a$ and $c \in \mathbb{R}$ are rationally independent and the function

$$(t,x,z) \mapsto g_t(x,z)$$

is jointly continuous, with:

$$g_{t+\lambda}(x,z) = g_t(x,z) + g_\lambda(x+at,z+ct)$$

$$(t,x+1,z) = g_t(x,z) \pmod{2} = g_t(x,z+1)$$

for all $x, z, s, t \in \mathbb{R}$.

For proof of minimality see, for example, [2] 6.19.2.6. Proof that

$\mathcal{O}(X,\mathbb{R})$ is the string of $\mathcal{A}^0$ is analogous to 12.10.

13.4 We outline the proof that the string of $\mathcal{A}^9$ is $\mathbb{R}$-admissable. (The proofs for $\mathcal{A}^3, \mathcal{A}^4$ are similar.)

If $\mathcal{O}(K^3,\mathbb{R})$ is the string of $\mathcal{A}^9$ then the action of $\mathbb{R}$ is of the form:

$$(k_1,k_2,k_3)^t = (k_1e^{a_1t}, k_2 e^{a_2t}, k_3 e^{a_3t})$$

for all $k_1, k_2, k_3 \in \mathbb{R}$ and $t \in \mathbb{R}$, where $a, b, c \in \mathbb{R}$ are rationally independent, and if

$$(k_1,k_2)^t = (k_1e^{a_1t},k_2 e^{a_2t})$$

then $g_{t+\lambda}(k_1,k_2) = g_t(k_1,k_2) + g_\lambda(k_1,k_2)$

for all $k_1, k_2 \in \mathbb{R}$ and $t, \lambda \in \mathbb{R}$.

A necessary and sufficient condition that $\mathcal{O}(K^3,\mathbb{R})$ be minimal and that

$\mathcal{O}(K^3,\mathbb{R})$ be the string of $\mathcal{A}^9$ is that there exist no continuous solution $f \in C(K^3,\mathbb{R})$ to the equation:

$$f(k_1 e^{a_1t}, k_2 e^{a_2t}) = f(k_1,k_2) e^{2a_1t} g_t(k_1,k_2) + s$$

for any $t \in \mathbb{Z}$ and $s \in \mathbb{R}$.

Writing $f(k_1,k_2) = k_1 k_2 e^{2a}_1 g(k_1,k_2)$ ($g \in C(K^3,\mathbb{R})$), the condition becomes that there is no continuous solution $f_1 \in C(K^3,\mathbb{R})$ to the equation:

$$f_1(k_1 e^{a_1t}, k_2 e^{a_2t}) = f_1(k_1,k_2) + k_1 k_2 e^{2a}_1 t$$

for any $t \in \mathbb{R}$. 

\[ \]
\[ \Phi(k_1, k_2, k_3) = (k_1, \Phi(k_2, k_3)), \quad \Phi_2(k_1, k_2) = (k_1, k_2) \]

such that \( \Phi: (K^3, \bar{s}) \to (K^3, s') \) is an isomorphism.

This implies \( \Phi_2((k_1, k_2), f(k_1, k_2)) = \Phi_2(k_1, k_2) \)

and hence \( (P\Phi_2)((k_1, k_2), f(k_1, k_2)) = P\Phi_2(k_1, k_2) \)

which contradicts (12.16.1) having no continuous solution.

So \( s_{\Phi} \in (X, P, \Phi, t) \). Q.E.D.

12.17 The previous paragraphs, in particular 12.14 - 12.16, indicate that the proof of 12.9, except for the strings A10, B3, B3 (see 12.11) is a consequence of the following proposition, which is analogous to a result of [4].

Proposition Let \( t \) be a minimal distal homeomorphism of \( X \). Let \( G \) be a compact connected Lie group and let \( C \) be a finite automorphism group of \( (X, G) \) (12.13) such that \( c(xt) = c(xt) \) for all \( c \in C, x \in X \). Let \( s_c \) denote the homeomorphism of \( Y = X \times G \) defined by:

\[
(x, g)s_c = (xt, gc(x)).
\]

Then \( \mathcal{O} \cap F \) is minimal if \( \mathcal{O} \) is dense in \( \mathcal{O} \) where \( \mathcal{O} \) is a closed subset of \( C(X, G) \) of one of the following forms (see 12.12, 12.13):

1. \( \mathcal{O} = \mathcal{O}_C \)
2. \( X = X^a 	imes B \) for \( a = 1 \) or \( 2 \) and \( B \) a finite set, \( G = X^a \) and \( \mathcal{O} = \mathcal{O}_C \cap \{ f : f = Pf \} \) where \( C \) is \( P \)-invariant.

Proof The following are true:

(a) Given an open cover \( \{ V_1, \ldots, V_n \} \) of \( G \), there exists an integer \( p \) such that if \( V_i \subseteq \{ V_1, \ldots, V_n \} (i = 1, \ldots, p) \) then \( G = V_1 \cup \ldots \cup V_p \).

(b) Given \( f \in \mathcal{O} \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) with the following property:

if \( x \in X \) is fixed and \( u : F 

\rightarrow G \) satisfies \( d(u(y), f(y)) \leq \delta \) for all \( y \in F \)

where \( F \) is a finite set with \( c \cdot F \cap F = \emptyset \) for all \( c \in C \), then there exists \( v \in \mathcal{O} \) with \( v \wedge F = u \), and \( \sup d(v(x), f(x)) \leq \varepsilon \), where \( d \) is a metric on \( G \).

(a) is proved in [4] Proposition 2. For the proof of (b) when \( \mathcal{O} \) is as in (i), see lemma 12.18. The proof of (b) when \( \mathcal{O} \) is as in (ii) is omitted.
Now fix $y_0 = (x_0, l) \in Y$. For $U$ open in $Y$, let

$E(U) = \{ f \in \mathcal{C} : \{ y \in \mathcal{C} : y \in U \} \cap \mathcal{C} \neq \emptyset \}$. Using (i) and (ii), use an argument similar to that of [3] lemma 2 to show that $E(U)$ is dense in $C(X, G)$. Then note that $\{ f \in \mathcal{O} : f \text{ is minimal} \} \cup \bigcup_{U \text{ open} \in Y} E(U)$, hence is dense in $\mathcal{O}$, since $Y$ has a countable basis of open sets.

12.18 Lemma (b) of 12.17 is true for $\mathcal{O}$ as in (i) of 12.17.

Proof Assume without loss of generality that the metric $d$ is $C$-invariant. Choose $\delta > 0$ and $S(g) \ (g \in G)$ such that:

$\{ g' : d(g, g') < \frac{\delta}{2} \leq S(g) \leq \{ g' : d(g, g') < \frac{\delta}{2} \}$, where $S(g)$ is homeomorphic to $\mathbb{R}^N$ for some $n$.

For $y \in F$, choose $U_y$, an open neighbourhood of $y$, such that:

$f(U_y) \subseteq \{ g' : d(g', f(y)) < \frac{\delta}{2} \}$, and $U_y \cap \mathcal{C}U_y = \emptyset$ for $c \in C$.

If $y = c.y_1 \ (y_1 \in F)$, define $U_y = cU_y$.

For $y \in \mathcal{C}F$, define $v_y(y) = c.y_1 \ (y_1 \in F)$ if $y = c.y_1 \ (y_1 \in F)$ and $v_y(x) = f(x)$, for $x$ in the boundary of $U_y$.

The different types of minimal distal $\mathcal{O}$-actions on compact connected topological manifolds of dimension $\leq 3$ were obtained by Bronstein [1], though not quite in the form given here.

13. Alt., the following lemma, quoted by Bronstein for roughly the same purpose, and easily verified, shows that all the strings in tables $A$ and $B$ except for $A_9$, $A_{10}$, $A_{13}$, $A_{14}$, are not $\mathcal{O}$-admissible.

Lemma [5] Let $(X, \mathcal{R})$ be a minimal periodic t.g. and $(Y, \mathcal{R})$ a minimal almost periodic extension of $(X, \mathcal{R})$. Then $(Y, \mathcal{R})$ is almost periodic.
12.15 If \( \mathcal{S}' \) is one of the strings of \( A9 - A23 \) then the proof of 12.9 for \( \mathcal{S}' \) reduces to proving:

\[
\mathcal{O}(Y_r; \mathcal{S}, t, t') \cap \{ s_r^t \in \mathcal{O}_r \} \text{ is dense in } \text{Hom}(Y_r; G, t, t') \cap \{ s_r^t \in \mathcal{O}_r \}
\]

where:

\( \mathcal{S} \) is the string of \( B1, B2, B5 \) or \( B6 \).

(i) \( C \) is a finite \( P \)-invariant automorphism group of \( (Y_r/G, G, t, t') \) such that the actions of \( C \) on \( Y_r/G \) and \( G_0 \) commute with those of \( K_r \) and \( G \) acts freely on \( Y_r/G \). Hence \( C \mathcal{H}_r \) is a finite \( P \)-invariant automorphism group of \( (Y_r/G, G) \).

12.16 Lemma If \( \mathcal{S} \) is one of the strings \( A1, A4, A9, B1, B2, B5, B6 \) and \( s_r^t \in \text{Hom}(Y_r; G, t, t') \) then a sufficient condition for \( \mathcal{S} \in \mathcal{O}(Y_r; \mathcal{S}, t, t') \) is that \( s_r^t \) be minimal.

Proof We indicate the proof only when \( \mathcal{S} \) is one of \( A1, A2, A5, B5, B6 \). For the strings \( B1, B2, B5, B6 \), we can assume \( \mathcal{S} \) is \( B1 \). This follows from the fact that if \( \mathcal{S}(K2, u) \) is \( n \) in \( B2, B5, B6 \), then \( \mathcal{S}(K^2, u^2) \) is as in \( B1 \).

Hence suppose \( \mathcal{S} \) is the string of \( B1 \), so that \( t = t' : K^2 \rightarrow K^2 \) is of the form:

\[
(k_1, k_2) t = \omega k_1, g(k_1)k_2), \quad \omega \in C(k, K),
\]

and \( (k_1, k_2, k_3) e_t = (\omega k_1, g(k_1)k_2, f(k_1, k_2)k_3) \).

Suppose \( s_r^t \) is minimal. By [14] Theorem 1.1, this is true if and only if there is no continuous solution \( \xi \) to the equation:

\[
(12.16.1) \quad \xi((k_1, k_2)t, (F(k_1, k_2)) \xi = \xi(k_1, k_2) \quad \text{for any } n \in \mathbb{Z} - 100.
\]

If this equation cannot hold for \( F_r \), it also cannot hold for \( s_r^t \), so that \( s_r^t \) is minimal.

Suppose \( s_r^t \notin \mathcal{O}(Y_r; \mathcal{S}, t) \). Then (5.5, 5.5, 5.7) \( (Y_r, s_r^t) = (Y_r, s_r^t) = (X^2, s_r^t) \) must be an almost periodic extension of \( (X, s) \) where \( X \) denotes the homeomorphism \( k_1 \mapsto \omega k_1, (k_1, K) \).

Hence \((X, s)\) is a \( G/H \)-extension of \( (X, s) \) where there exists a group \( L \) with \( H \triangleleft G \) and \( L/H \cong G/H \cong H \). This forces \( (G, K) = (X^2, [11]) \).

This means there exists a minimal homeomorphism \( s' \) of \( X^2 \) of the form:

\[
(k_1, k_2, k_3)s' = (\omega k_1, l(k_1)k_2, m(k_1)k_3) \quad \text{with } \omega, l \in C(K, K), \text{ and a homeomorphism}
\]
of $K^3$ of the form:
$G(k_1, k_2, k_3) = (k_1, f_2(k_1)k_2, f_3(k_1, k_2)k_3)$, where $g_1 \in C(k, k), g_2 \in C(k^2, k)$, such that $G : (K^3, s_0) \rightarrow (K^3, s')$ is an isomorphism.
This implies $g_3((k_1, k_2)t) . f_3(k_1, k_2) = g_3(k_1, k_2) . m(k_1)$
and hence $(f_1, f_2) ((k_1, k_2)t) . f(k_1, k_2) = P g_3(k_1, k_2)$
which contradicts (12.16.1) having no continuous solution.
So $s_0 \in G(Y, F, t)$. Q.E.D.

12.17 The previous paragraphs, in particular 12.14 - 12.16, indicate that
the proof of 12.9, except for the strings 110, 33, 33 (see 12.11) is a
consequence of the following proposition, which is analogous to a result of [4].
Proposition Let $t$ be a minimal distal homeomorphism of $X$. Let $G$ be a compact
connected Lie group and let $C$ be a finite automorphism group of $(X, G)$ (12.15)
such that $c(x) = c(x)$ for all $c \in C, x \in X$. Let $s_x$ denote the homomorphism
of $Y = X \times G$ defined by:

$$(x, g)_s = (xt, x'(x)).$$

Then $\mathcal{Q} \cap \{ s : s_x = \text{minimal} \}$ is dense in $\mathcal{Q}$ where $\mathcal{Q}$ is a closed subset
of $C(X, G)$ of one of the following forms (see 12.12, 12.13):
(i) $\mathcal{Q} = \mathcal{Q}_C$
(ii) $X = X \times B$ for $s = 1$ or 2 and $B$ a finite set, $G = X^F$ and
$\mathcal{Q} = \mathcal{Q}_G \cap \{ f : f = Pf \}$ where $C$ is $P$-invariant.

Proof The following are true:
(a) Given an open cover $\{ V_1, \ldots, V_n \}$ of $G$, there exists an integer $p$ such that
if $W_i \subseteq \{ V_1, \ldots, V_n \}$ ($i = 1, \ldots, p$) then $G = \coprod_{i=1}^n W_i$.
(b) Given $f \subseteq \mathcal{Q}$ and $E \supseteq \mathcal{Q}$ there exists $b > 0$ with the following property:
if $x \in X$ is fixed and $u : F \rightarrow G$ satisfies $d(u(y), f(y)) \leq \varepsilon$ for all $y \in F$
where $F$ is a finite set with $c.F \cap F = \emptyset$ for all $c \in C$, then there exists
$v \in \mathcal{Q}$ with $v |F = u$, and $\sup d(v(x), f(x)) \leq \varepsilon$, where $d$ is a metric on $G$.

(a) is proved in [4] Proposition 2. For the proof of (b) when $\mathcal{Q}$ is as
in (i), see lemma 12.18. The proof of (b) when $\mathcal{Q}$ is as in (ii) is omitted.
Now fix \( y_0 = (x_0, l) \in Y \). For \( U \) open in \( Y \), let
\[
E(U) = \{ x \in X : y_0^n x \in U \}.
\]
Using (i) and (ii), use an argument similar to that of [3] lemma 2 to show that \( E(U) \) is dense in \( C(X, G) \).

Then note that \( \{ f \in \Omega : f \text{ is minimal} \} = \bigcap \text{open}(U) \), hence is dense in \( \Omega \), in \( Y \).

since \( Y \) has a countable basis of open sets.

12.18 Lemma (b) of 12.17 is true for \( \Omega \) as in (i) of 12.17.

Proof Assume without loss of generality that the metric \( d \) is \( G \)-invariant.

Choose \( \delta > 0 \) and \( S(g) \) (\( g \in G \)) such that:
\[
\{ g' : d(g, g') < \delta \} = S(g) \subseteq \{ g' : d(g, g') < \varepsilon/2 \},
\]
where \( S(g) \) is homeomorphic to \( \mathbb{R}^n \) for some \( n \).

For \( y \in Y \), choose \( U_y \), an open neighbourhood of \( y \), such that:
\[
f(U_y) \subseteq \{ g' : d(g', f(y)) < \delta \}, \quad \text{and} \quad U_y \cap cU_y = \varnothing \quad \text{for } c \in C.
\]

If \( y = c y_1 \) (\( y_1 \in Y \)), define \( U_y = cU_y \).

For \( y \in \bigcup cF \), define \( v_y(y) = c u(y_1) \) if \( y = c y_1 \) for \( c \in C \), \( y_1 \in F \)
and extend \( v_y \) to a function \( v_y : U_y \rightarrow S(f(y)) \) such that \( v_y(c.x) = c v_y(x) \).

Then define \( v = v_y \) on \( U_y \) (\( y \in cF \))
and extend otherwise.

\( 13 \) \( \mathbb{R} \)-admissibility

13.1 The different types of minimal distal \( \mathbb{R} \)-actions on compact connected topological manifolds of dimension \( \geq 3 \) were obtained by Bronstein [1], though not quite in the form given here.

13.2 Clearly \( \mathbb{R} \) (with the usual topology) can only act minimally on a connected space. Then the following lemma, quoted by Bronstein for roughly the same purpose, and easily verified, shows that all the strings in tables A and B except for A9, A10, A13, A14, are not \( \mathbb{R} \)-admissible.

Lemma (5) Let \((X, \mathbb{R})\) be a minimal periodic \( \mathbb{R} \)- and \((Y, \mathbb{R})\) a minimal almost periodic extension of \((X, \mathbb{R})\). Then \((Y, \mathbb{R})\) is almost periodic.
(Note that an almost periodic action of \( K \) on \( \mathbb{R} \) must be periodic, hence the lemma implies a distal action of \( \mathbb{R} \) on a 2-dimensional manifold must be almost periodic.)

13.13 The string of \( A_10 \) is \( \mathbb{R} \)-admissible.

\( \mathcal{F}(X, \mathbb{R}) \) is the string of \( A_10 \) if and only if \( X = \mathbb{R}/\mathbb{Z}_n \) (8.2), and the action of \( \mathbb{R} \) is given by:

\[ [x, y, z] \rightarrow [x + \alpha t, y + \beta t + \omega t^2/2, z + \gamma t] \]

for all \( x, y, z, t \in \mathbb{R} \), where \( \alpha, \beta, \gamma \in \mathbb{R} \) are rationally independent and the function

\[ (t, x, z) \rightarrow \sigma_t(x, z) \]

is jointly continuous, with:

\[ \sigma_t(x, z) = \delta_t(x, z) + \sigma_0(x + \alpha t, z + \omega t) \]

\[ \sigma_t(x + 1, z) = \sigma_t(x, z) \quad \text{(mod } \mathbb{Z}) = \delta_t(x, z + 1) \quad \text{for all } x, z, t \in \mathbb{R} \).

For proof of minimality see, for example, [2] 6.19.2.6. Proof that

\( \mathcal{F}(X, \mathbb{R}) \) is the string of \( A_10 \) is analogous to 12.10.

13.4 We outline the proof that the string of \( A_9 \) is \( \mathbb{R} \)-admissible. (The proofs for \( A_{13}, A_{14} \) are similar.)

If \( \mathcal{F}(X, \mathbb{K}) \) is the string of \( A_9 \) then the action of \( \mathbb{K} \) is of the form:

\[ (k_1, k_2, k_3) \rightarrow \left( k_1 e^{2\pi i \alpha t}, k_2 e^{2\pi i \beta t}, k_3 e^{2\pi i \gamma t} \right) \]

for all \( k_1, k_2, k_3 \in \mathbb{K} \) and \( t \in \mathbb{R} \), where \( \alpha, \beta, \gamma \in \mathbb{K} \) are rationally independent, and if

\[ (k_1, k_2) \rightarrow \left( k_1 e^{2\pi i \alpha t}, k_2 e^{2\pi i \beta t} \right) \]

then \( \sigma_{t+\nu}(k_1, k_2) = \sigma_t(k_1, k_2) \cdot \sigma_\nu((k_1, k_2) \cdot t) \)

for all \( k_1, k_2 \in \mathbb{K} \) and \( t \in \mathbb{R} \).

A necessary and sufficient condition that \( (k_1, k_2) \) be admissible and that

\( \mathcal{F}(X, \mathbb{K}) \) be the string of \( A_9 \) is that there exist no continuous solution \( f \in C(\mathbb{K}, \mathbb{K}) \) to the equation:

\[ f(k_1 e^{2\pi i \alpha t}, k_2 e^{2\pi i \beta t}) = f(k_1, k_2) \cdot e^{2\pi i \delta t} \]

for any \( f \in C(\mathbb{K}, \mathbb{K}) \) and \( \delta \in \mathbb{K} \).

Writing \( f(k_1, k_2) = k_1^{\delta} \cdot k_2^{\delta} \) \( \in C(\mathbb{K}, \mathbb{K}) \), the condition becomes that there is no continuous solution \( f_1 \in C(\mathbb{K}, \mathbb{K}) \) to the equation:

\[ f_1(k_1 e^{2\pi i \alpha t}, k_2 e^{2\pi i \beta t}) = f_1(k_1, k_2) \cdot e^{\rho t} \]

for any \( \rho \in \mathbb{K} \).
Let $g_t(k_1, k_2) = \int h((k_1, k_2)u) \, du$.

By choosing $h$ with suitable Fourier coefficients, we can ensure that there is no continuous solution $f_1$ to (13.4.2).


In this appendix we give details of results which were omitted from §§ 1 - 13 for the purpose of brevity, since those sections were submitted for publication:

(i) We prove that the assumption of distality in proposition 5.5 is unnecessary (14.1 - 14.2).

(ii) We give the general "finite-dimensional" version of theorem 1.2 (see 1.4) with such details of the proof as seem necessary (14.3 - 14.12). Note that the assumption "$T \in \mathcal{I}$" (1.4) is not necessary after all.

(iii) We show that in theorem 1.2, the hypothesis that $X$ have finitely many arcwise-connected components can be replaced by the hypothesis that $X$ be locally connected (14.13 - 14.14) (see 1.3).

14.1 For the proof of the more general version of proposition 5.5, we need the following facts about distal extensions. A reference is [2].

For a group $T$, there exists a universal minimal set $(I, T)$ such that $(I, \mathcal{H}_p)$ is a compact Hausdorff topological semigroup with dense subgroup $T$, where $\mathcal{H}_p$ denotes the topology on $I$, the identity of $T$ is an idempotent of $I$, $I = u_1$ has no non-trivial ideals and:

$$q \mapsto pq \quad (p, q \in I)$$

$$q \mapsto qt \quad (q \in I, \quad t \in T)$$

are $\mathcal{H}_p$-continuous.

If $(X, T)$ is a minimal t.g. then there exist universal minimal distal and almost periodic extensions of $(X, T)$ denoted by $(X^*, T)$ and $(X^\#, T)$ respectively.

$$(X, T) \subsetneq (X^\#, T) \subsetneq (X^*, T) \subsetneq (I, T).$$

$(X, T)$ can be regarded as $\{[p]_X : p \in I\}$, where $[p]_X$ is the $\sim_X$-equivalence class of $p \in I$, where $\sim_X$ is a closed $T$-invariant equivalence
relation on I.

Write $G_X = \{g \in G : [g]_X = [u]_X\}$ where $G$ is the subgroup $Iu$ of $I$.

Then $G_x < G_X$, and $G_x^p < G_x$. Now let $(X, T)$ be fixed.

(a) If $(Y, T)$ is a distal minimal extension of $(X, T)$ with $(X, T) \leq (Y, T)$, then $g \mapsto [gp]_Y$ maps $G_X$ onto $\pi^{-1}(\pi([p]_Y))$.

Hence $(G_x/G_y, [p]_p)$ is homeomorphic to $(\pi^{-1}(\pi([u]_Y)), [p]_p)$.

(b) There exists a topology $\sigma \leq [p]_p$ on $G$ ($\sigma$ would be called the $T(G(X^*))$-topology in [2]) such that each of the following maps

$(G_x, \sigma) \rightarrow (G_x, [p]_p)$ is continuous ([2]) (11.17):

$p \mapsto qp \quad q : p \mapsto pq \quad (p, q \in G_x)$. 

$(G_x/G_y, \sigma)$ is compact $T_1$.

(c) For a $\sigma$-closed $H$, $G_x \leq H \leq G_X$, define:

$$\text{alg}(H) = \{f \in C(X^*) : f(hp) = f(p) \text{ for all } h \in H, p \in I^2\}.$$ 

Then $\text{alg}(H)$ is a $T$-invariant $C^*$-subalgebra of $C(X^*)$ containing $C(X)$.

For a $T$-invariant $C^*$-subalgebra $\mathcal{A}$, $G_x \leq \mathcal{A} \leq C(X^*)$, define

$$\text{gp}(\mathcal{A}) = \{h \in G_x : f(hp) = f(p) \text{ for all } f \in \mathcal{A}, p \in I^2\}.$$ 

Then $\text{gp}(\mathcal{A})$ is a $\sigma$-closed subgroup of $G_x$.

$$\text{alg}(\text{gp}(\mathcal{A})) = \mathcal{A} \text{ and } \text{gp}(\text{alg}(\mathcal{A})) = H \text{ ([2] Ch.13}).$$

(d) For a $\sigma$-closed $H$, $G_x \leq H \leq G_X$, $G_x^p \leq H$ if and only if $(G_x/H, [p]_p) = (G_x^p/H, [p]_p)$. For a $\sigma$-closed $H$, $G_x \leq H \leq G_x^p$ if and only if multiplication in $G_x^p/H$ is $[p]_p$-continuous in each variable. In this case, the left-action of $(G_x^p/H, [p]_p)$ on $(I/H, [p]_p)$ is continuous in each variable, where $I/H = \{hp : p \in I^2\}$. Since $(I/H, [p]_p)$ is compact Hausdorff by (c), this implies the left-action is jointly continuous ([18]).

14.2 Proposition Proposition 5.5 is true without the assumption that the $(Z_1, T)$ be distal.

Proof As in 5.5, we construct $\overline{\gamma} : E(Y_1) \rightarrow E(Y_2)$. Let $J_1$ be a minimal ideal of $E(Y_1)$, and $J_2 = \overline{\gamma}(J_1)$. In a similar manner to 5.5, we can make $(Z_1, T)$ a factor of $(Z_1, T)$ so that the following diagram
4.5 The statement of the general "finite-dimensional" version of theorem 1.2 is obtained from the statement of theorem 1.2 as follows:

Replace the hypothesis that $X$ have finitely many arcwise-connected components by the hypothesis that $X$ have finitely many connected components. Omit the sentence "These hypotheses...topological manifold". Omit conclusion (i). In conclusion (iii), omit the words "so that $G_1 = (Y_1, X_1, X_{i-1} - l, G_i, N_i, \pi_1, S_i, Y_i)$ is a fibre bundle (3.1) for $1 \leq i \leq r$".

Replace the words "manifold" and "Lie group", wherever they occur in the statement of the theorem, by "finite-dimensional space" and "finite-dimensional group" respectively.

The proof of the new version follows the lines of the proof of 1.2.
once we have proved the following:

14.4 Proposition Let \((X,T) \leq_{\#} (Y,T) \leq_{\#} (Z,T) \quad (\pi_1^T \pi_2 = \pi)\) where \((Z,T)\) is minimal, \(\pi^{-1}(x)\) is connected \((x \in X)\), \((Y,T)\) is an a.p. extension of \((X,T)\), and \((Z,T)\) is a finite a.p. extension of \((Y,T)\). Then \((Z,T)\) is an a.p. extension of \((X,T)\).

For the proof we need a sequence of lemmas. Proofs of the easier ones will be omitted.

14.5 Lemma If \((X,T) \leq (Y,T) \leq (Z,T)\) where \((Y,T)\) is a finite a.p. extension of \((X,T)\) and \((Z,T)\) is an a.p. extension of \((Y,T)\), then \((Z,T)\) is an a.p. extension of \((X,T)\).

14.6 Lemma For proposition 14.4, we may assume \(\pi^{-1}(x)\) is connected \((x \in X)\).

14.7 Lemma Let \(G\) be a compact topological group, \(H \leq G\), and suppose \(G/H\) is connected. Then if \(G_0\) denotes the connected component of \(1 \in G\), \(G_0H = HG_0 = G\).

14.8 Lemma Let \(G\) be a compact connected topological group. Let \(A\) be a finite group acting freely and continuously on the compact connected Hausdorff space \(X\) such that \(G\) identifies with the orbit space under the map \(\gamma : X \rightarrow G\). Suppose \(\gamma(xy_0) = 1\). Then \(X\) can be made a topological group in such a way that \(x_0\) is the identity and \(\gamma\) a group homomorphism. The group structure is the unique group structure on \(X\) making \(x_0\) the identity and \(\gamma\) a group homomorphism and the maps \(q_1 \rightarrow pq\) continuous for each \(p \in X\) (alternatively the maps \(q_1 \rightarrow qp\) continuous for each \(p \in X\)).

Proof \(G\) is the inverse limit of the net \(\left\{ \left\{ G_n \right\} \right\}_{n \in D} \) of compact connected Lie groups. Let \(\pi_n : G \rightarrow G_n\) be the limit map.

Let \(\pi_n = \pi_n \circ \gamma\). Then for each \(x \in X\), \(\pi_n^{-1}(\pi_n(x)) = \bigcap_{n \in D} \pi_n^{-1}(\pi_n(x))\).

For an index \(x\) on \(X\), let \(B_{\pi}(x) = \{ x' : (x,x') \in \pi \} \).
Let \( U_n(x) = \bigcup B_{\varepsilon_n}(x') \).

Choose a symmetric index \( \delta \) on \( X \) such that if \( \delta(x_1) = \delta(x_2) \) and \( (x_1,x_2) \in \delta \cdot \delta^{-1} \), then \( x_1 = x_2 \).

Choose a symmetric closed index \( \varepsilon \) on \( X \) such that \( (x_1,x_2) \in \varepsilon \) implies \( (ax_1,ax_2) \in \varepsilon \) for all \( a \in A \).

There exists \( n_0 \in \mathbb{D} \) such that \( \delta^{-1} \cdot \varepsilon_n(x) \subseteq U_n(x) \) for all \( x \in X \), \( n \geq n_0 \).

Define \( \sim_n \) by \( x \sim_n x' \) (\( n \geq n_0 \)) if and only if \( \delta_n(x) = \delta_n(x') \) and \( (x',x) \in \varepsilon \). This is a closed \( A \)-invariant equivalence relation on \( X \).

Write \( X_n = X/\sim_n \). \( A \) acts freely and continuously on \( X_n \) by \( a \cdot [x]_n = [ax]_n \).

Define \( \tau_n : X_n \rightarrow G_n \) by \( \tau_n([x]_n) = \delta_n(x) \).

Define \( \sigma_n : X \rightarrow X_n \) by \( \sigma_n(x) = [x]_n \) and \( \sigma_{nm} : X_m \rightarrow X_n \) (\( n \leq m \)) by \( \sigma_{nm}([x]_m) = [x]_n \). Then the following diagram commutes (\( n \leq m \leq n' \)):

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma_n} & X_n \\
\downarrow{\gamma} & & \downarrow{\tau_n} \\
G & \xrightarrow{\pi_n} & G_n
\end{array}
\]

Write \( x_n = [x]_n \). Then \( \sigma_{mn}(x_n) = x_m \) (\( m \leq n \)). For each \( n \geq n_0 \), there exists a unique topological group structure on \( X_n \) making \( x_n \) the identity and \( \tau_n \) a group homomorphism. Then each \( \sigma_{mn} \) (\( m \leq n \)) is a group homomorphism. Then \( \{X_n, \sigma_n\}_{n_0}^{\infty} \) is the inverse limit of the net \( \{X_n, \sigma_{nm}\}_{n_0}^{\infty} \) of groups, hence \( X \) can be given a topological group structure such that each \( \sigma_n \) is a group homomorphism, and \( x_0 \) is the identity. Then each \( \delta_n = \tau_n \cdot \sigma_n \) is a group homomorphism, and \( \delta \) is a group homomorphism.

The uniqueness statement of the lemma is the "unique lifting theorem" for covering spaces (see, for instance, [19]).
14.9 Proof of proposition 14.4.

Let \((X,T), (Y,T), (Z,T)\) be as in the statement of proposition 14.4. Use the notation of 14.1.

Let \(G' = G_x/G_{x^*}\), \(H' = G_y/G_{x^*}\), \(L' = G_z/G_{x^*}\).

Then \((G'/L', \bar{\gamma}_p)\) is connected and \(H'/L'\) is finite.

Put \(N' = \bigcap_{g \in G_x} g^{-1}H'g\). Then \((G'/N', \bar{\gamma}_p) = (G'/N', \bar{\sigma})\).

\(N'/(N' \cap L')\) is finite. Put \(M' = \bigcap_{N \in \mathfrak{N}} N^{-1}(N' \cap L')\). \(N'/M'\) is finite, since \(N'/M'\) acts effectively on \(N'/(N' \cap L')\).

We can assume that \(M' \triangleleft G'\), from which it will follow that \(M' = \bigcap_{g \in G_x} g^{-1}L'g\).

For let \(R' = \{g \in G' : gM' = M'g\}\). \(R'\) is \(\sigma\)-closed, and since \(N' \triangleleft G'\), \(R'\) is of finite index in \(G'\). If necessary, replace \(X\) by \(X^*/R'\), \(Y\) by \(X^*/(H' \cap R')\), and \(Z\) by \(X^*/(L' \cap R')\).

Now let \(B_1, B_2\) be the groups containing \(M'\) such that \(B_1/M'\) and \(B_2/M'\) are the \(J\) \(-\) connected and \(\sigma\)-connected components of \(M'\) in \(G'/M'\) respectively. Then \(B_1 = B_2 = B', \) say (14.10), and \(B'\) is \(\sigma\)-closed.

Write \(G = G'/M', N = N'/M', H = H'/M', L = L'/M', B = B'/M'\).

\(G\) inherits \(\sigma\)- and \(J\)-topologies from \(G'\).

To prove 14.4, we only have to show the maps:

\[ q \longmapsto pq \quad \text{and} \quad q \longmapsto qp \quad (p,q \in G) \text{ are } \bar{\gamma}_p\text{-continuous (14.1(d))}. \]

\((B, J_p)\) is a finite cover of \((B/N \cap B, J_p) = (B/N \cap B, \bar{\sigma})\). 14.8 implies there exists a topological group structure on \(B\) making \(l \in B\) the identity and the natural quotient map (relative to the original group structure) \(B \longrightarrow B/(N \cap L)\) a group homomorphism. The uniqueness clause of 14.8 implies that the topological group structure is the same as the original group structure. So \((B, J_p) = (B, \bar{\sigma})\) (essentially 14.1(d) - see also [2] Chs. 11-13).

\(B \triangleleft G\). So \(b \mapsto a^{-1}ba : (B, J_p) \longrightarrow (B, J_p)\) is continuous for each \(a \in L\).

\(G = BL = LB \) (14.11), so \((G/L, J_p) = (B/L, J_p)\), and the maps:
(G/L, J_p) \rightarrow (G/L, J_p) : Lg \mapsto Lg' \\
(G, J_p) \rightarrow (G/L, J_p) : g \mapsto Lg'g \quad (g, g' \in G) \text{ are continuous.}

Hence ([18]) the map:

(G/L × G, J_p × J_p) \rightarrow (G/L, J_p) : (Lg, g') \mapsto Lgg' \text{ is continuous.}

Let C(G/L, G/L) denote the topological semigroup of \( J_p \)-continuous maps of G/L into itself, where the multiplication is composition of functions and the topology is the topology of uniform convergence.

Let \( G \rightarrow C(G/L, G/L) : g \mapsto \mathcal{G}_g \) be defined by \((Lg')\mathcal{G}_g = Lg'g\).

Since this is a continuous injective homomorphism of G into C(G/L, G/L), multiplication in G is \( J_p \)-continuous in each variable, as required.

14.10 Lemma Let \( B_1, B_2 \) be the groups containing \( N' \) such that \( B_1/N' \) and \( B_2/N' \) are the \( J_p \)-connected and \( \sigma \)-connected components of \( M' = G'/M' \) respectively. Then \( B_1 = B_2 \).

Proof Clearly \( B_1 \subset B_2 \) and \( N'B_1 = N'B_2 \). So \( B_1 \) is of finite index in \( B_2 \). To show \( B_1 = B_2 \), it suffices to show \( B_1 \) is \( \sigma \)-closed.

Use the notation of 14.1. Let \((W, T) = (I/M', T)\) where \( I/M' = \{ M'p : p \in I \} \). \((W, T)\) is a distal extension of \((X, T)\), say \((X, T) \sim (W, T)\). Let \((V, T) = (W/\sim, T)\), where \( w_1 \sim w_2 \) if and only if \( \mathcal{G}(w_1) = \mathcal{G}(w_2) \) and \( w_1, w_2 \) lie in the same connected component of \( \mathcal{G}^{-1}(w_1) \). By 14.1(a) and (c), \( G_{\mathcal{V}} = B_1 \), so that \( B_1 \) is \( \sigma \)-closed as required.

14.11 Lemma \( BL = LB = G \).

Proof \( B/N \) is the connected component of the identity in \( G/N \). So (14.7) \( B/N = G/N \). So \( BH = HB = G \). So \( BL \) is of finite index in \( G \). So \( G/L \) is a finite union of cosets of \( B/L \), which are \( \sigma \)-closed, hence \( J_p \)-closed.

So, since \( G/L \) is \( J_p \)-connected, \( BL = G \).

14.12 In [20] an example is constructed of a minimal t.g. \((X, T)\) with totally disconnected phase space such that \((X, T)\) is a finite group
extension of an a.p. factor, but \((X,T)\) is not almost periodic.

14.13 Proposition. Let \((X,T)\) be minimal distal and let \(X\) be finite-dimensional and locally connected. Then \(X\) is a manifold.

Note. This was proved by Bronstein in [1]. As I was unable to understand the proof, I include one here.

It suffices to prove the following lemma, by analogue with § 7.

14.14 Lemma. Let \((W,T)\) be minimal distal, with \(W\) locally connected and connected. Let \((V,T) \leq \pi(W,T)\), with \(V\) a manifold. Then it is not possible to find a strictly increasing sequence \(\{V_n,T\}_{n=1}^\infty\) such that each \((V_n,T)\) is a finite extension of \((V,T)\) and \((W,T)\) the inverse limit of \(\{V_n,T\}\).

Proof. Suppose for contradiction that \((W,T)\) is the inverse limit of a strictly increasing sequence \(\{V_n,T\}_{n=1}^\infty\) as described in the statement of the lemma. Let \(U \subseteq V\) be a simply connected open set. Since each \(V_n\) is an open cover of \(V\), by passing to the limit we can find a map \(\sigma : U \longrightarrow \pi^{-1}(U)\) with \(\pi \sigma = \text{identity}\). Then \(\pi^{-1}(U)\) is homeomorphic to \(U \times G/H\) (3.2) where \((W,T)\) is a \(G/H\)-extension of \((V,T)\), and hence \(G/H\) is totally disconnected, infinite and perfect.

It follows that no open subset of \(\pi^{-1}(U)\) is connected, contradicting the fact that \(W\) is locally connected.
ON THE FIBRES OF A MINIMAL DISTAL EXTENSION OF A TRANSFORMATION GROUP

11. INTRODUCTION

Let \((X, T)\) be a minimal transformation group, and let \(f: X \to X\) be a factor map.

If \((Y, T)\) is an almost periodic extension of \((X, T)\) then all the fibres of \(f^{-1}(y)\), \(y \in Y\), are closed in a fixed homogeneous space \(\mathcal{X}\).

One might ask whether the fibres \(f^{-1}(y)\) are all homeomorphic if \((Y, T)\) is a minimal extension of \((X, T)\). The answer is, in general, 

yes, as is shown in \(\mathcal{X}\). However, if we assume that \(\mathcal{X}\) is arcwise-connected

(or, more generally, has infinitely many arcwise-connected components)

then theorems 6.1 gives a positive answer.

I should like to thank my supervisor, Professor E. Perry, and

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12. NOTATION

2:1. \((X, T)\) will denote a transformation group \((x, t) \mapsto tx\) on a compact hausdorff phase space \(X\). \(T\) will be a topological group acting on \(X\)

on the right, the action being jointly continuous.

2.2. If \((Y, T)\) is a factor of \((X, T)\), the factor map being \(\pi: (X, T) \to (Y, T)\)

we shall write \((X, T) \to (Y, T)\).

2.3. If \(G\) is a compact topological group acting on a compact hausdorff

space \(x\), the action being jointly continuous, then \(G\) will denote the compact hausdorff group whose elements are equivalence with the quotient topology.
11. INTRODUCTION

Let \((X, T)\) be a minimal quasi-separable transformation group, and an extension of \((Y, T)\), with \(\Pi: (X, T) \rightarrow (Y, T)\) as the factor map. If \((X, T)\) is an almost periodic extension of \((Y, T)\) then all the fibres \(\Pi^{-1}(y)\) \((y \in Y)\) are homeomorphic to a fixed homogeneous space [1].

One might ask whether the fibres \(\Pi^{-1}(y)\) \((y \in Y)\) are all homeomorphic if \((X, T)\) is a distal extension of \((Y, T)\). The answer is, in general, no, as is shown in \(\S 6\). However, if we assume that \(Y\) is arcwise-connected (or, more generally, has finitely many arcwise-connected components) then theorem 5.1 gives a positive answer.

I should like to thank my supervisor Professor W. Parry, and Dr. K. Schmidt, for helpful discussion. I should also like to thank the S.R.C. for financial support.

12. NOTATION

2.1. \((X, T)\) will denote a transformation group (t.g.) with compact Hausdorff phase space \(X\). \(T\) will be a topological group acting on \(X\) on the right, the action being jointly continuous.

2.2. If \((Y, T)\) is a factor of \((X, T)\), the factor map being \(\Pi: (X, T) \rightarrow (Y, T)\), we shall write \((Y, T) <_f (X, T)\).

2.3. If \(G\) is a compact topological group acting on a compact Hausdorff space \(Z\), the action being jointly continuous, then \(Z/G\) will denote the compact Hausdorff orbit space endowed with the quotient topology.
2.4. A word about diagrams: all arrows in diagrams will denote continuous surjective maps; and two-ended arrows will denote homeomorphisms; if the objects in a diagram are the phase spaces of transformation groups with respect to a group $T$, all maps in the diagram will be assumed to denote $T$-homeomorphisms; if $G$ is a compact topological group acting continuously on a compact Hausdorff space $Z$, then $Z \times G$ will denote the orbit map.

3. PRELIMINARIES ABOUT FIBRE BUNDLES

3.1. Definition For present purposes, a fibre bundle $S = (Z, X, Y, G, H, \pi, \sigma, \rho)$ satisfies

(i) $X, Y, Z$ are compact Hausdorff spaces, and $\pi, \sigma, \rho$ are continuous surjective maps.

(ii) $G$ is a compact Lie group with closed subgroup $H$. $\bigcap_{g \in G} g^{-1} H g = \{e\}$ and $G$ acts freely on the left of $Z$, the action being jointly continuous.

(iii) The following diagram is commutative.
Y is called the base of the bundle, and \( G \) the group of the bundle. This definition of fibre bundle is essentially the same, for a restricted class of bundles, as that in [4] Chapter 1 §2, since ([2], theorem 1 of §5.4) the free action of a compact Lie group on a compact Hausdorff space is locally trivial.

The following definition of bundle map essentially coincides, for the restricted class of bundles, with that of [4] Chapter 1, §2.

3.2. Definition Let \( \mathcal{B} = (Z,X,Y,G,H,\sigma,p) \) and \( \mathcal{B}' = (Z',X',Y',G,H',\sigma',p') \) be fibre bundles. \( \phi \) is a bundle map between \( \mathcal{B} \) and \( \mathcal{B}' \) (written \( \phi : \mathcal{B} \to \mathcal{B}' \)) if \( \phi \) is continuous, \( \phi : Z \to Z' \), and \( \phi(gz) = g\phi(z) \) for all \( g \in G, z \in Z \).

Then \( \phi \) induces maps \( \phi_1 : X \to X' \) and \( \phi_2 : Y \to Y' \) such that the following diagram is commutative:

![Diagram 3.2](attachment:image)
3.3. Definition. If $\mathcal{B} = (Z, X, Y, G, H, \pi, \sigma, \rho)$ and $I = \{0, 1\}$, then $\mathcal{B} \times I$ denotes the bundle $(Z \times I, X \times I, Y \times I, G, H, \Pi \times \text{identity}, 
abla \times \text{identity}, \rho \times \text{identity})$ where the action of $G$ on $Z \times I$ is defined in terms of the action of $G$ on $Z$ by $g.(z,t) = (g.z, t)$ for all $g \in G, z \in Z, t \in I$.

3.4. We shall need:


Let $\mathcal{G} = (Z, X, Y, G, H, \pi, \sigma, \rho)$ and $\mathcal{G}' = (Z', X', Y', G, H', \sigma', \rho')$ be bundles and let $\phi: \mathcal{G} \to \mathcal{G}'$ be a bundle map inducing $\phi: Y \to Y'$. Let $h: Y \times I \to Y'$ be a continuous map with $h(y, c) = \phi(y)$ for all $y \in Y$. Then there exists a bundle map $k: \mathcal{G} \times I \to \mathcal{G}'$ inducing $h: Y \times I \to Y'$ such that $k(z, 0) = \phi(z)$ for all $z \in Z$.

4. ON THE FURSTENBERG STRUCTURE THEOREM.

4.1. Definition. Let $(X, T)$ be minimal and $(Y, T) \leq_{R} (X, T)$. $(X, T)$ is a quotient (Lie) group extension of $(Y, T)$ if there exist a minimal t.c. $(Z, T)$ with $(X, T) \leq_{G} (Z, T)$ and a compact (Lie) topological group $G$ with closed subgroup $H$, $\bigcap_{g \in G} g^{-1}Hg = \{e\}$, such that $G$ acts freely on the left of $Z$, the action being jointly continuous, $(gz)t = g(zt)$ for all $g \in G, z \in Z, t \in T$, and the following diagram is commutative.

Diagram 4.1.

$$
\begin{array}{ccc}
Z & \xrightarrow{\sigma} & Y (\cong Z/G) \\
\downarrow & & \downarrow \\
X (\cong Z/H) & \xrightarrow{\sigma} & \\
\end{array}
$$
Remark. Note that if $G$ is Lie then $(Z_0, X_0, Y_0, G, \pi, \nu, o, o)$ is a fibre bundle where $Y_0$ is any closed subset of $Y$, $Z_0 = \rho^{-1}(Y_0)$ and $X_0 = \pi^{-1}(Y_0)$.

4.2. Definition. Let $(Y, T) \leq (X, T)$ with $(X, T)$ minimal. $(X, T)$ is a distal extension of $(Y, T)$ if given $x_1, x_2 \in X$ with $\pi(x_1) = \pi(x_2)$, the existence of a net $(t_n) \subseteq T$ with $\lim x_1 t_n = \lim x_2 t_n$ implies $x_1 = x_2$.

4.3. Definition. $(X, T)$ is quasi-separable if $C(X)$ is generated by its norm-separable $T$-invariant subalgebras. For instance, if $X$ is metric, or if $T$ is separable or $\sigma$-compact, then $(X, T)$ is quasi-separable.

4.4. The following modification of the Furstenburg structure theorem will be needed in the proof of theorem 5.1. Proof of the modification will not be given here. The proof of conclusions (1) - (v) with the word "Lie" omitted in (iv) can be found in [1] Chapters 14 and 15.

Theorem Let $(X, T)$ be a minimal quasi-separable t.g., $(Y, T) \leq (X, T)$, $(X, T)$ a distal extension of $(Y, T)$. Then there exist an ordinal $\alpha$, a family $\{(X_\beta, T)\}_{0 \leq \beta < \alpha}$ of factors of $(X, T)$ and $T$-homomorphisms $\\{\pi_{\beta, \gamma}\}_{0 \leq \beta, \gamma < \alpha}$ such that

1) $(X_\beta, T) \leq \pi_{\beta, \gamma}(X_T, T), 0 < \beta, \gamma < \alpha$.

2) $\pi_{\beta, \gamma} \circ \pi_{\gamma, \delta} = \pi_{\beta, \delta}, 0 < \beta, \gamma, \delta < \alpha$.

3) $\pi_{\alpha, \alpha} = \pi$.

4) $(X_\beta, T) = (Y, T)$, $(X_\alpha, T) = (X, T)$.

5) $(X_{\beta+1}, T)$ is a proper quotient Lie group extension of $(X_\beta, T)$ for $\beta < \alpha$.

6) If $\beta$ is a limit ordinal then $(X_\beta, T)$ is the inverse limit of $\{(X_T, T)\}_{0 < \gamma < \beta}$. 
$5.$ THE POSITIVE RESULT

5.1. **Theorem.** Let $(X,T)$ be a minimal quasi-separable t.q., $(Y,T) \prec_{c}(X,T).$ $(X,T)$ a distal extension of $(Y,T)$, and let $Y$ be arcwise-connected. Then for any $y_0, y_1 \in Y$, $\pi^{-1}(y_0)$ and $\pi^{-1}(y_1)$ are homeomorphic.

**Proof.** Let $\alpha_{\beta} = \{(X_{\beta},T)\}_{0 \leq \beta \leq \alpha}$ and $\{\pi_{\beta,\gamma}\}_{0 \leq \beta \leq \gamma \leq \alpha}$ satisfy (i) - (v) of (4.1). Write $I = [0,1]$.

Choose a path $h_0 : \{y_0\} \times I \rightarrow Y$ with $h_0(y_0,0) = y_0$ and $h_0(y_0,1) = y_1$.

Write $P_\beta = \pi_{\beta,\gamma}^{-1}(y_0)$, $Q_\beta = \pi_{\beta,\gamma}^{-1}(y_1)$, $R_\beta = \pi_{\beta,\gamma}^{-1}(h_0(\{y_0\} \times 1))$.

Hence $P_\beta \subseteq R_\beta$, $Q_\beta \subseteq h_\beta$, $P_\gamma = \pi^{-1}(y_0)$, $Q_\gamma = \pi^{-1}(y_1)$.

Find by transfinite induction on $\beta$ continuous maps $\{h_\beta\}_{0 \leq \beta \leq \alpha}$ such that:

(i) $h_\beta : P_\beta \times I \rightarrow \pi_{\beta,\gamma}^{-1}(y_0)$ restricts to a homeomorphism of $P_\beta \times \{1\}$ onto $Q_\beta$, and $h_\beta(x,0) = x$ for all $x \in P_\beta$.

(ii) The following diagram commutes for $\gamma \leq \beta$:

\[
\begin{array}{ccc}
P_\beta \times I & \xrightarrow{h_\beta} & R_\beta \\
\downarrow{\pi_{\gamma,\beta}} & & \downarrow{\pi_{\gamma,\beta}} \\
P_\gamma \times I & \xrightarrow{h_\gamma} & R_\gamma
\end{array}
\]

If (i) and (ii) are satisfied, $h_\beta$ will restrict to a homeomorphism of $P_\beta \times \{0\} = \pi^{-1}(y_0) \times \{0\}$ onto $Q_\alpha = \pi^{-1}(y_1)$ and the proof will be completed.

$h_0$ satisfies (i) and (ii), hence it remains to assume that $\{h_\beta\}_{0 \leq \beta \leq \alpha}$ have been constructed satisfying (i) and (ii) for $0 < \beta \leq \alpha$, and to construct $h_\alpha$. 

(1) Case \( \delta = n+1, \) some \( n. \)

Since \( X_\delta \) is a quotient Lie group extension of \( X_n \) (by 4.4(iv)) we have the following diagram for some minimal t.g. \((Z,T),\) compact Lie group \( G, \) with closed subgroup \( H \) and \( \bigcap _{g \in G} g^{-1}Hg = \{ e \}: \)

Diagram 5.1(b)

By the remark of (4.1), \( \mathcal{O} = (\mathfrak{g}, \cdots, P, G, H, \cdots, \sigma, \rho) \) and \( \mathcal{O}' = (\mathfrak{g}', \cdots, P', G, H, \cdots, \sigma, \rho) \) are fibre bundles where \( W = p^{-1}(P_n) \) and \( W' = p^{-1}(P_n) \) (so \( W \subseteq W' \)). By the inductive hypothesis we have a map \( h_n: P_n \times I \rightarrow R_n, \) satisfying (i) and (ii). By the Homotopy Covering Theorem 3.4, we can find a bundle map \( k: \mathcal{O} \times I \rightarrow \mathcal{O}' \) inducing \( h_n: P_n \times I \rightarrow R_n, \) and such that \( k(w,0) = w \) for all \( w \in W. \) Let \( h_c: P_\delta \times I \rightarrow R_\delta \) be the induced map as shown in the diagram:

Diagram 5.1(c)
Case 6 a limit ordinal

$h_\delta$ is well-defined and satisfies (i) and (ii) if we define
$h_\delta(v, t) (v \in P_\delta, t \in I)$ by

\[ \Pi_n(h_\delta(v, t)) = h_n(\Pi_n(v), t) \text{ for all } n < \delta. \]

§6. A COUNTEREXAMPLE

6.1. In this section we construct a connected minimal distal t.g. 
$(X, \mathcal{B})$ and a factor 
$(Y, \mathcal{D}) \leq (X, \mathcal{B})$ such that $Y$ has infinitely many 
arewise-connected components and such that not all the fibres 
$\Pi^{-1}(y) (y \in Y)$ are homeomorphic to each other. $X$ will be a connected 
metric space of covering dimension 3, and $(X, \mathcal{B})$ will be a group 
extension of an almost periodic t.g., so $(X, \mathcal{B})$ will be of degree 2 
(Cf [1], 15.1.2). $T$ will be the group $\mathbb{Z}$ of integers.

6.2. Definitions. Let $\mathbb{Z}, \mathbb{R}$ denote the additive groups of integers 
and reals respectively, let $\mathbb{K} = \mathbb{R}/\mathbb{Z}$ denote the circle group and $S_d$ 
a fixed solenoid which is the inverse limit of the sequence 

\[ \cdots \xrightarrow{n_3} \mathbb{K}_3 \xrightarrow{n_2} \mathbb{K}_2 \xrightarrow{n_1} \mathbb{K}_1 \]

where $\mathbb{K}_i = K$, $n_i$ denotes the homomorphism $x \mapsto n_i \cdot x (x \in \mathbb{K}_{i-1})$, $n_i$ 
being an integer $\geq 2$ for each $i$. All group operations will be 
written additively.

Let $n_{r,s} : \mathbb{K}_r = \mathbb{K}_s$ be the homomorphism $n_r \cdot n_{r+1} \cdots n_{s-1} (r < s)$.

Let $\chi_1 : S_d = \mathbb{K}_1$ be the inverse limit homomorphism.

6.3. The following facts about $S_d$ are known.

(1) Since $S_d$, being the character group of a subgroup of the reals 
with the discrete topology, is a continuous homomorphic image of the 
Bohr compactification of $\mathbb{R}$, ([3] 1.8), $S_d$ contains a dense 1-parameter 
subgroup $\Gamma$. 

B
(ii) $\chi^{-1}_1(0)$ is an infinite closed subgroup of $S^d$, hence uncountable.

Since $\Gamma \cap \chi^{-1}_1(0)$ is countable, $\Gamma$ has uncountable index in $S^d$.

(iii) For $\beta \in \Gamma$ and $x \in R$ we can uniquely define $x^\beta$ satisfying

(a) $x \mapsto x^\beta$ is a continuous homomorphism of $R$ onto $\Gamma$

(b) $1^\beta = \beta$.

(iv) $\Gamma$ is the arcwise-connected component of $0 \in S^d$. For let $x \in S^d$ be in the arcwise-connected component of $0 \in S^d$. Let $\sigma: [0,1] \to S^d$ be a path from $0$ to $x$. $\sigma$ is the unique lifting to $S^d$ of the path $\chi \cdot \sigma: [0,1] \to K$, with the property $\sigma(0) = 0$. Now $\chi \cdot \sigma$ must be homotopic to the path $\tau_2(t): Z + at$ for some $a \in R$ where $Z + a = \chi(x)$ lifts to a unique path in $S^d$ joining $0$ to $x$. But we can assume $\Gamma = \{\gamma(t): t \in R\}$ where $\gamma$ is a homomorphism such that $\chi \cdot \gamma(t) = Z + t$ for all $t \in R$, and then the path $t \mapsto \gamma(t)$ is the lifting of $\tau_2$, hence $x = \gamma(1), x \in \Gamma$.

(v) For each integer $n > 0$, $S^d$ contains at most $n$ elements $x$ such that $nx = 0$, since the same is true for each $\mathbb{R}^d$.

6. Definitions of some phase spaces

Let $X = (\mathbb{R} \times S^d^2)/\sim$ where $\sim$ is the smallest equivalence relation on $\mathbb{R} \times S^d^2$ such that $(x,y,z) \sim (x+1, y+\pi, z)$ for all $x \in \mathbb{R}, y, z \in S^d$.

Let $[x, y, z]$ denote the equivalence class of $(x, y, z)$. Then $X$ is compact metric and can be realized as an "inverse limit of nilmanifolds".

For $i = 1, 2, \ldots$ let $X_i = (\mathbb{R} \times S^d_{i-1})/\sim_i$ where $\sim_i$ is the smallest equivalence relation on $X_i$ such that $(x,y,z) \sim_i (x+1, y+\pi, z)$ for all $x \in \mathbb{R}, y, z \in S^d$. Let $[x, y, z]_i$ denote the $\sim_i$-equivalence class of $(x, y, z)$.
Define $\Pi_{i,j}: X_i \times X_j$ by $\Pi_{i,j}[x,y,z]_j = [x, n_{i,j} y, n_{i,j} z] (i < j)$. Define $\Pi_{i,p}: X \times X_1$ by $\Pi_{i,p}[x,y,z] = [x, x_1(y), x_1(z)]$.

Then $(X, \{\Pi_{i,p}\})$ is the inverse limit of $\{\{X_i, \Pi_{i,j}\}_{i<j}\}$. Moreover it can be shown that $X_1$ is a 3-dimensional nilmanifold whose fundamental group $\Pi_1(X_1)$ is isomorphic to the multiplicative group of matrices

\[ \begin{pmatrix} 1 & m & n \\ 0 & 1 & p \end{pmatrix} : m, n, p \in \mathbb{Z} \]

6.5. Definitions of some $Z$-actions.

For actions of $Z$ with phase space $W$, if $t$ is the homeomorphism of $W$ corresponding to the action of $1 \in Z$ on $W$, we shall denote the corresponding t.g. by $(W, t)$.

We now define minimal distal $Z$-actions on $X$ and $X_i (i = 1, 2, ...)$.

Choose $1, \alpha, \beta \in \mathbb{R}$ to be rationally independent and let $2 \epsilon \Gamma$ be in the inverse image under $\chi_1$ of $Z + \beta_1 \in \mathbb{R}$ such a $2 \epsilon \Gamma$ exists by considering a lifting (under $\chi_1$) to $Sd$ of a path in $K_1$ from $Z$ to $Z + \beta_1$.

Define $t: X \times X$ by $[x,y,z]_t = [x+\alpha, y+x^3, z+3] (6.3(iii))$. Then $(X_1, t_1) \circ \Pi_{i,j} (X_j, t_j) \circ \Pi_{j,p} (X, t) (i < j)$, where $t_i: X_i \times X_1$ is of the form

\[ [x, y_i, z_i]_i \times \frac{\chi_i(x)}{} = [x + \alpha, y_i + x y_i, z_i + 2 y_i] \] where $\gamma_i \in \mathbb{R}$ is the unique element of $\mathbb{R}$ for which

$\chi_i(x) = Z + x y_i$ for all $x \in \mathbb{R}$.

Since $(X, t)$ is the inverse limit of $(X_1, t_1)$, to show $(X, t)$ is minimal distal it suffices to show each $(X_1, t_1)$ is minimal distal.

Define a free action of $K$ on $X_1$ by

$w.[x, y_1, z_1]_1 = [x, y_1 + w, z_1]_1 (w \in K)$. 
The action of $K$ on $X_1$ commutes with $t_1$ and we have

$$(X_1, t_1) \longrightarrow X_1/K \cong (K^2, s_1) \quad (\text{see (2.4)}),$$

where $(x, z)s_1 = (x + a, z + \gamma_1)$, $(x, z \in K)$.

$a$ and $\gamma_1$ are rationally independent since $Z + \gamma_1 = Z + \beta$ and hence $\gamma_1$ and $\beta$ are rationally dependent. Thus $(K^2, s_1)$ is minimal and hence $(X_1, t_1)$ is minimal — for if not we could find a finite subgroup $H$ of $K$ such that $X_1/H$ was homeomorphic to $K^3$, which would imply that the commutator subgroup of $H(X)$ was of finite index (see, for example [1] (6.19.2.6)).

Clearly $(X_1, t_1)$ is also distal.

6.6. Definitions. Define $Y = S^d$ and $\Pi: X \to Y$ by $\Pi(x, y, z) = x$.

Then $(Y, \circ) \leq H (X, t)$ where $zs = z + \beta$.

Then $\Pi^{-1}(z) = (x, y, z) \sim x$ where $\sim$ is the smallest equivalence relation such that $(x, y) \sim_z (x + 1, y + 2)$ for all $x, y \in S^d$. Let $(x, y, z)$ denote the $\sim_z$-equivalence class of $(x, y) \in \mathbb{R} \times S^d$.

6.7. Proposition. Let $O$ denote the identity of $S^d = Y$. There exists $z \in Y$ such that $nz \neq n$ for any $n \in \mathbb{Z}$, $n \neq 0$. For such $a, z$, $\Pi^{-1}(z)$ and $\Pi^{-1}(O)$ are not homeomorphic.

Proof. Let $A = \{v \in S^d: av = 0, \text{some } a \in \mathbb{Z}, \text{a } \neq 0\}$. Then by (6.2(w)) $A$ is countable, hence by (6.3(ii)) $A + A \neq S^d$. But $n \in A$ for some $n \in \mathbb{Z}$, $n \neq 0$, if and only if $w \in A - A$. Hence $z$ exists.

Note that $\Pi^{-1}(w)$ is a quotient of $\mathbb{R} \times S^d$ by a discrete subgroup, so that its fundamental group $\Pi_1(\Pi^{-1}(w))$ is independent of the base-point, and $\mathbb{R} \times S^d$ is a covering of $\Pi^{-1}(w)$, so that loops in $\Pi^{-1}(w)$ based at $(0, 0)$ lift to paths in $\mathbb{R} \times S^d$ joining $(0, 0)$ to $(n, nw)$ $(n \in \mathbb{Z})$.

$(0, 0)$ can be joined to $(n, 0)$ for any $n \in \mathbb{Z}$. So $\Pi_1(\Pi^{-1}(0)) \neq \mathbb{Z}$.

$(0, 0)$ cannot be joined to $(n, nz)$ for any $n \in \mathbb{Z}$, $n \neq 0$ (6.3(iv)).
So $\prod_1^{n-1}(z) = 0$.

Therefore $\prod^{-1}(z)$ and $\prod^{-1}(0)$ are not homeomorphic.

§7. **UNCOUNTABLY MANY HOMOTOPIC TYPES OF FIBRES**

7.1 In this section we show that in the example of §6, where $(Y,T) \prec (X,T)$, and $X = (S^d \times \mathbb{R})/\sim$ and $Y = S^d$, the cardinality of the set of distinct homotopic (hence topological) types of the fibres $\pi^{-1}(z)$ ($z \in S^d = Y$) is the cardinality of the continuum. (In §6 it was shown merely that the cardinality was greater than one.)

Recall that $S^d$ is the inverse limit of the sequence:

$$\cdots K_4 \xrightarrow{n_3} K_3 \xrightarrow{n_2} K_2 \xrightarrow{n_1} K_1,$$

with limit maps $\chi_k: S^d \to K_1$.

We make the additional assumption that each $n_k = 2$.

The fibre $\pi^{-1}(z)$ is the compact connected abelian group $(S^d \times \mathbb{R})/N_z$ where $N_z$ is the discrete group generated by the element $(z,1)$. Let $A_z$ denote the character group of $(S^d \times \mathbb{R})/N_z$.

7.2 Proposition. $\pi^{-1}(z)$ and $\pi^{-1}(w)$ are homotopic if and only if $A_z$ and $A_w$ are isomorphic as groups.

We do not give a proof of this. It follows from either (1) $A_z$ is the first Čech cohomology group of $(S^d \times \mathbb{R})/N_z$ ([5]), or

or (2) It can be shown that a continuous map between two compact connected abelian groups is homotopic to a unique group homomorphism.

7.3 Description of $A_z$.

Fix $z \in S^d$.

$(S^d \times \mathbb{R})/N_z$ is the inverse limit of $(\{\chi_n(z)\}/N_{n,z}, \{\sigma_{n,z}^{n+1}\})$

where $N_{n,z}$ is generated by $(\chi_n(z),1)$ and:

$$\sigma_{n,n+1}((K_n \times \mathbb{R})/N_{n+1,z} \to (K_n \times \mathbb{R})/N_{n,z}$$

is defined by:

$$\sigma_{n,n+1}(N_{n+1,z} + (k,x)) = N_{n,z} + (k,x).$$
The limit map $\sigma_n : (Sd x k)/N_n \rightarrow (K_n^x k^2)/N_{n+1}$ is defined by:

$$\sigma_n(N_n + (y, x)) = N_{n+1} + (\chi(y), x).$$

Now $\chi_n(z) = e^{2\pi i \theta_n}$ for a unique $\theta_n = \theta_n(z) \in [0, 1)$. Then $\theta_{n+1} = \theta_n/2$ or $\theta_{n+1} = \theta_n/2 + 1/2$.

$\varphi_n : (K_n x k)/N_n \rightarrow k^2$ is a group isomorphism,

if $\varphi_n(N_n + (k, x)) = (ke, x)$. Then $(Sd x k)/N_n$ is the inverse limit of:

$$k^2 \rightarrow k^2 \rightarrow \cdots \rightarrow k^2 \rightarrow k^2$$

where $\xi_{n,n+1} = \varphi_n \circ \sigma_{n,n+1} \circ \varphi_n^{-1}$.

If $\theta_{n+1} = \theta_n/2$ then $\xi_{n,n+1}(k_1, k_2) = (k_1, k_2)$.

If $\theta_{n+1} = \theta_n/2 + 1/2$ then $\xi_{n,n+1}(k_1, k_2) = (k_1, k_2)$.

Let $A_{n,z}$ be an isomorphic copy of the character group of $k^2$. We have:

$$A_{1,z} \rightarrow A_{2,z} \rightarrow A_{3,z} \rightarrow \cdots$$

$A_n$ can be regarded as the inductive limit of this sequence. We shall now define $A_{n,z}$ so that $A_{n,z} \subseteq A_{n+1,z} \subseteq A_z \subseteq \mathbb{Q}^2$ and $\xi_{n,n+1}$ is the inclusion map.

Define $a_n = a_n(z) = 0$ if $\theta_{n+1} = \theta_n/2$

and $a_n = a_n(z) = 1$ if $\theta_{n+1} = \theta_n/2 + 1/2$.

Then let $A_{n,z}$ be generated by the elements $(0, 1)$ and $(1/2^{n-1}, -\sum_{i=1}^{n-1} a_{n-i}/2^i)$.

7.4 Lemma If $A$ and $B$ are subgroups of $\mathbb{Q}^n$ with rational span $\mathbb{Q}^n$ then any group isomorphism between $A$ and $B$ can be uniquely extended to a $\mathbb{Q}$-linear isomorphism of $\mathbb{Q}^n$.

We omit the proof. It follows that $A_z$ is isomorphic to only countably many distinct groups $A_w$ ($w \in Sd$). Thus we only have to prove:

7.5 Proposition If $A_z = \bigcup_{n=1}^{\infty} A_{n,z}$ where $A_{n,z} \subseteq \mathbb{Q}^2$ is generated by $(1/2^{n-1}, -\sum_{i=1}^{n-1} a_{n-i}/2^i)$ as described in 7.3, then the cardinality
of the set \( \{ A_z : z \in S \} \) is the cardinality of the continuum.

Proof (i) We claim \( A_z = A_w \) if and only if \( a_n(z) = a_n(w) \) for all \( n \).

For suppose \( A_z = A_w \). Then for each \( n \),
\[
(1/2^{n-1}, \frac{1}{i=1} a_{n-1}(z)/2^1) \in A_w \cup \bigcup_m A_m,w.
\]

Then \( (1/2^{n-1}, \frac{1}{i=1} a_{n-1}(z)/2^1) = (0,q) + q(1/2^{m-n}, \frac{1}{i=1} a_{m-1}(w)/2^i) \)
for some \( p, q \in \mathbb{Z}, \ n > 1 \). Then \( q = 2^{m-n}(m \geq n) \) and:
\[
p = 2^{m-n} \frac{1}{i=1} a_{m-1}(w)/2^1 = - \frac{1}{i=1} a_{n-1}(z)/2^1.
\]
The fractional parts are equal. So:
\[
- \frac{1}{i=1} a_{n-1}(w)/2^1 = - \frac{1}{i=1} a_{n-1}(z)/2^1.
\]
So \( a_n(z) = a_n(w) \) for \( 1 \leq i \leq n-1 \). Hence result, since \( n \) is arbitrary.

(ii) Let \( \{ b_n \}_{n=1}^{\infty} \) be any sequence of 0's and 1's. We claim there exists
\( z \in S \) with \( a_n(z) = b_n \) for all \( n \).

For take \( z \in \bigcap_{n=1}^{\infty} (\mathbb{C} \backslash 0 \mathbb{Q} \backslash \mathbb{Z} + (0, 2\pi \mathbb{N})) \) where \( \theta_1 = 0 \) and \( \theta_n \) is defined
inductively for \( n \geq 1 \) by:
\[
\theta_{n+1} = \theta_n / 2 \quad \text{if } b_n = 0
\]
\[
= \theta_n / 2 + 1/2 \quad \text{if } b_n = 1.
\]

(i) and (ii) show that the cardinality of \( \{ A_z : z \in S \} \) is the

Cardinality of the set of sequences of 0's and 1's, as required.
NON-CONJUGACY OF A MINIMAL DISTAL DIFFEOMORPHISM OF THE TORUS TO A $C^1$ SKEW-PRODUCT
NON-CONJUGACY OF A MINIMAL DISTAL Diffeomorphism of the Torus to a $C^1$ Skew-Product.

K. Msis

Introduction

Let $\text{Hom}^+(K^n)$ denote the set of orientation-preserving homeomorphisms of the $n$-dimensional torus $K^n = \mathbb{R}^n / \mathbb{Z}^n$. If $T$ is a minimal element of $\text{Hom}^+(K)$, then it is known that $T$ is topologically conjugate to an irrational rotation of $K$, which is, of course, $C^\omega$. Correspondingly, if $T$ is a minimal distal element of $\text{Hom}^+(K^2)$, it is known (see, for instance, [4]) that $T$ is topologically conjugate to a homeomorphism of $K^2$ of the form:

$$T_{x,g} : (x,y) \mapsto (x+\psi(y),y \cdot g(x))$$

where $g \in C(K,K)$ and $\psi$ is irrational. In this paper, it is shown that, contrary to what happens for the circle, or for almost periodic homeomorphisms in general, there is a minimal distal $C^\omega$ element of $\text{Hom}^+(K^2)$ which is not topologically conjugate to any $C^1$ homeomorphism of the form $T_{x,g}$.

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§1. Preliminaries.

1.1. If $f \in C(K^n, K^n)$, then there exists a unique element of $C(K^n, K^n)$, again denoted by $f$, such that $f(0) \in [0,1]^n$, and the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \\
\downarrow & & \downarrow \\
\mathbb{Z}^n & \xrightarrow{f} & \mathbb{Z}^n \\
\end{array}$$
For \( \text{Hom}^+(K) \), this correspondence reduces to a correspondence between 
\( \text{Hom}^+(K) \) and \( \{ f \in C(K,K) \mid f \text{ is a homeomorphism}, f(x+1) = f(x) + 1 \) 
for all \( x \in \mathbb{R} \), and \( f(0) \in [0,1] \).

Note that in what follows, for all equations (inequalities) involving 
elements of \( C(K^2,K^2) \) corresponding to elements of \( C(K^n,K^n) \), the equality (inequality) sign denotes real equality (inequality) and not equality (inequality) mod \( \mathbb{Z}^2 \).

1.2. Let \( \text{Hom}^+(K) \) be given the topology of uniform convergence. The rotation number function \( \xi : \text{Hom}^+(K) \to K \) is continuous.

If \( q \in \mathbb{Z} \) and \( f \in \text{Hom}^+(K) \), then \( \xi(f) = Z+(p/q) \) for some \( p \in \mathbb{Z} \) if and only if there exists \( x \in K \) with \( f^q(x) = x \). (See, for example, [3], [1] for definition and basic properties of \( \xi \).)

1.3. Definition. Let \( f \in \text{Hom}^+(K) \) with \( \xi(f) = Z+(p/q) \) with \( p, q \) coprime and positive, \( 0 < p < q \). We follow [1] in defining \( f \) to be semistable forward if:

\[ f^q(x) > x + p \text{ for all } x \in \mathbb{R}. \]

1.4. Denjoy's Theorem. (See, for example, [3].)

Let \( f \in \text{Hom}^+(K) \) be \( C^2 \) and \( \xi(f) = Z+\xi \in [0,1] \) and irrational. Then there exists a unique \( \xi' \in \text{Hom}^+(K) \) such that:

\[ \xi'(f(x)) = \xi'(x) + \xi \text{ for all } x \in \mathbb{R}, \quad \xi'(0) = 0. \]

\( \xi' \) is called the eigenfunction of \( f \) corresponding to \( \xi \). Note that, in particular, \( f \) is minimal almost periodic.

2. Reduction of the Problem.

Throughout this section, let \( f \in \text{Hom}^+(K) \) be \( C^\infty \) with \( \xi(f) = Z+\xi \), \( \xi \) irrational, \( \xi \in [0,1] \).

Let \( T \in \text{Hom}^+(K^2) \) be given by:

\[ T(x,y) = (f(x), x+y). \]

Then \( (K^2,T) \) is distal, and the maximal almost periodic factor is
(K, f). Since (K, f) is minimal by 1.4, (K, T) is minimal by 2.2.

Consider the following four statements. It will be shown that
2.4 \Rightarrow 2.3 \Rightarrow 2.2 \Rightarrow 2.1.

2.1. If T(x, y) = (f(x), x + y), then T is not conjugate to any C¹ homeomorphism of the form:

\[ T_{\beta, g} : (x, y) \mapsto (x + \beta, y + g(x)), \text{ where } \beta \in \mathbb{R} \text{ and } g \in C¹(K, K). \]

2.2. The equation:

\[ x - q(x) = \wp(\wp(x)) + \chi(f(x)) - \chi(x) + \rho \]

does not hold for any \( \wp \in C¹(\mathbb{R}, \mathbb{R}), \chi \in C(\mathbb{R}, \mathbb{R}), \rho \in \mathbb{R} \), where \( \wp \) and \( \chi \) have period 1, \( \int_0^1 \chi = 0 \), and \( \wp \) is the eigenfunction of \( f \) corresponding to \( \lambda \) (see 1.4).

2.3. For each \( \psi \in C¹(\mathbb{R}, \mathbb{R}) \) with period 1 and \( \int_0^1 \psi = 0 \), there exists a strictly increasing sequence \( \{ n_k \} \) of positive integers with:

(i) \[ \sup_{n} \sup_{x \in K} \left| \sum_{i=0}^{n-1} \psi(x + i \omega) \right| < \infty \]

(ii) The sequence \[ \left\{ \sup_{x \in K} \left| \sum_{i=0}^{n_{k+1}-1} (f^i(x) - 1 \omega) - (m_{k}/n_{k+1}) \sum_{i=0}^{n_{k+1}-1} (f^i(x) - 1 \omega) \right| \right\} \]

is unbounded.

2.4. There exists a constant \( B > 0 \), a sequence \( \{ q_n \} \) of positive integers with \( q_{n+1} > q_n^3 \) and a sequence \( \{ x_n \} \) of elements of \( K \) such that, if for each \( a \), \( m \) is any multiple of \( q_n \) with \( m < m \leq q_n^3 \), then:

(i) \[ \left| \frac{1 - e^{2 \pi i x_n}}{1 - e^{2 \pi i x_n}} \right| \leq 1 \text{ for } r > q_n^3, \text{ } r \text{ not a multiple of } q_n. \]

(ii) \[ \left\{ \frac{1}{m_{n+1}} \sum_{i=0}^{m_{n+1}-1} (f^i(x_n) - 1 \omega) \right\} = \left\{ \frac{1}{m_{n}} \sum_{i=0}^{m_{n}-1} (f^i(x_n) - 1 \omega) \right\} \bmod B^2. \]
2.2 $\Rightarrow$ 2.1. If $T$ is conjugate to a $C^1$ homeomorphism of the form $T_0^x$, we can assume $\beta = \pi$, and that the conjugacy is given by:

$$(x,y) \mapsto (g(x), h(x) + y)$$

where $h \in C(X,\mathbb{R})$ and $g$ is the eigenfunction of $f$ corresponding to $\omega$.

This is essentially because the group of eigenvalues is preserved under conjugacy, and a conjugacy must give 1-1 correspondences between the groups of eigenfunctions, and between the groups of generalized eigenfunctions of order 2. The result follows.

2.3 $\Rightarrow$ 2.2. Suppose 2.2 does not hold, i.e. the equation of 2.2 is satisfied by some $\psi$, $\chi$, $\mu$. Replacing $x$ by $f^k(x)$ in the equation, we obtain:

$$f^k(x) - i\omega - q(x) - \mu = \psi(g(x) + i\omega) + \chi(f^{k-1}(x)) - \chi(f^k(x))$$

Summing over $i$ from 0 to $n-1$, we obtain:

$$\sum_{i=0}^{n-1} f^k(x) - i\omega - q(x) - \mu = \sum_{i=0}^{n-1} \psi(g(x) + i\omega) + \chi(f^n(x)) - \chi(x).$$

Then (1) and (ii) of 2.3 cannot hold simultaneously for any sequence $\{n_i\}$.

2.4 $\Rightarrow$ 2.3. Suppose 2.4 holds.

Let $\psi \in C^2(\mathbb{R},\mathbb{R})$ have period 1, and $\int_{-1}^{1} \psi = 0$. It suffices to find a sequence $\{a_n\}$ with $a_n \leq a_1 \leq a_2$, $a_n$ a multiple of $a_1$, such that $\{a_n/a_n\}$ is unbounded, and:

$$\sup_{n} \sup_{x \in [0,1]} \left| \sum_{i=0}^{n-1} \psi(x+i\omega) \right| < \infty.$$
\[ \sum_{r=0}^{n-1} |a_r| e^{2\pi i r} \leq \sum_{l,k=\nu_n}^{3} |e| e^{2\pi i l/k} - 1 + \sum_{l,k=\nu_n}^{3} 2^{1/3} |a_r| \]

where \( \sum_{r} \) denotes that the \( r \)th term is omitted if \( r \) is a multiple of \( q_n \).

Then, by 2.4(1):
\[ \left| \sum_{r=0}^{n-1} e^{i(\pi x + \pi r)} \right| \leq \sum_{r=0}^{n-1} |a_r| + \sum_{l,k=\nu_n}^{3} |a_{lq_n}| \]

\[ \sum_{r=-\infty}^{\infty} 2^{1/3} |a_r| \leq \left\{ \sum_{r=-\infty}^{\infty} r^{-1/3} \right\}^{1/2} \times \left\{ \sum_{r=-\infty}^{\infty} |a_r|^2 r^2 \right\}^{1/2} < \infty. \]

Thus it suffices to find a sequence \( \{a_r\} \) such that:
\[ (2.5) \quad \left\{ \frac{m_n}{q_n^2} \right\} \text{ is unbounded, } q_n \leq m_n < q_n^2, \text{ and } m_n \text{ is a multiple of } q_n \text{ and:} \]
\[ \sup_{n} \frac{n}{m_n} \sum_{l=1}^{1} |a_{lq_n}| < \infty. \]

Now \[ \sum_{l=1}^{1} |a_{lq_n}| \leq \left\{ \sum_{l=1}^{1} q_n^2 |a_{lq_n}|^2 \right\}^{1/2} \times \left\{ \sum_{l=1}^{1} (1/t_e^2 q_n^2) \right\}^{1/2} \]

Write \( C = \left\{ \sum_{l=1}^{1} (1/t_e^2) \right\}^{1/2} \) and \( \delta (q_n) = \frac{\sum_{l=1}^{1} (1/t_e^2 q_n^2)}{q_n} \)

Then \( \delta (q_n) \to 0 \text{ as } n \to \infty \) and \[ \sum_{l=1}^{1} |a_{lq_n}| \leq C \delta (q_n)/q_n^2. \]

Now take \( n \) to be the greatest multiple of \( q_n \) which is not greater than \( \min(q_n/\delta (q_n), q_n^2) \), or take \( n = q_n \) if \( q_n \) is too small for such a multiple to exist. Then the sequence \( \{a_r\} \) satisfies (2.5), as required.

\section{Solution of the reduced problem}

We are now reduced to constructing a \( C^\infty f \in \text{Hom}^\infty (K) \) with \( \delta (f) = \delta_q \)

irrational, such that \( f, \delta_q \) satisfy the conditions of 2.4. The construction is similar to Arnold's construction \( [1] \) of a \( C^\infty f \in \text{Hom}^\infty (K) \) with

irrational rotation number and eigenfunction which is not absolutely
continuous.

The construction of \( f \).

Sequences \( \{f_n^2\}, \{p_n\}, \{q_n\}, \{x_n^2\} \) will be constructed such that:

3.1. Each \( f_n \) is defined and analytic in \( \{ z : |\text{Im } z | < \frac{1}{2} \} \), \( f_n(\mathbb{R}) \subseteq \mathbb{R} \),
\[ f_n(z + 1) = f_n(z) + 1 \text{ for all } z, \text{ } f_n(0) \in [0, 1), \text{ } f_n(x) > 1/2 \text{ for all } x \in \mathbb{C} , \]
(so that \( f_n[0, 1] \subseteq \text{Hom}^*(\mathbb{R}), f_n[0, 1] \subseteq \mathbb{R} \) and:
\[ \sup_{|z| < 1} |f_n(z) - f_{n+1}(z)| < 1/2^n. \]

3.2. \( p_n \) and \( q_n \) are coprime, \( 0 < p_n < q_n \), \( \gamma(f_n) = z(p_n/q_n) \), \( q_{n+1} > q_n^5 \)
and \( p_{n+1}/q_{n+1} - p_n/q_n = 1/q_n q_{n+1}. \)

3.3 \( f_n \) is asymptotically forward and has exactly one cycle i.e. exactly one
finite minimal \( f_n \)-invariant set (see 1.2).

3.4 - 3.6 hold for any sequence \( \{m_n\} \) of positive integers such that \( m_n \) is a multiple of \( q_n \) with \( q_n \leq m_n \leq q_n^2; \)

\[ \left|\frac{2 \text{Im} f_n^p/q_n}{1 - e^{2 \pi i r p/q_n}} - \frac{2 \text{Im} f_n^{p+1}/q_{n+1}}{1 - e^{2 \pi i r (p+1)/q_{n+1}}}\right| < 1/2^n, \]

for \( r \leq q_n^5, r \) not a multiple of \( q_n, s \leq n. \)

3.5. \( \sup_{p \in q_n} \left| \frac{1}{(1/a_n)} \sum_{i=0}^{m_i-1} (f_n^i(x) - 1p_i/q_n) \right| = \left( (1/a_n) \sum_{i=0}^{m_i} (f_n^i(x) - 1p_i/q_n) \right) \]
\[ \leq (1/2^{n+1}) c_n \text{ for } r = n. \]

3.6 \( \{x_n^2\} \) is a sequence in \( \mathbb{R} \) and:

\[ \left| (1/a_{n+1}) \sum_{i=0}^{m_{n+1}-1} (f_n^{i+1}(x_n) - 1p_{n+1}/q_{n+1}) \right| \leq \left( (1/a_{n+1}) \sum_{i=0}^{m_n} (f_n^i(x_n) - 1p_i/q_n) \right) + (1/a_n^2) \]

Then let \( f = \lim_n f_n, \text{ } f' = \lim_n p_n/q_n. \)

3.2 implies \( |p_n/q_n - p_{n+1}/q_{n+1}| < 1/2^n q_n^2 \) for sufficiently large \( n. \)
hence $\alpha$ is irrational ([1] §1).

Taking limits in 3.4 - 3.6 implies $f, \alpha$ satisfy 2.4, with $B = 1/6$ in 2.4(11). For 3.5 implies that:

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \left( f_\infty^i(x) - \frac{i}{q} \right) - \left( \frac{1}{n} \sum_{i=0}^{n-1} (f_\infty^i(x) - 1) \right) \right| < \frac{1}{16qr}$$

Now use this in 3.6 with $r = n$ and $r = n+1$, to get 2.4(11) with $B = 1/3$.

Let $p_1, q_1$ be arbitrary coprime integers, $0 < p_1 < q_1$, and take any $f_1$ satisfying 3.1 and 3.3 with $\tilde{f}(f_1) = p_1/q_1 + \mathbb{Z}$. (Use [1] lemma 7 to get a unique cycle for $f_1$.)

Suppose $f_n, p_n, q_n$ have been chosen and define $x_n, f_{n+1}, p_{n+1}, q_{n+1}$ as follows:

**Choice of $x_n$.** There are precisely $q_n$ points in any half-open interval of length one, which correspond to the points of the unique cycle of $f_n | \mathbb{R} \in \text{Hom}^+(K)$. Let $y, z \in \mathbb{R}$ correspond to points in the cycle with $y < z$, and such that if $y < w < z$, then $w$ does not correspond to a point in the cycle. Then for each $1, f_n^1(y)$ and $f_n^1(z)$ have the same property.

Choose $x_n$ with $y < x_n < z$ and such that:

$$0 < f_n^1(x_n) - f_n^1(y) < \frac{1}{q_n}(f_n^1(z) - f_n^1(y)), \ 0 < 1 - q_n.$$

Then if $m_n$ is any multiple of $q_n$ with $q_n \leq m_n < q_n^2$:

$$\frac{1}{q_n} \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y)) \geq \frac{1}{q_n} \sum_{i=0}^{m_n-1} (f_n^i(z) - f_n^i(y)) \geq \frac{1}{q_n} \sum_{i=0}^{m_n-1} (f_n^i(z) - f_n^i(y))$$

**Lemma.** $(1/q_n) \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y)) \rightarrow (1/q_n) \sum_{i=0}^{m_n-1} (f_n^i(z) - f_n^i(y))$ as $m_n \rightarrow \infty$.

**Proof.** Clearly, it suffices to show:

$$\frac{1}{q_n} \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y)) \rightarrow (1/q_n) \sum_{i=0}^{m_n-1} (f_n^i(z) - f_n^i(y))$$

But $(1/q_n) \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y)) = (1/q_n) \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y)) = \frac{1}{q_n} \sum_{i=0}^{m_n-1} (f_n^i(x_n) - f_n^i(y))$ as $r \rightarrow \infty$.
So it suffices to show that for each \( i, C < i \leq q_n - 1 \):
\[
\frac{r}{r} \sum_{s=0}^{r-1} (f^{-s}_n(x_n) - sp_n - i(p_n/q_n)) \rightarrow f^{-i}_n(z) - ip_n/q_n \quad \text{as} \quad r \rightarrow \infty.
\]

For this it suffices to show:
\[
f^{i+sq}_n(x_n) - sp_n - ip_n/q_n \rightarrow f^{-i}_n(z) - ip_n/q_n \quad \text{as} \quad s \rightarrow \infty.
\]

But this follows from there being no elements of the cycle of \( f_n \) between \( x_n \) and \( z \) \( (\text{[1] 5l}) \). Q.E.D.

Now choose \( t_n \geq q_n \) such that:
\[
\left| \frac{1}{r} \sum_{t=0}^{r-1} (f^{-t}_n(x_n) - ip_n/q_n) - \frac{1}{r} \sum_{i=0}^{q_n-1} (f^{-i}_n(z) - ip_n/q_n) \right| < \frac{1}{r} \epsilon_n
\]
for all \( t \geq t_n \). Then if \( t \geq t_n \):
\[
(3.8) \quad (1/t) \sum_{s=0}^{t-1} (f^{-s}_n(x_n) - ip_n/q_n) > (1/t) \sum_{s=0}^{t-1} (f^{-s}_n(z) - ip_n/q_n) - 1/r \epsilon_n
\]
\[
= (1/t) \sum_{s=0}^{t-1} (f^{-s}_n(z) - ip_n/q_n) - 1/r \epsilon_n
\]
\[
= (1/t) \sum_{s=0}^{t-1} (f^{-s}_n(z) - ip_n/q_n) + 1/r \epsilon_n
\]
\[
= \frac{1}{t} \epsilon_n \quad \text{by (3.7)}.
\]

where \( n \) is any multiple of \( q_n \) with \( n \leq n \leq q_n^2 \).

Choice of \( p_{n+1}, q_{n+1} \): Choose \( \frac{1}{2^n} > \delta_n > 0 \) such that if \( 0 < \lambda < \delta_n \),
\[
f_{n+1}(z) = f_n(z) + \lambda (\text{lim} x_n < 1), \text{and } f_{n+1} \text{ is semistable forward with}
\]
rotation number \( p_{n+1}/q_{n+1} \), then \( f_{n+1}, p_{n+1}, q_{n+1} \text{ satisfy conditions 3.4, 3.5. Choose } a, b \in \mathbb{Z} \text{ such that: } a \lambda_n - b = 1.
\]

Take \( q_{n+1} = b + uq_n, \ p_{n+1} = a + up_n, \text{ for u large enough to ensure }
\]
\( q_{n+1} \geq t_n, \text{ and such that } \frac{u}{2^n} = \frac{1}{q_{n+1}}. \text{ Then } p_{n+1}, q_{n+1} \text{ satisfy 3.2 and 3.4.}
\]

Choice of \( f_{n+1} \): Suppose \( f_{n+1}(z) = p_{n+1}/q_{n+1} \), where \( f_{n+1} \) is semistable forward. Such a \( \lambda_n \) exists and is unique \( (\text{[1] 5l}) \).

Choose \( f_{n+1}(z) = f_n(z) + \lambda_n + \xi_n(z) \) such that \( \xi_n(z) = p_{n+1}/q_{n+1} \),
\( \xi_n(z) \geq 0 \text{ for all x } \in \mathbb{Z} \). \( f_{n+1} \) has a unique cycle, and \( \xi_n \) is small enough.
to ensure 3.1 - 3.5 are satisfied ([1]§1).

**Verification that 3.6 is satisfied.**

\[
\frac{m_{n+1}^{-1}}{m_{n+1}} \sum_{i=0}^{q_{n+1}^{-1}} \left( f_{n+1}^i(x_n) - \frac{1}{p_{n+1}} q_n \right) \geq \frac{1}{q_{n+1}} \sum_{i=0}^{q_{n+1}^{-1}} \left( f_{n+1}^i(x_n) - \frac{1}{p_{n+1}} q_n \right)
\]

(since \( f_{n+1} \) is semistable forward)

\[
\frac{m_{n+1}}{m_n} \sum_{i=0}^{q_{n+1}^{-1}} \left( f_{n}^i(x_n) - \frac{1}{p_n} q_n \right) \geq \frac{3}{4 q_n} - \frac{1}{2 q_n}, \text{ by 3.8, 3.2 and because } q_n \geq t_n, \text{ where } m_n \text{ and } m_{n+1} \text{ are multiples of } q_n, q_{n+1} \text{ respectively.}
\]

The construction is completed.