GROUP RINGS WITH NON-ZERO SOCLE

James Stephen Richardson

A thesis submitted for the degree of Doctor of Philosophy

University of Warwick

Department of Mathematics

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CONTENTS

ACKNOWLEDGEMENTS

I should like to express my deep appreciation and gratitude to my supervisor Dr Brian Hartley for his unfailing help and encouragement: he has been a constant, cheerful source of inspiration, advice and hospitality to me through three years.

I should also like to thank Professor J.A. Green and Professor D.S. Passman for their useful suggestions, and my friend Kenny Brown for the many stimulating conversations which have so increased my appreciation of mathematics and of the real world.

I am greatly indebted to the Association of Commonwealth Universities and the British Council for the scholarship which has supported me at Warwick, and to the Universities of Virginia, Warwick and Wisconsin for additional financial assistance.

SUMMARY

The aim of this thesis is to investigate the circumstances under which group rings over fields have non-zero socle, i.e. contain minimal one-sided ideals.

After an introductory chapter, we consider the special case of a periodic abelian group and a non-modular field (that is, a field of characteristic prime to the orders of the elements of the group). This special case, and the background material contained in Chapter III, serve as preparation for our principal results, which concern locally finite groups.

We establish necessary and sufficient conditions on an arbitrary field K and a locally finite group G for the group ring KG to contain minimal one-sided ideals: the most important condition is that G should be a Černikov group. We then examine the structure of KG when these conditions are satisfied. We show that KG has a finite series of ideals each factor of which is a direct sum of quasi-Frobenius rings, and characterize the socle of KG. We also classify indecomposable KG-modules, and determine (for countable but not necessarily locally finite groups G) necessary and sufficient conditions for all indecomposable KG-modules to be irreducible.

In the final chapter we consider non-locally-finite groups, conjecturing that group rings of such groups never contain minimal one-sided ideals. We establish the truth of this conjecture for several classes of groups, and also consider semiartinian group rings.

NOTATION

(m,n)	highest common factor of integers m and n
$[m,n]$	lowest common multiple of m and n
$o(m,n)$	order of m modulo n (if $(m,n)=1$)
$\delta_{x,y}$	1 if $x=y$; 0 otherwise
K	a field
char K	its characteristic
K^*	its multiplicative group
$\bar{K}$	an algebraic closure of K
$\mathbb{Q}$	the rational field
$\mathbb{F}_{p^d}$	a finite field of order p^d
G	a group
$\pi(G)$	the set of primes p such that G has elements of order p
$\Delta(G)$	the FC-centre of G
$\Omega(G)$	the subgroup of G generated by all elements of prime order
$O_p(G)$	the largest normal p-subgroup of G
C_{p^n}	a cyclic group of order p^n
C_{p^∞}	a Prüfer p-group
KG, $K[G]$	a group ring
supp α	the support of $\alpha \in KG$ (see p. 2)
$J(KG)$	the Jacobson radical of KG

$N(KG)$	the nilpotent radical of KG
$K(G)$	a certain subfield of K (see p. 21)
$X \subseteq KG$	X is a subset of KG
$\ell_{KG}(X) = \{\alpha \in KG : \alpha X = 0\}$	
$r_{KG}(X) = \{\alpha \in KG : X\alpha = 0\}$	
$\mathcal{C}(I)$	the controller of an ideal I of KG (see p. 95)
V	a right KG-module
$C_G(V) = \{g \in G : vg = v \text{ for all } v \in V\}$	
$\mathrm{Ann}_{KG}(V) = \{\alpha \in KG : V\alpha = 0\}$	
$\mathrm{End}_{KG}(V)$	the ring of KG-endomorphisms of V
$\mathrm{So}(V)$	the socle of V
$\mathrm{So}_\alpha(V)$	the α-th term of the ascending Loewy series of V (see pp. 4, 119)
$H \leqslant G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
$V_H = V\|_H$	the restriction of V to KH
W	a right KH-module
$W^G = W\|^G = W \otimes_{KH} KG$, the induced module	
Min	the minimum condition on subgroups
$\underline{\underline{F}}, \underline{\underline{A}}, \ldots$	group classes (see p. 112)
$\llcorner, \flat, \ldots$	group-theoretical operations (see p. 112)

Chapter I

INTRODUCTION

1. Preamble

Let K be a field and G a group. Our aim is to investigate consequences of the supposition that the group ring KG contains a minimal one-sided ideal.

Our central results, which concern the case of a locally finite group G, occur in Chapter IV. In preparation for these we examine the special case of a periodic abelian group G and a non-modular field K (Chapter II), and set down some necessary background results of a more general nature (Chapter III). In Chapter V we consider non-locally-finite groups G. The contents of the various chapters will be described in more detail in the first sections thereof.

In Section 2 we investigate the behaviour of the socle of a group ring when either the group or the field is extended, while the remainder of this section is concerned with establishing some notation and definitions (see also the list of notation commencing on page (vii)).

Let G be a group. By $\pi(G)$ we denote the set of primes p such that G has elements of order p. If $\underline{\underline{X}}$ is a property of groups, we say that G is <u>almost</u> an $\underline{\underline{X}}$-group if G has a normal $\underline{\underline{X}}$-subgroup of finite index.

Let K be a field. We denote the group ring of G over K by KG, or sometimes K[G]. If

$$\alpha = \sum_{g \in G} \alpha_g g \in KG \qquad (\alpha_g \in K)$$

then the <u>support</u> of α is

$$\operatorname{supp} \alpha = \{g \in G : \alpha_g \neq 0\} ,$$

a finite subset of G.

Let V be a (right) KG-module; we always assume that V is unitary. We denote by $\operatorname{Ann}_{KG}(V)$ the annihilator of V in KG (an ideal of KG), and by $\operatorname{End}_{KG}(V)$ the ring of KG-endomorphisms of V. The <u>composition length</u> of V is the length of a composition series for V, provided a finite such series exists.

The <u>augmentation ideal</u> of KG is the kernel of the map

$$KG \longrightarrow K, \qquad \sum_{g \in G} \alpha_g g \mapsto \sum_{g \in G} \alpha_g \qquad (\alpha_g \in K)$$

(which is induced by the group homomorphism $G \to 1$). Augmentation ideals of group rings KG, KH, etc. will be denoted $\underline{\underline{g}}$, $\underline{\underline{h}}$, etc. If H is a normal subgroup of G then $\underline{\underline{h}}G = \underline{\underline{h}}.KG = KG.\underline{\underline{h}}$ is a two-sided ideal of KG, being the kernel of the map $KG \to K[G/H]$ induced by the canonical group homomorphism $G \to G/H$. We shall require the following well known result on the augmentation ideal of a group ring:

<u>Lemma 1.1</u> If K is a field and G a group with generating set $\{g_i\}$, then

$$\underline{g} = \sum_i (g_i-1)KG = \sum_i KG(g_i-1) .$$

<u>Proof</u> (see [18; 3.1.1]) We prove the first equality. Certainly, each $g_i-1 \in \underline{g}$. Conversely, suppose

$$\alpha = \sum_{g\in G} \alpha_g g \in \underline{g} \qquad (\alpha_g \in K) ,$$

so that $\sum_{g\in G} \alpha_g = 0$. Then

$$\alpha = \alpha - 0 = \sum_{g\in G} \alpha_g(g-1) ,$$

so if $A = \sum(g_i-1)KG$, it is enough to prove that $g-1 \in A$ for all $g \in G$. Let

$$H = \{g \in G : g-1 \in A\} .$$

If $h \in H$ then

$$g_i h - 1 = (h-1) + (g_i-1)h \in A$$

and $$g_i^{-1}h - 1 = (h-1) - (g_i-1)g_i^{-1}h \in A ,$$

so $g_i h, g_i^{-1}h \in H$. Thus for all i we have $g_i H = g_i^{-1}H = H$, whence $H = \langle g_i \rangle H = GH$. But $1 \in H$ so $H = G$ as required.

2. The socle: subgroups and field extensions

Let R be a ring and V a left or right R-module. The <u>socle</u> $So(V)$ of V is the sum of the minimal submodules of V. The <u>ascending Loewy series</u> of V is defined inductively by

$$So_0(V) = 0 ;$$

$$So_{n+1}(V)/So_n(V) = So(V/So_n(V)) \qquad (n = 0,1,2,\ldots) .$$

For the ring R itself we shall usually denote the right socle $So(R_R)$ by $So(R)$; the left socle will always be denoted $So({}_RR)$. A submodule W of V is <u>essential in V</u> if every non-zero submodule of V has non-zero intersection with W; we shall write W ess V when this occurs. The following result is well known:

<u>Lemma 2.1</u> The socle of V is the intersection of the essential submodules of V.

<u>Proof</u> If N is a minimal and W an essential submodule of V, then $0 \neq N \cap W \leq N$, so $N \leq W$. Thus

$$So(V) \leq \bigcap\{W : W \text{ ess } V\} = E ,$$

say.

Conversely, we show that every submodule M of E is complemented in E, so that E is completely reducible and therefore contained in $So(V)$. By Zorn's lemma there exists a submodule T of V maximal subject to $M \cap T = 0$. We claim that $M \oplus T$ is essential in V. For if $I \leq V$ with $(M \oplus T) \cap I = 0$,

then $N \oplus T \oplus I$ is a direct sum, so $N \cap (T + I) = 0$. Thus $I \leq T$ by choice of T, whence $I = 0$. Hence $E \leq N \oplus T$, so $E = N \oplus (E \cap T)$ as required.

It will be useful to know when the augmentation ideal of a group ring is essential.

<u>Lemma 2.2</u> Let K be a field and G a group. Then $\underline{g}$ is not essential in KG_{KG} if and only if G is of finite order not divisible by char K.

<u>Proof</u> If char $K \nmid |G| < \infty$ then KG_{KG} is completely reducible by Maschke's theorem, so no proper right ideal is essential. Conversely, suppose N is a non-zero right ideal with $N \cap \underline{g} = 0$. Then $N + \underline{g} = KG$ as $\underline{g}$ is maximal, so $N \cong KG/\underline{g}$ is the trivial KG-module. Let $0 \neq \alpha \in N$, and write

$$\alpha = \sum_{x \in G} \alpha_x x \qquad (\alpha_x \in K) .$$

If $g \in G$ then

$$\sum_{x \in G} \alpha_x x = \alpha = \alpha g^{-1} = \sum_{y \in G} \alpha_y y g^{-1} = \sum_{x \in G} \alpha_{xg} x ,$$

whence $\alpha_g = \alpha_1 \neq 0$. Thus $G = \operatorname{supp} \alpha$ is finite. Moreover, under the canonical map $KG \to K$,

$$\alpha = \alpha_1 . \sum_{x \in G} x$$

maps to $\alpha_1 |G|$. Since $\alpha \notin \underline{g}$, it follows that $|G| \neq 0$ in K.

Let KG be a group ring, H a subgroup of G, and F an extension field of K. We shall require a number of results relating the socles of KG, KH and FG.

<u>Lemma 2.3</u> Let K be a field, H a subgroup of a group G, V a right KH-module, and $V^G = V \otimes_{KH} KG$ the induced module.

(a) If V_λ are submodules of V, then

$$(\bigcap V_\lambda)^G = \bigcap V_\lambda^G .$$

(b) If W is a submodule of V and W^G ess V^G, then W ess V.

(c) If $H \trianglelefteq G$ and W ess V then W^G ess V^G.

(d) If $H \trianglelefteq G$, then $So(V^G) \leq So(V)^G$; in particular,

$$So(KG) \leq So(KH)KG .$$

<u>Proof</u> Let T be a right transversal to H in G.

(a) Since ${}_{KH}KG = \bigoplus_{x \in T} KHx$, we have $V^G = \bigoplus_{x \in T} V \otimes x$. Thus

$$\bigcap V_\lambda^G = \bigcap_\lambda (\bigoplus_{x \in T} V_\lambda \otimes x) = \bigoplus_{x \in T} (\bigcap_\lambda V_\lambda) \otimes x = (\bigcap_\lambda V_\lambda)^G .$$

(b) This follows immediately from (a).

(c) (see also [3; 2.5]) As H is normal, $V \otimes x$ is a KH-module for each $x \in T$, and $W \otimes x$ ess $V \otimes x$. Hence

$$W^G = \bigoplus_{x \in T} W \otimes x \quad \text{ess} \quad \bigoplus_{x \in T} V \otimes x = V^G$$

(as KH-submodule so a fortiori as KG-submodule).

(d) By Lemma 2.1,

$$So(V) = \bigcap \{ W : W \text{ ess } V \} ,$$

so by (a) and (c)

$$So(V)^G = \bigcap \{ W^G : W \text{ ess } V \}$$

$$\geq \bigcap \{ U : U \text{ ess } V^G \} = So(V^G) .$$

Putting $V = KH_{KH}$ we obtain the particular case cited.

The hypothesis in (c) and (d) that H is normal may be weakened: for example, H ascendant or locally subnormal in G is sufficient. The following easy but extremely useful lemma, due to Hannah and O'Meara [8], gives a variation of (c) with no such hypothesis on H.

<u>Lemma 2.4</u> Let K be a field and H an infinite subgroup of a locally finite group G. Then the augmentation ideal $\underline{\underline{h}}G$ is essential in KG_{KG}.

<u>Proof</u> Suppose there exists non-zero $\alpha \in KG$ with $\alpha KG \cap \underline{\underline{h}}G = 0$. Since $L = \langle \operatorname{supp} \alpha \rangle$ is finite but H is not, there exists a finite subgroup F of H with

$$|F| > |L| / (\dim_K \alpha KL) .$$

Let $D = \langle F, L \rangle$. Then $\alpha KD \cap \underline{\underline{f}}D = 0$, so

$$\begin{aligned} |D| &\geqslant \dim_K (\alpha KD \oplus \underline{\underline{f}}D) \\ &= \dim_K \alpha KD + \dim_K \underline{\underline{f}}D \\ &= |D|(\dim_K \alpha KL) / |L| + (|F| - 1)|D| / |F| \\ &> |D| , \end{aligned}$$

a contradiction.

<u>Lemma 2.5</u> Let K be a field and H a subgroup of a group G.

(a) If $So(KG) \cap KH \neq 0$ then $So(KH) \neq 0$.

(b) If $|G:H| < \infty$, then $So(KG) = 0$ if and only if $So(KH) = 0$.

<u>Proof</u> (a) Let $0 \neq \alpha \in So(KG) \cap KH$. Then αKG is cyclic and completely reducible, so has the minimum condition on KG-submodules. Now $\alpha KG \cong \alpha KH|^G$, so αKH has the minimum condition on KH-submodules, and in particular contains a minimal submodule. Hence $So(KH) \neq 0$.

(b) Since $\bigcap_{g \in G} H^g$ is normal and of finite index in both H and G, we may assume $H \trianglelefteq G$. Thus $So(KH) = 0$ implies $So(KG) = 0$ by Lemma 2.3(d). Suppose $So(KH) \neq 0$, and let I be a minimal right ideal of KH. Then the restriction $IG|_H \cong I^G|_H$ of IG to KH is a direct sum of $|G:H|$ irreducible KH-submodules, so has minimum condition. A fortiori IG has minimum condition on KG-submodules, so $0 \neq So(IG) \leq So(KG)$.

We may obtain more precise information on the behaviour of the socle under certain group and field extensions using the following results on 'relative projectivity'. Recall that an algebraic element of an extension of a field K is called <u>separable</u> if its minimal polynomial over K has no repeated roots; an algebraic field extension is separable if all its elements are separable.

<u>Lemma 2.6</u> Suppose either

(a) $A = KG$ and $B = KH$, where K is a field and H is a normal subgroup of a group G of finite index not divisible by char K; or

(b) B is an algebra over a field K and $A = B \otimes_K F$ where F is a finite separable extension of K.

If V is an A-module and W an A-submodule which as B-submodule is a direct summand of V, then W is already a direct summand as A-submodule. In particular, if V is completely reducible as B-module it is completely reducible as A-module.

<u>Proof</u> See [15; 15.2, 15.4] or [18; 7.2.2, 7.2.3]. Part (a) is Higman's version of Maschke's theorem.

<u>Lemma 2.7</u> Let K be a field and H a normal subgroup of a group G such that char $K \nmid |G:H| < \infty$. Then

(a) $So(KG) = So(KH)KG$;

(b) $So(KH) = So(KG) \cap KH$.

<u>Proof</u> (a) If I is a minimal right ideal of KH then $IG|_H$ is completely reducible; hence IG is completely reducible by Lemma 2.6(a). Thus $So(KH)KG \leq So(KG)$, and (a) follows by Lemma 2.3(d).

(b) This follows from (a) since if H is any subgroup (not necessarily normal) of G and S is a right ideal of KH, then $SG \cap KH = S$. For let T be a right transversal to H in G, with $1 \in T$; then $KG = \bigoplus_{x \in T} KHx$, so

$$\begin{aligned} SG \cap KH &= KH \cap S.\bigoplus KHx \\ &= KH.1 \cap \bigoplus Sx \\ &= S . \end{aligned}$$

<u>Lemma 2.8</u> Let F be an extension of a field K, and G a group.

(a) $So(FG) \leq So(KG)F$.

(b) If F is a finite separable extension of K, then

$$So(FG) = So(KG)F$$

and

$$So(KG) = So(FG) \cap KG .$$

<u>Proof</u> Let $\{\omega_i\}$ be a basis of F over K.

(a) A proof parallel to that of Lemma 2.3(d) may be applied, using the basis $\{\omega_i\}$ instead of a transversal, and noting that

$$FG_{KG} = \bigoplus KG\omega_i .$$

(b) Since

$$So(KG)F|_{KG} = \bigoplus So(KG)\omega_i$$

is a direct sum of $|F:K|$ copies of $So(KG)$, it is completely reducible. By Lemma 2.6(b) it follows that $So(KG)F$ is also completely reducible, so is contained in $So(FG)$. Hence by (a), $So(FG) = So(KG)F$. An argument similar to that of Lemma 2.7(b) now shows that

$$So(FG) \cap KG = So(KG)F \cap KG = So(KG) .$$

Chapter II

PERIODIC ABELIAN GROUPS

3. Preliminaries

In this chapter we investigate consequences of the supposition that the group ring KG has non-zero socle in the case when G is a periodic abelian group and K is a non-modular field for G (i.e. char $K \notin \pi(G)$). We establish two principal results, which will both be of use in the investigation in Chapter III of group rings of arbitrary locally finite groups over arbitrary fields. Firstly, we determine necessary and sufficient conditions for the socle of KG to be non-zero (Theorem 5.3). Secondly, assuming the socle non-zero we describe the ascending Loewy series in terms of augmentation ideals of certain subgroups of G (Corollary 6.3), and show in particular that the series reaches KG after a finite number of steps (Corollary 6.4).

The necessary and sufficient conditions we shall obtain for the socle of KG to be non-zero are the following:

S1: G satisfies Min, the minimum condition on subgroups;

S2: G is almost locally cyclic; and

S3: $|k(G) \cap K : k| < \infty$, where k is the prime field of K, and $k(G)$ is a certain algebraic extension of k, to be defined in Section 4.

The next two results provide information on the structure of abelian groups satisfying conditions S1 and S2. If G is an abelian group we denote by $\Omega(G)$ the subgroup of all elements of finite square-free order in G. A <u>Prüfer</u> (or <u>quasicyclic</u>) <u>group</u> is isomorphic to the multiplicative group of all p^n-th complex roots of unity, where $n = 0,1,2,\ldots$, for some fixed prime p; all proper subgroups of such a group (denoted C_{p^∞}) are finite.

<u>Theorem 3.1</u> If G is an abelian group, the following are equivalent:

(a) G satisfies Min;

(b) G is periodic and $\Omega(G)$ is finite;

(c) G has a decomposition

$$G = F \times P_1 \times \ldots \times P_m \quad (0 \leq m < \infty) ,$$

where F is finite and each P_i is a Prüfer group.

<u>Proof</u> See [6 ; 25.1, 3.1].

<u>Corollary 3.2</u> If G is an abelian group with Min, the following are equivalent:

(a) G has a finite subgroup F such that G/F is locally cyclic;

(b) G is almost locally cyclic;

(c) G has a decomposition

$$G = F \times P_1 \times \ldots \times P_m \quad (0 \leq m < \infty) ,$$

where F is finite and the P_i are Prüfer p_i-groups for distinct primes p_i.

<u>Proof</u> (a) $\Rightarrow$ (b) Let $n = |F| < \infty$. Since G is abelian, $G^n = \{g^n : g \in G\}$ is a quotient of G and indeed of G/F, as $F^n = 1$. Thus G^n like G/F is locally cyclic. But G/G^n has finite exponent and satisfies Min, so is finite by Theorem 3.1 (since a Prüfer group has infinite exponent). Hence G is almost locally cyclic.

(b) $\Rightarrow$ (c) By Theorem 3.1, since G satisfies Min, there is a decomposition

$$G = F \times P_1 \times \ldots \times P_m \quad (0 \leq m < \infty)$$

with F finite and each P_i a Prüfer group. Now $P_1 \times \ldots \times P_m$ like G is almost locally cyclic, but has no proper subgroup of finite index, so is itself locally cyclic. Thus no two P_i can be p-groups for the same prime p.

(c) $\Rightarrow$ (a) $G/F \cong P_1 \times \ldots \times P_m$ is locally cyclic.

We remark that (a) and (b) remain equivalent if G is any periodic abelian group.

To foreshadow the significance of condition 33, we observe that it always holds if G is finite or K is a finite extension of k, but if K is algebraically closed then 33 holds only if G is finite. When G is a locally cyclic group

with Lin, it is convenient to consider a condition equivalent both to B3 and to the existence of minimal ideals in KG: namely, the existence of <u>K-inductive subgroups</u> in G. We call a finite subgroup H of G K-inductive if every irreducible KH-module faithful for H remains irreducible when induced up to G. For our study of K-inductive subgroups in Section 4, we shall require a field-theoretic lemma (3.7). The next four results, and the associated definitions, are standard.

<u>Lemma 3.3</u> If E is a finite extension of a field F, the following are equivalent:

(a) E is a splitting field of some polynomial over F;

(b) every irreducible polynomial over F with a root in E splits as a product of linear factors over E.

<u>Proof</u> See [12; Theorem 10, p. 42].

When the equivalent conditions (a) and (b) hold, E is called a <u>normal</u> extension of F. Notice that it follows from (a) that if $F \leq K \leq E$ are fields with E normal over F, then E is also normal over K.

<u>Lemma 3.4</u> The separable elements in an algebraic extension form a subfield.

<u>Proof</u> See [12; Theorem 11, p. 46].

An extension E of a field F is <u>simple</u> if $E = F(\theta)$ is generated over F (as a field) by a single element θ.

<u>Lemma 3.5</u> Any finite separable field extension is simple.

<u>Proof</u> See [12; pp. 54, 59].

<u>Lemma 3.6</u> Suppose E_1 and E_2 are extensions of a field F lying in some common extension of F. Then the following are equivalent:

(a) The canonical map

$$E_1 \otimes_F E_2 \to E_1 E_2 \ , \quad \sum \alpha_i \otimes \beta_i \mapsto \sum \alpha_i \beta_i$$

is an isomorphism;

(b) there exists a basis of E_2 over F which is linearly independent over E_1;

(c) any subset of E_1 linearly independent over F is independent over E_2.

<u>Proof</u> (a) $\Rightarrow$ (b) Let $\{\omega_i\}$ be a basis of E_2 over F, so that $E_2 = \bigoplus F\omega_i$. Then

$$E_1 \otimes_F E_2 = \bigoplus (E_1 \otimes \omega_i) \ .$$

Applying the canonical isomorphism, we find that

$$E_1 E_2 = \bigoplus E_1 \omega_i \ ,$$

so $\{\omega_i\}$ is a basis of $E_1 E_2$ over E_1, and in particular linearly independent over E_1. Since any linearly independent set may be extended to a basis, we may prove similarly that (a) implies (c).

(b) $\Rightarrow$ (a) Let $\{\omega_i\}$ be a basis of E_2 over F which is linearly independent over E_1. As above

$$E_1 \otimes_F E_2 = \bigoplus (E_1 \otimes \omega_i) .$$

If $\sum \alpha_i \otimes \omega_i \in E_1 \otimes_F E_2$ $(\alpha_i \in E_1)$ maps to zero in E_1E_2, i.e. $\sum \alpha_i \omega_i = 0$, then each α_i is zero. Thus the canonical map (which is always onto) is an isomorphism. Similarly, (c) implies (a).

When (a)-(c) hold, E_1 and E_2 are said to be <u>linearly disjoint</u> over F.

<u>Lemma 3.7</u> Let D and E be subfields of some field, and suppose that E is a finite normal separable extension of $D \cap E$. Then

(a) D and E are linearly disjoint over $D \cap E$;

(b) if F is a subfield of E containing $D \cap E$ then $FD \cap E = F$.

<u>Proof</u> (a) By Lemma 3.5, E contains an element θ with $E = (D \cap E)(\theta)$. Let f be the minimal polynomial of θ over $D \cap E$. Then f is in fact irreducible over D. For if $f = gh$, where g and h are monic polynomials over D, then the roots of g and h are roots of f, so lie in E by Lemma 3.3(b). The coefficients of g and h are (plus or minus) elementary symmetric functions in the roots, so lie in $D \cap E$. But f is irreducible over $D \cap E$, so over D too.

If n is the degree of f, then $\{1,\theta,\ldots,\theta^{n-1}\}$ is a basis of E over $D\cap E$, consisting of elements which are linearly independent over D. So D and E are linearly disjoint over $D\cap E$.

(b) Let $\{\omega_i\}$ be a basis of D over $D\cap E$, with $\omega_1 = 1$. Then $FD = \sum F\omega_i$. By (a), the ω_i are linearly independent over E. Suppose

$$\beta = \sum \alpha_i\omega_i \in FD\cap E \quad (\alpha_i \in F) .$$

Then $$(\alpha_1 - \beta)\omega_1 + \sum_{i\neq 1}\alpha_i\omega_i = 0 \quad (\alpha_1-\beta, \alpha_i \in E)$$

so $\beta = \alpha_1 \in F$. Thus $FD\cap E = F$.

The next two lemmas will explain the usefulness of the assumption, made throughout this chapter, that K is a non-modular field for G. We say that a K-algebra satisfies a condition $\underline{\underline{X}}$ <u>locally</u> if every finite subset is contained in an $\underline{\underline{X}}$-subalgebra. In particular an algebra is <u>locally Wedderburn</u> if every finite subset lies in a semisimple artinian subalgebra.

<u>Lemma 3.8</u> If G is a locally finite group and K a field with char $K \notin \pi(G)$, then KG is locally Wedderburn.

<u>Proof</u> If S is a finite subset of KG, then

$$H = \langle \operatorname{supp} \alpha : \alpha \in S \rangle$$

is a finite subgroup of G. Then KH contains S and is semisimple artinian by Maschke's theorem.

Recall that an element e of a ring is an <u>idempotent</u> if $e^2 = e \neq 0$. Idempotents e and f are <u>orthogonal</u> if $ef = fe = 0$. An idempotent is <u>primitive</u> if it cannot be expressed as the sum of two orthogonal idempotents.

<u>Lemma 3.9</u> Let A be a locally Wedderburn algebra. Then

(a) every non-zero right ideal of A contains an idempotent;

(b) a right ideal is minimal if and only if it is generated by a primitive idempotent;

(c) $\mathrm{So}(A_A)$ contains and is generated by all primitive idempotents of A;

(d) if A is commutative then

$$\mathrm{So}(A) = \bigoplus \{eA : e \text{ is a primitive idempotent in } A\} .$$

<u>Proof</u> (a) Let I be a right ideal of A containing a non-zero element α, and choose a semisimple artinian subalgebra B containing α. Now (a) certainly holds in B (since every non-zero right ideal is a direct summand so is generated by an idempotent). Hence αB $(\subseteq \alpha A \leq I)$ contains an idempotent.

(b) Let e be a primitive idempotent in A and I a non-zero right ideal contained in eA. By (a), I contains an idempotent f. Then $f \in eA$, say $f = e\alpha$, whence $ef = e^2\alpha = e\alpha = f$. Now $e = fe + (e-fe)$, and we easily have $(fe)^2 = fe$, $(e-fe)^2 = e-fe$, $fe(e-fe) = (e-fe)fe = 0$. As e is primitive, either $fe = 0$ or $e-fe = 0$. If $fe = 0$ then $f = f^2 = fef = 0$, a contradiction.

Hence $e = fe \in I$, so $I = eA$. Thus eA is a minimal right ideal.

On the other hand, if I is a minimal right ideal of A, then by (a) I contains an idempotent e. Since $0 \neq eA \leq I$, we have $I = eA$. Moreover, if

$$e = e_1 + e_2, \quad e_1 e_2 = e_2 e_1 = 0, \quad e_i^2 = e_i \neq 0,$$

then $0 \neq e_1 = ee_1 \in I$, so $e \in I = e_1 A$, and $e_2 = e_2 e \in e_2 e_1 A = 0$, a contradiction. Thus e is primitive.

(c) Since $So(A_A)$ is the sum of the minimal right ideals, (c) follows immediately from (b).

(d) This follows from (c). The sum is direct since primitive idempotents e and f in a commutative ring are either equal or orthogonal: if $ef \neq 0$ then as $e = ef + e(1-f)$ we find that $e = ef$; similarly $f = ef$. Thus if $e_1, e_2, \ldots, e_n$ are distinct primitive idempotents, then

$$e_1 A \cap \sum_{i=2}^{n} e_i A \leq e_1 \cdot \left(\sum_{i=2}^{n} e_i A \right) = 0$$

(since if $\alpha \in e_1 A$ then $\alpha = e_1 \alpha$).

Thus we are led to investigate the primitive idempotents in KG: this is done in Section 5. As well as the question of the existence of primitive idempotents, we consider (for almost locally cyclic groups G with Min) the connection between primitive idempotents and irreducible KG-modules. When S3 holds, there is a one-to-one onto

correspondence between primitive idempotents in KG and isomorphism classes of irreducible KG-modules with finite centralizer (i.e. finite kernel in G); moreover there are only finitely many non-isomorphic such modules having any fixed finite subgroup of G as centralizer (Theorem 5.5). But if S3 fails to hold the situation is quite different: there are no primitive idempotents in KG, but given any finite subgroup C of G such that G/C is locally cyclic, there exist $2^{\aleph_0}$ non-isomorphic irreducible KG-modules with centralizer C (Theorem 5.3).

In Section 6, as mentioned above, we examine the ascending Loewy series of KG when S1, S2 and S3 hold.

4. K-Inductive subgroups

Let G be a periodic abelian group and K a field with char $K \notin \pi(G)$. Let $\bar{K}$ be an algebraic closure of K, and $\bar{K}^*$ its multiplicative group. We denote by $K(G)$ the K-subalgebra of $\bar{K}$ generated by all images of homomorphisms $G \to \bar{K}^*$; as G is periodic, $K(G)$ is in fact a subfield of $\bar{K}$. Since the torsion subgroup of $\bar{K}^*$ is a direct product of Prüfer groups, one for each prime not equal to char K, if G is locally cyclic then $\bar{K}^*$ has exactly one subgroup isomorphic to G; the elements of this subgroup generate $K(G)$ as a K-algebra, for any quotient of G is isomorphic (albeit unnaturally) to a subgroup of G.

<u>Lemma 4.1</u> Let H be a finite cyclic group and K a field with char $K \notin \pi(H)$. Then there exist irreducible KH-modules faithful for H, and all such modules have dimension $|K(H) : K|$ over K.

<u>Proof</u> $K(H)^*$ has a unique subgroup isomorphic to H, so we may choose a monomorphism $\theta : H \to K(H)^*$. Then $K(H)$ becomes a KH-module with H-action given by

$$v.h = vh^{\theta}, \quad v \in K(H), \quad h \in H .$$

If $0 \neq v \in K(H)$ then since H^{θ} generates $K(H)$ as K-algebra, $v.KH = vK(H) = K(H)$. Thus $K(H)$ is an irreducible KH-module; it is faithful for H as θ is one-to-one.

Let V be any irreducible KH-module faithful for H. Then V is isomorphic to KH/M for some maximal ideal M of KH. Now KH/M is a field, containing (since V is faithful) a multiplicative subgroup isomorphic to H which generates it over K. It follows that KH/M is algebraic over K, and thence isomorphic to the field $K(H)$. Thus

$$\dim_K V = \dim_K KH/M = |K(H):K| ,$$

completing the proof.

If K is a field, G a group, and V a KG-module, we write

$$C_G(V) = \{g \in G : vg = v \text{ for all } v \in V\} .$$

<u>Lemma 4.2</u> Let G be a periodic abelian group, H a subgroup of G containing $\Omega(G)$, and K a field with char $K \notin \pi(G)$. Let V be an irreducible KH-module faithful for H, and W a non-zero submodule of the induced module $V^G = V \otimes_{KH} KG$. Then W is faithful for G.

<u>Proof</u> Since G is abelian, the restriction $V^G|_H$ of V^G to H is a direct sum of copies of V. As V is irreducible, W_H is also a direct sum of copies of V. Suppose $1 \neq g \in C_G(W)$. There exists an integer n such that $1 \neq g^n \in \Omega(G) \leq H$. But then $1 \neq g^n \in C_H(W_H) = C_H(V)$, a contradiction as V is faithful for H. Hence W is faithful for G.

Let K be a field and G a locally cyclic group with Min such that char $K \notin \pi(G)$. A finite subgroup H of G will be called <u>K-inductive in G</u> if whenever V is an irreducible KH-module faithful for H, the induced module V^G is an irreducible KG-module.

<u>Lemma 4.3</u> A finite subgroup H of G is K-inductive if and only if the following two conditions are satisfied:

(a) H contains $\Omega(G)$;

(b) whenever L is a finite subgroup of G containing H, we have

$$|K(L) : K(H)| = |L : H| .$$

<u>Proof</u> Suppose H is K-inductive in G. By Lemma 4.1 there exists an irreducible KH-module V faithful for H; then V^G is irreducible.

(a) Suppose $\Omega(G) \nleq H$; then there exists a finite non-trivial subgroup L of G with $HL = H \times L$. Now $V^{H \times L}$ is reducible: indeed $\{\sum_{x \in L} v \otimes x : v \in V\}$ is a proper submodule. A fortiori V^G is reducible, a contradiction. So $\Omega(G) \leq H$.

(b) Let L be a finite subgroup of G containing H. Then V^L like V^G is irreducible; by (a) and Lemma 4.2 V^L is faithful for L. Hence using Lemma 1.1,

$$\begin{aligned} |K(L):K(H)| &= |K(L):K| \,/\, |K(H):K| \\ &= \dim_K V^L / \dim_K V \\ &= |L:H| \,, \end{aligned}$$

since $V^L = V \otimes_{KH} KL$.

Now suppose (a) and (b) hold. We may express G as the union of a chain

$$H = H_0 \leqslant H_1 \leqslant H_2 \leqslant \ldots \leqslant G$$

of finite subgroups. Let V be any irreducible KH-module faithful for H. By (a) and Lemma 4.2, any irreducible submodule of V^{H_i} is faithful for H_i, so has dimension $|K(H_i):K|$ by Lemma 4.1. But by (b) and Lemma 4.1,

$$\begin{aligned} |K(H_i):K| &= |K(H_i):K(H)|\,|K(H):K| \\ &= |H_i:H| \dim_K V \\ &= \dim_K V^{H_i} \quad . \end{aligned}$$

Hence V^{H_i} is itself irreducible. Now V^G may be regarded as the union of the V^{H_i}, so is also irreducible. Thus H is K-inductive in G.

<u>Corollary 4.4</u> A finite subgroup H of G is K-inductive if and only if there exists an irreducible KH-module V faithful for H and such that V^G is irreducible.

<u>Proof</u> If such a V exists then by the first half of the proof of Lemma 4.3 H satisfies (a) and (b); then by the second half H is K-inductive. The converse follows from Lemma 4.1.

<u>Corollary 4.5</u> If the finite subgroup H of G contains a K-inductive subgroup H_1, H itself is K-inductive.

<u>Proof</u> We have $\Omega(G) \leq H_1 \leq H$, and, for any finite L containing H,

$$\begin{aligned} |K(L) : K(H)| &= |K(L) : K(H_1)| \,/\, |K(H) : K(H_1)| \\ &= |L : H_1| \,/\, |H : H_1| \\ &= |L : H| \ . \end{aligned}$$

<u>Proposition 4.6</u> If $H \leq L \leq G$ and L is finite then in any case we have

$$|K(L) : K(H)| \leq |L : H| \ .$$

<u>Proof</u> If $m = |L : H|$ and the subgroup of $K(L)^*$ isomorphic to L is generated by ξ, then $\xi^m \in K(H)$, so the polynomial $f(X) = X^m - \xi^m$ has degree m over $K(H)$ and ξ as a root. Hence $|K(L) : K(H)| = |K(\xi) : K(H)| \leq m$.

<u>Lemma 4.7</u> Let F and K be subfields of some field. Then

$$|KF : F| \leq |K : K \cap F| \ .$$

(Here the ring KF may or may not be a field.)

<u>Proof</u> Any basis of K over $K \cap F$ also spans KF over F.

<u>Theorem 4.8</u> Let G be a locally cyclic group with Min, and K a field with char $K \notin \pi(G)$. If there exists any K-inductive subgroup in G, there exists a unique minimal K-inductive subgroup in G.

<u>Proof</u> Since K-inductive subgroups are finite, it is sufficient to show that if H_1 and H_2 are K-inductive in G, then so is $H_1 \cap H_2$. But let H_1 be K-inductive, and H_2 any subgroup of G. Then

$$\Omega(H_2) \leqslant \Omega(G) \cap H_2 \leqslant H_1 \cap H_2 .$$

Moreover, if L is a finite subgroup of H_2 containing $H_1 \cap H_2$, then $H_1 \cap H_2 = H_1 \cap L$, so

$$\begin{aligned} |K(L) : K(H_1 \cap H_2)| &= |K(L) : K(H_1 \cap L)| \\ &\geqslant |K(L) : K(H_1) \cap K(L)| \\ &\geqslant |K(L)K(H_1) : K(H_1)| \end{aligned}$$

by Lemma 4.7. Clearly $K(L)K(H_1) \leqslant K(LH_1)$, and in fact we have equality, since if $\theta : LH_1 \to \bar{K}^*$ is a homomorphism, then $(LH_1)^\theta \subseteq L^\theta H_1^\theta \subseteq K(L)K(H_1)$. So as H_1 is K-inductive in G,

$$\begin{aligned} |K(L) : K(H_1 \cap H_2)| &\geqslant |K(LH_1) : K(H_1)| \\ &= |LH_1 : H_1| \\ &= |L : H_1 \cap L| \\ &= |L : H_1 \cap H_2| . \end{aligned}$$

But $|K(L) : K(H_1 \cap H_2)| \leqslant |L : H_1 \cap H_2|$ by Proposition 4.6, so by Lemma 4.3 $H_1 \cap H_2$ is K-inductive in H_2.

Thus if V is an irreducible $K[H_1 \cap H_2]$-module faithful for $H_1 \cap H_2$, then V^{H_2} is irreducible, and faithful for H_2 by Lemma 4.2. If now H_2 is also K-inductive in G, then V^G is irreducible; hence $H_1 \cap H_2$ is K-inductive in G by Corollary 4.4. This completes the proof.

We shall now investigate more closely the conditions under which a locally cyclic group with Min contains inductive subgroups for various fields.

<u>Lemma 4.9</u> Let G be a locally cyclic group with Min. Then $\Omega(G)$ is $\mathbb{Q}$-inductive in G.

<u>Proof</u> Suppose L is a finite subgroup of G containing $H = \Omega(G)$, and let ε be a primitive $|L|$-th root of unity. Then

$$|\mathbb{Q}(L) : \mathbb{Q}| = |\mathbb{Q}(\varepsilon) : \mathbb{Q}| = \varphi(|L|) ,$$

where φ is the Euler function. Thus

$$\begin{aligned} |\mathbb{Q}(L) : \mathbb{Q}(H)| &= \varphi(|L|)/\varphi(|H|) \\ &= \varphi(|L:H||H|)/\varphi(|H|) \\ &= |L:H| , \end{aligned}$$

for $\pi(L) = \pi(H)$ and if p is a prime dividing an integer m, then $\varphi(pm) = p\varphi(m)$. Hence $\Omega(G) = H$ is $\mathbb{Q}$-inductive in G by Lemma 4.3.

If m and n are positive integers, their highest common factor is denoted by (m,n). If $(m,n) = 1$, we shall denote by $o(m,n)$ the order of m modulo n, i.e. the smallest positive integer r such that n divides $m^r - 1$. If G is a locally cyclic group with Min, say

$$G \cong C_{p_1^{n_1}} \times \ldots \times C_{p_k^{n_k}}$$

where the p_i are distinct primes and $1 \leq n_i \leq \infty$, then

$N = p_1^{n_1} \ldots p_k^{n_k}$ will be called the supernatural number associated with G. Evidently the concepts of divisibility and highest common factor extend to supernatural numbers.

The following is a slightly strengthened form of [9 ; 2.2]:

<u>Lemma 4.10</u> Let G be a locally cyclic group with Min, and $\mathbb{F}_{p^d}$ a finite field of order p^d, with $p \notin \pi(G)$. Let N be the supernatural number associated with G, and put

$$n = (N, 2^2.3.5.7....) ,$$

$$r = o(p^d, n) ,$$

$$m = (N, p^{dr}-1) .$$

Then the unique subgroup H of order m in G is $\mathbb{F}_{p^d}$-inductive in G.

<u>Proof</u> Since $n \mid p^{dr}-1$, we have $n \mid m$, whence $\Omega(G) \leq H$. Let L be a finite subgroup of G containing H. Then L is cyclic and $\mathbb{F}_{p^d}(L)$ is the smallest extension $\mathbb{F}_{p^{dt}}$ of $\mathbb{F}_{p^d}$ such that L may be embedded in $\mathbb{F}^*_{p^{dt}}$, i.e. such that $\ell = |L|$ divides $|\mathbb{F}^*_{p^{dt}}| = p^{dt}-1$. Hence t is the smallest positive integer such that $\ell \mid p^{dt}-1$, so we have

$$|\mathbb{F}_{p^d}(L) : \mathbb{F}_{p^d}| = t = o(p^d, \ell) .$$

By Lemma 4.3, to show that H is $\mathbb{F}_{p^d}$-inductive in G it is sufficient to prove that $|\mathbb{F}_{p^d}(L) : \mathbb{F}_{p^d}(H)| = |L : H|$, i.e. that if $n \mid \ell \mid m$ then

$$o(p^d,\ell)\,/\,o(p^d,m) \;=\; \ell/m\ .$$

Note that $o(p^d,m)=r$, for since $n\mid m$, $r=o(p^d,n)\mid o(p^d,m)$, while as $m\mid p^{dr}-1$, $o(p^d,m)\mid r$. We shall prove by induction on ℓ/m (more precisely, on the sum of the exponents in the prime power factors of ℓ/m) that if $o(p^d,\ell)=t$ and $p^{dt}-1=k\ell$, then $(k\,,\,N/m)=1$, and $t/r=\ell/m$.

Firstly, let $\ell=m$, so $t=r$. Write $p^{dr}-1=km$. Then $(km\,,\,N)=(p^{dr}-1\,,\,N)=m$, so $(k\,,\,N/m)=1$. Also $t/r=1=\ell/m$.

Now suppose that $m\mid\ell\mid\ell q\mid N$, where q is a prime. Let $t=o(p^d,\ell)$ and $p^{dt}-1=k\ell$. By induction we may assume that $(k\,,\,N/m)=1$ and $t/r=\ell/m$. We then have

$$\begin{aligned} p^{dtq} &= (1+k\ell)^q \\ &= 1+qk\ell+\tfrac{1}{2}q(q-1)(k\ell)^2+\ldots+(k\ell)^q\ . \end{aligned}$$

Let $q_1\mid N$ be prime. If $q_1\neq q$ then as $qq_1\mid\ell$ we have

$$p^{dtq}\equiv 1+qk\ell \pmod{\ell qq_1}\ .$$

If $q_1=q$ we have $q\mid\ell$ so (since $q\mid\binom{q}{s}$ for $s=2,\ldots,q-1$)

$$p^{dtq}\equiv 1+qk\ell+(k\ell)^q \pmod{\ell q^2}\ ,$$

whence $$p^{dtq}\equiv 1+qk\ell \pmod{\ell q^2}$$

provided $q>2$. But if $q=2$ then $2^2\mid\ell q\mid N$ whence $2^2\mid n\mid m\mid\ell$, and again we obtain

$$p^{dtq}\equiv 1+qk\ell \pmod{\ell q^2}\ .$$

In particular we see that $\ell q\mid p^{dtq}-1$, so $t'=o(p^d,\ell q)$ divides tq. Moreover, $\ell\mid\ell q$, so $t=o(p^d,\ell)\mid t'$. If

$\ell q \mid p^{dt}-1 = k\ell$, then $q \mid k$. But $m \mid \ell \mid \ell q \mid N$, so then q divides N/m, a contradiction as $(k, N/m) = 1$. Hence $\ell q \nmid p^{dt}-1$. Thus $t \mid t' \mid tq$, but $t \neq t'$, so $o(p^d, \ell q) = t' = tq$. We have

$$t'/r = tq/r = \ell q/m.$$

Now write $p^{dt'}-1 = k'\ell q$. By the above congruences, if q_1 is any prime divisor of N, we have

$$k'\ell q \equiv k\ell q \pmod{\ell q q_1},$$

whence

$$k' \equiv k \pmod{q_1}.$$

Thus if $q_1 \mid (k', N/m)$ then $q_1 \mid (k, N/m) = 1$, a contradiction. Hence $(k', N/m) = 1$. This completes the induction, and the proof.

The subgroup H we have constructed is in almost all cases minimal inductive, as we now show.

<u>Proposition 4.11</u> With notation as in Lemma 4.10, H is the minimal $\mathbb{F}_p$-inductive subgroup of G unless

(a) $|O_2(G)| = 4$;

(b) $p^d \equiv 3 \pmod 4$; and

(c) $o(p^d, m/4)$ is odd,

in which case the subgroup of index 2 in H is minimal inductive.

<u>Proof</u> We remark first that if (a) holds then $4 \mid n$, so $o(p^d, n) \mid o(p^d, n) = r$; thus 4 divides $(N, p^{dr}-1) = m$, and (c) makes sense.

Suppose that H is not minimal inductive. Then H contains a proper inductive subgroup L. By Corollary 4.5 we may choose L maximal in H, so that $q = |H : L|$ is prime. Let $\ell = |L| = m/q$.

Suppose $n \mid \ell$. Since $\ell \mid m$, we then have

$$r = o(p^d,n) \mid o(p^d,\ell) \mid o(p^d,m) = r$$

(see the proof of Lemma 4.10). Hence using Lemma 4.3(b),

$$\begin{aligned} q &= |H : L| \\ &= |\mathbb{F}_{p^d}(H) : \mathbb{F}_{p^d}(L)| \\ &= o(p^d,m) / o(p^d,\ell) \\ &= 1 , \end{aligned}$$

a contradiction. Thus $n \nmid \ell$. But by Lemma 4.3(a), $\Omega(G) \leq L$, whence $(H, 2.3.5.7....)$ divides ℓ. Hence we see that $2^2 \| n$ (that is, $(2^\infty,n) = 2^2$) but $2 \| \ell$. Since $n \mid m$ and q is prime, it follows that $q = 2$ and $2^2 \| m$. Of course, $p \neq 2$.

If $(x,y) = 1$ then $o(p^d,xy) = [o(p^d,x) , o(p^d,y)]$ (the least common multiple). Write $m = 2\ell = 2^2 z$, so that $2 \nmid z$. From above,

$$2 = q = \frac{o(p^d,m)}{o(p^d,\ell)} = \frac{[o(p^d,2^2) , o(p^d,z)]}{[o(p^d,2) , o(p^d,z)]}$$

whence (as $o(p^d,2) = 1$) we obtain

$$[o(p^d,2^2) , o(p^d,z)] = 2.o(p^d,z) .$$

Since the value of $o(p^d,2^2)$ must be either 1 or 2, we see

that $o(p^d,2^2) = 2$ (whence (b)) and $2 \nmid o(p^d,z)$ (whence (c)).

As $2^2 \mid n$, we have

$$2 = o(p^d,2^2) \mid o(p^d,n) = r ,$$

so $p^{2d}-1$ divides $p^{dr}-1$. But $2 \mid p^d-1$ and $2^2 \mid p^d+1$, so $2^3 \mid p^{2d}-1 \mid p^{dr}-1$. Now $2^2 \| m = (N , p^{dr}-1)$, so $2^2 \| N$, i.e. (a) holds.

Conversely suppose that (a), (b) and (c) hold, and let L be the subgroup of index 2 in H. Since $2^2 \mid n \mid m = |H|$, clearly $\Omega(G) \leq L$. Moreover, writing $|H| = 2|L| = 2^2 z$, so that $2 \nmid z$, we have

$$\begin{aligned} |\mathbb{F}_{p^d}(H) : \mathbb{F}_{p^d}(L)| &= o(p^d,|H|) / o(p^d,|L|) \\ &= \frac{[o(p^d,2^2),o(p^d,z)]}{[o(p^d,2) , o(p^d,z)]} \\ &= o(p^d,2^2) = 2 = |H : L| \end{aligned}$$

(by (c) then (b)). Since H is inductive, it follows by Lemma 4.3 that L is too.

Finally, if L_1 is an inductive subgroup of L, we see as before that $|L : L_1|$ is a power of 2. But $\Omega(G) \leq L_1$ whence 2 divides $|L_1|$, and $2 \| |L|$, so $L_1 = L$. Hence L is minimal inductive.

In passing from prime fields (covered by Lemmas 4.9 and 4.10) to arbitrary fields, we shall apply Lemma 3.7, the relevance of which is explained by the following:

<u>Lemma 4.12</u> If L is a finite cyclic group, k a field with char $k \notin \pi(\mathbf{L})$, and T a field with $k \leq T \leq k(L)$, then k(L) is a finite normal separable extension of T.

<u>Proof</u> As k(L) is the splitting field over T of the polynomial $X^{|L|}-1$, it is a finite normal extension of T. Since

$$(X^{|L|}-1 , \tfrac{d}{dX}(X^{|L|}-1)) = (X^{|L|}-1 , |L|X^{|L|-1}) = 1 ,$$

$X^{|L|}-1$ has no repeated roots. Now k(L) is generated over T by the roots of $X^{|L|}-1$, so by Lemma 3.4 k(L) is separable over T.

<u>Theorem 4.13</u> Let K be any field, k its prime field, and G a locally cyclic group satisfying Min with char $k \notin \pi(G)$. Then G has a K-inductive subgroup if and only if

$$|k(G) \cap K : k| < \infty.$$

(Here $k(G) \cap K$ is a subfield of $\bar{K}$, in which $\bar{k}$ and k(G) are embedded.)

<u>Proof</u> Suppose that H is a K-inductive subgroup of G and L is a finite subgroup of G containing H. Then by Proposition 4.6 we have

$$|k(L) : k(H)| \leq |L : H| = |K(L) : K(H)|$$

(as H is K-inductive). Now $K(L) = k(L)K(H)$, so by Lemma 4.7

$$\begin{aligned} |K(L) : K(H)| &= |k(L)K(H) : K(H)| \\ &\leq |k(L) : k(L) \cap K(H)| \\ &\leq |k(L) : k(H)| \end{aligned}$$

(as $k(H) \leq k(L) \cap K(H)$). We now have

$$|k(L) : k(L) \cap K(H)| = |k(L) : k(H)| ,$$

whence $\quad k(L) \cap K \leq k(L) \cap K(H) = k(H)$.

As G is locally finite it follows that $k(G) \cap K \leq k(H)$. Hence

$$|k(G) \cap K : k| \leq |k(H) : k| \leq |H| < \infty .$$

Conversely, suppose that $|k(G) \cap K : k| < \infty$: say $k(G) \cap K = k(\gamma_1, \ldots, \gamma_s)$ (in view of Lemma 3.5, we could actually assume that $s = 1$). By Lemma 4.9 or 4.10, as k is a prime field, G contains a k-inductive subgroup H_1. Since G is locally finite, there exists a finite subgroup H of G containing H_1 and such that $\gamma_1, \ldots, \gamma_s \in k(H)$. Then

$$k(G) \cap K = k(\gamma_1, \ldots, \gamma_s) \leq k(H) .$$

We shall show that H is K-inductive in G. Note first that $\Omega(G) \leq H_1 \leq H$ by Lemma 4.3(a).

Let L be a finite subgroup of G containing H. Then

$$k(L) \cap K \leq k(G) \cap K \leq k(H) .$$

Hence taking $D = K$, $E = k(L)$, and $F = k(H)$ in Lemma 3.7(b) (and applying Lemma 4.12), we obtain

$$K(H) \cap k(L) = k(H)K \cap k(L) = k(H) .$$

By Lemma 3.7(a), $K(H)$ $(=D)$ and $k(L)$ $(=J)$ are linearly disjoint over their intersection $k(H)$. Hence a basis for $k(L)$ over $k(H)$ also constitutes a basis for $K(L) = K(H)k(L)$ over $K(H)$. Thus

$$|K(L) : K(H)| = |k(L) : k(H)| = |L : H|$$

as $H \geqslant H_1$ is k-inductive by Corollary 4.5. By Lemma 4.3, H is K-inductive in G.

<u>Corollary 4.14</u> Let K be any field, k its prime field, and G a periodic abelian group with char $k \notin \pi(G)$. Suppose that $|k(G) \cap K : k| < \infty$. Then every locally cyclic quotient of G satisfying Min contains a K-inductive subgroup.

<u>Proof</u> If $\bar{G}$ is any quotient of G, every image of $\bar{G}$ in $\bar{K}^*$ is also an image of G, and therefore $k(\bar{G}) \leqslant k(G)$. Now apply Theorem 4.13.

5. Primitive idempotents in KG

Let G be an abelian group and K a field. If $\alpha \in KG$, we write

$$C_G(\alpha) = \{g \in G : \alpha g = \alpha\} .$$

Since G is abelian, $C_G(\alpha)$ is in fact the centralizer $C_G(\alpha KG)$ of αKG considered as a KG-module. If e is an idempotent in KG, we say that e is <u>faithful (for G)</u> if $C_G(e) = 1$.

<u>Lemma 5.1</u> Let G be a periodic abelian group and K a field with char $K \notin \pi(G)$. Suppose KG contains a primitive idempotent e. Then G satisfies Min and is almost locally cyclic. If e is faithful, G is locally cyclic, and $\langle \text{supp } e \rangle$ is K-inductive in G.

<u>Proof</u> Let $H = \langle \text{supp } e \rangle$, a finite subgroup of G. Then eKH is an irreducible KH-module and $eKH|^G = eKG$ an irreducible KG-module by Lemmas 3.8 and 3.9(b). As in the proof of Lemma 4.3, it follows that $\Omega(G) \leq H$, whence $\Omega(G)$ is finite and G satisfies Min (Theorem 3.1). If e is faithful for G so for H, then H is K-inductive in G by Corollary 4.4.

The group $C = C_G(e)$ is finite, since it acts faithfully (by multiplication) as a group of permutations on the finite set supp e. The irreducible KG-module eKG, considered as a ring, is actually a field F. The homomorphism $G \to F^*$, $g \mapsto eg$ has kernel C. Hence G/C embeds in F^* so is locally

cyclic. Thus G is almost locally cyclic by Corollary 3.2. If e is faithful then $C = 1$ and G itself is locally cyclic. This completes the proof.

We shall now investigate the circumstances under which KG contains primitive idempotents faithful for G, given that G is locally cyclic and satisfies Min. We shall need the following technical lemma (which will also be used in Sections 6 and 15).

<u>Lemma 5.2</u> Let G be a periodic abelian group and K a field with char $K \notin \pi(G)$. Let $\mathcal{L}$ be a family of finite subgroups of G such that every finite subset of G lies in some member of $\mathcal{L}$. Given $\{e_L : L \in \mathcal{L}\}$ such that for $L_1, L_2 \in \mathcal{L}$, e_{L_1} is a primitive idempotent in KL_1, and $e_{L_1} e_{L_2} \neq 0$, there exists a maximal ideal M of KG such that

(a) for each $L \in \mathcal{L}$, $M \cap KL = (1-e_L)KL$ (in particular, $e_L \notin M$);

(b) $C_G(KG/M) = \bigcup\{C_G(e_L) : L \in \mathcal{L}\}$.

<u>Proof</u> Let

$$M = \bigcup\{(1-e_L)KL : L \in \mathcal{L}\}.$$

We show first that M is an ideal in KG. If $L_1, L_2 \in \mathcal{L}$, there exists $L \in \mathcal{L}$ with $L_1 L_2 \leq L$. Since

$$e_L = e_{L_i} e_L + (1-e_{L_i}) e_L \qquad (i = 1,2)$$

is primitive in KL and $(e_{L_i} e_L)^2 = e_{L_i} e_L \neq 0$, we conclude that

cyclic. Thus G is almost locally cyclic by Corollary 3.2. If e is faithful then $C = 1$ and G itself is locally cyclic. This completes the proof.

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<u>Lemma 5.2</u> Let G be a periodic abelian group and K a field with char $K \notin \pi(G)$. Let $\mathcal{L}$ be a family of finite subgroups of G such that every finite subset of G lies in some member of $\mathcal{L}$. Given $\{e_L : L \in \mathcal{L}\}$ such that for $L_1, L_2 \in \mathcal{L}$, e_{L_1} is a primitive idempotent in KL_1, and $e_{L_1} e_{L_2} \neq 0$, there exists a maximal ideal M of KG such that

(a) for each $L \in \mathcal{L}$, $M \cap KL = (1-e_L)KL$ (in particular, $e_L \notin M$);

(b) $C_G(KG/M) = \bigcup\{C_G(e_L) : L \in \mathcal{L}\}$.

<u>Proof</u> Let

$$M = \bigcup\{(1-e_L)KL : L \in \mathcal{L}\}.$$

We show first that M is an ideal in KG. If $L_1, L_2 \in \mathcal{L}$, there exists $L \in \mathcal{L}$ with $L_1 L_2 \leq L$. Since

$$e_L = e_{L_i} e_L + (1-e_{L_i}) e_L \qquad (i = 1,2)$$

is primitive in KL and $(e_{L_i} e_L)^2 = e_{L_i} e_L \neq 0$, we conclude that

$e_{L_i} e_L = e_L$, whence $(1-e_{L_i})(1-e_L) = 1-e_{L_i}$. Thus

$$(1-e_{L_1})KL_1 + (1-e_{L_2})KL_2 \subseteq (1-e_L)KL \subseteq M .$$

Hence M is additively closed, and therefore clearly a K-subspace of KG. If $L_1 \in \mathcal{L}$ and $g \in G$, there exists $L \in \mathcal{L}$ with $\langle L_1, g\rangle \leqslant L$, and we have

$$\begin{aligned}(1-e_{L_1})KL_1 g &\subseteq (1-e_L)KLg \\ &= (1-e_L)KL \subseteq M ,\end{aligned}$$

whence M is indeed an ideal of KG.

Suppose for some $L \in \mathcal{L}$, $e_L \in M$. Then $e_L \in (1-e_{L_1})KL_1$ for some $L_1 \in \mathcal{L}$, and we have $e_L = (1-e_{L_1})e_L$ whence $e_{L_1} e_L = 0$, a contradiction. Thus

$$(1-e_L)KL \leqslant M \cap KL \lneqq KL .$$

Since char $K \notin \pi(L)$ and e_L is primitive in KL, $(1-e_L)KL$ is a maximal ideal of KL, so we have (a).

To show that M is a maximal ideal of KG, suppose that $\alpha \in KG-M$, and let $\operatorname{supp} \alpha \subseteq L \in \mathcal{L}$. Then $\alpha \in KL-(M \cap KL)$, so $1 \in (M \cap KL) + \alpha KL \subseteq M + \alpha KG$. Hence $M + \alpha KG = KG$ as required.

Let $L_1 \in \mathcal{L}$, $x \in C_G(e_{L_1})$, and $\alpha \in KG$. Choose $L \in \mathcal{L}$ with

$$\langle x, \operatorname{supp} \alpha, L_1 \rangle \leqslant L .$$

As before $e_{L_1} e_L = e_L$, so $x \in C_G(e_L)$. Thus

$$(\alpha x - \alpha)e_L = \alpha(xe_L - e_L) = 0 ,$$

whence $$\alpha x - \alpha \in (1-e_L)KL \subseteq M ,$$

i.e. $(\alpha + M)x = \alpha + M$. It follows that

$$\bigcup\{C_G(e_L) : L \in \mathcal{L}\} \leq C_G(KG/M) .$$

Conversely let $x \in C_G(KG/M)$, so that $x-1 \in M$, and there exists $L \in \mathcal{L}$ with $x-1 \in (1-e_L)KL$. Then $e_L(x-1) = 0$, so $x \in C_G(e_L)$. This completes the proof of (b).

<u>Theorem 5.3</u> Let G be a locally cyclic group with Min and K a field with char $K \notin \pi(G)$. Then the following are equivalent:

(a) KG contains a faithful primitive idempotent;

(b) G contains a K-inductive subgroup;

(c) there are only finitely many non-isomorphic irreducible KG-modules faithful for G;

(d) there do not exist $2^{\aleph_0}$ non-isomorphic irreducible KG-modules faithful for G;

(e) $|k(G) \cap K : k| < \infty$, where k is the prime field of K.

Furthermore, when (a)-(e) hold, there is a one-to-one onto correspondence between faithful primitive idempotents of KG and isomorphism classes of irreducible KG-modules faithful for G.

<u>Proof</u> (a) implies (b) by Lemma 5.1, and (b) is equivalent to (e) by Theorem 4.13.

Now suppose H is a K-inductive subgroup of G, and V is an irreducible KG-module faithful for G. Since H is finite, V_H is completely reducible, so it contains an

irreducible KH-submodule W say. Then $V_H = \sum_{x \in G} Wx$, and $Wx \cong W$ as KH-modules since G is abelian. Hence $C_H(W) = C_H(V_H) = 1$. So as H is K-inductive, W^G is irreducible. But there is a non-zero KG-map $W^G \to V$, $w \otimes x \mapsto wx$, so $V \cong W^G$. Thus every irreducible KG-module faithful for G is isomorphic to W^G for some irreducible KH-module W faithful for H. (Note that $W \cong eKH$ and $V \cong eKG$ for some idempotent e in KH which is faithful and primitive in KG.) There are only finitely many non-isomorphic such W, and therefore only finitely many non-isomorphic irreducible KG-modules faithful for G. Hence (b) implies (c). Trivially (c) implies (d).

The last part of the theorem now follows also. For if e is a faithful primitive idempotent in KG, then eKG is an irreducible KG-module faithful for G; as we have just shown, every such module arises in this way. If e and f are idempotents in KG and $eKG \cong fKG$, then if $\theta: eKG \to fKG$ is an isomorphism, we have $\theta(e) = f\theta(e) = \theta(e)f$; applying θ^{-1} we obtain $e = ef$. Similarly $f = fe$, so $e = f$.

To prove that (d) implies (a), we shall assume that KG contains no faithful primitive idempotent, and exhibit $2^{\aleph_0}$ non-isomorphic irreducible KG-modules faithful for G. Let

$$\Omega(G) = L_0 \leqslant L_1 \leqslant L_2 \leqslant \ldots \leqslant G$$

be a chain of finite subgroups with union G.

For $n = 0,1,2,\ldots$ let T_n denote the set of all n-tuples with each entry either 0 or 1. By induction we shall construct for each integer n a finite subgroup H_n of G and for each $\varphi \in T_n$ a faithful primitive idempotent e_φ in KH_n. Firstly, let $H_0 = L_0 = \Omega(G)$. By Lemma 4.1, KH_0 contains a faithful primitive idempotent e.

Now suppose inductively that we have constructed H_n and $\{e_\varphi : \varphi \in T_n\}$. By Lemma 4.2 each e_φ is faithful for G, so by hypothesis is not primitive in KG. Hence we may choose a finite subgroup H_{n+1} of G containing $H_n L_{n+1}$ and such that for each $\varphi \in T_n$, e_φ decomposes in KH_{n+1}; say

$$e_\varphi KH_{n+1} = e_{(\varphi,0)}KH_{n+1} \oplus e_{(\varphi,1)}KH_{n+1} \oplus \cdots ,$$

where $e_{(\varphi,0)}$ and $e_{(\varphi,1)}$ are primitive idempotents in KH_{n+1}. By Lemma 4.2, since $e_\varphi KH_{n+1} = e_\varphi KH_n|^{H_{n+1}}$, $e_{(\varphi,0)}$ and $e_{(\varphi,1)}$ are faithful for H_{n+1}. Thus we have chosen $e_{\varphi'}$ for each $\varphi' \in T_{n+1}$. This completes the inductive construction. Note that

$$\bigcup_{n=0}^{\infty} H_n = \bigcup_{n=0}^{\infty} L_n = G .$$

Let $\varphi = (a_1, a_2, a_3, \ldots)$ be an infinite sequence of 0's and 1's. Write $e_0(\varphi) = e$ and $e_n(\varphi) = e_{(a_1,\ldots,a_n)}$ $(n = 1,2,\ldots)$. If $1 \leq m \leq n$ then by our construction $e_m(\varphi)e_n(\varphi) = e_n(\varphi) \neq 0$. By Lemma 5.2 with $\mathcal{L} = \{H_0, H_1, \ldots\}$, there is a maximal ideal $M = M(\varphi)$ of KG with $1 - e_n(\varphi) \in M(\varphi)$ and $e_n(\varphi) \notin M(\varphi)$ for all n, and

$$C_G(KG/M(\varphi)) = \bigcup_{n=0}^{\infty} C_G(e_n(\varphi)) = 1 .$$

Thus $V(\varphi) = KG/M(\varphi)$ is an irreducible KG-module faithful for G.

If $\varphi \neq \psi$ then $V(\varphi)$ and $V(\psi)$ are not KG-isomorphic. For if φ and ψ differ first in the n-th place, then $e_n(\varphi)e_n(\psi) = 0$; hence $e_n(\psi) = e_n(\psi)(1 - e_n(\varphi)) \in M(\varphi)$, so $e_n(\psi)$ annihilates $V(\varphi)$. But $1-e_n(\psi) \in M(\psi)$, so $e_n(\psi)$ acts as the identity on $V(\psi)$. This completes the proof of the theorem.

In [1; 2.12] (see also [18; 14.4.3(ii)]) S.D. Berman proves a result related to part of Theorem 5.3; namely, if G is an infinite abelian p-group and K is a field with char $K \neq p$ and 'of the first kind with respect to p' (a condition equivalent to $|k(C_{p^\infty}) \cap K : k| < \infty$), then KG contains a primitive idempotent if and only if $G \cong C_{p^\infty} \times F$ where F is finite.

We now extend parts of Theorem 5.3 from locally cyclic to abelian almost locally cyclic groups. The result which we shall obtain (Theorem 5.5) is also a generalization of [9; 2.5]. We shall require:

<u>Lemma 5.4</u> Let K be a field, G a periodic abelian group with char $K \notin \pi(G)$, and C a finite subgroup of G. Then the canonical projection $\theta: KG \to K[G/C]$ determines a one-to-one map from the set of primitive idempotents e in KG with

$C_G(e) = C$ onto the set of faithful primitive idempotents in $K[G/C]$. (Both these sets might be empty.)

<u>Proof</u> If $\alpha \in KG$ we write $\theta(\alpha) = \bar{\alpha}$. Let ν denote the idempotent

$$\frac{1}{|C|} \sum_{x \in C} x ,$$

so that $\bar{\nu} = 1$. If $\alpha \in \underline{\underline{c}}KG \cap \nu KG$ then

$$\alpha = \nu\alpha \in \nu\underline{\underline{c}}KG = 0$$

(since if $x \in C$ then $\nu(x-1) = \nu x - \nu = 0$). Thus

$$\ker\theta \cap \nu KG = \underline{\underline{c}}KG \cap \nu KG = 0 .$$

(In fact it is easily seen that $KG = \underline{\underline{c}}KG \oplus \nu KG$.)

Let X be the set of idempotents e in KG with $C_G(e) \geqslant C$, and Y the set of all idempotents in $K[G/C]$. We claim that θ maps X bijectively onto Y. For suppose $e_i \in X$ $(i = 1,2)$. Since $C \leqslant C_G(e_i)$, $\nu e_i = e_i$, so if $\bar{e}_i = 0$ we have $e_i \in \ker\theta \cap \nu KG = 0$, a contradiction. Thus $\bar{e}_i \in Y$. If $\bar{e}_1 = \bar{e}_2$ then $e_1 - e_2 \in \ker\theta \cap \nu KG = 0$, so $e_1 = e_2$. If $\bar{\alpha} \in Y$ $(\alpha \in KG)$ put $f = \nu\alpha$. Then $\bar{f} = \bar{\nu}\bar{\alpha} = \bar{\alpha}$ (so $f \neq 0$); moreover

$$f - f^2 = \nu(\alpha - \alpha^2) \in \ker\theta \cap \nu KG = 0$$

and

$$C_G(f) = C_G(\nu\alpha) \geqslant C_G(\nu) \geqslant C ,$$

so $f \in X$.

We next claim that if $e \in X$ then $C_{\bar{G}}(\bar{e}) = C_G(e)/C$. For if $g \in C_G(e)$ then $\bar{e}\bar{g} = \overline{eg} = \bar{e}$, so $gC \in C_{\bar{G}}(\bar{e})$. Conversely, suppose $gC \in C_{\bar{G}}(\bar{e})$; then $\bar{e}\bar{g} = \bar{e}$, so $eg - e \in \ker\theta \cap \nu KG = 0$, whence

$g \in C_G(e)$. It follows that $C_G(e) = C$ if and only if $\bar{e}$ is faithful for G/C.

To complete the proof it is sufficient to show that $e \in X$ is not primitive in KG if and only if $\bar{e}$ is not primitive in $K[G/C]$. Thus suppose

$$e = e_1 + e_2 , \quad e_1 e_2 = 0 , \quad e_i^2 = e_i \neq 0 .$$

Since $ee_i = e_i$ we have $C \leq C_G(e) \leq C_G(e_i)$, so $e_i \in X$. Hence

$$\bar{e} = \bar{e}_1 + \bar{e}_2 , \quad \bar{e}_1 \bar{e}_2 = 0 , \quad \bar{e}_i^2 = \bar{e}_i \neq 0 .$$

Conversely, suppose

$$\bar{e} = \bar{\alpha}_1 + \bar{\alpha}_2 , \quad \bar{\alpha}_1 \bar{\alpha}_2 = 0 , \quad \bar{\alpha}_i^2 = \bar{\alpha}_i \neq 0 ,$$

and let $f_i = \nu \alpha_i$ as before. Then

$$e - f_1 - f_2 = \nu(e - \alpha_1 - \alpha_2) \in \ker \theta \cap \nu KG = 0 ,$$

and similarly $f_1 f_2 = \nu(\alpha_1 \alpha_2) = 0$. Hence

$$e = f_1 + f_2 , \quad f_1 f_2 = 0 , \quad f_i^2 = f_i \neq 0.$$

<u>Theorem 5.5</u> Let K be a field, k its prime field, and G an abelian almost locally cyclic group with Min such that char $k \notin \pi(G)$. If $|k(G) \cap K : k| = \infty$, then KG contains no primitive idempotents. Suppose that $|k(G) \cap K : k| < \infty$. If C is any finite subgroup of G such that G/C is locally cyclic, then KG contains a non-zero finite number of primitive idempotents e with $C_G(e) = C$, and there is a one-to-one onto correspondence between such idempotents and isomorphism classes of irreducible KG-modules V with $C_G(V) = C$.

<u>Proof</u> Suppose that KG contains a primitive idempotent e; we show that $|k(G)\cap K : k| < \infty$. Let $C = C_G(e)$. By Lemma 5.4, the image of e in $K[G/C]$ is a primitive idempotent faithful for G/C. Thus G/C is locally cyclic, and by Theorem 5.3 $|k(G/C)\cap K : k| < \infty$.

Since every image of G/C is an image of G, we have $k(G/C) \leq k(G)$. Now let $F = k(\prod O_p(G))$, where the product is taken over those primes p such that $O_p(G)$ is finite. Then $|F:k| < \infty$ since G satisfies Min. Moreover $k(G) = F.k(G/C)$. For k(G) is determined by the exponents of the primary components of G, and since C is finite, if $\exp O_p(G) = \infty$ then $\exp O_p(G/C) = \infty$. Hence by Lemma 4.7,

$$|k(G) : k(G/C)| = |F.k(G/C) : k(G/C)| \leq |F:k| < \infty .$$

Now k(G/C) is a union of finite normal separable extensions of $k(G/C)\cap K$ (see Lemma 4.12); Lemma 3.7(a) together with a local argument shows that k(G/C) and K are linearly disjoint over $k(G/C)\cap K$. In particular, any subset of $k(G)\cap K$ which is linearly independent over $k(G/C)\cap K$ is a subset of k(G) which is linearly independent over k(G/C), so

$$|k(G)\cap K : k(G/C)\cap K| \leq |k(G) : k(G/C)| < \infty .$$

We now have

$$|k(G)\cap K : k| = |k(G)\cap K : k(G/C)\cap K|\,|k(G/C)\cap K : k| < \infty .$$

Now suppose that $|k(G)\cap K : k| < \infty$, and that C is a

finite subgroup of G such that G/C is locally cyclic. Since $k(G/C) \leq k(G)$ we also have $|k(G/C) \cap K : k| < \infty$. In view of Lemma 5.4, an application of Theorem 5.3 to $K[G/C]$ yields the remaining statements of Theorem 5.5.

To conclude this section, we draw together the results we have obtained to give necessary and sufficient conditions for the existence of minimal ideals in the group ring of a periodic abelian group over a non-modular field.

<u>Theorem 5.6</u> Let K be a field with prime field k and G a periodic abelian group with char $k \notin \pi(G)$. Then So(KG) is non-zero if and only if

(a) G satisfies Min;

(b) G is almost locally cyclic; and

(c) $|k(G) \cap K : k| < \infty$.

<u>Proof</u> By Lemma 3.9, $So(KG) \neq 0$ if and only if KG contains a primitive idempotent. Hence if $So(KG) \neq 0$ then (a) and (b) hold by Lemma 5.1, and (c) holds by Theorem 5.5. Conversely, if (a), (b) and (c) hold then by Corollary 3.2 G has a finite subgroup C with G/C locally cyclic, so KG contains primitive idempotents by Theorem 5.5.

6. The Loewy series of KG

We now investigate the ascending Loewy series of the group ring of a periodic abelian group over a non-modular field. Since this series is of little interest if its terms are zero, we are led in the light of Theorem 5.6 and Corollary 3.2 to introduce the following hypothesis, which will be assumed throughout this section.

Hypothesis 6.1 K is a field with prime field k, and G is a periodic abelian group with char $k \notin \pi(G)$ and having a decomposition

$$G = F \times P_1 \times \ldots \times P_m \qquad (0 \leq m < \infty),$$

where F is finite and the P_i are Prüfer p_i-groups for distinct primes p_i. Finally, $|k(G) \cap K : k| < \infty$, so that $So(KG) \neq 0$.

We shall describe the ascending Loewy series of KG in terms of the augmentation ideals of the P_i. We commence with the socle itself.

Theorem 6.2 $So(KG) = \underline{P}_1 G \cap \ldots \cap \underline{P}_m G$.

Proof We remark that when $m = 0$ (so that G is finite and KG completely reducible) the empty intersection is to be interpreted as KG itself. Thus we shall assume that $m \geq 1$.

By Lemma 2.2, $\underline{p}_i$ is essential in KG_i for each i, so $\underline{p}_i G$ is essential in KG by Lemma 2.3(c). Thus $So(KG) \leqslant \bigcap \underline{p}_i G$ by Lemma 2.1.

Conversely, suppose that $0 \neq \alpha \in \bigcap \underline{p}_i G$. Let $H = \langle \operatorname{supp} \alpha \rangle$, and write

$$\alpha = \alpha e_1 + \ldots + \alpha e_r ,$$

where the e_j are orthogonal primitive idempotents in KH, and $\alpha e_j \neq 0$ for each j. Since $e_j KH$ is irreducible, $\alpha e_j KH = e_j KH$, so there exists $\beta_j \in KH$ such that $e_j = \alpha e_j \beta_j$; thus $e_j \in \bigcap \underline{p}_i G$. Hence it is sufficient to show that if H is a finite subgroup of G, e is a primitive idempotent in KH, and $e \in \bigcap \underline{p}_i G$, then $e \in So(KG)$, i.e. if $e \notin So(KG)$ then $e \notin \bigcap \underline{p}_i G$.

Choose a chain

$$H = H_0 \leqslant H_1 \leqslant \ldots \leqslant G$$

of finite subgroups with union G. If f is a primitive idempotent in KH_n for some $n \geqslant 0$, consider the set of all sequences $(f_n, f_{n+1}, \ldots)$ such that

(i) f_j is a primitive idempotent in KH_j for all $j \geqslant n$;

(ii) $f_n = f$;

(iii) $f_j f_{j+1} = f_{j+1}$ for all $j \geqslant n$.

If $r \geqslant 0$ we shall say that f is <u>r-stationary</u> if for all such sequences $(f_n, f_{n+1}, \ldots)$ and all $j \geqslant 0$ we have $f_{n+r} = f_{n+r+j}$. Note that if

$$f = f_1' + \ldots + f_t'$$

where the f_i' are orthogonal primitive idempotents in KH_{n+1}, then f is r-stationary (for $r \geqslant 1$) if and only if each f_i' is (r-1)-stationary. Moreover f is 0-stationary if and only if it is primitive in KG. Hence if f is r-stationary and we write f as a sum of orthogonal primitive idempotents in KH_{n+r}, then each such idempotent will be 0-stationary; thus by Lemma 3.9(c) we have $f \in So(KG)$.

Now let e be a primitive idempotent in KH with $e \notin So(KG)$. Then $e = e_0$ is not r-stationary for any r. Hence among the finitely many orthogonal primitive idempotents in KH_1 whose sum is e_0, there must exist one, say e_1, which is not r-stationary for any r. Similarly we may choose a primitive idempotent e_2 in KH_2 which satisfies $e_1 e_2 = e_2$ and is not r-stationary for any r, and so on. In this way we obtain a sequence $e_0 = e, e_1, e_2, \ldots$ such that e_i is a primitive idempotent in KH_i, and $e_i e_{i+1} = e_{i+1}$.

Consider the chain of subgroups $C_G(e_0) \leqslant C_G(e_1) \leqslant \ldots$, and suppose that $C = \bigcup_{i=0}^{\infty} C_G(e_i)$ is finite; then $C = C_G(e_n)$ for some n. For $i \geqslant n$, $e_i KH_i$ is an irreducible module faithful for H_i/C, so H_i/C is cyclic; hence G/C is locally cyclic. Thus by Corollary 4.14 G/C contains a K-inductive subgroup. Thus we may choose $s \geqslant n$ so that H_s/C is K-inductive in G/C (Corollary 4.5). But e_s is a primitive idempotent in KH_s

with $C_G(e_s) = C$, so e_s is primitive in KG, i.e. O-stationary, a contradiction. It follows that C is infinite, whence by Lemma 5.2 (with $\mathcal{L} = \{H_0, H_1, \ldots\}$) there is a maximal ideal M of KG such that $e = e_0 \notin M$ and $C_G(KG/M) = C$ is infinite. Then $C_G(KG/M)$ contains P_i for some i, whence $\underline{\underline{p}}_i G \leq M$. Thus $e \notin \bigcap \underline{\underline{p}}_i G$, as required.

As an example we may take G to be a Prüfer group C_{p^∞} and K a subfield of the complex numbers with $|\mathbb{Q}(C_{p^\infty}) \cap K : \mathbb{Q}| < \infty$; then $So(KG) = \underline{\underline{g}}$, a result obtained by Müller in [14].

<u>Corollary 6.3</u> For $0 < i \leq m$,

$$So_i(KG) = \bigcap_{|I|=i} \sum_{j \in I} \underline{\underline{p}}_j G ,$$

where the intersection is taken over all subsets I of $\{1, \ldots, m\}$ with i elements.

<u>Proof</u> We proceed by induction on i: the case $i = 1$ is the theorem we have just proved.

The canonical maps $KG \to K[G/P_j] \cong KG/\underline{\underline{p}}_j G$ induce a KG-map

$$KG \longrightarrow \bigoplus_{j=1}^{m} KG/\underline{\underline{p}}_j G$$

with kernel $\bigcap \underline{\underline{p}}_j G = So(KG)$. Hence we have a KG-monomorphism

$$\psi : KG/So(KG) \longrightarrow \bigoplus_{j=1}^{m} KG/\underline{\underline{p}}_j G .$$

Suppose $1 < i \leq m$. Then

$$So_i(KG)/So(KG) = So_{i-1}(KG/So(KG))$$
$$= \varphi^{-1}\left(\bigoplus_{j=1}^{m} So_{i-1}(KG/\underline{\underline{p}}_jG)\right) .$$

Hence

$$So_i(KG) = \bigcap_{j=1}^{m} \{\alpha \in KG : \alpha + \underline{\underline{p}}_jG \in So_{i-1}(KG/\underline{\underline{p}}_jG)\} .$$

By induction on i, since $G/P_j \cong F \times P_1 \times \ldots \times P_{j-1} \times P_{j+1} \times \ldots \times P_m$,

$$So_{i-1}(KG/\underline{\underline{p}}_jG) = \bigcap_{|I_j|=i-1} \sum_{\ell \in I_j} (\underline{\underline{p}}_\ell G + \underline{\underline{p}}_jG)/\underline{\underline{p}}_jG$$

where the intersection is taken over all subsets I_j of $\{1,\ldots,m\} - \{j\}$ with i-1 elements. Hence we have

$$So_i(KG) = \bigcap_{j=1}^{m} \bigcap_{|I_j|=i-1} \sum_{\ell \in I_j} (\underline{\underline{p}}_\ell G + \underline{\underline{p}}_jG) ,$$

an expression easily seen to be equal to the one desired.

<u>Corollary 6.4</u> The ascending Loewy series of KG reaches KG after exactly m+1 steps, i.e. $So_m(KG) \neq KG = So_{m+1}(KG)$.

<u>Proof</u> By the previous corollary with $i = m$, we have

$$So_m(KG) = \sum_{j=1}^{m} \underline{\underline{p}}_jG .$$

Let $A = P_1 \times \ldots \times P_m = \langle P_1,\ldots,P_m \rangle$, so that by Lemma 1.1 we have

$$\underline{\underline{a}}G = \sum_{j=1}^{m} \sum_{x \in P_j} (x-1)KG = \sum_{j=1}^{m} \underline{\underline{p}}_jG .$$

Thus $So_m(KG) = \underline{\underline{a}}G \neq KG$. Moreover,

$$KG/So_m(KG) = KG/\underline{\underline{a}}G \cong K[G/A] .$$

By Maschke's theorem, since $G/A \cong F$ is finite, $K[G/A]$ is completely reducible as $K[G/A]$-module, and therefore also as

KG-module. Hence

$$So(KG/So_m(KG)) = KG/So_m(KG) ,$$

i.e. $So_{m+1}(KG) = KG$.

We remark that the ascending Loewy series of KG enables us to classify irreducible KG-modules as follows. For a given irreducible KG-module M there is a unique integer $\lambda \in \{0,\ldots,m\}$ such that M is a composition factor of $So_{\lambda+1}(KG)/So_\lambda(KG)$. Further, λ is equal to the number of Prüfer factors $P_1,\ldots,P_m$ which are contained in $C_G(M)$. We also remark that every indecomposable KG-module is irreducible. The proofs of these results will be given in a more general setting in Section 15.

Chapter III

SOME BACKGROUND RESULTS

7. On groups

In this chapter we record a number of results which will be needed in our study of the socle in group rings of locally finite groups (Chapter IV) and non-locally-finite groups (Chapter V). In Section 8 we present the material required on rings and algebras, and in Section 9 we consider group rings specifically, while this section deals with the necessary group theory, mentioning FC-groups, Černikov groups, and linear groups. For the most part we are content to state results only, referring the reader to the literature for proofs.

An <u>FC-group</u> is a group in which each element has only a finite number of conjugates. We define the <u>FC-centre</u> of a group G as

$$\Delta(G) = \{x \in G : |G:C_G(x)| < \infty\} .$$

The following result is well known:

<u>Lemma 7.1</u> If G is any group, $\Delta(G)$ is a characteristic subgroup of G. The torsion elements of $\Delta(G)$ form a locally finite subgroup with torsion-free abelian quotient.

<u>Proof</u> See [13; 4.1.6] or [15: 19.3].

A <u>Černikov group</u> is an almost abelian group satisfying Min. By Theorem 3.1 we see that Černikov groups may be characterized as finite extensions of direct products of finitely many Prüfer groups. In determining those locally finite groups whose group rings may have non-zero socle (Section 12) we shall require the following deep result of Šunkov:

<u>Theorem 7.2</u> If G is a locally finite group every abelian subgroup of which satisfies Min, then G is a Černikov group.

<u>Proof</u> See [13; 5.8].

When considering group rings over fields of positive characteristic, the full force of Šunkov's theorem will not be needed: the following far more elementary special case will suffice.

<u>Lemma 7.3</u> If G is a nilpotent group every abelian subgroup of which satisfies Min, then G is a Černikov group.

<u>Proof</u> See [13; 1.G.4 (or even 1.G.3)].

If E is a division ring, a <u>linear group over E</u> is a group of linear transformations of a finite-dimensional vector space over E.

<u>Theorem 7.4</u> Let G be a finitely generated linear group over a field K. Then

(a) if char $K = 0$ then for any prime q, G is almost residually finite-q';

(b) if char $K = p > 0$ then G is almost residually finite-p.

<u>Proof</u> This follows immediately from [24; 4.7].

<u>Theorem 7.5</u> (Schur) A periodic group which is linear over a field is locally finite.

<u>Proof</u> See [24; 4.9].

8. On rings and algebras

In this section we discuss quasi-Frobenius rings, separable algebras, a theorem of Kaplansky, locally Wedderburn algebras, and strongly prime rings.

If X is a subset of a ring A, we denote by $\ell_A(X)$ and $r_A(X)$ respectively the left and right annihilators of X in A. When confusion is unlikely the subscript $_A$ will be omitted.

Proposition 8.1 If A is a right and left artinian ring, the following are equivalent:

(a) A_A is injective;

(b) $_AA$ is injective;

(c) for every right ideal R and left ideal L of A we have

$$r(\ell(R)) = R \ , \quad \ell(r(L)) = L \ .$$

Proof See [22; XIV.3.1, XIV.3.3].

An artinian ring A satisfying (a)-(c) is called quasi-Frobenius. Note that from (c) it follows that taking annihilators induces an inclusion-reversing bijection between the lattices of right and left ideals of A.

Proposition 8.2 Every irreducible right module for a quasi-Frobenius ring is isomorphic to a minimal right ideal.

Proof See [22; XIV.3.2, XI.5.1].

<u>Proposition 8.3</u> The following properties of a right module M over a quasi-Frobenius ring A are equivalent:

(a) M is injective;

(b) M is projective;

(c) $M \cong \bigoplus_i e_i A$ for a family $\{e_i\}$ of primitive idempotents in A.

<u>Proof</u> See [22; XIV.3.6].

We next consider separable algebras. An algebra A over a field K is called <u>separable</u> if $A \otimes_K F$ is semisimple for every field extension F of K. (We remark that if A is an algebraic field extension of K, this definition agrees with that given in Section 2: see [4; 71.9].) Note that a separable algebra is in particular semisimple: take $F = K$. Recall that a field is <u>perfect</u> if every finite extension is separable; in particular, prime fields and fields of characteristic zero are perfect.

<u>Proposition 8.4</u> Every semisimple algebra over a perfect field is separable.

<u>Proof</u> See [18; 7.3.9] or [2; §7, No. 5].

<u>Proposition 8.5</u> A finite-dimensional K-algebra A is separable if and only if there exists an extension F of K such that $A \otimes_K F$ is isomorphic to a direct sum of full matrix algebras over F.

Proof See [4; 71.2].

The importance for our purposes of separable algebras derives from the following corollary to a theorem of Bourbaki:

Theorem 8.6 The tensor product of two separable algebras is again separable.

Proof See [18; 7.3.10] or [4; 71.10].

Recall that an algebra A over a field K is said to satisfy a polynomial identity if there is a non-zero polynomial $f(X_1,\ldots,X_m)$ in non-commuting indeterminates $X_1,\ldots,X_m$ over K such that $f(\alpha_1,\ldots,\alpha_m)=0$ for all $\alpha_1,\ldots,\alpha_m \in A$.

Lemma 8.7 The ring $M_n(K)$ of $n\times n$-matrices over a field K satisfies a polynomial identity.

Proof See [18; 5.1.6]. (In fact, K could be any commutative ring.)

The next theorem, which characterizes primitive polynomial-identity algebras, is due to Kaplansky.

Theorem 8.8 Suppose an algebra A over a field K satisfies a polynomial identity and has a faithful irreducible module V. Let E be the division algebra $\mathrm{End}_A(V)$. Then $t=\dim_E V$ is finite, and A is isomorphic to the ring $M_t(E)$ of $t\times t$-matrices over E.

<u>Proof</u> See [18; 5.3.4] or [15; 6.4].

Kaplansky's theorem has the following corollary, which is probably well known.

<u>Corollary 8.9</u> Let A be a locally Wedderburn algebra (with unit element) satisfying a polynomial identity, and let M_A be a module with a finite composition series. Then M is completely reducible.

<u>Proof</u> Since the property of being a locally Wedderburn algebra (like that of being semisimple artinian) is inherited by epimorphic images, we may assume that M_A is faithful. Let

$$0 = M_0 < M_1 < \ldots < M_r = M$$

be a composition series, and set

$$T_i = \mathrm{Ann}_A(M_i/M_{i-1}) \qquad (i = 1,\ldots,r)$$

and $T = \bigcap T_i$. Then $MT^r = 0$ so $T^r = 0$, whence $T = 0$ by Lemma 3.9(a). Each A/T_i is primitive and satisfies a polynomial identity so is artinian by Theorem 8.8. Hence A, which is isomorphic to an A-submodule of $\oplus A/T_i$, is semisimple artinian. Thus M_A is completely reducible.

We define the <u>endomorphism dimension</u> of an irreducible module to be the dimension of the module over its endomorphism ring (which is a division ring by Schur's lemma).

<u>Lemma 8.10</u> (Farkas and Snider) Let A be a locally Wedderburn algebra and V an irreducible right A-module of finite endomorphism dimension. Then V is an injective A-module.

<u>Proof</u> [5; Lemma 3] Assume that V is not injective, so that by Baer's criterion [22; I.6.5] there is a right ideal I of A and an A-map $\varphi : I \to V$ which cannot be lifted to A.

Let $\mathcal{S}$ be the set of finite-dimensional semisimple subalgebras of A. Let $B \in \mathcal{S}$, and put

$$D(B) = \{v \in V : \varphi(a) = va \text{ for all } a \in I \cap B\} .$$

Then $D(B) \neq \emptyset$ since V_B (like every B-module) is injective. If $w \in D(B)$ and

$$\ell_V(I \cap B) = \{v \in V : va = 0 \text{ for all } a \in I \cap B\}$$

(a B-submodule of V), then easily

$$D(B) = w + \ell_V(I \cap B) .$$

Since A is locally Wedderburn, every element of I lies in some member of $\mathcal{S}$, so our assumption is that

$$\bigcap\{D(B) : B \in \mathcal{S}\} = \emptyset .$$

Let $E = \mathrm{End}_A(V)$. Since $\dim_E V$ is finite, we may choose $B_0 \in \mathcal{S}$ such that $d = \dim_E \ell_V(I \cap B_0)$ is minimal. By the empty intersection there exists $B_1 \in \mathcal{S}$ with

$$D(B_0) \not\subseteq D(B_1) .$$

Now B_0 and B_1 are finite-dimensional and A is locally Wedderburn, so there exists $B_2 \in \mathcal{S}$ with $B_0 \cup B_1 \subseteq B_2$. Then

$$\emptyset \neq D(B_2) \subseteq D(B_0) \cap D(B_1) \subsetneq D(B_0) .$$

Thus if $w \in D(B_2)$ we have

$$\begin{aligned} w + \ell_V(I \cap B_2) &= D(B_2) \\ &\subsetneq D(B_0) = w + \ell_V(I \cap B_0) , \end{aligned}$$

contradicting the minimality of d.

The following technical result of Hartley [10; Theorem C1] will be used in Section 15.

<u>Theorem 8.11</u> Let A be a locally Wedderburn algebra of countable dimension, and V an irreducible A-module. Then exactly one of the following alternatives holds:

(i) V has finite endomorphism dimension and is injective;

(ii) V has infinite endomorphism dimension and may be embedded in an indecomposable A-module of composition length two.

<u>Proof</u> The first alternative comes from Lemma 8.10. For the construction of the indecomposable A-module of the second alternative, see [10].

Recall that a ring R is <u>prime</u> if whenever $\alpha, \beta \in R$ and $\alpha R \beta = 0$ either α or β is zero, or equivalently, if $r(\alpha R) = 0$ for all non-zero $\alpha \in R$. Handelman and Lawrence [7] call R <u>(right) strongly prime</u> if for each non-zero $\alpha \in R$ there is a finite subset X of R with $r(\alpha X) = 0$. The next result is [7; IV, Corollary 2]; we give a different proof.

<u>Lemma 8.12</u> If R is strongly prime then So(R) is either 0 or R.

<u>Proof</u> Suppose that $So(R) \neq 0$, and let αR be a minimal right ideal. Let X be a finite subset of R with $r(\alpha X) = 0$. Then the obvious map

$$R_R \longrightarrow \bigoplus_{\xi \in X} \alpha\xi R$$

is one-to-one. Since $\alpha\xi R \leq \alpha R$, $\alpha\xi R$ is either zero or αR, and it follows that R_R is completely reducible, i.e. $So(R) = R$.

9. On group rings

We commence this section of background material on group rings with a series of miscellaneous elementary and well known lemmas.

Lemma 9.1 Let K be a field, H a subgroup of a group G, and $\alpha \in KH$. Then

(a) $r_{KG}(\alpha) = r_{KH}(\alpha)KG$;

(b) α is regular (i.e. not a zero-divisor) in KG if and only if it is regular in KH.

Proof (a) Certainly $r_{KH}(\alpha)KG \leq r_{KG}(\alpha)$. Conversely, suppose $\beta \in r_{KG}(\alpha)$. Let T be a right transversal to H in G, and write

$$\beta = \sum_{x \in T} \beta_x x \qquad (\beta_x \in KH) .$$

Then $$0 = \alpha\beta = \sum_{x \in T} \alpha\beta_x x \qquad (\alpha\beta_x \in KH) ,$$

so $\alpha\beta_x = 0$ for each x, whence $\beta_x \in r_{KH}(\alpha)$ and $\beta \in r_{KH}(\alpha)KG$.

(b) Part (b) follows at once from (a) and its left-hand analogue.

Lemma 9.2 (Wallace [23; 2.1]) Let K be a field and G a group with a family $\{H_\lambda : \lambda \in \Lambda\}$ of normal subgoups satisfying

(a) $\bigcap_{\lambda \in \Lambda} H_\lambda = 1$; and

(b) if $\lambda, \mu \in \Lambda$ then there exists $\nu \in \Lambda$ with $H_\nu \leq H_\lambda \cap H_\mu$.

Then $\bigcap_{\lambda \in \Lambda} \underline{\underline{h}}_\lambda G = 0$.

Proof Suppose $0 \neq \alpha \in \bigcap \underline{\underline{h}}_\lambda$; we may assume $1 \in \operatorname{supp} \alpha$ (by replacing α by αx^{-1} where $x \in \operatorname{supp} \alpha$). Since $\operatorname{supp} \alpha$ is finite, by (a) and (b) there exists $o \in \Lambda$ with $H_o \cap \operatorname{supp} \alpha = \{1\}$. Now $\alpha \in \underline{\underline{h}}_o G$, so if $\alpha = \sum_{g \in G} \lambda_g g$ ($\lambda_g \in K$) then in $K[G/H_o]$ we have

$$0 = \sum \lambda_g g H_o = \lambda_1 H_o + \sum_{gH_o \neq H_o} \lambda_g g H_o ,$$

whence $\lambda_1 = 0$, a contradiction.

Lemma 9.3 Let L be an extension of a field K, G a group, and V a KG-module of finite K-dimension. Then

$$\operatorname{End}_{LG}(V \otimes_K L) \cong \operatorname{End}_{KG}(V) \otimes_K L .$$

Proof See [4; 29.5].

The next result is certainly well known: see for example [3; 2.5], where it is stated without proof.

Lemma 9.4 Let K be a field, H a subgroup of a group G, and V an injective right KH-module. Then

(a) $M = \operatorname{Hom}_{KH}(KG, V)$ is an injective right KG-module;

(b) if $|G:H| < \infty$, V^G is isomorphic to M, so is also injective.

Proof (The action of KG on M is as usual given by

$$(\mu\gamma)(\alpha) = \mu(\gamma\alpha) \qquad (\mu \in M;\ \gamma, \alpha \in KG) \quad .)$$

(a) By Baer's criterion for injectivity [22; I.6.5], it is sufficient to show that any KG-map $\varphi : I \to M$ from a right

ideal I of KG is the restriction of a map $KG \to M$. Define a KH-map $\sigma : I_H \to V$, $\alpha \mapsto \varphi(\alpha)(1)$. Since V is injective, there is a KH-map τ making

$$\begin{tikzcd} 0 \arrow[r] & I_H \arrow[r] \arrow[d, "\sigma"'] & KG|_H \arrow[dl, dashed, "\tau"] \\ & V & \end{tikzcd}$$

commute. Now $\tau \in M$; define a KG-map $\hat{\varphi} : KG_{KG} \to M$, $\gamma \mapsto \tau\gamma$. Then if $\alpha \in I$ and $\beta \in KG$ we have

$$\begin{aligned} \hat{\varphi}(\alpha)\ (\beta) &= (\tau\alpha)(\beta) \\ &= \tau(\alpha\beta) \\ &= \sigma(\alpha\beta) \qquad \text{as } \alpha\beta \in I \\ &= \varphi(\alpha\beta)(1) \\ &= (\varphi(\alpha)\beta)(1) \\ &= \varphi(\alpha)(\beta) \quad , \end{aligned}$$

so that $\hat{\varphi}|_I = \varphi$ as required.

(b) Let T be a right transversal to H in G, so that

$$V^G = \bigoplus_{x \in T} V \otimes x$$

and

$$KG = \bigoplus_{x \in T} x^{-1}KH \quad .$$

A routine verification shows that

$$V^G \to M, \quad \sum_{x \in T} v_x \otimes x \mapsto \left(\sum_{x \in T} x^{-1}\alpha_x \mapsto \sum_{x \in T} v_x \alpha_x \right)$$

$$(v_x \in V,\ \alpha_x \in KH)$$

and

$$M \to V^G, \quad \varphi \mapsto \sum_{x \in T} \varphi(x^{-1}) \otimes x$$

(where the last sum is meaningful since $|T| = |G:H| < \infty$) are mutually inverse KG-maps.

We remark that the proof of part (a) actually gives a more general result: if R is a subring of a ring S (with the same 1) and V is an injective right R-module, then $\mathrm{Hom}_R(S_R, V)$ is an injective right S-module.

<u>Corollary 9.5</u> If K is a field and G a finite group, then KG is a quasi-Frobenius ring.

<u>Proof</u> KG is finite-dimensional so artinian. Taking $H = 1$ in Lemma 9.4(b) we see that $KG \cong K1|^G$ is right self-injective.

Recall that if G is a finite group, a field L is a <u>splitting field for G</u> if $\mathrm{End}_{LG}(V) = L$ for every irreducible LG-module V.

<u>Theorem 9.6</u> Let G be a finite group and K any field. Then K has a finite separable extension L which is a splitting field for G.

<u>Proof</u> This result is proved in [4; 69.11] under the additional hypothesis that K is perfect. If K has characteristic zero this hypothesis is of course satisfied. Suppose K has characteristic $p > 0$. By the result cited (applied to the perfect field $\mathbb{F}_p$) there is a finite field F of characteristic p which is a splitting field for G. Let L

be a composite of K and F (i.e. a quotient of $K \otimes_{\mathbb{F}_p} F$ by a maximal ideal). Then L is a finite separable extension of K, since it is generated over K (like F over $\mathbb{F}_p$) by roots of unity. Moreover L, which contains a copy of F, is a splitting field for G.

The next three results concern the Jacobson radicals of group rings.

<u>Lemma 9.7</u> Let K be a field, and H a normal subgroup of finite index n in a group G. Then

$$(J(KG))^n \leqslant J(KH)KG \leqslant J(KG) .$$

<u>Proof</u> See [18;7.2.7] or [15; 16.6].

<u>Theorem 9.8</u> Let K be a field and G a soluble group with char $K \notin \pi(G)$. Then $J(KG) = 0$.

<u>Proof</u> See [18; 7.4.6] or [15; 18.9]. (We shall only require the simpler case of an abelian group.)

If G is a locally finite group and p a prime, we denote by $O_p(G)$ the unique largest normal p-subgroup of G.

<u>Lemma 9.9</u> Let G be a locally finite group, K a field of characteristic $p > 0$, and V an irreducible KG-module. Then

$$O_p(G) \leqslant C_G(V) .$$

<u>Proof</u> Let $P = O_p(G)$. We must show that $V.pG = 0$ for every

irreducible V, i.e. that $\underline{p}G \leqslant J(KG)$. Hence it is sufficient to show that $\underline{p}G$ is a nil ideal. Let

$$\alpha = \sum_{i=1}^{n} \lambda_i(x_i-1)g_i \in \underline{p}G \qquad (\lambda_i \in K,\ x_i \in P,\ g_i \in G),$$

and put $H = \langle x_i, g_i : i = 1,\ldots,n\rangle$. Since $x_i \in P \cap H \leqslant O_p(H)$, we may assume that $G = H$ is finite. As $P \trianglelefteq G$ we have $(\underline{p}G)^n = \underline{p}^n G$, so it is enough to prove that if G is a finite p-group then $\underline{g}$ is nilpotent.

We proceed by induction on $|G|$ (following [18; 3.1.6]). If $|G| = p$ and $G = \langle x\rangle$ then

$$\begin{aligned} \underline{g}^p &= ((x-1)KG)^p && \text{by Lemma 1.1} \\ &= (x-1)^p KG && \text{as } KG \text{ is commutative} \\ &= (x^p-1)KG && \text{since char } K = p \\ &= 0. \end{aligned}$$

If $|G| = p^m$ $(m>1)$ let H be a central subgroup of G of order p. The image of $\underline{g}$ under $KG \to K[G/H]$ lies in the augmentation ideal of $K[G/H]$, which is nilpotent by induction. Hence for some t we have $\underline{g}^t \leqslant \underline{h}G$. But by the above and as H is central in G, $(\underline{h}G)^p = \underline{h}^p G = 0$. Hence $\underline{g}^{tp} = 0$ as required.

The next lemma is an early example of a class of group ring results known as 'intersection theorems'.

<u>Lemma 9.10</u> Let G be a group with a normal abelian subgroup A, and put

$$H = \{x \in G : |A:C_A(x)| < \infty\}.$$

(a normal subgroup of G containing A). If K is a field and I a non-zero ideal of KG, then $I \cap KH \neq 0$.

Proof See [18; 7.4.9] or [15; 21.1].

We shall require two results relating group rings and polynomial identities.

Lemma 9.11 Let K be a field and G an almost abelian group. Then KG satisfies a polynomial identity.

Proof See [18; 5.1.11] or [15; 5.1]. The crux of the proof is that if A is an abelian normal subgroup of G of finite index n, then KG may be embedded in the n×n-matrix ring over the commutative ring KA: cf. Lemma 8.7.

A (right) annihilator ideal of a ring is a two-sided ideal which is the right annihilator of some subset of the ring.

Theorem 9.12 (Passman) Let K be a field and G a group. Then the following are equivalent:

(a) KG has an annihilator ideal $A \neq KG$ such that KG/A satisfies a polynomial identity;

(b) $|G : \Delta(G)| < \infty$ and $|\Delta(G)'| < \infty$.

Proof See [18; 5.2.11] or [17; Theorem 1].

Next we consider injectivity and endomorphism dimension of irreducible KG-modules.

<u>Lemma 9.13</u> (Farkas and Snider) Let K be a field and G a group. The trivial KG-module K is injective if and only if G is locally finite and char $K \notin \pi(G)$.

<u>Proof</u> See [18; 3.2.12] or [5; Theorem 1].

<u>Lemma 9.14</u> Let K be a field and G an almost abelian group. Then every irreducible KG-module has finite endomorphism dimension.

<u>Proof</u> Use Lemma 9.11 and Theorem 8.8.

Lemma 9.14 has a partial converse:

<u>Theorem 9.15</u> (Hartley) Let K be a field and G a locally finite group with char $K \notin \pi(G)$. Then every irreducible KG-module has finite endomorphism dimension if and only if G is almost abelian.

<u>Proof</u> See [18; 12.4.16] or [10; Theorem B].

This section's penultimate result is due to Handelman and Lawrence [7; Proposition III.3].

<u>Lemma 9.16</u> Let K be a field and $G = A*B$ the free product of non-trivial groups A and B. Then KG is strongly prime.

<u>Proof</u> Let $1 \neq a \in A$, $1 \neq b \in B$, and put $X = \{aa, ab, ba, bb\} \subseteq G$. We shall show that $r(\gamma X) = 0$ whenever $0 \neq \gamma \in KG$. (Thus KG is actually 'uniformly' strongly prime.)

We say that a non-trivial element g of G has <u>type</u> AA and <u>length</u> $\lambda(g) = 2n+1$ if g may be written in the form (necessarily unique)

$$g = a_1 b_1 a_2 b_2 \cdots a_n b_n a_{n+1} \quad (1 \neq a_i \in A,\ 1 \neq b_i \in B) .$$

We define elements of types AB, BA and BB, and their lengths, similarly. Any non-trivial element of G falls into exactly one of the four types. We define $\lambda(1) = 0$.

Suppose $0 \neq \gamma, \delta \in KG$ but $\gamma X \delta = 0$. Choose elements v of $\operatorname{supp} \gamma$ and w of $\operatorname{supp} \delta$ of maximal length; clearly $v, w \neq 1$. Suppose v is of type ?A (i.e. AA or BA) and w is of type A? (there are three other cases, which may be handled similarly). Now $\gamma bb\delta \in \gamma X \delta = 0$, so $vbbw \notin \operatorname{supp} \gamma bb\delta$; hence there must exist $v_1 \in \operatorname{supp} \gamma$ and $w_1 \in \operatorname{supp} \delta$ with $v_1 \neq v$, $w_1 \neq w$, but $v_1 bbw_1 = vbbw$. Then

$$\begin{aligned} \lambda(v) + 2 + \lambda(w) &= \lambda(vbbw) \\ &= \lambda(v_1 bbw_1) \\ &\leq \lambda(v_1) + 2 + \lambda(w_1) \\ &\leq \lambda(v) + 2 + \lambda(w) , \end{aligned}$$

whence $\lambda(v_1) = \lambda(v)$ and $\lambda(w_1) = \lambda(w)$. Since $v_1 bbw_1 = vbbw$, it follows from the uniqueness of the reduced form expression that $v_1 = v$ and $w_1 = w$, a contradiction.

In fact, as Handelman and Lawrence show, the coefficient ring K need not be a field: it suffices for K to be

strongly prime. The modification required in the proof is elementary.

<u>Lemma 9.17</u> Let G be a group, K a field with $|K| > |G|$, and V an irreducible right KG-module. Then $E = \mathrm{End}_{KG}(V)$ is algebraic over K.

<u>Proof</u> (see [18; 7.1.2, 9.1.6]) If $0 \neq v \in V$, then $E \to V$, $e \mapsto ev$ is a K-monomorphism; moreover, V is an image of KG_{KG}. Hence

$$\dim_K E \leq \dim_K V \leq \dim_K KG = |G| < |K| \quad .$$

Thus if $e \in E-K$, the elements $\{(e-a)^{-1} : a \in K\}$ of E are linearly dependent over K: say

$$\sum_{i=1}^{n} b_i (e-a_i)^{-1} = 0 \qquad (a_i, b_i \in K) \ ,$$

where the a_i are distinct. Since the $e-a_i$ and their inverses commute, we find by multiplying by the common denominator that e satisfies the polynomial

$$f(X) = \sum_{i=1}^{n} b_i \prod_{j \neq i} (X-a_j) \quad ,$$

which is non-zero since $f(a_1) \neq 0$. Hence e is algebraic over K.

Chapter IV

LOCALLY FINITE GROUPS

10. Preamble

In this chapter we examine consequences of supposing that KG contains a minimal one-sided ideal N in the case where G is a locally finite group and K is an arbitrary field.

We commence in Section 11 by investigating properties of the endomorphism ring of N, using a local technique.

Then in Section 12 we consider consequences of the existence of N for the structure of G. We find that G must be a Černikov group (Theorem 12.1), and then use the results of Chapter II to deduce necessary and sufficient conditions for the existence of a minimal one-sided ideal (Theorem 12.2): namely, that G should have a normal abelian subgroup A of finite index such that K is non-modular for A and A satisfies conditions S1, S2 and S3 of Section 3.

In Section 13 we investigate consequences of the existence of N for the structure of KG itself. We show in Lemma 13.3 that the ascending Loewy series of KG reaches KG after finitely many steps; it follows that every non-zero KG-module has non-zero socle (i.e. KG is 'semiartinian').

The principal result of the section is that KG has a finite series of ideals each factor in which is a direct sum of quasi-Frobenius rings (Theorem 13.4). We also show that the socle of KG is a direct sum of minimal two-sided ideals (Theorem 13.5).

Most of Section 14 is devoted to the determination of the 'controller' of the socle of KG, that is, the smallest normal subgroup C of G for which there is an ideal in KC which generates the socle of KG. Of course, if the socle is zero, this subgroup is trivial; otherwise it is a certain easily described subgroup of G depending only on the characteristic of K (Theorem 14.8). We use this result to obtain, in Theorem 14.9, an expression for the socle of KG. This expression is quite explicit except that it involves the socle of a finite-group algebra, and is therefore the best obtainable until the problem of characterizing such socles is solved.

In Section 15 we use the knowledge of the structure of KG gleaned in Section 13 to classify indecomposable KG-modules, in a manner analogous to the partitioning of indecomposables for a finite-group algebra into blocks; we also describe the injective and projective indecomposable KG-modules. Finally, we determine (for countable but not

necessarily locally finite G) the conditions under which all indecomposable KG-modules are irreducible.

It is convenient at this point to remark on the relationship between the left and right socles of KG. Since for any group G and field K, $g \mapsto g^{-1}$ induces an anti-automorphism of KG, the left socle of KG is zero if and only if the right socle is also zero. When G is locally finite, we have the following stronger result (stated, but not completely proved, in [14; §2]).

<u>Proposition 10.1</u> Let K be a field, G a locally finite group, and $\alpha \in KG$. Then αKG is a minimal right ideal if and only if $KG\alpha$ is a minimal left ideal. In particular,

$$So({}_{KG}KG) = So(KG_{KG}) .$$

<u>Proof</u> Suppose αKG is a minimal right ideal, and let H be any finite subgroup of G containing supp α. Then αKH is a minimal right ideal of KH (since $\alpha KG \cong \alpha KH|^G$). Moreover, $KG\alpha$ is the union of $KH\alpha$ over all such H, so it is enough to show that $KH\alpha$ is minimal. Now $\alpha KH \cong KH/r(\alpha)$, so $r(\alpha) = r(KH\alpha)$ is a maximal right ideal. Since KH is quasi-Frobenius (Corollary 9.5), its left and right submodule lattices are anti-isomorphic (see Proposition 8.1), and it follows that $KH\alpha = \ell r(KH\alpha)$ is a minimal left ideal. This completes the proof.

Note that the last part of this proof may be extended to show that if H is finite then $So_n({}_{KH}KH) = So_n(KH_{KH})$ for all values of n. For if αKH has a series of length n with completely reducible factors, then so does $KH\alpha$. However, the preceding local argument has no obvious analogue when $n>1$, and it is an open question whether for G locally finite $So_n({}_{KG}KG) = So_n(KG_{KG})$ for all n.

11. Endomorphism rings

Let G be a locally finite group, K a field, and αKG (for some $\alpha \in KG$) a minimal right ideal of KG. In this section we examine the division ring $E_G = \mathrm{End}_{KG}(\alpha KG)$.

Let $H = \langle \mathrm{supp}\, \alpha \rangle$, a finite subgroup of G. Then

$$\mathcal{L} = \{F \geqslant H : F \text{ is a finite subgroup of } G\}$$

is a directed set of subgroups of G, i.e. any two members of $\mathcal{L}$ are both contained in some common third member. Moreover, $\bigcup \mathcal{L} = G$. If $L \in \mathcal{L} \cup \{G\}$ then αKL is a minimal right ideal of KL, so $E_L = \mathrm{End}_{KL}(\alpha KL)$ is a division algebra over K by Schur's lemma. If $F, L \in \mathcal{L} \cup \{G\}$ and $F \leqslant L$, then $\alpha KL \cong \alpha KF \otimes_{KF} KL$. Hence there is a K-algebra map $E_F \to E_L$, $\varphi \mapsto \varphi^L$, where for $\beta_i \in KF$, $\gamma_i \in KL$,

$$\begin{aligned} \varphi^L = \varphi \otimes_{KF} KL : \quad \alpha KF \otimes_{KF} KL &\longrightarrow \alpha KF \otimes_{KF} KL \\ \sum \alpha\beta_i \otimes \gamma_i &\longmapsto \sum \varphi(\alpha\beta_i) \otimes \gamma_i \quad . \end{aligned}$$

Since $\varphi^L|_{\alpha KF} = \varphi$, the map $\varphi \mapsto \varphi^L$ is one-to-one. Furthermore, if also $M \in \mathcal{L} \cup \{G\}$ and $F \leqslant L \leqslant M$, the diagram

$$\begin{tikzcd} E_F \arrow[r] \arrow[dr] & E_L \arrow[d] \\ & E_M \end{tikzcd}$$

commutes, since if $\varphi \in E_F$ then

$$\begin{aligned} (\varphi \otimes_{KF} KL) \otimes_{KL} KM &= \varphi \otimes_{KF} (KL \otimes_{KL} KM) \\ &= \varphi \otimes_{KF} KM \quad . \end{aligned}$$

Thus the E_F $(F \in \mathcal{L})$ and the $E_F \to E_L$ form a directed system

of K-algebras and K-algebra maps.

<u>Lemma 11.1</u> $E_G = \varinjlim \{E_F : F \in \mathcal{L}\}$ is the direct limit of this system.

<u>Proof</u> It remains to be shown that given a K-algebra A and K-algebra maps $\theta_F : E_F \to A$ ($F \in \mathcal{L}$) such that all diagrams

$$\begin{tikzcd} E_F \arrow[r] \arrow[dr, "\theta_F"'] & E_L \arrow[d, "\theta_L"] \\ & A \end{tikzcd} \qquad (F, L \in \mathcal{L}\ ;\ F \leq L)$$

commute, there is a unique map $\theta : E_G \to A$ making all diagrams

$$\begin{tikzcd} E_F \arrow[r] \arrow[dr, "\theta_F"'] & E_G \arrow[d, "\theta"] \\ & A \end{tikzcd} \qquad (F \in \mathcal{L})$$

commute. Thus let $\varphi \in E_G$. Then $\varphi(\alpha) \in \alpha KG$, so since G is locally finite we may choose $F \in \mathcal{L}$ with $\varphi(\alpha) \in \alpha KF$. Then $\varphi|_{\alpha KF} \in E_F$, and we have $(\varphi|_{\alpha KF})^G = \varphi$, since both are elements of E_G mapping α to $\varphi(\alpha)$, so they agree on αKG. Now define $\theta(\varphi) = \theta_F(\varphi|_{\alpha KF})$. This is independent of the choice of F by the commutativity of the first diagram above; for the same reason, θ is a K-algebra map. If $F \in \mathcal{L}$ and $\psi \in E_F$, then

$$\theta(\psi^G) = \theta_F(\psi^G|_{\alpha KF}) = \theta_F(\psi) ,$$

so the second diagram above commutes. To show that θ is unique, suppose that $\zeta : E_G \to A$ is another K-algebra map making

$$\begin{tikzcd} E_F \arrow[r] \arrow[dr, "\theta_F"'] & E_G \arrow[d, "\zeta"] \\ & A \end{tikzcd} \qquad (F \in \mathcal{L})$$

commute. If $\varphi \in E_G$ then choosing F as above

$$\zeta(\varphi) = \zeta((\varphi|_{\alpha KF})^G) = \theta_F(\varphi|_{\alpha KF}) = \theta(\varphi) ,$$

so $\zeta = \theta$ as required.

We remark that this result may be generalized: if H is any finite subgroup of G, $\mathcal{L}$ is as above, and V is a finitely generated KH-module, then

$$\mathrm{End}_{KG}(V^G) = \varinjlim \{\mathrm{End}_{KF}(V^F) : F \in \mathcal{L}\} .$$

Lemma 11.1 enables us to reduce certain questions concerning E_G to the corresponding questions about E_F, an improvement since F is finite. This is illustrated in the following:

<u>Theorem 11.2</u> Let K be a field, G a locally finite group, αKG a minimal right ideal of KG, and E_G the division ring $\mathrm{End}_{KG}(\alpha KG)$. Then

(a) E_G is locally a finite-dimensional separable K-algebra;

(b) if char $K = p > 0$, E_G is a field;

(c) αKG is finite-dimensional over E_G.

<u>Proof</u> (a) By Lemma 11.1, any finite subset of E_G lies in the image of the map $E_F \to E_G$ for some $F \in \mathcal{L}$. Since the map is one-to-one, this image is a subalgebra of E_G isomorphic to E_F, so it is sufficient to prove that E_F is a finite-dimensional separable K-algebra.

Since E_F is isomorphic to a subalgebra of $KF/J(KF)$ and F is finite, E_F is finite-dimensional over K. By Theorem 9.6 there exists a finite separable extension L of K which is a splitting field for F. Then by Lemma 9.3, $E_F \otimes_K L \cong \mathrm{End}_{LF}(\alpha LF)$. But αKF is irreducible so αLF is completely reducible by Lemma 2.8(b). Since every irreducible LF-module has endomorphism ring L, we see that $\mathrm{End}_{LF}(\alpha LF)$ is isomorphic to a direct sum of full matrix rings over L. Hence E_F is a separable K-algebra by Proposition 8.5.

(b) (This part, which is well known in the finite case, is a modification of [5; Lemmas 8 and 9].)

Since E_G is a division ring, we need only show that it is commutative. By Lemma 11.1 any two elements of E_G lie in a subalgebra isomorphic to E_F for some $F \in \mathcal{L}$, so we may assume that $G = F$ is finite.

Let $\mathbb{F}_p$ be the prime field of K. Since $J(\mathbb{F}_pG)$ is nilpotent, we have $J(\mathbb{F}_pG).K \leq J(KG)$. On the other hand, by proposition 8.4 $\mathbb{F}_pG/J(\mathbb{F}_pG)$ is a separable $\mathbb{F}_p$-algebra, so

$$KG/J(\mathbb{F}_pG).K \cong (\mathbb{F}_pG/J(\mathbb{F}_pG)) \otimes_{\mathbb{F}_p} K$$

is semisimple, and $J(KG) \leq J(\mathbb{F}_pG).K$. By Wedderburn's theorem on finite division algebras, $\mathbb{F}_pG/J(\mathbb{F}_pG)$ is a direct sum of matrix rings over fields. If L is one of these fields then by Proposition 8.4 again L is a separable $\mathbb{F}_p$-algebra, so

$L \otimes_{\mathbb{F}_p} K$ is semisimple and therefore a direct sum of fields.

Hence

$$KG/J(KG) = KG/J(\mathbb{F}_pG).K \cong (\mathbb{F}_pG/J(\mathbb{F}_pG)) \otimes_{\mathbb{F}_p} K$$

is also a direct sum of matrix rings over fields. Thus $E_G = \mathrm{End}_{KG/J(KG)}(\alpha KG)$ is a field.

(c) (cf. [11]) If $F \in \mathcal{L}$ then by Wedderburn's (other) theorem the dimension of αKF over E_F is equal to the multiplicity of αKF as a right-module direct summand of $KF/J(KF)$. Hence

$$\begin{aligned} \dim_{E_F} \alpha KF &\leq \dim_K(KF/J(KF)) / \dim_K \alpha KF \\ &\leq |F| / (|F:H| \dim_K \alpha KH) \\ &\leq |H| . \end{aligned}$$

We now show that also $\dim_{E_G} \alpha KG \leq |H| = n$ say, i.e. that any $\beta_1, \ldots, \beta_{n+1} \in \alpha KG$ are linearly dependent over E_G. For there exists $F \in \mathcal{L}$ with $\beta_1, \ldots, \beta_{n+1} \in \alpha KF$, and then there exist $\varphi_1, \ldots, \varphi_{n+1} \in E_F$ (not all zero) with $\sum_{i=1}^{n+1} \varphi_i(\beta_i) = 0$. Applying the K-algebra monomorphism $E_F \to E_G$, $\varphi \mapsto \varphi^G$, and recalling that $\varphi^G|_{\alpha KF} = \varphi$, we see that there exist $\varphi_1^G, \ldots, \varphi_{n+1}^G \in E_G$ (not all zero) with $\sum_{i=1}^{n+1} \varphi_i^G(\beta_i) = 0$, as required.

<u>Corollary 11.3</u> Let K be a field and G a locally finite group with $So(KG) \neq 0$. Then G contains a finite normal subgroup C such that G/C is linear over a division ring (which is a field if char $K > 0$).

<u>Proof</u> Let αKG be a minimal right ideal of KG. Then

$C = C_G(\alpha KG)$ is a normal subgroup of G, and acts faithfully (by right multiplication) as a group of permutations of the finite set $\operatorname{supp} \alpha$, so is finite. Now G/C acts faithfully on αKG, which by Theorem 11.2(c) is a finite-dimensional vector space over the division ring $E_G = \operatorname{End}_{KG}(\alpha KG)$; that is, G/C is linear over E_G. If char $K > 0$, E_G is a field by Theorem 11.2(b).

12. Structure of G

In this section we obtain necessary and sufficient conditions for KG to contain minimal one-sided ideals when G is a locally finite group and K is any field. We commence by singling out the most difficult step.

Theorem 12.1 Let K be a field and G a locally finite group. If $So(KG) \neq 0$ then G is a Černikov group.

Proof We show first that any residually finite subgroup H of G is finite. For let $\{H_\lambda\}$ be the set of all normal subgroups of H of finite index. The intersection of any two such subgroups is a third, and $\bigcap H_\lambda = 1$; so by Lemma 9.2, $\bigcap \underline{\underline{h}}_\lambda H = 0$. By Lemma 2.3(a), $\bigcap \underline{\underline{h}}_\lambda G = 0$, so as $So(KG) \neq 0$ is contained in every essential right ideal, $\underline{\underline{h}}_\lambda G$ is not essential in KG_{KG} for some λ. By Lemma 2.4, H_λ is finite, and therefore H is too.

If B is any abelian subgroup of G, then $\Omega(B)$ is a direct product of elementary abelian groups, so is residually finite, so finite by the above. By Theorem 3.1, B satisfies Min.

It follows by Šunkov's theorem (7.2) that G is a Černikov group.

When K has positive characteristic, it is possible

to avoid this appeal to Šunkov's theorem (the proof of which relies on many of the deepest results of finite group theory); instead we use an approach similar to that of [16; 3.2]. Thus suppose char $K>0$. Let αKG be a minimal right ideal of KG, and $A = r(\alpha KG)$ its right annihilator (a two-sided ideal). By Theorem 11.2, αKG is of finite dimension n say over $E_G = \mathrm{End}_{KG}(\alpha KG)$, which is a field. Each element of KG acts E_G-linearly on αKG by right multiplication, so there is a K-algebra map $KG \to \mathrm{End}_{E_G}(\alpha KG)$. This map has kernel A, so KG/A embeds in (and by the Jacobson density theorem is even isomorphic to) $\mathrm{End}_{E_G}(\alpha KG) \cong M_n(E_G)$. Thus by Lemma 8.7, KG/A satisfies a polynomial identity. By Theorem 9.12, since A ($\neq KG$) is an annihilator ideal, we have $|G:\Delta(G)| < \infty$ and $|\Delta(G)'| < \infty$.

Let $C = C_{\Delta(G)}(\Delta(G)')$. Then C' ($\leq \Delta(G)'$) is central in C, so C is nilpotent of class 2. From above, every abelian subgroup B of C satisfies Min, so C is a Černikov group by Lemma 7.3. Now $\Delta(G)/C$ acts as a group of automorphisms of $\Delta(G)'$, so is finite; hence C has finite index in G, and G too is a Černikov group.

We now deduce the necessary and sufficient conditions sought.

<u>Theorem 12.2</u> Let K be a field with prime field k and G a locally finite group. Then KG contains minimal right ideals if and only if:

(a) G is a Černikov group with characteristic divisible abelian subgroup A of finite index;

(b) char $K \notin \pi(A)$;

(c) A is locally cyclic; and

(d) $|k(A) \cap K : k| < \infty$.

<u>Proof</u> In view of Theorem 12.1 we may restrict our attention to groups G satisfying (a). Since by Lemma 2.5(b) $So(KG) \neq 0$ if and only if $So(KA) \neq 0$, it suffices to show that $So(KA) \neq 0$ if and only if A satisfies (b), (c) and (d).

Suppose $So(KA) \neq 0$, and let αKA be a minimal (right) ideal. If char $K = p > 0$ then by Lemma 9.9, $O_p(A)$ is contained in $C_A(\alpha KA)$, which is finite (since it acts faithfully on supp α). Since A is divisible, $O_p(A) = 1$, i.e. (b) holds (as of course it does if char $K = 0$). By Theorem 5.6, (d) holds, and A is almost locally cyclic; since it is divisible, we have (c).

Conversely, if (b), (c) and (d) hold, then $So(KA) \neq 0$ by Theorem 5.6.

13. Structure of KG

In this section we investigate the structure of the group ring KG when K is a field and G a locally finite group such that $So(KG) \neq 0$. In the light of Theorem 12.2, we introduce the following hypothesis, which will be assumed (except where specifically noted) throughout Section 13.

Hypothesis 13.1 K is a field with prime field k and characteristic $p \geqslant 0$, and G is a Černikov group with characteristic divisible abelian subgroup A of finite index n. The group A satisfies $p \notin \pi(A)$ and has a direct decomposition

$$A = P_1 \times \ldots \times P_m \qquad (m \geqslant 0)$$

where the P_i are Prüfer groups for distinct primes p_i. Finally, $|k(A) \cap K : k| < \infty$, so $So(KG) \neq 0$.

Lemma 13.2 Let M be a right KA-module.

(a) If M is irreducible, M^G has composition length at most $n = |G:A|$.

(b) If M is completely reducible, $So_n(M^G) = M^G$.

Proof (a) Since $A \trianglelefteq G$, $M^G|_A$ is a direct sum of n irreducible KA-modules, so has composition length n. A fortiori, M^G has composition length at most n.

(b) Since So_n and $-^G$ preserve direct sums, we may assume M irreducible. The result then follows from (a).

The bound on the 'Loewy height' of M^G in (b) may be improved: a Maschke-type argument shows that n may be replaced by $|G/A : O_{p'}(G/A)|$. A similar remark applies to the next lemma.

<u>Lemma 13.3</u> If V is any right KG-module then

$$So_{n(m+1)}(V) = V .$$

In particular, if $V \neq 0$ then So(V) is essential in V.

<u>Proof</u> Since the first property in question is inherited by images and direct sums, it is sufficient to verify it for $V = KG_{KG}$. By Lemma 13.2(b), if $i \geqslant 0$ then

$$\frac{So_{i+1}(KA)KG}{So_i(KA)KG} \cong \left.\frac{So_{i+1}(KA)}{So_i(KA)}\right|^G$$

has a series of length n with completely reducible factors. By Corollary 6.4, $So_{m+1}(KA) = KA$. Hence KG_{KG} has a series of length n(m+1) with completely reducible factors.

If $V \neq 0$ and W is a non-zero submodule of V, then $So_{n(m+1)}(W) = W$, so $W \cap So(V) = So(W) \neq 0$. Hence So(V) is essential in V.

We shall write

$$S_i = So_i(KA)KG \qquad (0 \leqslant i \leqslant m+1) ,$$

so that each S_i is an ideal of KG, and $S_{m+1} = KG$. We now show that each factor S_{i+1}/S_i (considered as a ring, generally without unit element) of the series

$$0 = S_0 \leqslant S_1 \leqslant \ldots \leqslant S_{m+1} = KG$$

is a direct sum of quasi-Frobenius rings. Recall that a <u>centrally primitive idempotent</u> in a ring is a primitive idempotent of the centre of the ring.

<u>Theorem 13.4</u> For $0 \leqslant i \leqslant m$,

(a) if ε is a centrally primitive idempotent in KG/S_i then $\varepsilon(KG/S_i)$ is a quasi-Frobenius ring;

(b) $S_{i+1}/S_i = \bigoplus\left\{\varepsilon(KG/S_i) : \varepsilon \text{ is a centrally primitive idempotent in } KG/S_i\right\}$.

<u>Proof</u> Let $Q = KG/S_i$ and $R = KA/So_i(KA)$. We preface the proof with three observations. Firstly, consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & So_i(KA) & \longrightarrow & KA & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & So_i(KA)|^G & \longrightarrow & KA|^G & \longrightarrow & R^G & \longrightarrow & 0 \\
 & & \| & & \| & & \wr\| & & \\
0 & \longrightarrow & So_i(KA)KG & \longrightarrow & KG & \longrightarrow & Q & \longrightarrow & 0
\end{array} .$$

Here the first row is exact, so the second row, obtained from the first by tensoring with the flat module ${}_{KA}KG$, is also exact; in other words, $R^G \cong Q$ as KG-modules. The vertical arrows are KA-module embeddings of the form

$$M \cong M \otimes 1 \longrightarrow M \otimes_{KA} KG = M^G .$$

Now the first two vertical maps are K-algebra morphisms, so the embedding $R \to R^G \cong Q$ is also a K-algebra morphism; we shall identify R with its image in Q under this embedding.

Secondly, suppose that M is any KA-module and $m \in M$, so that mKA is a submodule of M. Since ${}_{KA}KG$ is flat, $mKA|^G$ is a submodule of M^G, and we have

$$\begin{aligned} mKA|^G &= mKA \otimes_{KA} KG = m \otimes_{KA} KG \\ &= (m \otimes 1)KG = mKG \leqslant M^G . \end{aligned}$$

Thirdly, suppose that e is a primitive idempotent in R. Now G acts on KA by conjugation, leaving $So_i(KA)$ invariant, so G acts on R. Let T be a right transversal in G to $N_G(e) = \{g \in G : e^g = e\}$; then $|T| \leqslant n$ since $A \leqslant N_G(e)$. Let $\hat{e} = \sum_{x \in T} e^x$; then $\hat{e}$ is independent of the choice of T, and (since distinct primitive idempotents in R are orthogonal) is an idempotent in R. By the first observation above, we may consider $\hat{e}$ and each e^x as idempotents in Q; since G leaves $\hat{e}$ invariant, $\hat{e}$ is central in Q. In the KA-module R, we have

$$\hat{e}KA = \bigoplus_{x \in T} e^x KA ;$$

therefore, by the second observation above (taking $M = R$, $m = \hat{e}$)

$$\hat{e}Q = \hat{e}KG = \hat{e}KA|^G = \bigoplus_{x \in T} e^x KA|^G .$$

Now R is an epimorphic image of KA, so is locally Wedderburn (Lemma 3.8); thus by Lemma 3.9(b), $e^x KA = e^x R$ is irreducible for each $x \in T$. Hence by Lemma 13.2(b), $\hat{e}Q$ has composition length (as right KG-module) at most n^2. Similarly $\hat{e}Q$ has

finite composition length as left KG-module. Since $\hat{e}Q$ is an epimorphic image of the ring KG, and its KG- and $\hat{e}Q$-submodule lattices coincide, it follows that $\hat{e}Q$ is an artinian ring. Furthermore, each $e^x KA$ is an injective KA-module (Lemmas 9.14, 3.8, 8.10), so by Lemma 9.4(b), $\hat{e}Q$ is injective as right KG-module. Since any $\hat{e}Q$-module may be considered as a KG-module, we conclude that $\hat{e}Q$ is right self-injective, and therefore a quasi-Frobenius ring.

We now turn to the proof of the theorem. Let ε be a centrally primitive idempotent in Q. By Lemma 13.3, there exists non-zero $\alpha \in So(\varepsilon Q|_{KG})$. Then by Lemma 2.3(d),

$$\alpha \in So(Q_{KG}) = So(R^G) \leq So(R)^G ,$$

so by Lemma 3.9(d) there is a primitive idempotent $e \in R$ such that in Q we have $e\alpha \neq 0$. Since $\hat{e}e = e$ and $\varepsilon\alpha = \alpha$ we have $\hat{e}\varepsilon \neq 0$, whence $\hat{e}\varepsilon = \varepsilon$ as ε is centrally primitive. Hence εQ is a ring direct summand of $\hat{e}Q$, so is quasi-Frobenius. Thus we have (a). Furthermore,

$$\hat{e} \in So(R) = So(KA/So_i(KA)) = So_{i+1}(KA) / So_i(KA) \quad :$$

say $\hat{e} = \beta + So_i(KA)$ where $\beta \in So_{i+1}(KA) \leq S_{i+1}$. Then in Q,

$$\varepsilon = \hat{e}\varepsilon = (\beta + S_i)\varepsilon \in S_{i+1}/S_i .$$

To complete the proof of (b), note that by Lemma 3.9(d) So(R) is the direct sum of subrings $\hat{e}R$ as e runs over a system $\mathcal{E}$ of representatives of the G-conjugacy classes of primitive idempotents in R. Hence (using the second observ-

ation above)

$$S_{i+1}/S_i = So(R)^G = \bigoplus\{\hat{e}Q : e \in \mathcal{E}\} .$$

Each $\hat{e}Q$ is artinian, so may be written in the form $\varepsilon_1 Q \oplus \ldots \oplus \varepsilon_s Q$ where the ε_j are centrally primitive idempotents in Q.

<u>Theorem 13.5</u> Let K be any field and G any locally finite group. Then So(KG) is a direct sum of minimal (two-sided) ideals.

<u>Proof</u> We may assume that $So(KG) \neq 0$, and hence that Hypothesis 13.1 holds. Let Q be a homogeneous component of $So(KG_{KG})$. Then Q is an ideal, and by Proposition 10.1, ${}_{KG}Q$ is completely reducible. Let P be a homogeneous component of ${}_{KG}Q$, again an ideal. As So(KG) is the direct sum of such ideals P, it is sufficient to show that P is a direct sum of minimal ideals.

Now P_{KG} is a direct sum of copies of some minimal right ideal V. By Theorem 13.4(b), as

$$V \leqslant So(KG) \leqslant So(KA)KG = S_1 ,$$

there is a centrally primitive idempotent $\varepsilon \in KG$ with $V\varepsilon = V$. Then $P = P\varepsilon \leqslant \varepsilon KG$, which is artinian by Theorem 13.4(a). Hence P_{KG} is a direct sum of finitely many copies of V. Similarly ${}_{KG}P$ is a direct sum of finitely many copies of some minimal left ideal W. Let $B = KG/Ann_{KG}(V)$, $C = KG/Ann_{KG}(W)$, and let

C^{op} be the opposite ring of C. Then P considered as a KG-bimodule has the same structure as P considered as a right $B \otimes_K C^{op}$-module, so it is sufficient to show that the latter module is completely reducible.

As G is almost abelian KG satisfies a polynomial identity (Lemma 9.11). Hence B is primitive and satisfies a polynomial identity, so by Theorem 8.8 is isomorphic to a matrix ring $M_t(E)$ over $E = \mathrm{End}_B(V) = \mathrm{End}_{KG}(V)$. Similarly, $C^{op} \cong M_u(F)$ say, where $F = \mathrm{End}_{KG}(W)$. By Theorem 11.2(a), each of E and F is locally a finite-dimensional separable K-algebra. By Theorem 8.6 the tensor product of separable algebras is semisimple, so $E \otimes_K F$ is a locally Wedderburn algebra. Hence $B \otimes_K C^{op} \cong M_{tu}(E \otimes_K F)$ is also locally Wedderburn. Let G^{op} denote the opposite group of G. Then $B \otimes_K C^{op}$ is an epimorphic image of $KG \otimes_K KG^{op} \cong K[G \times G^{op}]$, which satisfies a polynomial identity as $G \times G^{op}$ is almost abelian. The conclusion now follows from Corollary 8.9, since P has a composition series as B-module so a fortiori as $B \otimes_K C^{op}$-module.

Theorem 13.4 has another consequence (which can also be demonstrated more directly: see [20; 3.2]). Note that KG is semiprime if and only if G has no finite normal subgroup of order divisible by the characteristic of K (cf. Theorem 14.4).

<u>Corollary 13.6</u> Let K be any field and G any locally finite group such that KG is semiprime. Then $So(KG) \neq 0$ if and only if KG has a ring direct summand which is isomorphic to a full matrix ring over a division ring D.

<u>Proof</u> If ε is a central idempotent in KG such that $\varepsilon KG \cong M_t(D)$, then $0 \neq So(\varepsilon KG) \leq So(KG)$. Conversely, if $So(KG) \neq 0$ we may assume Hypothesis 13.1, and then by Theorem 13.4 (with $i = 0$) KG contains a centrally primitive idempotent ε such that εKG is quasi-Frobenius. Then εKG is semiprime (like KG) and artinian, and contains no central idempotents other than ε. Hence εKG is isomorphic to a matrix ring $M_t(D)$ over a division ring D.

We remark that if K has positive characteristic then by Theorem 11.2(b) D is necessarily a field. In any case, if $So(KG) \neq 0$ then by Theorem 12.1 G is almost abelian, so D satisfies a polynomial identity (Lemma 9.11), and is therefore finite-dimensional over its centre ([18; 5.3.4] or [15; 6.4]).

14. A characterization of the socle

We now turn to the problem of finding an explicit characterization of the socle of KG when G is locally finite. Since no such characterization is known in the case of a finite group, the expression we obtain (in Theorem 14.9) involves the socle of a finite-group algebra. A major step towards this expression is the determination (in Theorem 14.8) of the 'controller' of the socle. The concept of the controller of an ideal in a group ring was introduced by Passman [18;§8.1]; for convenience we shall prove two of his results, on which the idea is based.

If H is a subgroup of a group G and K is any field, it is easy to see that the map

$$\pi_H : KG \longrightarrow KH, \qquad \sum_{g\in G} \lambda_g g \mapsto \sum_{g\in H} \lambda_g g \qquad (\lambda_g \in K)$$

is a KH-bimodule homomorphism.

<u>Lemma 14.1</u> Let K be a field, H a normal subgroup of a group G, and I an ideal of KG. Then

$$(I \cap KH)KG \leqslant I \leqslant \pi_H(I)KG .$$

Furthermore, if either inclusion is an equality then both are.

<u>Proof</u> (cf. [18; 1.1.5, 1.1.6]) The first inclusion is clear. Suppose $\alpha \in I$, and let T be a transversal to H in G. Then α may be written in the form

$$\alpha = \sum_{x\in T} \alpha_x x \qquad (\alpha_x \in KH) .$$

If $x,y \in T$ then $\pi_H(xy^{-1}) = \delta_{x,y}$. Since π_H is a left KH-module map, we have

$$\pi_H(\alpha y^{-1}) = \sum_{x\in T} \alpha_x \pi_H(xy^{-1}) = \alpha_y .$$

Thus $$\alpha = \sum_{x\in T} \pi_H(\alpha x^{-1})x \in \pi_H(I)KG ,$$

since $\alpha x^{-1} \in Ix^{-1} = I$. This establishes the second inclusion.

If $I = \pi_H(I)KG$ then $\pi_H(I) \subseteq I \cap KH$, whence

$$I = \pi_H(I)KG \leqslant (I \cap KH)KG .$$

Conversely, if $I = (I \cap KH)KG$ then

$$\pi_H(I) = (I \cap KH)\pi_H(KG) \subseteq I ,$$

whence $I \geqslant \pi_H(I)KG$.

When $(I \cap KH)KG = I = \pi_H(I)KG$, we say that H <u>controls</u> I.

<u>Lemma 14.2</u> [18; 8.1.1] Let K be a field, G a group, and I an ideal of KG. Then there exists a unique normal subgroup $\mathcal{C}(I)$ of G such that $H \trianglelefteq G$ controls I if and only if $H \geqslant \mathcal{C}(I)$.

<u>Proof</u> Let W be the intersection of all normal subgroups of G which control I. We shall show that $\pi_W(I) \subseteq I$. Let $\alpha \in I$ and suppose

$$\operatorname{supp}\alpha - W = \{g_1,\ldots,g_n\} \qquad (0 \leqslant n < \infty) .$$

For each $i = 1,\ldots,n$ there exists a normal subgroup H_i controlling I such that $g_i \notin H_i$. Then

$$\begin{aligned}\pi_W(\alpha) &= \pi_{H_1 \cap H_2 \cap \ldots \cap H_n}(\alpha) \\ &= \pi_{H_1}(\pi_{H_2}(\ldots \pi_{H_n}(\alpha)\ldots)) \\ &\in I \ ,\end{aligned}$$

since $\pi_{H_i}(I) \subseteq I$ for each i. By Lemma 14.1, W controls I, and is therefore clearly the unique minimal controlling subgroup for I.

If H is any normal subgroup of G containing W then

$$I \geqslant (I \cap KH)KG \geqslant (I \cap KW)KG = I \ ,$$

so H controls I. The result now follows with $\mathcal{C}(I) = W$.

The subgroup $\mathcal{C}(I)$ is called the <u>controller</u> of the ideal I. We shall need:

<u>Lemma 14.3</u> Let I be an ideal of KG and $L = \ell(I)$ its left annihilator. Then $\mathcal{C}(L) \leqslant \mathcal{C}(I)$.

<u>Proof</u> It is enough to show that $H = \mathcal{C}(I)$ controls L, i.e. that $\pi_H(L) \subseteq L$. Now

$$\begin{aligned}\pi_H(L)I &= \pi_H(L)\pi_H(I)KG && \text{since H controls I} \\ &= \pi_H(L.\pi_H(I))KG && \text{since } \pi_H \text{ is a right KH-map} \\ &\leqslant \pi_H(LI)KG && \text{since H controls I} \\ &= 0 \ ,\end{aligned}$$

so $\pi_H(L) \subseteq \ell(I) = L$.

Passman has determined the controller of the nilpotent radical N(KG) of KG:

<u>Theorem 14.4</u> Let K be a field of characteristic $p \geq 0$ and G any group. Then $\mathcal{C}(N(KG)) = \Delta^p(G)$, where

$$\Delta^p(G) = \langle x \in \Delta(G) : |x| \text{ is a power of } p \rangle .$$

<u>Proof</u> When $p > 0$ this is [18; 8.1.9(i)]. When $p = 0$, $N(KG) = 0$ [18; 4.2.13] so $\mathcal{C}(N(KG)) = 1 = \Delta^p(G)$.

Since if G is finite the socle and the nilpotent radical of KG are each other's annihilators, it follows from Lemma 14.3 that in this case $\Delta^p(G)$ is also the controller of the socle. When G is merely locally finite, the situation is more complicated, since in the light of condition (d) of Theorem 12.2, we must expect $\mathcal{C}(So(KG))$ to depend on K itself and not just on the characteristic. However, this dependence turns out to be rather crude: for a group G satisfying conditions (a)-(c) of Theorem 12.2, $\mathcal{C}(So(KG))$ can take only two values - 1 (iff K is so large that $So(KG) = 0$) or $A\Delta^p(G)$. Before investigating this we prove two general lemmas.

<u>Lemma 14.5</u> Let K be a field and G a group. Suppose $So(KG)$ is essential in KG_{KG}, and controlled by $H \trianglelefteq G$. Then

$$So(KG) = So(KH)KG ,$$

and

$$So(KG) \cap KH = So(KH) \text{ ess } KH_{KH}.$$

<u>Proof</u> By Lemma 2.3(b), since $(So(KG) \cap KH)KG = So(KG)$, $So(KG) \cap KH$ is essential in KH_{KH}, so contains $So(KH)$. Thus

$So(KG) \geqslant So(KH)KG$, and equality holds by Lemma 2.3(d). Hence also $So(KG) \cap KH = So(KH)KG \cap KH = So(KH)$.

<u>Lemma 14.6</u> If K is a field of characteristic $p \geqslant 0$ and G is a group, then the finite-p' residual

$$\bigcap \{ N \trianglelefteq G : p \nmid |G:N| < \infty \}$$

of G controls So(KG).

<u>Proof</u> By Lemmas 2.7 and 14.2.

For the remainder of this section, we again assume Hypothesis 13.1: in view of Theorem 12.2, this assumption entails no loss of generality.

<u>Lemma 14.7</u> $\Delta^p(G)$ is finite.

<u>Proof</u> We may easily reduce to the case where $G = \Delta^p(G)$. In particular G is an FC-group, so its minimal subgroup A of finite index is central. If x and y are p-elements of G with $xA = yA$, then there exists $g \in A$ with $xg = y$. Since g is a central p'-element, $\langle x \rangle = \langle y \rangle$. But G is generated by its p-elements, so may be generated by $|G:A|$ (or fewer) elements. Hence G is finite.

<u>Theorem 14.8</u> Assume Hypothesis 13.1, and let $D = \Delta^p(G)$. Then

$$\mathcal{C}(So(KG)) = AD.$$

Moreover,
$$So(KG) = So(K[AD])KG$$
and
$$So(KG) \cap K[AD] = So(K[AD]) \quad .$$

<u>Proof</u> We show first that $J(KG)$ is the left annihilator of $So(KG)$. Certainly in view of Proposition 10.1 we have $J(KG).So(KG) = 0$. For the converse it is sufficient to show that $\ell(So(KG))$ is a nil ideal. Thus let $\alpha \in \ell(So(KG))$ and put $H = \langle \operatorname{supp} \alpha \rangle$. Now $r_{KG}(\alpha) \geqslant So(KG)$, so by Lemma 13.3, $r_{KG}(\alpha)$ is essential in KG_{KG}. By Lemma 9.1(a), $r_{KG}(\alpha) = r_{KH}(\alpha)KG$; hence $r_{KH}(\alpha)$ is essential in KH_{KH} (Lemma 2.3(b)) so contains $So(KH)$. But H is finite, so by Corollary 9.5 and Proposition 8.2, $So(KH)$ contains a copy of every irreducible left KH-module. It follows that $\alpha \in J(KH)$, whence α is nilpotent as required.

However, by Lemma 9.7 and Theorem 9.8, since G is almost abelian-p', $J(KG)$ is nilpotent. Thus

$$\ell(So(KG)) = J(KG) = N(KG) .$$

Hence by Theorem 14.4 and Lemma 14.3,

$$D = \mathcal{C}(N(KG)) \leqslant \mathcal{C}(So(KG)) .$$

Recall that $A = P_1 \times \ldots \times P_m$, where the P_i are Prüfer groups. By Lemma 2.4, $\underline{\underline{p}}_i G$ is essential in KG_{KG}, whence $So(KG) \leqslant \underline{\underline{p}}_i G$. Let $C = \mathcal{C}(So(KG))$ and $T = C \cap P_i$; then

$$So(KG) \cap KC \leqslant \underline{\underline{p}}_i G \cap KC = \underline{\underline{t}} C$$

(where the equality holds since $\underline{\underline{p}}_i G$ is the set of elements of KG whose coefficient sum on each right coset of P_i is zero). Hence by Lemmas 13.3 and 14.5, $\underline{\underline{t}} C$ is essential in KC_{KC}, whence $\underline{\underline{t}}$ is essential in KT by Lemma 2.3(b). By Lemma

2.2, T is not a finite p'-group, so must be infinite. Therefore $P_i = T \leq C$. We have now shown that $AD \leq \mathcal{C}(So(KG))$.

We next prove that $C_G(A) = H$ say controls $So(KG)$. Let I be a minimal ideal of KG. Since A has no proper subgroup of finite index, it follows from Lemma 9.10 that $I \cap KH \neq 0$. Hence as I is minimal, $(I \cap KH)KG = I$, so $\pi_H(I) \subseteq I$ by Lemma 14.1. Since So(KG) is a direct sum of minimal ideals (Theorem 13.5), we have $\pi_H(So(KG)) \subseteq So(KG)$, i.e. H controls So(KG) as required.

Since A is the minimal subgroup of finite index in G, and abelian,

$$H = C_G(A) = \Delta(G) \geq AD .$$

Furthermore, H/AD is a finite p'-group since $D = \Delta^p(G)$ contains all p-elements of H. Hence AD controls So(KH) by Lemma 2.7. By Lemma 14.5 twice we now have

$$So(KG) = So(KH)KG = So(K[AD])KG$$

and $$So(K[AD]) = So(KH) \cap K[AD] = So(KG) \cap K[AD] .$$

Thus AD controls So(KG), and the proof is complete.

We are now ready to give our characterization of the socle of KG.

<u>Theorem 14.9</u> Assume Hypothesis 13.1 and let $D = \Delta^p(G)$. Then

$$\begin{aligned} So(KG) &= So(KA)So(KD)KG \\ &= \left(\bigcap_{i=1}^{m} \underline{p}_i A\right) So(KD)KG . \end{aligned}$$

<u>Proof</u> Note that the second equality holds by Theorem 6.2.

Let $\gamma \in So(KG)$. By Lemma 2.3(d), $\gamma \in So(KA)KG$, so by Lemma 3.9(d) there is an idempotent e (not necessarily primitive) in So(KA) with $e\gamma = \gamma$. By Lemma 2.3(d) again, $\gamma \in So(KD)KG$, so

$$\gamma = e\gamma \in So(KA)So(KD)KG \quad .$$

It remains to be shown that $So(KA)So(KD)KG \leqslant So(KG)$. By Theorem 14.8, $So(KG) = So(K[AD])KG$, so we may assume that $G = AD$. Since by Lemma 14.7 D is finite, there exists a finite separable extension F of K which is a splitting field for D (Theorem 9.6). By Lemma 2.8(b) we have

$$\begin{aligned} So(KA)So(KD)KG &= So(KA)So(KD)FG \cap KG \\ &= So(FA)So(FD)FG \cap KG \end{aligned}$$

and $$So(FG) \cap KG = So(KG) \quad ,$$

so we may assume that $K = F$. Let M and N be minimal right ideals of KA and KD respectively; we must show that $MN \leqslant So(KG)$.

We claim that $M \otimes_K N$ is a minimal right ideal of $KA \otimes_K KD \cong K[A \times D]$. Let V be a non-zero submodule of $M \otimes_K N$: say $\sum m_i \otimes n_i \in V$, where $\{n_i\}$ is a (finite) K-basis of N, $m_i \in M$, and $m_1 \neq 0$. As K is a splitting field for D, $End_{KD}(N) = K$, so by the Jacobson density theorem the map $KD \to End_K(N)$ is onto. Hence for each j there exists $\delta_j \in KD$ with $n_1\delta_j = n_j$ and

$n_i\delta_j = 0$ $(i \neq 1)$. Thus for each j, $m_1 \otimes n_j \in V$. As $M = m_1KA$, clearly $V = M \otimes_K N$ as required.

Since $G = AD$ and $D \leq \Delta(G) = C_G(A)$, there is a K-algebra epimorphism $\theta: KA \otimes_K KD \to KG$, induced by $a \otimes d \mapsto ad$ $(a \in A,\ d \in D)$. Thus $MN = \theta(M \otimes_K N)$ is either a minimal right ideal of KG or zero, and is contained in So(KG) in either case.

15. Indecomposable modules

In this section we classify indecomposable KG-modules when K is a field and G a locally finite group such that $So(KG) \neq 0$, in a manner which generalizes the classification into blocks of indecomposable modules for a finite-group algebra. We also describe the injective and projective indecomposable KG-modules. To conclude the section, we consider a more general question: for arbitrary K and G, when is every indecomposable KG-module irreducible?

In view of Theorem 12.2, we shall again assume Hypothesis 13.1, until further notice. As in Section 13, we set

$$S_i = So_i(KA)KG \qquad (0 \leq i \leq m+1) \quad .$$

Proposition 15.1 Let M be an indecomposable right KG-module.

(a) There exists a unique integer $\lambda = \lambda(M) \in \{0,\ldots,m\}$ such that $MS_\lambda = 0$ but $MS_{\lambda+1} = M$.

(b) There exists a unique centrally primitive idempotent $\varepsilon \in KG/S_\lambda$ such that $M\varepsilon = M$.

(c) If M is injective than M has finite composition length and is isomorphic to a direct summand of $(KG/S_\lambda)_{KG}$; conversely each indecomposable direct summand of $(KG/S_\lambda)_{KG}$ is injective.

(d) If M is projective then M is also injective, and

$\lambda(M) = 0$. Thus the projective indecomposable KG-modules are exactly the indecomposable direct summands of KG_{KG}.

<u>Proof</u> Firstly we remark that if N is an indecomposable direct summand of $(KG/S_\lambda)_{KG}$ $(0 \leq \lambda \leq m)$ then by Lemmas 13.3 and 2.3(d), $0 \neq So(N) \leq So(KG/S_\lambda) \leq S_{\lambda+1}/S_\lambda$, whence by Theorem 13.4(b) there exists a centrally primitive idempotent η in KG/S_λ with $So(N)\eta \neq 0$. Since $N = N\eta \oplus N(1-\eta)$ is indecomposable, $N = N\eta$ is a direct summand of $\eta(KG/S_\lambda)$. In particular, N like $\eta(KG/S_\lambda)$ is an injective KG-module of finite composition length (see the proof of Theorem 13.4(a)); furthermore $NS_{\lambda+1} = N$ since $\eta \in S_{\lambda+1}/S_\lambda$.

(a,b) Let λ be the greatest integer such that $MS_\lambda = 0$; then $\lambda \leq m$ as $S_{m+1} = KG$. Now M may be considered as a KG/S_λ-module, and $M(S_{\lambda+1}/S_\lambda) \neq 0$. Thus there exists a centrally primitive idempotent $\varepsilon \in KG/S_\lambda$ with $M\varepsilon \neq 0$, and then $M = M\varepsilon$ since M is indecomposable. Hence $MS_{\lambda+1} = M$. The uniqueness of λ and ε is clear.

(c) Since $\varepsilon(KG/S_\lambda)$ is an epimorphic image of KG, M is injective (as well as indecomposable) when considered as an $\varepsilon(KG/S_\lambda)$-module. By Proposition 8.3, M is isomorphic to a right direct summand of $\varepsilon(KG/S_\lambda)$, and hence to a direct summand of $(KG/S_\lambda)_{KG}$. The remaining assertions of (c) follow from the above remark.

(d) M is projective and indecomposable when considered as an $\varepsilon(KG/S_\lambda)$-module, and hence by Proposition 8.3 M is cyclic, as $\varepsilon(KG/S_\lambda)$- or KG-module. Thus M is isomorphic to a direct summand of KG_{KG}. By the above remark, M is injective, and $MS_1 = M$, whence $\lambda(M) = 0$.

For an irreducible KG-module M we can provide an alternative characterization of the integer $\lambda(M)$.

<u>Proposition 15.2</u> Let M be an irreducible right KG-module and i an integer with $0 \leq i \leq m$. Then the following are equivalent:

(a) $i = \lambda(M)$;

(b) M is isomorphic to a submodule of $(S_{i+1}/S_i)_{KG}$;

(c) M is isomorphic to a composition factor of $(S_{i+1}/S_i)_{KG}$;

(d) the kernel $C_A(M)$ of M in A contains exactly i of the Prüfer direct factors $P_1, \ldots, P_m$ of A.

<u>Proof</u> (a) $\Rightarrow$ (b) We have $MS_i = 0$ but $MS_{i+1} = M$, whence M is an irreducible KG/S_i-module with $M(S_{i+1}/S_i) = M$. By Theorem 13.4(b) there is a centrally primitive idempotent $\varepsilon \in KG/S_i$ with $M\varepsilon = M$. Then M is an irreducible $\varepsilon(KG/S_i)$-module, so by Proposition 8.2 (since $\varepsilon(KG/S_i)$ is quasi-Frobenius), M is isomorphic to a right ideal of $\varepsilon(KG/S_i)$, whence to a submodule of $(S_{i+1}/S_i)_{KG}$.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (a) Suppose $M \cong U/V$ where $S_i \leq V \leq U \leq S_{i+1}$. Since $US_i \leq S_i \leq V$, we have $MS_i \cong (U/V)S_i = 0$, so $\lambda(M) \geq i$. If $u \in U/S_i$ then by Theorem 13.4(b) there exist distinct (and therefore orthogonal) centrally primitive idempotents $\varepsilon_1, \ldots, \varepsilon_k$ in KG/S_i with

$$u = u\varepsilon_1 + \ldots + u\varepsilon_k \in (U/S_i)(S_{i+1}/S_i) ,$$

since each ε_j lies in S_{i+1}/S_i. Thus $US_{i+1} = U$, whence $MS_{i+1} \cong (U/V)S_{i+1} = U/V \neq 0$, and $\lambda(M) \leq i$.

(a) $\Leftrightarrow$ (d) Note that (a) holds if and only if i is the greatest integer such that $MS_i = 0$, i.e. such that $S_i \leq \mathrm{Ann}_{KG}(M)$. Since M is irreducible, $\mathrm{Ann}_{KG}(M)$ is a prime ideal. By Corollary 6.3,

$$S_i = So_i(KA)KG = \bigcap_{|I|=i} \sum_{j \in I} \underline{p}_j G .$$

Hence $S_i \leq \mathrm{Ann}_{KG}(M)$ if and only if for at least i values of j, $\underline{p}_j G \leq \mathrm{Ann}_{KG}(M)$, i.e. $P_j \leq C_A(M)$.

We now cease to assume Hypothesis 13.1, and consider, for arbitrary K and G, the question of when all indecomposable KG-modules are irreducible. In [1; Theorem 2.7] Berman shows that it is sufficient for G to be periodic abelian and K non-modular for G. We extend his result in the following:

<u>Theorem 15.3</u> Let G be a periodic almost abelian group and K a field with char $K \notin \pi(G)$. Then every indecomposable KG-module is irreducible.

<u>Proof</u> Let A be a normal abelian subgroup of finite index in G, and V an indecomposable right KG-module.

Suppose F is a finite normal subgroup of G contained in A, and e is a primitive idempotent in KF. As in the proof of Theorem 13.4, we let T be a right transversal in G to $N_G(e) = \{g \in G : e^g = e\}$; then T is finite since $A \leqslant N_G(e)$. Let $\hat{e} = \sum_{x \in T} e^x$; then $\hat{e}$ is independent of the choice of T, central in KG, and (since the e^x are distinct primitive idempotents in KF, so orthogonal) an idempotent. Since G/A is finite, we may choose, among all finite F in A normal in G and all primitive idempotents e in KF satisfying $V\hat{e} \neq 0$, an F and an e with $N_G(e)$ minimal. Since V is indecomposable, $V\hat{e} = V$, so $\hat{e}$ acts as the identity on V.

Let

$$\mathcal{L} = \{L \leqslant A : F \leqslant L \trianglelefteq G,\ |L| < \infty\} ;$$

since $|G:A|$ is finite, every finite subset of A lies in some member of $\mathcal{L}$. We shall construct primitive idempotents f_L in KL ($L \in \mathcal{L}$) to which we may apply Lemma 5.2. Let $L \in \mathcal{L}$, and consider the various idempotents in KL of the form $\hat{f}$, where f is a primitive idempotent in KL. Since these idempotents are central in KG, and have sum 1, and since V is indecomposable, there is exactly one such idempotent, say η, such that $V\eta \neq 0$. Then η acts as the identity on V, so $\hat{e}\eta \neq 0$.

If $e\eta = 0$, then for $x \in G$, $0 = (e\eta)^x = e^x\eta$, whence $\hat{e}\eta = 0$, a contradiction. Thus $e\eta \neq 0$, so since KL is semisimple artinian we may choose a primitive idempotent f_L in KL with $f_L e\eta \neq 0$. In particular $f_L\eta \neq 0$; since η is the sum of some G-conjugacy class of primitive idempotents in KL, it follows that $\hat{f}_L = \eta$, so $\hat{f}_L$ acts as the identity on V. Also $f_L e \neq 0$, whence

$$f_L e = f_L$$

as f_L is primitive; hence $N_G(e) \geq N_G(f_L)$. For if $g \in G$ and $f_L^g = f_L$, then $f_L e^g = (f_L e)^g = f_L^g = f_L = f_L e$, so $e^g e \neq 0$ whence $e^g = e$. By the minimality of $N_G(e)$, we have $N_G(e) = N_G(f_L)$.

Suppose $L_1, L_2 \in \mathcal{L}$ with $L_1 \leq L_2$. Then $\hat{f}_{L_1}$ and $\hat{f}_{L_2}$ both act as the identity on V, so $\hat{f}_{L_1}\hat{f}_{L_2} \neq 0$. Hence $f_{L_1}\hat{f}_{L_2}$ (like $e\eta$ above) is also non-zero. Thus for some $x \in G$, $f_{L_1} f_{L_2}^x \neq 0$, and then $f_{L_1} f_{L_2}^x = f_{L_2}^x$ as $f_{L_2}^x$ is primitive in KL_2. Since $f_{L_2} e = f_{L_2}$ (from above), we have

$$f_{L_1} f_{L_2}^x e^x = f_{L_1} f_{L_2}^x \neq 0 ,$$

so $f_{L_1} e^x \neq 0$ whence $f_{L_1} e^x = f_{L_1}$. But from above $f_{L_1} e = f_{L_1}$, so $ee^x \neq 0$ whence $x \in N_G(e) = N_G(f_{L_2})$. Thus

$$f_{L_1} f_{L_2} = f_{L_1} f_{L_2}^x \neq 0 ,$$

whence $f_{L_1} f_{L_2} = f_{L_2}$.

Now given any $L_1, L_2 \in \mathcal{L}$, let $L_1 L_2 \leq L \in \mathcal{L}$. Then

$f_{L_1} f_L = f_L = f_{L_2} f_L$, so $f_{L_1} f_{L_2} \neq 0$. Thus we may apply Lemma 5.2 to obtain a maximal ideal M of KA such that for all $L \in \mathcal{L}$,

$$M \cap KL = (1 - f_L)KL \quad .$$

Let T be a (finite) right transversal to $N_G(e)$ in G. We claim that

$$\mathrm{Ann}_{KA}(V) \geqslant \bigcap_{x \in T} M^x \quad .$$

For let $\alpha \in \bigcap_{x \in T} M^x$, and say $\mathrm{supp}\,\alpha \subseteq L \in \mathcal{L}$. Then for $x \in T$,

$$\begin{aligned} \alpha \in M^x \cap KL &= (M \cap KL)^x \\ &= (1 - f_L^x)KL \quad , \end{aligned}$$

so $f_L^x \alpha = 0$. But $N_G(f_L) = N_G(e)$, so $\hat{f}_L = \sum_{x \in T} f_L^x$, whence $\hat{f}_L \alpha = 0$. Since $\hat{f}_L$ acts as the identity on V, we have $\alpha \in \mathrm{Ann}_{KA}(V)$.

Thus $KA/\mathrm{Ann}_{KA}(V)$ is an image of the completely reducible KA-module $KA / \bigcap_{x \in T} M^x$, so is a semisimple artinian K-algebra. Thus its module V_A is completely reducible. By Lemma 2.6(a) V is completely reducible as KG-module; since V is indecomposable, it is irreducible.

We now consider necessary conditions for indecomposable KG-modules to be irreducible, commencing with:

<u>Lemma 15.4</u> Let K be a field and G a group such that every indecomposable KG-module is irreducible. Then G is locally finite and char $K \notin \pi(G)$.

<u>Proof</u> The injective hull of the trivial KG-module K is indecomposable so irreducible; that is, K is injective. Now use Lemma 9.13.

When G is countable, we can establish necessary and sufficient conditions. The following result extends a theorem of Hartley:

<u>Theorem 15.5</u> If K is a field and G is a countable group, the following are equivalent:

(a) G is periodic and almost abelian, and char $K \notin \pi(G)$;

(b) every indecomposable KG-module is irreducible;

(c) every irreducible KG-module is injective.

<u>Proof</u> (The equivalence of (a) and (c) is [10; Theorem A].)

(a) $\Rightarrow$ (b) This is Theorem 15.3.

(b) $\Rightarrow$ (c) If (b) holds then by Lemmas 15.4 and 3.8, KG is locally Wedderburn, so Theorem 8.11 applies. But (b) precludes alternative (ii) of that theorem from occurring, so we have (c).

(c) $\Rightarrow$ (a) [10] Given (c), Lemma 9.13 shows that G is locally finite and char $K \notin \pi(G)$. By (c) and Theorem 8.11, every irreducible KG-module has finite endomorphism dimension, so G is almost abelian by Theorem 9.15.

Chapter V

NON-LOCALLY-FINITE GROUPS

16. A conjecture

In this chapter we investigate the existence of minimal right ideals in group rings of groups which are not locally finite. The results we shall obtain all provide evidence in support of

Conjecture 16.1 Let G be a non-locally-finite group and K a field. Then $So(KG) = 0$.

In Section 17 we show that this conjecture is valid for certain group classes, in particular for a class of generalized FC-soluble groups, which includes all radical and all locally soluble groups (Theorem 17.3), and for free products (Proposition 17.4). We also show that if K has characteristic p ($\geqslant 0$) then residually finite-p' groups G satisfy Conjecture 16.1 (Proposition 17.5); we deduce that groups linear over a field of characteristic zero or not equal to p also satisfy the conjecture (Corollary 17.7).

A ring is called (right) semiartinian if every non-zero right module has non-zero socle. Recalling from Lemma 13.3 that if a group ring of a locally finite group has non-zero socle then it is semiartinian, we are led to consider a

weaker form of Conjecture 16.1:

<u>Conjecture 16.2</u> If K is a field and G a group such that KG is semiartinian, then G is locally finite.

We establish some special cases of this second conjecture in Theorem 18.4.

In Section 17 we shall employ the notation of group-theoretical classes and operations (see [21; Section 1.1]). The group classes we mention include the following:

$\underline{\underline{F}}$	:	finite groups
$\underline{\underline{F}}_{p'}$	:	finite p'-groups (where p is a prime)
$\underline{\underline{F}}_{0'}$	:	finite groups
$\underline{\underline{A}}$	:	abelian groups
$\underline{\underline{B}}$	:	FC-groups
(G)	:	the class of all groups isomorphic to a fixed group G, together with all trivial groups.

We shall use a number of group-theoretical operations. If $\underline{\underline{X}}$ is a group class, we define the following group classes:

$L\underline{\underline{X}}$	:	locally-$\underline{\underline{X}}$ groups (i.e. groups in which every finite subset lies in an $\underline{\underline{X}}$-subgroup)
$R\underline{\underline{X}}$	:	residually-$\underline{\underline{X}}$ groups
$P\underline{\underline{X}}$	:	groups with an ascending (transfinite) series with each factor in $\underline{\underline{X}}$
$S\underline{\underline{X}}$	:	subgroups of groups in $\underline{\underline{X}}$.

Each of these operations is a closure operation, i.e. satisfies $A^2\underline{\underline{X}} = A\underline{\underline{X}}$ for all $\underline{\underline{X}}$. We also require the closure operation $<\acute{P},L>$, whose closed classes are the classes which are both $\acute{P}$- and L-closed [21; p. 5].

We shall need an easy lemma concerning products of group classes:

<u>Lemma 16.3</u> If $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ are group classes then $\underline{\underline{Y}}.L\underline{\underline{X}} \subseteq L(\underline{\underline{Y}}\underline{\underline{X}})$.

<u>Proof</u> Let $G \in \underline{\underline{Y}}.L\underline{\underline{X}}$, so that G has a normal subgroup $H \in \underline{\underline{Y}}$ with $G/H \in L\underline{\underline{X}}$. If $\{g_1,\ldots,g_n\}$ is a finite subset of G then $\{g_1H,\ldots,g_nH\}$ $(\subseteq G/H)$ is contained in some $\underline{\underline{X}}$-subgroup W/H of G/H. Then $\{g_1,\ldots,g_n\} \subseteq W \in \underline{\underline{Y}}\underline{\underline{X}}$ as required.

17. Some well-behaved group classes

Abusing set-theoretic notation, we define

$$\underline{\underline{S}} = \llcorner\underline{\underline{F}} \cup \{G : So(KG) = 0 \text{ for all fields } K\} ;$$

thus Conjecture 16.1 is true if and only if $\underline{\underline{S}}$ is the class of all groups.

Lemma 17.1 $\llcorner\underline{\underline{F}}.\underline{\underline{A}} \subseteq \underline{\underline{S}}$.

Proof Suppose that $G \in \llcorner\underline{\underline{F}}.\underline{\underline{A}} - \underline{\underline{S}}$, so that $G \notin \llcorner\underline{\underline{F}}$ contains a locally finite normal subgroup H with G/H abelian, and there exists a field K with $So(KG) \neq 0$. Since $G/H \notin \llcorner\underline{\underline{F}}$, G contains an element x of infinite order modulo H. Now $\langle x,H\rangle \trianglelefteq G$, so by Lemma 2.3(d) we may assume $G = \langle x,H\rangle = \langle x\rangle H$. By Lemma 2.3(d) again, $So(KH) \neq 0$. Hence by Theorem 12.2, H contains a locally cyclic subgroup A of finite index such that char $K \notin \pi(A)$. Then also $\langle x,A\rangle$ has finite index in G, so by Lemma 2.5(b) we may assume that $H = A$. Then in particular every subgroup of H is characteristic, and char $K \notin \pi(G)$.

By Lemma 14.6, since $G/H \cong \langle x\rangle$ is residually finite-p' for any p, H controls $So(KG)$, so there exists non-zero $\alpha \in So(KG) \cap KH$. Then αKG is completely reducible and cyclic, so has the minimum condition on KG-submodules. Since $\langle \text{supp } \alpha\rangle$ is a finite characteristic subgroup of H, there exists $r > 0$ such that $\alpha^{x^r} = \alpha$. Now

$$\alpha KG \geqslant \alpha(x^r - 1)KG \geqslant \dots \geqslant \alpha(x^r - 1)^t KG \geqslant \dots ,$$

so by the minimum condition $\alpha(x^r-1)^t KG = \alpha(x^r-1)^{t+1}KG$ for some $t \geqslant 0$. Then (since α and x^r commute) there exists $\gamma \in KG$ with $(x^r-1)^t\alpha = (x^r-1)^{t+1}\alpha\gamma$.

Let $W = \langle x^r \rangle$. Then if $0 \neq \delta \in \ell_{KW}(x^r-1) = \ell_{KW}(\underline{\underline{W}})$ (by Lemma 1.1), we find as in the proof of Lemma 2.2 that $\operatorname{supp}\delta = W$, which is impossible. Thus x^r-1 is regular in KW, so too in KG (by Lemma 9.1(b)). Hence $\alpha = (x^r-1)\alpha\gamma$. Since $\alpha \neq 0$, also $\alpha\gamma \neq 0$: write

$$\alpha\gamma = \sum_{n=M}^{N} x^n\beta_n \qquad (M \leqslant N;\ \beta_n \in KH;\ \beta_M, \beta_N \neq 0) \quad .$$

Then
$$KH \ni \alpha = (x^r - 1)\sum_{n=M}^{N} x^n\beta_n$$
$$= x^{r+N}\beta_N + \ldots - x^M\beta_M ,$$

where we have shown only the greatest and least powers of x. Hence $r+N = M = 0$; so $r = M-N \leqslant 0$, a contradiction.

The following rather technical lemma allows us to improve on Lemma 17.1.

<u>Lemma 17.2</u> Let $\underline{\underline{X}}$ be a group class such that $L\underline{\underline{F}}.\underline{\underline{X}} \subseteq \underline{\underline{S}}$. Then

(a) $L\underline{\underline{F}}.L\underline{\underline{X}} \subseteq \underline{\underline{S}} = L\underline{\underline{S}}$;

(b) $\underline{\underline{S}}\underline{\underline{X}} \subseteq \underline{\underline{S}}$;

(c) $L\underline{\underline{F}}.\acute{P}\underline{\underline{X}} \subseteq \underline{\underline{S}}$;

(d) if $\underline{\underline{X}} = s\underline{\underline{X}}$ then $\underline{\underline{S}}.\langle \acute{P}, L\rangle\underline{\underline{X}} \subseteq \underline{\underline{S}}$.

<u>Proof</u> (a) By Lemma 16.3, $L\underline{\underline{F}}.L\underline{\underline{X}} \subseteq L(L\underline{\underline{F}}.\underline{\underline{X}}) \subseteq L\underline{\underline{S}}$, so it is sufficient to show that $L\underline{\underline{S}} = \underline{\underline{S}}$. Thus let $G \in L\underline{\underline{S}}$ and suppose

there exists a field K with $So(KG) \neq 0$: we must show that $G \in L\underline{\underline{F}}$. Let $g_1, \ldots, g_n \in G$ and $0 \neq \alpha \in So(KG)$, and put $H = \langle g_1, \ldots, g_n, \operatorname{supp} \alpha \rangle$. Since $G \in L\underline{\underline{S}}$, there exists $L \leqslant G$ with $H \leqslant L \in \underline{\underline{S}}$. Now $0 \neq \alpha \in So(KG) \cap KL$, so by Lemma 2.5(a), $So(KL) \neq 0$. Hence $L \in L\underline{\underline{F}}$, whence $\langle g_1, \ldots, g_n \rangle \in \underline{\underline{F}}$ as required.

(b) Let $G \in \underline{\underline{SX}}$: say $H \trianglelefteq G$ with $H \in \underline{\underline{S}}$, $G/H \in \underline{\underline{X}}$. Suppose K is a field with $So(KG) \neq 0$; then by Lemma 2.3(d), $So(KH) \neq 0$, so $H \in L\underline{\underline{F}}$. Hence $G \in L\underline{\underline{F}}.\underline{\underline{X}} \subseteq \underline{\underline{S}}$.

(c) Let $G \in L\underline{\underline{F}}.\flat\underline{\underline{X}}$, so that there is an ordinal ρ and an ascending series

$$G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_\alpha \trianglelefteq G_{\alpha+1} \trianglelefteq \ldots \trianglelefteq G_\rho = G$$

such that $G_0 \in L\underline{\underline{F}}$ and $G_{\alpha+1}/G_\alpha \in \underline{\underline{X}}$ for all $\alpha < \rho$. We proceed by induction on ρ. Suppose first that ρ is not a limit ordinal; then by induction $G_{\rho-1} \in \underline{\underline{S}}$, so $G \in \underline{\underline{SX}} \subseteq \underline{\underline{S}}$ by (c). Now assume that $\rho(>0)$ is a limit ordinal, and let H be a finitely generated subgroup of G. Then $H \leqslant G_\alpha$ for some $\alpha < \rho$, and by induction $G_\alpha \in \underline{\underline{S}}$. Hence $G \in L\underline{\underline{S}} = \underline{\underline{S}}$ by (a).

(d) Let

$$\underline{\underline{T}} = \{G : L\underline{\underline{F}}.s(G) \subseteq \underline{\underline{S}}\}$$

(where $s(G)$ is the class of groups isomorphic to subgroups of G). Then clearly $s\underline{\underline{T}} = \underline{\underline{T}}$ and $L\underline{\underline{F}}.\underline{\underline{T}} \subseteq \underline{\underline{S}}$. Let $G \in L\underline{\underline{T}}$. Then

$$L\underline{\underline{F}}.s(G) \subseteq L\underline{\underline{F}}.sL\underline{\underline{T}} \subseteq L\underline{\underline{F}}.LS\underline{\underline{T}} = L\underline{\underline{F}}.L\underline{\underline{T}} \subseteq \underline{\underline{S}}$$

by (a). Hence $\mathfrak{L}\underline{\underline{T}} = \underline{\underline{T}}$. Similarly,

$$\mathfrak{L}\underline{\underline{F}}.\mathfrak{S}\acute{\mathfrak{P}}\underline{\underline{T}} \subseteq \mathfrak{L}\underline{\underline{F}}.\acute{\mathfrak{P}}\mathfrak{S}\underline{\underline{T}} = \mathfrak{L}\underline{\underline{F}}.\acute{\mathfrak{P}}\underline{\underline{T}} \subseteq \underline{\underline{S}}$$

by (c), whence $\acute{\mathfrak{P}}\underline{\underline{T}} = \underline{\underline{T}}$. We conclude that $\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{T}} = \underline{\underline{T}}$.

Now $\mathfrak{L}\underline{\underline{F}}.\mathfrak{S}\underline{\underline{X}} = \mathfrak{L}\underline{\underline{F}}.\underline{\underline{X}} \subseteq \underline{\underline{S}}$, so $\underline{\underline{X}} \subseteq \underline{\underline{T}}$. Therefore

$$\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{X}} \subseteq \langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{T}} = \underline{\underline{T}} \quad .$$

In particular, $\mathfrak{L}\underline{\underline{F}}.\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{X}} \subseteq \underline{\underline{S}}$. Thus by (b), $\underline{\underline{S}}.\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{X}} \subseteq \underline{\underline{S}}$.

<u>Theorem 17.3</u> If $\underline{\underline{B}}$ is the class of FC-groups, then

$$\underline{\underline{S}}.\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{B}} \subseteq \underline{\underline{S}} \quad .$$

<u>Proof</u> In an FC-group the periodic elements form a locally finite normal subgroup with abelian quotient group (Lemma 7.1). Hence $\mathfrak{L}\underline{\underline{F}}.\underline{\underline{B}} = \mathfrak{L}\underline{\underline{F}}.\underline{\underline{A}}$, and the theorem follows from Lemmas 17.1 and 17.2(d).

We remark that the class $\langle\acute{\mathfrak{P}},\mathfrak{L}\rangle\underline{\underline{B}}$ contains, for example, all radical (i.e. hyper-(locally nilpotent)) groups, and all locally (FC-)soluble or (FC-)hypercentral groups.

<u>Proposition 17.4</u> Let $G = A*B$ be the free product of non-trivial groups A and B. Then $G \in \underline{\underline{S}}$.

<u>Proof</u> If K is any field then by Lemma 9.16 KG is strongly prime. Hence by Lemma 8.12, $\mathfrak{Z}o(KG)$ equals 0 or KG. The latter case is impossible by Lemma 2.2, as G is infinite.

We now consider residually finite groups. By definition, $\underline{\underline{F}}_{0'} = \underline{\underline{F}}$.

<u>Proposition 17.5</u> Let K be a field of characteristic $p \geq 0$, and $G \in {}_R\underline{\underline{F}}_{p'}$. If $So(KG) \neq 0$ then $G \in \underline{\underline{F}}$.

<u>Proof</u> By Lemma 14.6, the identity subgroup of G controls $So(KG)$, i.e. $So(KG) = (So(KG) \cap K)KG$. Thus $So(KG) \cap K \neq 0$, whence $K \subseteq So(KG)$, i.e. $So(KG) = KG$. Hence G is finite by Lemma 2.2.

<u>Corollary 17.6</u> If p and q are distinct primes, then

$$ {}_R\underline{\underline{F}}_{p'} \cap {}_R\underline{\underline{F}}_{q'} \subseteq \underline{\underline{S}} \ . $$

It follows both from Proposition 17.4 and from Corollary 17.6 that free groups lie in $\underline{\underline{S}}$.

<u>Corollary 17.7</u> Let G be a linear group over a field F.

(a) If char $F = 0$ then $G \in \underline{\underline{S}}$.

(b) If there is a field K with char $K \neq$ char F and $So(KG) \neq 0$, then $G \in {}_L\underline{\underline{F}}$.

<u>Proof</u> In case (a) suppose K is any field with $So(KG) \neq 0$. In either case put $q = \text{char } K$ (≥ 0). By Theorem 7.4, $G \in {}_L({}_R\underline{\underline{F}}_{q'} \cdot \underline{\underline{F}})$. If $g_1, \ldots, g_n \in G$ and $0 \neq \alpha \in So(KG)$ then

$$ H = \langle g_1, \ldots, g_n, \text{ supp } \alpha \rangle \in {}_R\underline{\underline{F}}_{q'} \cdot \underline{\underline{F}} \ , $$

and $So(KH) \neq 0$ by Lemma 2.5(a). Thus $H \in \underline{\underline{F}}$ by Lemma 2.3(d) and Proposition 17.5. Hence $G \in {}_L\underline{\underline{F}}$.

18. Semiartinian group rings

We now consider semiartinian group rings. Firstly, we note that Handelman and Lawrence [7] prove, for a field K and a group G, that if KG is strongly prime then G has no non-trivial locally finite normal subgroup; they conjecture that the converse also holds. If this is correct then Conjecture 16.2 is a consequence. For suppose KG is semiartinian, and let L(G) be the product of all locally finite normal subgroups of G. Then L(G) is locally finite, and

$$L(G/L(G)) = 1 .$$

Now $K[G/L(G)]$ is an image of KG so has non-zero socle; if it is strongly prime we conclude from Lemmas 8.12 and 2.2 that G/L(G) is finite, whence G is locally finite.

Semiartinian rings may be characterized in terms of their transfinite ascending Loewy series. For a (right) module V we define $So_0(V) = 0$, and

$$So_{\alpha+1}(V)/So_\alpha(V) = So(V/So_\alpha(V)) ,$$

$$So_\lambda(V) = \bigcup_{\beta<\lambda} So_\beta(V)$$

for any ordinal α and any limit ordinal λ. Note that the property $So_\alpha(V) = V$ is equivalent to the condition that V has an ascending series of type α with completely reducible factors, so is inherited by submodules, images, and direct sums.

<u>Lemma 18.1</u> The ring R is semiartinian if and only if $So_\alpha(R_R) = R$ for some ordinal α.

<u>Proof</u> If α is an ordinal of cardinal larger than $|R|$ then $So_{\alpha+1}(R) = So_\alpha(R)$, i.e. $So(R/So_\alpha(R)) = 0$, so if R is semiartinian then $R/So_\alpha(R) = 0$. Conversely, if $So_\alpha(R) = R$ then we see that $So_\alpha(V) = V$ first for free and then for arbitrary right R-modules V. Thus if $V \neq 0$ then $So(V) \neq 0$.

<u>Lemma 18.2</u> Given a group G and a field K, suppose for some ordinal α that $So_\alpha(\underline{\underline{g}}_{KG}) = \underline{\underline{g}}$. Then either G is locally finite or $So_\alpha(KG_{KG}) = KG$.

<u>Proof</u> Suppose that G is not locally finite, so that there exist $g_1, \ldots, g_n \in G$ such that $H = \langle g_1, \ldots, g_n \rangle$ is infinite. The obvious map

$$\varphi : KG_{KG} \longrightarrow \bigoplus_{i=1}^{n} (g_i - 1)KG$$

has kernel $r_{KG}(\{g_1 - 1, \ldots, g_n - 1\}) = r_{KG}(G\underline{\underline{h}})$ (by Lemma 1.1). If $0 \neq \gamma \in r_{KG}(G\underline{\underline{h}})$ we find (as in Lemma 2.2) that $\operatorname{supp} \gamma \supseteq H$, a contradiction. Hence φ is a monomorphism. For each i, $g_i - 1 \in \underline{\underline{g}}$, so $So_\alpha((g_i - 1)KG) = (g_i - 1)KG$. It follows that $So_\alpha(KG_{KG}) = KG$.

<u>Lemma 18.3</u> Let K be a field and H a normal subgroup of a group G.

(a) If V is a right KH-module, α is an ordinal, and

$So_\alpha(V^G) = V^G$, then $So_\alpha(V) = V$.

(b) If KG is semiartinian then so is KH.

<u>Proof</u> (a) We show by induction that for all ordinals α, $So_\alpha(V^G) \leq So_\alpha(V)^G$: the desired result is an immediate consequence. The case $\alpha = 1$ is Lemma 2.3(d), and if α is a limit ordinal the proof is clear. Suppose that α is not a limit ordinal, and that $So_{\alpha-1}(V^G) \leq So_{\alpha-1}(V)^G$. Then

$$(So_\alpha(V^G) + So_{\alpha-1}(V)^G) / So_{\alpha-1}(V)^G$$

is an image of $So_\alpha(V^G)/So_{\alpha-1}(V^G)$, and is therefore completely reducible. Thus

$$\begin{aligned}
(So_\alpha(V^G) + So_{\alpha-1}(V)^G) / So_{\alpha-1}(V)^G &\leq So(V^G/So_{\alpha-1}(V)^G) \\
&= So((V/So_{\alpha-1}(V))^G) \\
&\leq So(V/So_{\alpha-1}(V))|^G \quad \text{by } 2.3(d) \\
&= (So_\alpha(V)/So_{\alpha-1}(V))|^G \\
&= So_\alpha(V)^G/So_{\alpha-1}(V)^G ,
\end{aligned}$$

and $So_\alpha(V^G) \leq So_\alpha(V)^G$ as required.

(b) This follows from part (a) and Lemma 18.1.

We can now prove some special cases of Conjecture 16.2, if we impose two rather stringent conditions on K.

<u>Theorem 18.4</u> Let G be a group and K an algebraically closed field with $|K| > |G|$. Suppose that KG is semiartinian, and that at least one of the following conditions holds:

(a) char $K = 0$; or

(b) G is periodic; or

(c) G is finitely generated and has no proper subgroup of finite index.

Then G is finite.

<u>Proof</u> Let α be the least ordinal such that $So_\alpha(KG) = KG$ (see Lemma 18.1). Since $1 \in KG$, α is not a limit ordinal. We proceed by induction on α. If $\alpha = 1$ then KG_{KG} is completely reducible, so G is finite by Lemma 2.2.

Thus suppose $\alpha > 1$, and let $T = So_{\alpha-1}(KG)$. Now KG/T is completely reducible: say

$$KG/T = V_1 \oplus \ldots \oplus V_r ,$$

where the V_i are irreducible right KG- (and KG/T-) modules, and r is finite since $1 \in KG$. Since KG/T is semisimple artinian, V_i is finite-dimensional over its endomorphism ring E_i for each i. By Lemma 9.17, since $|K| > |G|$, each E_i is algebraic over K; but K is algebraically closed, so $E_i = K$. Now $G/C_G(V_i)$ acts faithfully by right multiplication on the finite-dimensional K-space V_i, so is linear over K.

Let $H = C_G(KG/T) = \bigcap_{i=1}^{r} C_G(V_i)$. Then G/H embeds in the direct product of the groups $G/C_G(V_i)$, so is also linear over K. In case (a), since $So(K[G/H]) = So(KG/\underline{g}H) \neq 0$, G/H is locally finite by Corollary 17.7(a). In case (b), G/H is

locally finite by Theorem 7.5. In case (c), since G/H is finitely generated, G/H is (almost) residually finite by Theorem 7.4(b); thus $H = G$ since G has no proper subgroup of finite index.

Since H acts trivially on KG/T we have

$$\underline{\underline{h}}G \leq T = So_{\alpha-1}(KG) ,$$

whence $So_{\alpha-1}(\underline{\underline{h}}G) = \underline{\underline{h}}G$. By Lemma 18.3(a), $So_{\alpha-1}(\underline{\underline{h}}) = \underline{\underline{h}}$. Then by Lemma 18.2, either H is locally finite, or $So_{\alpha-1}(KH) = KH$. In the latter case H is actually finite, by induction on α. (Note that H satisfies the same hypotheses as G: KH is semi-artinian by Lemma 18.3(b); in case (c) we have already seen that $H = G$.)

Thus in any case both H and G/H are locally finite, so G is too. If k is the prime field of K and A is any infinite periodic abelian group with char $k \notin \pi(A)$, then $|k(A) \cap K : k| = |k(A) : k| = \infty$, since K is algebraically closed. Hence it follows from Theorem 12.2 that G is finite.

REFERENCES

1. S.D. Berman, 'Group algebras of countable abelian p-groups', Publ. Math. Debrecen 14 (1967) 365-405 (Russian).

2. N. Bourbaki, Algèbre, Chapter 8, Hermann, Paris, 1958.

3. W.D. Burgess, 'Rings of quotients of group rings', Canad. J. Math. 21 (1969) 865-875.

4. C.W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, 1962.

5. D.R. Farkas and R.L. Snider, 'Group algebras whose simple modules are injective', Trans. Amer. Math. Soc. 194 (1974) 241-248.

6. L. Fuchs, Infinite Abelian Groups, Volume I, Academic Press, New York and London, 1970.

7. D. Handelman and J. Lawrence, 'Strongly prime rings', Trans. Amer. Math. Soc. 211 (1975) 209-223.

8. J. Hannah and K.C. O'Meara, 'Maximal quotient rings of prime group algebras', Proc. Amer. Math. Soc. (to appear).

9. B. Hartley, 'A class of modules over a locally finite group I', J. Austral. Math. Soc. 16 (1973) 431-442.

10. B. Hartley, 'Injective modules over group rings', Quarterly J. Math. (1977).

11. B. Hartley and J.S. Richardson, 'The socle in group rings' (submitted).

12. N. Jacobson, Lectures in Abstract Algebra, Volume III - Theory of Fields and Galois Theory, Van Nostrand, Princeton, New Jersey, 1964.

13. O.H. Kegel and B.A.F. Wehrfritz, Locally Finite Groups, North-Holland, Amsterdam, 1973.

14. W. Müller, 'Radikal und Sockel in Gruppenalgebren über lokalendlichen Gruppen', Arch. Math. (Basel) 25 (1974) 476-482.

15. D.S. Passman, Infinite Group Rings, Marcel Dekker, New York, 1971.

16. D.S. Passman, 'Minimal ideals in group rings', Proc. Amer. Math. Soc. 31 (1972) 81-86.

17. D.S. Passman, 'The type I part of a group ring', Communications in Alg. 2 (1974) 469-473.

18. D.S. Passman, The Algebraic Structure of Group Rings, Wiley-Interscience, 1977.

19. J.S. Richardson, 'Primitive idempotents and the socle in group rings of periodic abelian groups', Compositio Math. 32 (1976) 203-223.

20. J.S. Richardson, 'Group rings with non-zero socle' (submitted).

21. D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Part 1, Springer-Verlag, Berlin, 1972.

22. B. Stenström, Rings of Quotients, Springer-Verlag, Berlin, 1975.

23. D.A.R. Wallace, 'The Jacobson radicals of the group algebras of a group and of certain normal subgroups', Math. Z. 100 (1967) 282-294.

24. B.A.F. Wehrfritz, Infinite Linear Groups, Springer-Verlag, Berlin, 1973.