IRREDUCIBLE MODULES AND THEIR INJECTIVE HULLS OVER GROUP RINGS.

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Contents

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Declaration.

I have not submitted any of the material of this thesis for any degree or qualification before. The results of chapter 2 are based on my recent paper in the Proceedings of the Cambridge Philosophical Society. Some of the results of chapter 3 have been proved independently by A.V. Jategaonkar and announced in the Notices of the American Mathematical Society.

Summary

If R is a ring, and V any R module, then there is a unique minimal injective module, $E_R(V)$ containing V. The module $E_R(V)$ is called the injective hull of V. Our aim in this thesis is to study the injective hull of irreducible modules over various group rings.

Chapter 1 contains some preliminary results which are used in later chapters. In chapter 2 we study the group algebra of a locally finite group G over a field k. A module E is said to be $\sum$-injective if any direct sum of copies of E is injective. We characterize $\sum$-injective kG modules and provide necessary and sufficient conditions for the injective hull of every irreducible kG module to be $\sum$-injective.

The remaining three chapters concern group rings, SG of polycyclic groups. If R is a commutative Noetherian ring, and V an irreducible R module, it is known that $E_R(V)$ is artinian. In chapters 3 and 4, we study analogues of this result. Chapter 3 covers all the cases where we know that $E_{SG}(V)$ is artinian, while in chapter 4, we examine situations in which $E_{SG}(V)$ is not artinian.

In fact we show that $E_{SG}(V)$ can fail to be locally artinian. This answers a question of Jategaonkar concerning (two sided) Noetherian rings.

The main result of chapter 5 concerns irreducible modules over polycyclic group algebras, kG. We shall show that any polycyclic-by-finite group G has a characteristic abelian-by-finite subgroup A, known as the plinth socle of G, such that, if V is an irreducible kG module, the restriction of V to A is generated by finite dimensional kA modules. The motivation is partly a theorem of P. Hall to which the above result reduces when G is nilpotent.

Also the condition arose quite naturally in chapter 4, and some applications are given to problems studied there.

A detailed introduction is given seperately for each of the chapters 2-5.

Chapter 1. Preliminary Results.

1. Introduction.

In this section we give an account of the basic definitions and theory which will be needed later. Roughly speaking the aim is to develop enough background so that the material on locally finite groups and that on polycyclic groups may be read independently. We assume that all rings have identity elements and all modules are unital. Unless otherwise stated all modules will be right modules.

2. Groups.

We shall be concerned mainly with two generalizations of finite groups, namely the classes of locally finite groups and of polycyclic-by-finite groups.

Definitions 2.1. Let G be a group and p a prime.

i) G is said to be periodic if every element of G has finite order.

ii) G is said to be locally finite if every finitely generated subgroup of G is finite.

iii) A set Σ of subgroups of G is a local system of G if $G = \bigcup_{S \in \Sigma} S$ and for every pair $S, T \in \Sigma$ there is a subgroup $U \in \Sigma$ such that $S, T \subseteq U$.

iv) An element $g \in G$ is a p element if the order of g is a power of p and a p' element if the order of g is prime to p. G is a p group (respectively p' group) if all its elements are p elements (p' elements).

v) If G is a locally finite group there exists a unique maximal normal p subgroup of G denoted by $O_p(G)$ and a unique maximal normal p' subgroup $O_{p'}(G)$. We refer the reader to (28) CH 1 §B for further details.

Since we will consider group algebras over fields of both positive characteristic and characteristic zero, it is convenient to regard zero as a prime in the statement of certain theorems. To do this we require the following definitions in the case of locally finite groups,

namely, any locally finite group is a O' group and the only O group is the identity group.

We shall need the following results on locally finite groups.

Theorem 2.2. A periodic linear group is locally finite.

Proof: See (28) 1.L.1.

Theorem 2.3. (Brauer-Feit) There exists an integer-valued function $f(n,p,|P|)$ of three variables such that any periodic linear group of degree n over a field ofcharacteristic $p \geqslant 0$, that contains a finite maximal p subgroup P , has an abelian normal p' subgroup of finite index bounded by $f(n,p,|P|)$.

Proof: (4).

Theorem 2.4. (Hall-Kulatilaka) Every infinite locally finite group contains an infinite abelian subgroup.

Proof: (28) 2.5.

Definition 2.5. Let $\mathfrak{X}, \mathfrak{Y}$ be classes of groups.

i) A group G is said to be an $\mathfrak{X}$-by-$\mathfrak{Y}$ group if there is a normal $\mathfrak{X}$ subgroup N of G such that G/N is a $\mathfrak{Y}$ group.

Thus we may speak for example of the classes of abelian-by-finite, nilpotent-by-finite and (abelian-by-nilpotent)-by-finite groups.

ii) A group G is said to be a poly$\mathfrak{X}$ group if there is a finite subnormal series $1 = G_0 \triangleleft G_1 \triangleleft \dots G_n = G$ such that G_{i+1}/G_i is an $\mathfrak{X}$ group for $i = 0,\dots n-1$.

We shall make frequent use of the following result.

Lemma 2.6. Any poly (cyclic or finite) group contains a characteristic subgroup of finite index which is poly (infinite cyclic).

Proof: (36) 10.2.5.

Hence we may refer to this class of groups as polycyclic-by-finite. If G is polycyclic-by-finite and $1 = G_0 \triangleleft G_1 \dots G_n = G$ is a subnormal series with G_{i+1}/G_i either finite or cyclic, then the number of quot-

ients G_{i+1}/G_i which are infinite cyclic is known to be an invariant, the Hirsch number, $h(G)$ of G.

Definitions 2.7.

i) If G is a group, the upper central series $\{Z_d(G)\}_{d \geq 1}$ is defined by the rules $Z_1(G) = Z(G)$, the centre of G and $Z_{d+1}(G)/Z_d(G) = Z(G/Z_d(G))$.

This series can of course be continued transfinitely, but we shall not need to do so.

ii) A group is said to be nilpotent if $Z_c(G) = G$ for some integer c and the least such c is called the class of G.

iii) If G is polycyclic-by-finite, there is a unique maximal finite normal subgroup the finite radical, $F(G)$ and a unique maximal nilpotent normal subgroup, the Fitting subgroup $Fit(G)$ of G. (These are consequences of the fact that G has the maximum condition on subgroups).

iv) If G is a polycyclic-by-finite group, the Zalesskii subgroup, $Zal(G)$ is defined by the rule $F(G) \subseteq Zal(G)$, $Zal(G)/F(G) = Z(Fit(G/F(G))$.

We assemble the following well known facts about nilpotent and polycyclic-by-finite groups.

Lemma 2.8. Let G be a nilpotent group.

i) If $N \triangleleft G$, $N \neq 1$, then $N \cap Z(G) \neq 1$.

ii) The finite radical, $F(G)$ contains all of the elements of finite order in G, and $G/F(G)$ is torsion free.

iii) If $Z(G)$ is torsion free, then the upper central factors $Z_{d+1}(G)/Z_d(G)$ are also torsion free.

iv) Every subgroup of G is subnormal in G.

v) If G is finitely generated, then G is polycyclic.

vi) If G is finitely generated, and $Z(G)$ is finite, then G is finite.

Proof: Most of these results can be proved by induction on the class of G. For i) we refer the reader to (36) 11.4.3, and for ii) and iii) to (36) 11.1.2 and 11.1.3, respectively. A proof of iv) can be found in (17), theorem 10.3.3, while v) and vi) are contained in (36), 11.4.3 and (39), theorem 2.24.

Lemma 2.9. Let G be a polycyclic-by-finite group.

i) If $N \triangleleft G$, $N \neq 1$, then $N \cap Zal(G) \neq 1$.

ii) If G is infinite then it has a characteristic infinite abelian torsion free subgroup.

iii) If Zal(G) is finite, then G is finite.

iv) G/Fit(G) is abelian-by-finite.

v) G is residually finite.

Proof: i) is a remark of Roseblade's (42), p. 390. For ii) see (36) 10.2.9, iii) is an easy consequence of ii) and the definition.

For iv) see (39) 3.25 and for v), (36) 10.2.11.

Definitions 2.10.

i) Let G be a finite group and p a prime, $p > 0$. Then G is said to be p soluble if for every homomorphic image $\overline{G}$ of G, either $O_p(\overline{G}) \neq 1$, or $O_{p'}(\overline{G}) \neq 1$. We say that G is p nilpotent if $G/O_{p'}(G)$ is a p group.

ii) A polycyclic-by-finite group is p nilpotent, provided every finite homomorphic image of G is p nilpotent.

In view of lemma 2.9 v) p nilpotence is a reasonably restrictive property. However we have the following result.

Lemma 2.11.

i) Any polycyclic-by-finite group contains a characteristic p nilpotent subgroup of finite index.

ii) Normal subgroups and homomorphic images of p nilpotent groups are p nilpotent.

Proof: For i) see (36) 11.2.6.

Clearly a homomorphic image of a p nilpotent group is p nilpotent.

Let H be a normal subgroup of the p nilpotent group G, and H/N a finite image of H with $|H:N| = n$. Then H^n is a characteristic subgroup of finite index in H, and $H^n \subset N$. Since $H^n \triangleleft G$, we may assume that $H^n = 1$. Since G is residually finite, there exists a normal subgroup K of finite index in G such that $K \cap H = 1$. Thus $H = H/H \cap K \cong HK/K \leq G/K$, and so H is p nilpotent.

3. Rings.

Definition 3.1. Let R be a ring.

i) The Jacobson radical, J(R) of R is the intersection of the maximal right ideals of R.

ii) R is said to be semisimple if J(R) = 0.

ii) We say that R is semilocal, local or scalar local if R/J(R) is semisimple artinian, simple artinian or a division ring, respectively.

Definition 3.2. A ring is locally artinian, provided every finitely generated subring is artinian.

Definition 3.3. A ring R is said to be (von Neumann) regular if either of the following equivalent conditions hold:

i) for $a \in R$, there exists $x \in R$, such that $axa = a$.

ii) every finitely generated right ideal of R is generated by an idempotent.

We refer the reader to (36) 3.1.3 for proof of the equivalence.

Definition 3.4. A Noetherian ring R is said to be polycentral if in every homomomorphic ring image of R, any non-zero ideal contains a non-zero central element.

Some examples of rings with some of these properties are given in the following result.

Proposition 3.5. Let k be a field of characteristic $p \geq 0$, G a group and kG the group ring of G over k.

i) kG is locally artinian if and only if G is locally finite.

ii) kG is regular if and only if G is a locally finite p' group.

iii) kG is a polycentral ring if and only if G is polycyclic-by-finite and is a finite p'-by-(nilpotent-by-finite p) group.

Proof: i) follows from Connell's result, (36) 10.1.1 that a group ring kG is artinian if and only if G is a finite group. For ii) see (36) 3.1.5, and for iii), (36) 11.3.12. Note that according to the conventions of § 2, the condition in iii) is that G is finite-by-

nilpotent and finitely generated if p is the prime zero.

Definition 3.6. Let I be an ideal in a ring R.

i) I is said to be nilpotent if $I^n = 0$, for some integer n.

ii) I is nil if each of its elements is nilpotent, that is for each $a \in I$, there exists an integer n such that $a^n = 0$.

Examples of such ideals are given by the Jacobson radical of a (locally) artinian ring.

Lemma 3.7. i) If R is an artinian ring, J(R) is nilpotent.

ii) If R is a locally artinian ring, J(R) is a nil ideal.

Proof: i) is well known, see (29) § 3.5 cor 1., and ii) follows from i).

Definition 3.8. If T is a subset of a ring R, we write r(T) for the right annihilator of T, $\{r \in R \mid Tr = 0\}$. The left annihilator of T, l(T) is defined similarly.

We shall use the appropriate underlined small letter to denote the augmentation ideal of a group ring. Thus if R = SG is a group ring of a group G over some ring S, and H is a subgroup of G, then $\underline{h}$ will denote the ideal of SH generated by the set $\{h-1 \mid h \in H\}$ and $\underline{h}G$ will denote the right ideal of R generated by $\underline{h}$.

If H is a finite subgroup of G, we shall write $\hat{H} = \sum_{h \in H} h \in SH$.

As an illustration of these concepts there is the following result.

Lemma 3.9. Let H be a finite subgroup of a group G. Then we have

i) $r(\underline{h}) = \hat{H}kG$

ii) $r(\hat{H}) = \underline{h}G$. Similar results hold for left annihilators.

Proof: See (36) 3.1.2.

Of course if H is a normal subgroup of a group G, then $\underline{h}G$ will be a two sided ideal of SG.

4. Modules.

Notation and definitions 4.1.

If R is a ring and W an R module, $\mathrm{End}_R(W)$ will denote the ring of R endomorphisms of W. Homomorphisms of modules will generally be denoted by Greek letters. If V is a submodule of W and $\varphi(V) \subseteq V$, for all $\varphi \in \mathrm{End}_R(W)$, we say V is a fully invariant submodule of W.

Definitions 4.2. Let T be a subset of a ring R, V an R module and W a subset of V.

i) We write $\mathrm{ann}_R W = \{r \in R \mid Wr = 0\}$ and $\mathrm{ann}_V T = \{v \in V \mid vT = 0\}$, the annihilators of W in R and of T in V respectively.

ii) If $\mathrm{ann}_R W = 0$, V is said to be a faithful R module.

iii) If V satisfies the minimum condition on annihilators of subsets of R we shall write V satisfies min-ann.

Definition 4.3. If $R = SG$ a group ring and V is an R module, we write $C_G(V)$ for $\{g \in G \mid g-1 \in \mathrm{ann}_R V\}$, the centraliser of V in G.

Definition 4.4. Let $0 \neq V \subseteq W$ be R modules. W is an essential extension of V if for every non-zero submodule T of W we have $T \cap V \neq 0$. If in addition W properly contains V we say the extension is proper.

If W is an essential extension of V we shall sometimes say V is an essential submodule of W.

Proposition 4.5. Let E be an R module. The following are equivalent:

i) E is a direct summand of any module containing it.

ii) E has no proper essential extensions.

iii) The functor $\mathrm{Hom}_R(\ ,E)$ is exact.

iv) If A is a submodule of an R module B and $\Theta: A \to E$ is an R homomorphism, then there is an R homomorphism $\bar{\Theta}: B \to E$ such that $\bar{\Theta}(a) = \Theta(a)$ for $a \in A$. (We say that $\bar{\Theta}$ extends Θ or that the restriction of $\bar{\Theta}$ to A equals Θ and write $\bar{\Theta}_A = \Theta$).

v) If I is a right ideal of R and $\Theta: I \to E$ a homomorphism then there is a homomorphism $\bar{\Theta}: R \to E$ which extends Θ.

Proof: See (46) proposition 2.1, and theorems 2.15 and 2.17.

Any module satisfying the conditions of proposition 4.5 is said to be injective.

Theorem 4.6. Let V be an R module. There exists a module $E_R(V) = E$ containing V and satisfying the following equivalent conditions:

i) E is an essential extension of V and is injective.

ii) E is injective, and whenever E' is a proper submodule of E containing V, then E' is not injective.

iii) E is an essential extension of V, and whenever E' is an R module properly containing E, then E' is not an essential extension of V.

Moreover the module $E_R(V)$ is unique up to isomorphism.

Proof: (46) theorem 2.21.

We shall drop the reference to the ring and write $E_R(V) = E(V)$ when no confusion is likely to arise. E(V) is the injective hull of V.

If E is an injective module, then any direct product of copies of E is injective, (46) proposition 2.2. However a direct sum of copies of E need not be injective. This motivates the idea of a Σ-injective module.

Proposition 4.7. Let E be an injective R module. Then the following conditions are equivalent:

i) E satisfies min-ann.

ii) R has the maximum condition (max.) on annihilators of subsets of E.

iii) Any direct sum of copies of E is injective.

ii) Any countable direct sum of copies of E is injective.

A module satisfying the above conditions is said to be Σ-injective.

Proof: See Faith (11).

If V is an R module, we write Soc(V) for the socle of V, that is the sum of the irreducible submodules of V.

Definition 4.8. A module V is finitely embedded if Soc(V) is a finitely generated essential submodule of V. It is easy to see that V

is finitely embedded if and only if there exist finitely many irreducible modules $V_1, \ldots V_n$ such that V is isomorphic to a submodule of $\bigoplus_{(n)}^{n} E(V_i)$, see (46) proposition 3.18.

For example, it is well known that any module with finite composition length is finitely embedded. We note the following generalization.

Theorem 4.9. Let V be an R module. Then V is artinian if and only if every homomorphic image of V is finitely embedded.

Proof: (46) theorem 3.21.

Definitions 4.10. i) A ring R is self-injective if the right regular module R_R is injective.

ii) A ring R is a quasi-Frobenius ring (q.F. ring) if R is artinian and self-injective.

Examples are provided by group rings of finite groups.

Theorem 4.11. If S is a q.F. ring and G a finite group, then the group ring SG is a q.F. ring.

Proof: (9) p.402 exercise 1 (d), or (36) 3.2.8 and 10.1.1.

The irreducible modules over a q.F. ring are very easily described.

Proposition 4.12. Let V be an irreducible module over a q.F. ring R. Then V is isomorphic to a minimal right ideal of R.

Proof: (9) corollary 58.13.

Lemma 4.13. Let G be a finite p group and k a field of characteristic p. Then the injective hull of the trivial kG module is kG.

Proof: kG is injective by 4.11 above. Also since 0 and 1 are the only idempotents kG is indecomposable and so contains a unique minimal submodule which must be the trivial module. It is readily seen that any element of kG which is invariant under all elements of G must be a scalar multiple of $\hat{G}$.

Lemma 4.14. Let S be a subring of a ring R.

i) If V is an injective S module, then the coinduced module $\mathrm{Hom}_S(R,V)$ is an injective R module.

ii) If V is an injective R module, the restriction V_S is an injective S module.

<u>Proof</u> i)(c.f. (7) p. 866 or (9) 37.7).

We make $\mathrm{Hom}_S(R,V)$ into a right R module by defining

$$(fr)(s) = f(rs) \quad \text{for } f \in \mathrm{Hom}_S(R,V) \quad r,s \in R.$$

It is easily seen that $\mathrm{Hom}_S(R,\ )$ is a functor from right S modules to right R modules. If $\Theta: A \to B$ is a map between S modules we define $\Theta^R: \mathrm{Hom}_S(R,A) \to \mathrm{Hom}_S(R,B)$ by $(\Theta^R(f))(x) = \Theta(f(x))$ for $f \in \mathrm{Hom}_S(R,A), x \in R$.

Now suppose that V is an injective S module, $M \subseteq N$ are R modules and $\psi: M \to \mathrm{Hom}_S(R,V)$ an R map. Then we define an S map $\Psi: M \to V$ by $\Psi(m) = \psi(m)(1)$. Then since V is injective Ψ extends to an S map $\Psi_1: N \to V$. Finally we define an R map $\psi_1: N \to \mathrm{Hom}_S(R,V)$ by $\psi_1(n)r = \Psi_1(nr)$, for $n \in N$, $r \in R$. It is easily checked that ψ_1 extends ψ.

ii) To any diagram of S modules,

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\Theta"] & N \\ & V & \end{tikzcd}$$

there corresponds a diagram of R modules

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_S(R,M) \arrow[r] \arrow[d, "\Theta^R"] & \mathrm{Hom}_S(R,N) \\ & \mathrm{Hom}_S(R,V_S) & \end{tikzcd}$$

Also since V is an injective R module and $V \cong \mathrm{Hom}_R(R,V)$ is an R submodule of $\mathrm{Hom}_S(R,V)$, V is isomorphic to a direct summand of $\mathrm{Hom}_S(R,V)$ and combining Θ^R with the projection map gives an R map from $\mathrm{Hom}_S(R,M)$ to V. Then since V is injective as an R module we can extend Θ^R to an R map $\overline{\Theta^R}$, say from $\mathrm{Hom}_S(R,N)$ to V, and finally restricting $\overline{\Theta^R}$ to the S submodule $\mathrm{Hom}_S(S,N) = N$ gives an S map extending the original map Θ.

<u>Lemma 4.15</u> Let H be a subgroup of a group G, and S any ring. If V is an SH module then $\mathrm{Hom}_{SH}(SG,V)$ contains a copy of the induced module $V \otimes_{SH} SG$. If $|G:H|$ is finite, then $V \otimes_{SH} SG \cong \mathrm{Hom}_{SH}(SG,V)$.

<u>Proof:</u> {If $s_i \mid i \in I$} is a right transversal to H in G, then $\{s_i^{-1} \mid i \in I\}$ is a left transversal, and for

$$v = \sum v_i \otimes s_i \in V \otimes_{SH} SG \quad \text{and} \quad a = \sum s_j^{-1} a_j \in SG,$$

we define $f_v \in \mathrm{Hom}_{SH}(SG,V)$ by $f_v(a) = \sum v_k a_k$.

The map $v \longrightarrow f_v$ is an embedding $V \otimes_{SH} SG \longrightarrow \mathrm{Hom}_{SH}(SG,V)$ whose image consists of the SH maps which are zero on all but finitely many cosets of SH. The result follows easily.

<u>Lemma 4.16.</u> Let $S \subseteq R$ be rings such that as a left S module R has a free basis $\{r_\lambda \mid \lambda \in \Lambda\}$, and for $s \in S$ $\exists s' \in S$ with $r_\lambda s = s' r_\lambda$. If V_2 is an essential extension of the right S module V_1, then $V_2 \otimes_S R$ is an essential extension of the right R module $V_1 \otimes_S R$.

<u>Proof:</u> As an S module, we have $V_i \otimes_S R = \bigoplus V_i \otimes r_\lambda$, for $i = 1,2$ and $V_2 \otimes r_\lambda$ is an essential extension of $V_1 \otimes r_\lambda$. Therefore $V_2 \otimes_S R$ is an essential extension of $V_1 \otimes_S R$ as an S module and a fortiori as an R module.

We intend to apply lemma 4.16 in situations where H is a normal subgroup of a group G, T a ring and $S = TH$, $R = TG$, or where K is a field extension of k and $S = kG$, $R = KG$.

<u>Notation 4.17.</u> Although we shall not really use the Ext machinery, it will be convenient to use the symbol $\mathrm{Ext}(V,U) = 0$ as an abbreviation for the statement that any short exact sequence $0 - U - W - V - 0$ splits.

<u>Lemma 4.18.</u> Let R be a ring and U,V R modules which have a series of submodules $0 = U_0 \leq U_1 \leq \dots \leq U_s = U$, $0 = V_0 \leq V_1 \leq \dots \leq V_t = V$, and denote by $\overline{U}_i$, $\overline{V}_j$ the factor modules U_i/U_{i-1}, V_j/V_{j-1}, respectively.

If $\mathrm{Ext}(\overline{V}_j, \overline{U}_i) = 0$ for all i,j then $\mathrm{Ext}(V,U) = 0$.

<u>Proof:</u> We use induction on s+t, the result being true for s+t = 2 by hypothesis.

Let $0 - U - W \xrightarrow{\pi} V - 0$ be exact with U,V as above. Suppose that $s+t > 2$, and $t > 1$. Let $W_1 = \pi^{-1}(V_1)$.

Then $0 - U - W_1 - V_1 - 0$ is exact and by induction there is a submodule T of W_1 such that $T + U = W_1$, $T \cap U = 0$.

Now $0 - U \to W/T \longrightarrow W/(T \oplus U) - 0$ is exact and $W/(T \oplus U) \cong V/V_1$, so by induction there is a submodule S of W such that $S + U = W$ and $S \cap U \subseteq T$.

Therefore $S \cap U = T \cap U = 0$. Hence we may suppose that t = 1 and $s > 1$.

Now $0 \longrightarrow U/U_1 \rightarrow W/U_1 \rightarrow V \longrightarrow 0$ is exact and so by induction there is a submodule T of W such that $T + U = W$ and $T \cap U \subseteq U_1$. By the Modular law we have $(T + U_1) \cap U = U_1 + (T \cap U) = U_1$ and we may assume that $T \cap U = U_1$.

Finally $0 \longrightarrow U_1 \rightarrow T \rightarrow T/(T \cap U) \longrightarrow 0$ is exact and $T/(T \cap U) \cong W/U \cong V$ so by induction there is a submodule S of T such that $S \cap U_1 = 0$ and $S + U_1 = T$.

Therefore $S + U = S + U + U_1 = T + U = W$, and $S \cap U \subseteq T \cap U \subseteq U_1$, and so $S \cap U = 0$.

We shall need the following versions of Clifford's theorem.

<u>Theorem 4.19.</u> Let H be a normal subgroup of a group G, k a field and V an irreducible kG module. Suppose that either

i) $\dim_k V < \infty$

or ii) $|G:H| < \infty$.

Then V is a completely reducible kH module.

Moreover if $V = U_1 \oplus \ldots \oplus U_s$, where the U_i are the homogeneous components of V as a kH module, and $K = \{g \in G \mid U_1 g = U_1\}$, then $H \subseteq K \subseteq G$, $|G:K| < \infty$ and $V = U_1 \otimes_{kK} kG$.

<u>Proof:</u> The result in this generality seems to be well known, see (51) 1.7 for a proof under assumption i) and (36) 7.2.16 for a proof using ii).

Finally we quote Mackey's theorem.

<u>Theorem 4.20.</u> Let K and H be subgroups of a group G, k a field and W a kK module. For each (K,H) double coset KaH, $W_a = W \otimes a$ is a kK_a module where $K_a = K^a \cap H$, and $(W_a|_{K_a})^H$ is a kH module which depends only on the double coset KaH. Moreover $(W^G)_H = \oplus (W_a|_{K_a})^H$ where the sum is taken over all (K,H) double cosets KaH in G.

<u>Proof:</u> see (9) 44.2.

Chapter 2. Locally Finite Groups.

1. Introduction.

Two recent results relate the existence of injective modules for group algebras which are 'small' in some sense to the structure of the group.

(1) The trivial kG module is injective if and only if G is a locally finite p' group (13), where k is a field of characteristic $p \geq 0$.

(2) If G is a countable group, then every irreducible kG module is injective if and only if G is a locally finite p' group which is abelian-by-finite (13) and (22).

In this chapter we investigate several situations in which kG has injective modules with countable k dimension. The main result can be seen as a generalization of (2) above.

Theorem 4.7. If G is a locally finite group and k is a field of characteristic $p \geq 0$, then the injective hull of every irreducible kG module has countable k dimension if and only if G is abelian-by-finite and has no infinite p subgroup.

However, we begin in section 2 by studying the centralizer $C_G(V)$ of an arbitrary injective kG module V. This is always a locally finite p' group (compare (1) above) and furthermore, if W is any module with $C_G(W)$ a locally finite p' group, then W is injective as a kG module if and only if it is injective as a $kG/C_G(W)$ module.

In section 3 we study Σ-injective modules, or equivalently by proposition 1.4.7, injective modules satisfying min-ann. In fact we find the latter condition easier to work with and in lemma 3.1 (adapted from an argument due to D.S. Passman), we show that if R is a k algebra and V is an injective R module with countable k dimension, then V satisfies min-ann.

Suppose now that V is a right R module satisfying min-ann. and set

$R_o = \mathrm{ann}_R V$. Then R/R_o has max on right annihilator ideals (lemma 3.2, see also (12), corollary 11). If in addition, we suppose that V is irreducible and R is locally artinian, then R/R_o is a primitive ring and every right ideal contains a non-zero idempotent (lemma 3.3). This forces R/R_o to be simple artinian and so $R/R_o = M_n(E)$ where $E = \mathrm{End}_R V$ and $n = \dim_E V$. In particular V is finite dimensional as a (left) module over its endomorphism ring.

However we can improve this result by lifting idempotents in locally artinian rings to show that if R is a locally artinian ring and the injective hull of an irreducible R module V satisfies min-ann., then again $R/R_o = M_n(E)$, where E, R_o and n are as in the previous paragraph (theorem 3.6).

In section 4 we specialize to the case where $R = kG$, the group algebra of a locally finite group has an injective module V satisfying min-ann. If $R_o = \mathrm{ann}_R V$ and $\overline{R} = R/R_o$, then using the arguments of section 3 we see that $S = \overline{R}/J(\overline{R})$ is semisimple artinian, so $S = \bigoplus M_{n_i}(k_i)$ for certain division algebras k_i. If char $k = p > 0$, or if k contains all roots of one, we see that all the division algebras which occur are fields (this follows from lemma 4.3).

Now there is a natural homomorphism from $G/C_G(V)$ to $\prod GL(n_i, k_i)$ and from the assumption that V satisfies min-ann. we can deduce easily that G has no infinite p subgroups. It follows from the theorem of Brauer and Feit quoted in chapter 1, that the image of $G/C_G(V)$ in each factor $GL(n_i, k_i)$ is abelian-by-finite and hence the image of $G/C_G(V)$ in $\prod GL(n_i, k_i)$ is also abelian-by-finite. Also the kernel of this homomorphism is a finite p subgroup and in fact we show that $G/C_G(V)$ is abelian-by-finite. Assembling these results we obtain a characterization of Σ-injective modules over group algebras of locally finite groups (theorem 4.4). 2·3

However, we do not need the full force of this characterization in order to prove theorem 4.7. In fact it follows from theorem 3.6 that if G is a locally finite group such that the injective hull of every irreducible kG module has countable k dimension, then G is a restricted group, that is every irreducible kG module is finite dimensional over its endomorphism ring, and we show that a restricted locally finite group with no infinite p subgroups is abelian-by-finite, thus proving the harder part of theorem 4.7. Without the assumption that G has no infinite p subgroups, Hartley (unpublished) has shown that if G is a locally finite restricted group in characteristic p, then $G/O_p(G)$ is abelian-by-finite.

Finally in section 5 we apply some of the above techniques to study certain related areas, namely the injective hull of the regular module, and the existence of finite dimensional injective modules for arbitrary group algebras.

The results of this chapter are based on (33).

2. The centralizer of an injective module.

Lemma 2.1. Let G be any group and k a field of characteristic $p \geq 0$. If V is an injective kG module, then $C_G(V)$ is a locally finite p' group.

Proof: Let $C = C_G(V)$. Then by lemma 1.4.14, the restriction V_C is injective and is trivial as a kC module. Therefore the one-dimensional trivial module k is a direct summand of V_C and so is injective. Thus C is a locally finite p'group by (13), theorem 1.

Lemma 2.2. If V is an injective kG module then V is injective as a $kG/C_G(V)$ module.

Proof: trivial.

Lemma 2.3. Suppose that H is a normal locally finite p' subgroup of a group G. If V is an injective kG/H module, then V is injective as a kG module when H is allowed to act trivially.

Proof: Let I be a right ideal of kG and $\phi : I \to V$ a kG map.

If r is any element of $I \cap \underline{h}G$ we can write $r = \sum_{(i)}^{n} a_i(g_i - 1)$ where $a_i \in kG$ and $g_1, \ldots, g_n$ are finitely many elements of H.

Now as we noted in 1.3.5 kH is von Neumann regular, and hence there is an idempotent $e \in kH$ such that

$$kH(g_1 - 1) + \ldots + kH(g_n - 1) = kHe.$$

Since $(g_i - 1) \in kHe$ we have

$$(g_i - 1) = (g_i - 1)e \quad \text{for } i = 1, \ldots, n.$$

Therefore $r = \sum a_i(g_i - 1) = \sum a_i(g_i - 1)e = re$ and so $\phi(r) = \phi(r)e = 0$, as $\phi(r) \in V$, e is in the augmentation ideal of kH and H acts trivially on V.

Hence we can extend ϕ to a map from $I + \underline{h}G$ to V by setting $\phi(\underline{h}G) = 0$ and this gives a kG/H map

$$\phi : \frac{I + \underline{h}G}{\underline{h}G} \longrightarrow V.$$

Since V is injective as a kG/H module $\bar{\phi}$ extends to kG/H and then composition with the natural map $kG \to kG/H$ gives a map $kG \to V$ which extends ϕ.

<u>Remark:</u> The previous two lemmas could be compared to the following result, (36) 3.2.9 iii). Let R be a regular ring, R_0 an ideal of R and V an R module with $VR_0 = 0$. Then V is injective as an R module if and only if V is injective as an R/R_0 module.

3. Modules satisfying min-ann.

<u>Lemma 3.1.</u> (c.f. (36) 3.2.11). If R is a k algebra and V an injective R module with $\dim_k V$ countable, then V satisfies min-ann.

<u>Proof:</u> Suppose $V_1 > V_2 > \ldots$ is a strictly descending chain of annihilators where $V_i = \mathrm{ann}_V S_i$, say for certain subsets S_i of R.

If we set $I_i = \mathrm{ann}_R V_i$, then $V_i = \mathrm{ann}_V I_i$ and so we may work with I_i in place of S_i. Note that I_i is a right ideal and we have a strictly ascending chain $I_1 < I_2 < \ldots$

For each integer n choose an element $v_n \in V_n \setminus V_{n+1}$.

Then $v_n I_{n+1} \neq 0$, so $\exists\, a_{n+1} \in I_{n+1}$ such that $v_n a_{n+1} \neq 0$. It follows

that $a_{n+1} \notin I_n = \mathrm{ann}_R V_n$.

Set $W = \bigcap V_n$ a k subspace of V and let $T = k^{\mathbb{N}}$ be the k space of countably infinite sequences $(t_1, t_2, \ldots)$ of elements t_i in k. We aim to construct a 1-1 k linear transformation $\phi : T \longrightarrow V/W$. Since $\dim_k V$ is countable and $\dim_k T$ is uncountable, this will give a contradiction.

Set $I = \bigcup I_i$ a right ideal of R. If $(t_1, t_2, \ldots) = t \in T$ we define $f_t : I \longrightarrow V$ by $f_t(r) = \sum_{n=1}^{\infty} v_n t_n r$.

If $r \in I$, then $r \in I_j$ for some j and hence $v_n t_n r = 0$ for all $n \geq j$ and the above sum is finite for all $r \in I$.

Now f_t is a module homomorphism and so extends to an R map $f_t^* : R \longrightarrow V$, since V is injective. Suppose that $g : R \longrightarrow V$ also extends f_t. Then

$$g(1)r = g(r) = f_t^*(r) = f_t^*(1)r \quad \text{for all } r \in I.$$

Therefore $(f_t^*(1) - g(1)) \in \mathrm{ann}_V I_i$ for all i, and so $f_t^*(1) - g(1) \in W$.

Hence we have a well-defined map $\phi : T \longrightarrow V/W$ given by

$$\phi(t) = f_t^*(1) + V/W.$$

Now if $a, b \in k$ and $s, t \in T$ then $f_{as+bt} = af_s + bf_t$ and so $af_s^* + bf_t^*$ extends f_{as+bt} and $\phi(as+bt) = a\phi(s) + b\phi(t)$. Therefore ϕ is a k linear transformation.

Now suppose $t = (t_1, t_2, \ldots) \in \ker \phi$. Then $f_t^*(1) \in W$ so

$$f_t(r) = f_t^*(1)r = 0 \text{ for all } r \in I.$$

If $t \neq 0$ choose j minimal with $t_j \neq 0$. Then

$$0 = f_t(a_{j+1}) = \sum v_n t_n a_{j+1} = t_j v_j a_{j+1}.$$

Hence $v_j a_{j+1} = 0$ contradicting the choice of a_{j+1}. Therefore ϕ is 1-1 and the lemma is proved.

<u>Lemma 3.2.</u> If R is a ring, V a right R module satisfying min-ann. and $R_o = \mathrm{ann}_R V$, then $\bar{R} = R/R_o$ has max. on right annihilators. In particular $\bar{R}_R$ has max. on direct summands.(c.f. (12) cor.11).

<u>Proof:</u> Suppose that $I_1 < I_2 < \ldots$ is a strictly ascending chain of right annihilator ideals in R. By replacing I_n by $r(l(I_n))$ we may

assume that $I_n = r(J_n)$ where $J_1 > J_2$ and J_n is the left annihilator ideal of I_n.

Set $W_n = VJ_n$, then clearly $W_n \subseteq \mathrm{ann}_V I_n$. If $j \in J_n \setminus J_{n+1}$, then since V is a faithful $\overline{R}$ module $\exists v \in V$ such that $vjI_{n+1} \neq 0$. Hence $W_n \not\subseteq \mathrm{ann}_V I_{n+1}$ and it follows that $\mathrm{ann}_V I_{n+1} \subsetneq \mathrm{ann}_V I_n$ for all n contradicting the assumption that V satisfies min-ann.

Since $\overline{R}$ has a 1 any direct summand of $\overline{R}_R$ has the form $e\overline{R}$ for some idempotent $e \in \overline{R}$ and $e\overline{R} = r(\overline{R}(1-e))$. Therefore $\overline{R}_R$ has max. on direct summands.

<u>Lemma 3.3.</u> Let R be a locally artinian semisimple ring. If I is a non-zero right ideal of R, then I contains a non-zero idempotent.

<u>Proof:</u> Since R is semisimple, I is not nil so there is an element $a \in R$ which is not nilpotent. If S is an artinian subring of R containig a, then $I \cap S$ is an ideal of S which is not nil. Hence $I \cap S \not\subseteq J(S)$. It follows from (1), theorem 2.4A, that $I \cap S$ contains a non-zero idempotent.

<u>Lemma 3.4.</u> Let M be an R module such that:

i) M has max. on direct summands.

ii) Every non-zero submodule of M contains a non-zero direct summand.

Then M is a finite direct sum of irreducible R modules.

<u>Proof:</u> Let N be a non-zero submodule of M and choose a direct summand K of M maximal subject to being contained in N. Suppose that $K \oplus L = M$.

If the intersection $L \cap N$ is non-zero it contains a non-zero direct summand K_1 of M, where $K_1 \oplus L_1 = M$ say.

However since $K_1 \subseteq L$ we have

$$K_1 + (L \cap L_1) = L \cap (K_1 + L_1) = L \text{ by the Modular}$$

law and hence $(K + K_1) \oplus (L \cap L_1) = M$, contradicting the maximality of K. Therefore $L \cap N = 0$ and $K = N$ is a direct summand of M.

Hence the lattice of submodules of M is complemented and the result

follows from (29) §3.3, proposition 2.

We are now in a position to show that if R is a locally artinian ring and V an irreducible R module satisfying min-ann. then V is finite dimensional as a vector space over its endomorphism ring. First, however we prove a lemma on idempotent lifting which will give a stronger result.

<u>Lemma 3.5.</u> Let I be a two sided ideal in a ring R. Let $\overline{R} = R/I$ and write $\overline{x}$ for the image of x in the factor ring $\overline{R}$. Suppose that either

i) I is a nil ideal

or ii) R is locally artinian.

Then we may lift ascending (and descending) chains of principal right ideals generated by idempotents over I. More precisely, if $\overline{e}_1\overline{R} \subsetneq \overline{e}_2\overline{R} \quad \ldots$ is a strictly ascending chain of ideals in $\overline{R}$ where each $\overline{e}_i$ is an idempotent, then there exist idempotents f_i in R such that $\overline{f}_i = \overline{e}_i$ and $f_1R \subsetneq f_2R \quad \ldots$

Similarly for the other chains of ideals.

<u>Proof:</u> Note that $\overline{e}_1\overline{R} \subsetneq \overline{e}_2\overline{R}$ if and only if $\overline{R}(1 - \overline{e}_1) \supsetneq \overline{R}(1 - \overline{e}_2)$. Hence by symmetry it suffices to prove the result for ascending chains of right ideals.

(i) Let I be a nil ideal. Suppose first that $\overline{e}$ is an idempotent in $\overline{R}$. Then $e^2 - e \in I$, and so $(e^2 - e)^k = 0$ for some integer k.

Now, by the Binomial theorem,

$$1 = \{e + (1-e)\}^{2k} = \sum_{r=0}^{2k}\binom{2k}{r}e^{2k-r}(1-e)^r$$

$$= \sum_{r=0}^{k-1}\binom{2k}{r}e^{2k-r}(1-e)^r + \binom{2k}{k}e^k(1-e)^k + \sum_{r=k+1}^{2k}\binom{2k}{r}e^{2k-r}(1-e)^r.$$

Notice that the middle term is zero by choice of k.

Set
$$\lambda(e) = \sum_{r=0}^{k-1}\binom{2k}{r}e^{k-r}(1-e)^r,$$
$$\mu(e) = \sum_{r=k+1}^{2k}\binom{2k}{r}e^{2k-r}(1-e)^{r-k}$$

and
$$\sigma(e) = e^k\lambda(e), \quad \tau(e) = (1-e)^k\mu(e).$$

Then λ, μ, σ and τ are all polynomials in e,

$$1 = \sigma(e) + \tau(e)$$

and $$\sigma(e)\tau(e) = \tau(e)\sigma(e) = e^{k}(1-e)^{k}\lambda(e)\mu(e) = 0.$$

Hence $$\sigma(e) = \sigma(e)\{\sigma(e) + \tau(e)\} = \sigma(e)^2.$$

Now $$\sigma(e) = e^{2k} + \binom{2k}{1}e^{2k-1}(1-e) + \dots$$

$$\equiv e^{2k} \equiv e \text{ modulo } I.$$

Similarly $\tau(e)$ is an idempotent which is congruent to $(1-e)$ modulo I.

(The above argument was shown to me by T.K. Carne of Trinity College, Cambridge).

Now suppose that $\bar{e}_1\bar{R} < \bar{e}_2\bar{R}$ ... is an ascending chain of right ideals in $\bar{R}$, with $\bar{e}_i$ idempotents.

Then, clearly $\bar{R}(1-\bar{e}_i)$ is the left annihilator of $\bar{e}_i\bar{R}$ in $\bar{R}$ and so $\bar{R}(1-\bar{e}_1) > \bar{R}(1-\bar{e}_2)$ is a descending chain with $1-\bar{e}_i$ idempotents. We lift $1-\bar{e}_1$ to an idempotent f_1 in R as above.

Then $\bar{R}(1-\bar{e}_1) > \bar{R}(1-\bar{e}_2)$ gives $(1-\bar{e}_2) = (1-\bar{e}_2)(1-\bar{e}_1) = (1-\bar{e}_2)\bar{f}_1$.

So replacing $1-e_2$ by $(1-e_2)f_1$ we may assume that $(1-e_2) \in Rf_1$.

Next we lift $1-\bar{e}_2$ to an idempotent $f_2 \in R$, so that f_2 is a polynomial in $1-e_2$ and hence $f_2 \in Rf_1$ and so $Rf_2 < Rf_1$.

Therefore $(1-f_1)R < (1-f_2)R$ and $1-f_i$ is an idempotent in R which is congruent to e_i modulo I, for $i = 1,2$.

Repeating this process gives the result when I is a nil ideal.

(ii) Now let I be any ideal in a locally artinian ring R, and $\bar{e}_1, \bar{e}_2$ idempotents in $\bar{R}$ such that $\bar{e}_1\bar{R} \lneq \bar{e}_2\bar{R}$. Then e_1 and e_2 are contained in some artinian subring S of R and $I \cap S$ is an ideal of S.

Therefore in order to lift e_1 and e_2 we may assume that R = S, i.e. that R is artinian.

First suppose that $I \cap J(R) = 0$.

Then $$I \cong \frac{I}{I \cap J(R)} \cong \frac{I + J(R)}{J(R)}$$

and $$\frac{I + J(R)}{J(R)} \oplus \frac{K}{J(R)} = \frac{R}{J(R)}$$ for some ideal K of R, since $R/J(R)$

is semisimple artinian.

Therefore $I + K = R$, and $I \cap K \subseteq J(R) \cap I = 0$. Hence $I \oplus K = R$. Now for $i = 1,2$, $e_i = x_i + f_i$ where $x_i \in I$ and $f_i \in K$ and so $\bar{e}_i = \bar{f}_i$.

Also $\quad \bar{x}_i + \bar{f}_i = \bar{e}_i = \bar{e}_i^2 = (\bar{x}_i + \bar{f}_i)^2 = \bar{x}_i^2 + \bar{f}_i^2,$

so $\quad f_i - f_i^2 \in (x_i^2 - x_i) + I = I.$

Hence $f_i - f_i^2 = 0$, since $I \cap K = 0$, and f_i is an idempotent which is congruent to e_i modulo I.

We claim that $f_1R \subseteq f_2R$, i.e. that $f_1 = f_2f_1$. This is certainly true modulo I since $\bar{f}_1 - \bar{f}_2\bar{f}_1 = \bar{e}_1 - \bar{e}_2\bar{e}_1 = 0$.

Therefore $f_1 - f_2f_1 \in I \cap K = 0$.

The inclusion is strict since $f_1R = f_2R$ gives $f_2 = f_1f_2$ and $\bar{e}_2 = \bar{f}_2 = \bar{f}_1\bar{f}_2 = \bar{e}_1\bar{e}_2$, and so $\bar{e}_1\bar{R} = \bar{e}_2\bar{R}$.

In general, if $I \cap J(R) \neq 0$ we can lift $\bar{e}_1$, $\bar{e}_2$ to idempotents f_1, f_2 in $R/I \cap J(R)$ such that $f_1(R/I \cap J(R)) \subset f_2(R/I \cap J(R))$, but if R is artinian, $I \cap J(R)$ is nilpotent and part i) allows us to lift f_1 and f_2 to idempotents in R.

<u>Theorem 3.6.</u> Let R be a locally artinian ring and $W \subseteq V$ right R modules such that W is irreducible and V satisfies min-ann.

Let $R_0 = \mathrm{ann}_R W$ and $E = \mathrm{End}_R W$. Then W has finite dimension n, say as a (left) module over E and $\bar{R} = R/R_0 = M_n(E)$, the ring of $n \times n$ matrices over E.

In particular this occurs if V is the injective hull of W and is Σ-injective (by proposition 1.4.7).

<u>Proof:</u> Let $R_1 = \mathrm{ann}_R V$. Then by lemma 3.2 R/R_1 has max. on right ideals generated by an idempotent.

Now R_0/R_1 is an ideal in the locally artinian ring R/R_1 and lemma 3.5 enables us to conclude that $\bar{R}$ also has max. on right ideals generated by an idempotent.

However, $\bar{R}$ is a locally artinian, primitive ring and hence by lemma 3.3 any non-zero right ideal contains a non-zero idempotent. Lemma 3.4

applied to the right regular module then shows that $\overline{R}$ is artinian and hence simple artinian.

Finally W is a faithful irreducible module for $\overline{R}$ and so $\overline{R} = M_n(E)$, where $E = \mathrm{End}_{\overline{R}} W$ and $n = \dim_E W$ by the Jacobson density theorem (9), 26.8 , (29),3.1, proposition 3.

We remark that if $R = kG$ is the group algebra of a locally finite group, and if V is any Σ-injective kG module then $kG/\mathrm{ann}_{kG}(V)$ is artinian (theorem 4.4). We have been unable to decide whether any locally artinian ring with a faithful Σ-injective module is actually artinian.

4. Group algebras of locally finite groups.

Lemma 4.1. Let G be a locally finite group, and k a field of characteristic $p > 0$. If kG has a non-zero injective module V satisfying min-ann., then G has no infinite p subgroups.

Proof: If the result is false than G has a strictly ascending chain $G_1 \subsetneq G_2 \subsetneq \dots$ of finite p subgroups.

If $V_n = \mathrm{ann}_V\, g_n$, then clearly $V_{n+1} \subsetneq V_n$.

Consider the restriction of V to the finite subgroup G_{n+1} . $V_{G_{n+1}}$ is a module for the finite dimensional algebra kG_{n+1} and so contains a non-zero finitely generated submodule.

Hence $V_{G_{n+1}}$ must contain an irreducible submodule which is the trivial module since G_{n+1} is a p group.

Now $V_{G_{n+1}}$ is injective by lemma 1.4.14 and so contains a copy of the injective hull of k which is isomorphic to kG_{n+1}.

We identify kG_{n+1} with this submodule of $V_{G_{n+1}}$.

Then by lemma 1.3.9, $\hat{G}_n \cdot g_n = 0$ but $\hat{G}_n(g-1) \neq 0$ for any $g \in G_{n+1} \setminus G_n$

This shows that $V_1 \supsetneq V_2 \supsetneq \dots$ is a strictly descending chain of annihilators and this is a contradiction. Hence G has no infinite p subgroups.

Lemma 4.2. Let H be a finite group, k a field of characteristic $p > 0$. Then $kH/J(kH)$ is isomorphic to a direct sum of matrix rings over comm-

utative fields.

Proof: Let $\mathbb{F}_p$ denote the prime subfield of k. We have $\mathbb{F}_pH/J(\mathbb{F}_pH) = \bigoplus M_{n_i}(k_i)$ where the k_i are commutative fields by Wedderburn's theorem on finite division rings.

Now $k \otimes_{\mathbb{F}_p} J(\mathbb{F}_pH)$ is a nilpotent ideal in kH and so

$$k \otimes_{\mathbb{F}_p} J(\mathbb{F}_pH) \leq J(kH).$$

On the other hand, $\mathbb{F}_pH/J(\mathbb{F}_pH)$ is a separable $\mathbb{F}_pH$ algebra by (3), § 7, no. 5, and hence $J(kH) \leq k \otimes_{\mathbb{F}_p} J(\mathbb{F}_pH)$.

Therefore

$$\frac{kH}{J(kH)} = \frac{\mathbb{F}_pH \otimes_{\mathbb{F}_p} k}{J(\mathbb{F}_pH) \otimes_{\mathbb{F}_p} k} = \frac{\mathbb{F}_pH}{J(\mathbb{F}_pH)} \otimes_{\mathbb{F}_p} k$$

is a direct sum of matrix algebras over commutative fields.

Lemma 4.3. Let G be a locally finite group, and V an irreducible kG module which is finite dimensional as a module over $E = \mathrm{End}_H V$. Suppose that either i) k is a splitting field for all finite subgroups of G,

or ii) characteristic $k = p > 0$.

Then E is a field.

Proof: Case i) is proved by Farkas and Snider (13).

Case ii) Let $R = kG$, and $R_0 = \mathrm{ann}_R V$.

If $\dim_E V = n$, then $\overline{R} = R/R_0 = M_n(E)$ by the Jacobson density theorem, see (9), 26.8. Therefore there is an idempotent $\overline{e}$ in $\overline{R}$ such that $\overline{eRe} \cong E$.

By lemma 3.5, we may lift $\overline{e}$ to an idempotent e in R.

Let $\overline{x}, \overline{y} \in \overline{eRe}$, $x = ere$, $y = ese$ say, and choose a finite subgroup H of G containing the supports of e, r, and s.

Now let e* denote the image of e in kH/J(kH), and set

$$A = \frac{ekHe}{ekHe \cap J(kH)} \cong \frac{ekHe + J(kH)}{J(kH)} \leq \frac{kH}{J(kH)} .$$

Then $A \cong e^*(kH/J(kH))e^*$, and kH/J(kH) is a direct sum of matrix algebras over commutative fields by lemma 4.2. It follows that A is also a direct sum of matrix algebras over fields.

Now consider the combined map $\theta : ekHe \hookrightarrow ekGe \to E$, given by the

inclusion of H in G followed by the projection of H onto $\overline{H}$. Since E has no zero divisors and $ekHe \cap J(kH)$ is nilpotent we have

$$\theta(ekHe \cap J(kH)) = 0.$$

Hence we have a map $\overline{\theta}: A \to E$ and by construction the elements x and y lie in the image of A. However, since A is a direct sum of matrix algebras over fields and E has no zero divisors, it follows that the image of A under $\overline{\theta}$ is a field. In particular $\overline{x}$ and $\overline{y}$ commute, but $\overline{x}$ and $\overline{y}$ were arbitrary elements of E. Hence E is a field as claimed.

We now obtain the characterization of Σ-injective kG modules promised in the introduction to this chapter.

<u>Theorem 4.4.</u> Let G be a locally finite group, k a field of characteristic $p \geqslant 0$. If $p = 0$ we assume that k contains all roots of one.

If V is any kG module the following are equivalent:

(1) V is injective and satisfies min-ann.

(2) V is Σ-injective.

(3) There is a normal p' subgroup H of G, which is contained in $C_G(V)$ and such that G/H is a finite extension of an abelian p' group, and finitely many isomorphism types $W_1, W_2, \ldots, W_s$, of irreducible kG/H modules such that V is isomorphic to a direct sum of the injective hulls of the $W_1, \ldots W_s$. (Notice that by the results of section 2 it is immaterial whether we form these injective hulls in the category of kG modules or the category of kG/H modules).

Moreover, if (1) - (3) hold then kG induces an artinian ring of transformations on V, that is $kG/\mathrm{ann}_{kG}V$ is artinian.

<u>Proof:</u> (1) and (2) are equivalent by 1.4.7.

(1) $\Rightarrow$ (3). Let $H = C_G(V)$. Then H is a normal p' subgroup of G by lemma 2.1. Also V is a Σ-injective module for kG/H and we may assume that H = 1 and then require to prove that G is a finite extension of an abelian p' group.

If $R = kG$, $R_0 = \mathrm{ann}_R V$, $\overline{R} = R/R_0$ then by lemma 3.2 $\overline{R}$ has the maximum

condition on right ideals generated by an idempotent. Moreover $S = \overline{R}/J(\overline{R})$ is a locally artinian semisimple ring, and by lemma 3.5, the chain condition on idempotently generated right ideals lifts to S.

Therefore by lemmas 3.3 and 3.4, S is semisimple artinian.

If U is an irreducible S module we can regard U as an irreducible kG module and since U is finite dimensional over $\mathrm{End}_S U$ and $\mathrm{End}_S U = \mathrm{End}_{kG} U$, we conclude from lemma 4.3 that $\mathrm{End}_S U$ is a field.

Hence, by the Artin-Wedderburn theorem $S \cong \oplus M_{n_i}(k_i)$ a direct sum of matrix rings over fields. Let $U(\overline{R})$, $U(S)$ denote the groups of units of $\overline{R}$ and S respectively. Then as $C_G(V) = 1$, G embeds in $U(\overline{R})$ and we have a group homomorphism $G \hookrightarrow U(\overline{R}) \xrightarrow{\psi} U(S) \hookrightarrow \prod GL(n_i, k_i)$.

Here ψ is obtained from the natural homomorphism of $\overline{R}$ onto S.

If char $k = 0$, R is semisimple by (35) theorem 18.7, and since R is locally artinian, any factor ring of R is semisimple. In particular $\overline{R} = S$.

If char $k = p > 0$, and $a \in \ker \psi$, then $a = 1-j$ for some $j \in J(\overline{R})$, and since $\overline{R}$ is locally artinian, $J(\overline{R})$ is locally nilpotent and $\exists r$ such that $j^r = 0$. Choose $s \geqslant 1$ such that $p^s \geqslant r$, then $(1-j)^{p^s} = 1-j^{p^s} = 1$. Therefore the kernel of this group homomorphism is a p group P which must be finite.

Now, G has no infinite p subgroups by lemma 4.1, and it follows from the theorem of Brauer and Feit (theorem 1.2.3), that the image of G in each factor $GL(n_i, k_i)$ is abelian-by-finite, and therefore the image of G in $\prod GL(n_i, k_i)$ is abelian-by-finite. Therefore G/P is abelian-by-finite.

Let A be a normal subgroup of finite index in G such that A/P is an abelian p' group and set $C = C_A(P)$. Then $|G:C| < \infty$ and C is nilpotent (of class two) and so $C = O_p(C) \times O_{p'}(C)$.

Again since G has no infinite p subgroups we have $|C:O_{p'}(C)| < \infty$ and so $|G:O_{p'}(G)| < \infty$.

Set $K = O_{p'}(G)$. Then $P \cap K = 1$ and so $K = K/P \cap K \cong PK/K \leqslant G/P$ and K is abelian-by-finite and so is a finite extension of an abelian p' group.

We show next that $\overline{R}$ is artinian. Since K is a locally finite p' group

kK is von Neumann regular and hence so is

$$T = kK/kK \cap R_o \cong kK + R_o/R_o \leq R/R_o = \overline{R}.$$

Now $\overline{R}$ contains no infinite set of orthogonal idempotents by lemma 3.2, and so by lemma 3.4, T is semisimple artinian, but $\overline{R}$ is a finitely generated module over T (it can be generated by the image of a transversal to K in G) and therefore $\overline{R}$ is artinian.

A result of Cailleau (8) states that any Σ-injective module is a direct sum of indecomposable Σ-injective modules. This is easily seen in the present case where V is a Σ-injective module for the artinian ring $\overline{R}$, for the socle M of V is a direct sum of irreducible submodules, and since V is injective $E(M) \leq V$. If the inclusion were strict then as $E(M)$ is injective it would have a complement N in V, but then since $\overline{R}$ is artinian N would have an irreducible submodule intersecting M trivially. This contradiction shows that V is a direct sum of injective hulls of irreducible $\overline{R}$ modules. Again since $\overline{R}$ is artinian only finitely many isomorphism types $W_1, W_2, \ldots, W_s$ can occur. Clearly each W_i is an irreducible module for kG.

(3) $\Rightarrow$ (2). Since H is a normal p' subgroup of G, any injective kG/H module is injective when regarded as a kG module with H acting trivially by lemma 2.3. Hence we may assume that H = 1.

Therefore G is a finite extension of an abelian p' group K. Let W be any irreducible kG module. Then the restriction W_K is a direct sum of of finitely many irreducible kK modules by Clifford's theorem. Now any irreducible kK module has countable dimension over k (see (21) p.123) and so $\dim_k W$ is countable. In addition any irreducible kK module is injective by the proof of (13) theorem 3, and hence W_K is injective since it is the direct sum of finitely many injective modules.

Again since the index $|G:K|$ is finite the induced module $W \otimes_{kK} kG$ is an injective module containing W by lemma 1.4.15 and has countable

dimension over k. Hence dim $E(W)$ is countable and by lemma 3.1, $E(W)$ satisfies min-ann and so is $\sum$-injective.

Clearly any direct sum of copies of $E(W)$ is $\sum$-injective, and if $W_1, W_2, \ldots W_s$ are finitely many irreducible kG modules and V is any direct sum of their injective hulls then V is $\sum$-injective. This completes the proof of theorem 4.4.

Notice that if we have uncountably many isomorphic copies $\{W_i\}$ of an irreducible kG module, where G is an abelian p' group, then $\bigoplus_i E(W_i)$ is a $\sum$-injective module with uncountable dimension over k, so the converse of lemma 3.1 fails to hold.

We also record the following result.

<u>Corollary 4.5.</u> If G is a locally finite group and k a field of characteristic $p > 0$, then kG has a $\sum$-injective module if and only if $|G:O_{p'}(G)| < \infty$.

We now study the group algebra kG of a locally finite group, which has the property that the injective hull of every irreducible kG module has countable dimension over k. We show that any such group is abelian-by-finite. This provides a generalization of (22), theorem A. The technique will be to reduce to linear groups.

<u>Lemma 4.6.</u> i) Suppose H is a subgroup of G and that there is an irreducible kH module W such that $H/C_H(W)$ is not abelian-by-finite. Then there is an irreducible kG module V such that $G/C_G(V)$ is not abelian-by-finite.

ii) Suppose G has a locally finite p' subgroup H of finite index and that G is not abelian-by-finite. Let k be a field of characteristic $p \geq 0$. Then there is an irreducible kG module V such that $G/C_G(V)$ is not abelian-by-finite.

<u>Proof:</u> i) By (35) lemma 10.2 i), there is an irreducible kG module W such that W is a kH submodule of V_H.

Hence $C_G(V) \cap H = C_H(V) \subseteq C_H(W)$. Now $H/C_H(W)$ is not abelian-by-finite and so neither is $H/C_G(V) \cap H \cong HC_G(V)/C_G(V)$.

Therefore $G/C_G(V)$ is not abelian-by-finite.

ii) This follows from part i) and (22), lemma 2.3.

Let k be a field of characteristic $p \geqslant 0$. For brevity we say that a locally finite group G is <u>restricted over k</u>, if every irreducible kG module is finite dimensional over its endomorphism ring.

<u>Theorem 4.7.</u> Let G be a locally finite group, k a field of characteristic $p \geqslant 0$. The injective hull of every irreducible kG module has countable dimension over k if and only if G is a finite extension of an abelian p' group.

<u>Proof:</u> We have shown in the proof of theorem 4.4 that if G is a finite extension of an abelian p' group then the injective hull of every irreducible kG module has countable dimension over k.

Now suppose that G is a locally finite group such that the injective hull of every irreducible kG module has countable dimension over k. Then G is restricted over k by theorem 3.6. If char $k = 0$, the result follows from (22), theorem B. If char $k = p > 0$, then by lemmas 3.1 and 4.1, G has no infinite p subgroups and the result follows from the following theorem.

<u>Theorem 4.8.</u> Suppose char $k = p > 0$ and that G is a locally finite restricted group over k with no infinite p subgroups. Then G is abelian-by-finite.

<u>Proof: Step 1. If G has a p' subgroup of finite index then G is abelian-by-finite.</u>

Otherwise by lemma 4.6 ii) kG has an irreducible module V such that $G/C_G(V)$ is not abelian-by-finite. However if $E = \mathrm{End}_{kG}V$ and $n = \dim_E V$, then $G/C_G(V)$ embeds in $GL(n,E)$ and E is a field by lemma 4.3 ii). This is impossible by the Brauer-Feit theorem.

<u>Step 2.</u> We may suppose that $O_p(G) = 1$.

Notice that $G/O_p(G)$ is a restricted group with no infinite p subgroups. Suppose that we have shown $G/O_p(G)$ to be abelian-by-finite and let $A/O_p(G)$ be an abelian normal subgroup with finite index and

Therefore $G/C_G(V)$ is not abelian-by-finite.

ii) This follows from part i) and (22), lemma 2.3.

Let k be a field of characteristic $p \geqslant 0$. For brevity we say that a locally finite group G is <u>restricted over k.</u> if every irreducible kG module is finite dimensional over its endomorphism ring.

<u>Theorem 4.7.</u> Let G be a locally finite group, k a field of characteristic $p \geqslant 0$. The injective hull of every irreducible kG module has countable dimension over k if and only if G is a finite extension of an abelian p' group.

<u>Proof:</u> We have shown in the proof of theorem 4.4 that if G is a finite extension of an abelian p' group then the injective hull of every irreducible kG module has countable dimension over k.

Now suppose that G is a locally finite group such that the injective hull of every irreducible kG module has countable dimension over k. Then G is restricted over k by theorem 3.6. If char $k = 0$, the result follows from (22), theorem B. If char $k = p > 0$, then by lemmas 3.1 and 4.1, G has no infinite p subgroups and the result follows from the following theorem.

<u>Theorem 4.8.</u> Suppose char $k = p > 0$ and that G is a locally finite restricted group over k with no infinite p subgroups. Then G is abelian-by-finite.

<u>Proof: Step 1. If G has a p' subgroup of finite index then G is abelian-by-finite.</u>

Otherwise by lemma 4.6 ii) kG has an irreducible module V such that $G/C_G(V)$ is not abelian-by-finite. However if $E = \mathrm{End}_{kG}V$ and $n = \dim_E V$, then $G/C_G(V)$ embeds in $GL(n,E)$ and E is a field by lemma 4.3 ii). This is impossible by the Brauer-Feit theorem.

<u>Step 2.</u> We may suppose that $O_p(G) = 1$.

Notice that $G/O_p(G)$ is a restricted group with no infinite p subgroups. Suppose that we have shown $G/O_p(G)$ to be abelian-by-finite and let $A/O_p(G)$ be an abelian normal subgroup with finite index and

$B/O_p(G) = O_{p'}(A/O_p(G))$.

Then $O_p(G)$ is a finite normal maximal p subgroup of B and so $C = C_B(O_p(G))$ has finite index in B. We claim that C has a p' subgroup Q of finite index. Then Q will have finite index in G and G will be abelian-by-finite by step 1.

Set $P = C \cap O_p(G)$, then P is a central Sylow p subgroup of any finite subgroup S such that $P \leq S \leq C$ and therefore $S = P \times O_{p'}(S)$ for any such S by the Schur-Zassenhaus theorem. It follows that the p' elements of S form a subgroup and hence the p' elements of C form a subgroup.

Therefore $C = P \times O_{p'}(C)$ and $O_{p'}(C)$ has finite index in C and so in G.

<u>Step 3. If G is residually finite and satisfies the conditions of the theorem, then G is abelian-by-finite.</u>

Let P be a maximal p subgroup of G. For each $x \in P$, $x \neq 1$, we choose a normal subgroup N_x of finite index in G such that $x \notin N_x$.

Set $N = \bigcap N_x$, then N is a normal subgroup with finite index in G and $P \cap N = 1$ by construction. Therefore since any two maximal p subgroups of G are conjugate by (28), 1.D.12, N is a p' subgroup of G and G is abelian-by-finite by step 1.

<u>Step 4. Completion of the proof.</u>

Let G be any group satisfying the conditions of the theorem and suppose in addition that $O_p(G) = 1$.

If $g-1 \in J(kG)$, then $(g-1)^{p^n} = 0$ for some n. Therefore $g^{p^n} = 1$ and so $G \cap (1 + J(kG)) \leq O_p(G) = 1$. Hence there exist irreducible kG modules $\{V_i \mid i \in I\}$ such that $\bigcap_{i \in I} C_G(V_i) = 1$.

Now each $G/C_G(V_i)$ is a linear group over a field and so is abelian-by-finite as in step 1.

Let J_i be a normal subgroup with finite index in G such that $J_i/C_G(V_i)$ is abelian.

If $J = \bigcap J_i$, then $J' \leq \bigcap C_G(V_i) = 1$, so J is abelian. Clearly G/J is residually finite and so abelian-by-finite by step 3. Hence G is meta-

belian-by-finite. Let L be a metabelian subgroup of finite index in G, P a maximal p subgroup of L and Σ the local system consisting of all finite subgroups S of L containing P.

If $S \in \Sigma$, P is a Sylow p subgroup of S and by Hall's theorem, S has a p' complement with index $|P|$.

Hence $|S:O_{p'}(S)| \leq |P|!$ for all $S \in \Sigma$, and an inverse limit argument such as (28) 1.K.2, shows that $|L:O_{p'}(L)| \leq |P|!$.

Therefore G has a p' subgroup of finite index and is abelian-by-finite by step 1.

<u>Remark.</u> The above proof is substantially due to B. Hartley. Without the assumption that G has no infinite p subgroups he has shown that if G is a locally finite restricted group over a field of characteristic $p \geq 0$, then $G/O_p(G)$ is abelian-by-finite.

We also have the following variant of theorem 4.7.

<u>Theorem 4.9.</u> Let G be a countable locally finite group and k a field of characteristic $p \geq 0$. The injective hull of every irreducible kG module has finite composition length if and only if G is a finite extension of an abelian p' group.

<u>Proof:</u> Since G is a countable group, any irreducible kG module has countable dimension, since it is a factor module of kG. Hence any kG module with a finite (or even countable) composition series has countable dimension over k. The result follows from theorem 4.7 with minor modifications.

5. Some related results.

We begin by studying the injective hull of the regular module. The techniques of section 3 will be applied to obtain a short proof of Lawrence's result that a countable dimensional self-injective ring is quasi-Frobenius, (30), and to show that if G is locally finite and $\dim_k E_{kG}(kG)$ is countable then G is finite.

For non-locally finite groups the situation may be rather complex- we

have the following example.

Proposition 5.1. Let $C_\infty = \langle x \mid - \rangle$ denote the infinite cyclic group. Then dim $E_{kC_\infty}(kC_\infty)$ is countable if and only if k is a countable field. $E_{kC_\infty}(kC_\infty)$ is always Σ-injective.

Proof: Notice that kC_∞ is an Ore domain whose quotient ring Q may be identified with the ring of rational functions in the variables x and x^{-1}. Hence $Q \cong E_{kC_\infty}(kC_\infty)$ by (29), § 4.6, proposition 2, and § 4.3, proposition 3.

Now if k is countable, then kC_∞ and Q are countable rings and so dim $E_{kC_\infty}(kC_\infty)$ is countable.

On the other hand, if k is uncountable it is easily checked that the elements $1/(x+a)$ as a ranges over k, are linearly independent.

Any injective module over a Noetherian ring is Σ-injective by (46), theorem 4.1. In particular $E_{kC_\infty}(kC_\infty)$ is Σ-injective.

Theorem 5.2. If R is a k algebra with a countable dimensional, faithful injective right module V, then R has the maximum condition on right annihilator ideals.

Proof: By lemma 3.1, V satisfies min-ann. and so by lemma 3.2, R satisfies max. on right annihilator ideals.

By a result of Faith (11), theorem 5.2, a right self-injective ring satisfying max. on right annihilator ideals is quasi-Frobenius. Hence we have the following corollary.

Corollary 5.3. (Lawrence (30)). A countable dimensional right self-injective ring is quasi-Frobenius.

For group algebras we have a slightly stronger result.

Theorem 5.4. Suppose kG has a faithful, injective module with countable dimension over k. Then G has no infinite locally finite subgroups.

Proof: By lemmas 3.1 and 3.2 kG has max. on right ideals generated by an idempotent.

If G has an infinite locally finite subgroup, then by 1.2.4 G has

an infinite locally finite abelian subgroup A. Since by lemma 4.1, G has no infinite locally finite p subgroups, where p = char k, it follows that $H = C_{p'}(A)$ is an infinite locally finite p' subgroup of G.

Therefore H has a strictly ascending chain $H_1 \leqslant H_2 \leqslant \ldots$ of finite p' subgroups of G. If $e_i = \hat{H}_i/|H_i|$, then $(1-e_1)kG \leqslant (1-e_2)kG \leqslant \ldots$ is a strictly ascending chain of right ideals generated by idempotents. This contradiction establishes the result.

<u>Corollary 5.5.</u> (Renault (37)). If kG is self-injective, then G is finite.

<u>Proof:</u> It is easily seen that G is locally finite, as in (36), lemmas 3.2.5 and 3.2.7.

If kG is self-injective then so is kH whenever H is a subgroup of G, since kG_{kH} is injective and free as a kH module, and so kH_{kH} is a direct summand of an injective module and hence is injective.

Therefore by corollary 5.3 or theorem 5.4 every countable subgroup of G is finite. If G is not finite, let S be a countably infinite subset of G, then $\langle S \rangle$ is a countable subgroup of G and so is finite. This is a contradiction. Hence G is finite.

It seems likely that if any group algebra kG has an irreducible injective module V, then G is locally finite. To conclude this chapter we show that this is the case if $\dim_k V < \infty$.

<u>Lemma 5.6.</u> Let H be an infinite cyclic subgroup of a group G, and k any field. Then no non-zero element of kH is a zero divisor in kG.

<u>Proof:</u> Let $H = \langle x \rangle$. Clearly we can assume k is algebraically closed, and it follows that if some non-zero element of kH is a zero divisor, then $(x-\lambda)$ is a zero divisor for some $\lambda \in k^*$. Let $(x-\lambda)\alpha = 0$, and $y = \lambda^{-1}x$, $\beta = \lambda\alpha$ and $H_1 = \langle y \rangle$, then $(y-1)\beta = 0$, and so $(y^n-1)\beta = 0$ for all $n \in Z$. Therefore $\beta \in r(H_1) = 0$ by (36) 3.1.2. Hence $\alpha = 0$.

<u>Theorem 5.7.</u> If G is any group, k a field of characteristic $p \geqslant 0$, and kG has an injective module V of finite dimension n, then G is locally finite, and $G/C_G(V)$ has an abelian p' subgroup of finite index bounded by a function of p and n.

<u>Proof:</u> We show first that G is periodic. If x is an element of infinite order in G, then clearly $Vf(x) = 0$ for some non-zero polynomial $f(x)$ in x. Let v be a non-zero element of V, and consider the map $f(x)kG \longrightarrow V$ given by $f(x)\alpha \longrightarrow v\alpha$ for $\alpha \in kG$. This is well defined since by lemma 5.5, $f(x)$ is a non zero divisor in kG. Since V is injective, there exists $w \in V$ such that $wf(x) = v$. This is impossible.

Now, by lemma 2.1, $C_G(V)$ is a locally finite p' group and we may assume that $C_G(V) = 1$. Therefore G is a periodic subgroup of $Gl(n,k)$, and so is locally finite by 1.2.2. Clearly V satisfies min-ann. and so G has no infinite p subgroups. In fact the argument of lemma 4.1 shows that if P is any p subgroup of G, then $|P| \leqslant p^n$. Therefore by the Brauer-Feit theorem, 1.2.1, G has an abelian p' subgroup of finite index bounded by a function of p and n.

Chapter 3. Polycyclic Groups I - Artinian Modules.

1. Introduction.

In this chapter we begin the study of injective modules over polycyclic group rings.

For an ideal I in an arbitrary ring R, we denote by $\hat{R}_I$, or simply $\hat{R}$, the I-adic completion $\varprojlim R/I^n$ of R.

Some well known results of Matlis (31) state that if R is a commutative Noetherian ring, V an irreducible R module and $I = \mathrm{ann}_R V$, then

(1) E(V) is artinian.

(2) V is, up to isomorphism the only composition factor of E(V).

(3) $\hat{R}_I \cong \mathrm{End}\, E(V)$.

The main results of this chapter generalise (1) - (3) above to group rings of polycyclic groups over suitable coefficient rings.

Main Theorem. Let S be the ring of integers or a field of characteristic $p > 0$. Let $R = SG$ be the group ring of a polycyclic group G, and V a finitely generated R module such that $|G:C_G(V)| < \infty$ and $Vp = 0$ if $S = \mathbb{Z}$. Let $M = \mathrm{ann}_R(V)$. Then we have

(1) E(V) is artinian.

(2) E(V) has only finitely many isomorphism types of composition factors.

(3) If G is p nilpotent, then $\hat{R}_M = M_n(\mathrm{End}\, E(V))$ a full matrix ring over End E(V), where n denotes the multiplicity of V in the socle of R/M.

The main theorem applies in particular if the coefficient ring is $\mathbb{Z}$ or an absolute field, and V is an irreducible SG module. For by (36) 12.2.9, if $S = \mathbb{Z}$, then there is a prime p such that $Vp = 0$, and by (36) 12.3.3, $|G:C_G(V)| < \infty$.

Simple examples show that the exact analogues of results (2) and (3) of Matlis do not hold. If $G = S_3$ the symmetric group of degree 3, and

k the algebraic closure of the field of 3 elements, then since G has 2 3' conjugacy classes, there are 2 irreducible kG modules k,V. E(k) has composition length 3 with factors k, V, k and $\dim \mathrm{End}_{kG} E(k) = 2$. On the other hand since $G = G'G^{(3)}$, we have $\mathfrak{g} = \mathfrak{g}^2$. Hence $\widehat{kG}_{\mathfrak{g}} = kG/\cap \mathfrak{g}^n = k$.

If k is the algebraic closure of the field of 2 elements then since $G = S_3$ has 2 2' classes, there are two irreducible kG modules k,V. Now V is injective and occurs with multiplicity 2 in kG. Also since k is algebraically closed $\mathrm{End}_{kG} V = k$. Hence if $M = \mathrm{ann}_{kG} V$ we have by the main theorem $\widehat{kG}_M = M_2(k)$.

Our proof of the main theorem requires the existence of ideals of R with certain Artin-Rees properties. If I is an ideal in a ring R, we say that <u>I has AR1</u> if the ring $R^*(I) = R \oplus IX \oplus I^2X^2 \oplus \ldots$ is Noetherian, and that <u>I has AR2</u> if for all finitely generated R modules M and submodules U, there exists an integer n such that $MI^n \cap U \subseteq UI$. It is well known that I has AR2 whenever it has AR1, see (36) 11.2.1.

The first result of Matlis quoted above may be deduced from the last and the fact that $\hat{R}_I$ is Noetherian (2) 10.26, by exhibiting a lattice anti-isomorphism between the submodules of E(V) and the ideals of $\hat{R}_I$.

We follow this approach quite closely, and accordingly the main part of this chapter is divided into 3 sections.

Suppose that V is an irreducible SG module such that $|G:C_G(V)| < \infty$, and $Vp = 0$, if $S = Z$. Then by (5) 2.3.5 and 2.3.6, there exists an ideal of the form $I = pSG + qG$ with AR1 contained in $\mathrm{ann}_{SG} V$. Furthermore q may be chosen to be a normal subgroup of finite index in G. Hence V may be regarded as an irreducible module over the q.F. ring $SG/I = S/pS(G/Q)$ and by 1.4.12 V is isomorphic to a minimal right ideal of SG/I. More generally in section 2 we view any right ideal W of SG/I as an SG module with I acting trivially and show that there exists an idempotent $e \in \widehat{SG}_I$ such that $e\widehat{SG}e = \mathrm{End}\, E(W)$ (theorem 2.2).

Some special cases of this result are discussed. In particular if

$W = SG/I$ then $e = 1$, and if W is a minimal right ideal of SG/I then e is a primitive idempotent. Moreover if G is p nilpotent we can use a result on idempotent lifting (proposition 2.4) to show that if W is a two sided ideal of SG/I, then e is central, and finally by letting W range over the indecomposable two sided ideals of SG/I we obtain a decomposition of $\widehat{SG}_I$ as a direct sum of matrix rings over complete scalar local rings (theorems 2.6 and 2.7).

Section 3 is devoted to showing that certain completions $\widehat{SG}_M$ of SG are Noetherian. In particular this holds whenever $S = \mathbb{Z}$ or an absolute field, G is p nilpotent, and M a maximal ideal of SG containing the prime p. (compare (24) corollary 9, and (32)). We also show that End $E(V)$ is Noetherian whenever $S = \mathbb{Z}$ or a field of characteristic $p > 0$, G is polycyclic-by-finite and V an irreducible SG module satisfying $|G:C_G(V)| < \infty$.

In section 4 we deduce that $E(V)$ is artinian from the fact that $\widehat{SG}_I$ has Morita duality with injective cogenerator $E_{SG}(SG/I) = E_{\widehat{SG}}(SG/I)$. We leave the definitions until §4 and for the moment we merely remark that this is a formalisation of the lattice anti-isomorphism used by Matlis. We also show in section 4 that $E(V)$ has only finitely many isomorphism types of composition factors.

In section 5 we indicate briefly how our results can be modified to show that if k is any field of characteristic $p > 0$, and V is an irreducible kG module with $\dim_k E(V) < \infty$ then $E(V)$ is an artinian module with only finitely many isomorphism types of composition factors.

Some of the results in this chapter have been proved independently by Jategaonkar, and announced in (27).

2. Endomorphism rings and completions.

Theorem 2.2 is proved by constructing a suitable sequence of commutative diagrams and taking inverse limits. These diagrams are constructed inductively using the following lemma. However we introduce some notation first. If M is a right R module, we consider M as a left module over its endomorphism ring. If e is an idempotent in R, then End $eR \cong eRe$ via the map $\beta \mapsto \beta(e)$ for $\beta \in$ End eR. Further, if $\varphi: M \longrightarrow N$

is a monomorphism of right R modules and $\varphi(M)$ is a fully invariant submodule of N, then we denote by $\varphi^*: \mathrm{End}\, N \longrightarrow \mathrm{End}\, M$ the induced map of endomorphism rings given by $[\varphi^*\beta](m) = \varphi^{-1}\beta\varphi(m)$, for $\beta \in \mathrm{End}\, N$.

<u>Lemma 2.1.</u> Let R be a self-injective ring containing ideals J and K which are mutual annihilators on both sides, and such that $J \leq J(R)$ and $K = aR$ for some a which is central in R. Let e be an idempotent in R, and let $\bar{\ }: R \longrightarrow R/J = \bar{R}$ denote the natural map.

Let N be an injective R module and suppose that $M = \mathrm{ann}_N J$ is essential in N and that $\varphi: M \longrightarrow \overline{eR}$ is an isomorphism of right R modules.

Then there exists an isomorphism $\psi: N \longrightarrow eR$ such that the induced diagram

$$\begin{tikzcd} eRe \arrow[r, "\psi^*"] \arrow[d, "\text{natural map (nat.)}"'] & \mathrm{End}\, N \arrow[d, "\text{restriction map (res.)}"] \\ \overline{eRe} \arrow[r, "\varphi^*"'] & \mathrm{End}\, M \end{tikzcd}$$

commutes.

<u>Proof:</u> Notice that since M is a fully invariant submodule of the injective module N, the restriction map is defined and is surjective.

The hypothesis on J and K imply that the map $\lambda: \bar{R} \longrightarrow K$, $\lambda(\bar{r}) = ar$ is an isomorphism of right R modules. Clearly $\lambda: \overline{eR} \longrightarrow aeR$ is an isomorphism, and since a is central, every endomorphism of eR preserves aeR, so we have a diagram of ring homomorphisms

$$\begin{tikzcd} \mathrm{End}\, eR \arrow[r] \arrow[d, "\text{res.}"'] & eRe \arrow[d, "\text{nat.}"] \\ \mathrm{End}\, aeR \arrow[r] & \overline{eRe} \end{tikzcd} \qquad (*).$$

It is easily seen that the diagram commutes.

Since λ and φ are isomorphisms the map $\lambda\varphi: M \longrightarrow aeR$ is an isomorphism and since R is self-injective we can extend $\lambda\varphi$ to a map $\psi: N \longrightarrow eR$. ψ is injective because $M \cap \ker\psi = \ker\lambda\varphi = 0$. It follows that $eR = \psi(N) \oplus N'$, for some R submodule N'.

Now $aeR \leq \psi(N)$, so as a is central $N'a = 0$. Hence $N' \leq J \leq J(R)$ and since N' is generated by an idempotent we must have $N' = 0$ and so ψ is an isomorphism.

We have a commutative diagram

$$\begin{tikzcd} N \arrow[r, "\psi"] \arrow[d, phantom, "\cup"] & eR \arrow[dr, "\nu"] & \\ M \arrow[r, "\varphi"'] & \overline{eR} \arrow[r, "\lambda"'] & aeR \end{tikzcd}$$

of R modules and hence a commutative diagram of endomorphism rings.

$$\begin{tikzcd} \mathrm{End}\, N \arrow[d] & eRe \arrow[l, "\psi^*"'] \arrow[r] \arrow[d, dotted] & \mathrm{End}\, aeR \arrow[dl, "\lambda^*"] \\ \mathrm{End}\, M & \overline{eRe} \arrow[l, "\varphi^*"] & \end{tikzcd}$$

However by (*) we can fill in the dotted map with the natural homomorphism to make the right hand triangle commute.

Hence the square

$$\begin{tikzcd} \mathrm{End}\, N \arrow[d, "\text{res.}"'] & eRe \arrow[l, "\psi^*"'] \arrow[d, "\text{nat.}"] \\ \mathrm{End}\, M & \overline{eRe} \arrow[l, "\varphi^*"] \end{tikzcd}$$

commutes as required.

We intend to apply lemma 2.1 in situations where G is a finite group, H a normal p subgroup of G and either

i) $R = kG$, where k is a field of characteristic $p > 0$, $J = \underline{h}G$, $K = \hat{H}kG$

or ii) $R = (Z/p^{n+1}Z)G$, $J = p^n(Z/p^{n+1}Z)G + \underline{h}G$ and $K = p(Z/p^{n+1}Z)\hat{H}G$.

<u>Theorem 2.2.</u> Let G be a polycyclic-by-finite group, Q a normal subgroup of finite index in G and S the ring of integers or a field of characteristic $p > 0$. Suppose that the ideal $I = pSG + \underline{q}G$ has AR2 and let W be any non-zero right ideal of SG/I. We view W as a right SG module with I acting trivially. Then there exists an idempotent $e \in \hat{SG}_I$ such that $\mathrm{End}\, E \cong e\hat{SG}e$, where $E = E(W)$.

<u>Proof:</u> Suppose $v \in E$, then $M = (vSG + W)$ is a finitely generated submodule of E containing W. Since I has AR2 there is an integer n such that $MI^n \cap W \subseteq WI = 0$. As M is an essential extension of W, we have $MI^n = 0$. Therefore $vI^n = 0$ and $E = \bigcup_k \mathrm{ann}_E I^k$.

Let $T_k = \langle x^{p^k} \mid x \in Q \rangle$, $J_k = \underline{t}_k G + p^{k+1}SG$. We claim that the systems $\{I^k\}$ and $\{J_k\}$ are cofinal and show first by induction that $J_k \subseteq I^k$ for all $k \geq 0$. For $k = 0$ this is clear and the inductive step follows

from the identity

$$x^{p^{k+1}} - 1 = (x^{p^k} - 1 + 1)^p - 1 = (x^{p^k} - 1)^p + \sum_{r=1}^{p-1} \binom{p}{r} (x^{p^k} - 1)^r, \text{ since}$$

all the Binomial coefficients are divisible by p.

We fix an integer k. It remains to show that there exists an integer r such that $I^r \leq J_k$. Let $\widetilde{SQ} = SQ/J_k \cap SQ = S/p^{k+1}S(Q/T_k)$ and denote by $\tilde{\underline{q}}$ the image of $\underline{q}$ in this ring. Then by (34) chapter 5, $\tilde{\underline{q}}$ is nilpotent and hence so too is the ideal $p\widetilde{SQ} + \tilde{\underline{q}}$. Hence there exists an integer r such that $(pSQ + \underline{q})^r \leq (p^{k+1}SQ + \underline{t}_kQ)$. Therefore,

$$I^r = (pSG + \underline{q}G)^r = (pSQ + \underline{q})^r G \leq (p^{k+1}SQ + \underline{t}_kQ)G = J_k.$$

It follows that $\widehat{SG}_I = \widehat{SG}_{\underset{\sim}{J}}$, the completion of SG at the filtration $\underset{\sim}{J} = \{J_k\}$ and also that $E = \bigcup E_k$, where $E_k = \mathrm{ann}_E J_k$.

Now by (46), proposition 2.27, E_k is the injective hull of W as an SG/J_k module, and $SG/J_k = (S/p^{k+1}S)G/T_k$ is a self-injective ring.

As SG/J_0 is self-injective, it contains an idempotent e_0 such that $E_0 \cong e_0SG/J_0$ and we have an induced isomorphism $\mathrm{End}\, E_0 \longleftarrow e_0SG/J_0e_0$ in the usual way.

Suppose we have an idempotent $e_k \in SG/J_k$ and an isomorphism $\Theta_k : E_k \longrightarrow e_kSG/J_k$ which induces an isomorphism $\Theta_k^* : \mathrm{End}\, E_k \longleftarrow e_kSG/J_ke_k$. Then as $J_k/J_{k+1} \leq J(SG/J_{k+1})$ there exists by lemma 2.3.5 an idempotent $e_{k+1} \in SG/J_{k+1}$ such that e_k is the image of e_{k+1} under the natural map $SG/J_{k+1} \longrightarrow SG/J_k$.

Let $\overline{S} = S/p^{k+2}S$. The hypotheses of lemma 2.1 hold with $N = E_{k+1}$, $M = E_k$, $R = SG/J_{k+1} = \overline{S}G/T_{k+1}$, $J = J_k/J_{k+1}$ and $K = \widehat{pT_k/T_{k+1}\overline{S}(G/T_{k+1})}$.

Hence there exists an isomorphism $\Theta_{k+1} : E_{k+1} \longrightarrow e_{k+1}SG/J_{k+1}$ making the induced diagram

$$\begin{array}{ccc} \mathrm{End}\, E_{k+1} & \xleftarrow{\Theta_{k+1}^*} & e_{k+1}SG/J_{k+1}e_{k+1} \\ \text{res.} \downarrow & & \downarrow \text{nat.} \\ \mathrm{End}\, E_k & \xleftarrow[\Theta_k^*]{} & e_kSG/J_ke_k \end{array}$$

commute.

Therefore by taking inverse limits

$$\mathrm{End}\, E = \varprojlim \mathrm{End}\, E_k \cong \varprojlim e_kSG/J_ke_k = L. \text{ The sequence}$$

$(e_1, e_2, \ldots) = e$ is an idempotent in $\varprojlim SG/J_k = \widehat{SG}_{\underset{\sim}{J}} = \widehat{SG}_I$ and $L \leq \widehat{SG}_I$.

Clearly $e\widehat{SG}e \leq L$, and if $l = (l_1, l_2, \ldots) \in L$, then $l = ele = (e_1 l_1 e_1, e_2 l_2 e_2, \ldots) \in e\widehat{SG}e$.

Hence $L = e\widehat{SG}e \cong \mathrm{End}_{SG} E(W)$, and the theorem is proved.

We now discuss some special cases of this result.

<u>Corollary 2.3.</u> If in the notation of theorem 2.2 we have

a) $W = SG/I$, then the idempotent $e = 1$.

b) W a minimal right ideal of SG/I, then the idempotent e is primitive.

<u>Proof:</u> a) This follows since at every stage in the lifting process we must have $e_k = 1$. For, as $I = J_0$, $e_0 = 1$, and then if $e_k = 1$, since J_k/J_{k+1} is a nil ideal in the ring SG/J_{k+1}, the only idempotent mapping onto e_k is $e_{k+1} = 1$.

b) If W is a minimal right ideal of SG/I, then W is irreducible as an SG module. Hence by (46) p.49, corollary 2, $E(W)$ is indecomposable. Therefore by (46), proposition 3.12, $\mathrm{End}\, E(W)$ is a scalar local ring and so has only trivial idempotents. Since $\mathrm{End}\, E(W) \cong e\widehat{SG}e$ by theorem 2.2, e is a primitive idempotent in $e\widehat{SG}e$ and so also in $\widehat{SG}$.

It seems that before we can prove any results for W a proper two sided ideal of SG/I we must impose the additional hypothesis that G is p nilpotent in view of the following result.

<u>Proposition 2.4.</u> a) If M is a nil ideal in a commutative ring S, and G any group then every central idempotent in $(S/M)G$ is the image of a central idempotent in SG.

b) If the prime p is nilpotent in the commutative ring S and N is a normal subgroup of a group G, then every central idempotent in $S(G/N)$ is the image of a central idempotent in SG provided

either i) N is a finite p' subgroup of G,

or ii) G is a finite p nilpotent group and S/pS is a field of characteristic p.

c) Suppose that k is a field of characteristic $p > 0$, and G a finite p soluble group. If all the central idempotents in the group ring of every proper factor group of G over k are the images of central idempotents in kG, then G is p nilpotent.

<u>Proof:</u> First note that if J is a nil ideal in a ring R and e is a central idempotent in R/J, and e is the image of an element z which is central in R, then e is the image of a central idempotent of R. For by lemma 2.3.5, we can choose a polynomial f in z which is an idempotent in R and whose image under the natural map is e.

a) This follows from the above remark since the centres of S/MG and SG are spanned by the class sums.

b i) This follows since $SG = pG \oplus SG\hat{N}/|N|$ is a direct sum of two sided ideals. However we can write down the form of the idempotents quite explicitly. Note that since p is nilpotent and is coprime to $|N|$, $|N|$ is invertible in S and we write $a = |N|^{-1}$. Let $\bar{G} = G/N$.

Let $e = \sum \lambda_g \bar{g}$ be a central idempotent in $S\bar{G}$ and suppose that g is an element of G mapping onto $\bar{g}$. Set $f = \sum \lambda_g g.\hat{N}a$. Clearly f is independent of the choice of the elements g since each element in the same coset of N occurs with the same coefficient in the above sum. It is also clear that $\bar{f} = e$.

Hence $\overline{f^2 - f} = 0$ and so $f^2 - f \in pG$, but $f = f\hat{N}a$ and therefore $f^2 - f = 0$.

Now let $h \in G$. Since $\bar{f} = e$ is central we have $\overline{fh - hf} = 0$. Therefore $fh - hf \in pG$ and it follows as above that $fh - hf = 0$ and so f is a central idempotent in SG mapping onto e.

ii) By i) it suffices to handle the case where N is a p subgroup of the finite p nilpotent group G. Let e be a central idempotent in SG/N. By Osima's theorem (36) 4.3.11, the image $\tilde{e}$ of e in $S/pS(G/N)$ has support consisting of p' conjugacy classes.

Hence $\tilde{e} = \tilde{f}$ where f is in the additive subgroup of the centre of SG/N

spanned by the p' classes. Since G is p nilpotent this is in fact a subring, and since we can find a polynomial in f, which is an idempotent mapping onto $\tilde{f}$, we can assume that f itself is an idempotent.

Then f-e is nilpotent, fe is an idempotent and $g = f - fe = f(f-e)$ is nilpotent.

Now $fe = f - g$, so $(f-g)^2 = f-g$. Therefore $g = g^2$ and $g = 0$, so $f = fe = e$.

In other words the support of e consists of p' conjugacy classes.

If a is a p' element of G, let C(a) denote its class sum, and $\bar{a}$ the image of a in $\bar{G} = G/N$. Then we have $C(\bar{a}) = \overline{C(a)}$, since $\bar{a} = \bar{a}^g$ implies $(a,g) \in N \cap O_{p'}(G) = 1$. Hence we may lift e to a central element of SG and by the remark at the beginning of the proof we may lift e to a central idempotent of SG.

c) Suppose now that G is a finite p soluble group, k a field of characteristic $p > 0$, and that central idempotents may be lifted over $N = O_{p',p}(G)$, then by projecting onto $G/O_{p'}(G)$, it is easily seen that central idempotents in kG/N are the images of central idempotents in $kG/O_{p'}(G)$. Hence we may suppose that $O_{p'}(G) = 1$, and have to show that G is a p group. Let $N = O_p(G)$.

It is well known that $C_G(N) \subseteq N$, see for example (17), theorem 18.4.4. Hence if g is a p' element of G and $g \neq 1$, there exists an element $a \in N$ such that $a^g \neq a$, so $g \neq g^a$.

Suppose that e is a central idempotent in kG. Then by Osima's theorem cited above we have $e = \sum_{\substack{g \text{ a } p' \\ \text{element}}} \lambda_g C(g)$.

We claim that if $g \neq 1$, then $\overline{C(g)} = 0$, where the overbar denotes the image in kG/N. To see this we define an equivalance relation $\sim$ on **Supp** C(g) by $g_1 \sim g_2$ if and only if $g_1^a = g_2$ for some $a \in N$. Let $C_1, \ldots C_s$ be the equivalance classes under $\sim$ and set $\hat{C}_i = \sum_{g \in C_i} g$. Then $C(g) = \sum_{i=1}^{s} \hat{C}_i$. Now if g_i is a representative of C_i then since N permutes the elements of C_i transitively, we have $|C_i| = |N:C_N(g_i)|$ a power of

p. Also $|C_1| \neq 1$ by the second paragraph of the proof, and the elements of C_1 all lie in the same coset of N. Hence $\overline{\hat{C}_1} = 0$, and $\overline{C(g)} = 0$, if g is a p' element of G and $g \neq 1$. Therefore $\bar{e} = 0$ or 1.

However if G is not a p group, $N = O_p(G) \lneq G$. Let $\bar{G} = G/N$ and $Q = O_{p'}(\bar{G})$. Then it is easily seen that $\hat{Q}/|Q|$ is a non-trivial central idempotent in $k\bar{G}$. This contradiction shows that G is a p group.

Before proving the next result we make the following general remark. If e is a central idempotent in $R = (S/p^kS)G$ a group ring of a finite group, then the composition factors of $I = eR$ and $Soc(I)$ are identical. For since R is quasi-Frobenius, any irreducible R module is isomorphic to a minimal right ideal of R. However $Soc(R) = Soc(I) \oplus Soc((1-e)R)$ and if V is a composition factor of I (as a right R module), then $Ve = V$, and hence V is isomorphic to a submodule of $Soc(I)$.

<u>Theorem 2.5.</u> With the hypothesis of theorem 2.2, suppose in addition that G is p nilpotent, and that $\mathcal{J}$ is a two sided ideal of SG/I.

Let $M = ann_{SG}\mathcal{J}$, then there exists a central idempotent $e \in \hat{SG}_I$ such that $End\ E \cong e\,\hat{SG}_I \cong \hat{SG}_M$.

<u>Proof:</u> The existence of the central idempotent e and the first isomorphism follow from lemma 2.1, theorem 2.2 and proposition 2.4, since at each stage the idempotent e_k can be chosen to be central.

Recall that in the notation of theorem 2.2, $J_k = t_kG + p^{k+1}SG$, $E_k = ann_E J_k$, and E_k is the injective hull of $\mathcal{J}$ as an SG/J_k module with $SG/J_k = [S/p^{k+1}S]G/T_k$. We have $E_k = e_kSG/J_k$ and $Soc(E_k) \leq \mathcal{J}$.

Therefore by the remarks preceding the theorem $M^r \leq ann_{SG}E_k$ for some integer r.

On the other hand $ann_{[S/p^{k+1}S]G/T_k}E_k = (1-e_k)(S/p^{k+1}S)G/T_k$ and $(1-e_k)$ is contained in the image $\bar{M}$ of M in $(S/p^{k+1}S)G/T_k$.

Since $(1-e_k)$ is an idempotent, we have $(1-e_k) \in \cap \bar{M}^r$.

In particular $(1-e_k) \in \bar{M}^k$, and as $J_k \leq I^k \leq M^k$, we have $M^k/J_k = (M/J_k)^k$ and it follows that $ann_{SG}E_k \leq M^k$.

Hence the filtrations $\{\mathrm{ann}_{SG}E_k\}$, and $\{M^k\}$ induce the same topology on SG.

By a similar argument to that used in proving theorem 2.2 we can now show that $\widehat{SG}_I \cong \mathrm{End}_{SG}E$ by taking inverse limits.

We next obtain a decomposition of $\widehat{SG}_I$ by allowing W to range over each 'block' in SG/I and M to range over the corresponding annihilators.

<u>Theorem 2.6.</u> With the hypothesis of theorem 2.2, suppose in addition that G is p nilpotent and that

$SG/I = S/pS(G/Q) = \bigoplus_{i=1}^{r} z_i S/pS(G/Q)$ is the decomposition of SG/I into indecomposable two sided ideals. Here the z_i are centrally primitive idempotents. Let $W_i = \mathrm{Soc}(z_i S/pS(G/Q))$, $M_i = \mathrm{ann}_{SG}W_i$ and $E_i = E_{SG}(W_i)$.

Then there exist centrally primitive idempotents $e_1, e_2, \ldots e_r$ in $\widehat{SG}_I$ such that

$$\begin{array}{ccccccccc}
\widehat{SG}_I & \cong & e_1\widehat{SG}_I & \oplus & e_2\widehat{SG}_I & \oplus & \cdots & \oplus & e_r\widehat{SG}_I \\
 & & \cong & & \cong & & & & \cong \\
 & & \widehat{SG}_{M_1} & \oplus & \widehat{SG}_{M_2} & \oplus & \cdots & \oplus & \widehat{SG}_{M_r} \\
 & & \cong & & \cong & & & & \cong \\
 & & \mathrm{End}\ E_1 & \oplus & \mathrm{End}\ E_2 & \oplus & \cdots & \oplus & \mathrm{End}\ E_r.
\end{array}$$

<u>Proof:</u> The vertical isomorphisms have been obtained in the preceding theorem. It merely remains to verify the direct sum decomposition. This can be seen in a number of ways.

1) At the kth stage of the lifting process of theorem 2.2 we obtain centrally primitive idempotents $e_{k,1}, e_{k,2}, \ldots e_{k,r}$ such that $S/p^{k+1}S(G/T_k) \cong \bigoplus_{i=1}^{r} e_{k,i} S/p^{k+1}S(G/T_k)$.

The result follows on taking inverse limits.

2) It is easily seen that $\widehat{SG}_I \cong \mathrm{End}\ E_{SG}(SG/I) = \mathrm{End} \bigoplus_{i=1}^{r} E_i = \bigoplus_{i=1}^{r} \mathrm{End}\ E_i$ since the E_i arise from distinct blocks in the finite group rings $S/p^{k+1}S(G/T_k)$.

We next show that each endomorphism ring End E_i is a full matrix ring over a scalar local ring.

<u>Theorem 2.7.</u> Let $z = z_i$ be one of the centrally primitive idempotents of theorem 2.6, and $V = \mathrm{Soc}(zS/pS(G/Q)) = \bigoplus_{j=1}^{n} V_j$, the direct sum of n

isomorphic copies of an irreducible right SG module V, and let $D = \mathrm{End}\, E(V)$.

Then $\mathrm{End}\, E(\mathcal{V}) = M_n(D)$, and D is a complete scalar local ring.

<u>Proof:</u> The fact that the V_j are isomorphic, which is well known when k is algebraically closed, (47) lemma 6, follows easily from the fact that $M = \mathrm{ann}_{SG} V$ has AR2 by (6) or (43) for any irreducible SG/I module V. For every submodule of $E_{SG/I}(V)$ is annihilated by some power of M, and so every composition factor is isomorphic to V.

Hence if V and V' are non-isomorphic irreducible SG/I modules then $\mathrm{Hom}_{SG}(E(V), E(V')) = 0$, and so V and V' lie in different blocks.

Therefore $E_{SG}(\mathcal{V}) \cong \bigoplus E_{SG}(V_j)$.

Let $\pi_j : E(\mathcal{V}) \longrightarrow E(V_j)$, $\eta_i : E(V_i) \longrightarrow E(\mathcal{V})$ denote the projection and inclusion maps.

If $\alpha \in \mathrm{End}\, E(\mathcal{V})$, then

$\alpha_{ij} = \eta_i \alpha \pi_j \in \mathrm{Hom}_{SG}(E(V_i), E(V_j)) \cong D$, and it is easily seen that the map $\alpha \longrightarrow (\alpha_{ij})$ is an isomorphism,

$$\mathrm{End}\, E(\mathcal{V}) \longrightarrow M_n(D).$$

By lemma 3.10 and proposition 3.12 of (46), $\mathrm{End}\, E(V)$ is a scalar local ring with maximal ideal $J = \{\theta \in \mathrm{End}\, E(V) \mid \theta(V)\} = 0$. It is easily seen that $\mathrm{End}\, E(V)$ is complete in the topology induced by $\{J^n\}_{n \geq 0}$.

<u>Remarks i)</u> If G is a finite p' group and k is a field of characteristic p, the preceding two results with $I = 0$ follow from the familiar Artin-Wedderburn theorem. It is also interesting to compare these results with work of Goldie (15) on complete local rings Q having a unique maximal ideal M such that Q/M is simple artinian and $\bigcap M^n = 0$. In this situation we have $Q/M \cong M_n(D)$ and $Q \cong M_n(L)$, where D is a division ring and L is a complete scalar local ring, (15) theorem 4.5.

ii) Part 3) of the main theorem is now immediate from theorems 2.5–2.7.

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Then $\mathrm{End}\, E(\mathcal{V}) = M_n(D)$, and D is a complete scalar local ring.

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ii) Part 3) of the main theorem is now immediate from theorems 2.5-2.7.

3. Certain completions are Noetherian.

If I is an ideal in a Noetherian ring R, and I has AR2, we do not know whether the completion $\hat{R}$ is Noetherian, or whether the ideal $\hat{I}$ of $\hat{R}$ generated by the image of I also has AR2. As in the commutative case we have $\hat{I} \subseteq J(\hat{R})$, (2) 10.15. If R is Noetherian and semilocal, with $\hat{I} = J(\hat{R})$, then by (25), theorem 4.1, $\hat{I}$ has AR2 if and only if every right ideal of $\hat{R}$ is closed in the $\hat{I}$-adic topology, that is

$$\bigcap_{n \geq 1} (M + \hat{I}^n) = M \quad \text{for all } M \trianglelefteq \hat{R}.$$

We note that the closure of 0 is $\bigcap \hat{I}^n$ and hence if Jacobson's conjecture holds, 0 is always a closed ideal. Also we do not require that $\bigcap I^n = 0$ for our results, that is the I-adic topology on R need not be Hausdorff and R need not embed in $\hat{R}$. However this condition, which is required by some authors, see Goldie (15), can be imposed without much difficulty if R = SG, the group ring of a polycyclic group G over a prime ring S, for G has a normal subgroup H of finite index such that SH is a prime ring, and then if I is an ideal of SH with AR2, we have $\bigcap I^n = 0$, by (36) 11.2.13.

Before stating the first main result of this section we collect some results from the literature whose proofs apply in more general situations than those originally considered.

<u>Proposition 3.1.</u> i) Let I be an ideal in a ring R, and suppose that I has AR1. Then the completion $\hat{R}_I$ is Noetherian.

ii) Let I be an ideal in a ring R, and suppose that every right ideal of R is closed in the I-adic topology and that the graded ring $\bigoplus_{n \geq 0} I^n/I^{n+1}$ is Noetherian, then I has AR2.

<u>Proof:</u> i) is proved exactly as in the commutative case (2) 10.26, or (24) theorem [illegible].

ii) may be proved in the same manner as (15) theorem 5.4.

<u>Theorem 3.2.</u> Suppose that I is an ideal in a Noetherian ring R, and that I has [illegible] and that R/I is artinian. Then the ideal $\hat{I}$ of $\hat{R}$ generated

by the image of I has ARZ.

We first prove a lemma.

<u>Lemma 3.3.</u> Let I be an ideal in a ring R such that $\cap I^n = 0$. If $\hat{R} = \varprojlim R/I^n$, and $\hat{I}$ denotes the image of I in R, then $\cap \hat{I}^n = 0$.

<u>Proof:</u> Since $\cap I^n = 0$, we may regard $\hat{R}$ as the ring of Cauchy sequences with respect to the filtration $\{I^n\}$, and $\hat{I}^n$ as the ideal of Cauchy sequences of elements of I^n.

Suppose $a = (a_1, a_2, \ldots) \in \cap \hat{I}^n$. We claim that for any integer n, there exists an integer f(n) such that $a_m \in I^n$, for $m \geqslant f(n)$. If this were not the case then $a_i \in I^n$ for infinitely many values of i. However this contradicts the fact that a can be written as a Cauchy sequence of elements of I^n.

This shows that $a = (a_1, a_2, \ldots)$ is in fact a Null sequence in the I-adic topology and since $\cap I^n = 0$, we have $a = 0$.

<u>Proof of theorem 3.2:</u> Suppose first that $\bigcap_{n=1}^{\infty} I^n = 0$. By lemma 3.3, 0 is closed in the $\hat{I}$-adic topology. Now $\hat{I} \leqslant J(\hat{R})$ by (2) 10.15, so $\hat{I}^n \leqslant J(\hat{R})^n$ for all n. On the other hand $\hat{R}/\hat{I}^n$ is artinian for all n and hence there exists an integer k such that $J(\hat{R})^k \leqslant \hat{I}^n$.

Therefore the $\hat{I}$-adic and $J(\hat{R})$-adic topologies on $\hat{R}$ are the same, and 0 is closed in the $J(\hat{R})$-adic topology. Hence $\hat{R}$ is a semilocal ring in the sense of Himohara, and by (23) theorem 1, right ideals of $\hat{R}$ are closed in the $J(\hat{R})$-adic topology and so in the $\hat{I}$-adic topology also. By proposition 3.1 applied to the ideal $\hat{I}$ of the ring $\hat{R}$, it will suffice to show that the graded ring $\oplus\, \hat{I}^n/\hat{I}^{n+1}$ is Noetherian, but as in the commutative case, (2) 10.15, we have $\hat{I}^n/\hat{I}^{n+1} \cong I^n/I^{n+1}$ and so $\oplus\, \hat{I}^n/\hat{I}^{n+1} \cong \oplus\, I^n/I^{n+1}$ and this is a factor ring of $R^*(I)$ which is

assumed to be Noetherian. The result follows in this case.

In general let $I^{\infty} = \bigcap I^n$, $T = R/I^{\infty}$, and $\bar{I} = I/I^{\infty}$.

Then $\hat{T}_{\bar{I}} = \varprojlim T/\bar{I}^n \cong \varprojlim R/I^m = \hat{R}_I$, and $\bar{I}\hat{T} \cong \hat{I}$. Now the ring T and the ideal $\bar{I}$ satisfy the hypothesis of the theorem and we have in addition that $\bigcap \bar{I}^n = 0$. Hence $\bar{I}\hat{T} \cong \hat{I}$ has AR2.

The existence of ideals satisfying the hypothesis of theorem 3.2 is guaranteed by the following result of K.A.Brown.

<u>Theorem 3.4.</u> ((5) theorems 2.3.5 and 2.3.6, see also (24) theorem 6).

Let S be the ring of integers or a field of characteristic $p > 0$, and let G be a polycyclic-by-finite group. If H is a normal subgroup of finite index in G, there exists a normal subgroup Q of finite index in G such that $Q \leq H$ and $I = pSG + \underline{q}G$ has AR1.

<u>Corollary 3.5.</u> Let SG and I be as in theorem 3.4. Then $\widehat{SG}_I$ is Noetherian, and $\hat{I}$ has AR2.

<u>Proof:</u> Immediate from proposition 3.1 i) and theorem 3.2.

<u>Theorem 3.6.</u> Let G be a polycyclic-by-finite group, S the ring of integers or a field of characteristic $p > 0$, and V an irreducible SG module such that $|G:C_G(V)| < \infty$. Then End E(V) is a Noetherian ring.

<u>Proof:</u> Let $H = C_G(V)$. By (36) 12.2.9, if $S = \mathbb{Z}$, there is a prime p such that $Vp = 0$. By theorem 3.4 there exists a normal subgroup Q of finite index in G such that $Q \leq H$ and $I = pSG + \underline{q}SG$ has AR1. We may regard V as a right ideal of SG/I and by theorem 2.2, there exists an idempotent $e \in \widehat{SG}_I$ such that End $E(V) \cong e\widehat{SG}e$. By corollary 3.5, $\widehat{SG}$ is Noetherian. Let $I_1 \leq I_2 \leq \ldots$ be an ascending chain of right ideals of $e\widehat{SG}e$. Then there exists an integer k such that $I_k\widehat{SG} = I_{k+1}\widehat{SG} = \ldots$

However $I_t = I_t e\widehat{SG}e = I_t\widehat{SG}e$ and so $I_k = I_{k+1} = \ldots$

<u>Theorem 3.7.</u> Let G be a p nilpotent group and S the ring of integers or a field of characteristic $p > 0$. Let M be a maximal ideal of SG such that $\underline{q}G \leq M$, for some normal subgroup H of finite index in G. If $S = \mathbb{Z}$ assume in addition that $p \in M$. Then $\widehat{SG}_M$ is Noetherian.

Proof: By theorem 3.4 there exists a normal subgroup H of finite index in G such that $H \leq N$ and $I = pSG + qSG$ has ARl. Thus by corollary 3.5, $\widehat{SG}_I$ is Noetherian. However by theorem 2.6, $\widehat{SG}_H$ is a direct summand of $\widehat{SG}_I$. Hence $\widehat{SG}_H$ is Noetherian.

Remark: The above result should be compared with (24) corollary 9. Also if G is an extension of a finite p' group by a finitely generated nilpotent-by-(finite p) group, the result follows from (32), at least if S is a field of characteristic p, since then by 1.3.5, kG is a polycentral ring.

4. Morita duality.

The aim of this section is to show that certain completions of polycyclic-by-finite group rings have Morita duality. Parts 1) and 2) of the main theorem will then follow easily.

If R is a ring, an R module E is said to be an <u>injective cogenerator</u> of R if E is injective and for every R module A and every non-zero element a of A, there is a module homomorphism $\phi : A \longrightarrow E$ such that $\phi(a) \neq 0$. We shall make use of the following criterion.

Lemma 4.1. Let E be an injective module over a right Noetherian ring R. Then E is an injective cogenerator if and only if $Soc(E)$ contains an isomorphic copy of every irreducible right R module.

Proof: Suppose that the stated condition holds. Let A be a right R module and $a \in A$, $a \neq 0$. Let B be a maximal proper submodule of the Noetherian module aR. Then aR/B is irreducible and hence by assumption there is a right module monomorphism $\phi : aR/B \longrightarrow Soc(E)$. Now ϕ lifts to a module homomorphism $\phi_1 : aR \longrightarrow Soc(E)$ and since E is injective ϕ_1 extends to a right module homomorphism $\phi_2 : A \longrightarrow E$ such that $\phi_2(a) \neq 0$.

Conversely suppose that E is an injective cogenerator, and let V be an irreducible R module, and $v \in V$, $v \neq 0$. Let $\phi : V \longrightarrow E$ be a module homomorphism such that $\phi(v) \neq 0$. Clearly $\phi(V)$ is an isomorphic copy of V contained in $Soc(E)$.

Now another definition. If R is a ring, then by (50) theorem 3.3 R is <u>Morita dual</u> to itself if and only if R has a bimodule E such that E is an injective cogenerator on both sides and $R \cong \mathrm{End}_R(E_R)$ and $R \cong \mathrm{End}_R({}_RE)$, and in fact we shall use this as our definition.

As a first step we show that if $I = qG + pSG$ is an ideal of SG chosen as in theorem 3.4, then $\widehat{SG}_I$ is Morita dual to itself and has $E_{SG}(SG/I) = E$ as an injective cogenerator.

Let $I^{\infty} = \bigcap I^n$, $T = SG/I^{\infty}$ and $\overline{I} = I/I^{\infty}$. We first note that since I has AR2, $E = \bigcup \mathrm{ann}_E I^k$ and E may be regarded in a natural way as a T module, since $EI^{\infty} = 0$. Suppose that $a = (a_1, a_2, \ldots)$ is a Cauchy sequence of elements of T in the $\overline{I}$-adic topology. By passing to a subsequence if necessary, we may assume that $a_j - a_i \in \overline{I}^i$ for all $j \geqslant i$.

If $v \in E$, then $vI^k = 0$ for some k and we set $va = va_k$. Since $\hat{T}_{\overline{I}} \cong \widehat{SG}_I$, this makes E into a right $\widehat{SG}_I$ module.

Since $T \leq \hat{T}_{\overline{I}}$ we have $\mathrm{End}_{\widehat{SG}}(E_{\widehat{SG}}) \cong \mathrm{End}_{\hat{T}_{\overline{I}}}(E_{\hat{T}}) \leq \mathrm{End}_T(E_T) = \mathrm{End}_{SG}(E_{SG})$. On the other hand suppose $\phi \in \mathrm{End}_{SG}(E_{SG})$, $v \in E$, $vI^k = 0$ and $a \in \widehat{SG}_I \cong \hat{T}_{\overline{I}}$ as above. Then $\phi(va) = \phi(va_k) = \phi(v)a_k = \phi(v)a$.

Hence $\mathrm{End}_{\widehat{SG}}(E_{\widehat{SG}}) \cong \mathrm{End}_{SG}(E_{SG}) \cong \widehat{SG}_I$, by corollary 2.3.

Now E may be regarded as a left module over its endomorphism ring, $\widehat{SG}_I$ and by taking inverse limits as in theorem 2.2 we see that $\mathrm{End}_{\widehat{SG}}({}_{\widehat{SG}}E) \cong \widehat{SG}_I$. We outline the details. Let $W = SG/I$. If $\phi \in \hat{I}$ is an endomorphism of E_{SG} then clearly $\phi(W) = 0$. Hence $\hat{I}W = 0$.

Since by theorem 3.2, $\hat{I}$ has AR2 we have ${}_{\widehat{SG}}E = \bigcup \mathrm{ann}_E \hat{I}^k$.

Let $T_k = \langle x^{p^k} \mid x \in Q \rangle$, $J_k = t_kG + p^{k+1}SG$, and $\hat{J}_k$, the two sided ideal of $\widehat{SG}$ generated by the image of J_k. Then as in theorem 2.2 the systems $\{\hat{I}^k\}$ and $\{\hat{J}_k\}$ are cofinal and we have ${}_{\widehat{SG}}E = \bigcup E_k$, where $E_k = \mathrm{ann}_E \hat{J}_k$ is the injective hull of W as an $\widehat{SG}/\hat{J}_k \cong SG/J_k$ module. As in the proof of theorem 2.2 we get a commutative diagram of ring homomorphisms.

$$\begin{tikzcd} \mathrm{End}\,({}_{\widehat{SG}}E_{k+1}) \arrow[d, "\text{res.}"'] & SG/J_{k+1} \arrow[l] \arrow[d, "\text{nat.}"] \\ \mathrm{End}\,({}_{\widehat{SG}}E_k) & SG/J_k \arrow[l] \end{tikzcd}$$

and it follows that $\mathrm{End}_{\widehat{SG}}({}_{\widehat{SG}}E) \cong \widehat{SG}$, by taking inverse limits.

To show that $\widehat{SG}_I$ is Morita dual to itself it remains to show that $E_{\widehat{SG}}(W) = E_{SG}(W)$ is an injective cogenerator.

<u>Lemma 4.2.</u> Let $I = qG + pSG$ be an ideal of SG chosen as in theorem 3.4 and let $W = SG/I$. Then $E_{\widehat{SG}}(W) = E_{SG}(W) = E$ is an injective cogenerator as an $\widehat{SG}_I$ bimodule.

<u>Proof:</u> Suppose first that L is a right ideal of $\widehat{SG}_I$. By proposition 3.1, L is finitely generated. Suppose $\phi : L \longrightarrow E$ is a right $\widehat{SG}$ module homomorphism. Then $\phi(L) \subseteq \mathrm{ann}_E I^k$ for some k, and therefore $\phi(L)I^k = 0$, and so $\phi(L)\hat{I}^k = 0$. Since $\hat{I}$ satisfies AR2, there exists an integer n such that $\widehat{SG}\hat{I}^n \cap L \subseteq L\hat{I}^k$.

Hence $\phi(\hat{I}^n \cap L) = 0$ and ϕ induces a map

$\phi_1 : L/L \cap \hat{I}^n \cong L + \hat{I}^n/\hat{I}^n \longrightarrow E$. Now $L + \hat{I}^n/\hat{I}^n$ is a right ideal in $\widehat{SG}/\hat{I}^n$, but the image of SG is dense in $\widehat{SG}$ so $SG + \hat{I}^n = \widehat{SG}$, and $SG \cap \hat{I}^n = I^n$ for all n.

Therefore $\widehat{SG}/\hat{I}^n = SG + \hat{I}^n/\hat{I}^n \cong SG/SG \cap \hat{I}^n = SG/I^n$. Now $\mathrm{ann}_E I^n$ is injective as an SG/I^n module and so we get a map

$\phi_2 : \widehat{SG}/\hat{I}^n \longrightarrow \mathrm{ann}_E I^n \subseteq E$, extending ϕ.

In fact the above proof shows that E is also injective as a left $\widehat{SG}$ module since as we saw just before the lemma, ${}_{\widehat{SG}}E = \cup E_k$, where $E_k = \mathrm{ann}_E \hat{I}^k$.

Now $\hat{I} = I\widehat{SG} \subseteq J(\widehat{SG})$ by (2) 10.15 and so any irreducible right or left $\widehat{SG}$ module is also irreducible as an $\widehat{SG}/\hat{I} \cong SG/I$ module. Since $W = SG/I$, $\mathrm{Soc}(E)$ contains an isomorphic copy of every irreducible $\widehat{SG}$ module. Therefore by lemma 4.1 E is an injective cogenerator as an $\widehat{SG}$ bimodule.

Theorem 4.3. Let S be the ring of integers or a field of characteristic $p > 0$, and suppose $I = qG + pSG$ is an ideal of SG chosen as in theorem 3.4. Then $\widehat{SG}_I$ is Morita dual to itself.

Proof: This is now immediate from what we have proved above.

In more detail, let $W = SG/I$, $E = E_{SG}(W) = E_{\widehat{SG}}(W)$. Then by lemma 4.1 E is an injective cogenerator as an $\widehat{SG}_I$ bimodule and we have $\mathrm{End}_{\widehat{SG}}(E_{\widehat{SG}}) \cong \widehat{SG}_I$ and $\mathrm{End}_{\widehat{SG}}({}_{\widehat{SG}}E) \cong \widehat{SG}_I$. A right or left $\widehat{SG}_I$ module U is said to be E-reflexive if $\mathrm{Hom}_{\widehat{SG}}(\mathrm{Hom}_{\widehat{SG}}(U,E),E)$ is naturally isomorphic to U. Theorem 3.3 of (50) states that the categories ${}_{\widehat{SG}}\mathcal{N}$, $\mathcal{N}_{\widehat{SG}}$ of E-reflexive left and right $\widehat{SG}$ modules are closed under submodules, factor modules and finite direct sums and contain all finitely generated modules.

For the sake of completeness, we calculate the dual category to the category of finitely generated (left) $\widehat{SG}$ modules.

Suppose that U is a finitely generated left $\widehat{SG}$ module. Then for some integer s $\widehat{SG}^{(s)} \to U \to 0$ is exact. Applying the functor $\mathrm{Hom}_{\widehat{SG}}(\ ,E): {}_{\widehat{SG}}\mathcal{N} \to \mathcal{N}_{\widehat{SG}}$ gives $0 \to \mathrm{Hom}_{\widehat{SG}}(U,E) \to \mathrm{Hom}_{\widehat{SG}}(\widehat{SG}^{(s)},E) \cong E^{(s)}$.

Now since W is artinian $E = E_{\widehat{SG}}(W)$ has finitely generated essential socle and so is finitely embedded. Hence the finite direct sum $E^{(s)}$ is finitely embedded and so too is the submodule $\mathrm{Hom}_{\widehat{SG}}(U,E)$. Similarly one can show that if U is finitely embedded then $\mathrm{Hom}_{\widehat{SG}}(U,E)$ is finitely generated.

The above result is an analogue of the result of Matlis (31) for commutative, Noetherian, complete local rings. It is also possible to prove our theorem 4.3 by showing directly that the categories of finitely generated (left) $\widehat{SG}$ modules and finitely embedded (right) $\widehat{SG}$ modules are dual without using (50), and this is the approach adopted by Matlis.

A variant of theorem 4.3 can be obtained for maximal ideals in group rings of p nilpotent groups using lemma 4.2 and theorem 2.6 to see that $E_{SG}(SG/M)$ is injective as an $\widehat{SG}_M$ bimodule.

Lemma 4.4. Let G be a p nilpotent group, S the ring of integers or a field of characteristic $p > 0$. Let M be a maximal ideal of SG and suppose that M contains an ideal of the form $I = qG + pSG$ as in theorem 2.2, and that I has AR1. Then $E_{\hat{SG}}(SG/M)$ is an injective cogenerator as an $\hat{SG}_M$ bimodule.

Proof: We adopt the notation of theorem 2.6 and suppose that $SG/M = W = W_1$, $M = M_1$ etc.

Then $E_{\hat{SG}}(SG/I) = \oplus E_{\hat{SG}_I}(W_i)$, and $\hat{SG}_I \cong \oplus \hat{SG}_{M_i}$ by theorem 2.6 and $\hat{SG}_M$ is a direct summand of $\hat{SG}_I$ and any right (left) ideal L of $\hat{SG}_M$ can be regarded as a right (left) ideal of $\hat{SG}_I$ and so any map from L to $E_{\hat{SG}}(W)$ extends to $\hat{SG}_I$ and hence to $\hat{SG}_M$. The fact that $E_{\hat{SG}}(W)$ is a cogenerator is proved as before using lemma 4.1.

We note that the hypotheses on M in lemma 4.4 are satisfied whenever $S = Z$ or an absolute field of characteristic $p > 0$ and M is a maximal ideal of SG containing the prime p. This follows from theorem 3.4 and (41) theorem A.

Theorem 4.5. Let M and SG be as in lemma 4.4. Then

a) $\hat{SG}_M$ is Morita dual to itself.

b) $\hat{M}$ has AR2 as an ideal of $\hat{SG}_M$.

Proof: a) This is similar to theorem 4.3, and we omit the proof.

b) By a result of Hartley (unpublished), since $\hat{SG}_M$ is Noetherian, it suffices to show that if W is a finitely generated essential extension of an $\hat{SG}_M$ module V such that $V\hat{M} = 0$, then $W\hat{M}^n = 0$ for some integer n.

Since $V\hat{M} = 0$, V is an $\hat{SG}/\hat{M}$ module, and since $\hat{SG}/\hat{M} \cong SG/M$ is an artinian ring and V is a finitely generated module we may assume that V is irreducible. Hence W is isomorphic to a submodule of $E_{SG}(V)$, and $E_{SG}(V) = E_{\hat{SG}_M}(V)$ by lemma 4.4. Hence, since M has AR2 by (6), or (43), it follows that $WM^n = 0$ for some integer n and that $W\hat{M}^n = 0$.

We next aim to prove parts 1) and 2) of the main theorem. Since we have occasionally passed to subgroups of finite index it is helpful to

have the following result.

<u>Lemma 4.6</u> (see also (48) p.225, corollary). Let S be a commutative ring, H a normal subgroup of finite index in a group G, and W an SG module. Then $E_{SH}(W)$ is artinian if and only if $E_{SG}(W)$ is artinian.

<u>Proof:</u> Suppose that $E = E_{SH}(W)$ is artinian. Then by lemma 1.4.15, $E \otimes_{SH} SG$ is an injective SG module containing W. As an SH module, $E \otimes_{SH} SG = \bigoplus_{i=1}^{n} E \otimes g_i$, where $g_1, \ldots g_n$ is a transversal to H in G, and this is artinian as an SH module. Hence $E \otimes_{SH} SG$ is artinian as an SG module, and so is the submodule $E_{SG}(W)$.

Conversely if $E_{SG}(W)$ is artinian as an SG module, then by (45) lemma 8, it is artinian as an SH module.

However $E_{SG}(W)$ is injective as an SH module by lemma 1.4.14, and hence contains an isomorphic copy of $E_{SH}(W)$. It follows that $E_{SH}(W)$ is artinian.

<u>Proof of the main theorem, part 1:</u> We have S = Z or a field of characteristic $p > 0$, G a polycyclic-by-finite group and V a finitely generated SG module such that $|G:C_G(V)| < \infty$ and $Vp = 0$ and we have to prove that $E_{SG}(V)$ is artinian.

The conditions on V imply that $SG/\mathrm{ann}_{SG}V$ is an artinian ring, and since V is finitely generated we may assume that V is irreducible.

By theorem 3.4, we can choose an ideal of the form $I = QG + pSG$ contained in $\mathrm{ann}_{SG}V$, with Q a normal subgroup of finite index in G such that I has ARI. By theorem 4.3 $\widehat{SG}_I$ is Morita dual to itself with $D_{SG}(W)$ as an injective cogenerator, where $V \subseteq W = SG/I$. In this duality $E_{SG}(W)$ corresponds to $\widehat{SG}_I$ and we have a lattice anti-isomorphism between the submodules of $E_{SG}(W)$ and (left) ideals of $\widehat{SG}_I$.

However by proposition 3.1 i) $\widehat{SG}_I$ is Noetherian. Hence $E_{SG}(W)$ and $E_{SG}(V)$ are artinian modules.

Alternatively, since $\widehat{SG}_I$ has Morita duality, factor modules of finitely embedded modules are finitely embedded by (50) theorem 3.3, and

$E_{SG}(V)$ is artinian by theorem 1.4.9.

Proof of the main theorem, part 2: We continue with the notation adopted above. If $X = A/B$ is a composition factor of $E_{SG}(V) = E$, then $A = aSG + B$, for some $a \in A$. Since I has AR2, $aI^n = 0$ for some n. Hence $XI^n = 0$, and since X is irreducible $XI = 0$. Therefore X is an irreducible SG/I module and since SG/I is artinian there are, up to isomorphism only finitely many possibilities for X.

If in addition G is p nilpotent and $M = \mathrm{ann}_{SG}V$, then since M has AR2, and SG/M is a simple artinian ring, the same proof with M in place of I shows that V is, up to isomorphism the only composition factor of $E_{SG}(V)$.

5. Finite dimensional modules.

In this section we indicate briefly how the main theorem may be applied to show that if k is any field of characteristic $p > 0$, G a polycyclic-by-finite group and V a finite dimensional kG module, then $E_{kG}(V)$ is an artinian module with finitely many isomorphism types of composition factors.

Theorem 5.1. If k is a field of characteristic $p > 0$, G a polycyclic-by-finite group and V a finite dimensional kG module, then E(V) is an artinian module with only finitely many isomorphism types of composition factors.

Proof: By lemma 1.4.14, we may assume that k is algebraically closed.

Let $\mathrm{Soc}(V) = V_1 \oplus V_2 \oplus \dots \oplus V_n$ where the V_i are irreducible. Then $E(V) = \bigoplus_{i=1}^{n} E(V_i)$ and we may assume that V is irreducible.

Let $H = C_G(V)$. Then by Mal'cev's theorem on soluble linear groups, (49) lemma 3.5, G/H is abelian-by-finite. Therefore by lemma 4.6, we may assume that G/H is torsion free abelian. As k is algebraically closed, we have $\dim_k V = 1$.

Let $V = k\langle v\rangle$ for some $v \in V$. If $g \in G$, $vg = \chi(g)v$ for some $\chi(g) \in k^*$, and we have a corresponding character $\chi : G \to k^*$ of G.

Let $G_1 = \{g\chi(g)^{-1} \mid g \in G\}$ a subgroup of the group of units of kG. Then we have a group homomorphism $G \longrightarrow G_1$ sending g to $g\chi(g)^{-1}$. This is clearly surjective, and if $g\chi(g)^{-1} = 1$, then $g = \chi(g) \in k^*$, so $g = 1$.

Hence $G \cong G_1$ and clearly the subalgebra of kG spanned by G_1 is just the group algebra kG. Of course V can be regarded as a kG_1 module in a natural way, and as such it is trivial for G_1. Thus since $E_{kG_1}(k)$ is an artinian module with finitely many isomorphism types of composition factors, the same holds for $E_{kG}(V)$.

In more detail, suppose that W is any kG module containing V. Then W is a kG_1 module via $w(g\chi(g)^{-1}) = \chi(g)^{-1}wg$ and clearly W is an essential extension of V as a kG module if and only if W is a kG_1 essential extension of k. The result now follows from the main theorem.

It seems appropriate to mention at this point that Steve Donkin (10) has recently shown that if k is a field of characteristic zero, G a polycyclic-by-finite group, and V a finite dimensional kG module, then $E_{kG}(V)$ is artinian. The situation does differ somewhat from that considered above in view of the absence of ideals with Artin-Rees properties. If k is a field of characteristic 0, and $\underline{g}$ the augmentation ideal of the polycyclic-by-finite group G over k, then by (36) 11.2.5 $\underline{g}$ has AR2 if and only if G is finite-by-nilpotent, and Donkin has shown that $E_{kG}(k)$ has finitely many isomorphism types of composition factors if and only if G is nilpotent-by-finite.

Let $G_1 = \{g\chi(g)^{-1} \mid g \in G\}$ a subgroup of the group of units of kG. Then we have a group homomorphism $G \longrightarrow G_1$ sending g to $g\chi(g)^{-1}$. This is clearly surjective, and if $g\chi(g)^{-1} = 1$, then $g = \chi(g) \in k^*$, so $g = 1$.

Hence $G \cong G_1$ and clearly the subalgebra of kG spanned by G_1 is just the group algebra kG. Of course V can be regarded as a kG_1 module in a natural way, and as such it is trivial for G_1. Thus since $E_{kG_1}(k)$ is an artinian module with finitely many isomorphism types of composition factors, the same holds for $E_{kG}(V)$.

In more detail, suppose that W is any kG module containing V. Then W is a kG_1 module via $w(g\chi(g)^{-1}) = \chi(g)^{-1}wg$ and clearly W is an essential extension of V as a kG module if and only if W is a kG_1 essential extension of k. The result now follows from the main theorem.

It seems appropriate to mention at this point that Steve Donkin (10) has recently shown that if k is a field of characteristic zero, G a polycyclic-by-finite group, and V a finite dimensional kG module, then $E_{kG}(V)$ is artinian. The situation does differ somewhat from that considered above in view of the absence of ideals with Artin-Rees properties. If k is a field of characteristic 0, and $\underline{g}$ the augmentation ideal of the polycyclic-by-finite group G over k, then by (36) 11.2.5 $\underline{g}$ has AR2 if and only if G is finite-by-nilpotent, and Donkin has shown that $E_{kG}(k)$ has finitely many isomorphism types of composition factors if and only if G is nilpotent-by-finite.

Chapter 4. Polycyclic Groups II - Non-artinian Modules.

1. Introduction.

We have seen that if G is a polycyclic-by-finite group, k a field and V a finite dimensional irreducible kG module, then the injective hull $E_{kG}(V)$ of V is artinian. However if k is a non-absolute field, every irreducible kG module is finite dimensional if and only if G is abelian-by-finite (18). In contrast with the results of chapter 3, we prove the following.

Main Theorem. If k is a non-absolute field, and G a polycyclic-by-finite group, then the injective hull of every irreducible kG module is artinian if and only if G is abelian-by-finite.

Suppose first that k is any field and that the polycyclic-by-finite group G has an abelian normal subgroup A of finite index. Let V be an irreducible kG module. Then by lemma 3.4.6 $E_{kG}(V)$ is artinian if and only only if $E_{kA}(V)$ is artinian. Now V_A is completely reducible by Clifford's theorem, and the injective hull of every irreducible kA module is artinian since kA is a commutative Noetherian ring, see (31). It follows that $E_{kA}(V)$ and $E_{kG}(V)$ are artinian modules.

For the remainder of this chapter k will denote a non-absolute field.

We prove the main theorem by taking a polycyclic group G which is not abelian-by-finite and constructing non-artinian essential extensions of an irreducible kG module V. However, we need to employ different constructions according to whether or not G is nilpotent-by-finite.

In section 3 we suppose that G is nilpotent-by-finite but not abelian-by-finite and show that there is an irreducible kG module V such that $\mathrm{Ext}(U_i,V) \neq 0$, for an infinite sequence of pairwise non-isomorphic irreducible kG modules $U_1, U_2, \ldots$ In particular the socle of $E_{kG}(V)/V$ is not artinian.

In section 4 we suppose that the polycyclic-by-finite group G is not nilpotent-by-finite and construct an irreducible kG module V and a finitely generated essential extension W of V such that W is not artin-

ian. This result (theorem 4.1) may be of interest from a ring theoretic point of view, in connection with attempts to prove Jacobson's conjecture (see (26) especially the question at the top of page 116). We know of no other example of a two sided Noetherian ring R which has an irreducible module V such that $E_R(V)$ is not locally artinian.

However it seems that still worse behaviour can occur and in fact we deduce theorem 4.1 from the following result. We postpone the definitions of the krull dimension, k-dim(W) of a module W, and the eccentric plinth length, e(G) until sections 2 and 4 respectively.

<u>Theorem 4.2.</u> Let G be an abelian-by-nilpotent-by-finite, polycyclic-by-finite group. There exists an irreducible kG module V and a finitely generated essential extension W of V such that

$$k\ \dim(W) \geqslant e(G).$$

For the moment we remark that e(G) = 0 if and only if G is nilpotent-by-finite, and the non-zero module W is artinian if and only if k-dim(W) = 0.

The irreducible modules V which we consider are induced from a one-dimensional module v for a self centralising abelian normal subgroup A of G. If we assume that v is faithful for A it will be automatic that V is irreducible (lemma 4.3).

As an initial special case, suppose that G is abelian-by-(infinite cyclic), $G = \langle A,x\rangle$ and that the abelian normal subgroup A is an eccentric plinth for G. An example of such a group is the group with the presentation $G = \langle a,b,x \mid (a,b) = 1,\ a^x = b,\ b^x = ab \rangle$. Since $x - 1$ is a non zero divisor in kG there exists an element $w \in E_{kG}(V)$ such that $w(x - 1) = v$. If $W = wkG$, then W is a proper essential extension of V. Let $\overline{W} = W/V = \overline{w}kG$, where $\overline{w}(x - 1) = 0$. We have $\overline{W} = \overline{w}kA$ and so as a kA module $\overline{W} \cong kA/I$, where $I = \mathrm{ann}_{kA}\overline{w}$ and it follows from a theorem of Bergman (36) 9.3.9 that either $\dim_k \overline{W} < \infty$, or $\overline{W} \cong kA$. The first case is eventually eliminated and then we see that

ian. This result (theorem 4.1) may be of interest from a ring theoretic point of view, in connection with attempts to prove Jacobson's conjecture (see (26) especially the question at the top of page 116). We know of no other example of a two sided Noetherian ring R which has an irreducible module V such that $E_R(V)$ is not locally artinian.

However it seems that still worse behaviour can occur and in fact we deduce theorem 4.1 from the following result. We postpone the definitions of the krull dimension, k-dim(W) of a module W, and the eccentric plinth length, e(G) until sections 2 and 4 respectively.

<u>Theorem 4.2.</u> Let G be an abelian-by-nilpotent-by-finite, polycyclic-by-finite group. There exists an irreducible kG module V and a finitely generated essential extension W of V such that

$$k\ \dim(W) \geqslant e(G).$$

For the moment we remark that $e(G) = 0$ if and only if G is nilpotent-by-finite, and the non-zero module W is artinian if and only if $k\text{-}\dim(W) = 0$.

The irreducible modules V which we consider are induced from a one-dimensional module v for a self centralising abelian normal subgroup A of G. If we assume that v is faithful for A it will be automatic that V is irreducible (lemma 4.3).

As an initial special case, suppose that G is abelian-by-(infinite cyclic), $G = \langle A, x\rangle$ and that the abelian normal subgroup A is an eccentric plinth for G. An example of such a group is the group with the presentation $G = \langle a,b,x \mid (a,b) = 1,\ a^x = b,\ b^x = ab\rangle$. Since $x - 1$ is a non zero divisor in kG there exists an element $w \in E_{kG}(V)$ such that $w(x - 1) = v$. If $W = wkG$, then W is a proper essential extension of V. Let $\overline{W} = W/V = \overline{w}kG$, where $\overline{w}(x - 1) = 0$. We have $\overline{W} = \overline{w}kA$ and so as a kA module $\overline{W} \cong kA/I$, where $I = \mathrm{ann}_{kA}\overline{w}$ and it follows from a theorem of Bergman (36) 9.3.9 that either $\dim_k \overline{W} < \infty$, or $\overline{W} \cong kA$. The first case is eventually eliminated and then we see that

$\overline{W} > \overline{W}_{\mathfrak{g}} > \overline{W}_{\mathfrak{g}}^2$ ... is a strictly descending chain of kG submodules of W.

In general if G is not nilpotent-by-finite, we can find a section of G of the form $\langle A,x\rangle$ as above and induce the essential extension we have obtained in the special case to obtain an essential extension of an irreducible kG module.

This argument would suffice to prove theorem 4.1. To obtain theorem 4.2 we apply a result of Roseblade, (42) theorem C1 in place of Bergman's theorem.

In the penultimate section of this chapter we construct, for any given odd prime p, an example of a metabelian polycyclic group G with normal subgroups A, H such that $A \leq H$, H/A and G/H are infinite cyclic, $A = C_G(A)$ $e(G) = 1$ and $e(H) = p$. The point of the example is that by theorem 4.2 there is an irreducible kH module V and a finitely generated essential extension W of V such that $k\text{-dim}(W) \geq p$. Here V is induced from a one-dimensional kA module which is faithful for A. Now the induced module W^G is a finitely generated essential extension of V^G with $k\text{-dim}(W^G) \geq p$, and V^G is irreducible by lemma 4.3. However $e(G) = 1$ and so the krull dimension of a finitely generated essential extension of of an irreducible kG module cannot be bounded above by a function of e(G).

Finally in section 6 we indicate another approach to theorem 4.2.

2. Normal subgroups of finite index.

If R is a ring and M a right R module, k-dim(M) is defined recursively as follows : if $M = 0$ then $k\text{-dim}(M) = -1$ and if q is an integer such that $k\text{-dim}(M) \nleq q - 1$, then $k\text{-dim}(M) = q$ if and only if there is no infinite descending chain $M = M_0 > M_1 > \ldots$ such that $k\text{-dim}(M_i/M_{i+1}) \nleq q - 1$. If no such integer q exists we could put $k\text{-dim}(M) = \infty$. However if G is a polycyclic-by-finite group and M is a finitely generated kG module, then it follows from a theorem of P.F.Smith (47) that $k\text{-dim}(M) \leq h(G)$, the Hirsch number of G.

<u>Lemma 2.1.</u> Let H be a normal subgroup of finite index in a group G.

a) Suppose that there is an irreducible kH module V and an infinite sequence $U_1, U_2, \ldots$ of pairwise non-isomorphic irreducible kH modules such that $\mathrm{Ext}(U_i, V) \neq 0$, then there is an irreducible kG module V', and an infinite sequence $U_1', U_2', \ldots$ of pairwise non-isomorphic irreducible kG modules such that $\mathrm{Ext}(U_i', V') \neq 0$ for each i.

b) Suppose there is an irreducible kH module V and a finitely generated essential extension W of V such that $k\text{-dim}(W) = n$. Then there is an irreducible kG module V' and a finitely generated essential extension W' of V' such that $k\text{-dim}(W') = n$.

<u>Proof:</u> a) If $g_1, \ldots, g_n$ is a transversal to H in G, then as a kH module $U_j^G = \bigoplus_{i=1}^{n} U_j \otimes g_i$, and we may assume that $U_2 \not\cong U_1 g_i$ for any i. Then U_1^G and U_2^G have no kH composition factor in common and hence no kG composition factor in common. Similarly, by choosing a subsequence if necessary we may assume that U_i^G and U_j^G have no kG composition factors in common whenever $i \neq j$. Now it follows from lemma 1.4.16 that $\mathrm{Ext}(U_i^G, V^G) \neq 0$ and so by lemma 1.4.18 that $\mathrm{Ext}(U_i^G, V') \neq 0$ for some kG composition factor V' of V^G. Now since V^G has finite length, some composition factor V' must have $\mathrm{Ext}(U_i^G, V') \neq 0$ infinitely many times and we may suppose that $\mathrm{Ext}(U_i^G, V') \neq 0$ for all i. Now by lemma 1.4.18 $\mathrm{Ext}(U_i, V') \neq 0$ for some kG composition factor U_i' of U_i^G.

b) By a result of Segal (45) lemma 8, if M is any kG module, the krull dimensions of M as a kH module and as a kG module are equal. By lemma 1.4.16 W^G is an essential extension of V^G which is an essential extension of its socle $\bigoplus_{i=1}^{k} V_i$ as a kG module. Therefore we may consider W^G as a submodule of $E_{kG}(\bigoplus_{i=1}^{k} V_i) = \bigoplus_{i=1}^{k} E(V_i)$. Let W_i be the image of the projection of W^G onto $E(V_i)$. Then $W^G \leq \bigoplus_{i=1}^{k} W_i$ and $k\text{-dim}(W^G) = \sup k\text{-dim}(W_i)$. Therefore there exists a W_i for which $k\text{-dim}(W_i) = n$, and W_i is a finitely generated essential extension of the irreducible kG module V_i.

3. Nilpotent-by-finite groups.

In this section we prove the main theorem for finitely generated nilpotent-by-finite groups.

Theorem 3.1. Let G be a finitely generated nilpotent-by-finite group which is not abelian-by-finite. There exists an irreducible kG module V and an infinite sequence $U_1, U_2, \ldots$ of pairwise non-isomorphic irreducible kG modules such that $\mathrm{Ext}(U_i, V) \neq 0$, for each i.

In particular $\mathrm{Soc}(E_{kG}(V)/V)$ is not artinian.

By lemma 2.1 a) this will follow from

Theorem 3.2. Let G be a finitely generated nilpotent group which is not abelian-by-finite. There exist infinite sets $\mathcal{S}, \mathcal{T}$ of pairwise non-isomorphic irreducible kG modules such that $\mathrm{Ext}(U,V) \neq 0$ and $\mathrm{Ext}(V,U) \neq 0$ for all $U \in \mathcal{S}$, $V \in \mathcal{T}$.

Preliminary reduction for theorem 3.2. If $\bar{G}$ is a homomorphic image of G and $k\bar{G}$ has sets of irreducible modules $\mathcal{S}, \mathcal{T}$ satisfying the conclusion of the theorem, then we may regard $\mathcal{S}, \mathcal{T}$ as kG modules and kG satisfies the conclusion of the theorem. Hence since G has the maximum condition on subgroups we may suppose that G is not abelian-by-finite, but that every proper homomorphic image of G is abelian-by-finite. As noted by D.L.Harper (19), this forces G to have a presentation of the form

$$G = \left\langle x_1, \ldots, x_n, y_1, \ldots, y_n, z \;\middle|\; \begin{array}{l} (x_i,x_j) = (y_i,y_j) = (x_i,z) = (y_i,z) = 1 \\ (x_i,y_j) = 1 \text{ if } i \neq j \quad (x_i,y_i) = z^{m_i} \end{array} \right\rangle \quad (\dagger)$$

where $m_i \in Z$, and $m_1 | m_2 | \ldots | m_n$.

To see this, note first that G must be torsion free, since the torsion elements of G form a normal subgroup T and if G/T is abelian-by-finite it is easily seen that G is abelian-by-finite. Hence by lemma 1.2.8 iii),

the upper central factors of G are torsion-free and so if $G = \zeta_c(G)$ then $c \geqslant 2$, and $G/\zeta_{c-2}(G)$ is not abelian-by-finite. Hence $\zeta_{c-2}(G) = 1$ and G has class two. If $Z = Z(G)$ were not infinite cyclic then it would contain the direct product $Z_1 \times Z_2$ of two infinite cyclic groups and

G would embed in $G/Z_1 \times G/Z_2$ and so would be abelian-by-finite. Therefore $Z = \langle z \rangle$ is infinite cyclic and the commutator bracket defines a non-singular antisymmetric bilinear form on the free abelian group G/Z. A theorem of Frobenius asserts that G/Z has a Z basis of the desired form.

Our irreducible modules are constructed using the following lemma.

<u>Lemma 3.3.</u> Let A be a self-centralising abelian normal subgroup of a group G which is nilpotent of class two. If $Z = Z(G)$ and u is a linear kA module such that $C_Z(u) = 1$, then $U = u \otimes_{kA} kG$ is an irreducible kG module.

<u>Proof:</u> Let $\{g_\lambda \mid \lambda \in \Lambda\}$ be a transversal to A in G. Then U has a k basis of the form $\{u \otimes g_\lambda \mid \lambda \in \Lambda\}$. Let U' be a non-zero submodule of U and choose a non-zero element w which has shortest length on this basis.

We may assume that w has the form

$$w = \mu_0 u \otimes g_0 + \dots \mu_l u \otimes g_l \quad \text{where } \mu_i \in k,\ \mu_0, \mu_l \neq 0, \text{ and } g_0 = 1.$$

Now suppose that $l \geq 1$. Since $g_l \notin A = C_G(A)$, there exists $a \in A$ such that $(g_l, a) \neq 1$. Since G has class two $(g_l, a) = z \in Z$. Now by hypothesis $ua = \chi u$ and $uz = \zeta u$, with $\chi, \zeta \in k$ and $\zeta \neq 1$. Therefore

$u \otimes g_0 a = \chi u \otimes g_0$, $u \otimes g_l a = \zeta \chi u \otimes g_l$ and it is easily seen that $wa - \chi w$ is a non-zero element of U' of shorter length. Therefore $l = 0$ and $u \in U'$, so $U' = U$.

<u>Proof of theorem 3.2.</u> Let G be the group defined by (†), and let $H_1 = \langle x_1, z \rangle$, $H_2 = \langle y_1, z \rangle$, $H = H_1 H_2$, $Y = \langle y_2, \dots, y_n \rangle$, $A = H_1 Y$, $B = H_2 Y$ and $C = HY$.

We fix a monomorphism $\Theta : \langle z \rangle \to k^*$, and let $\zeta = \Theta(z)$. Since k^* has infinite rank, it is possible to extend Θ to H_1 in infinitely many different ways. Let $\phi_1, \phi_2, \dots$ be homomorphisms from H_1 to k^* which coincide with Θ on $\langle z \rangle$ and such that $\phi_i(x_1) \neq \zeta^m \phi_j(x_1)$ for any $m \in \mathbb{Z}$ and $i \neq j$ (‡).

Let $v_1, v_2, \dots$ be the corresponding kH_1 modules determined by

$v_i h = \phi_i(h)v_i$ for $h \in H_1$. We can also regard $v_1, v_2, \ldots$ as kA modules by allowing Y to act trivially.

Let $\mathcal{S} = \{ v_i \otimes_{kA} kG \}_{i \geq 1}$. By lemma 3.3, the modules in $\mathcal{S}$ are irreducible and by the condition (‡) they are pairwise non-isomorphic.

The modules in $\mathcal{T}$ are constructed similarly. Let $\psi_1, \psi_2, \ldots$ be homomorphisms from H_2 to k^* which coincide with Θ on $\langle z \rangle$ and satisfying $\psi_i(y_1) \neq \zeta^m \psi_j(y_1)$ for any $m \in \mathbb{Z}$ and $i \neq j$. We can regard the corresponding kH_2 modules $u_1, u_2, \ldots$ also as kB modules by allowing Y to act trivially and then set $\mathcal{T} = \{u_i \otimes_{kB} kG\}_{i \geq 1}$.

To simplify notation let us choose representative modules $V = v \otimes_{kA} kG \in \mathcal{S}$, $U = u \otimes_{kB} kG \in \mathcal{T}$. Here v,u are one-dimensional kA and kB modules arising from extensions ϕ, ψ of Θ to H_1, H_2 respectively. We aim to show that $\mathrm{Ext}(U,V) \neq 0$.

By lemma 3.3 $V' = v \otimes_{kH_1} kH$ is an irreducible kH module. Furthermore $\mathrm{ann}_{kH} V' = (z - \zeta)kH$ and we may regard V' as an irreducible $R = kH/\mathrm{ann}_{kH} V'$ module. Since R is a domain, we have a well-defined module homomorphism $(y_1 - \psi(y_1))R \longrightarrow V' \subseteq E(V')$ given by

$$(y_1 - \psi(y_1)) \longrightarrow v.$$

Therefore, there exists $w \in E_R(V')$ such that $w(y_1 - \psi(y_1)) = v$. Clearly $w \notin V'$ and so if $W = wR$ we have a non-split exact sequence $0 \longrightarrow V' \longrightarrow W \longrightarrow W/V' = U' \longrightarrow 0$ of R modules.

We can now regard this as a sequence of kH modules with z acting as ζ and since U' is generated by a copy of the one-dimensional kH_2 module u, it is a homomorphic image of $u \otimes_{kH_2} kH$. However this induced module is irreducible by lemma 3.3 again, so $U' \cong u \otimes_{kH_2} kH$.

Now we have an exact sequence $0 \longrightarrow V' \longrightarrow W \longrightarrow U' \longrightarrow 0$ of kH modules such that W is an essential extension of V' and we can regard this sequence as a sequence of kC modules by allowing Y to act trivially, and then induce this sequence to kG. By lemma 1.4.16, $W \otimes_{kC} kG$ is an essential extension of $V' \otimes_{kC} kG$, and $V' \otimes_{kC} kG = v \otimes_{kA} kG = V$. Similarly $U' \otimes_{kC} kG = u \otimes_{kA} kG = U$. This shows that $\mathrm{Ext}(U,V) \neq 0$ and a similar argument shows

that $\mathrm{Ext}(V,U) \neq 0$.

4. Non-nilpotent-by-finite groups.

In (42) Roseblade introduces an invariant for polycyclic groups, namely the eccentric plinth length which, in some sense measures how far a group is from being nilpotent-by-finite. This invariant will be crucial to our discussion and we repeat the appropriate definitions.

A free abelian subgroup, A of a polycyclic-by-finite group G is said to be a <u>plinth</u> if there exists a subgroup G_o of G containing A such that

i) $|G:G_o| < \infty$ and $A \trianglelefteq G_o$

ii) $A \otimes_Z Q$ is an irreducible QH module whenever H is a subgroup of finite index in G_o.

It is known that any infinite normal subgroup of a polycyclic-by-finite group G contains a plinth of G, (41) lemma 2.

We note that if A is a plinth in G we do not require that A is normal in G, only that $|G:N_G(A)| < \infty$.

If $\dim_Q(A \otimes_Z Q) = 1$, A is said to be a <u>centric</u> plinth, and if $\dim_Q(A \otimes_Z Q) \geq 2$, A is an <u>eccentric</u> plinth.

Now suppose that A is a free abelian normal subgroup of a polycyclic-by-finite group G, and consider $V = A \otimes_Z Q$ as a QG module. We can find a subgroup H of finite index in G such that the composition length of V as a QH module is maximal. The length of a composition series for V as a QH module will be called the <u>G-plinth length of A</u> and written $pl_G(A)$. The number of composition factors whose dimension over Q exceeds one is also an invariant, the <u>eccentric G-plinth length of A</u>, $e_G(A)$.

Any polycyclic-by-finite group G has a normal poly-(infinite cyclic) subgroup H of finite index, and H has a series

$1 = H_o < H_1 < \ldots H_s = H$, with $H_i \trianglelefteq H$ and free abelian factors H_i/H_{i-1} for $i = 1, \ldots, s$ and we set

$$pl(G) = pl(H) = \sum_{i=1}^{t} pl_{H/H_{i-1}}(H_i/H_{i-1})$$

$$e(G) = e(H) = \sum_{i=1}^{t} e_{H/H_{i-1}}(H_i/H_{i-1})$$

It is known that pl(G) and e(G) are invariants of G, the <u>plinth length</u> and <u>eccentric plinth length</u> of G. Moreover G is nilpotent-by-finite if and only if e(G) = 0.

The aim of this section is to prove the main theorem for polycyclic groups with e(G) > 0. However in this case we can prove more and we may now state

<u>Theorem 4.1.</u> Let G be a polycyclic-by-finite group which is not nilpotent-by-finite. There exists an irreducible kG module V and a finitely generated essential extension W of V such that W is not artinian.

<u>Theorem 4.2.</u> Let G be a polycyclic-by-finite, abelian-by-nilpotent-by-finite group. There exists an irreducible kG module V and a finitely generated essential extension W of V such that

$$k\text{-dim}(W) \geq e(G).$$

<u>Deduction of theorem 4.1:</u> If any homomorphic image of G has a module W satisfying the conclusion of theorem 4.1, then we may regard W as a kG module. Hence since G has the maximum condition on subgroups we may suppose that G is not nilpotent-by-finite but that every proper homomorphic image of G is nilpotent-by-finite.

Since an infinite polycyclic-by-finite group contains an infinite abelian normal subgroup by (36) 10.2.9 this clearly forces G to be abelian-by-nilpotent-by-finite. In fact G must be metabelian-by-finite. Indeed if F is the Fitting subgroup of G, then G/F' is not nilpotent-by-finite. For if K/F' is the Fitting subgroup of G/F', then as K/F' and F are nilpotent, K is nilpotent by (39) 2.27. Thus K = F and G/F' is not nilpotent-by-finite. Our assumption on G forces F' = 1. Now F is abelian, and G/F is abelian-by-finite, and so G is metabelian-by-finite. Now e(G) > 0 since G is not nilpotent-by-finite (in fact we must have e(G) = 1). The result follows from theorem 4.2 and the defin-

itions.

The irreducible module V in theorem 4.2 is constructed using the following result (compare (14) lemma 1.1).

<u>Lemma 4.3.</u> Suppose that A is a non-trivial self-centralising abelian normal subgroup of an arbitrary group G. Suppose that k is a field such that there is a faithful homomorphism $\Theta: A \to k^*$, and let v be the corresponding linear kA module given by $va = \Theta(a)v$ for $a \in A$. Then the induced module $V = v \otimes_{kA} kG$ is irreducible.

<u>Proof:</u> This is similar to lemma 3.3, and we omit it.

Before proving theorem 4.2 we need some preliminary results.

<u>Lemma 4.4.</u> Let R be a ring and $0 - M - N - L - 0$ (*) an exact sequence of R modules such that N has finite composition length and no composition factor of M is isomorphic to one of L.

Suppose that either

a) R is artinian and polycentral

b) $R = kG$ the group algebra of a finitely generated nilpotent group, and that $\dim_k N < \infty$. Then the sequence (*) splits.

<u>Proof:</u> b) follows from a) applied to the artinian ring $kG/\mathrm{ann}_{kG}N$, since kG is polycentral by 1.3.5.

a) We may assume that M and L are irreducible by lemma 1.4.18 . Let J be the Jacobson radical of R. If $NJ = 0$, then (*) is an exact sequence of R/J modules and splits since R/J is semisimple artinian. Therefore we may assume that $NJ = M$.

Clearly $MJ = LJ = 0$ and we may assume that N is faithful for R. Choose a non-zero central element z in J. Then the map $N/M \to M$ given by $(n + M) \to nz$ is a well defined non-zero R module homomorphism. Hence by Schur's lemma $M \cong L$. This contradiction proves the lemma.

We remark that the above result may be false under slightly weaker hypotheses. For, as we saw in section 3, if G is a finitely generated

nilpotent-by-finite group which is not abelian-by-finite then there is an irreducible kG module V and an infinite sequence $U_1, U_2, \ldots$ of pairwise non-isomorphic kG modules such that $\mathrm{Ext}(U_i, V) \neq 0$ for each i. Of course V and U_i are infinite dimensional in this case. On the other hand S. Donkin (10) has shown that if k is a field of characteristic zero, and G is a polycyclic-by-finite group, then $E_{kG}(k)$ has only finitely many isomorphism types of composition factors if and only if G is nilpotent-by-finite.

Lemma 4.4 has various consequences for the plinth structure of an abelian-by-nilpotent-by-finite group G. For suppose that A is a free abelian normal subgroup of G and that $\overline{G} = G/A$ is nilpotent. Let $W = A \otimes_Z \mathbb{Q}$ and let $\overline{H} = H/A$ be a subgroup of finite index in $\overline{G}$ such that the composition length of W as a $\mathbb{Q}\overline{H}$ module is maximal among subgroups of finite index. Then by lemma 4.4 the composition factors of W as a $\mathbb{Q}\overline{H}$ module are the same as the composition factors of $V = \mathrm{Soc}_{\mathbb{Q}\overline{H}}(W)$, and the composition factors of V correspond to the plinths of G which are subgroups of A. We note in particular the following corollary.

<u>Corollary 4.5.</u> With the notation of the previous paragraph, suppose that a) no plinth for G contained as a subgroup in A is eccentric then G is nilpotent-by-finite.

b) every plinth for G contained as a subgroup of A is eccentric, then $e(G) = e_G(A) = pl_G(A)$.

<u>Proof:</u> a) The assumption implies that all the composition factors of V as a $\mathbb{Q}\overline{H}$ module are one-dimensional, and so the composition factors of W as a $\mathbb{Q}\overline{H}$ module are also one-dimensional. This means we can find a subgroup of finite index in H which centralizes every composition factor of W. Therefore H and so G are nilpotent-by-finite.

b) Since A is abelian we have $e(G) = e(G/A) + e_G(A)$. However G/A is nilpotent so $e(G/A) = 0$. It remains to show that $e_G(A) = pl_G(A)$. By the hypothesis every composition factor of V as a $\mathbb{Q}\overline{H}$ module has dimension

greater than one. By lemma 4.4, the same is true for the $Q\bar{H}$ composition factors of W. It follows that $pl_G(A) = e_G(A)$.

Part a) of the preceding corollary has been obtained independently by D.L.Harper, (20) proposition 4.3.

We need another consequence of lemma 4.4.

<u>Proposition 4.6.</u> Let G be any group, k a field and

$0 \longrightarrow V \longrightarrow W \xrightarrow{\pi} U \longrightarrow 0$ (*) an exact sequence of kG modules.

Suppose that A is a finitely generated nilpotent normal subgroup of G such that V_A is locally finite. If $C_V(A) = 0$ and $C_U(A) = U$, then the sequence of kG modules (*) splits.

A module is said to be <u>locally finite</u>, if every element generates a finite dimensional submodule. The restriction of the kG module V to A is denoted by V_A. Thus to say that V_A is locally finite means that for all $v \in V$, vkA is finite dimensional.

Also $C_V(A)$ denotes the fixed points of V as a kA module, that is the submodule $\{v \in V \mid va = v \text{ for all } a \in A\}$.

<u>Proof:</u> We show first that (*) splits as a sequence of kA modules.

Now U is trivial as a kA module, that is $U \cong \bigoplus_{i \in I} U_i$ for some index set I, where U_i is a trivial one-dimensional kA module.

Let $W_i = \pi^{-1}(U_i)$. Then $0 \longrightarrow V \longrightarrow W_i \xrightarrow{\pi} U_i \longrightarrow 0$ is exact for each $i \in I$ and we can assume that $\dim_k U = 1$.

Suppose $A = \langle a_1, \ldots, a_n \rangle$ and choose $u \in W \setminus V$. Then $W = V + k\langle u \rangle$ and $ua_j^{\epsilon} = u + v_{j\epsilon}$ for $j = 1, \ldots, n$ and $\epsilon = \pm 1$ and certain elements $v_{j\epsilon} \in V$.

By hypothesis $v_{j\epsilon}$ is contained in a finite dimensional kA submodule $V_{j\epsilon}$ of V. Let $T = \sum V_{j\epsilon} + k\langle u \rangle$. Then T is a finite dimensional kA submodule of V and we have an exact sequence $0 - T \cap V \longrightarrow T \longrightarrow T/T \cap V \longrightarrow 0$ of kA modules. Now by construction every composition factor of $T/T \cap V$ is trivial, while no composition factor of $T \cap V$ is trivial, since $C_V(A) = 0$. Therefore by lemma 4.4 $T = (T \cap V) \oplus T_1$ where T_1 is some kA submodule of T isomorphic to $T/T \cap V$. It is now clear that $T_1 = C_T(A)$.

Hence $W = V + \Gamma = V + \{(T \cap V) + C_T(A)\} = V + C_T(A) = V + C_W(A)$ and $V \cap C_W(A) = 0$, so $W = V \oplus C_W(A)$. Since A is a normal subgroup of G, $C_W(A)$ is a kG submodule of W and the result is proved.

<u>Lemma 4.7.</u> Let R be a Noetherian domain. Then every non-zero ideal in R contains a non-zero product of prime ideals.

<u>Proof:</u> If the lemma is false let I be a maximal counterexample in R. Then I is not prime, and so there are ideals A,B strictly containing I such that $AB \subseteq I$. By the maximality of I there exist prime ideals $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ of R such that $0 \neq P_1P_2 \ldots P_n \subseteq A$, $0 \neq Q_1Q_2 \ldots Q_m \subseteq B$.

Therefore $P_1P_2 \ldots P_nQ_1Q_2 \ldots Q_m \subseteq AB \subseteq I$ and this is a non-zero product of prime ideals since R is a domain.

An important special case in the proof of theorem 4.2 will occur when G is abelian-by-(infinite cyclic) and to reduce to this case in general we need to find a single element x which acts in a suitable manner on a free abelian normal subgroup. This is done in the next two lemmas. The first is a generalization of a result of D.S.Passman, (36), 12.3.1.

<u>Lemma 4.8.</u> Let H be a finitely generated abelian group, k a field and $V = V_1 \oplus \ldots \oplus V_s$ a kH module such that for all subgroups H_1 of finite index in H, V_i is irreducible as a kH_1 module. Then there exists an element x in H such that V_i is an irreducible $k\langle x^n \rangle$ module for all $n \geq 1$ and $i = 1, \ldots, s$.

<u>Proof:</u> We may assume that $C_H(V) = 1$. Since H is abelian and the V_i are irreducible kH modules, there exist maximal ideals $M_i \lhd kH$ such that $kH/M_i \cong V_i$ for $i = 1, \ldots, s$.

Now H embeds in $kH/M_1 \oplus \ldots \oplus kH/M_s$ via the diagonal map $\phi(h) = (h + M_1, \ldots, h + M_s)$ for if $\phi(h) = \phi(1)$ then $h - 1 \in M_i$ for all i and so $h \in C_H(V) = 1$.

Now kH/M_i is a field F_i and we can identify the additive groups of F_i and V_i. The map ϕ gives rise to the original module action on F_i, for

if $(\bar{v}_1, \ldots, \bar{v}_s) \in V_1 \oplus \ldots \oplus V_s$, let $\bar{v}_i = v_i + M_i$ for elements $v_i \in kH$, then

$$(v_1 + M_1, \ldots, v_s + M_s)(h + M_1, \ldots, h + M_s)$$
$$= (v_1 h + M_1, \ldots, v_s h + M_s) = (\bar{v}_1, \ldots, \bar{v}_s)h.$$

Hence H acts as subgroup of the multiplicative group, $(F_1 + \ldots + F_s)^\circ$ of $F_1 + \ldots + F_s$, and if H_1 is a subgroup of finite index in H then each F_i is an irreducible kH_1 module under this action.

Now by the Nullstellensatz, F_i is a finite algebraic extension of k, and as in Passman's lemma F_i/k is separable. This is clear in characteristic zero, so suppose char $k = p > 0$. Let L/k be a maximal separable extension of k inside F_i, so that F_i/L is purely inseparable. Then F_i°/L° is a p group, and hence so too is $(F_1 + .. F_i + .. + F_s)^\circ/(F_1 + .. L + .. + F_s)^\circ$.

Let $H_1 = H \cap (F_1 + .. + L + .. F_s)^\circ$. As H is finitely generated and H/H_1 is a p group we conclude that H_1 has finite index in H, and the additive group of $F_1 + .. + L + .. + F_s$ is a kH_1 submodule of V containing $V_1 \oplus .. V_{i-1} \oplus V_{i+1} \oplus .. \oplus V_s$ and having non-zero intersection with V_i. Hence by our assumption on subgroups of finite index $(F_1 + .. + L + \ldots + F_s)^+ = (F_1 + .. + F_i + .. + F_s)^+$ and so $L = F_i$ and F_i is separable.

Now let $\mathcal{K}$ be the set of subrings of $F_1 + .. + F_s$ of the form $K_1 + .. + K_s$ where K_i is a subfield of F containing k, and at least one K_i is properly smaller than F_i. Since F_i/k is a finite separable extension the set $\mathcal{K}$ is finite.

If $(K_1 + .. + K_s)^\circ$ is the group of units of $K_1 + .. + K_s$ then $H_1 = H \cap (K_1 + .. + K_s)^\circ$ is a subgroup of H and since H is abelian so is the isolator of H_1 in H namely $i_H(H_1) = \{ h \in H \mid h^n \in H_1$ for some $n \geq 1\}$.

Clearly $i_H(H_1)/H_1$ is the torsion subgroup of H/H_1.

Suppose that $|H : i_H(H_1)| < \infty$. Then since $H/i_H(H_1)$ is torsion free we

have $H = i_H(H_1)$ and so since H is finitely generated $|H:H_1| < \infty$, but then $(K_1 + .. + K_s)$ is a proper kH_1 submodule of V having non-zero intersection with each V_i and this contradicts our hypothesis.

Therefore by (36), lemma 4.2.1 there exists an element x in H such that $x \notin i_H(H \cap (K_1 + \ldots + K_s)^o)$ for any subring $K_1 + \ldots + K_s$ belonging to the set $\mathcal{K}$. This implies that $x^n \notin (K_1 + \ldots + K_s)^o$ for all $n \geqslant 1$ and subrings belonging to $\mathcal{K}$, and hence $k\langle x^n\rangle = F_1 + \ldots + F_s$ for all $n \geqslant 1$. Therefore $k\langle x^n\rangle$ acts irreducibly on F_i for each $n \geqslant 1$, and the lemma is proved.

The group theoretic interpretation of the above lemma is as follows: Suppose that A is a normal free abelian subgroup of the polycyclic-by-finite group G and that $V = A \otimes_Z Q$ is completely reducible as a $QG/C_G(A)$ module. Then $G/C_G(A)$ is abelian-by-finite by (51) lemma 3.5, and as far as the G-plinth structure of A is concerned we can suppose that $G/C_G(A)$ is abelian. Then lemma 4.8 asserts that there is an element x in $G/C_G(A)$ such that the G-plinths of A are precisely the $\langle x\rangle$-plinths of A.

In certain cases we can eliminate the hypothesis of complete reducibility.

<u>Lemma 4.9.</u> Let A be a free abelian normal subgroup of a polycyclic-by-finite group G, and suppose that $G/C_G(A)$ is nilpotent. Then there is an element $x \in G$ such that the G-plinths of A are precisely the $\langle x\rangle$-plinths of A.

<u>Proof:</u> Let $W = A \otimes_Z Q$ and choose a normal subgroup H of finite index in G such that the composition length of W as a QH module is maximal among subgroups groups of finite index. Let V be the socle of W as a QH module. Now by lemma 4.8 there exists an element x in H such that the composition factors of V as a QH module are the same as the composition factors of V as a $Q\langle x\rangle$ module for all $n \geqslant 1$. Now by lemma 4.4, the composition factors of W as a QH module are the same as the composition factors of V, since $H/C_H(W)$ is nilpotent. It follows that the G-plinths of A are precisely the $\langle x\rangle$-plinths of A.

Note: We refer the reader to (42) for the definition and properties of orbitally sound groups.

Note: We refer the reader to (42) for the definition and properties of orbitally sound groups.

Finally, before we prove theorem 4.2 we need a result which will enable us to find a lower bound for the krull dimensions of our essential extensions.

<u>Lemma 4.10.</u> Let A be a free abelian normal subgroup of a polycyclic-by-finite group G and let W be a non-zero kG module which is free as a kA module. Then $k\text{-dim}(W) \geqslant pl_G(A)$.

<u>Proof:</u> Let $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$, and choose a normal subgroup H of finite index in G such that the composition length of V as a $\mathbb{Q}H$ module is maximal among subgroups of finite index. By (45) lemma 8, the krull dimensions of W as a kG and as a kH module are equal. Thus we may replace G with H in which case $pl_G(A)$ is the maximum length of a chain of normal subgroups of G contained in A with infinite factors. Now if $B \neq 1$ is an infinite normal subgroup of G contained in A, then we have a descending chain $W > W\underline{b} > W\underline{b}^2 > \ldots$ of kG submodules of W with non-zero factors $W\underline{b}^n/W\underline{b}^{n+1}$, since W is free as a kA module. Thus by induction it suffices to show that $W\underline{b}^n/\underline{b}^{n+1}$ is free as a kA/B module, and to do this we may assume that $W = kA$. Let $\{g_\lambda \mid \lambda \in \Lambda\}$ be a transversal to B in A. Any element of $\underline{b}^nA/\underline{b}^{n+1}A$ can be written in the form $\sum_\lambda a_\lambda g_\lambda + \underline{b}^{n+1}A$ for certain elements $a_\lambda \in \underline{b}^n$ and the map from $\underline{b}^nA/\underline{b}^{n+1}A$ to $\underline{b}^n/\underline{b}^{n+1} \otimes_k kA/B$ given by $\sum a_\lambda g_\lambda + \underline{b}^{n+1}A \longrightarrow \sum (a_\lambda + \underline{b}^{n+1}) \otimes g_\lambda$ is easily seen to be an isomorphism of kA/B modules, and since $\underline{b}^n/\underline{b}^{n+1} \otimes kA/B$ is free as a kA/B module with non-zero rank equal to $\dim_k \underline{b}^n/\underline{b}^{n+1}$ the result follows.

<u>Proof of theorem 4.2.</u> We have an abelian-by-nilpotent-by-finite group G and must construct an irreducible kG module V and a finitely generated essential extension W of V such that $k\text{-dim}(W) \geqslant e(G)$. Now we may assume that $\Delta(G) = 1$ by factoring out some centric plinths if necessary, since if any factor group of G has a finitely generated essential extension W of an irreducible module, then W can also be regarded as an irreducible kG module. Also by lemma 2.1 we may pass to a subgroup of finite index in G to assume that G is torsion free and orbitally sound and that A is a maximal abelian normal subgroup of G with G/A nilpotent. By lemma 4.9

Note: $\Delta = \Delta$

Note: $\Delta = \Delta(H)$ is the F.C. subgroup of H, i.e. the subgroup $\{h \in H \mid |H : C_H(h)| < \infty\}$.

there exists an element x in G such that the G-plinths of A are the same as the $\langle x \rangle$ plinths of A.

If $A \subsetneq C_G(A)$, then $C_G(A)/A \cap Z(G/A) \neq 1$, since G/A is nilpotent and this would contradict the maximality of the abelian normal subgroup A. Hence $A = C_G(A)$, and if $H = \langle A, x \rangle$, then $A = C_H(A)$.

Let V be an irreducible kG module constructed by the method of lemma 4.3. That is $V = v \otimes_{kA} kG$, where v is a one-dimensional kA module which is faithful for A. Then the kH module $V_1 = v \otimes_{kA} kH$ is also irreducible.

Now consider the map $\varphi : (x - 1)kH \longrightarrow V_1 \subseteq E_{kH}(V_1)$ given by

$$(x - 1)\alpha \longrightarrow v\alpha .$$

This is well defined since $(x - 1)$ is a non zero divisor. Since $E_{kH}(V_1)$ is injective, φ extends to a kH map $\bar{\varphi} : kH \longrightarrow E_{kH}(V_1)$. If $w = \bar{\varphi}(1)$, then $w(x - 1) = \varphi(x - 1) = v$. It is easily seen that w cannot lie in V_1, therefore if $W_1 = wkH$ then W_1 is a proper essential extension of V_1

Now consider the kH module $\bar{W} = W_1/V_1$. This is generated by an element $\bar{w}$ such that $\bar{w}(x - 1) = 0$ and therefore $\bar{W} = \bar{w}kA$. Now since kA is commutative we have $\bar{W} \cong kA/J$, where $J = \mathrm{ann}_{kA}\bar{w}$ as kA modules. Suppose that $J \neq 0$, so that $I = JkH$ is a non-zero ideal of kH. By lemma 4.7, I contains a non-zero product $P_1P_2 \cdots P_n$ of prime ideals of kH. Let $P_i^{\dagger} = \{g \in H \mid g - 1 \in P_i\} \trianglelefteq H$. If $P_i^{\dagger} = 1$, for some i then we have $P_i = (P_i \cap k\Delta)kH = 0$, by theorem C1 of (41) and this is impossible.

Therefore $P_i \neq 1$ and by lemma 1.2.9, $P_i^{\dagger}$ has non-trivial intersection with the Zalesskii subgroup of H, which is A and so $P_i^{\dagger}$ contains an infinite abelian normal subgroup A_i of H with $A_i \subseteq A$.

Therefore if $\mathfrak{a}_i$ denotes the augmentation ideal of A_i we have $\mathfrak{a}_iH \subseteq P_i$, and so $\mathfrak{a}_1 \cdots \mathfrak{a}_nH \subseteq I$. Suppose that the integer n is chosen such that $\mathfrak{a}_1 \cdots \mathfrak{a}_nH \subseteq I$, but $\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}H \nsubseteq I$. If $n = 1$, this is to be interpreted as meaning $\mathfrak{a}_1H \subsetneq I \subsetneq kH$. Then $w\mathfrak{a}_1 \cdots \mathfrak{a}_{n-1}H = W_2$ is a non-zero kH submodule of W_1, and since W_1 is an essential extension of V_1 and V_1 is irreducible we have $V_1 \subseteq W_2$. Also $\bar{W}_2 = W_2/V_1 \neq 0$ by the choice of n.

Therefore we have a non-split exact sequence $0 \longrightarrow V_1 \longrightarrow W_2 \longrightarrow \overline{W}_2 \longrightarrow 0$ of kH modules. Now V_1 was induced from a one-dimensional kA module v which was faithful for A and so the restriction of V_1 to A_n is a direct sum of one-dimensional modules, and $C_{V_1}(A_n) = 0$. Also $W_2 \underline{a}_n = 0$, so $C_{\overline{W}_2}(A_n) = \overline{W}_2$. This provides a contradiction to proposition 4.6 and so $J = 0$. Therefore $\overline{W} \cong kA$ and so by lemma 4.10 $k\text{-dim}(\overline{W}) \geqslant pl_H(A)$. However x was chosen so that the G-plinths of A are precisely the $\langle x \rangle$-plinths of A, so $pl_H(A) = pl_G(A)$. Also since G/A is nilpotent and $\Delta(G) = 1$ we have by corollary 4.5 b) that $pl_G(A) = e(G)$. Therefore $k\text{-dim}(W_1) = k\text{-dim}(\overline{W}) \geqslant e(G)$ and W_1 is a finitely generated essential extension of the irreducible kH module V_1. Now since G/A is nilpotent, H is a subnormal subgroup of G and therefore by repeated application of lemma 1.4.16, $W = W_1 \otimes_{kH} kG$ is a finitely generated essential extension of $V_1 \otimes_{kH} kG = v \otimes_{kA} kG = V$, which is an irreducible kG module. This completes the proof of theorem 4.2.

5. Upper bounds.

Let G be a polycyclic-by-finite group and k a non-absolute field. To save repetition, let us write m(kG) =

$$\sup \left\{ k\text{-dim}(W) \;\middle|\; \begin{array}{l} W \text{ is a finitely generated essential extension of} \\ \text{an irreducible } kG \text{ module} \end{array} \right\}.$$

By a result of Smith (47), $m(kG) \leqslant h(G)$. However if G is abelian-by-finite then $m(kG) = 0$ and this bound is too large. If G is abelian-by-nilpotent-by-finite, then theorem 4.2 states that $e(G) \leqslant m(kG)$ and we thought for some time that possibly $e(G) = m(kG)$ always. This hope is dashed by the examples we construct next. The problem here is really the construction of the groups which are split extensions of subrings of the ring of algebraic integers in an algebraic number field by units in these rings. We note that by Dirichlet's theorem, (38) page 148, the group of algebraic units can have arbitrarily large torsion free rank, and the proof of (36) 12.3.1 shows that we can always find an algebraic

unit, all of whose powers act irreducibly on the ring of algebraic integers, so such examples should be commonplace. However, we prefer to look at specific examples, from which we could eventually derive presentations of the desired groups.

Theorem 5.1. Let p be a prime number greater than 2. There exists a metabelian, polycyclic group G with normal subgroups A and H such that

i) $A \subseteq H$, and G/H and H/A are infinite cyclic, ii) $A = C_G(A)$, iii) $e(H) = p$ and $pl(H) = p + 1$, iv) $e(G) = 1$ and $pl(G) = 3$.

We shall need some preparatory results. Let Θ be the real pth root of 2 and $K = \mathbb{Q}(\Theta)$. Since $(\Theta - 1)(1 + \Theta + \dots \Theta^{p-1}) = 1$, $y = \Theta - 1$ is a unit in the ring of algebraic integers of K.

Lemma 5.2. For all integers $r \geqslant 1$, $K = \mathbb{Q}(y^r)$.

Proof: If $y^r \in \mathbb{Q}$ for some $r \geqslant 1$, then $y^r = \pm 1$, since y^r and y^{-r} are algebraic integers which are rational. This is impossible as y is a real number which is greater than 1.

Now let $x = \sqrt{2} + 1$ and $L = \mathbb{Q}(\sqrt{2})$. Then x is a unit in the ring of algebraic integers of L, and it is easy to see that $L = \mathbb{Q}(x^r)$ for all $r \geqslant 1$. Let $F = \mathbb{Q}(\sqrt{2}, \Theta)$.

Lemma 5.3. The degree of F over $\mathbb{Q}$ is 2p, and $\langle x,y \rangle$ is a free abelian subgroup of rank two of the group of units in the ring of algebraic integers of F.

Proof: The first statement is clear. If $x^n y^m = 1$ for some integers n and m, then $x^n \in \mathbb{Q}(y^m)$ and therefore $x^n \in \mathbb{Q}$. Hence $n = 0$ and so $m = 0$.

Proof of theorem 5.1: The group $\langle x,y \rangle$ acts on the lattice of algebraic integers of F and stabilises the sublattice generated by $a_i = \Theta^i$, and $b_i = \sqrt{2}\Theta^i$, where $0 \leq i \leq p-1$. Explicitly we have

$$a_i.x = \theta^i(\sqrt{2}+1) = a_i + b_i \qquad \text{for } i = 0,1,\ldots,p-1.$$
$$b_i.x = \sqrt{2}\,\theta^i(\sqrt{2}+1) = 2a_i + b_i$$
$$a_i.y = (\theta^{i+1} - \theta^i) = a_{i+1} - a_i \qquad \text{for } i = 0,1,\ldots,p-2$$
$$b_i.y = \sqrt{2}(\theta^{i+1} - \theta^i) = b_{i+1} - b_i$$

and $a_{p-1}.y = 2a_o - a_{p-1}$, $b_{p-1}.y = 2b_o - b_{p-1}$.

We let $G = \langle a_i, b_i, x, y \rangle_{i=0,\ldots,p-1}$ be the split extension of this lattice by the free abelian group $\langle x,y \rangle$. The above relations may be used to write a presentation for G.

For $i = 0, \ldots, p-1$, let $A_i = \langle a_i, b_i \rangle$ and let $A = A_o \times \ldots \times A_{p-1}$. Then $A \in N_G(A_i)$ and $A_i \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to the additive group of L as a $\mathbb{Q}\langle x \rangle$ module . Let $H = \langle A, x \rangle$. Since x^r is a primitive field element for L, for all $r \geqslant 1$, we have $e(H) = p$, $pl(H) = p + 1$.

Now let $C = \langle a_o, \ldots, a_{p-1} \rangle$ and $D = \langle b_o, \ldots, b_{p-1} \rangle$. Then $C \otimes_{\mathbb{Z}} \mathbb{Q}$ and $D \otimes_{\mathbb{Z}} \mathbb{Q}$ are isomorphic to the additive group of K as $\mathbb{Q}\langle y \rangle$ modules, and since y^s is a primitive field element for K, for all $s \geqslant 1$, the only $\mathbb{Q}\langle y^s \rangle$ submodules of $A \otimes_{\mathbb{Z}} \mathbb{Q}$ are $C \otimes_{\mathbb{Z}} \mathbb{Q}$ and $D \otimes_{\mathbb{Z}} \mathbb{Q}$. Hence if $r \geqslant 1$, $s \geqslant 1$, then $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible as a $\mathbb{Q}\langle x^r, y^s \rangle$ module. It follows that $e(G) = 1$, and $pl(G) = 3$.

Clearly $x^n y^m \in C_G(a_o)$ implies $x^n y^m = 1$, and it follows that $A = C_G(A)$.

<u>Corollary 5.4.</u> Let G be the group constructed in theorem 5.1. Then $m(kG) \geqslant p$.

<u>Proof:</u> We retain our previous notation. By theorem 4.2 there is an irreducible kH module V and a finitely generated essential extension $\mathcal{V}$ of V such that $k\text{-dim}(\mathcal{V}) \geqslant p$. Furthermore we can choose V to be induced from a one-dimensional kA module which is faithful for A. Now by lemma 1.4.16, the induced module $\mathcal{V}^G$ is a finitely generated essential extension of V^G, and since $A = C_G(A)$, V^G is irreducible by lemma 4.3. Clearly $k\text{-dim}(\mathcal{V}^G) \geqslant p$ and the result follows.

The above result shows that m(kG) cannot be bounded above by any function of e(G) and pl(G). We remark that the group ring kG has other

interesting properties, for by (42) theorem G2, kH has primitive length p, while kG has primitive length 1. The situation is in striking contrast with that for commutative Noetherian rings, or fully bounded Noetherian rings, where there is a strong connection between indecomposable injective modules and prime ideals, see (31) and (26) theorems 3.5 and 5.3.

It seems that it would be rather difficult to obtain a realistic estimate for m(kG) in terms of the structure of G. We do not even know whether m(kG) = 0 when G is nilpotent-by-finite, that is whether a finitely generated essential extension of an irreducible kG module is artinian for a nilpotent-by-finite group G. However we do have the following result.

<u>Proposition 5.5</u> Let R be a prime ring with k-dim(R) = n. Then either R is simple artinian or k-dim(W) < n whenever W is a finitely generated essential extension of an irreducible R module.

<u>Proof:</u> Let V be an irreducible R module, and $W = \sum w_i R$ a finitely generated submodule of $E_R(V)$. Then k-dim(W) = sup k-dim($w_i R$) and we may assume that W is cyclic, so there is a right ideal I of R such that $W \cong R/I$. If I is not essential in R, there is a non-zero right ideal T of R such that $I \cap T = 0$. Then $T \cong I + T/I \subseteq E(V)$ and so since V is irreducible and T is an essential extension of V, T contains a copy of V. Therefore X = Soc(R) $\neq$ 0. Since R is a right Goldie ring by (16),3.4, X has finite composition length and therefore $R \cong \mathrm{End}_R X$ is simple artinian.

However if I is essential in R, then by (16), theorem 6.1 we have k-dim(R/I) < k-dim(R).

<u>Corollary 5.6.</u> Let R be a prime ring of krull dimension 1. If V is an irreducible R module, then E(V) is locally artinian.

<u>Proof:</u> Immediate.

<u>Corollary 5.7.</u> Let G be a polycyclic-by-finite group and h(G) the Hirsch number of G. Then m(kG) $\leq$ h(G) - 1, for all fields k.

<u>Proof:</u> The group G has a normal subgroup H of finite index which is poly-(infinite cyclic) and kH is a prime ring by (36) 4.2.10, and by

(45) $k\text{-dim}(kH) = h(H)$. It is easy to see that $m(kG) = m(kH)$ and therefore by proposition 5.5, $m(kG) \leqslant h(G) - 1$.

6. An alternative approach to theorem 4.2.

In the proof of theorem 4.2 given in § 4, we use the existence of the injective hull of the irreducible module V_1 to find an essential extension $W_1 = vkH$ of V_1 such that $v(x - 1) = v$. The bulk of the work then involved showing that $k\text{-dim}(W_1) \geqslant e(H) = e(G)$.

We can also prove theorem 4.2 by constructing the module W_1 by writing down the action of H on a vector space basis. It will then be clear that $k\text{-dim}(W_1) \geqslant e(H)$ and our work lies in showing that W_1 is an essential extension of V_1. We assume the notation of the first three paragraphs of the proof of theorem 4.2 given on pages 72-74.

Alternative proof of theorem 4.2: Let ϕ be the embedding $\phi : A \hookrightarrow k^*$ corresponding to the one-dimensional kA module v. Consider a vector space W_1 over k with basis $\{w_a \mid a \in A\} \cup \{vx^i \mid i \in \mathbb{Z}\}$. We make W_1 into a kH module as follows

$$vx^i.x^j = vx^{i+j} \quad , \quad vx^i.a = \phi(a^{x^{-i}})vx^i, \text{ and}$$

$$w_a.b = w_{ab} \quad , \quad w_a.x = w_{a^x} + va^x, \text{ for } a,b \in A,\ i,j \in \mathbb{Z}.$$

Clearly the subspace spanned by the set $\{vx^i \mid i \in \mathbb{Z}\}$ is a kH module isomorphic to $V_1 = v \otimes_{kA} kH$. The fact that W_1 is actually a kH module follows easily from the equations

$$w_a.xb = (w_{a^x} + va^x)b = w_{a^x b} + va^x b = w_{ab^{x^{-1}}x} = w_a.b^{x^{-1}}x.$$

Now W_1/V_1 is free of rank one as a kA module and so by lemma 4.10 $k\text{-dim}(W_1) \geqslant pl_H(A) = e(H)$. It remains to show that W_1 is an essential extension of V_1.

Suppose that T is a non-zero submodule of W_1 such that $T \cap V_1 = 0$. Then $W_2 = W_1/(T \oplus V_1)$ is a proper factor module of $W_1/V_1 \cong kA$ and so $J = \text{ann}_{kA} W_2 \neq 0$ and $I = JkH \neq 0$. By lemma 4.7, I contains a non-zero product of prime ideals of kH, and by an argument used before I contains a non-zero product $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n$ of augmentation ideals of infinite abelian

(45) k-dim(kH) = h(H). It is easy to see that n(kG) = n(kH) and therefore by proposition 5.5, n(kG) $\leqslant$ h(G) - 1.

6. An alternative approach to theorem 4.2.

In the proof of theorem 4.2 given in § 4, we use the existence of the injective hull of the irreducible module V_1 to find an essential extension $W_1 = wkH$ of V_1 such that $w(x - 1) = v$. The bulk of the work then involved showing that k-dim(W_1) $\geqslant$ e(H) = e(G).

We can also prove theorem 4.2 by constructing the module W_1 by writing down the action of H on a vector space basis. It will then be clear that k-dim(W_1) $\geqslant$ e(H) and our work lies in showing that W_1 is an essential extension of V_1. We assume the notation of the first three paragraphs of the proof of theorem 4.2 given on pages 72-74.

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$$w_a.b = w_{ab} \quad , \quad w_a.x = w_{a^x} + va^x, \text{ for } a,b \in A,\ i,j \in \mathbb{Z}.$$

Clearly the subspace spanned by the set $\{vx^i \mid i \in \mathbb{Z}\}$ is a kH module isomorphic to $V_1 = v \otimes_{kA} kH$. The fact that W_1 is actually a kH module follows easily from the equations

$$w_a.xb = (w_{a^x} + va^x)b = w_{a^x b} + va^x b = w_{ab^{x^{-1}}x} = w_a.b^{x^{-1}}x.$$

Now W_1/V_1 is free of rank one as a kA module and so by lemma 4.10 k-dim(W_1) $\geqslant$ $pl_H(A)$ = e(H). It remains to show that W_1 is an essential extension of V_1.

Suppose that T is a non-zero submodule of W_1 such that $T \cap V_1 = 0$. Then $W_2 = W_1/(T \oplus V_1)$ is a proper factor module of $W_1/V_1 \cong kA$ and so $J = \mathrm{ann}_{kA} W_2 \neq 0$ and $I = JkH \neq 0$. By lemma 4.7, I contains a non-zero product of prime ideals of kH, and by an argument used before I contains a non-zero product $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n$ of augmentation ideals of infinite abelian

(45) $k\text{-dim}(kH) = h(H)$. It is easy to see that $n(kG) = n(kH)$ and therefore by proposition 5.5, $n(kG) \leqslant h(G) - 1$.

6. An alternative approach to theorem 4.2.

In the proof of theorem 4.2 given in § 4, we use the existence of the injective hull of the irreducible module V_1 to find an essential extension $W_1 = WkH$ of V_1 such that $w(x - 1) = v$. The bulk of the work then involved showing that $k\text{-dim}(W_1) \geqslant e(H) = e(G)$.

We can also prove theorem 4.2 by constructing the module W_1 by writing down the action of H on a vector space basis. It will then be clear that $k\text{-dim}(W_1) \geqslant e(H)$ and our work lies in showing that W_1 is an essential extension of V_1. We assume the notation of the first three paragraphs of the proof of theorem 4.2 given on pages 72-74.

<u>Alternative proof of theorem 4.2:</u> Let ϕ be the embedding $\phi : A \rightarrow k^*$ corresponding to the one-dimensional kA module v. Consider a vector space W_1 over k with basis $\{w_a \mid a \in A\} \cup \{vx^i \mid i \in \mathbb{Z}\}$. We make W_1 into a kH module as follows

$$vx^i.x^j = vx^{i+j} \quad , \quad vx^i.a = \phi(a^{x^{-i}})vx^i, \text{ and}$$

$$w_a.b = w_{ab} \quad , \quad w_a.x = w_{a^x} + va^x, \text{ for } a,b \in A,\ i,j \in \mathbb{Z}.$$

Clearly the subspace spanned by the set $\{vx^i \mid i \in \mathbb{Z}\}$ is a kH module isomorphic to $V_1 = v \otimes_{kA} kH$. The fact that W_1 is actually a kH module follows easily from the equations

$$w_a.xb = (w_{a^x} + va^x)b = w_{a^x b} + va^x b = w_{ab^{x^{-1}}x} = w_a.b^{x^{-1}}x.$$

Now W_1/V_1 is free of rank one as a kA module and so by lemma 4.10 $k\text{-dim}(W_1) \geqslant pl_H(A) = e(H)$. It remains to show that W_1 is an essential extension of V_1.

Suppose that T is a non-zero submodule of W_1 such that $T \cap V_1 = 0$. Then $W_2 = W_1/(T \oplus V_1)$ is a proper factor module of $W_1/V_1 \cong kA$ and so $J = \text{ann}_{kA} W_2 \neq 0$ and $I = JkH \neq 0$. By lemma 4.7, I contains a non-zero product of prime ideals of kH, and by an argument used before I contains a non-zero product $\mathfrak{a}_1\mathfrak{a}_2 \cdots \mathfrak{a}_n$ of augmentation ideals of infinite abelian

normal subgroups A_i of H. We suppose that the number n of the a_i has been chosen minimal with respect to this property.

Consider the exact sequence $0 \longrightarrow V_1 \longrightarrow W_1/T \longrightarrow W_1/(T \oplus V_1) = W_2 \longrightarrow 0$. Now W_2 has a series of submodules

$$T \oplus V_1 \subseteq T \oplus V_1 + W_1 a_1 \cdots a_{n-1}H \subseteq \cdots \subseteq T \oplus V_1 + W_1 a_1 H \subseteq W_1,$$ each factor of which is annihilated by some $a_i H$. It follows from lemma 1.4.18 and proposition 4.6 that this sequence splits. That is there is a submodule S of W_1 such that $S + V_1 = W_1$ and $S \cap V_1 \subseteq T$. However $T \cap V_1 = 0$ and so the assumption that W_1 is not an essential extension of V_1 implies that it is a split extension. We show directly that this is untenable.

Since $S \cong W_1/V_1$ there is an element $s \in S$ such that $s(x-1) = 0$. Let $s = \sum \mu_a w_a + \sum \lambda_i v x^i$, then we have

$$0 = s(x-1) = \sum \mu_a (w_a x - w_a) + \sum \mu_a v a^x + \sum (\lambda_{i-1} - \lambda_i) v x^i$$

$$= \sum (\mu_a x^1 - \mu_a) w_a + \sum \mu_a v a^x + \sum (\lambda_{i-1} - \lambda_i) v x^i.$$

Since the vx^i are linearly independent we see that $\lambda_i = 0$ for all i. Also $\mu_a = \mu_a x = \mu_a x^2 = \ldots$ If $\mu_a \neq 0$ for some $a \in A$, we must have $a^{x^n} = a$ for some $n \geq 1$, otherwise the expression for s would have infinitely many non-zero coefficients. However by our choice of x, $a^{x^n} = a$ holds only if $a = 1$. Therefore if $a \neq 1$, $\mu_a = 0$.

Hence $s = \mu_1 w_1$ and $0 = \mu_1 w_1 (x-1) = \mu_1 v$. This is a contradiction and shows that W_1 is an essential extension of V_1 as a kH module. Therefore the induced module $W = W_1^G$ is an essential extension of the module V_1^G by lemma 1.4.18, V_1^G is irreducible by lemma 4.3, and $k\text{-dim}(W) \geq e(H) = e(G)$.

Chapter 5. The Plinth Socle.

1. Introduction.

The material of this chapter goes somewhat beyond our main theme of injective hulls of irreducible modules. However it is motivated in part by the results of chapter 4. Proposition 4.4.6 gave a sufficient condition for an exact sequence $0 \to V \to W \to U \to 0$ of modules over a group ring kG to split. The rather technical hypothesis required the existence of a nilpotent normal subgroup A of G such that V_A is locally finite. In this chapter we show that if G is a polycylic-by-finite group, there is a characteristic abelian-by-finite subgroup A of G such that if V is any irreducible kG module, V_A is locally finite.

This result can also be seen in relation to a result of P. Hall (18), see also (36) 12.2.10 which states that if G is a finitely generated nilpotent group with centre Z, and V is an irreducible kG module, then the image of kZ in $\mathrm{End}_{kG}V$ is algebraic over k. It follows easily that V_Z is locally finite.

If G is a polycyclic-by-finite group, we consider the subgroup generated by all of the plinths, and set $\mathrm{plsoc}(G) = \langle A \mid A \text{ is a plinth in } G \rangle$, the <u>plinth socle of G</u>. Clearly this is a characteristic subgroup.

<u>Theorem 2.8.</u> If G is a polycyclic-by-finite group, k a field and V an irreducible kG module, then V_A is locally finite, where $A = \mathrm{plsoc}(G)$.

This result is proved by induction on the Hirsch number of G, using a result, theorem 2.5, essentially due to D.L.Harper, that under certain circumstances an irreducible kG module is induced from a module for a subgroup of smaller Hirsch number.

In section 3 we consider some applications of theorem 2.8. Following the terminology of Farkas and Snider (14), we shall call a kG module V <u>finitely induced</u> if $V = W \otimes_{kK} kG$ where K is a subgroup of G and W is a finite dimensional kK module. If k is an absolute field then of course every irreducible kG module is finite dimensional, and so we assume that k is non-absolute. Segal (44) has shown that if G is a finitely generated

nilpotent-by-finite group, then every irreducible kG module is finitely induced if and only if G is abelian-by-finite. We prove the following result.

Theorem 3.1 Let G be a polycyclic-by-finite group and k a non-absolute field. Every irreducible kG module is finitely induced if and only if every nilpotent subgroup of G is abelian-by-finite.

We next consider essential extensions of irreducible kG modules for G a polycyclic group. Using proposition 4.4.6 we show

Theorem 3.2. If G is a polycyclic group, k a field and V,U irreducible kG modules such that V is infinite dimensional and U is finite dimensional, then Ext(U,V) = 0.

By the results of chapter 3, and those of Donkin (10), we know that if V is a finite dimensional irreducible kG module, for G polycyclic-by-finite then $E_{kG}(V)$ is artinian, and it seems likely that if V is an infinite dimensional irreducible kG module, then $E_{kG}(V)$ is not artinian. We can show this in particular cases.

Corollary 3.5. Let G be a polycyclic-by-finite group and suppose that kG is a primitive ring with V a faithful irreducible module. Then $E_{kG}(V)$ is not locally artinian.

Finally in section 4 we discuss the relationship of the plinth socle to the Zalesskii subgroup and give some examples.

2. Locally finite modules.

We quote some results from Harper's thesis.

Theorem 2.1.((20) theorem 3.13). Suppose that G is a polycyclic-by-finite group, k any field and A an infinite abelian normal subgroup of G. Then no irreducible kG module can be torsion free as a kA module.

This is deduced from the corresponding result for absolute fields, (36) 12.3.6, (41) theorem E.

If a subgroup B of a polycyclic group G is a plinth, we do not require B to be normal in G, only $|G:N_G(B)| < \infty$. It is therefore helpful to have

the following result.

Lemma 2.2 ((20) lemma 3.15) If B is a plinth in the polycyclic-by-finite group G, then there exists an abelian normal subgroup A of G such that $A \cap B > 1$.

In fact if Zal(G) denotes the Zalesskii subgroup of G, it is not hard to see that $Zal(G) \cap B > 1$, and we may take A to be the centre of Zal(G).

If A is a subgroup of a group G, we denote by $\mathfrak{a}_s$ the augmentation ideal of the subgroup $A^s = \langle a^s \mid a \in A \rangle$.

Lemma 2.3. ((20) lemma 3.7). Let A be an eccentric plinth in the polycyclic-by-finite group G, and suppose that $A \trianglelefteq G$. If k is a field and P a prime ideal of kA such that $|G:N_G(P)| < \infty$ then there is a positive integer s such that $\mathfrak{a}_s \leq P$.

Since $A \trianglelefteq G$, G acts by conjugation on kA and we set $N_G(P) = \{g \in G \mid P^g = P\}$. We notice that Bergman's theorem (36) 9.3.9 shows that $\dim_k kA/P < \infty$.

Harper was concerned to show that certain group algebras of polycyclic-by-finite groups could not have primitive irreducible modules (a module is said to be primitive if it cannot be induced from a module for the group algebra of a proper subgroup). The technique is to use a result of Roseblade in conjunction with the results stated above.

If S is any ring and V an S module, then for any subset X of S we denote by ${}^*X$ the annihilator of X in V namely

${}^*X = \{v \in V \mid vx = 0 \text{ for all } x \in X\}$. This of course is a departure from our previous notation.

We denote by $\Pi_S(V)$ the set of all ideals of S which are maximal with respect to ${}^*P > 0$. If S satisfies the maximal condition on ideals, then $\Pi_S(V)$ is non-empty and it is easily seen that any member of $\Pi_S(V)$ is a prime ideal.

We shall consider the case where A is an abelian normal subgroup of the polycyclic group G and V an irreducible kG module. By theorem 2.1, V

cannot be torsion free as a kA module, so if $P \in \pi_{kA}(V)$, then $P > 0$.

<u>Lemma 2.4.</u> ((41)lemma 3). Let k be a field, and R the group ring of a group G over k. Let S be the group ring of a normal subgroup H of G over k, and V a kG module. If $P \in \pi_S(V)$ and T is a right transversal to $N_G(P)$ in G then $({}^*P)R = \bigoplus_{t \in T} {}^*Pt$.

If V is irreducible, then as ${}^*P > 0$ we must have $({}^*P)R = V$ and hence $V = {}^*P \otimes_{kN} kG$, where $N = N_G(P)$.

The proof of the next result is adapted from (20) theorem 4.2.

<u>Theorem 2.5.</u> Let G be a polycyclic -by-finite group, k a field , V an irreducible kG module, and suppose that B is an eccentric plinth in G such that $B \cap C_G(V) = 1$.

Then there exists a subgroup K of G such that $h(K) < h(G)$, $|B:B \cap K| < \infty$ and $V = V_1 \otimes_{kK} KG$, where V_1 is an irreducible kK module.

<u>Proof:</u> By lemma 2.2 there exists an abelian normal subgroup A of G such that $A \cap B > 1$. Therefore $\mathrm{rank}(B) = \mathrm{rank}(A \cap B)$, and if K is any subgroup of G such that $|A \cap B : A \cap B \cap K| < \infty$, then $|B : A \cap B \cap K| < \infty$ and so $|B : B \cap K| < \infty$.

Therefore by replacing B by $A \cap B$, we may assume that $B \leq A$.

Let $H = \mathrm{core}_G(N_G(B))$, that is the largest normal subgroup of G contained in $N_G(B)$. Then $|G:H| < \infty$, and $A \leq H$.

Now by Clifford's theorem $V_H = U_1 \oplus \ldots \oplus U_r$ where the U_i are the homogeneous components of V as a kH module.

If $B \cap C_H(U_i) \neq 1$ for each $i = 1, \ldots r$, then $B \cap C_H(V) \neq 1$, which goes against our hypothesis. Hence $B \cap C_H(U_i) = 1$ for some i.

Let $H_1 = \{g \in G \mid U_i g = U_i\}$, the stabiliser of U_i. Then again by Clifford's theorem $V = U_i \otimes_{kH_1} kG$, and we may replace G with H_1 and V with U_i to assume that V has only one homogeneous component as a kH module.

Therefore $V_H = W_1 \oplus \ldots \oplus W_s$ where the W_i are irreducible kH modules, which are all isomorphic to a fixed kH module W, say.

By theorem 2.1, W is not torsion free as a kB module. Choose $P \in \pi_{kB}(W)$,

then $P \neq 0$ and ${}^*P \neq 0$.

Therefore ${}^*(PkA) > 0$ and we can choose $Q \in \pi_{kA}(V)$ such that $PkA \leq Q$. Then $P \leq Q \cap kB$.

Since the W_i are isomorphic, and Q has non-zero annihilator in V, we deduce that Q has non-zero annihilator in W. Hence $Q \cap kB$ has non-zero annihilator in W.

By the maximality of P we have $P = Q \cap kB$. Now let $K = N_G(Q)$, $L = N_H(P)$. Then by lemma 2.4 we have $V = {}^*Q \otimes_{kK} kG$ and $W = {}^*P \otimes_{kL} kH$. Let $V_1 = {}^*Q$, an irreducible kK module.

We have $B \leq A$ and Q is an ideal of kA, so $B \leq N_G(Q) = K$.

It only remains to show $h(K) < h(G)$.

If this is not the case then $|G:K| < \infty$. Now $H \cap K \leq L$ and so $|H:L| < \infty$. Now by lemma 2.3, $\underline{b}_s \leq P$ for some $s \geq 1$.

Therefore ${}^*P\underline{b}_s = 0$ and since B^s is characteristic in B, it is normal in H and so $({}^*P \bullet h)\underline{b}_s = 0$ for each $h \in H$. Therefore $W\underline{b}_s = 0$, but this contradicts the fact that $B \cap C_H(W) = 1$.

<u>Lemma 2.6.</u> Let B_1, B_2 be subgroups of a group G and V a kG module.

i) If $B_1 \trianglelefteq G$ and V_{B_i} is locally finite for $i = 1,2$ then V_B is locally finite where $B = B_1B_2$.

ii) If $B_1 \leq B_2$, $|B_2:B_1| < \infty$ and V_{B_1} is locally finite, then V_{B_2} is locally finite.

<u>Proof:</u> i) It suffices to show that if $v \in V$, then $\dim_k vkB < \infty$. By assumption $\dim_k vkB_1 < \infty$, so let $v_1, \ldots, v_n$ be a k-basis of vkB_1. Then $vkB \leq \sum_{i=1}^{n} v_i kB_2$ which is finite dimensional.

ii) Let $v \in V$, we must show $\dim vkB_2 < \infty$. If $g_1, \ldots, g_n$ is a left transversal to B_1 in B_2, then any element of kB_2 can be written as $\sum_{i=1}^{n} g_i \beta_i$, where $\beta_i \in kB_1$. Therefore $vkB_2 \leq \sum_{i=1}^{n} vg_i kB_1$ and this is finite dimensional.

<u>Proposition 2.7.</u> Let B be a plinth in the polycyclic-by-finite group G and V an irreducible kG module. Then V_B is locally finite.

Proof: We use induction on $h(G)$. If $h(G) \leq 2$, G is abelian-by-finite, so $\dim_k V < \infty$ and the result is trivial.

Suppose first that $B \cap C_G(V) \neq 1$. Then since B is a plinth, we have $|B:B \cap C_G(V)| < \infty$. Now $V_{B \cap C_G(V)}$ is a direct sum of trivial modules, so is certainly locally finite, and the result follows from lemma 2.6 ii).

If $\mathrm{rank}(B) = 1$, the result follows by methods of P. Hall (10). There is a normal subgroup H of finite index in G which centralises B. As a kH module $V = \bigoplus_{i=1}^{r} V_i$ where V_i is an irreducible kH module.

Let $\langle z \rangle = B \cap H$. Then by lemma 2.1 or (36) 12.2.8 there is a polynomial $f_i(z)$ in $k\langle z \rangle$ such that $V_i f_i(z) = 0$. Hence each V_i is a locally finite $k\langle z \rangle$ module and the result follows from lemma 2.6 since $|B:\langle z \rangle| < \infty$.

Finally, let us suppose that $B \cap C_G(V) = 1$, and $\mathrm{rank}(B) \geq 2$. Then by theorem 2.5 there is a subgroup K of G such that $h(K) < h(G)$, $|B:B \cap K| < \infty$ and $V = V_1 \otimes_{kK} kG$ where V_1 is an irreducible kK module.

Since $h(K) < h(G)$ we can apply the inductive hypothesis to the plinths of K.

Consider the $\mathbb{Q}G$ module $B \otimes_{\mathbb{Z}} \mathbb{Q}$. It is a consequence of Mal'cev's theorem on irreducible soluble linear groups, (36) 12.1.3 that as B is a plinth $N_G(B)/C_G(B)$ is abelian-by-finite. Hence there is a normal subgroup G_0 of finite index in G containing B and such that $B \trianglelefteq G_0$ and $G_0/C_{G_0}(B)$ is abelian. Now if $K_0 = K \cap G_0$ then

$$K_0 C_{G_0}(B)/C_{G_0}(B) \trianglelefteq G_0/C_{G_0}(B)$$ and by Clifford's theorem it follows that $B \otimes_{\mathbb{Z}} \mathbb{Q}$ is completely reducible as a $\mathbb{Q}K_0$ module.

Hence there is a subgroup $B_1 \times \ldots \times B_n$ of finite index in B such that each B_i is a plinth in K. By induction $V_1 |_{B_i}$ is locally finite for each $i = 1, \ldots, n$. Therefore by lemma 2.6i) $V_1|_B$ is locally finite.

Now, we can in fact assume that $B \trianglelefteq G$. For, if $H = \mathrm{core}_G(N_G(B))$, then $|G:H| < \infty$ and V is completely reducible as a kH module, and it suffices to show that each kH submodule of V is locally finite as a $k(H \cap B)$ module.

Since $V_1|_B$ is locally finite we can conclude that $V_1 \otimes G|_B$ is locally

finite for all $g \in G$ and therefore $V = V_1 \otimes_{kB} kG$ is locally finite as a kB module.

<u>Theorem 2.8.</u> If G is a polycyclic-by-finite group, k a field and V an irreducible kG module, then V_A is locally finite, where $A = \mathrm{plsoc}(G)$.

<u>Proof:</u> We have $A = \langle B \mid B$ is a plinth in $G \rangle$. Now since A is finitely generated there are finitely many plinths $B_1, \ldots, B_r$ such that

$A = \langle B_1, \ldots, B_r \rangle$.

Let $G_o = \mathrm{core}_G(\bigcap_{i=1}^{r} N_G(B_i))$, so that G_o is a normal subgroup of finite index in G which normalises each B_i, and let $C_i = B_i \cap G_o \trianglelefteq G_o$, and $C = \langle C_1, \ldots, C_r \rangle$. Then $|B_i : C_i| < \infty$ and by (41) lemma 1 $|A:C| < \infty$.

Hence by lemma 2.6 ii) it suffices to show that V is locally finite as a kC module.

Now by Clifford's theorem V is completely reducible as a kG_o module and by proposition 2.7 V_{C_i} is locally finite. Since $C_i \trianglelefteq G_o$, $C = C_1C_2 \ldots C_r$ and lemma 2.6 i) shows that V_C is locally finite. This completes the proof.

We have as a consequence an 'intersection theorem' for maximal right ideals, which could be compared with (36) 7.4.9 and Bergman's theorem (36) 9.3.9.

<u>Corollary 2.9.</u> If G is a polycyclic-by-finite group, k a field, M a maximal right ideal of kG and $A = \mathrm{plsoc}(G)$, then $\dim_k kA/M \cap kA < \infty$.

<u>Proof:</u> As kG/M is an irreducible kG module, it is locally finite as a kA module. Hence for all $\alpha \in kG$, $\dim_k (M + \alpha kA)/M < \infty$. Putting $\alpha = 1$ gives the result.

3. Applications.

We first study finitely induced modules. Let $\mathfrak{X}$ be the class of polycyclic-by-finite groups all of whose nilpotent subgroups are abelian-by-finite. We need to know that the class $\mathfrak{X}$ is closed under taking homomorphic images. Suppose that $G \in \mathfrak{X}$, $H \trianglelefteq G$ and K/H is a nilpotent subgroup of G/H. Then K is a subgroup of G and so $K \in \mathfrak{X}$. It suffices to show that

if $G \in \mathfrak{X}$, then every nilpotent factor group of G is abelian-by-finite. This follows from a result of Zaicev (52) which states that if G is polycyclic-by-finite, $H \trianglelefteq G$ with G/H nilpotent, then there is a nilpotent subgroup X of G such that $|G:HX| < \infty$. Now, if $G \in \mathfrak{X}$, then X is abelian-by-finite, and so G/H is abelian-by-finite. We remark that Zaicev's theorem follows easily from a result of Robinson, (40) theorem 5.

<u>Theorem 3.1.</u> Let G be a polycyclic-by-finite group and k a non-absolute field. Every irreducible kG module is finitely induced if and only if $G \in \mathfrak{X}$.

<u>Proof:</u> Suppose that $G \in \mathfrak{X}$ and V is an irreducible kG module. We use induction on $h(G)$ to show that V is finitely induced. Since the class $\mathfrak{X}$ is closed under taking homomorphic images, we may assume $C_G(V) = 1$.

Now $Fit(G)$, the Fitting subgroup of G is abelian-by-finite and hence G is metabelian-by-finite. If every plinth in G is centric, then by corollary 4.4.5, or (20), proposition 4.3, G is nilpotent-by-finite and so abelian-by-finite, and V is finite dimensional.

If B is an eccentric plinth in G, then since $B \cap C_G(V) = 1$, there is a subgroup K of G such that $h(K) < h(G)$ and $V = V_1 \otimes_{kK} kG$ by theorem 2.5.

Now V_1 is an irreducible kK module, and hence by induction, V_1 and therefore V are finitely induced.

Conversely suppose that the polycyclic-by-finite group G does not belong to $\mathfrak{X}$. We claim G has a subgroup H isomorphic to a free nilpotent group of class two on two generators, $\langle x,y,z \mid (x,y) = z,\ (x,z) = (y,z) = 1\rangle$.

Now G has a nilpotent subgroup H_1 which is not abelian-by-finite, and H_1 has a subgroup H_2 of finite index which is torsion free, but not abelian-by-finite. Let $Z_1 = Z(H_2)$, $Z_2 = Z(H_2/Z_1)$, and choose $x \in Z_2 \setminus Z_1$, then $(x,y) \neq 1$ for some $y \in H_2$ and $(x,y) = z \in Z_1$. $H = \langle x,y,z\rangle$ is the desired subgroup.

We use Harper's construction, (19) of a primitive irreducible kH module. Let ζ be an element of k^* which is not a root of one, and $M = (z - \zeta)kH + (x+y+1)kH$, a right ideal of kH. In (19) Harper shows

that $U = kH/K$ is a primitive irreducible module.

Let V be an irreducible kG module such that $U \subseteq V_H$. Suppose if possible that $V = W \otimes_{kK} kG$, where $K \subseteq G$, and W is a finite dimensional kK module.

Then by Mackay's theorem $V_H = \oplus\ (W_a|_{K_a})^H$, where $K_a = K^a \cap H$, $W_a = W \otimes a$, a module for kK^a, and the sum is taken over all double cosets KaH. Now since U is irreducible and $U \subseteq V_H$, we see that $U \subseteq (W_a|_{K_a})^H$ for some $a \in G$. Let $L = K_a$ and $W' = W_a$ so that $U \subseteq W' \otimes_{kL} kH$.

Let $Z = \langle z \rangle$. If $L \cap Z = 1$, let $\{g_i \mid i \in I\}$ be a transversal to LZ in H. Then as a transversal to L in H, we may take $\{z^j g_i \mid j \in Z, i \in I\}$ and so as a kZ module, $W' \otimes_{kL} kH$ is free of rank $|I|.\dim_k W'$. Since U has non-zero annihilator in kZ, it cannot be embedded in a free kZ module.

Therefore $|LZ:L| < \infty$ and we may assume that $Z \subseteq L$.

Now U is generated by an element u such that $u(z-\zeta) = u(x+y+1) = 0$.

Suppose that $x_1 = x^n y^m \in L$ where $x_1 \neq 1$, then as W' is finite dimensional $W'f(x_1) = 0$, for some non-zero polynomial $f(x_1) \in k[x_1]$. Now x_1 is contained in the abelian normal subgroup $A = \langle x_1, z \rangle$ of H, and so commutes with each of its conjugates in H, so as $u \in W' \otimes_{kL} kH$, $ug(x_1) = 0$, for some non-zero polynomial $g(x_1)$. Therefore the ideal $I = g(x_1)kA + (z-\zeta)kA$ has non-zero annihilator in U, and we may choose $P \in \Pi_{kA}(U)$ containing I.

Then, by lemma 2.4 $U = P \otimes_{kN} kH$, where $N = N_H(P)$. Since U is a primitive module $N = H$, but then PkH is a two sided ideal of kH properly containing $(z-\zeta)kH$, which is impossible as $(z-\zeta)kH$ is primitive and so by (42), theorem G1 a maximal ideal in kH. Therefore $L = Z$.

If $w_1, \ldots, w_t$ is a vector space basis of W', then

$$\{w_i x^j y^k \mid i = 1, \ldots, t,\ j,k \in \mathbb{Z}\} \text{ is a basis of } W' \otimes_{kZ} kH.$$

Now if we express u as a linear combination of these basis elements and use the fact that $u(x+y+1) = 0$, we easily obtain a contradiction to their linear independence. This shows that the irreducible kG module V is not finitely induced.

Particular cases of the above result have been obtained by Harper, (20) corollary 4.9, and Segal (44), theorem A. In (14), Farkas and Snider show that any primitive ideal in the group algebra of a polycyclic group is the

annihilator of an irreducible finitely induced module.

We now examine essential extensions of irreducible modules

<u>Theorem 3.2.</u> If G is a polycyclic-by-finite group, k a field and V,U irreducible kG modules such that V is infinite dimensional and U is finite dimensional, then $\mathrm{Ext}(U,V) = 0$.

<u>Proof:</u> Let $0 \to V \to W \to U \to 0$ (*) be an exact sequence of kG modules with V,U as above. We have to show that this sequence splits.

Clearly we may assume that $C_G(W) = 1$. Also, if $H \trianglelefteq G$, with $|G/H| < \infty$ and (*) splits as a sequence of kH modules, then $W = V \oplus U'$ where $U' \cong_{kH} U$, and $U'kG$ is a finite dimensional kG submodule of W, and therefore $U'kG \cap V = 0$ and (*) splits as a sequence of kG modules.

Let $C = C_G(U)$ and $D = C_G(V)$. If $a,b \in C \cap D$ and $w \in W$, then

$w((a-1)(b-1) - (b-1)(a-1)) = 0$, so $(a,b) \in C_G(W) = 1$ and so $C \cap D$ is abelian. Hence, if F denotes the Fitting subgroup of G, then $C \cap D \leq C \cap F$. Now G/C and G/F are abelian-by-finite and so $G/C \cap F$ is abelian-by-finite. If $|C \cap F : C \cap D| < \infty$, then $G/C \cap D$ would be abelian-by-finite. However this contradicts the fact that V is an infinite dimensional irreducible module.

Therefore $C \cap F/C \cap D$ is an infinite normal subgroup of $G/C \cap D$ and so contains a plinth $A/C \cap D$. Dropping to a subgroup of finite index in G we may assume that A is a normal subgroup of G.

Now A is a nilpotent normal subgroup of G, and it is easily seen that $C_V(\underline{a}) = 0$, and $C_U(\underline{a}) = U$. Moreover since V is an irreducible $kG/C \cap D$ module and $A/C \cap D$ is a plinth in $G/C \cap D$, proposition 2.7 shows that V_A is locally finite. Therefore by proposition 4.4.6, the sequence (*) splits.

As our final application, we consider the injective hull $E_{kG}(V)$ of an irreducible kG module V, for G a polycyclic-by-finite group.

If $\dim_k V$ is finite, then as we have remarked $E_{kG}(V)$ is artinian, and it seems likely that the converse holds.

If A is a plinth in G, then by (36) lemma 12.3.1, we can find an

element $x \in N_G(A)$ such that A is a plinth in $\langle A,x\rangle$. In contrast with theorems 2.1 and 2.2 we have the following result.

<u>Lemma 3.3.</u> With G, A, and x as above, suppose that V is an irreducible kG module and A is an eccentric plinth in G such that $A \cap C_G(V) = 1$. Then $V_{\langle x\rangle}$ is torsion free.

<u>Proof:</u> Let $H = \mathrm{core}(N_G(A))$. Then H is a normal subgroup with finite index in G which normalises A. Clifford's theorem shows that $V_H = \bigoplus_{i=1}^{r} V_i$ a direct sum of irreducible kH modules, and it suffices to show each V_i is torsion free. Hence we may suppose that $A \trianglelefteq G$.

If $V_{\langle x\rangle}$ is not torsion free, it has a finite dimensional $k\langle x\rangle$ submodule. Since $B = \langle A,x\rangle$ is a split extension of A by x, and V_A is locally finite it follows that V has a finite dimensional kB submodule, by arguments similar to those used in proving lemma 2.6.

Hence V has a finite dimensional irreducible kB submodule V_1.
Let $P = \mathrm{ann}_{kA} V_1 \neq 0$ since $\dim_k V_1 < \infty$. Hence by lemma 2.3, $\underline{a}_s \leq P$ for some $s \geq 1$.

Therefore $V_1\underline{a}_s = 0$, and since $A \trianglelefteq G$, $V_1 g\underline{a}_s = 0$ for all $g \in G$. Since V is an irreducible kG module, this shows $V\underline{a}_s = 0$, contradicting our assumption that $A \cap C_G(V) = 1$.

<u>Theorem 3.4.</u> If G is a polycyclic-by-finite group, V an irreducible kG module and A an eccentric plinth in G such that $A \cap C_G(V) = 1$. Then $E_{kG}(V)$ is not locally artinian.

<u>Proof:</u> Since V_A is locally finite, it contains a finite dimensional irreducible kA submodule $U_0 = u_0kA$ for some $u_0 \in U_0$.

Now there exists $x \in N_G(A)$ such that A is a plinth in $\langle A,x\rangle = H$. We consider the kH submodule $U = u_0kH$ of V.

By lemma 3.3, the elements $\{u_0x^i \mid i \in \mathbb{Z}\}$ are linearly independent. Let $u_i = u_0x^i$ and $U_i = u_ikA = U_0x^i$.

Suppose that $U_{n+1} \cap \sum_{i=0}^{n} U_i \neq 0$. Then $U_{n+1} \subseteq \sum_{i=0}^{n} U_i$, since U_{n+1} is an irreducible kA module. Hence $u_0x^m \in \sum_{i=0}^{n} U_i$ for all $m \geq 0$ and so these elements cannot all be linearly independent. It follows that the sum

$\sum_{i\in\mathbb{Z}} U_i$ is direct and therefore $U = U_o \otimes_{kA} kH$, and if $v_1, \ldots, v_n$ is a basis for U_o, then $\{v_i x^j \mid i = 1, \ldots, n, j \in \mathbb{Z}\}$ is a basis for U.

Now, since $(x-1)$ is a non zero divisor, there exists an element $w \in E_{kH}(U)$ such that $w(x-1) = v_1$. If $\overline{V} = wkH$ it is easily seen that $\overline{V}$ is a proper essential extension of U. We have an exact sequence $0 \to U \to \overline{V} \to \overline{V} \to 0$ of kH modules and using Bergman's theorem as in chapter 4 we see that either $\dim_k \overline{V} < \infty$ or $\overline{V}$ is free as a kA module. The first alternative is impossible by theorem 3.2.

Now $w \in E_{kH}(U) \subseteq E_{kH}(V) \subseteq E_{kG}(V)$. Let $W_1 = wkG$, a finitely generated essential extension of V. If W_1 is artinian, then it has a composition series of finite length, and hence by proposition 2.7, the restriction of W_1 to A is locally finite. This is patently not the case, since W_1 has a non-zero free kA module as a factor module of a submodule.

We have shown that $E_{kG}(V)$ is not locally artinian.

<u>Corollary 3.5.</u> Let G be a non-trivial polycyclic-by-finite group and suppose that kG is a primitive ring with V a faithful irreducible module. Then $E_{kG}(V)$ is not locally artinian.

<u>Proof:</u> Clearly $C_G(V) = 1$. Also since kG is primitive, $\Delta(G) = 1$ and no plinth in G can be centralised by a subgroup of finite index. Hence any plinth in G is eccentric and the result follows.

4. Properties of the plinth socle.

Throughout this section, G will denote a polycyclic-by-finite group, $F = F(G)$ the finite radical of G, H/F the Fitting subgroup of G/F and Zal(G)/F the centre of H/F. Zal(G) is the Zalesskii subgroup of G.

As noted by Roseblade (42) p.390, every non-trivial normal subgroup of G meets Zal(G) non-trivially. Since any infinite normal subgroup of G contains a plinth in G, any infinite normal subgroup has non-trivial intersection with plsoc(G), and so any non-trivial normal subgroup of G meets plsoc(G)F(G) non-trivially.

Lemma 4.1. If G is a finitely generated nilpotent group, then

$$Zal(G) = plsoc(G)F(G).$$

Proof: Take $b \in Zal(G)$. If b has finite order $b \in F(G)$. Otherwise $B = \langle b \rangle$ is a plinth in G. Conversely, suppose that B is a plinth in G. Then BF/F is a plinth in G/F, and so is centralized by a subgroup of finite index in G. However centralizers are isolated in a torsion free nilpotent group and so $BF/F \leq Zal(G)/F$. Therefore $BF \leq Zal(G)$.

Lemma 4.2. If B is a plinth in a polycyclic-by-finite group G, and $G_0 = N_G(B)$, then $B \leq Zal(G_0)$.

Proof: Since the Zalesskii subgroup meets any non-trivial normal subgroup of G_0, we have $|B:B \cap Zal(G_0)| < \infty$. Let $F = F(G_0)$ and H/F the Fitting subgroup of G_0/F. If $(B,H) \neq 1$, then $(B,h) \neq 1$ for some $h \in H$.

The map $\Theta : b \to (b,h)$ is an endomorphism of B. Moreover $B^n \leq B \cap Zal(G_0)$, for some $n \geqslant 1$, and if $b \in B \cap Zal(G_0)$, $h \in H$, then $(b,h) \in B \cap F = 1$.

Therefore $B^n \leq \ker \Theta$ and hence (B,h) has finite order.

Therefore $(B,h) = 1$.

We remark that if L is any subgroup of a polycyclic group G having the property of the plinth socle expressed in theorem 2.8, then L must be abelian-by-finite. For any irreducible kL module V can be embedded in an irreducible kG module W, and if W_L is locally finite then V must be finite dimensional. If this is true for all fields k, then L must be abelian-by-finite.

Finally, we give two examples of polycyclic groups G such that $Zal(G)/plsoc(G)$ is infinite.

Example 4.3. Let

Let $x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ act on a free Z module $A = \langle a,b,c,d \rangle$ of rank 4.

Thus $a.x = b$, $b.x = a + b$, $c.x = c + d$, $d.x = d$. We form the split extension $G = A \rtimes x$ and regard A as a subgroup of G. Let $A_1 = \langle a,b \rangle$,

$A_2 = \langle c,d \rangle$ and $A_3 = \langle d \rangle = Z(G)$.

Since A is an abelian normal subgroup of G, we have $A \leq \mathrm{Fit}(G)$. Since no power of x acts nilpotently on A_1 (this can be checked by computing the eigenvalues of x), $A = \mathrm{Fit}(G) = \mathrm{Zal}(G)$.

Now, if we regard $A \otimes_{\mathbb{Z}} \mathbb{L}$ as a $\mathbb{L}\langle x^n \rangle$ module for $n \geq 1$, we see that there are just two irreducible submodules $A_1 \otimes_{\mathbb{Z}} \mathbb{L}$ and $A_3 \otimes_{\mathbb{Z}} \mathbb{L}$.

Hence $\mathrm{plsoc}(G) = A_1 \times A_3$, with A_1 an eccentric and A_3 a centric plinth.

We remark that in this example it is possible to construct examples of irreducible kG modules V such that V_A is not locally finite where $A = \mathrm{Zal}(G)$.

<u>Example 4.4.</u> Let H be a free abelian group of finite rank ≥ 1, and A_1 a $\mathbb{Z}H$ module such that $A_1 \otimes_{\mathbb{Z}} \mathbb{L}$ is an irreducible $\mathbb{Q}H$ module with dimension ≥ 2 and suppose that $C_H(A_1) = 1$.

By theorem 1.5.7, $A_1 \otimes_{\mathbb{Z}} \mathbb{L}$ cannot be injective as a $\mathbb{Q}H$ module. Now by (10) the injective hull $E_{\mathbb{Q}H}(A_1 \otimes_{\mathbb{Z}} \mathbb{L})$ is artinian and all the composition factors are identical.

Hence we can choose a proper essential extension A_2 of A_1 such that $A_2/A_1 \cong A_1$ as $\mathbb{Q}H$ modules. Let G be the split extension of A_2 by H.

Now A_2 is abelian and so $A \leq \mathrm{Fit}(G)$. The condition $C_H(A_1) = 1$ implies that $A_2 = \mathrm{Fit}(G) = \mathrm{Zal}(G)$. On the other hand clearly $\mathrm{plsoc}(G) = A_1$ by construction.

This method yields many explicit examples. Let $H = \langle x \rangle$ be infinite cyclic and let A_1 be the $\mathbb{L}\langle x \rangle$ module $\mathbb{Q}[x]/(x^2 - x - 1)$.

Then A_1 is an irreducible $\mathbb{Q}\langle x^n \rangle$ module for each $n \geq 1$, and we can choose a $\mathbb{Z}$ basis $\{a,b\}$ such that $a.x = b$, $b.x = a + b$.

There exists an element $c \in E_{\mathbb{Q}\langle x \rangle}(A_1)$ such that $c(x^2 - x - 1) = a$.

If $d = c.x$, then $d.x = a + c + d$. Let A be the $\mathbb{Z}\langle x \rangle$ module $\langle a,b,c,d \rangle$ and $G = A \rtimes x$. Then $\mathrm{Zal}(G) = A$ and $\mathrm{plsoc}(G) = A_1$.

Using the fact that any irreducible kG module V is finitely induced for this group, it is possible to show that V_A is locally finite.

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