ON THE THEORY OF FITTING CLASSES OF FINITE GROUPS

by

Owen John Brison

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Declaration

I declare that none of the material in this thesis has been used by me before.

Material which is not specifically attributed, or described as "well-known" or "familiar" is original. In this regard, Chapter 1 consists of background material for reference purposes and is thus not original. In Chapter 4 , we give a new proof of a recent theorem of Berger ; what we claim as original in that case is our approach to the proof and not of course the result itself : this is discussed in the introduction to Chapter 4.

SUMMARY

Chapter 1 comprises background material.

Chapters 2 and 3 deal with "Hall π-properties" of Fitting classes of finite soluble groups. Given a Fitting class $\mathfrak{F}$ and a set of primes π, Lockett defines a Fitting class $\mathcal{L}_\pi(\mathfrak{F})$ to comprise those groups in which an $\mathfrak{F}$-injector contains a Hall π-subgroup; we also have $\mathcal{K}_\pi(\mathfrak{F})$, the class of groups whose Hall π-subgroups belong to $\mathfrak{F}$. We find that $\mathcal{L}_\pi(\ )$ and $\mathcal{K}_\pi(\ )$ commute with Lockett's upper-star operation, while $\mathcal{K}_\pi(\ )$ is multiplicative. We characterize normal Fitting classes in terms of $\mathcal{L}_\pi(\ )$. Closure of classes under taking Hall-subgroups ("Hall-closure") is connected with $\mathcal{K}_\pi(\ )$; we introduce closure under Hall-subgroups containing the Fitting subgroup: the class of groups with central socle is so closed, but not Hall-closed. If $\mathfrak{F}$ is a primitive saturated formation of finite defect n, then $G \in \mathfrak{F}$ if and only if all Hall π-subgroups for $|\pi| \leq n$ belong to $\mathfrak{F}$.

Let $\mathfrak{F} \leq \mathcal{S}$ be a Fitting class of characteristic π, where π is not the set of all primes, and let $\mathcal{H}$ be the smallest normal Fitting class. Then $\mathfrak{F} * \mathcal{H}$ is Hall-closed if and only if $\mathfrak{F}^* = \mathcal{S}_\pi$. We show that $\mathcal{D} * \mathcal{H} = \mathcal{S}_\pi * \mathcal{H}$ if and only if $\mathcal{D}^* = \mathcal{S}_\pi$, and that if $\mathcal{D} * \mathcal{H} = \mathcal{G} * \mathcal{H}$, where $\mathcal{G}$ has odd characteristic, then $\mathcal{D}^* = \mathcal{G}^*$.

Chapter 4 contains a new proof of a theorem of Berger showing how transfer maps "determine" $\mathcal{H}$ (as above). Our proof is in the context of the Lausch group, and we believe is simpler than Berger's. Using similar arguments we prove a result on generation of Fitting classes. Examples are considered and we show that a class of Hawkes is generated by a single group.

CHAPTER 1

Background for Fitting class theory.

The theory of Fitting classes of finite (soluble) groups is currently at a stage where there are many important concepts and results scattered throughout the literature, few of which have yet found their way into a textbook.

An excellent general introduction to the subject as it was in 1969 is contained in the mimeographed notes by J. C. Looker of lectures given in Canberra by Professor Gaschütz, [26]. More recently, an outline of the subject as it was in 1973 is contained in the survey article [17] by Cossey. The projected book by Doerk and Hawkes, [21], will give a thorough treatment of the subject.

We have thus found it convenient to use this chapter as a repository for some of the more important concepts and results of the theory of Fitting classes; in most cases we will refer to the literature for the relevant proofs. We also introduce in this chapter our notation as well as various assumed results from group theory, and set up the formalism for certain constructions we will use, such as the wreath product. We hope that our notation will be found to coincide with that in general use, and that any notation which might not be explicitly defined will be sufficiently well-known to be clear.

The only claim for originality in this chapter is that we draw attention to a lacuna in the statement of a recent lemma of Berger, (1.9.3).

1.1 Group-theoretical preliminaries.

1.1.1 Notation, conventions and basic assumptions.

(a) Let G be a group. Then $H \leq G$ will indicate that H is a subgroup of G, $H \trianglelefteq G$ that H is a normal subgroup of G, $H \text{ sn } G$ that H is a subnormal subgroup of G, and $H \text{ char } G$ that H is a characteristic subgroup of G. We may write $H \lneq G$ to emphasise that H is a proper subgroup of G, and $G \geq H$ will be equivalent to $H \leq G$. If $H \leq G$, then $N_G(H)$, $C_G(H)$ will denote the normalizer, centralizer, respectively, of H in G. The automorphism group of G will be denoted by $\underline{\mathrm{Aut}(G)}$. If G and H are isomorphic groups, we will write $G \cong H$. In general, groups will be denoted by upper-case Roman (or, occasionally, Greek) letters, while group elements will be denoted by lower-case Roman, or Greek, letters; $H \trianglelefteq\cdot G$;$(H \cdot\!\trianglelefteq G)$ denotes that H is maximal (minimal) normal in G.

(b) The set of all (positive, rational) prime numbers will be denoted by $\mathbb{P}$.

Suppose that $\pi \subseteq \mathbb{P}$. Then π' will denote the set $\mathbb{P} \setminus \pi$. A <u>π-number</u> will be a natural number all of whose prime divisors belong to π, while a <u>π-group</u> will be a finite group whose order is a π-number. If $\pi = \{p\}$, where $p \in \mathbb{P}$, we shall usually omit the braces when referring to the sets $\{p\}$ and $\{p\}'$. Sets of primes will usually be denoted by lower-case Greek letters.

(c) Suppose that G is a finite group and that $\pi \subseteq \mathbb{P}$. Then $|G|$ will denote the order of G, while $|G|_\pi$ will denote the π-part of $|G|$; that is, the unique natural number m such that $G = mn$, with m a π-number and n a π'-number. The set of prime divisors of $|G|$ will be denoted by $\underline{\mathrm{char}(G)}$, while $\underline{\mathrm{comp}(G)}$ will denote

the following set (which coincides with char(G) if G is soluble)

$$\mathrm{comp}(G) = \{p \in \mathbb{P} : G \text{ possesses a composition factor of order } p\}.$$

Suppose that H is a subgroup of G. The index of H in G will be denoted by $|G : H|$. If H is a π-number and $|G : H|$ is a π'-number, then H will be called a <u>Hall π-subgroup of G</u>; in the case that $\pi = \{p\}$, H will be called a <u>Sylow p-subgroup of G</u>. The set of all Hall π-subgroups of G will be denoted by $\underline{\mathrm{Hall}_\pi(G)}$; the set of all Sylow p-subgroups will be denoted by $\underline{\mathrm{Syl}_p(G)}$.

The π-socle of G, that is, the join of all minimal normal abelian π-subgroups of G, will be denoted by $\underline{\pi\text{-soc}(G)}$.

(d) If $n \in \mathbb{N}$ (the set of all natural numbers), then $\underline{\mathrm{Sym}(n)}$ will denote the symmetric group on n symbols, $\underline{\mathrm{Alt}(n)}$ the alternating group on n symbols, and $\underline{C_n}$ the cyclic group of order n.

(e) If G is a finite group, then $\underline{F(G)}$ will denote the Fitting subgroup of G; that is, the largest nilpotent normal subgroup of G. That this definition is well-posed is of course a consequence of Fitting's theorem that the product of two nilpotent normal subgroups of a group is again nilpotent (and normal): see [41; III.4.1]. The Frattini subgroup of G will be denoted by $\underline{\Phi(G)}$. The nilpotent residual of G, that is, the smallest normal subgroup N of G such that G/N is nilpotent, will be denoted by $\underline{G^{\mathcal{N}}}$ - it is well-known that this is indeed a well-posed definition.

(f) We assume the well-known fact that if G is a finite soluble group, then $F(G) \gneq 1$, and in fact $C_G(F(G)) \leq F(G)$.

(g) We assume knowledge of the Sylow theorems (see [41; I.7]), and of P. Hall's fundamental theorems on the existence and conjugacy of Hall subgroups in finite soluble groups (see [41; VI.1.8]).

(h) If G is a finite group and S is a set of elements of G, then $\langle S \rangle$ will denote the subgroup of G generated by S; that is, the smallest subgroup G containing S. We will use $\langle g \rangle$ to denote the cyclic group generated by g (of some prescribed order). If $H, K \leq G$, then $[H, K] = \langle h^{-1} h^{k} : h \in H, k \in K \rangle \leq G$, while if A is a group of operators on G, then $[G, A]$ will denote the subgroup $\langle g^{-1} g^{\alpha} : g \in G, \alpha \in A \rangle$ of G.

(i) If G is a group and we write $G = N] H$, it will be understood that $N \trianglelefteq G$, $H \leq G$ with $N \cap H = 1$, and $G = NH$; that is, G is the semi-direct product of N by H. If N is a group admitting the group H of operators, we recall the procedure for constructing the abstract semi-direct product of N by H, also to be denoted by $N] H$, as described in [28; section 2.5]. We often use the "] " notation only at the first occurrence of a particular group, or for emphasis.

(j) A <u>characteristic conjugacy class</u>, $\mathcal{C}$, of a group G is a conjugacy class of subgroups of G such that if $\alpha \in \mathrm{Aut}(G)$, then we have $\mathcal{C}^{\alpha} = \mathcal{C}$.

The following well-known and elementary arithmetical result will be fundamental for our purposes, and so we include a proof.

<u>1.1.2 Lemma</u>. Let G be a finite group, and π be a set of primes.

(a) If $N \trianglelefteq G$ and $H \in \mathrm{Hall}_{\pi}(G)$, then $H \cap N \in \mathrm{Hall}_{\pi}(N)$ and $HN/N \in \mathrm{Hall}_{\pi}(G/N)$.

(b) Suppose that $G = N_1 N_2$, where $N_1, N_2 \trianglelefteq G$. Suppose that $H_i \in \mathrm{Hall}_{\pi}(N_i)$, for $i = 1, 2$, and that $H_1 H_2 \subseteq H \leq G$, where H is a π-group. Then $H_1 H_2 = H \in \mathrm{Hall}_{\pi}(G)$.

<u>Proof</u>. (a) Since $N \trianglelefteq G$, then $HN \leq G$, and so $|HN|_\pi \mid |G|_\pi$. But $|HN|_\pi = |H|_\pi |N|_\pi / |H \cap N|_\pi$, whence $|N|_\pi = |H \cap N|_\pi$ and $H \cap N \in \mathrm{Hall}_\pi(N)$. Further, $|G/N|_\pi = |H|_\pi / |H \cap N|_\pi = |H/H \cap N|_\pi = |HN/N|_\pi$, and so $HN/N \in \mathrm{Hall}_\pi(G/N)$.

(b) We have $|G|_\pi = |N_1|_\pi |N_2|_\pi / |N_1 \cap N_2|_\pi$. Now, $|H_1| = |N_1|_\pi$, $|H_2| = |N_2|_\pi$ and $|H_1H_2| = |N_1|_\pi |N_2|_\pi / |H_1 \cap H_2| \leq |G|_\pi$. But $H_1 \cap H_2$ is a π-subgroup of $N_1 \cap N_2$, and so $|H_1 \cap H_2| \mid |N_1 \cap N_2|_\pi$. It follows that $|H_1 \cap H_2| = |N_1 \cap N_2|_\pi$ and that $|H_1H_2| = |G|_\pi$, giving the result.

We record the following for reference.

<u>1.1.3 Proposition</u>. (a) ([28; 2.2.1 iii]) Suppose that G is a group and that H, K are subgroups of G. Then $[H, K] \trianglelefteq \langle H, K \rangle$.
(b) ([28; 5.3.2 and 5.3.6]) Suppose that A is a p'-group of automorphisms of the p-group P. Then we have

(i) if P is abelian, then $P = [P, A] \times C_P(A)$; and

(ii) $[P, A, A] = [P, A]$.

<u>1.1.4 Definition</u>. Let G and H be groups, and let $G \times H$ be their external direct product; that is, the group $\{(g,h) : g \in G, h \in H\}$, with "component-wise" multiplication.

Then $G \times H$ has subgroups

$$\{(g,1) : g \in G\} \quad \text{and} \quad \{(1,h) : h \in H\} ,$$

which we customarily denote by $G \times 1$ and $1 \times H$, respectively.

The subgroup K of $G \times H$ will be called <u>subdirect</u> in $G \times H$ if $K(G \times 1) = K(1 \times H) = G \times H$.

All groups considered in Chapters 1, 2 and 3 will be finite; those in Chapters 2 and 3 will also be soluble. In Chapter 4, we shall consider certain infinite groups (the so-called "Lausch group" and associated groups), but these will be used as tools , and not as an end of study in themselves. The finite groups considered in Chapters 1 and 4 will not necessarily be soluble.

1.2 Closure operations and classes of groups.

A (group-theoretical) class $\mathfrak{X}$ is a class of groups containing all groups isomorphic to any of its members, and also containing all groups of order 1.

1.2.1 Notation. (a) We will fix the following notation for some of the more commonly-encountered classes of groups. Let π be a set of primes, and define

(1)	:	the class of all groups of order 1;
$\mathcal{E}$	:	the class of all finite groups;
$\mathcal{S}$	:	the class of all finite soluble groups;
$\mathcal{N}$	:	the class of all finite nilpotent groups;
$\mathcal{A}$	:	the class of all finite abelian groups;
$\mathcal{E}_\pi$	:	the class of all finite π-groups;
$\mathcal{S}_\pi$	:	the class $\mathcal{S} \cap \mathcal{E}_\pi$; and
$\mathcal{N}_\pi$	:	the class $\mathcal{N} \cap \mathcal{E}_\pi$.

(b) The following classes will be introduced at the references given:

$\mathcal{C}_\pi$	:	(1.3.16) ;
$e_\pi(\mathfrak{F})$	:	(1.3.16) ; and
$\mathcal{H}$	:	(1.4.5) .

(c) If $\mathfrak{X}$ is a class of groups then an $\mathfrak{X}$-group will simply be a group belonging to $\mathfrak{X}$. Parentheses, ( ), will be used to denote classes, as distinct from braces, { }, for sets. If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, then $\mathfrak{X}\setminus\mathfrak{Y}$ will denote the class $(G \in \mathfrak{X} : G \notin \mathfrak{Y})$.

(d) We will use P. Hall's concept of a <u>closure operation</u> on classes of groups, as introduced in [31]. A closure operation, C, maps classes of groups to classes of groups and satisfies

$$\mathfrak{X} \subseteq C\mathfrak{X} = C(C\mathfrak{X}) \subseteq C\mathfrak{Y} ,$$

whenever $\mathfrak{X} \subseteq \mathfrak{Y}$ are classes of groups. (We will use $\subseteq$ to denote inclusion for classes, and $\subsetneqq$ for strict inclusion).

If $\mathfrak{X}$ is a class of groups and C is a closure operation, then $\mathfrak{X}$ is said to be <u>C-closed</u> if $\mathfrak{X} = C\mathfrak{X}$. If B and C are closure operations, we define an operation BC by $BC\mathfrak{X} = B(C\mathfrak{X})$; in general, BC will not be a closure operation. We also define the operation $\langle B , C\rangle$ by stipulating that $\langle B , C\rangle\mathfrak{X}$ be the smallest class containing $\mathfrak{X}$ which is both B-closed and C-closed (see (1.2.4) below). It is well-known that $\langle B , C\rangle$ is a closure operation, and that

$$\langle B , C\rangle\mathfrak{X} = (G \in \mathcal{E} : G \in (BC)^n\mathfrak{X} \text{ for some ordinal } n).$$

(e) We will use the following standard notation for closure operations. Let $\mathfrak{X}$ be a class of groups, and define

$S\mathfrak{X} = (G \in \mathcal{E} : \exists\, B \in \mathfrak{X}$ with $G \cong A \leqslant B)$;

$S_n\mathfrak{X} = (G \in \mathcal{E} : \exists\, B \in \mathfrak{X}$ with $G \cong A$ sn $B)$;

$S_F\mathfrak{X} = (G \in \mathcal{E} : \exists\, B \in \mathfrak{X}$ with $G \cong A \leqslant B$ such that $A^{\mathcal{N}}$ sn $B)$;

$Q\mathfrak{X} = (G \in \mathcal{E} : \exists\, B \in \mathfrak{X}$ with $G \cong B/A$ for some $A \trianglelefteq B)$;

$N_o\mathfrak{X} = (G \in \mathcal{E} : \exists\, N_1,\ldots,N_r$ sn G with each $N_i \in \mathfrak{X}$ and $G = \langle N_1,\ldots,N_r\rangle)$;

$D_o\mathfrak{X} = (G \in \mathcal{E} : \exists\, N_1,\ldots,N_r \trianglelefteq G$ with each $N_i \in \mathfrak{X}$ and $G = N_1\times\cdots\times N_r)$;

$R_o\mathfrak{X} = (G \in \mathcal{E} : \exists\, N_1,\ldots,N_r \trianglelefteq G$ with each $G/N_i \in \mathfrak{X}$ and $\bigcap_i N_i = 1)$;

and

$$E_{\phi}\mathfrak{X} = (G \in \mathcal{E} : \exists\ K \trianglelefteq G \text{ with } K < \Phi(G) \text{ such that } G/K \in \mathfrak{X}) .$$

It is well-known that S , S_n , S_F , Q , N_o , D_o , R_o and E_{ϕ} are all closure operations: for proofs in the case of R_o and E_{ϕ} , see [15; lemma 1.1] ; for a proof that S_F is a closure operation, see [39; lemma 1] .

We note that an S-closed class is S_F-closed, and an S_F-closed class is S_n-closed, while an N_o-closed class is D_o-closed. It is well-known, and easy to check, that an $\langle S , D_o \rangle$-closed class is R_o-closed.

We have the following familiar definitions of the classes which will form the objects of our study.

<u>1.2.3 Definition</u>. Let $\mathfrak{X}$ be a class of finite groups.

(a) If $\mathfrak{X} = \langle S_n , N_o \rangle \mathfrak{X}$, then $\mathfrak{X}$ is said to be a <u>Fitting class</u>.

(b) If $\mathfrak{X} = \langle S_F , N_o \rangle \mathfrak{X}$, then $\mathfrak{X}$ is said to be a <u>Fischer class</u>.

(c) If $\mathfrak{X} = \langle Q , R_o \rangle \mathfrak{X}$, then $\mathfrak{X}$ is said to be a <u>formation</u>.

(d) If $\mathfrak{X} = \langle Q , R_o , E_{\phi} \rangle \mathfrak{X}$, then $\mathfrak{X}$ is said to be a <u>saturated formation</u>.

We note that a Fischer class is a Fitting class, and that an S-closed Fitting class is a Fischer class.

<u>1.2.4 Observation</u>. Suppose that $\{\mathfrak{X}_{\alpha}\}_{\alpha \in A}$ is a collection of C-closed classes, where C is a closure operation. Then $\bigcap_{\alpha} \mathfrak{X}_{\alpha}$ is C-closed.

1.2.5 Remarks. The concepts of a formation and of a saturated formation are due to Gaschütz, [25]. The concepts of a Fitting class and of a Fischer class are due to Fischer (see [22]), and were introduced as "dualizations" of the concept of a formation. Originally, these concepts were introduced within the universe $\mathcal{S}$ of finite soluble groups; indeed, Gaschütz' introduction of the theory of saturated formations unified the theories of the Hall and Carter subgroups in (finite) soluble groups, while Fitting classes gave rise to further such characteristic conjugacy classes of subgroups, the so-called "injectors". (An account of Gaschütz' theory of formations can be found in section VI.7 of [41]). The direction of research in recent years has, however, tended to concentrate on an investigation of the classes themselves, rather than on facts which can be deduced about the structure of finite soluble groups from a knowledge of the classes. This trend is emphasised by the fact that research is now being undertaken into classes within the universe $\mathcal{E}$ of all finite groups: see, for example, [43] and [4] . Within $\mathcal{E}$ we can lose the characteristic conjugacy classes of subgroups which first led to the study of formations: for example, Alt(5) possesses no Hall $\{3,5\}$-subgroup.

In this thesis, we shall be mainly interested in Fitting classes of finite soluble groups; indeed, in chapters 2 and 3 we shall be dealing exclusively with classes contained in the universe $\mathcal{S}$. In chapter 4, we will give a new proof of a recent theorem of Berger, and will investigate some consequences of this theorem. The theorem itself will be proved within the universe $\mathcal{E}$, although most of the consequences will be for classes within $\mathcal{S}$.

1.3 First properties of Fitting classes and formations.

If π is a set of primes, it is well-known that the classes $\mathcal{E}$, $\mathcal{E}_\pi$, $\mathcal{S}$, $\mathcal{S}_\pi$, $\mathcal{N}$ and $\mathcal{N}_\pi$ are all both Fitting classes and (saturated) formations. That $\mathcal{N}$ is a Fitting class is a consequence of the theorem of H. Fitting, [24], referred to previously (see [41; III.4.1]). Fitting classes receive their name because of this result.

We recall some familiar definitions.

1.3.1 Definition. Suppose that $\mathfrak{X}$ is a class of finite groups, that $\mathfrak{F}$ is a Fitting class and that $\mathfrak{G}$ is a formation. Let G be a finite group.

(a) If $T \leq G$ and $T \in \mathfrak{X}$, we say that T is an <u>$\mathfrak{X}$-subgroup of G</u>.

(b) If $T \leq G$ is maximal subject to being an $\mathfrak{X}$-subgroup of G, we say that T is <u>$\mathfrak{X}$-maximal in G</u>.

(c) The subgroup T of G is said to be an <u>$\mathfrak{X}$-injector of G</u> if $T \cap N$ is $\mathfrak{X}$-maximal in N for all N sn G.

(d) The <u>$\mathfrak{F}$-radical</u> of G, denoted by $G_{\mathfrak{F}}$, is the join of all normal $\mathfrak{F}$-subgroups of G. Since $\mathfrak{F}$ is N_0-closed, then $G_{\mathfrak{F}}$ is in fact the (unique) largest normal $\mathfrak{F}$-subgroup of G.

(e) The $\mathfrak{G}$-residual of G, denoted by $G^{\mathfrak{G}}$, is the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{G}$. Since $\mathfrak{G}$ is R_0-closed, $G^{\mathfrak{G}}$ is in fact the (unique) smallest normal subgroup M of G such that $G/M \in \mathfrak{G}$.

<u>1.3.2 Remarks</u>. Let $\mathfrak{X}$, $\mathfrak{F}$, $\mathfrak{G}$ and G be as in (1.3.1). The following are well-known, and are not hard to prove.

(a) If T is an $\mathfrak{X}$-injector of G and $\alpha \in \mathrm{Aut}(G)$, then T^{α} is also an $\mathfrak{X}$-injector of G. Further, if N sn G, then $T \cap N$ is an $\mathfrak{X}$-injector of N.

(b) The subgroups $G_{\mathfrak{F}}$ and $G^{\mathfrak{G}}$ are characteristic in G.

(c) Using (b), an inductive argument shows that if N sn G, then $N \leq G_{\mathfrak{F}}$, and so $G_{\mathfrak{F}}$ is the join of all <u>subnormal</u> $\mathfrak{F}$-subgroups of G.

(d) We note that $G_{\mathfrak{N}} = F(G)$. If π is a set of primes, we customarily denote $G_{\mathfrak{S}_\pi}$ and $G^{\mathfrak{S}_\pi}$ by $O_\pi(G)$ and $O^\pi(G)$, respectively.

<u>1.3.3 Definition</u>. Let $\mathfrak{F}$ be a Fitting class. The <u>characteristic</u> of $\mathfrak{F}$, denoted by <u>char($\mathfrak{F}$)</u>, is defined as follows.

$$\mathrm{char}(\mathfrak{F}) = \{p \in \mathbb{P} : \exists\, G \in \mathfrak{F} \text{ such that } G \text{ has a composition factor of order } p\}$$

$$= \{p \in \mathbb{P} : \exists\, G \in \mathfrak{F} \text{ with } p \in \mathrm{comp}(G)\}.$$

<u>1.3.4 Proposition (Hartley, [33; page 204])</u>. Let $\mathfrak{F}$ be a Fitting class with $\mathrm{char}(\mathfrak{F}) = \pi \subseteq \mathbb{P}$. Then $\mathfrak{N}_\pi \subseteq \mathfrak{F}$. If, further, $\mathfrak{F} \subseteq \mathfrak{S}$, then also $\mathfrak{F} \subseteq \mathfrak{S}_\pi$.

Since $\mathfrak{S}$ is a Fitting class, it is reasonable to consider Fitting class theory within the universe $\mathfrak{S}$ if we so desire. The following theorems of Fischer, Gaschütz and Hartley are fundamental to the theory of Fitting classes within the universe $\mathfrak{S}$; we refer to [23] for the proofs.

<u>1.3.2 Remarks.</u> Let $\mathfrak{X}$, $\mathfrak{F}$, $\mathfrak{G}$ and G be as in (1.3.1). The following are well-known, and are not hard to prove.

(a) If T is an $\mathfrak{X}$-injector of G and $\alpha \in \mathrm{Aut}(G)$, then T^{α} is also an $\mathfrak{X}$-injector of G. Further, if N sn G, then $T \cap N$ is an $\mathfrak{X}$-injector of N.

(b) The subgroups $G_{\mathfrak{F}}$ and $G^{\mathfrak{G}}$ are characteristic in G.

(c) Using (b), an inductive argument shows that if N sn G, then $N \leq G_{\mathfrak{F}}$, and so $G_{\mathfrak{F}}$ is the join of all <u>subnormal</u> $\mathfrak{F}$-subgroups of G.

(d) We note that $G_{\mathcal{N}} = F(G)$. If π is a set of primes, we customarily denote $G_{\mathcal{S}_\pi}$ and $G^{\mathcal{S}_\pi}$ by $O_\pi(G)$ and $O^\pi(G)$, respectively.

<u>1.3.3 Definition.</u> Let $\mathfrak{F}$ be a Fitting class. The <u>characteristic</u> of $\mathfrak{F}$, denoted by <u>char($\mathfrak{F}$)</u>, is defined as follows.

$$\mathrm{char}(\mathfrak{F}) = \{p \in \mathbb{P} : \exists\, G \in \mathfrak{F} \text{ such that } G \text{ has a composition factor of order } p\}$$

$$= \{p \in \mathbb{P} : \exists\, G \in \mathfrak{F} \text{ with } p \in \mathrm{comp}(G)\}.$$

<u>1.3.4 Proposition (Hartley, [33; page 204]).</u> Let $\mathfrak{F}$ be a Fitting class with $\mathrm{char}(\mathfrak{F}) = \pi \subseteq \mathbb{P}$. Then $\mathcal{N}_\pi \subseteq \mathfrak{F}$. If, further, $\mathfrak{F} \subseteq \mathcal{S}$, then also $\mathfrak{F} \subseteq \mathcal{S}_\pi$.

Since $\mathcal{S}$ is a Fitting class, it is reasonable to consider Fitting class theory within the universe $\mathcal{S}$ if we so desire. The following theorems of Fischer, Gaschütz and Hartley are fundamental to the theory of Fitting classes within the universe $\mathcal{S}$; we refer to [23] for the proofs.

1.3.5 Theorem (Fischer, Gaschütz and Hartley [23: Satz 17).

(a) Suppose that $\mathfrak{F}$ is a Fitting class of soluble groups and that $G \in \mathcal{S}$. Then G possesses $\mathfrak{F}$-injectors and all such are conjugate in G.

(b) Suppose that $\mathfrak{F}$ is a class of finite soluble groups and suppose that if $G \in$ then G possesses $\mathfrak{F}$-injectors. Then $\mathfrak{F}$ is a Fitting class.

We remark that Fitting classes are contained in $\mathcal{E}$ by definition.

1.3.6 Theorem (Fischer, Gaschütz and Hartley [23: Satz 2 & Korollar]).

Suppose that $\mathfrak{F}$ is a Fitting class of soluble groups, and that $G \in \mathcal{S}$.

(a) If $N \trianglelefteq G$ with $G/N \in \mathcal{N}$, and if T is $\mathfrak{F}$-maximal in G such that $T \cap N$ is an $\mathfrak{F}$-injector of N, then T is an $\mathfrak{F}$-injector of G.

(b) If T is an $\mathfrak{F}$-injector of G and $T \leq H < G$, then T is an $\mathfrak{F}$-injector of H.

1.3.7 Theorem (Fischer, see [26: theorem 8.7]).

Suppose that $\mathfrak{F}$ is a Fitting class of soluble groups, and that $G \in \mathcal{S}$. Suppose that T is an $\mathfrak{F}$-injector of G, and that M/N is a chief factor of G. Then T either covers or avoids M/N; that is, either $M \leq TN$ or else $M \cap T \leq N$, respectively.

Next come two well-known technical results which we will tend to use without specific reference.

1.3.8 Lemma. Let $\mathfrak{F}$ be a Fitting class. Suppose that $G \in \mathcal{E}$ and that T is an $\mathfrak{F}$-injector of G with T sn G. Then T char G.

Proof. Suppose firstly that $T \trianglelefteq G$, and let $\alpha \in \mathrm{Aut}(G)$. Then T

and T^{α} are both normal $\mathfrak{F}$-injectors of G, by (1.3.2a). Thus, $T \leq TT^{\alpha} \in N_0\mathfrak{F} = \mathfrak{F}$, and so by $\mathfrak{F}$-maximality of T, we have $T = T^{\alpha}$. It now follows (by an easy induction) that if T sn G, then T char G.

<u>1.3.9 Lemma</u>. Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes, and that $\mathfrak{X}$ and $\mathfrak{Y}$ are formations.

(a) If G is a group of minimal order subject to belonging to $\mathfrak{G} \setminus \mathfrak{F}$, then G has a unique maximal normal subgroup, M say, and $M = G_{\mathfrak{F}}$.

(b) If G is a group of minimal order subject to belonging to $\mathfrak{Y} \setminus \mathfrak{X}$, then G has a unique minimal normal subgroup, N say, and $N = G^{\mathfrak{X}}$.

Part (a) of the above lemma will often be used in conjunction with the following.

<u>1.3.10 Lemma</u>. Suppose that G is a finite group and that G has a unique maximal normal subgroup, M say. Then either G/M is a non-abelian simple group or else $|G : M| = p$, G/G' is a cyclic p-group and $G' = G^{\mathcal{N}} = O^p(G)$.

<u>Proof</u>. Suppose that G/M is abelian; then certainly $G/M \cong C_p$ for some $p \in \mathbb{P}$. Since any normal subgroup of $G/G^{\mathcal{N}}$ is the image of a normal subgroup of G, then $\bar{G} = G/G^{\mathcal{N}}$ must be a p-group and $G/G' \cong \bar{G}/(\bar{G})'$ must be a cyclic p-group. But since $\Phi(\bar{G}) \geq \bar{G}'$, then $\bar{G}/\Phi(\bar{G})$ is cyclic, and so $\bar{G}$ must be cyclic, by [28; 5.1.2]. Thus $G' \leq G^{\mathcal{N}}$, and the result follows.

We now come to the concepts of products for Fitting classes and for formations.

<u>1.3.11 Definition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, $\mathfrak{X}$ and $\mathfrak{Y}$ be formations, and $\mathfrak{U}$ and $\mathfrak{V}$ be classes of finite groups. Define

(a) $\mathfrak{F} * \mathfrak{G} = (G \in \mathcal{E} : G/G_{\mathfrak{F}} \in \mathfrak{G})$;

(b) $\mathfrak{X} \circ \mathfrak{Y} = (G \in \mathcal{E} : G^{\mathfrak{Y}} \in \mathfrak{X})$; and

(c) $\mathfrak{U}\mathfrak{V} = (G \in \mathcal{E} : \exists N \trianglelefteq G$ with $N \in \mathfrak{U}$ and $G/N \in \mathfrak{V})$;

<u>1.3.12 Proposition</u>. Let $\mathfrak{D}$, $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, and let $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ be formations. Then

(a) $\mathfrak{F} * \mathfrak{G}$ is a Fitting class, contained in $\mathcal{S}$ if $\mathfrak{F}, \mathfrak{G} \subseteq \mathcal{S}$;

(b) $(\mathfrak{D} * \mathfrak{F}) * \mathfrak{G} = \mathfrak{D} * (\mathfrak{F} * \mathfrak{G})$;

(c) $G_{\mathfrak{F}*\mathfrak{G}}/G_{\mathfrak{F}} = (G/G_{\mathfrak{F}})_{\mathfrak{G}}$ for all $G \in \mathcal{E}$;

(d) $\mathfrak{X} \circ \mathfrak{Y}$ is a formation, contained in $\mathcal{S}$ if $\mathfrak{X}, \mathfrak{Y} \subseteq \mathcal{S}$;

(e) $(\mathfrak{X} \circ \mathfrak{Y}) \circ \mathfrak{Z} = \mathfrak{X} \circ (\mathfrak{Y} \circ \mathfrak{Z})$;

(f) $G^{\mathfrak{X} \circ \mathfrak{Y}} = (G^{\mathfrak{Y}})^{\mathfrak{X}}$ for all $G \in \mathcal{E}$; and

(g) $\mathfrak{F} * \mathfrak{G} = \mathfrak{F} \circ \mathfrak{G}$ if $\mathfrak{F}$ and $\mathfrak{G}$ are also formations.

<u>Proof</u>. These facts are consequences of relatively easy calculations. For (a), (b) and (c), see [43; 1.2.5]; for (d), (e) and (f) see [23; 7.15]; for (g) see [23; page 57] .

<u>1.3.13 Proposition (Cossey, [18; 3.1]; Beidleman [1; 2.4])</u>.

Let $\mathfrak{D}$, $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes. Then

(a) $\mathfrak{F} \subseteq \mathfrak{F} * \mathfrak{G}$, but we need not have $\mathfrak{G} \subseteq \mathfrak{F} * \mathfrak{G}$;

(b) if $\mathfrak{D} \subseteq \mathfrak{G}$, then $\mathfrak{F} * \mathfrak{D} \subseteq \mathfrak{F} * \mathfrak{G}$; and

(c) if $\mathfrak{D} \subseteq \mathfrak{G}$, and if also $\mathfrak{F}$ is Q-closed, then $\mathfrak{D} * \mathfrak{F} \subseteq \mathfrak{G} * \mathfrak{F}$.

We will see later (1.7.6) that if in (1.3.13 c) we have that $\mathfrak{F}$ is not Q-closed, then it can happen that $\mathfrak{D} * \mathfrak{F} \nsubseteq \mathfrak{G} * \mathfrak{F}$.

<u>1.3.14 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, and let $\mathfrak{X}$ and $\mathfrak{Y}$ be formations.

(a) [47; 2.6.6]. If $\mathfrak{F}$ and $\mathfrak{G}$ are Fischer classes, then so also is $\mathfrak{F} * \mathfrak{G}$.

(b) If $\mathfrak{F}$ and $\mathfrak{G}$ are S-closed and $\mathfrak{G}$ is also Q-closed, then $\mathfrak{F} * \mathfrak{G}$ is S-closed.

(c) If $\mathfrak{X}$ and $\mathfrak{Y}$ are S-closed, then so also is $\mathfrak{X} \circ \mathfrak{Y}$.

Parts (b) and (c) of the above proposition are well-known, and are consequences of straightforward calculations. It is not known whether the hypothesis that $\mathfrak{G}$ be Q-closed in (b) is necessary; indeed, no S-closed Fitting class which is not already Q-closed is known.

As a consequence of (1.2.4), we note that if $\{\mathfrak{F}_\alpha\}_{\alpha \in A}$ is a family of Fitting classes, then $\bigcap_\alpha \mathfrak{F}_\alpha$ is also a Fitting class. This motivates the following familiar definition.

<u>1.3.15 Definition</u>. Let $\mathfrak{X}$ be a class of finite groups. Then the <u>Fitting class generated by $\mathfrak{X}$</u>, denoted by $\langle \mathfrak{X} \rangle_{\text{Fitt}}$, is defined to be the smallest Fitting class containing $\mathfrak{X}$; we have

$$\langle \mathfrak{X} \rangle_{\text{Fitt}} = \bigcap \{ \mathfrak{F} : \mathfrak{F} \text{ is a Fitting class and } \mathfrak{F} \supseteq \mathfrak{X} \}$$
$$= (G \in \mathfrak{E} : G \in (S_n N_o)^m \mathfrak{X} \ : m \in \mathbb{N}) .$$

We note that a Fitting class may be regarded as a set of isomorphism classes of finite groups; the family of all Fitting classes may thus be indexed by a set, justifying the above.

In general, it is an extremely difficult problem to determine the Fitting class generated by a given class. For example, the Fitting

class generated by (the isomorphism class of) the group Sym(3) has not been determined in the sense that there is no general procedure for telling whether a given group belongs to $\langle \mathrm{Sym}(3)\rangle_{\mathrm{Fitt}}$. It was proved by Camina, [13], however, that $\langle \mathrm{Sym}(3)\rangle_{\mathrm{Fitt}}$ is a proper sub-class of $\mathcal{S}_3 * \mathcal{S}_2$. As Hartley points out on page 204 of [33], if p is a prime, then $\langle C_p \rangle_{\mathrm{Fitt}} = \mathcal{S}_p$. It is well-known that if G is a finite simple group, then $D_o(G)$ is $\langle S_n, N_o\rangle$-closed, and so $D_o(G) = \langle G\rangle_{\mathrm{Fitt}}$.

The following definitions appear to have been due to Gaschütz. We recall the definition of π-soc(G) from (1.1.1c).

<u>1.3.16 Definition (Gaschütz)</u>. Let π be a set of primes, and $\mathfrak{F}$ be a Fitting class of soluble groups. Define

(a) $\mathcal{C}_\pi = (G \in \mathcal{S} : \pi\text{-soc}(G) \leq Z(G))$; and

(b) $e_\pi(\mathfrak{F}) = (G \in \mathcal{S} :$ the π-G-chief factors of $G_{\mathfrak{F}}$ are G-central) ;

where if $\pi = \emptyset$, we understand that both $\mathcal{C}_\pi$ and $e_\pi(\mathfrak{F})$ coincide with $\mathcal{S}$; they are both contained in $\mathcal{S}$ for arbitrary π.

<u>1.3.17 Proposition</u>. Let π be a set of primes, and $\mathfrak{F}$ be a Fitting class of soluble groups.

(a) The class $\mathcal{C}_\pi$ is an R_o-closed Fitting class which is neither S_F- nor Q- closed provided that $\pi \neq \emptyset$.

(b) The class $e_\pi(\mathfrak{F})$ is a Fischer class.

(c) (Lockett) For each $G \in \mathcal{S}$, we have $G_{\mathcal{C}_\pi} = C_G(\pi\text{-soc}(G))$.

<u>Proof</u>. (a) That $\mathcal{C}_\pi$ is a Fitting class is proved in [26; 6.9d],

and also [47; 2.3.1], noting that $\mathfrak{b}_\pi = \bigcap_{p \in \pi} \mathfrak{b}_p$. That $\mathfrak{b}_\pi$ is R_0-closed was pointed out by Bryce and Cossey on page 267 of [9]. That $\mathfrak{b}_\pi$ is neither S_F- nor Q-closed (if $\pi \neq \emptyset$) can be proved by examples similar to those of [47; 2.3.2] and [26; 6.9d], respectively.

(b) This is proved in [47; 2.2.1] (and a following remark).

(c) This is [47; 2.3.3].

1.4 Normal Fitting classes.

The concept of a normal Fitting class was introduced (within the universe $\mathcal{S}$) by Blessenohl and Gaschütz [6]; we outline some of their theory.

1.4.1 Definition (Blessenohl and Gaschütz [6; 1.1]). The Fitting class $\mathfrak{F}$ of finite soluble groups is said to be normal if $\mathfrak{F} \neq (1)$ and if for each $G \in \mathcal{S}$, $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of G (and so the unique $\mathfrak{F}$-injector of G, by conjugacy of injectors).

An important result of [6] is the following.

1.4.2 Theorem (Blessenohl and Gaschütz, [6; Satz 5.3]).

Let $\mathfrak{F}$ be a non-trivial (i.e., not equal to (1)) Fitting class of soluble groups. Then $\mathfrak{F}$ is normal if, and only if, for each $G \in \mathcal{S}$ we have $G_{\mathfrak{F}} \geq G'$.

1.4.3 Theorem (Cossey, see [6; Satz 5.1]).

If $\mathfrak{F}$ is a normal Fitting class, then $\mathfrak{F} \geq \mathcal{N}$.

1.4.4 Theorem (Blessenohl and Gaschütz [6; Satz 5.1]).

Let $\underline{\underline{F}}$ be a family of normal Fitting classes. Then $\mathfrak{G} = \bigcap_{\mathfrak{F} \in \underline{\underline{F}}} \mathfrak{F}$ is also a normal Fitting class (and, in particular, is non-trivial).

1.4.5 Remarks. (a) The non-triviality of $\mathfrak{G}$ in (1.4.4) is a consequence of (1.4.3); indeed, it follows that there exists a unique smallest normal Fitting class, customarily denoted by $\mathfrak{H}$, or $\mathfrak{S}_*$ (see (1.5.8d)).. Also, the Fitting class $\mathfrak{F}$ is normal if and only if $\mathfrak{F} \supseteq \mathfrak{H}$.
(b) The concept of a normal Fitting pair, due to Blessenohl and Gaschutz [6], and Lausch's contribution to this theory, will be introduced in Chapter 4.

1.4.6 Theorem (Blessenohl and Gaschütz [6; Satz 3.2]).

Let $\mathfrak{F}$ denote the class

$\mathfrak{F} = (G \in \mathfrak{S}$: G induces (by conjugation) a group of even permutations on the elements of $O_{2'}(G))$.

Then $\mathfrak{F}$ is a normal Fitting class; indeed $|G : G_{\mathfrak{F}}| = 1$ or 2 for all $G \in \mathfrak{S}$.

This result leads to the observation that radicals for Fitting classes need not "respect" direct products. Thus, taking $G = \mathrm{Sym}(3)$, it is clear that $G_{\mathfrak{F}} = \mathrm{Alt}(3) \leq \mathrm{Sym}(3)$. On the other hand, $|G \times G : (G \times G)_{\mathfrak{F}}| \leq 2$, and so $(G \times G)_{\mathfrak{F}} \gneqq G_{\mathfrak{F}} \times G_{\mathfrak{F}}$.

1.4.6 Proposition (Cossey [18: 3.2]). Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes of soluble groups. Then $\mathfrak{F} * \mathfrak{G}$ is normal if either $\mathfrak{F}$ or $\mathfrak{G}$ is normal.

1.4.8 Corollary (Cossey [18; 3.6]). Suppose that $G \in \mathcal{S}$ and that $G/G_{\mathfrak{F}} \notin \mathcal{H}$ for some Fitting class $\mathfrak{F} \subseteq \mathcal{S}$, where $\mathcal{H}$ is as in (1.4.5a). Then $G \notin \mathcal{H}$.

1.5 Lockett's "Upper-star" operation.

In his paper [49], Lockett investigates the phenomenon concerning radicals and direct products mentioned in the previous section. We describe some of Lockett's results here. In this section, we shall be working within the universe $\mathcal{E}$. Lockett proves his results within the universe $\mathcal{S}$, but most of his proofs are entirely formal and carry over to $\mathcal{E}$. We will indicate the occasions on which solubility is needed.

1.5.1 Definition (Lockett [49; page 133]). Let $\mathfrak{F}$ be a Fitting class. Define the class $\mathfrak{F}^*$ by

$$\mathfrak{F}^* = (G \in \mathcal{E} : (G \times G)_{\mathfrak{F}} \text{ is subdirect in } G \times G),$$

where "subdirect" is as in (1.1.4).

1.5.2 Lemma (Lockett, [49; 2.1b]). Let $\mathfrak{F}$ be a Fitting class and let G be a finite group. Then

$$G \in \mathfrak{F}^* \text{ if and only if } (G \times G)_{\mathfrak{F}} = G_{\mathfrak{F}} \times G_{\mathfrak{F}} \langle (g, g^{-1}) : g \in G \rangle$$

$$\text{if and only if } (G \times G) = (G \times G)_{\mathfrak{F}} (1 \times G).$$

1.5.3 Theorem (Lockett, [49; references below]). Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, and let G and H be finite groups. Then

(a) $\mathfrak{F}^*$ is a Fitting class ;

(b) $\mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}^*)^* \subseteq \mathfrak{F}\mathcal{A}$;

(c) if A is a group of operators on G, then $[G_{\mathfrak{F}^*}, A] \leq G_{\mathfrak{F}}$;

(d) $(G \times G)_{\mathfrak{F}} = G_{\mathfrak{F}} \times G_{\mathfrak{F}} < (g, g^{-1}) : g \in G_{\mathfrak{F}^*} >$;

(e) if $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{F}^* \subseteq \mathfrak{G}^*$;

(f) if $\{\mathfrak{F}_\alpha\}_{\alpha \in A}$ is a family of Fitting classes, then

$$\left(\bigcap_{\alpha \in A} \mathfrak{F}_\alpha \right)^* = \bigcap_{\alpha \in A} \mathfrak{F}_\alpha^* \quad ;$$

(g) $\mathfrak{F}$ is normal if and only if $\mathfrak{F}^* = \mathfrak{S}$;

(h) if $\mathfrak{F}$ is either Q-closed, R_0-closed or S_F-closed, then $\mathfrak{F} = \mathfrak{F}^*$.

(i) $(G \times H)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times H_{\mathfrak{F}^*}$; and

(j) if $\mathfrak{F} \subseteq \mathfrak{S}$ and $G, H \in \mathfrak{S}$, then if T is an $\mathfrak{F}^*$-injector of $G \times H$, we have $T = (T \cap G) \times (T \cap H)$, the direct product of certain $\mathfrak{F}^*$-injectors of G and of H .

<u>References for the proof</u>. Parts (a), (b), (g) and (h) are the content of [49; 2.2] ; in proving that $\mathfrak{F} = \mathfrak{F}^*$ if $\mathfrak{F} = R_0\mathfrak{F}$ or $\mathfrak{F} = S_F\mathfrak{F}$ in part (h), we may make use of the fact that $\mathfrak{F}^* \subseteq \mathfrak{F}\mathfrak{A}$ from part (b) to avoid Lockett's assumption of solubility.

Parts (c) and (d) are proved in [49; 2.1] ; parts (e) and (f) are the content of [49; 2.3] , while parts (i) and (j) are the content of [49; 3.1] .

<u>1.5.4 Terminology and remarks</u>. (a) A Fitting class $\mathfrak{F}$ with $\mathfrak{F} = \mathfrak{F}^*$ is known as a <u>Lockett class</u>. Because of (1.5.3i), we see that radicals for a Lockett class "respect" direct products. In fact, this property characterizes Lockett classes among Fitting classes : see (1.5.5) below.

(b) Suppose that $\mathfrak{F}$ is a Fitting class and that π is a set of primes. By (1.5.3h), the classes $\mathcal{N}_\pi$, $\mathfrak{S}_\pi$, $\mathcal{E}_\pi$, $\mathcal{C}_\pi$ and $e_\pi(\mathfrak{F})$ are all Lockett classes, the final two because of (1.3.17) .

(c) By (1.5.3e), if $\mathfrak{F} \subseteq \mathfrak{S}$ then $\mathfrak{F}^* \subseteq \mathfrak{S}$.

<u>1.5.5 Proposition (from [21])</u>. Let $\mathfrak{F}$ be a Fitting class, and suppose that for all $G \in \mathfrak{F}\mathfrak{A}$, we have $(G \times G)_{\mathfrak{F}} = G_{\mathfrak{F}} \times G_{\mathfrak{F}}$. Then $\mathfrak{F}$ is a Lockett class. In particular, if $\mathfrak{F} \subseteq \mathfrak{S}$ and $(G \times G)_{\mathfrak{F}} = G_{\mathfrak{F}} \times G_{\mathfrak{F}}$ for all $G \in \mathfrak{S}$, then $\mathfrak{F}$ is a Lockett class.

<u>Proof</u>. Let $G \in \mathfrak{F}^*$. By (1.5.3b), $G \in \mathfrak{F}\mathfrak{A}$, and so $(G \times G)_{\mathfrak{F}} = G_{\mathfrak{F}} \times G_{\mathfrak{F}}$. Since $G \in \mathfrak{F}^*$, then $(G \times G)_{\mathfrak{F}}$ is subdirect in G, and it follows that $G = G_{\mathfrak{F}}$; the result follows.

<u>1.5.6 Lemma</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes and suppose that G is a soluble group of minimal order subject to belonging to $\mathfrak{G} \setminus \mathfrak{F}^*$. Then $G_{\mathfrak{F}} \geqslant G'$.

<u>Proof</u>. By (1.3.9a), $G_{\mathfrak{F}^*}$ is the unique maximal normal subgroup of G, and has prime index since G is soluble. By (1.5.3c), $[G_{\mathfrak{F}^*}, G] \leq G_{\mathfrak{F}}$, and so $G/G_{\mathfrak{F}}$, being "central-by-cyclic", is abelian; the result follows.

<u>1.5.7 Definition (Lockett, [49])</u>. If $\mathfrak{F}$ is a Fitting class, define

$$\mathfrak{F}_* = \cap \{ \mathfrak{X} : \mathfrak{X} \text{ is a Fitting class and } \mathfrak{X}^* = \mathfrak{F}^* \} .$$

<u>1.5.8 Proposition (Lockett, [49; page 135])</u>. Let $\mathfrak{F}$ and $\mathfrak{X}$ be Fitting classes. Then

(a) $\mathfrak{F}_*$ is a Fitting class ;

(b) $(\mathfrak{F}_*)_* = \mathfrak{F}_* = (\mathfrak{F}^*)_* \subseteq \mathfrak{F} \subseteq \mathfrak{F}^* = (\mathfrak{F}_*)^*$;

(c) $\mathfrak{F}_* \subseteq \mathfrak{X} \subseteq \mathfrak{F}^* \iff \mathfrak{X}^* = \mathfrak{F}^* \iff \mathfrak{X}_* = \mathfrak{F}_*$; and

(d) $\mathfrak{S}_* = \mathfrak{H}$, the smallest normal Fitting class (because of (1.5.3g).

<u>1.5.9 Lemma (Bryce and Cossey [10; 3.5])</u>. Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes with $\mathfrak{F} \subseteq \mathfrak{G}$. Then $\mathfrak{F}_* \subseteq \mathfrak{G}_*$.

<u>Proof</u>. The following proof, shorter than that of Bryce and Cossey, is well-known.

Since $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathfrak{F}^* \subseteq \mathfrak{G}^*$, by (1.5.3e). Now,

$$(\mathfrak{F}_* \cap \mathfrak{G}_*)^* = (\mathfrak{F}_*)^* \cap (\mathfrak{G}_*)^* = \mathfrak{F}^* \cap \mathfrak{G}^* = \mathfrak{G}^*$$

by (1.5.3f) and (1.5.8b). Thus, $\mathfrak{F}_* \subseteq \mathfrak{F}_* \cap \mathfrak{G}_*$ and so $\mathfrak{F}_* \subseteq \mathfrak{G}_*$.

The following terminology is becoming standard.

<u>1.5.10 Definition</u>. Let $\mathfrak{F}$ be a Fitting class. Define the <u>Lockett section of $\mathfrak{F}$</u>, denoted by <u>Locksec($\mathfrak{F}$)</u>, as follows

$$\mathrm{Locksec}(\mathfrak{F}) = \{\mathfrak{G} : \mathfrak{G} \text{ is a Fitting class and } \mathfrak{G}^* = \mathfrak{F}^*\}.$$

<u>1.5.11 Remarks</u>. (a) Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes. By (1.5.8), the following statements are equivalent :

(i) $\mathfrak{G} \in \mathrm{Locksec}(\mathfrak{F})$

(ii) $\mathfrak{F}_* \subseteq \mathfrak{G} \subseteq \mathfrak{F}^*$

(iii) $\mathrm{Locksec}(\mathfrak{F}) = \mathrm{Locksec}(\mathfrak{G})$.

(b) By (a), the Lockett sections form a partition of the family of all Fitting classes.

(c) Suppose that $\mathfrak{F}$ is a Fitting class, and that $G \in \mathfrak{F}^*$. Then, in $G \times G$, the subgroup $G \times 1$ is a projection of, and hence a homomorphic image of, $(G \times G)_{\mathfrak{F}}$. Thus, $\mathfrak{F}^* \subseteq Q\mathfrak{F}$. It follows that $\mathrm{char}(\mathfrak{F}^*) \subseteq \mathrm{char}(\mathfrak{F})$. Since $\mathfrak{F} \subseteq \mathfrak{F}^*$, we conclude that $\mathrm{char}(\mathfrak{F}^*) = \mathrm{char}(\mathfrak{F})$.

(d) Suppose that $\pi \subseteq \mathbb{P}$. By (1.5.3h), $\mathcal{N}_\pi = \mathcal{N}_\pi{}^*$. On the other

hand, by (c) above, $\mathrm{char}((\mathcal{N}_\pi)_*) = \pi$, and so $\mathcal{N}_\pi \subseteq (\mathcal{N}_\pi)_*$. Thus, $(\mathcal{N}_\pi)_* = \mathcal{N}_\pi = (\mathcal{N}_\pi)^*$.

<u>1.5.12 Definition (Lockett, [47; 5.1])</u>. Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes of soluble groups. We say that $\mathfrak{F}$ is <u>strongly contained in</u> $\mathfrak{G}$, written $\mathfrak{F} << \mathfrak{G}$, if for every $G \in \mathcal{S}$, an $\mathfrak{F}$-injector of G is contained in some $\mathfrak{G}$-injector of G.

<u>1.5.13 Proposition (Lockett, [49; 3.2])</u>. Let $\mathfrak{F}$ be a Fitting class of soluble groups, and let $G \in \mathcal{S}$. If T^* is an $\mathfrak{F}^*$-injector of G, then $(T^*)_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of G. In particular (by conjugacy of injectors), we have $\mathfrak{F} << \mathfrak{F}^*$.

<u>Remarks on the proof</u>. Lockett's proof of this result uses results of Mann and Alperin, and of Fischer, appearing as (4.2), and (4.8), respectively, of [48]. Bryce and Cossey have given a more direct proof in [11; 2.1].

<u>1.5.14 The Lockett conjecture</u>. On page 135 of [49], Lockett asks whether for every Fitting class $\mathfrak{F} \subseteq \mathcal{S}$ there exists a normal Fitting class $\mathfrak{X}$ such that $\mathfrak{F} = \mathfrak{F}^* \cap \mathfrak{X}$. Bryce and Cossey have shown [10; 3.6] that if $\mathfrak{F}_* = \mathfrak{F}^* \cap \mathcal{H}$, then there does exist such a class $\mathfrak{X}$.

For convenience of terminology, we shall say that the Fitting class $\mathfrak{F}$ of finite soluble groups <u>satisfies the Lockett conjecture</u> if $\mathfrak{F}_* = \mathfrak{F}^* \cap \mathcal{H}$. Thus, the property of satisfying the Lockett conjecture or not becomes an invariant of each Lockett section.

In [5], Berger and Cossey construct a Lockett class $\mathfrak{F}$ with $\mathfrak{F}_* \subsetneq \mathfrak{F} \cap \mathcal{H}$, thus answering Lockett's question negatively for $\mathfrak{F}_*$.

1.6 Primitive saturated formations.

All classes considered in this section will be assumed to lie in $\mathcal{S}$. We assume familiarity with the concept of the local definition of a saturated formation (of finite soluble groups), and with the Gaschütz-Luberseder theorem that every such saturated formation can be locally defined (see section VI.7 of [41]).

A Fitting formation will be a class which is both a Fitting class and a formation.

1.6.1 Definition (Hawkes. [37; section 4]).

Let $\underline{\underline{F}}_0$ denote the family comprising the empty set, $\emptyset$, and the class $\mathcal{S}$ (considered as a formation). If $i > 0$, define $\underline{\underline{F}}_i$ inductively by $\mathfrak{F} \in \underline{\underline{F}}_i$ if either $\mathfrak{F} \in \underline{\underline{F}}_{i-1}$ or $\mathfrak{F} = \mathfrak{F}(f)$ is a saturated formation with local definition $\{f(p)\}$, where $f(p) \in \underline{\underline{F}}_{i-1}$ for all $p \in \mathbb{P}$.

Let $\underline{\underline{F}}$ denote the family comprising all formations $\mathfrak{F}$ such that $\mathfrak{F} = \bigcup_j \mathfrak{F}_j$, where each $\mathfrak{F}_j \in \bigcup_i \underline{\underline{F}}_i$ and $\mathfrak{F}_j \subseteq \mathfrak{F}_{j+1}$ for all j.

Then $\underline{\underline{F}}$ is in fact a family of saturated formations (see [37; page 585]). If $\mathfrak{G} \in \underline{\underline{F}}$, we say that $\mathfrak{G}$ is a primitive saturated formation. If for some $n \in \mathbb{N}$ we have $\mathfrak{G} \in F_n \setminus F_{n-1}$, we say that $\mathfrak{G}$ has (finite) defect n.

1.6.2 Remarks. The family $\underline{\underline{F}}_1$ contains $\emptyset$ together with the classes $\mathcal{S}_\pi$ for $\pi \subseteq \mathbb{P}$ (including $\mathcal{S}_\emptyset = (1)$). The family $\underline{\underline{F}}_2$ contains, for example, the classes $\mathcal{N}_\pi$ for $\pi \subseteq \mathbb{P}$. The family $\underline{\underline{F}}_{n+1}$ contains the class of soluble groups of p-length at most n, where $p \in \mathbb{P}$, as well as

the classes $\mathcal{N}_\pi^r$ ($= \mathcal{N}_\pi * \cdots * \mathcal{N}_\pi$: r factors). Hawkes, [37], notes that the formation locally defined by $\{f(p) = \mathcal{N}^p : p \in \mathbb{P}\}$ lies in $\underline{\underline{F}}$ but in no $\underline{\underline{F}}_n$.

<u>1.6.3 Proposition (Hawkes, [37; page 588])</u>. If $\mathfrak{F}$ is a primitive saturated formation, then $\mathfrak{F}$ is a subgoup-closed Fitting formation.

<u>1.6.4 Proposition (Hawkes, [35; page 180])</u>. If $\mathfrak{F}$ is a Fitting formation contained in $\mathcal{N}^2$, then $\mathfrak{F}$ is both subgroup-closed and saturated.

Bryce and Cossey proved the following important converse to (1.6.3).

<u>1.6.5 Theorem (Bryce and Cossey, [8; Theorem 4])</u>. If $\mathfrak{F} \subseteq \mathcal{S}$ is a subgroup-closed Fitting formation, then $\mathfrak{F}$ is a primitive saturated formation.

As previously remarked, the only known S-closed Fitting classes are also Q-closed and, being automatically R_0-closed (by a remark on page 8), are formations; indeed, Bryce and Cossey have proved, [9] and [12] , that every S-closed Fitting class contained in $\mathcal{N}^3$ is Q-closed.

Bryce and Cossey have also proved that the Lockett conjecture is true for primitive saturated formations.

<u>1.6.6 Theorem (Bryce and Cossey, [10; 4.17])</u>. Let $\mathfrak{F}$ be a primitive saturated formation. Then $\mathfrak{F}_* = \mathfrak{F} \cap \mathfrak{H}$.

We shall have more to say about this result in later chapters.

1.7 Some constructions.

We firstly describe the formalism of the wreath product construction, which will be of considerable use to us. Our "standard references" on this topic will be section I.15 of [41], also [54] .

1.7.1 The wreath product.

Let G and H be finite groups, and let Ω be a finite set with n elements (usually identified with $\{1,\ldots,n\}$). Suppose that H possesses a representation σ as a group of permutations on Ω . We define the wreath product, denoted by $G \underset{\sigma}{wr} H$, of G by H with respect to σ as

$$G \underset{\sigma}{wr} H = \{(f,h) : f \text{ is a map from } \Omega \text{ into } G,\ h \in H\} ,$$

with multiplication

$$(f_1 , h_1)(f_2, h_2) = (g , h_1h_2) ,$$

where $(i)g = (i)f_1 \cdot (i^{(h_1\sigma)})f_2$, for $i \in \Omega$.

It is well-known that $W = G \underset{\sigma}{wr} H$ forms a group under this multiplication : see [41; I.15.1] . The set

$$G^* = \{(f,1) : f \text{ is a map from } \Omega \text{ to } G\}$$

forms a normal subgroup of W , known as the base group. In fact, G^* is isomorphic to the direct power of $n = |\Omega|$ copies of G (see [41; 1.15.2]). If we identify Ω with $\{1,\ldots,n\}$, then we may identify $(f,1) \in G^*$ with the element $((1)f,\ldots(i)f,\ldots,(n)f) \in G^n$; we shall usually make this identification without comment.

We define the ith coordinate subgroup, G_i , of G^* as

$$\begin{aligned} G_i &= \{(f,1) : (j)f = 1 \text{ if } j \neq i\} \\ &= \{(1,\ldots,\underset{i}{g},\ldots,1) : g \in G\} . \end{aligned}$$

Then $G \cong G_i$, via the isomorphism

$$\nu_i : G \to G_i \quad : \quad g \mapsto (\hat{g},1) \quad : \quad (i)\hat{g} = g \ , \ (j)\hat{g} = 1$$
$$= (1,\ldots,\underset{i}{g},\ldots,1)$$

If $L \leq G$, we define the subgroup L^* of G as

$$L^* = \{ (f,1) : (i)f \in L \text{ for all } i \in \Omega \}$$
$$= \{ (k_1,\ldots,k_n) : k_i \in L \} .$$

Then $L^* \cong L^n$, the direct power of n copies of L .

Define

$$\hat{H} = \{(e,h) : (i)e = 1 \text{ for all } i \in \Omega \ ; \ h \in H \} .$$

Then $\hat{H} \leq W$, $\hat{H} \cong H$ and $\hat{H}$ is in fact a complement to G^* in W . We shall refer to $\hat{H}$ as the <u>standard complement</u> of W , and will often identify $\hat{H}$ and H .

The subgroup $\hat{H}$ permutes the coordinate subgroups $G_i \leq G^* \trianglelefteq W$ according to the action of H on Ω (see 41; I.15.2). Thus, if $\hat{h} = (e,h) \in \hat{H}$, then

$$\hat{h} : G_i \to (G_i)^{\hat{h}} = G_{i(h\sigma)} \quad : \quad (1,\ldots,\underset{i}{g},\ldots,1) \mapsto (1,\ldots,\underset{i(h\sigma)}{g},\ldots,1)$$

If $L \leq G$, we note that L^* is $\hat{H}$-invariant .

If $|\Omega| = |H|$ and if σ is the regular permutation representation, ρ , of H , we obtain the <u>regular wreath product</u>, denoted by $G \underset{\rho}{wr} H$. If we simply write G wr H , the regular wreath product will be understood.

<u>1.7.2 Lemma</u>. Let G and H be (finite) groups, with $N \trianglelefteq G$ and $L \leq G$. Then (a) $(G \underset{\sigma}{wr} H)/N^* \cong (G/N) \underset{\sigma}{wr} H$; and

(b) $L \underset{\sigma}{wr} H \cong L^* H \leq G \underset{\sigma}{wr} H$.

The above lemma is well-known, and follows by direct calculation.

<u>1.7.3 Proposition (Cossey, [18; lemma 2.2])</u>. Let $\mathfrak{F}$ be a Lockett class and suppose that G and H are groups with $G \notin \mathfrak{F}$. Then

$$(G \text{ wr } H)_{\mathfrak{F}} = (G_{\mathfrak{F}})^* ,$$

in the above notation.

<u>Proof</u>. Let W denote G wr H. Since $\mathfrak{F}$ is a Lockett class, then $(G^*)_{\mathfrak{F}} = (G_{\mathfrak{F}})^*$. Suppose that $w \in W_{\mathfrak{F}} \setminus (G_{\mathfrak{F}})^*$. Then $w = gh$, where $g \in G^*$ and $h \in \hat{H} \setminus 1$. Now, $[G^*, \langle w \rangle] \leq G^* \cap W_{\mathfrak{F}} = (G_{\mathfrak{F}})^*$, and so $G_i^h (G_{\mathfrak{F}}) = G_i (G_{\mathfrak{F}})^*$, where G_i is the ith coordinate subgroup.

But $(G \text{ wr } H)/(G_{\mathfrak{F}})^* \cong (G/G_{\mathfrak{F}}) \text{ wr } H$, and since we have the regular wreath product and $h \neq 1$, then $G_i^h (G_{\mathfrak{F}})^* \neq G_i (G_{\mathfrak{F}})^*$ for all i. Thus, $W_{\mathfrak{F}} \leq (G_{\mathfrak{F}})^*$, and so $W_{\mathfrak{F}} = (G_{\mathfrak{F}})^*$, completing the proof.

We mention the "Twisted wreath product" construction: since this construction will be used once only (in(3.4.1)), we refer to [53], [38], or [41; I.15.10] for details.

The second important construction to be described is a well-known procedure for constructing finite soluble groups with unique chief series: this formalism appears in Lockett [47; 1.2.4].

<u>1.7.4 Lemma</u>. Suppose that G is a finite soluble group with a unique <u>minimal</u> normal subgroup N, where N is a p-group ($p \in \mathbb{P}$). If q is a prime distinct from p, then G possesses a faithful irreducible representation over $GF(q)$.

<u>Proof</u>. Let U denote the regular $GF(q)\,G$-module. If every G-composition factor of U were non-faithful, then they would all be

centralized by N. But then by [28; 5.3.2], N would centralize U, which is impossible since U is the regular G-module.

<u>1.7.5 Proposition</u>. Let $(p_n,\ldots,p_1)$ be a sequence of consecutively-distinct primes. Then there exists a group $G \in \mathcal{S}_{p_n} * \cdots * \mathcal{S}_{p_1}$ with unique chief series

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G ,$$

and elementary abelian subgroups $E_n,\ldots,E_1$, such that $E_i \in \mathcal{S}_{p_i}$, $E_iG_i = G_{i-1}$, $|E_i| = |G_{i-1}/G_i|$ and E_i is normalized by each E_j for $j \leq i$. Further, G is expressible as a repeated semi-direct product

$$G = E_n](E_{n-1}](\ \ldots](E_1)\ldots\ )) .$$

<u>Proof</u>. We proceed by induction on n. If $n = 1$, we take $G = E_1 = \langle x : x^{p_1} = 1\rangle$. Suppose that $n > 1$ and that $H = E_{n-1} \ldots E_1$ $\mathcal{S}_{p_{n-1}} * \cdots * \mathcal{S}_{p_1}$ has been constructed with the desired properties. Then $E_{n-1} \in \mathcal{S}_{p_{n-1}}$ is the unique minimal normal subgroup of H, and so by (1.7.4), since $p_{n-1} \neq p_n$, there exists a faithful irreducible $GF(p_n)$-module for H, say E_n. Let G denote the abstract semi-direct product $E_n]H$. Since $C_H(E_n) = 1$, then E_n is the unique minimal normal subgroup of $E_n]H$, and, since H has by induction a unique chief series, it follows that G has a unique chief series. The other statements are evidently satisfied.

<u>1.7.6 Notation and remarks</u>. (a) If $G \in \mathcal{S}_{p_n} * \cdots * \mathcal{S}_{p_1}$ has been constructed in the manner of (1.7.5), we shall say that <u>G is of type $M(p_n,\ldots,p_1)$</u>. Such a group G is not necessarily unique.

If the chief factor $G_{i-1}/G_i \cong E_i$ has order $p_i^{\alpha_i}$, we may say that G is of type $M(p_n^{\alpha_n},\ldots,p_1^{\alpha_1})$; of course, $\alpha_1 = 1$ by our construction.

(b) If $1 \leq r \leq n$ we have, in the notation of (1.7.5),

$$G_{\mathcal{N}^r} = G_{n-r} = G_{\mathfrak{X}} \text{ , where } \mathfrak{X} = \mathcal{S}_{p_n} * \cdots * \mathcal{S}_{p_{n-r}} \text{ ; and}$$

$$G^{\mathcal{N}^r} = G_r = G^{\mathcal{Y}} \text{ , where } \mathcal{Y} = \mathcal{S}_{p_r} * \cdots * \mathcal{S}_{p_1}$$

(c) If $(p_n,\ldots p_m,\ldots,p_1)$ is a sequence of consecutively-distinct primes and we are given a group H of type $M(p_m,\ldots,p_1)$, we may, in the manner of the proof of (1.7.5), "extend" H to a group G of type $M(p_n,\ldots,p_1)$ such that $G/G_m \cong H$.

(d) In referring to a group G of type $M(p_n^{\alpha_n},\ldots,p_1^{\alpha_1})$, we may say that G has chief factors of orders $p_1^{\alpha_1},\ldots,p_n^{\alpha_n}$,"<u>reading from the top</u>".

(e) We note that a group G of type $M(3,2,3)$ does not lie in $\mathcal{S}_2 * \mathcal{S}_3 * \mathcal{C}_2$. However, $\mathcal{S}_2 \subseteq \mathcal{S}_2 * \mathcal{S}_3$, while $\mathcal{S}_2 * \mathcal{C}_2 = \mathcal{S}$, clearly. This provides an example promised on page 14.

1.8 Dark's Fitting class.

In the paper [19] , Dark introduced a method of constructing Fitting classes which has proved to be a very important source of examples; for instance, the Berger-Cossey counterexample to the Lockett conjecture was constructed using Dark's method: see [5] .

Here we just quote Dark's theorem: no direct use will be made of it, although in Chapter 4 some similar classes constructed by Hawkes will be investigated.

<u>1.8.1 Theorem (Dark. [19; theorem 4])</u>. Suppose that Y is a finite soluble group and that π is a set of primes such that

(I) Y has a unique minimal normal subgroup, Y is odd and $Z(Y) = O_\pi(Y) = 1$; and

(II) π contains all prime divisors of $|Y/Y'|$.

Since $Z(Y) = 1$, there exists a natural embedding $\nu : Y \to \mathrm{Inn}(Y) \trianglelefteq \mathrm{Aut}(Y)$. Let α be an element of $\mathrm{Aut}(Y)$ of prime order; let $P = \langle\alpha\rangle$ and $X = Y\nu\langle\alpha\rangle$. Assume that the following conditions are satisfied:

(III) $P \cap Y\nu = 1$, $|P| = p \in \pi$ and $(p, |Y|) = 1$;

(IV) $[P , Y\nu] = Y\nu$;

(V) if $Y_1 \leq \mathrm{Aut}(Y)$, $Y_1 \cong Y$ and $Y\nu$ normalizes Y_1, then $Y_1 = Y\nu$;

(VI) if $\varphi \in \mathrm{Aut}(Y)$ and $\langle\alpha, \alpha^\varphi\rangle$ is a p-group, then $\langle\alpha\rangle = \langle\alpha^\varphi\rangle$.

If $G \in \mathcal{S}$, define $\beta(G) = O^{P'}(G/O_\pi(G))$.

Let $\mathfrak{X} = (G \in \mathcal{S} : \beta(G) \in S_n D_o(X))$.

Then $\mathfrak{X}$ is a Fitting class .

1.9 The "Quasi-R_o-lemma"

<u>1.9.1 Lemma</u>. Suppose that A and B are groups and that G_o is subdirect in $A \times B$ (see (1.1.4)) . Suppose that there exist subgroups X of A and Y of B with $G_o \leq X \times Y$. Then $X = A$, $Y = B$.

<u>Proof</u>. Suppose that there exists $a \in A$ with $a \notin X$. Since $A \times B = G_o B$, then there exists $g_o = (x,y) \in G_o$ and $b \in B$ with $(a,1) = (x,y)(1,b)$. But then $(ax^{-1},1) = (1,yb) \in A \times 1 \cap 1 \times B = 1$. hus $a = x \in X$, contrary to choice, and so $X = A$. Similarly, $Y = B$.

The following lemma, part (a) of which has come to be known as the "Quasi-R_0-Lemma", has assumed some importance in the theory of Fitting classes. Part (b), due to P. Hauck, was quoted by him in a talk at Warwick in 1977. We include a proof .

<u>1.9.2 Lemma ("Quasi-R_0-Lemma")</u>. Let $\mathfrak{F}$ be a Fitting class and G be a finite group. Suppose that there exist N , $M \trianglelefteq G$ with $N \cap M = 1$ and $G/NM \in \mathcal{N}$.

(a) (Lockett, [47; 2.6.7]). Suppose that $G/M \in \mathfrak{F}$. Then $G \in \mathfrak{F}$ if and only if $G/N \in \mathfrak{F}$.

(b) (Hauck, [34; 1.11]). Suppose that $\mathfrak{F}$ is a Lockett class and that $G \in \mathfrak{F}$. Then G/N , $G/M \in \mathfrak{F}$.

<u>Proof</u>. Let A denote G/N and B denote G/M . Now, $A \trianglerighteq U = \{mN : m \in M\} \cong M$, and $B \trianglerighteq V = \{nM : n \in N\} \cong N$, and $A \times B$ has a naturally-embedded copy of G , to wit $G_0 = \{(gN, gM) : g \in G\}$. Since $G_0 \geqslant U \times V$ and $A \times B/(U \times V) \cong A/U \times B/V \in \mathcal{N}$, then G_0 sn $A \times B$. Since $(gN, 1M) = (gN, gM)\,(1N, gM)^{-1}$ for all $g \in G$, then $A \times B = G_0 B$ and G_0 is subdirect in $A \times B$.

(a) Here, $B = G/M \in \mathfrak{F}$. If $G/N \in \mathfrak{F}$, then $A \times B \in \mathfrak{F}$, and so $G_0 \in S_n N_0 \mathfrak{F} = \mathfrak{F}$. If $G \in \mathfrak{F}$,then $A \times B = G_0 B \in N_0 \mathfrak{F}$, and so $G/N = A \in \mathfrak{F}$.

(b) Here, $\mathfrak{F}$ is a Lockett class and $G_0 \cong G \in \mathfrak{F}$, and since G_0 sn $A \times B$ then by (1.3.2c), $G_0 \leqslant (A \times B)_{\mathfrak{F}} = A_{\mathfrak{F}} \times B_{\mathfrak{F}}$. But then by (1.9.1), $A_{\mathfrak{F}} = A$ and $B_{\mathfrak{F}} = B$, as required.

The following lemma is due to Berger, [4]. Since it is not yet published, and will be important for us, mainly in Chapter 4 but also

for one result in Chapter 3, we will supply the proof. The proof given is that of Berger, with several details added; the result may also be proved by repeated application of (1.9.2a). We prove a second part of this lemma in (4.1.16); in that proof, we will refer to the internal details of the proof here.

<u>1.9.3 Lemma (Berger, 4; 3.2)</u>. Let $\mathfrak{F}$ be a Fitting class. Suppose that $G \in \mathcal{E}$, and that G contains subgroups V and W such that $V \trianglelefteq G$, $M \in \mathcal{N} \cap \mathfrak{F}$ and $G = VM$ (not-necessarily semi-direct).

Let U denote a group isomorphic with V , and let $\rho : V \to U$ be an isomorphism. Then ρ induces a homomorphism $\varphi : G \to \mathrm{Aut}(U)$ given by

$$\varphi : g \mapsto \{u \mapsto ((u\rho^{-1})^{g})\rho \ : u \in U\} \qquad ; \text{ for } g \in G ,$$

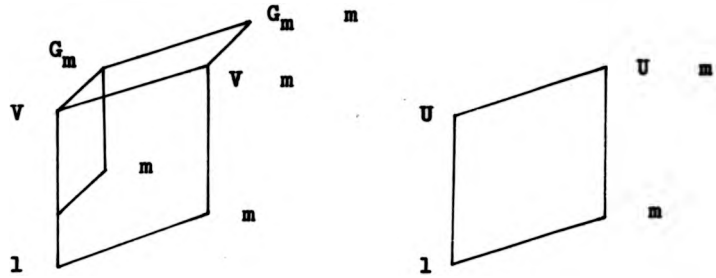
where the expression in braces represents the automorphism of U whose action on an arbitrary element $u \in U$ is as shown.

Form the abstract semi-direct product $U]\mathrm{Aut}(U)$, which has a subgroup $U](M\varphi)$. Then $G \in \mathfrak{F}$ if and only if $U](M\varphi) \in \mathfrak{F}$.

<u>Proof</u>. It is standard, and not hard to check, that φ is a homomorphism from G to $\mathrm{Aut}(U)$. Since $M \in \mathcal{N}$, then any subgroup H with $V \leqslant H \leqslant G$ is subnormal in G , and it follows that $G \in \mathfrak{F} = \langle S_n, N_o \rangle \mathfrak{F}$ if and only if $V\langle m\rangle \in \mathfrak{F}$ for all $m \in M$. Since $M\varphi \in \mathcal{N}$, it similarly follows that $U](M\varphi) \in \mathfrak{F}$ if and only if $U\langle m\varphi\rangle \in \mathfrak{F}$ for all $m \in M$.

Choose $m \in M$ and let G_m denote $V\langle m\rangle$ sn G . Now, m induces an inner automorphism, $\hat{m}$ say, of G_m , and we may form the abstract semi-direct product $G_m]\langle\hat{m}\rangle$. Since $V \trianglelefteq G_m$, then V is certainly $\langle\hat{m}\rangle$-invariant, and so $G_m]\langle\hat{m}\rangle$ has a subgroup $V]\langle\hat{m}\rangle$.

fig. 1.



Since $\hat{m}$ is the inner automorphism induced by m on $G_m = V\langle m\rangle$, then $\hat{m}$ acts faithfully on V and centralizes m. With this in mind, we may check that the map $\psi : V]\langle\hat{m}\rangle \to U]\langle m\varphi\rangle$ given by

$$\psi : v\hat{m}^i \mapsto (v\rho)(m^i\varphi)$$

is an isomorphism. Thus

$$U]\langle m\varphi\rangle \in \mathfrak{F} \text{ if and only if } V]\langle\hat{m}\rangle \in \mathfrak{F} . \qquad \ldots(1)$$

Again since m induces $\hat{m}$ on G_m, then $m\hat{m}^{-1}$ centralizes G_m and also $\hat{m}$. Thus, $G_m \cdot \langle m\hat{m}^{-1}\rangle$ and $V\langle\hat{m}\rangle \cdot \langle m\hat{m}^{-1}\rangle$ are both normal products; it is easy to check that they both coincide with $G_m]\langle\hat{m}\rangle$. Since $\hat{m}$ centralizes m, then $\mathrm{char}(\langle m\hat{m}^{-1}\rangle) \leq \mathrm{char}(\langle m\rangle)$. But $M \in \mathcal{N} \cap \mathfrak{F}$, and so $\langle m\hat{m}^{-1}\rangle \in \mathfrak{F}$. Thus, $G_m \in \mathfrak{F}$ if and only if $V]\langle\hat{m}\rangle \in \mathfrak{F}$, and so by (1), and the remarks in the first paragraph, $G \in \mathfrak{F}$ if and only if $U](M\varphi) \in \mathfrak{F}$, completing the proof.

1.9.4 Remark. In the original statement of the above result, Berger omits the condition that the nilpotent group M belong to $\mathfrak{F}$; however some such condition is necessary.

"if" : Let $\mathfrak{F} = \mathcal{S}_3$ and take $G \cong C_6$, with $V = U$ the subgroup of order 3 and M the subgroup of order 2. Then $U](M\varphi) \cong C_3 \in \mathfrak{F}$, but $G \notin \mathfrak{F}$.

"only if": If G is soluble and $G \in \mathfrak{F}$, then $M \in \mathcal{N}_{char(\mathfrak{F})} \subseteq \mathfrak{F}$. However, take $G = V \cong Alt(5)$, and let $\mathfrak{F} = \langle Alt(5) \rangle_{Fitt} = D_o(G)$ (see (1.3.18)). Let M be a subgroup of G of order 2. Then $G = VM \in \mathfrak{F}$, while $U\rfloor(M\varphi)$ has order 120 and so cannot belong to $D_o(G)$.

CHAPTER 2.

THE OPERATORS $\mathcal{K}_\pi(\ )$ AND $\mathcal{L}_\pi(\ )$.

In this chapter, we shall be working entirely within the universe $\mathcal{S}$ of finite, soluble, groups.

Given a Fitting class $\mathfrak{F}$ and a set of primes π, Lockett [48] defines a new class $\mathfrak{F}^\pi$ to consist of those (finite, soluble) groups G in which an $\mathfrak{F}$-injector contains a Hall π-subgroup of G, and proves that $\mathfrak{F}^\pi$ is a Fitting class. A reason for introducing the classes $\mathfrak{F}^\pi$ is that they assist in determining the injectors for a product of Fitting classes (see (2.3.3) below).

Hawkes has re-denoted $\mathfrak{F}^\pi$ by $\mathcal{L}_\pi(\mathfrak{F})$, regarding $\mathcal{L}_\pi(\ )$ as an "operator" on the family of all Fitting classes (of finite, soluble, groups). If we define $\mathcal{K}_\pi(\mathfrak{F})$ as the class of all (finite, soluble) groups in which the Hall π-subgroups belong to $\mathfrak{F}$, then $\mathcal{K}_\pi(\mathfrak{F})$ is again a Fitting class and so $\mathcal{K}_\pi(\ )$ gives another "operator". The $\mathcal{K}_\pi(\ )$ operator seems to have been known for some time, especially in the context of formation theory.

In this chapter, the operators $\mathcal{K}_\pi(\ )$ and $\mathcal{L}_\pi(\ )$ are investigated, together with various related topics concerned with the role of Hall subgroups in Fitting class theory.

After presenting some elementary properties in the first section, and giving a description of the $\mathcal{K}_\pi(\mathfrak{F})$radical in the second, we go on in section three to show that $\mathcal{K}_\pi(\mathfrak{F} * \mathfrak{G}) = \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{K}_\pi(\mathfrak{G})$ for all Fitting classes $\mathfrak{F}$ and $\mathfrak{G}$, and give sufficient conditions for this to hold for $\mathcal{L}_\pi(\ )$.

In section four, it is shown that $\mathcal{L}_\pi(\ )$ and $\mathcal{K}_\pi(\ )$ both commute with Lockett's "Upper-star" operation.

The operator $\mathcal{K}_\pi(\ )$ is closely related to what we call "Hall-closure"; that is, the closure of a given class under taking Hall subgroups of groups in the class. Our result that $\mathcal{K}_\pi(\ )$ and the upper-star operation commute leads to the observation that if $\mathfrak{F}$ is a Hall-closed Fitting class, then so also are $\mathfrak{F}^*$ and $\mathfrak{F}_*$. This yields as a special case a result of Bryce and Cossey that the smallest normal Fitting class, $\mathcal{S}_*$, is Hall-closed .

In section five, we give a new characterization of normal Fitting classes in terms of the $\mathcal{L}_\pi(\ )$ operator, while in section six we modify an example of Cossey. In the seventh section, the Fitting class generated by the so-called "π-perfect" groups is determined.

The eighth section is devoted to proving that the classes $\mathcal{C}_\pi$ and $e_\pi(\mathfrak{F})$ are closed under taking Hall subgroups containing the Fitting subgroup, a concept introduced in section one and related to $\mathcal{K}_\pi(\ )$.

In the final section, it is shown that if $\mathfrak{F}$ is a primitive saturated formation of finite defect n , then the (soluble) group G belongs to $\mathfrak{F}$ if, and only if, all Hall π-subgroups of G for $|\pi| \leq n$ belong to $\mathfrak{F}$.

We stress that in this chapter, the class of all finite soluble groups, $\mathcal{S}$, is taken as universe, and all classes and groups considered will be assumed to lie in $\mathcal{S}$ without further comment.

The operator $\mathcal{K}_\pi(\ )$ is closely related to what we call "Hall-closure"; that is, the closure of a given class under taking Hall subgroups of groups in the class. Our result that $\mathcal{K}_\pi(\ )$ and the upper-star operation commute leads to the observation that if $\mathfrak{F}$ is a Hall-closed Fitting class, then so also are $\mathfrak{F}^*$ and $\mathfrak{F}_*$. This yields as a special case a result of Bryce and Cossey that the smallest normal Fitting class, $\mathcal{S}_*$, is Hall-closed .

In section five, we give a new characterization of normal Fitting classes in terms of the $\mathcal{L}_\pi(\ )$ operator, while in section six we modify an example of Cossey. In the seventh section, the Fitting class generated by the so-called "π-perfect" groups is determined.

The eighth section is devoted to proving that the classes $\mathcal{C}_\pi$ and $e_\pi(\mathfrak{F})$ are closed under taking Hall subgroups containing the Fitting subgroup, a concept introduced in section one and related to $\mathcal{K}_\pi(\ )$.

In the final section, it is shown that if $\mathfrak{F}$ is a primitive saturated formation of finite defect n , then the (soluble) group G belongs to $\mathfrak{F}$ if, and only if, all Hall π-subgroups of G for $|\pi| \leq n$ belong to $\mathfrak{F}$.

We stress that in this chapter, the class of all finite soluble groups, $\mathcal{S}$, is taken as universe, and all classes and groups considered will be assumed to lie in $\mathcal{S}$ without further comment.

2.1 Definitions and elementary properties.

2.1.1 Definition and Conventions. Let $\mathcal{G}$ be a Fitting class.

(a) (Lockett, [48]). Let $\mathfrak{F}$ be a Fitting class, and define

$$\mathcal{L}_{\mathcal{G}}(\mathfrak{F}) = (G \in \mathcal{S} : \text{An } \mathfrak{F}\text{-injector of } G \text{ contains a } \mathcal{G}\text{-injector of } G) .$$

(b) Let $\mathfrak{X}$ be a class of groups, and define

$$\mathcal{K}_{\mathcal{G}}(\mathfrak{X}) = (G \in \mathcal{S} : \text{The } \mathcal{G}\text{-injectors of } G \text{ belong to } \mathfrak{X}).$$

(c) If $\mathcal{G} = \mathcal{S}_{\pi}$, where $\pi \subseteq \mathbb{P}$, in the above definitions (in which case a $\mathcal{G}$-injector is a Hall π-subgroup), we write $\mathcal{L}_{\pi}(\mathfrak{F})$ and $\mathcal{K}_{\pi}(\mathfrak{X})$, respectively, and omit the braces if $\pi = \{p\}$. If $\pi = \emptyset$, then by convention, $\mathcal{L}_{\emptyset}(\mathfrak{F}) = \mathcal{K}_{\emptyset}(\mathfrak{X}) = \mathcal{S}$, for all relevant $\mathfrak{F}$ and $\mathfrak{X}$.

If a given $\mathfrak{F}$-injector of a group G contains a $\mathcal{G}$-injector of G, then every $\mathfrak{F}$-injector contains a $\mathcal{G}$-injector, by conjugacy of injectors.

The following propositon of Lockett, [48], will be important for our work and so the proof of the first three parts is sketched, and the required examples for the fourth part are provided.

2.1.2 Proposition (Lockett, [48; 3.1]). Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then

(a) $\mathcal{L}_{\pi}(\mathfrak{F})$ is a Fitting class;

(b) $\mathfrak{F}, \mathcal{S}_{\pi'} \subseteq \mathfrak{F} * \mathcal{S}_{\pi'} \subseteq \mathcal{L}_{\pi}(\mathfrak{F}) = \mathcal{L}_{\pi}(\mathfrak{F}) * \mathcal{S}_{\pi'} = \mathcal{L}_{\pi}(\mathcal{L}_{\pi}(\mathfrak{F}))$;

(c) the following are equivalent:

(i) $\mathfrak{F} = \mathcal{L}_{\pi}(\mathfrak{F})$

(ii) $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$

(iii) for each $G \in \mathcal{S}$, the $\mathfrak{F}$-injectors of G have π-index in G; and

and

(d) if $\mathfrak{F}$ is a Fischer class, then so is $\mathcal{L}_\pi(\mathfrak{F})$, while if $\mathfrak{F}$ is either Q- or S-closed, it does not follow that $\mathcal{L}_\pi(\mathfrak{F})$ need be respectively Q- or S-closed.

<u>Proof</u>.(a) If $G \in \mathcal{L}_\pi(\mathfrak{F})$, $N \trianglelefteq G$ and V is an $\mathfrak{F}$-injector of G, then V contains some Hall π-subgroup H of G, while $V \cap N$ is an $\mathfrak{F}$-injector of N. But then $H \cap N \leq V \cap N$ and $H \cap N \in \mathrm{Hall}_\pi(N)$, whence $N \in \mathcal{L}_\pi(\mathfrak{F})$. It follows that $\mathcal{L}_\pi(\mathfrak{F}) = S_n\mathcal{L}_\pi(\mathfrak{F})$.
Now suppose that $G = N_1N_2$, where $N_i \trianglelefteq G$ and $N_i \in \mathcal{L}_\pi(\mathfrak{F})$, $i = 1,2$. Let V be an $\mathfrak{F}$-injector of G and let $K \in \mathrm{Hall}_\pi(V)$. Now $V \cap N_i$ is an $\mathfrak{F}$-injector of N_i, for $i = 1,2$, and since $K \cap N_i \in \mathrm{Hall}_\pi(V \cap N_i)$, then $K \cap N_i \in \mathrm{Hall}_\pi(N_i)$ since $N_i \in \mathcal{L}_\pi(\mathfrak{F})$. But now K contains Hall π-subgroups of both N_1 and N_2, and using (1.1.2), we may check that $K \in \mathrm{Hall}_\pi(G)$, whence $G \in \mathcal{L}_\pi(\mathfrak{F})$. Thus, $\mathcal{L}_\pi(\mathfrak{F})$ is N_0-closed and is therefore a Fitting class.

(b) Since $\mathcal{S}_{\pi'}$ is Q-closed, then by (1.3.13) and the definition, we may check that $\mathfrak{F}, \mathcal{S}_{\pi'} \subseteq \mathfrak{F} * \mathcal{S}_{\pi'}$, $\mathfrak{F} \subseteq \mathcal{L}_\pi(\mathfrak{F})$, $\mathfrak{F} * \mathcal{S}_{\pi'} \subseteq \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$, $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$, and $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathcal{L}_\pi(\mathfrak{F}))$.

Suppose that $G \in \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$. If V is an $\mathfrak{F}$-injector of G, then $V \cap G_{\mathcal{L}_\pi(\mathfrak{F})}$ is an $\mathfrak{F}$-injector of $G_{\mathcal{L}_\pi(\mathfrak{F})}$ and so contains a Hall π-subgroup of $G_{\mathcal{L}_\pi(\mathfrak{F})}$, and thus contains a Hall π-subgroup of G, since $G/G_{\mathcal{L}_\pi(\mathfrak{F})} \in \mathcal{S}_{\pi'}$. Thus, $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$.

Suppose that $\mathcal{L}_\pi(\mathfrak{F}) \subsetneqq \mathcal{L}_\pi(\mathcal{L}_\pi(\mathfrak{F}))$, and let G be a group of minimal order in $\mathcal{L}_\pi(\mathcal{L}_\pi(\mathfrak{F})) \setminus \mathcal{L}_\pi(\mathfrak{F})$. Then G has a unique maximal

normal subgroup $M = G_{\mathcal{L}_\pi(\mathfrak{F})}$ of prime index, p say, and M must be an $\mathcal{L}_\pi(\mathfrak{F})$-injector of G. Thus, M contains a Hall π-subgroup, H say, of G. Since $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$ and $G \notin \mathcal{L}_\pi(\mathfrak{F})$, then $p \in \pi$, contrary to $M \geqslant H$.

(c) The implication (i) $\Rightarrow$ (ii) follows from (b).

(ii) $\Rightarrow$ (iii). Assume that $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$, and let G be a group of minimal order such that if V is an $\mathfrak{F}$-injector of G, then $|G : V|$ is not a π-number. Let $M \triangleleft G$; then $M \cap V$ is an $\mathfrak{F}$-injector of M, and so $|M : M \cap V|$ is a π-number. If $V \nleqslant M$, then $|G : V| = |M : M \cap V|$ is a π-number. Thus $V \leqslant M$, and $|G : M| = q$, where $q \in \mathbb{P} \setminus \pi$. Now, V is also an $\mathfrak{F}$-injector of M, whence by conjugacy of injectors and the Frattini argument we have $G = M N_G(V)$. Choosing a q-element $x \in N_G(V) \setminus M$ so that $G = M\langle x\rangle$, we have $V\langle x\rangle \in \mathfrak{F} * \mathcal{S}_{\pi'}$, since $q \notin \pi$, contrary to the $\mathfrak{F}$-maximality of V.

(iii) $\Rightarrow$ (i). By (b), $\mathfrak{F} \subseteq \mathcal{L}_\pi(\mathfrak{F})$. Suppose that $\mathfrak{F} \subsetneqq \mathcal{L}_\pi(\mathfrak{F})$, and let G be a group of minimal order in $\mathcal{L}_\pi(\mathfrak{F}) \setminus \mathfrak{F}$. Then G has a unique maximal normal subgroup $M = G_{\mathfrak{F}}$, which must be an $\mathfrak{F}$-injector of G. Since $G \in \mathcal{L}_\pi(\mathfrak{F})$, then $|G : M| \notin \pi$, contrary to (iii).

(d) We refer to [47; 2.6.1d] for this. But note that Sym(4) belongs to $\mathcal{L}_2(\mathcal{N})$, while Sym(3), which appears as both subgroup and factor group of Sym(4), does not; thus $\mathcal{L}_\pi(\ )$ need not respect Q- or S-closure.

<u>2.1.3 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes. Then

(a) (Lockett) $\mathcal{K}_{\mathfrak{G}}(\mathfrak{F}) = \mathcal{L}_{\mathfrak{G}}(\mathfrak{F} \cap \mathfrak{G})$; and

(b) $\mathcal{K}_\pi(\mathfrak{F})$ is a Fitting class, for all $\pi \subseteq \mathbb{P}$.

<u>Proof</u>. (a) Suppose firstly that $G \in \mathcal{K}_{\mathfrak{G}}(\mathfrak{F})$. Let V be a $\mathfrak{G}$-injector of G, so that $V \in \mathfrak{F}$, while if $N \operatorname{sn} G$ then $V \cap N$ is

$\mathcal{G}$-maximal in N. But $V \cap N$ belongs to $\mathfrak{F}$, and so to $\mathfrak{F} \cap \mathcal{G}$. But then $V \cap N$ must be $\mathfrak{F} \cap \mathcal{G}$-maximal in N, and so V is an $\mathfrak{F} \cap \mathcal{G}$-injector of G. Thus $G \in \mathcal{L}_{\mathcal{G}}$

Suppose next that $G \in \mathcal{L}_{\mathcal{G}}(\mathfrak{F} \cap \mathcal{G})$. Let V be an $\mathfrak{F} \cap \mathcal{G}$-injector of G, so that V contains a $\mathcal{G}$-injector of G. But now V must itself be a $\mathcal{G}$-injector of G since $V \in \mathcal{G}$. Since $V \in \mathfrak{F}$, then $G \in \mathcal{K}_{\mathcal{G}}(\mathfrak{F})$.

(b) This follows from (a) and (2.1.2a).

In general, the classes $\mathcal{L}_{\mathcal{G}}(\mathfrak{F})$ and $\mathcal{K}_{\mathcal{G}}(\mathfrak{F})$ need not be Fitting classes: we refer to (2.1.17) below for remarks on this topic.

<u>2.1.4 Omnibus lemma</u>. Let $\mathfrak{X}$ and $\mathcal{Y}$ be classes of groups, $\mathfrak{F}$ and $\mathcal{G}$ be Fitting classes, and let π and ρ be sets of primes. Then we have the following.

(a) $\mathcal{K}_{\pi}(\mathfrak{X}) = \mathcal{S}_{\pi'} * \mathcal{K}_{\pi}(\mathfrak{X}) * \mathcal{S}_{\pi'} = \mathcal{K}_{\pi}(\mathcal{K}_{\pi}(\mathfrak{X})) = \mathcal{K}_{\pi}(\mathfrak{X} \cap \mathcal{S}_{\pi})$.

(b) $\mathcal{K}_{\pi}(\mathfrak{X}) \cap \mathcal{S}_{\pi} = \mathfrak{X} \cap \mathcal{S}_{\pi}$.

(c) It is not necessary that $\mathfrak{X} \subseteq \mathcal{K}_{\pi}(\mathfrak{X})$, in contrast to (2.1.2b).

(d) If $\mathfrak{X} \subseteq \mathcal{Y}$, then $\mathcal{K}_{\pi}(\mathfrak{X}) \subseteq \mathcal{K}_{\pi}(\mathcal{Y})$, while if $\mathfrak{F} \subseteq \mathcal{G}$ it is not necessary that $\mathcal{L}_{\pi}(\mathfrak{F}) \subseteq \mathcal{L}_{\pi}(\mathcal{G})$.

(e) If $\mathfrak{X}$ is C-closed where $C \in \{S, S_n, S_F, Q, N_0, R_0\}$, then $\mathcal{K}_{\pi}(\mathfrak{X})$ is C-closed.

(f) If $\{\mathfrak{X}_{\alpha}\}_{\alpha \in A}$ is a family of classes, then $\mathcal{K}_{\pi}(\bigcap_{\alpha} \mathfrak{X}_{\alpha}) = \bigcap_{\alpha} \mathcal{K}_{\pi}(\mathfrak{X}_{\alpha})$, while this can fail for $\mathcal{L}_{\pi}(\ )$.

(g) $\mathcal{K}_{\pi}(\mathcal{K}_{\rho}(\mathfrak{X})) = \mathcal{K}_{\pi \cap \rho}(\mathfrak{X}) = \mathcal{K}_{\rho}(\mathcal{K}_{\pi}(\mathfrak{X}))$.

(h) If $\rho \subseteq \pi$, then $\mathcal{L}_{\pi}(\mathfrak{F}) \subseteq \mathcal{L}_{\rho}(\mathfrak{F})$.

(i) $\mathcal{L}_{\pi\cup\rho}(\mathfrak{F}) = \mathcal{L}_{\pi}(\mathfrak{F}) \cap \mathcal{L}_{\rho}(\mathfrak{F})$.

(j) $\mathrm{Char}(\mathcal{K}_{\pi}(\mathfrak{F})) = (\pi \setminus \mathrm{char}(\mathfrak{F})) \doteq \pi' \cup \mathrm{char}(\mathfrak{F})$.

<u>Proof</u>.(a) Suppose that $G \in \mathcal{S}$ and that $M \leq N \trianglelefteq G$ with $M \trianglelefteq N$. Suppose also that $|M|$ and $|G : N|$ are π'-numbers. Then $G \in \mathcal{K}_{\pi}(\mathfrak{X})$ if and only if $M/N \in \mathcal{K}_{\pi}(\mathfrak{X})$, since a Hall π-subgroup of G is isomorphic to one of M/N. Thus $\mathcal{K}_{\pi}(\mathfrak{X}) = \mathcal{S}_{\pi'} * \mathcal{K}_{\pi}(\mathfrak{X}) * \mathcal{S}_{\pi'}$.

The other assertions follow easily from the definition of $\mathcal{K}_{\pi}(\ )$.

(b) This follows at once from the definition.

(c) A group G of type $M(5,3,2)$, as constructed in (1.7.5), lies in both $\mathcal{C}_3$ and $e_3(\mathcal{N})$ (see (1.3.16)), while a Hall $\{2,3\}$-subgroup of G does not. We will expand upon this example in (2.8.5).

(d) If $\mathfrak{X} \subseteq \mathfrak{Y}$, then certainly $\mathcal{K}_{\pi}(\mathfrak{X}) \subseteq \mathcal{K}_{\pi}(\mathfrak{Y})$. For the required example, let G denote $\mathrm{Sym}(4)$. An $\mathcal{S}_2$-injector of G belongs to $\mathrm{Syl}_2(G)$, while an $\mathcal{S}_2 * \mathcal{S}_3$-injector is the subgroup $\mathrm{Alt}(4)$. Thus $G \in \mathcal{L}_2(\mathcal{S}_2) \setminus \mathcal{L}_2(\mathcal{S}_2 * \mathcal{S}_3)$, while $\mathcal{S}_2 \subseteq \mathcal{S}_2 * \mathcal{S}_3$.

(e) If $G \in \mathcal{K}_{\pi}(\mathfrak{X})$, then a subgroup/normal subgroup/quotient group of G has a Hall π-subgroup isomorphic to a subgroup/normal subgroup/quotient group, respectively, of a Hall π-subgroup of G, whence $\mathcal{K}_{\pi}(\ )$ preserves S-, S_n-, and Q-closure, respectively.

Now suppose that $\mathfrak{X}$ is N_0-closed. Suppose that the group G is such that $G = N_1N_2$, where $N_i \trianglelefteq G$ and $N_i \in \mathcal{K}_{\pi}(\mathfrak{X})$, $i = 1,2$. Let $H \in \mathrm{Hall}_{\pi}(G)$. The assertion follows from (1.1.2).

Next suppose that $\mathfrak{X}$ is R_0-closed. Suppose that the group G possesses normal subgroups N_1, N_2 with $N_1 \cap N_2 = 1$ and $G/N_i \in \mathcal{K}_{\pi}(\mathfrak{X})$, $i = 1,2$. Let $H \in \mathrm{Hall}_{\pi}(G)$. Then $H \cap N_i \trianglelefteq H$,

$(H \cap N_1) \cap (H \cap N_2) = 1$ and $H/H \cap N_i$ is isomorphic with a Hall π-subgroup of G/N_i, for $i = 1, 2$. It follows that $H \in R_0\mathfrak{X} = \mathfrak{X}$, and so $G \in \mathfrak{K}_\pi(\mathfrak{X})$.

Finally suppose that $\mathfrak{X}$ is S_F-closed. Suppose that $L \leqslant G$ and $G \in \mathfrak{K}_\pi(\mathfrak{X})$ and that $L/\mathrm{core}_G(L) \in \mathfrak{N}$. Let $K \in \mathrm{Hall}_\pi(L)$ and suppose that $K \leqslant H \in \mathrm{Hall}_\pi(G)$. Now $K \geqslant \mathrm{core}_G(L) \cap H \trianglelefteq H$ and $K/(\mathrm{core}_G(L) \cap H) \in \mathfrak{N}$, whence $K \in S_F\mathfrak{X} = \mathfrak{X}$, and so $L \in \mathfrak{K}_\pi(\mathfrak{X})$.

(f) $G \in \mathfrak{K}_\pi(\bigcap_\alpha \mathfrak{X}_\alpha) \Leftrightarrow \mathrm{Hall}_\pi(G) \subseteq \bigcap_\alpha \mathfrak{X}_\alpha$ $\qquad \mathcal{L}_2(\mathcal{S}_2 \cap \mathcal{S}_2 * \mathcal{S}_3) = \mathcal{S}$

$\Leftrightarrow G \in \bigcap_\alpha \mathfrak{K}_\pi(\mathfrak{X}_\alpha)$. $\qquad$ but $\mathcal{L}_2(\mathcal{S}_2) \cap \mathcal{L}_2(\mathcal{S}_2 * \mathcal{S}_3) \not\ni \mathrm{Sym}(3)$.

(g) $G \in \mathfrak{K}_\pi(\mathfrak{K}_\rho(\mathfrak{X})) \Leftrightarrow$ the Hall ρ-subgroups of Hall π-subgroups of G belong to $\mathfrak{X}$

$\Leftrightarrow G \in \mathfrak{K}_{\pi \cap \rho}(\mathfrak{X})$.

(h) If $\rho \subseteq \pi$ and $G \in \mathcal{L}_\pi(\mathfrak{F})$, then certainly $G \in \mathcal{L}_\rho(\mathfrak{F})$, since a Hall π-subgroup always then contains a Hall ρ-subgroup.

(i) $G \in \mathcal{L}_{\pi\rho}(\mathfrak{F}) \Leftrightarrow$ an $\mathfrak{F}$-injector of G contains a Hall $(\pi \cup \rho)$-subgroup of G

$\Leftrightarrow G \in \mathcal{L}_\pi(\mathfrak{F}) \cap \mathcal{L}_\rho(\mathfrak{F})$.

(j) This is clear from the definition.

<u>2.1.5 Definition</u>. Let $\mathfrak{X}$ be a class of groups and $\mathfrak{F}$ be a Fitting class. Define $H\mathfrak{X} = (G \in \mathcal{S}$: G is isomorphic to a Hall-subgroup of an $\mathfrak{X}$-group) ; and

$H_{\mathfrak{F}}\mathfrak{X} = (G \in \mathcal{S}$: G is isomorphic to a Hall-subgroup H of a group $X \in \mathfrak{X}$, such that $H \geqslant X_{\mathfrak{F}})$.

It is clear that H and $H_{(1)}$ coincide, and that $H_{\mathfrak{F}} = H_{\mathfrak{N}_{\mathrm{char}}(\mathfrak{F})}$.

<u>2.1.6 Lemma</u>. The operations H and $H_{\mathfrak{F}}$ are closure operations.

<u>Proof</u>. Since $H = H_{(1)}$, we prove the result for $H_{\mathfrak{F}}$.

Certainly $\mathfrak{X} \subseteq H_{\mathfrak{F}}\mathfrak{X} \subseteq H_{\mathfrak{F}}\mathcal{Y}$, if $\mathfrak{X} \subseteq \mathcal{Y}$ are classes of groups.

Suppose that $G \in H_{\mathfrak{F}}(H_{\mathfrak{F}}\mathfrak{X})$. Then there exists $Y \in H_{\mathfrak{F}}\mathfrak{X}$ and $G_o \in \mathrm{Hall}(Y)$ with $G_o \geqslant Y_{\mathfrak{F}}$ such that $G \cong G_o$, while there exists $X \in \mathfrak{X}$ and $Y_o \in \mathrm{Hall}(X)$ with $Y_o \geqslant X_{\mathfrak{F}}$ such that $Y \cong Y_o$: we will identify G with $G_o \leqslant Y$ and Y with $Y_o \leqslant X$. Then G_o, being a Hall subgroup of Y_o, is one of X. Since $Y_o \geqslant X_{\mathfrak{F}}$, then $(Y_o)_{\mathfrak{F}} \geqslant X_{\mathfrak{F}}$. But $G_o \geqslant (Y_o)_{\mathfrak{F}}$ and so $G_o \geqslant X_{\mathfrak{F}}$, showing that $G_o \in H_{\mathfrak{F}}\mathfrak{X}$ and thus that $H_{\mathfrak{F}}(H_{\mathfrak{F}}\mathfrak{X}) = H_{\mathfrak{F}}\mathfrak{X}$; completing the proof.

If $\mathfrak{X} = H\mathfrak{X}$ we shall, in view of the lemma, say that $\mathfrak{X}$ is <u>Hall-closed</u>.

<u>2.1.7 Lemma</u>. Let $\mathfrak{X}$ be a class of groups and $\mathfrak{F}$ be a Fitting class. Let π be a set of primes. Then

(a) $\mathfrak{X} = H\mathfrak{X} \iff \mathfrak{X} \subseteq \mathcal{K}_\tau(\mathfrak{X})$ for all $\tau \subseteq \mathbb{P}$;

(b) $\mathfrak{X} = H_{\mathcal{N}}\mathfrak{X} \iff \mathcal{S}_{\tau'} * \mathfrak{X} \subseteq \mathcal{K}_\tau(\mathfrak{X})$ for all $\tau \subseteq \mathbb{P}$;

(c) $\mathfrak{Y} \subseteq \mathcal{K}_\pi(\mathfrak{F}) \implies \mathcal{L}_\pi(\mathfrak{Y}), \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathfrak{Y}) \subseteq \mathcal{K}_\pi(\mathfrak{Y})$;

(d) $\mathfrak{X} = \langle Q, H_{\mathcal{N}}\rangle\mathfrak{X} \implies \mathfrak{X} = H\mathfrak{X}$; and

(e) $\bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_\sigma(\mathfrak{X})$ is the largest Hall-closed class contained in $\mathfrak{X}$.

<u>Proof</u>. Part (a) is clear from the definitions.

(b) Suppose that $\mathfrak{X} = H_{\mathcal{N}}\mathfrak{X}$, and let $G \in \mathcal{S}_{\tau'} * \mathfrak{X}$. If $H \in \mathrm{Hall}_\tau(G)$, then $H \cong H\,O_{\tau'}(G)/O_{\tau'}(G) \in \mathrm{Hall}_\tau(G/O_{\tau'}(G))$. Now, $F(G/O_{\tau'}(G))$ is clearly a τ-group and so is contained in $H\,O_{\tau'}(G)/O_{\tau'}(G)$. By definition of

the operation $H_{\mathcal{N}}$, it follows that $H\,O_{\tau'}(G)/O_{\tau'}(G) \in \mathfrak{X}$, whence $H \in \mathfrak{X}$ and $G \in \mathcal{K}_{\tau}(\mathfrak{X})$, as required.

Now suppose that $\mathcal{S}_{\tau'} * \mathfrak{X} \subseteq \mathcal{K}_{\tau}(\mathfrak{X})$ for all $\tau \subseteq \mathbb{P}$, and suppose that there exists $G \in \mathfrak{X}$ with $F(G) \leq H \in \mathrm{Hall}(G)$ and $H \notin \mathfrak{X}$, where $\mathrm{char}(H) = \rho$ say. Then $O_{\rho'}(G) = 1$, whence $G \in \mathcal{S}_{\rho'} * \mathfrak{X} \subseteq \mathcal{K}_{\rho}(\mathfrak{X})$, and so $H \in \mathfrak{X}$, a contradiction.

(c) Suppose that $\mathfrak{Y} \subseteq \mathcal{K}_{\pi}(\mathfrak{Y})$, and let $G \in \mathcal{L}_{\pi}(\mathfrak{Y})$. If T is an $\mathfrak{Y}$-injector of G , then T must contain a Hall π-subgroup H of G. Since $\mathfrak{Y} \subseteq \mathcal{K}_{\pi}(\mathfrak{Y})$ and $H \in \mathrm{Hall}_{\pi}(T)$, then $H \in \mathfrak{Y}$, and $G \in \mathcal{K}_{\pi}(\mathfrak{Y})$. Thus $\mathcal{L}_{\pi}(\mathfrak{Y}) \subseteq \mathcal{K}_{\pi}(\mathfrak{Y})$. By (1.3.13) and (2.1.4), it follows that

$$\mathcal{S}_{\pi'} * \mathcal{L}_{\pi}(\mathfrak{Y}) \subseteq \mathcal{S}_{\pi'} * \mathcal{K}_{\pi}(\mathfrak{Y}) = \mathcal{K}_{\pi}(\mathfrak{Y}) .$$

(d) If $\mathfrak{X} = \langle Q , H_{\mathcal{N}} \rangle \mathfrak{X}$ and $\tau \subseteq \mathbb{P}$, then $\mathfrak{X} \subseteq \mathcal{S}_{\tau'} * \mathfrak{X}$ by (1.3.13), while by (b) above, $\mathcal{S}_{\tau'} * \mathfrak{X} \subseteq \mathcal{K}_{\tau}(\mathfrak{X})$. By (a), we have $\mathfrak{X} = H\,\mathfrak{X}$.

(e) Let $\mathcal{Y}$ denote $\bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_{\sigma}(\mathfrak{X})$. Since $\mathfrak{X}$ consists of finite groups, then $\mathcal{Y} \subseteq \mathfrak{X}$. We have

$$\begin{aligned} \mathcal{K}_{\pi}(\mathcal{Y}) &= \mathcal{K}_{\pi}\Big(\bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_{\sigma}(\mathfrak{X})\Big) \\ &= \bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_{\pi \cap \sigma}(\mathfrak{X}) && \text{by (2.1.4f,g)} \\ &\supseteq \bigcap_{\tau \subseteq \mathbb{P}} \mathcal{K}_{\tau}(\mathfrak{X}) = \mathcal{Y} , && \text{an intersection over "more" classes.} \end{aligned}$$

It follows by part (a) that $\mathcal{Y}$ is Hall-closed.

Now suppose that $\mathcal{Z} \subseteq \mathfrak{X}$ and that $\mathcal{Z}$ is Hall-closed. Then $\mathcal{Z} \subseteq \bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_{\sigma}(\mathcal{Z}) \subseteq \bigcap_{\sigma \subseteq \mathbb{P}} \mathcal{K}_{\sigma}(\mathfrak{X}) = \mathcal{Y}$, by (2.1.4d).

The result follows.

We observe that if $\mathfrak{X}$ is C-closed where $C \in \{S, S_n, S_F, Q, N_o, R_o\}$, then by (2.1.4e) and (1.2.4), it follows that the largest Hall-closed class contained in $\mathfrak{X}$ is also C-closed.

In section 2.8, examples of classes which are $H_{\mathcal{N}}$-closed but not Hall-closed will be given. The next proposition, which is primarily to investigate certain relationships between $\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{S}_{\pi'} * \mathcal{K}_\pi(\mathfrak{F})$ and $\mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathfrak{F})$, includes examples of classes which are not $H_{\mathcal{N}}$-closed.

<u>2.1.8 Proposition</u>. (a) $\mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}) = \mathcal{K}_\pi(\mathcal{N})$, for all $\pi \subseteq \mathbb{P}$.

(b) $\mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}^2) \subsetneqq \mathcal{K}_\pi(\mathcal{N}^2)$, whenever $\phi \neq \pi \subsetneqq \mathbb{P}$.

(c) If $p \in \mathbb{P}$, then $e_p(\mathcal{N}^2)$ is not $H_{\mathcal{N}}$-closed; in particular, there exists $\pi \subseteq \mathbb{P}$ with $\mathcal{S}_{\pi'} * \mathcal{L}_\pi(e_p(\mathcal{N}^2)) \nsubseteq \mathcal{K}_\pi(e_p(\mathcal{N}^2))$.

<u>Proof</u>. (a) Since $\mathcal{N}$ is S-closed, and so Hall-closed, then by (2.1.7c), $\mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}) \subseteq \mathcal{K}_\pi(\mathcal{N})$. Let $G \in \mathcal{K}_\pi(\mathcal{N})$, $H \in \mathrm{Hall}_\pi(G)$ and let $L = O_{\pi'}(G)$. Then $HL/L \in \mathrm{Hall}_\pi(G/L) \cap \mathcal{N}$. Clearly $F(G/L) \in \mathcal{S}_\pi$ and so $F(G/L) \leq HL/L$. By a well-known theorem of Fischer (see [41; VI. 7.18] (and [26; 6.18] or [47; 2.1.1/2] for a proof)), the $\mathcal{N}$-injectors of a group are precisely the $\mathcal{N}$-maximal subgroups containing the Fitting subgroup; in particular, HL/L is contained in an $\mathcal{N}$-injector of G/L and so $G/L \in \mathcal{L}_\pi(\mathcal{N})$ and $G \in \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N})$.

(b) Choose $p \in \pi$ and $q \notin \pi$. Let G be a group of type $M(p,q,p)$ (see (1.7.6)). A Hall π-subgroup of G is a Sylow p-subgroup, whence $G \in \mathcal{K}_\pi(\mathcal{N}^2)$. Now, $G^{\mathcal{N}}$, of index p in G, belongs to $\mathcal{N}^2$, and so must be an $\mathcal{N}^2$-injector of G, since $G \notin \mathcal{N}^2$. Since $O_{\pi'}(G) = 1$, then $G \notin \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}^2)$. (As in (a), the left-hand side is certainly contained in the right-hand).

(c) Choose primes q, r and s so that $|\{p, q, r, s\}| = 4$. Let G be a group of type $M(s^\gamma, r^\beta, p^\alpha, q^\delta)$, and take $\pi = \{p, q, s\}$.

Since $|G_{\mathcal{N}^2}| = s^{\gamma} r^{\beta}$, then $G \in e_p(\mathcal{N}^2)$. Let $H \in \mathrm{Hall}_{\pi}(G)$. Then $|H| = s^{\gamma} p^{\alpha} q$, and it is not hard to check that $|H_{\mathcal{N}^2}| = s^{\gamma} p^{\alpha}$, and that $H \notin e_p(\mathcal{N}^2)$. However, $H \geqslant G_{\mathcal{N}}$ and so $e_p(\mathcal{N}^2)$ is not $H_{\mathcal{N}}$-closed, while $G \in \mathcal{S}_{\pi'} * \mathcal{L}_{\pi}(e_p(\mathcal{N}^2)) \setminus \mathcal{K}_{\pi}(e_p(\mathcal{N}^2))$, as required.

We recall Lockett's result (2.1.2c) that $\mathfrak{F} = \mathcal{L}_{\pi}(\mathfrak{F})$ if and only if $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$. Since $\mathcal{K}_{\pi}(\mathfrak{F}) = \mathcal{S}_{\pi'} * \mathcal{K}_{\pi}(\mathfrak{F}) * \mathcal{S}_{\pi'}$, we might ask whether $\mathfrak{F} = \mathcal{S}_{\pi'} * \mathfrak{F} * \mathcal{S}_{\pi'}$ is sufficient to ensure that $\mathfrak{F} = \mathcal{K}_{\pi}(\mathfrak{F})$. It is not: taking $\mathfrak{F} = \mathcal{S}_{\pi'} * \mathcal{S}_{\pi} * \mathcal{S}_{\pi'}$, we have $\mathcal{K}_{\pi}(\mathfrak{F}) = \mathcal{S} \neq \mathfrak{F}$. To find necessary and sufficient conditions for $\mathfrak{F} = \mathcal{K}_{\pi}(\mathfrak{F})$, we define some new closure operations.

<u>2.1.9 Definition</u>. Let $\mathfrak{X}$ be a class of groups, and π be a set of primes. Let

$$S^{\pi}\mathfrak{X} = (G \in \mathcal{S} : \exists\, X \in \mathfrak{X} \text{ with } G \cong H \leqslant X \text{ and } |X : H| \text{ a } \pi\text{-number})\ ;$$

and $$O_{\pi}\mathfrak{X} = (G \in \mathcal{S} : \exists\, X \leqslant G,\ X \in \mathfrak{X}, \text{ with } |G : X| \text{ a } \pi\text{-number})\ .$$

It is not hard to check that S^{π} and O_{π} are closure operations.

<u>2.1.10 Proposition</u>. Let $\mathfrak{X}$ be a class of groups and π be a set of primes. Then $\mathfrak{X} = \mathcal{K}_{\pi}(\mathfrak{X})$ if and only if $\mathfrak{X} = \langle S^{\pi'}, O_{\pi'}\rangle\mathfrak{X}$.

<u>Proof</u>. It follows from the definitions that $\mathcal{K}_{\pi}(\mathfrak{X})$ is $\langle S^{\pi'}, O_{\pi'}\rangle$-closed. On the other hand, if $\mathfrak{X} = S^{\pi'}\mathfrak{X}$, then $\mathfrak{X} \subseteq \mathcal{K}_{\pi}(\mathfrak{X})$, while if $\mathfrak{X} = O_{\pi'}\mathfrak{X}$, then $\mathcal{K}_{\pi}(\mathfrak{X}) \subseteq \mathfrak{X}$, and the assertion follows.

We next investigate conditions under which inclusions of the type $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{G})$ or $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$, and so on, may occur. We recall from (2.1.4d) that if $\mathfrak{F} \subseteq \mathfrak{G}$, then $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{G})$, but it is not necessary that $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{G})$.

<u>2.1.11 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes and let π be a set of primes. Then $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{G})$ if and only if $\mathfrak{F} \cap \mathcal{S}_\pi \subseteq \mathfrak{G} \cap \mathcal{S}_\pi$.

<u>Proof</u>. If $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{G})$ then by (2.1.4b),

$$\mathfrak{F} \cap \mathcal{S}_\pi = \mathcal{K}_\pi(\mathfrak{F}) \cap \mathcal{S}_\pi \subseteq \mathcal{K}_\pi(\mathfrak{G}) \cap \mathcal{S}_\pi = \mathfrak{G} \cap \mathcal{S}_\pi .$$

If $\mathfrak{F} \cap \mathcal{S}_\pi \subseteq \mathfrak{G} \cap \mathcal{S}_\pi$ then by (2.1.4a,d),

$$\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F} \cap \mathcal{S}_\pi) \subseteq \mathcal{K}_\pi(\mathfrak{G} \cap \mathcal{S}_\pi) = \mathcal{K}_\pi(\mathfrak{G}) .$$

<u>2.1.12 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class and let π and τ be sets of primes. Then $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\tau(\mathfrak{F})$ if and only if the following two conditions both hold.

(a) $O_{\tau \setminus \pi}(\mathfrak{F} \cap \mathcal{S}_{\pi \cap \tau}) \subseteq \mathfrak{F}$; and

(b) $\mathfrak{F} \cap \mathcal{S}_\pi \subseteq \mathcal{K}_\tau(\mathfrak{F} \cap \mathcal{S}_\pi)$.

<u>Proof</u>. Suppose that $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\tau(\mathfrak{F})$. Let $G \in \mathcal{S}$ with $H \leq G$, $|G : H|$ a $(\tau \setminus \pi)$-number and $H \in \mathfrak{F} \cap \mathcal{S}_{\pi \cap \tau}$. Then $H \in \mathrm{Hall}_\pi(G)$, and so $G \in \mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\tau(\mathfrak{F})$. But $G \in \mathcal{S}_\tau$, whence $G \in \mathfrak{F}$, giving (a). Next let $G \in \mathfrak{F} \cap \mathcal{S}_\pi$, and let $H \in \mathrm{Hall}_\tau(G)$. Since $G \in \mathcal{K}_\pi(\mathfrak{F})$, which is contained in $\mathcal{K}_\tau(\mathfrak{F})$, then $H \in \mathfrak{F}$, and since $G \in \mathcal{S}_\pi$, then $H \in \mathfrak{F} \cap \mathcal{S}_\pi$, giving (b).

Suppose that (a) and (b) hold. Let $G \in \mathcal{K}_\pi(\mathfrak{F})$, and choose $H \in \mathrm{Hall}_\pi(G)$ and $K \in \mathrm{Hall}_\tau(G)$ such that $J = H \cap K \in \mathrm{Hall}_{\pi \cap \tau}(G)$. Since $G \in \mathcal{K}_\pi(\mathfrak{F})$, then $H \in \mathfrak{F} \cap \mathcal{S}_\pi$ and so $\mathcal{S}_{\pi \cap \tau} \ni J \in \mathfrak{F} \cap \mathcal{S}_\pi$, by (b), whence by (a), $K \in \mathfrak{F}$, giving $G \in \mathcal{K}_\tau(\mathfrak{F})$.

<u>2.1.13 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class, and let π and τ be sets of primes. Then $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$ if and only if $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\tau\setminus\pi}$.

<u>Proof</u>. Suppose that $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$. Certainly $\mathfrak{F} \subseteq \mathfrak{F} * \mathcal{S}_{\tau\setminus\pi}$; thus suppose that this inclusion is strict, and for a contradiction let G be a group of minimal order belonging to $\mathfrak{F} * \mathcal{S}_{\tau\setminus\pi} \setminus \mathfrak{F}$. Then G has a unique maximal normal subgroup $M = G_{\mathfrak{F}}$ of prime index, q say, where $q \in (\tau\setminus\pi)$. Further, M must be an $\mathfrak{F}$-injector of G. Since $|G : M| \notin \pi$, then M contains all Hall π-subgroups of G, and so $G \in \mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$. But then M must contain a Hall τ-subgroup of G, which is impossible since $|G : M| = q \in \tau$. It follows that $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\tau\setminus\pi}$.

Next suppose that $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\tau\setminus\pi}$, and for a contradiction let G be a group of minimal order in $\mathcal{L}_\pi(\mathfrak{F}) \setminus \mathcal{L}_\tau(\mathfrak{F})$. Then G has a unique maximal normal subgroup $M = G_{\mathcal{L}_\tau(\mathfrak{F})}$, of prime index, q say. Since $\mathcal{L}_\tau(\mathfrak{F}) = \mathcal{L}_\tau(\mathfrak{F}) * \mathcal{S}_{\tau'}$, then $q \in \tau$. Let T be an $\mathfrak{F}$-injector of G. Then $T \cap M$ is an $\mathfrak{F}$-injector of $M \in \mathcal{L}_\tau(\mathfrak{F})$, and so contains a Hall τ-subgroup of M. If $T \gneqq T \cap M$, then T must contain a Hall τ-subgroup of G, contrary to $G \notin \mathcal{L}_\tau(\mathfrak{F})$. Thus $T \leqslant M$. Since $G \in \mathcal{L}_\pi(\mathfrak{F})$, then $|G : T| = |G : M||M : T|$ is a π'-number, and so $q \notin \pi$. By the Frattini argument, $G = M\,N_G(T)$, and we may find a q-element $x \in N_G(T) \setminus M$. But then $T\langle x\rangle \in \mathfrak{F} * \mathcal{S}_{\tau\setminus\pi} = \mathfrak{F}$, since $q \in \tau\setminus\pi$, contradicting the $\mathfrak{F}$-maximality of T. This completes the proof.

<u>2.1.14 Corollary</u>. Let $\mathfrak{F}$ be a Fitting class, and let π and τ be sets of primes. Then $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\tau(\mathfrak{F})$ if and only if $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_\Delta$, where Δ is the symmetric difference $(\pi \setminus \tau) \cup (\tau \setminus \pi) = \pi \cup \tau \setminus \pi \cap \tau$.

<u>Proof</u>. We observe that if $\alpha, \beta \subseteq \mathbb{P}$ and $\mathfrak{G}$ is a Fitting class, then $\mathfrak{G} = \mathfrak{G} * \mathcal{S}_{\alpha \cup \beta}$ if and only if $\mathfrak{G} = \mathfrak{G} * \mathcal{S}_\alpha = \mathfrak{G} * \mathcal{S}_\beta$. This is because any group in $\mathcal{S}_{\alpha \cup \beta}$ belongs to $\mathcal{S}_\alpha * \mathcal{S}_\beta * \cdots * \mathcal{S}_\alpha * \mathcal{S}_\beta$ for a sufficiently large (finite) number of factors. The result now follows from (2.1.13).

The inclusion $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{G})$ has not lent itself to a complete analysis. Recalling Lockett's relation of strong inclusion " $<<$ ", (see (1.5.12)), we have the following.

<u>2.1.15 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes and let π be a set of primes. Then we have

(a) if $\mathfrak{F} << \mathfrak{G}$, then $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{G})$, but the converse need not hold ; and

(b) if $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{G})$, then $\mathfrak{F} \cap \mathcal{S}_\pi \subseteq \mathfrak{G} \cap \mathcal{S}_\pi$, but the converse need not hold.

<u>Proof</u>. (a) It is clear from the definitions that if $\mathfrak{F} << \mathfrak{G}$ then $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{G})$. For the required example, take $\pi = \{2, 3\}$, $\mathfrak{F} = \mathcal{S}_2 * \mathcal{S}_3$ and $\mathfrak{G} = \mathcal{S}_2 * \mathcal{S}_3 * \mathcal{S}_2$. Then $\mathfrak{F} \subseteq \mathfrak{G} \subseteq \mathcal{S}_\pi$ and $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{G}) = \mathcal{L}_\pi(\mathfrak{G})$ by (2.1.3a) and (2.1.4d). Now let G be a group of type $M(2^\gamma, 3^\beta, 2^2, 3)$, as in (1.7.6) A $\mathfrak{G}$-injector of G must coincide with the maximal normal subgroup of G of order $2^\gamma 3^\beta 2^2$, while an $\mathfrak{F}$-injector must have order $2^\gamma 3^{\beta+1}$,

subgroup of order $2^{\gamma} 3^{\beta}$ and have order $2^{\gamma} 3^{\beta+1}$; it follows that $\mathfrak{F} \not\leq \mathfrak{G}$.

(b) Suppose that $\mathcal{L}_{\pi}(\mathfrak{F}) \leq \mathcal{L}_{\pi}(\mathfrak{G})$, and let $H \in \mathfrak{F} \cap \mathcal{S}_{\pi}$. Then $H \leq \mathcal{L}_{\pi}(\mathfrak{F}) \leq \mathcal{L}_{\pi}(\mathfrak{G})$, and $H \in \mathfrak{G} \cap \mathcal{S}_{\pi}$. For the required example, take $\mathfrak{F} = \mathcal{S}_2$, and $\mathfrak{G} = \mathcal{S}_2 * \mathcal{S}_3$, so that $\mathfrak{F} \cap \mathcal{S}_2 = \mathfrak{G} \cap \mathcal{S}_2$ and $\mathcal{L}_2(\mathfrak{F}) = \mathcal{S}$. However, the group G in (a) above does not belong to $\mathcal{L}_2(\mathfrak{G})$.

It is desirable to relate injectors and radicals for $\mathcal{L}_{\pi}(\mathfrak{F})$ and $\mathcal{R}_{\pi}(\mathfrak{F})$ to the corresponding entities for $\mathfrak{F}$ in some way. In the next section, we will determine the $\mathcal{R}_{\pi}(\mathfrak{F})$-radical in this way. In the meantime, we quote a result of Lockett concerning the $\mathcal{L}_{\pi}(\mathfrak{F})$-injectors for certain classes $\mathfrak{F}$. This will have some bearing on (2.1.15).

<u>2.1.16 Proposition (Lockett [48:4.4])</u>. Let $\mathfrak{F}$ be a Fitting class and let G be a group. If the $\mathfrak{F}$-injector V of G permutes with the Hall π'-subgroup H of G , then $T = VH$ is an $\mathcal{L}_{\pi}(\mathfrak{F})$-injector of G .

Lockett, [48; page 109] , calls a Fitting class $\mathfrak{F} \leq \mathcal{S}$ <u>permutable</u> if it has the property that in each group $G \in \mathcal{S}$, an $\mathfrak{F}$-injector of G permutes with some Hall π'-subgroup of G for each $\pi \subseteq \mathbb{P}$. Thus, if $\mathfrak{F}$ is a permutable Fitting class, then an $\mathcal{L}_{\pi}(\mathfrak{F})$-injector of the group G is the product of an $\mathfrak{F}$-injector of G and a suitable Hall π'-subgroup of G. Lockett uses a result of Fischer to show [48;4.5] that if $\mathfrak{F}$ is a Fischer class then $\mathfrak{F}$ is permutable, and also shows, in section 5 of [48], how Dark's theorem (1.8.1) can be used to provide an

example of a non-permutable class. Indeed, Lockett also shows that (2.1.16) can fail for this class.

Because of (2.1.3a), then (2.1.16) can be used to describe the $\mathcal{K}_\pi(\mathfrak{F})$-injectors if $\mathfrak{F} \cap \mathcal{S}_\pi$ is permutable; we note that, by (2.1.2c), injectors for $\mathcal{L}_\pi(\mathfrak{F})$ and $\mathcal{K}_\pi(\mathfrak{F})$ always have π-index in G .

Using (2.1.16) and a result of Mann-Alperin quoted as 4.2 in [48] , we may show that if $\mathfrak{F} << \mathfrak{G}$ are permutable Fitting classes and π is a set of primes, then $\mathcal{L}_\pi(\mathfrak{F}) << \mathcal{L}_\pi(\mathfrak{G})$. We shall not be concerned with this aspect, and so omit the proof.

<u>2.1.17 Concluding remarks</u>. We defined $\mathcal{L}_{\mathfrak{G}}(\mathfrak{F})$ and $\mathcal{K}_{\mathfrak{G}}(\mathfrak{F})$ for arbitrary Fitting classes $\mathfrak{G} \subseteq \mathcal{S}$, in (2.1.1) , but our main interest is in the case that $\mathfrak{G} = \mathcal{S}_\pi$, where $\pi \subseteq \mathbb{P}$.

Lockett, on page 16 of his thesis [47] , observes that Sym(3) $\times$ C_2 may be regarded as a normal product of two copies of Sym(3) with their Sylow 3-subgroups coincident, and thus that

$$\mathrm{Sym}(3) \times C_2 \in N_0(\mathcal{K}_{\mathcal{N}}(\mathcal{S}_3)) \setminus \mathcal{K}_{\mathcal{N}}(\mathcal{S}_3) ,$$

so that $\mathcal{K}_{\mathfrak{G}}(\mathfrak{F})$ and $\mathcal{L}_{\mathfrak{G}}(\mathfrak{F})$ need not be Fitting classes.

In his dissertation [34] , P. Hauck investigates conditions under which $\mathcal{K}_{\mathfrak{G}}(\mathfrak{F})$ is a Fitting class; he finds that if there exists $\pi \subseteq \mathbb{P}$ such that $\mathfrak{G} = \mathfrak{G} * \mathcal{S}_\pi$ and $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$, then $\mathcal{K}_{\mathfrak{G}}(\mathfrak{F})$ is a Fitting class.

We note that if we define

$$\mathcal{J}_\pi(\mathfrak{F}) = (G \in \mathcal{S} : \text{ Each Hall } \pi\text{-subgroup of } G \text{ contains an } \mathfrak{F}\text{-injector of } G) ,$$

then, as above, Sym(3) $\times$ $C_2 \in N_0(\mathcal{J}_3(\mathcal{N})) \setminus \mathcal{J}_3(\mathcal{N})$, and so $\mathcal{J}_\pi(\mathfrak{F})$ need not be a Fitting class.

2.2 The $\mathcal{K}_\pi(\mathfrak{F})$-radical.

In this section, we show that the $\mathcal{K}_\pi(\mathfrak{F})$-radical of a group G has a very close relationship with the $\mathfrak{F}$-radical of a Hall π-subgroup of G. We give two descriptions of the $\mathcal{K}_\pi(\mathfrak{F})$-radical of G; one, (2.2.2a), in terms of the normal closure of the $\mathfrak{F}$-radical of a Hall π-subgroup of G; the other, (2.2.3), an iterative construction in terms of the cores of certain subgroups of factor groups of G.

<u>2.2.1 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class, π be a set of primes, G be a (soluble) group and H a member of $\mathrm{Hall}_\pi(G)$.

Then $G_{\mathcal{K}_\pi(\mathfrak{F})} \cap H = H_{\mathfrak{F}}$.

<u>Proof</u>. Let $K = G_{\mathcal{K}_\pi(\mathfrak{F})}$. By (1.1.2), $H \cap K \in \mathrm{Hall}_\pi(K) \subseteq \mathfrak{F}$. Since $H \cap K \trianglelefteq H$, it follows that $H \cap K \leq H_{\mathfrak{F}}$.

Let $F/K = F(G/K)$. Since $\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$ by (2.1.4a), then F/K must be a π-group, and so $F/K \leq HK/K \in \mathrm{Hall}_\pi(G/K)$, since $F/K \trianglelefteq G/K$. Because $H \cap K \leq H_{\mathfrak{F}}$, it follows by a straightforward argument on orders that $H_{\mathfrak{F}} \in \mathrm{Hall}_\pi(K.H_{\mathfrak{F}})$, whence $K.H_{\mathfrak{F}} \in \mathcal{K}_\pi(\mathfrak{F})$. Since $F \trianglelefteq G$, then $K.H_{\mathfrak{F}} \cap F \in S_n\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F})$, while since F/N is nilpotent then the group $K.H_{\mathfrak{F}} \cap F$ is subnormal in G, and is thus contained in K. By above, $F \leq HK$ and so $F \trianglelefteq HK$. Since $K.H_{\mathfrak{F}}$ is also normal in HK, then $[K.H_{\mathfrak{F}}, F] \leq K.H_{\mathfrak{F}} \cap F \leq K$. Thus, $H_{\mathfrak{F}}/K$ centralizes $F(G/K)$, whence by (1.1.1f) it follows that $H_{\mathfrak{F}} \leq H \cap K.H_{\mathfrak{F}} \cap F \leq H \cap K$, and the result follows.

The above proposition has some interesting consequences. We recall some familiar definitions.

Let A be a subgroup of the group G.

If for all $g \in G$ we have that A and A^g are conjugate in $\langle A, A^g \rangle$, then A is said to be <u>pronormal</u> in G. (See exercises 4 and 6 on pages 13 and 14 of [28]).

If π is a set of primes, A is said to be <u>π-normally embedded</u> in G (written <u>A π-ne G</u>) if a Hall π-subgroup of A is a Hall π-subgroup of a suitable normal subgroup of G. If $\pi = \{p\}$, we omit the braces as usual. Lockett [47; page 55] calls a subgroup which is p-normally embedded for all primes p <u>strongly pronormal</u>, and shows that such a subgroup is pronormal (see [47; 3.2.1]).

Finally, we recall from [29; page 205] that if $A \leqslant B \leqslant G$, then A is said to be <u>strongly closed in B with respect to G</u> if for all $g \in G$ we have $B \cap A^g \leqslant A$.

We note that a subgroup can be p-normally embedded for all $p \in \pi$, where $\pi \subseteq \mathbb{P}$, while not being π-normally embedded. For take $G = \mathrm{Alt}(4) \times \mathrm{Sym}(3)$. Then G has a cyclic subgroup, C, of order 6, which complements G', such that $C = (C \cap \mathrm{Alt}(4)) \times (C \cap \mathrm{Sym}(3))$. Now $C \cap \mathrm{Alt}(4) \in \mathrm{Syl}_3(C) \cap \mathrm{Syl}_3(\mathrm{Alt}(4))$, and

$$C \cap \mathrm{Sym}(3) \in \mathrm{Syl}_2(C) \cap \mathrm{Syl}_2(\mathrm{Sym}(3)),$$

whence C 2-ne G and C 3-ne G. But C is certainly not $\{2,3\}$-normally embedded in G.

<u>2.2.2 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class, π be a set of primes, G be a group and H a member of $\mathrm{Hall}_\pi(G)$. Then

(a) $G_{\mathcal{K}_\pi(\mathfrak{F})} = K$, where $K/\langle H_{\mathfrak{F}}^G\rangle = O_{\pi'}(G/\langle H_{\mathfrak{F}}^G\rangle)$;

(b) $H_{\mathfrak{F}} = H \cap \langle H_{\mathfrak{F}}^G\rangle \in \mathrm{Hall}\,(\langle H_{\mathfrak{F}}^G\rangle)$;

(c) $H_{\mathfrak{F}}$ τ-ne G for all $\tau \subseteq \pi$ and $H_{\mathfrak{F}}$ p-ne G for all $p \in \mathbb{P}$;

(d) $H_{\mathfrak{F}}$ is strongly closed in H with respect to G ; and

(e) $G = G_{\mathcal{K}_\pi(\mathfrak{F})} \cdot N_G(H_{\mathfrak{F}})$;

where $\langle H_{\mathfrak{F}}^G\rangle$ denotes (as usual) the normal closure of $H_{\mathfrak{F}}$ in G .

<u>Proof</u>. (a) Since $H_{\mathfrak{F}} \leq G_{\mathcal{K}_\pi(\mathfrak{F})} \trianglelefteq G$ by (2.2.1), then $\langle H_{\mathfrak{F}}^G\rangle \leq G_{\mathcal{K}_\pi(\mathfrak{F})}$. Define $K \trianglelefteq G$ by $K/\langle H_{\mathfrak{F}}^G\rangle = O_{\pi'}(G/\langle H_{\mathfrak{F}}^G\rangle)$. Since $\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$, then $K \leq G_{\mathcal{K}_\pi(\mathfrak{F})}$; on the other hand, by (2.2.1), $H_{\mathfrak{F}} \in \mathrm{Hall}_\pi(G_{\mathcal{K}_\pi(\mathfrak{F})})$, and so $G_{\mathcal{K}_\pi(\mathfrak{F})}/\langle H_{\mathfrak{F}}^G\rangle$, being thus a π'-group, is contained in $O_{\pi'}(G/\langle H_{\mathfrak{F}}^G\rangle)$.

(b) Since $H_{\mathfrak{F}} \leq \langle H_{\mathfrak{F}}^G\rangle \leq G_{\mathcal{K}_\pi(\mathfrak{F})}$ and $H_{\mathfrak{F}} = H \cap G_{\mathcal{K}_\pi(\mathfrak{F})}$ by (2.2.1), then $H_{\mathfrak{F}} = H \cap \langle H_{\mathfrak{F}}^G\rangle$.

(c) Suppose that $\tau \subseteq \pi$ and that $T \in \mathrm{Hall}_\tau(H_{\mathfrak{F}})$. Now, $T \leq \langle H_{\mathfrak{F}}^G\rangle \trianglelefteq G$ and $|\langle H_{\mathfrak{F}}^G\rangle : H_{\mathfrak{F}}|$ is a π'-number by (a). But $|H_{\mathfrak{F}} : T|$ is a τ'-number and since $\tau \subseteq \pi$ then $T \in \mathrm{Hall}_\tau(\langle H_{\mathfrak{F}}^G\rangle)$, whence $H_{\mathfrak{F}}$ τ-ne G . Suppose that $p \in \mathbb{P}$. If $p \in \pi$, then H p-ne G by the preceding. If $p \notin \pi$ then a Sylow p-subgroup of $H_{\mathfrak{F}}$ is trivial and so normal in G , completing the proof.

(d) If $g \in G$, then $H \cap H_{\mathfrak{F}}^g \leq H \cap \langle H_{\mathfrak{F}}^G\rangle = H_{\mathfrak{F}}$, by (b).

(e) This follows by the Frattini argument.

We next give an iterative construction which provides an alternative description of $G_{\mathcal{K}_\pi(\mathfrak{F})}$ to that of (2.2.2a) .

<u>2.2.3 Construction</u>. Let $\mathfrak{F}$ be a Fitting class, π be a set of primes, G be a group and H be a member of $\mathrm{Hall}_\pi(G)$.

Define $J_1, K_1 \trianglelefteq G$ by

$$J_1 = O_{\pi'}(G) \quad ; \text{ and}$$

$$K_1/J_1 = \mathrm{Core}_{G/J_1}(H_{\mathfrak{F}}J_1/J_1) .$$

Note that $K_1 \cap H \leq (H_{\mathfrak{F}}J_1) \cap H = H_{\mathfrak{F}}(J_1 \cap H) = H_{\mathfrak{F}}$, since $J_1 \cap H = 1$.

Now suppose that $J_i, K_i \trianglelefteq G$ have been defined, and that $K_i \cap H \leq H_{\mathfrak{F}}$. Define $J_{i+1}, K_{i+1} \trianglelefteq G$ by

$$J_{i+1} = O_{\pi'}(G/K_i) \quad ; \text{ and}$$

$$K_{i+1}/J_{i+1} = \mathrm{Core}_{G/J_{i+1}}(H_{\mathfrak{F}}J_{i+1}/J_{i+1}) .$$

Note that $(K_{i+1} \cap H) \leq (H_{\mathfrak{F}}J_{i+1}) \cap H = H_{\mathfrak{F}}(J_{i+1} \cap H)$. But $J_{i+1}/K_i \in \mathcal{S}_{\pi'}$, and so

$$(J_{i+1} \cap H)/(K_i \cap H) \cong K_i(J_{i+1} \cap H)/K_i \in \mathcal{S}_{\pi'} \cap \mathcal{S}_\pi ,$$

whence $(J_{i+1} \cap H) \leq (K_i \cap H) \leq H_{\mathfrak{F}}$ and $(K_{i+1} \cap H) \leq H_{\mathfrak{F}}$.

Since G is finite, there exists an integer n such that $J_n = K_n = K$, say. We claim that $K = G_{\mathcal{K}_\pi(\mathfrak{F})}$. Certainly $K \trianglelefteq G$, and by construction $K \cap H \leq H_{\mathfrak{F}}$, whence $K \cap H \in \mathfrak{F}$. But $K \cap H$ belongs to $\mathrm{Hall}_\pi(K)$, and so $K \in \mathcal{K}_\pi(\mathfrak{F})$ and $K \leq G_{\mathcal{K}_\pi(\mathfrak{F})}$. Suppose for a contradiction that $K \lneq G_{\mathcal{K}_\pi(\mathfrak{F})}$. Since $K = K_n = J_n$, then $O_{\pi'}(G/K)$ is trivial. Thus there exists $X \trianglelefteq G$ with X/K a G-chief π-factor and $X \leq G_{\mathcal{K}_\pi(\mathfrak{F})}$. But then $X \in \mathcal{K}_\pi(\mathfrak{F})$ and, since $X \cap H \in \mathrm{Hall}_\pi(X)$, it follows that $X \cap H \in \mathfrak{F}$. But $X \cap H \trianglelefteq H$ and so $X \cap H \leq H_{\mathfrak{F}}$.

However, $X/K \leq HK/K \triangleleft \mathrm{Hall}_\pi(G/K)$, and so $X = X \cap HK = K(X \cap H) \leq H_{\mathfrak{F}}K$, whence $1 \lneq X/K \leq H_{\mathfrak{F}}K/K$, while $\mathrm{Core}_{G/K}(H_{\mathfrak{F}}K/K) = 1$ by choice of K . This is a contradiction, and so the subgroup K as constructed must coincide with $G_{\mathcal{K}_\pi(\mathfrak{F})}$.

2.3 Multiplicative properties of $\mathcal{K}_\pi(\ )$ and $\mathcal{L}_\pi(\ )$.

2.3.1 Theorem. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, and let π be a set of primes. Then $\mathcal{K}_\pi(\mathfrak{F} * \mathfrak{G}) = \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{K}_\pi(\mathfrak{G})$.

Proof. Suppose that $G \in \mathcal{S}$ and that $H \in \mathrm{Hall}_\pi(G)$. By (2.2.1) , $H_{\mathfrak{F}} = H \cap G_{\mathcal{K}_\pi(\mathfrak{F})}$, and so $H/H_{\mathfrak{F}} \cong (H\, G_{\mathcal{K}_\pi(\mathfrak{F})})/G_{\mathcal{K}_\pi(\mathfrak{F})}$, which belongs to $\mathrm{Hall}_\pi(G/G_{\mathcal{K}_\pi(\mathfrak{F})})$. But then $G/G_{\mathcal{K}_\pi(\mathfrak{F})} \in \mathcal{K}_\pi(\mathfrak{G})$ if and only if $H/H_{\mathfrak{F}} \in \mathfrak{G}$, and it follows that $G \in \mathcal{K}_\pi(\mathfrak{F} * \mathfrak{G})$ if and only if $G \in \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{K}_\pi(\mathfrak{G})$, completing the proof .

The situation for $\mathcal{L}_\pi(\ )$ is more complicated; we start with some examples.

2.3.2 Examples. (a) $\mathcal{L}_\pi(\mathcal{N}) * \mathcal{L}_\pi(\mathcal{N}) \nsubseteq \mathcal{L}_\pi(\mathcal{N}^2)$ if $\varnothing \neq \pi \subsetneq \mathbb{P}$.

(b) $\mathcal{L}_\pi(\mathcal{N} * e_p(\mathcal{N}^2)) \nsubseteq \mathcal{L}_\pi(\mathcal{N}) * \mathcal{L}_\pi(e_p(\mathcal{N}^2))$ if $\{p\} \subsetneq \pi \subsetneq \mathbb{P}$.

Proof. (a) Let π be as specified, and suppose that $\mathcal{L}_\pi(\mathcal{N}) * \mathcal{L}_\pi(\mathcal{N})$ is contained in $\mathcal{L}_\pi(\mathcal{N}^2)$. Then by (2.3.1), (2.1.8a) and (2.1.2b) ,

$$\mathcal{K}_\pi(\mathcal{N}^2) = \mathcal{K}_\pi(\mathcal{N}) * \mathcal{K}_\pi(\mathcal{N}) = \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}) * \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}) =$$

$$\mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}) * \mathcal{L}_\pi(\mathcal{N}) \subseteq \mathcal{S}_{\pi'} * \mathcal{L}_\pi(\mathcal{N}^2)$$, contrary to (2.1.8b) .

(b) Let $q \in \pi \setminus \{p\}$ and $r \in \pi' \setminus \pi$. Let G be a group of type $M(q^{\delta}, r^{\gamma}, q^{\beta}, p^{\alpha}, q)$, as in (1.7.5/6). Then $|(G/F(G))_{\mathcal{N}^2}| = r^{\gamma} q^{\beta}$, and so $G/F(G) \in e_p(\mathcal{N}^2)$. Thus $G \in \mathcal{N} * e_p(\mathcal{N}^2) \subseteq \mathcal{L}_{\pi}(\mathcal{N} * e_p(\mathcal{N}^2))$. Since $q \in \pi$ and $r \notin \pi$, then $|G_{\mathcal{L}_{\pi}(\mathcal{N})}| = q^{\delta} r^{\gamma}$, while $G/G_{\mathcal{L}_{\pi}(\mathcal{N})}$, of type $M(q^{\beta}, p^{\alpha}, q)$, is a π-group and does not belong to $e_p(\mathcal{N}^2)$ since the factor of order p^{α} is not central. Thus $G \notin \mathcal{L}_{\pi}(\mathcal{N}) * \mathcal{L}_{\pi}(e_p(\mathcal{N}^2))$, and the assertion is proved.

To prove our positive results on the multiplicative properties of $\mathcal{L}_{\pi}(\ )$, we will need Lockett's determination of the injectors for a product of Fitting classes, as follows.

<u>2.3.3 Theorem (Lockett. [48;3.2])</u>.. Let $\mathfrak{F}$ and $\mathcal{G}$ be Fitting classes and G be a group. Suppose that $\mathrm{char}(\mathcal{G}) = \tau$. Let T be an $\mathfrak{F}$-injector of $G_{\mathcal{L}_{\tau}(\mathfrak{F})}$. By the Frattini argument and the definition of $\mathcal{L}_{\tau}(\mathfrak{F})$, T is normalized by some Hall τ-subgroup, H say, of G. Let V/T be a $\mathcal{G}$-injector of TH/T. Then V is an $\mathfrak{F} * \mathcal{G}$-injector of G, and each $\mathfrak{F} * \mathcal{G}$-injector of G is of this form.

<u>2.3.4 Lemma.</u> In the situation of (2.3.3), and letting L denote $G_{\mathcal{L}_{\tau}(\mathfrak{F})}$, we have

(a) $L \cap H = T \cap H$, $L \cap TH = T$, and $LH/L \cong TH/T$; and

(b) LV/L is a $\mathcal{G}$-injector of G/L.

<u>Proof</u>. Since T is an $\mathfrak{F}$-injector of $L \in \mathcal{L}_{\tau}(\mathfrak{F})$, then $\mathrm{Hall}_{\tau}(T) \subseteq \mathrm{Hall}_{\tau}(L)$. Now, $T \trianglelefteq TH$, and so $H \cap T \in \mathrm{Hall}_{\tau}(T) \subseteq \mathrm{Hall}_{\tau}(L)$, while

since $H \cap L \in \mathcal{S}_\tau$ then $L \cap H = T \cap H$. Thus $L \cap TH = T(L \cap H) = T$. Further, $TH/T = TH/(L \cap TH) \cong LTH/L = LH/L$, proving (a).

Now, $T \leq V \leq TH$, and in the above isomorphism, $V/T \to LV/L$. It follows that LV/L is a $\mathcal{G}$-injector of LH/L. If now W/L is any $\mathcal{G}$-injector of G/L then, since $\mathrm{char}(\mathcal{G}) = \tau$, W/L must be contained in some Hall τ-subgroup K/L of G/L, and so W/L is a $\mathcal{G}$-injector of K/L. But $LH/L \in \mathrm{Hall}_\tau(G/L)$. Thus by Sylow's theorem and the conjugacy of injectors, LV/L is conjugate in G/L to W/L, and is thus a $\mathcal{G}$-injector of G/L, proving (b).

<u>2.3.5 Proposition</u>. Let $\mathcal{F}$ and $\mathcal{G}$ be Fitting classes with $\mathrm{char}(\mathcal{G}) = \tau$, and let π be a set of primes. Then

(a) $\mathcal{L}_\pi(\mathcal{F} * \mathcal{G}) \subseteq \mathcal{L}_\tau(\mathcal{F}) * \mathcal{L}_\pi(\mathcal{G})$; and

(b) if either (1) $\mathcal{L}_\pi(\mathcal{G})$ is Q-closed, or

(2) $\mathcal{L}_\pi(\mathcal{F}) \subseteq \mathcal{L}_\tau(\mathcal{F})$,

then $\mathcal{L}_\pi(\mathcal{F} * \mathcal{G}) \subseteq \mathcal{L}_\pi(\mathcal{F}) * \mathcal{L}_\pi(\mathcal{G})$.

<u>Proof</u>. Let $G \in \mathcal{L}_\pi(\mathcal{F} * \mathcal{G})$ and let $L = G_{\mathcal{L}_\tau(\mathcal{F})}$. Let T be an $\mathcal{F}$-injector of L. By the Frattini argument, $G = L\,N_G(T)$ and, since $|G : N_G(T)| = |L : L \cap N_G(T)|$, which is a divisor of the τ'-number $|L : T|$, then $N_G(T)$ must contain some $H_\tau \in \mathrm{Hall}_\tau(G)$.

Let V/T be a $\mathcal{G}$-injector of TH/T so that by (2.3.3) V is an $\mathcal{F} * \mathcal{G}$-injector of G. Since $G \in \mathcal{L}_\pi(\mathcal{F} * \mathcal{G})$, there exists $H_\pi \in \mathrm{Hall}_\pi(G)$ with $H_\pi \leq V \leq T\,H_\tau$.

By (2.3.4b), LV/L is a $\mathcal{G}$-injector of G/L, and since $H_\pi \leq V$, then $LV/L \geq L\,H_\pi/L \in \mathrm{Hall}_\pi(G/L)$. Thus $G/L \in \mathcal{L}_\pi(\mathcal{G})$, and so $G \in \mathcal{L}_\tau(\mathcal{F}) * \mathcal{L}_\pi(\mathcal{G})$, proving (a).

Using (2.3.4a) and the fact that $H_\pi \leq T H_\tau$, we obtain $T = L \cap T H_\tau \geq L \cap H_\pi \in \mathrm{Hall}_\pi(L)$, whence $L \in \mathcal{L}_\pi(\mathfrak{F})$, and so $L = G_{\mathcal{L}_\tau(\mathfrak{F})} \leq G_{\mathcal{L}_\pi(\mathfrak{F})}$.

If now $\mathcal{L}_\pi(\mathfrak{G})$ is Q-closed, then $G \in \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathfrak{G})$, while if $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$, then $L = G_{\mathcal{L}_\pi(\mathfrak{F})}$ and so again we have $G \in \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathfrak{G})$; completing the proof.

<u>2.3.6 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes with $\mathrm{char}(\mathfrak{G}) = \tau$, and let π be a set of primes. Suppose that $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$. Then $\mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathfrak{G}) \subseteq \mathcal{L}_\pi(\mathfrak{F} * \mathfrak{G})$.

<u>Proof</u>. Suppose for a contradiction that G is a group of minimal order in $\mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathfrak{G}) \setminus \mathcal{L}_\pi(\mathfrak{F} * \mathfrak{G})$. Then G has a unique maximal normal subgroup $M = G_{\mathcal{L}_\pi(\mathfrak{F} * \mathfrak{G})}$ of prime index, q say, in G. By (2.1.2b), $q \in \pi$.

Suppose that $G = G_{\mathcal{L}_\tau(\mathfrak{F})}$. Let T be an $\mathfrak{F}$-injector of G; then T contains, and so is normalized by, a group $H_\tau \in \mathrm{Hall}_\tau(G)$, and it follows by (2.3.3) that in this case T is an $\mathfrak{F} * \mathfrak{G}$-injector of G. Since $G \notin \mathcal{L}_\pi(\mathfrak{F} * \mathfrak{G})$ then T cannot contain a Hall π-subgroup of G, and so $G \notin \mathcal{L}_\pi(\mathfrak{F})$. Since $M \in \mathcal{L}_\pi(\mathfrak{F} * \mathfrak{G})$, the $\mathfrak{F} * \mathfrak{G}$-injector $T \cap M$ of M contains a Hall π-subgroup of M. If $T \nleq M$, then T must contain a Hall π-subgroup of G, contrary to the above. Thus $T < M$. But now T is an $\mathfrak{F}$-injector of M containing a Hall π-subgroup of M, whence $M \in \mathcal{L}_\pi(\mathfrak{F})$, and so $M = G_{\mathcal{L}_\pi(\mathfrak{F})}$, since $G \notin \mathcal{L}_\pi(\mathfrak{F})$. But $G \in \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathfrak{G})$, and so $C_q \cong G/M \in \mathcal{L}_\pi(\mathfrak{G})$. Since $q \in \pi$,

then $C_q \in \mathcal{G}$ and $q \in \tau$. This contradicts the fact that $G \in \mathcal{L}_\tau(\mathfrak{F})$, since the $\mathfrak{F}$-injector T of G is contained in M. Thus $G \notin \mathcal{L}_\tau(\mathfrak{F})$.

Since $G \notin \mathcal{L}_\tau(\mathfrak{F})$ and M is the unique maximal normal subgroup of G, then $G_{\mathcal{L}_\tau(\mathfrak{F})} = M_{\mathcal{L}_\tau(\mathfrak{F})} = L$, say. We will construct $\mathfrak{F} * \mathcal{G}$-injectors of M and G. Thus, let T be an $\mathfrak{F}$-injector of L and choose $H_\tau \in \mathrm{Hall}_\tau(G)$ with $H_\tau \leq N_G(T)$, as in (2.3.3). Let $H_\tau^o = M \cap H_\tau \leq N_M(T)$. Let V/T be a $\mathcal{G}$-injector of $T H_\tau/T$, whence $V^o/T = V/T \cap (T H_\tau^o)/T$ is a $\mathcal{G}$-injector of $(T H_\tau^o)/T$. By (2.3.3), V is an $\mathfrak{F} * \mathcal{G}$-injector of G and V^o is an $\mathfrak{F} * \mathcal{G}$-injector of M; of course, $V^o = V \cap M$. Since $M \in \mathcal{L}_\pi(\mathfrak{F} * \mathcal{G})$, then V^o contains a Hall π-subgroup of M, and so $V^o \cap L$ contains a Hall π-subgroup of $L \trianglelefteq M$. But $T \leq V^o \cap L \leq V \cap L \leq TH_\tau \cap L = T$, by (2.3.4a), and so $L \in \mathcal{L}_\pi(\mathfrak{F})$. Since $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$ by hypothesis, then $G_{\mathcal{L}_\tau(\mathfrak{F})} = L = G_{\mathcal{L}_\pi(\mathfrak{F})}$, and since $G \in \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathcal{G})$ we have $G/L \in \mathcal{L}_\pi(\mathcal{G})$. But by (2.3.4b), LV/L is a $\mathcal{G}$-injector of G/L, whence LV must contain a Hall π-subgroup of G. Now $|LV : V| = |L : L \cap V| = |L : T|$ is a π'-number, and so V must contain a Hall π-subgroup of G, contrary to $G \notin \mathcal{L}_\pi(\mathfrak{F} * \mathcal{G})$. This completes the proof.

<u>2.3.7 Proposition</u>. Let $\mathfrak{F}$ and $\mathcal{G}$ be Fitting classes with $\mathrm{char}(\mathcal{G}) = \tau$, and let π be a set of primes. Suppose that $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$. Then $\mathcal{L}_\pi(\mathfrak{F} * \mathcal{G}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathcal{G})$.

<u>Proof</u>. This follows immediately from (2.3.5b) and (2.3.6).

2.3.8 Remarks. (a) We saw in (2.1.13) that $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$ if and only if $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\tau \setminus \pi}$. From this, or (2.1.4h), it follows that if $\tau \subseteq \pi$, then $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\tau(\mathfrak{F})$. Now let $\mathfrak{X}$ and $\mathcal{Y}$ be arbitrary Fitting classes, and let $\mathfrak{F} = \mathfrak{X} \cap \mathcal{S}_\pi$, $\mathcal{G} = \mathcal{Y} \cap \mathcal{S}_\pi$, where $\pi \subseteq \mathbb{P}$. It is not hard to check that $\mathfrak{F} * \mathcal{G} = (\mathfrak{X} * \mathcal{Y}) \cap \mathcal{S}_\pi$. Since now $\mathrm{char}(\mathcal{G}) \subseteq \pi$, (2.3.7) is applicable, and with (2.1.3) we obtain $\mathcal{K}_\pi(\mathfrak{X} * \mathcal{Y}) = \mathcal{L}_\pi((\mathfrak{X} * \mathcal{Y}) \cap \mathcal{S}_\pi) = \mathcal{L}_\pi(\mathfrak{F} * \mathcal{G}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_\pi(\mathcal{G}) = \mathcal{K}_\pi(\mathfrak{X}) * \mathcal{K}_\pi(\mathcal{Y})$, leading to an alternative proof of (2.3.1).

(b) The opposite inclusion to that of (2.3.5a) need not hold. For take $\mathfrak{F} = \mathcal{G} = \mathcal{S}_3$, $\tau = \{3\}$ and $\pi = \{2\}$. Then $\mathrm{Sym}(3) \notin \mathcal{L}_2(\mathcal{S}_3 * \mathcal{S}_3)$, while $\mathcal{L}_\tau(\mathfrak{F}) * \mathcal{L}_\pi(\mathcal{G}) = \mathcal{S} * \mathcal{S}_{2'} = \mathcal{S}$, as required.

Theorem (2.3.1) may be used in a "New from old" construction of Hall- and $H_{\mathcal{N}}$-closed Fitting classes, as follows.

2.3.9 Proposition. Let $\mathfrak{F}$ and $\mathcal{G}$ be Fitting classes.

(a) If $\mathfrak{F}$ and $\mathcal{G}$ are Hall-closed and if $\mathcal{G}$ is also Q-closed, then $\mathfrak{F} * \mathcal{G}$ is Hall-closed.

(b) If $\mathfrak{F}$ and $\mathcal{G}$ are $H_{\mathcal{N}}$-closed and if $\mathcal{G}$ is also Q-closed, then $\mathfrak{F} * \mathcal{G}$ is $H_{\mathcal{N}}$-closed.

Proof. (a) Let $\pi \subseteq \mathbb{P}$ be arbitrary. Since $\mathcal{G}$ is Q-closed and $\mathfrak{F}$ and $\mathcal{G}$ are Hall-closed, then

$$\begin{aligned} \mathfrak{F} * \mathcal{G} &\subseteq \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{G} && \text{by (2.1.7a) and (1.3.13c)} \\ &\subseteq \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{K}_\pi(\mathcal{G}) && \text{by (2.1.7a) and (1.3.13b)} \\ &= \mathcal{K}_\pi(\mathfrak{F} * \mathcal{G}) && \text{by (2.3.1)}, \end{aligned}$$

as required, in view of (2.1.7a).

(b) Let $\pi \subseteq \mathbb{P}$ be arbitrary. Since $\mathcal{G}$ is $\langle Q, H_{\mathcal{N}} \rangle$-closed, $\mathcal{G}$ is also Hall-closed by (2.1.7d). Since $\mathfrak{F}$ is $H_{\mathcal{N}}$-closed and $\mathcal{G}$ is Q-closed, then by (2.1.7b) and (1.3.13c), $\mathcal{S}_{\pi'} * \mathfrak{F} * \mathcal{G} \subseteq \mathcal{K}(\mathfrak{F}) * \mathcal{G}$. Since $\mathcal{G}$ is Hall-closed, $\mathcal{K}_{\pi}(\mathfrak{F}) * \mathcal{G} \subseteq \mathcal{K}_{\pi}(\mathfrak{F}) * \mathcal{K}_{\pi}(\mathcal{G}) = \mathcal{K}_{\pi}(\mathfrak{F} * \mathcal{G})$, by (2.1.7a), (1.3.13b) and (2.3.1), proving the assertion in view of (2.1.7a).

In (3.5.1) it is shown that $\mathcal{N}_{\pi} * \mathfrak{A}$ is not $H_{\mathcal{N}}$-closed if $|\pi| \geq 2$, even though $\mathcal{N}_{\pi}$ and $\mathfrak{A}$ (the smallest normal Fitting class) are both Hall-closed, and so some such condition as $\mathcal{G}$ is Q-closed would seem to be necessary in the above.

2.4 Lockett's "Upper-star" operation.

We recall from section 1.5 the definitions of Lockett's upper star operation and of a Lockett class.

In his dissertation [34], P. Hauck proves (Satz 6.6) that if $\mathfrak{F}$ is a Lockett class, $\mathcal{G}$ is a Fitting class and π is a set of primes such that $\mathcal{G} * \mathcal{S}_{\pi}$ is a Lockett class, then $\mathcal{K}_{\mathcal{G} * \mathcal{S}_{\pi}}(\mathfrak{F} * \mathcal{S}_{\pi'})$ is also a Lockett class, in our notation of (2.1.1). If we take $\mathcal{G} = (1)$ in this result, it follows that $\mathcal{K}_{\pi}(\mathfrak{F} * \mathcal{S}_{\pi'})$ is a Lockett class if $\mathfrak{F}$ is. But application of (2.1.4a) shows that $\mathcal{K}_{\pi}(\mathfrak{F} * \mathcal{S}_{\pi'}) = \mathcal{K}_{\pi}(\mathfrak{F})$, and so it follows that $\mathcal{K}_{\pi}(\mathfrak{F})$ is a Lockett class if $\mathfrak{F}$ is.

Hauck also observes [34; 8.3] that it follows from Lockett's result (2.1.16) together with a "converse" (similar to (1.5.5)) to (1.5.3j) that

if $\mathfrak{F}$ is a permutable (see page 51) Lockett class, then $\mathcal{L}_\pi(\mathfrak{F})$ is also a Lockett class.

We start this section by showing that if $\mathfrak{F}$ is an arbitrary Lockett class, then $\mathcal{L}_\pi(\mathfrak{F})$ is again a Lockett class, which implies the corresponding result for $\mathcal{K}_\pi(\mathfrak{F})$. We go on to show that, in fact, both $\mathcal{L}_\pi(\ )$ and $\mathcal{K}_\pi(\ )$ "commute" with the upper star operation, results which seem to have been previously unsuspected, and draw several conclusions. We end the section with another result on the connection between the upper-star operation and what might be termed the "Hall π-properties" of Fitting classes by showing that if $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes of co-prime characteristics, then $(\mathfrak{F}*\mathfrak{G})^* = \mathfrak{F}^* * \mathfrak{G}^*$.

<u>2.4.1 Proposition</u>. Let $\mathfrak{F}$ be a Lockett class and π be a set of primes. Then

(a) $\mathcal{L}_\pi(\mathfrak{F})$ is a Lockett class ; and

(b) $\mathcal{K}_\pi(\mathfrak{F})$ is a Lockett class..

<u>Proof</u>. (a) Suppose for a contradiction that $\mathcal{L}_\pi(\mathfrak{F}) \subsetneqq (\mathcal{L}_\pi(\mathfrak{F}))^*$, and let G be a group of minimal order in $(\mathcal{L}_\pi(\mathfrak{F}))^* \setminus \mathcal{L}_\pi(\mathfrak{F})$. By (1.3.9a), G has a unique maximal normal subgroup $M = G_{\mathcal{L}_\pi(\mathfrak{F})}$, which has prime index, p say, in G. Since $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$, by (2.1.2b), and $G \notin \mathcal{L}_\pi(\mathfrak{F})$, it follows that $p \in \pi$.

Since $G \in (\mathcal{L}_\pi(\mathfrak{F}))^*$, then $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} = (M \times M)\langle (g, g^{-1}) : g \in G\rangle$ by (1.5.2). If $(g, g^{-1}) \in M \times M$, then $g \in M$. Since $M \lneqq G$, it follows that $M \times M \lneqq (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$; indeed, since

$|G \times G : M \times M| = p^2$, we have $|(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} : M \times M| = p$.

Let J be an $\mathfrak{F}$-injector of G. Suppose that $J \nleq M$, so that $G = MJ$. Now $M \cap J$ is an $\mathfrak{F}$-injector of $M \in \mathcal{L}_\pi(\mathfrak{F})$, and so $J \cap M$ contains a Hall π-subgroup of M. Thus J contains a Hall π-subgroup of $G = MJ$, contrary to the fact that $G \notin \mathcal{L}_\pi(\mathfrak{F})$. It follows that $J \leq M$.

Since $\mathfrak{F}$ is a Lockett class, then by (1.5.3j), $J \times J$ is an $\mathfrak{F}$-injector of $G \times G$. Since $J \times J \leq M \times M \leq (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$, then $J \times J$ is an $\mathfrak{F}$-injector of $M \times M$ and $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$ by (1.3.6b), and because $M \times M$ and $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$ both belong to $\mathcal{L}_\pi(\mathfrak{F})$, then $J \times J$ contains a Hall π-subgroup of each. But this is impossible since $J \times J \leq M \times M$, which has index $p \in \pi$ in $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$, completing the proof of (a).

(b) By (2.1.3), $\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F} \cap \mathcal{S}_\pi)$. By (1.5.3f), $(\mathfrak{F} \cap \mathcal{S}_\pi)^* = \mathfrak{F}^* \cap \mathcal{S}_\pi^* = \mathfrak{F} \cap \mathcal{S}_\pi$, since $\mathfrak{F}$ and $\mathcal{S}_\pi$ are Lockett classes, whence $\mathfrak{F} \cap \mathcal{S}_\pi$ is a Lockett class. But now $\mathcal{K}_\pi(\mathfrak{F})$ is a Lockett class, by (a).

<u>2.4.2 Lemma.</u> Let $\mathfrak{F}$ be a Fitting class, π be a set of primes, and G be a group. Suppose that $G_{\mathcal{L}_\pi(\mathfrak{F})}$ is $\mathcal{L}_\pi(\mathfrak{F})$-maximal in G, and let V be an $\mathfrak{F}$-injector of G. Then $V \leq G_{\mathcal{L}_\pi(\mathfrak{F})}$.

<u>Proof.</u> By (1.5.3f) V is also an $\mathfrak{F}$-injector of $G_{\mathcal{L}_\pi(\mathfrak{F})}$ $V \leq G$, while $V \cap G_{\mathcal{L}_\pi(\mathfrak{F})}$ is an $\mathfrak{F}$-injector of $G_{\mathcal{L}_\pi(\mathfrak{F})}$, and so contains a Hall π-subgroup of $G_{\mathcal{L}_\pi(\mathfrak{F})}$. But then V contains a Hall π-subgroup

of $G_{\mathcal{L}_\pi(\mathfrak{F})}V$, which thus belongs to $\mathcal{L}_\pi(\mathfrak{F})$. Thus $V \leq G_{\mathcal{L}_\pi(\mathfrak{F})}$, by the $\mathcal{L}_\pi(\mathfrak{F})$-maximality of $G_{\mathcal{L}_\pi(\mathfrak{F})}$, as asserted.

<u>2.4.3 Theorem</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $\mathcal{L}_\pi(\mathfrak{F}^*) = (\mathcal{L}_\pi(\mathfrak{F}))^*$.

<u>Proof</u>. By (1.5.13), $\mathfrak{F} \ll \mathfrak{F}^*$, and so $\mathcal{L}_\pi(\mathfrak{F}) \subseteq \mathcal{L}_\pi(\mathfrak{F}^*)$ by (2.1.15). By (1.5.3e), $(\mathcal{L}_\pi(\mathfrak{F}))^* \subseteq \mathcal{L}_\pi(\mathfrak{F}^*)$, since $\mathcal{L}_\pi(\mathfrak{F}^*)$ is a Lockett class by (2.4.1). It remains to prove the opposite inclusion.

Suppose for a contradiction that $\mathcal{L}_\pi(\mathfrak{F}^*) \nsubseteq (\mathcal{L}_\pi(\mathfrak{F}))^*$, and let G be a group of minimal order in $\mathcal{L}_\pi(\mathfrak{F}^*) \setminus (\mathcal{L}_\pi(\mathfrak{F}))^*$. Then G has a unique maximal normal subgroup $M = G_{(\mathcal{L}_\pi(\mathfrak{F}))^*}$ of prime index p, by (1.3.9a), while by (1.3.10) and (1.5.6), G/G' is a cyclic p-group and $G_{\mathcal{L}_\pi(\mathfrak{F})} \geq G'$. (We may assume (1.3.9/10) without reference in future).

Let L denote $G_{\mathcal{L}_\pi(\mathfrak{F})}$. Since $G \notin \mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{L}_{\pi'}$ and $|G : L|$ is a power of p, it follows that $p \in \pi$.

Let V be an $\mathfrak{F}^*$-injector of G. Since $G \in \mathcal{L}_\pi(\mathfrak{F}^*)$, then V contains a Hall π-subgroup of G, and so $G = LV$ since $|G : L|$ is a π-number.

By (1.5.3j), $V \times V$ is an $\mathfrak{F}^*$-injector of $G \times G$, while by (1.5.13), $(V \times V)_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of $G \times G$. Since $(G \times G)' = G' \times G' \leq L \times L$, then $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})}$ is $\mathcal{L}_\pi(\mathfrak{F})$-maximal in $G \times G$, and so by (2.3.3),

$$(V \times V)_{\mathfrak{F}} \leq (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} \qquad \ldots (1).$$

Since $V \in \mathfrak{F}^*$, then by (1.5.2), we obtain

$$V \times V = (V \times V)_{\mathfrak{F}} (1 \times V) \qquad \cdots (2) .$$

Since $L = G_{\mathcal{L}_\pi(\mathfrak{F})}$, we obtain

$$1 \times L \leq L \times L \leq (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} \qquad \cdots (3) .$$

Remembering that $G = LV$, we now have

$$\begin{aligned}
(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} (1 \times G) &= (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} (1 \times L)(1 \times V) \\
&= (G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} (1 \times V) && \text{by (3)} \\
&\geqslant (V \times V) \quad (1 \times V) && \text{by (1)} \\
&= V \times V && \text{by (2)} .
\end{aligned}$$

Thus $(G \times G)_{\mathcal{L}_\pi(\mathfrak{F})} (1 \times G)$ contains both $L \times L$ and $V \times V$, and so coincides with $G \times G$. But then by (1.5.2), $G \in (\mathcal{L}_\pi(\mathfrak{F}))^*$, a contradiction. This completes the proof.

Since $\mathcal{K}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F} \cap \mathcal{S}_\pi)$ and $(\mathfrak{F} \cap \mathcal{S}_\pi)^* = \mathfrak{F}^* \cap \mathcal{S}_\pi$, it follows from (2.4.3) that $\mathcal{K}_\pi(\mathfrak{F}^*) = (\mathcal{K}_\pi(\mathfrak{F}))^*$. However, the proof of (2.4.3) makes use of Lockett's result (1.5.13). It is possible to give a proof of the result for $\mathcal{K}_\pi(\ )$ using only the simpler properties of the star operation, together with our results about $\mathcal{K}_\pi(\ )$, replacing (1.5.13) by (2.1.4d). We sketch this proof.

<u>2.4.4 Theorem</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $\mathcal{K}_\pi(\mathfrak{F}^*) = (\mathcal{K}_\pi(\mathfrak{F}))^*$.

<u>Proof</u>. Since $\mathfrak{F} \subseteq \mathfrak{F}^*$, by (1.5.3b), then $\mathcal{K}_\pi(\mathfrak{F}) \subseteq \mathcal{K}_\pi(\mathfrak{F}^*)$ by (2.1.4d). By (2.4.1), $\mathcal{K}_\pi(\mathfrak{F}^*)$ is a Lockett class, and so $(\mathcal{K}_\pi(\mathfrak{F}))^* \subseteq \mathcal{K}_\pi(\mathfrak{F}^*)$,

by (1.5.3e). It remains to prove the opposite inclusion.

Thus, suppose for a contradiction that G is a group of minimal order in $\mathcal{K}_\pi(\mathfrak{F}^*) \setminus (\mathcal{K}_\pi(\mathfrak{F}))^*$. Then G has a unique maximal normal subgroup $M = G_{(\mathcal{K}_\pi(\mathfrak{F}))^*}$, of prime index p, and G/G' is a cyclic p-group. Let $L = G_{\mathcal{K}_\pi(\mathfrak{F})}$; by (1.5.6), $L \geqslant G'$, and $p \in \pi$, since $G \notin \mathcal{K}_\pi(\mathfrak{F}) = \mathcal{K}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'}$.

Let $H \in \mathrm{Hall}_\pi(G)$, so that $H \times H \in \mathrm{Hall}_\pi(G \times G)$. Since $G \in \mathcal{K}_\pi(\mathfrak{F}^*)$ then $H \in \mathfrak{F}^*$, whence by (1.5.2), $(H \times H)_{\mathfrak{F}} = (H_{\mathfrak{F}} \times H_{\mathfrak{F}})\langle (h,h^{-1}) : h \in H \rangle$. Since $\mathcal{K}_\pi(\mathfrak{F}) \subseteq (\mathcal{K}_\pi(\mathfrak{F}))^*$, then $(G \times G)_{\mathcal{K}_\pi(\mathfrak{F})} \leqslant M \times M = (G \times G)_{(\mathcal{K}_\pi(\mathfrak{F}))^*}$. By (2.2.1), $(G \times G)_{\mathcal{K}_\pi(\mathfrak{F})} \cap (H \times H) = (H \times H)_{\mathfrak{F}}$, whence (h,h^{-1}) belongs to $M \times M$ for all $h \in H$. But then $H \leqslant M$, a contradiction, since $|G : M| = p \in \pi$. This completes the proof.

As corollaries to (2.4.4), we have the following two results.

<u>2.4.5 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class. If $\mathfrak{F}$ is Hall-closed, then so also are $\mathfrak{F}^*$ and $\mathfrak{F}_*$. In particular, $\mathfrak{F}^*$ is Hall-closed if and only if $\mathfrak{F}_*$ is.

<u>Proof</u>. Suppose that $\mathfrak{F}$ is Hall-closed, and let $\pi \subseteq \mathbb{P}$ be arbitrary. Then $\mathfrak{F} \subseteq \mathcal{K}_\pi(\mathfrak{F})$ by (2.1.7a), and so by (1.5.3e) and (1.5.9), we have both $\mathfrak{F}^* \subseteq (\mathcal{K}_\pi(\mathfrak{F}))^*$ and $\mathfrak{F}_* \subseteq (\mathcal{K}_\pi(\mathfrak{F}))_*$.

By (2.4.4), $(\mathcal{K}_\pi(\mathfrak{F}))^* = \mathcal{K}_\pi(\mathfrak{F}^*)$, and so $\mathfrak{F}^* \subseteq \mathcal{K}_\pi(\mathfrak{F}^*)$, and it follows by (2.1.7a) that $\mathfrak{F}^*$ is Hall-closed, since π was arbitrary.

By (2.4.4) and the fact that $(\mathfrak{F}_*)^* = \mathfrak{F}^*$ from (1.5.8b), we have

$$(\mathcal{K}_\pi(\mathfrak{F}_*))^* = \mathcal{K}_\pi((\mathfrak{F}_*)^*) = \mathcal{K}_\pi(\mathfrak{F}^*) = (\mathcal{K}_\pi(\mathfrak{F}))^* ,$$

and so, by definition of the "lower star", $(\mathcal{K}_\pi(\mathfrak{F}))_* \subseteq \mathcal{K}_\pi(\mathfrak{F}_*)$, whence $\mathfrak{F}_* \subseteq \mathcal{K}_\pi(\mathfrak{F}_*)$, by the first paragraph, and so $\mathfrak{F}_*$ is Hall-closed, since $\pi \subseteq \mathbb{P}$ was arbitrary.

<u>2.4.6 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class, and π be a set of primes. Suppose that $\mathcal{S}_\pi \subseteq \mathfrak{F}$. Then $(\mathcal{S}_\pi)_* = \mathcal{S}_\pi \cap \mathfrak{F}_*$.

<u>Proof</u>. By (1.5.8b/9), $(\mathcal{S}_\pi)_* \subseteq \mathcal{S}_\pi \cap \mathfrak{F}_*$, and it remains to prove the opposite inclusion. Clearly $\mathcal{S} = \mathcal{K}_\pi(\mathcal{S}_\pi) = (\mathcal{K}_\pi((\mathcal{S}_\pi)_*))^*$ by (2.4.4) and (1.5.8b), since $\mathcal{S}_\pi = \mathcal{S}_\pi^*$ by (1.5.3h). Thus by (1.5.3f), we have

$$\mathfrak{F}^* = \mathfrak{F}^* \cap (\mathcal{K}_\pi((\mathcal{S}_\pi)_*))^* = (\mathfrak{F} \cap \mathcal{K}_\pi((\mathcal{S}_\pi)_*))^* ,$$

and so $\mathfrak{F}_* \subseteq \mathfrak{F} \cap \mathcal{K}_\pi((\mathcal{S}_\pi)_*)$ by definition of the "lower star". But then $\mathcal{S}_\pi \cap \mathfrak{F}_* \subseteq \mathcal{S}_\pi \cap \mathfrak{F} \cap \mathcal{K}_\pi((\mathcal{S}_\pi)_*) = \mathcal{S}_\pi \cap \mathcal{K}((\mathcal{S}_\pi)_*) = (\mathcal{S}_\pi)_*$, by (2.1.4b), completing the proof.

Taking $\mathfrak{F} = \mathcal{S}$ (which is certainly Hall-closed) in (2.4.5/6) , we obtain the following results of Bryce and Cossey.

<u>2.4.7 Corollary (Bryce and Cossey, [10; 4.15 & 4.17])</u>.

(a) The smallest normal Fitting class, $\mathcal{H} = \mathcal{S}_*$, is Hall-closed.

(b) If $\pi \subseteq \mathbb{P}$, then $(\mathcal{S}_\pi)_* = \mathcal{S}_\pi \cap \mathcal{S}_*$.

Part (b) of the above corollary shows that the Lockett conjecture (see (1.5.14)) is true for $\mathcal{S}_\pi$. Of course, Bryce and Cossey prove that the Lockett conjecture is true for any primitive saturated formation.

In his dissertation [34; 6.5] , Hauck gives another, essentially different, proof of (2.4.7a) : he uses the so-called "Wreath product property" of Makan, [51] , to show that $\mathcal{K}_\pi(\mathcal{S}_*)$ is normal for $\pi \subseteq \mathbb{P}$.

We will see in Chapter 3 that while $\mathcal{S}$ and $\mathcal{S}_*$ are Hall-closed, there exist normal Fitting classes which are not Hall-closed, and so the converse to (2.4.5) need not hold.

Since $\mathcal{S}_* \neq \mathcal{S}$, then $\mathcal{S}_*$ is not a Lockett class and so is not subgroup-closed; thus there exists a Hall-closed class which is not already subgroup-closed. We will see in Chapter 4 that a certain class defined by Hawkes in [39] is an example of a Fischer class (and so a Lockett class) which is Hall-closed but not subgroup-closed.

<u>2.4.8 Observations</u>. Let $\mathfrak{F}$ be a Fitting class; we recall from (1.5.10) the definition of the "Lockett section" of $\mathfrak{F}$, denoted by Locksec($\mathfrak{F}$) .

Suppose that $\mathfrak{G} \in$ Locksec($\mathfrak{F}$). By (2.4.3) and (2.4.4), we have

$(\mathcal{L}_\pi(\mathfrak{G}))^* = \mathcal{L}_\pi(\mathfrak{G}^*) = \mathcal{L}_\pi(\mathfrak{F}^*) = (\mathcal{L}_\pi(\mathfrak{F}))^*$; and

$(\mathcal{K}_\pi(\mathfrak{G}))^* = \mathcal{K}_\pi(\mathfrak{G}^*) = \mathcal{K}_\pi(\mathfrak{F}^*) = (\mathcal{K}_\pi(\mathfrak{G}))^*$.

It follows that

$$\mathcal{L}_\pi(\mathfrak{G}) \in \text{Locksec}(\mathcal{L}_\pi(\mathfrak{F})) \quad ; \text{ and}$$

$$\mathcal{K}_\pi(\mathfrak{G}) \in \text{Locksec}(\mathcal{K}_\pi(\mathfrak{F})) \quad .$$

Thus $\mathcal{L}_\pi(\ )$ and $\mathcal{K}_\pi(\ )$ map Locksec($\mathfrak{F}$) into Locksec($\mathcal{L}_\pi(\mathfrak{F})$) and Locksec($\mathcal{K}_\pi(\mathfrak{F})$) , respectively. Suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes with $\mathfrak{F}_* \subseteq \mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{F}^*$. It follows easily from (1.5.13) that $\mathfrak{X} << \mathfrak{Y}$, and from (2.1.15a) that $\mathcal{L}_\pi(\mathfrak{X}) \subseteq \mathcal{L}_\pi(\mathfrak{Y})$, while $\mathcal{K}_\pi(\mathfrak{X}) \subseteq \mathcal{K}_\pi(\mathfrak{Y})$ by (2.1.4d). Thus, $\mathcal{L}_\pi(\ )$ and $\mathcal{K}_\pi(\ )$ induce inclusion-preserving maps from a given Lockett section to the appropriate

image sections. We will see, by consideration of the Lockett section of $\mathcal{S}$ (that is, the family of normal Fitting classes), that there is no necessity for these maps to be either one-to-one or onto. Indeed, if $\mathfrak{F}$ is any normal Fitting class, then $\mathcal{N} \subseteq \mathfrak{F}$ by (1.4.3), and it follows that $\mathcal{K}_p(\mathfrak{F}) = \mathcal{S}$, where $p \in \mathbb{P}$.

We will use $\mathcal{H}$ to denote $\mathcal{S}_*$, as previously mentioned.

Since $\mathcal{H} \subseteq \mathcal{K}_\pi(\mathcal{H})$ for all $\pi \subseteq \mathbb{P}$, by (2.4.7), then by (2.1.2b) and (2.1.7c), it follows that $\mathcal{H} \subseteq \mathcal{L}_\pi(\mathcal{H}) \subseteq \mathcal{K}_\pi(\mathcal{H}) \subseteq \mathcal{S}$. In fact, the action of $\mathcal{L}_\pi(\ )$ on normal Fitting classes is relatively easy to determine explicitly, as we shall see in the next result.

<u>2.4.9 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class, $\mathcal{G}$ be a normal Fitting class, and π be a set of primes. Then

(a) $\mathcal{L}_\pi(\mathfrak{F}) \cap \mathfrak{F}^* = (\mathfrak{F} * \mathcal{S}_{\pi'}) \cap \mathfrak{F}^*$;

(b) $\mathcal{L}_\pi(\mathcal{G}) = \mathcal{G} * \mathcal{S}_{\pi'}$;

(c) if $\mathcal{D}$ is a Fitting class such that $\mathcal{G} \subseteq \mathcal{D} \subseteq \mathcal{G} * \mathcal{S}_{\pi'}$, then $\mathcal{L}_\pi(\mathcal{D}) = \mathcal{L}_\pi(\mathcal{G})$; and

(d) the normal Fitting class $\mathcal{B}$ satisfies $\mathcal{B} = \mathcal{L}_\pi(\mathcal{B})$ if and only if $\mathcal{B} \supseteq \mathcal{L}_\pi(\mathcal{H})$.

<u>Proof</u>. (a) By (2.1.2b), $\mathfrak{F} * \mathcal{S}_{\pi'} \subseteq \mathcal{L}_\pi(\mathfrak{F})$, and so $(\mathfrak{F} * \mathcal{S}_{\pi'}) \cap \mathfrak{F}^*$ is contained in $\mathcal{L}_\pi(\mathfrak{F}) \cap \mathfrak{F}^*$; it remains to prove the opposite inclusion.

Suppose that $G \in \mathcal{L}_\pi(\mathfrak{F}) \cap \mathfrak{F}^*$. By (1.5.3c), $G_{\mathfrak{F}} \geqslant G'$, and so $G_{\mathfrak{F}}$ must be an $\mathfrak{F}$-injector of G. But then $G_{\mathfrak{F}}$ contains a Hall π-subgroup of G, whence $G/G_{\mathfrak{F}} \in \mathcal{S}_{\pi'}$ and $G \in \mathfrak{F} * \mathcal{S}_{\pi'}$, as required.

(b) Here, $\mathcal{G}^* = \mathcal{S} \supseteq \mathcal{L}_\pi(\mathcal{G})$, and the result follows from (a) .

(c) By (1.3.13c), $\mathcal{G} * \mathcal{S}_{\pi'} \subseteq \mathcal{D} * \mathcal{S}_{\pi'} \subseteq (\mathcal{G} * \mathcal{S}_{\pi'}) * \mathcal{S}_{\pi'} = \mathcal{G} * \mathcal{S}_{\pi'}$, since $\mathcal{S}_{\pi'}$ is Q-closed. Because $\mathcal{D} \supseteq \mathcal{G}$ is normal, then by (b) ,

$$\mathcal{L}_\pi(\mathcal{D}) = \mathcal{D} * \mathcal{S}_{\pi'} = \mathcal{G} * \mathcal{S}_{\pi'} = \mathcal{L}_\pi(\mathcal{G}) .$$

(d) If $\mathcal{H} \subseteq \mathcal{B}$ and $\mathcal{B} = \mathcal{L}_\pi(\mathcal{B})$, then by (b) and (1.3.13c), we have $\mathcal{L}_\pi(\mathcal{H}) = \mathcal{H} * \mathcal{S}_{\pi'} \subseteq \mathcal{B} * \mathcal{S}_{\pi'} = \mathcal{L}_\pi(\mathcal{B})$. Next suppose that $\mathcal{L}_\pi(\mathcal{H}) \subseteq \mathcal{B}$, and let $G \in \mathcal{B} * \mathcal{S}_{\pi'}$. Since $G_{\mathcal{B}} \geqslant G_{\mathcal{L}_\pi(\mathcal{H})}$, which is an $\mathcal{L}_\pi(\mathcal{H})$-injector of G and so has π-index in G by (2.1.2c) , then $G/G_{\mathcal{B}} \in \mathcal{S}_\pi \cap \mathcal{S}_{\pi'} = (1)$, and so $G \in \mathcal{B}$, as required.

It will be convenient to quote here a result of Berger, [2] , which we shall need from time to time. Berger proves the theorem as a result of certain normal Fitting class constructions; these constructions have recently been generalised by Laue, Lausch and Pain, [44] , and have been further generalised by Berger, [4]. We shall discuss these classes in sections 3.7 and 4.3 , but will use the following result without further comment.

<u>2.4.10 Theorem (Berger, [2; Proposition 3])</u>. Suppose that r and t are primes with $r \mid t-1$, and let G be a non-abelian group of order $t^n r$, having a cyclic (normal) Sylow t-subgroup of order t^n (see [28; 5.4.1]). Then $G \notin \mathcal{H} = \mathcal{S}_*$.

<u>2.4.11 Corollary</u>. Let π be a non-empty set of primes. Then $\mathcal{H} \subsetneqq \mathcal{H} * \mathcal{S}_\pi$.

<u>Proof</u>. Let $r \in \pi$. By Dirichlet's theorem ([32; Theorem 15]) , there exists $t \in \mathbb{P}$ with $r \mid t-1$. Let G be a non-abelian group of order tr : it is well-known that such a group exists, is unique up to isomorphism, and possesses a normal subgroup of order t . Since by (1.4.3), then $G \in \mathfrak{A} * \mathcal{S}_r \subseteq \mathfrak{A} * \mathcal{S}_\pi$, since $r \in \pi$. However, by (2.4.10), $G \notin \mathfrak{A}$. Since $\mathfrak{A} \subseteq \mathfrak{A} * \mathcal{S}_\pi$ by (1.3.13a), then $\mathfrak{A} \subsetneqq \mathfrak{A} * \mathcal{S}_\pi$.

Since $\mathcal{L}_\pi(\mathfrak{A}) = \mathfrak{A} * \mathcal{S}_{\pi'}$, then if $\pi \subsetneqq \mathbb{P}$, it follows that $\mathfrak{A} \subsetneqq \mathcal{L}_\pi(\mathfrak{A}) \subseteq \mathcal{K}_\pi(\mathfrak{A})$, and so neither $\mathcal{L}_\pi(\ )$ nor $\mathcal{K}_\pi(\ )$ can map any Lockett section onto the Lockett section of $\mathcal{S}$, in view of (2.4.9d). By (2.4.9c), $\mathcal{L}_\pi(\ )$ does not induce a one to one map of Locksec($\mathcal{S}$) to Locksec($\mathcal{S}$) .

We will see in (2.6.5) that if $\emptyset \neq \pi \subsetneqq \mathbb{P}$, then $\mathcal{L}_\pi(\mathfrak{A}) \subsetneqq \mathcal{K}_\pi(\mathfrak{A})$, and in section 3.7 that if $|\pi| \geqslant 2$, then $\mathcal{K}_\pi(\mathfrak{A}) \subsetneqq \mathcal{S}$.

This section ends with another example of the relationship between Lockett's star operation and the "Hall π-properties" of Fitting classes.

In his dissertation [34] , Hauck investigates whether, if $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes it follows that $(\mathfrak{F} * \mathfrak{G})^* = \mathfrak{F}^* * \mathfrak{G}^*$. He gives examples to show that this is not the case in general, but also gives some sufficient conditions on $\mathfrak{F}$ and $\mathfrak{G}$ under which it is true (see, for example, (2.5.3) below). Here we give a new sufficient condition for $(\mathfrak{F} * \mathfrak{G})^* = \mathfrak{F}^* * \mathfrak{G}^*$, namely, that $\mathfrak{F}$ and $\mathfrak{G}$ be of co-prime characteristics.

<u>2.4.12 Lemma</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes, and let G be a group in which $G_{\mathfrak{F}} = G_{\mathfrak{F}^*}$. Then $G \in \mathfrak{F}^* * \mathfrak{G}^*$ if and only if $G \in (\mathfrak{F} * \mathfrak{G})^*$.

<u>Proof</u>. Let L denote $G_{\mathfrak{F}} = G_{\mathfrak{F}^*}$. In $G \times G$, we have $G_{\mathfrak{F}^*} \times G_{\mathfrak{F}^*} \in \mathfrak{F}$, and so $(G \times G)_{\mathfrak{F}} = L \times L$. We note that there exists a natural isomorphism, ν say, from $(G \times G)/(L \times L)$ to $G/L \times G/L$. Now,

$G \in \mathfrak{F}^* * \mathfrak{G}^*$

$\Leftrightarrow \quad G/L \in \mathfrak{G}^*$

$\Leftrightarrow \quad (G/L \times G/L) = (G/L \times G/L)_{\mathfrak{G}} \; (L/L \times G/L)$ (by (1.5.2))

$\Leftrightarrow \quad (G \times G)/(L \times L) = ((G \times G)/(L \times L))_{\mathfrak{G}} \; ((L \times G)/(L \times L))$ (because of the isomorphism ν)

$\Leftrightarrow \quad (G \times G)/(G \times G)_{\mathfrak{F}} = ((G \times G)/(G \times G)_{\mathfrak{F}})_{\mathfrak{G}} \; ((1 \times G)(G \times G)_{\mathfrak{F}}/(G \;\; G)_{\mathfrak{F}})$

(since $L \times L = (G \times G)_{\mathfrak{F}}$)

$\Leftrightarrow \quad (G \times G)/(G \times G)_{\mathfrak{F}} = ((G \times G)_{\mathfrak{F} * \mathfrak{G}}/(G \times G)_{\mathfrak{F}}) \; ((1 \times G) \; (G \times G)_{\mathfrak{F}}/(G \times G)_{\mathfrak{F}})$

(by (1.3.12c))

$\Leftrightarrow \quad G \times G = (G \times G)_{\mathfrak{F} * \mathfrak{G}} \; (1 \times G)$ (because $(G \times G)_{\mathfrak{F}} \leq (G \times G)_{\mathfrak{F} * \mathfrak{G}}$)

$\Leftrightarrow \quad G \in (\mathfrak{F} * \mathfrak{G})^*$ (by (1.5.2)),

as claimed.

The following lemma seems to be well-known.

<u>2.4.13 Lemma</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Lockett classes. Then $\mathfrak{F} * \mathfrak{G}$ is a Lockett class.

<u>Proof</u>. Let G be a group; since $\mathfrak{F} = \mathfrak{F}^*$ and $\mathfrak{G} = \mathfrak{G}^*$, we may apply (2.4.12) to conclude that $G \in \mathfrak{F} * \mathfrak{G}$ if and only if $G \in (\mathfrak{F} * \mathfrak{G})^*$, as required.

<u>2.4.14 Proposition</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes with $\mathrm{char}(\mathfrak{F}) = \sigma$ and $\mathrm{char}(\mathfrak{G}) = \tau$, where $\sigma \cap \tau = \phi$. Then $(\mathfrak{F} * \mathfrak{G})^* = \mathfrak{F}^* * \mathfrak{G}^*$.

<u>Proof</u>. We firstly show that $(\mathfrak{F} * \mathfrak{G})^* \subseteq \mathfrak{F}^* * \mathfrak{G}^*$. Thus, suppose for a contradiction that G is a group of minimal order in $(\mathfrak{F} * \mathfrak{G})^* \setminus \mathfrak{F}^* * \mathfrak{G}^*$. Then G has a unique maximal normal subgroup M, of prime index p say, and $G/G' \in {}_p$ by (1.3.10). By (1.5.3c), $G_{\mathfrak{F} * \mathfrak{G}} \geqslant G'$, and so $G/G_{\mathfrak{F} * \mathfrak{G}} \in \mathcal{S}_p$.

By (1.3.13), $\mathfrak{F} * \mathfrak{G} \subseteq \mathcal{S}_\sigma * \mathcal{S}_\tau$. By (2.4.12), $\mathcal{S}_\sigma * \mathcal{S}_\tau$ is a Lockett class, and since $G \in (\mathfrak{F} * \mathfrak{G})^*$, then by (1.5.3e), $G \in \mathcal{S}_\sigma * \mathcal{S}_\tau$. Thus, G has a normal Hall σ-subgroup, H say, and $G_{\mathfrak{F}} \leqslant G_{\mathfrak{F}^*} \leqslant H$, since $\mathrm{char}(\mathfrak{F}) = \sigma$.

If $p \in \sigma$, then $H \nleqslant M$, contradicting the uniqueness of M, and so $p \in \tau$. Since $G/G_{\mathfrak{F} * \mathfrak{G}} \in \mathcal{S}_p$, then $H \leqslant G_{\mathfrak{F} * \mathfrak{G}}$. Since $\sigma \cap \tau = \emptyset$, it now follows that $H \leqslant G_{\mathfrak{F}} \leqslant G_{\mathfrak{F}^*}$, and so $G_{\mathfrak{F}} = G_{\mathfrak{F}^*}$ because of the preceding paragraph. But since $G \in (\mathfrak{F} * \mathfrak{G})^*$, it follows from (2.4.12) that $G \in \mathfrak{F}^* * \mathfrak{G}^*$, contrary to choice. Thus, $(\mathfrak{F} * \mathfrak{G})^* \subseteq \mathfrak{F}^* * \mathfrak{G}^*$.

We next show that $\mathfrak{F}^* * \mathfrak{G}^* \subseteq (\mathfrak{F} * \mathfrak{G})^*$. Suppose for a contradiction that G is a group of minimal order in $\mathfrak{F}^* * \mathfrak{G}^* \setminus (\mathfrak{F} * \mathfrak{G})^*$ Then G has a unique maximal normal subgroup M, of prime index p say. Since $\mathfrak{F}^* \subseteq (\mathfrak{F} * \mathfrak{G})^*$, by (1.5.3e), then $G \in \mathfrak{F}^* * \mathfrak{G}^* \setminus \mathfrak{F}^*$, and it follows that $p \in \tau$. By (1.5.6), $G_{\mathfrak{F} * \mathfrak{G}} \geqslant G'$, and so $G/G_{\mathfrak{F} * \mathfrak{G}} \in \mathcal{S}_\tau$ by (1.3.10). Let L denote $G_{\mathfrak{F} * \mathfrak{G}}$. Then $(G \times G)/(L \times L) \cong G/L \times G/L$, which is a τ-group. Since $L \times L \leqslant (G \times G)_{\mathfrak{F} * \mathfrak{G}}$, it follows that $(G \times G)/(G \times G)_{\mathfrak{F} * \mathfrak{G}} \in \mathcal{S}_\tau$. Now, by (1.3.12c), $(G \times G)_{\mathfrak{F} * \mathfrak{G}} / (G \times G)$ is equal to $((G \times G)/(G \times G)_{\mathfrak{F}})_{\mathfrak{G}}$, which belongs to $\mathcal{S}_\tau$, because $\mathrm{char}(\mathfrak{G}) = \tau$. It follows that $(G \times G)/(G \times G)_{\mathfrak{F}} \in \mathcal{S}_\tau$.

Now, $(G\times G)_{\mathfrak{F}^*}/(G\times G)_{\mathfrak{F}} \in \mathcal{S}_\sigma$, since $\mathfrak{F} \subseteq \mathfrak{F}^* \subseteq \mathcal{S}_\sigma$, and since $\sigma \cap \tau = \phi$, and $(G\times G)/(G\times G)_{\mathfrak{F}} \in \mathcal{S}_\tau$, it follows that

$$(G\times G)_{\mathfrak{F}} = (G\times G)_{\mathfrak{F}^*} = G_{\mathfrak{F}^*} \times G_{\mathfrak{F}^*} .$$

Thus $G_{\mathfrak{F}^*} \in \mathfrak{F}$, whence $G_{\mathfrak{F}^*} = G_{\mathfrak{F}}$. But since $G \in \mathfrak{F}^* * \mathcal{G}^*$, it follows by (2.4.11) that $G \in (\mathfrak{F} * \mathcal{G})^*$, contrary to choice. This completes the proof.

<u>2.4.15 Remark</u>. Let $\mathfrak{F}$ and $\mathcal{G}$ be Fitting classes with $(\mathfrak{F} * \mathcal{G})^* = \mathfrak{F}^* * \mathcal{G}^*$, and let $\pi \in \mathbb{P}$. Then $(\mathcal{K}_\pi(\mathfrak{F}) * \mathcal{K}_\pi(\mathcal{G}))^* = (\mathcal{K}_\pi(\mathfrak{F}))^* * (\mathcal{K}_\pi(\mathcal{G}))^*$.

<u>Proof</u>. Apply in order (2.3.1), (2.4.4), (2.3.1) and (2.4.4) .

<u>2.5 The relationship between $\mathcal{L}_\pi(\mathfrak{F})$ and $\mathfrak{F} * \mathcal{S}_{\pi'}$</u>.

Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. We recall Lockett's result (2.1.2) which states that

(α) $\mathfrak{F} \subseteq \mathfrak{F} * \mathcal{S}_{\pi'} = \mathcal{L}_\pi(\mathfrak{F} * \mathcal{S}_{\pi'}) \subseteq \mathcal{L}_\pi(\mathfrak{F}) = \mathcal{L}_\pi(\mathfrak{F}) * \mathcal{S}_{\pi'} = \mathcal{L}_\pi(\mathcal{L}_\pi(\mathfrak{F}))$;

and

(β) $\mathfrak{F} = \mathcal{L}_\pi(\mathfrak{F})$ if and only if $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$.

This prompts the following.

<u>2.5.1 Question</u>. For what Fitting classes $\mathfrak{F}$ and sets of primes π do we have $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$?

By (α) , $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$ if $\mathfrak{F}$ is already of the form $\mathfrak{F} = \mathcal{L}_\pi(\mathcal{G}) = \mathfrak{F} * \mathcal{S}_{\pi'}$, for some Fitting class $\mathcal{G}$. On the other hand, $\mathcal{L}_2(\mathcal{S}_2) = \mathcal{S} \neq \mathcal{S}_2 * \mathcal{S}_{2'}$ ($\not\ni$ Sym(3)) .

We show that this question leads to a new characterization of normal Fitting classes, (2.5.2), and also that $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$ if and only if $\mathcal{L}_\pi(\mathfrak{F}^*) = \mathfrak{F}^* * \mathcal{S}_{\pi'}$, (2.5.4), so that (2.5.1) need only be answered for Lockett classes.

<u>2.5.2 Theorem</u>. Let $\mathfrak{F}$ be a Fitting class. Then $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$ for all $\pi \subseteq \mathbb{P}$ if, and only if, either $\mathfrak{F} = (1)$ or $\mathfrak{F}$ is normal.

<u>Proof</u>. If $\mathfrak{F} = (1)$, then $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{S}_{\pi'}$ for all $\pi \subseteq \mathbb{P}$. Suppose that $\mathfrak{F}$ is normal and that $\pi \subseteq \mathbb{P}$. Then $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$, by (2.4.9b).

Now suppose that $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$ for all $\pi \subseteq \mathbb{P}$, and suppose for a contradiction that $\mathfrak{F}$ is not normal and that $\mathfrak{F} \neq (1)$.

Let G be a group of minimal order in which $G_{\mathfrak{F}}$ is not an $\mathfrak{F}$-injector of G. Let V be an $\mathfrak{F}$-injector of G. Then $G_{\mathfrak{F}} \lneqq V \ntrianglelefteq G$; indeed, by (1.3.8), V is not even subnormal in G, and so $G/G_{\mathfrak{F}}$ cannot be nilpotent. Let $F \trianglelefteq G$ be such that $F/G_{\mathfrak{F}} = F(G/G_{\mathfrak{F}})$, so that $F \lneqq G$, and choose $M \lhd\cdot G$ with $F \leq M$.

If $V \leq U < G$, then, since V is an $\mathfrak{F}$-injector of U by (1.3.6b), we conclude by the minimality of G that $V \trianglelefteq U$ and $U \leq N_G(V)$. It follows that V must be contained in precisely one maximal subgroup of G, say W, and that $W = N_G(V)$. Since V char W, then $W \neq M \trianglelefteq G$ and $G = MV$, while $|V : M \cap V| = |G : M| = p \in \mathbb{P}$. Since $M \cap V$ is an $\mathfrak{F}$-injector of M, then $M \cap V = M_{\mathfrak{F}} \leq G_{\mathfrak{F}} \lneqq V$, and so $G_{\mathfrak{F}} = M \cap V$.

If J is any normal subgroup of G with $G_{\mathfrak{F}} \leq J < M$, then it follows by an argument on orders that JV must be a proper subgroup of G, since $G = MV$. Thus $JV \leq W$ and $J \leq M \cap W$.

Let σ denote the set of prime divisors of $|G : V|$. Then V contains a Hall σ'-subgroup of G , and so $G \in \mathcal{L}_{\sigma'}(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\sigma}$, by our hypothesis on $\mathfrak{F}$. Thus, $G/G_{\mathfrak{F}} \in \mathcal{S}_{\sigma}$, and so $p = |G : M|$ divides $|G : V| = |M : G_{\mathfrak{F}}|$. Since $G/G_{\mathfrak{F}} \notin \mathcal{N}$, from above, there must exist $q \in \mathbb{P} \setminus \{p\}$ such that $q \mid |M : G_{\mathfrak{F}}|$.

If now $F = M$, then $M/G_{\mathfrak{F}}$ is nilpotent and has order divisible by at least the two distinct primes p and q ; in particular, there exist normal subgroups J and K of G with $G_{\mathfrak{F}} \leq J$, $K \nleq M$ and $M = JK$, contrary to the fact that $J , K \leq M \cap W \nleq M$ from above. Thus, since $G_{\mathfrak{F}} \leq F \nleq M$ and $F \trianglelefteq G$, we have $F \leq M \cap W$. But, since $W = N_G(V)$, then $[M \cap W , V] \leq M \cap W \cap V = G_{\mathfrak{F}}$, and so the non-trivial group $V/G_{\mathfrak{F}}$ is disjoint from and centralizes $F(G/G_{\mathfrak{F}})$, which is impossible in a soluble group. This completes the proof.

<u>2.5.3 Lemma (Hauck. [34; Satz 7.10d])</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $(\mathfrak{F} * \mathcal{S}_{\pi})^* = \mathfrak{F}^* * \mathcal{S}_{\pi}$.

<u>Proof</u>. We may either appeal to Hauck's result, which states that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes with $\mathfrak{Y} = \langle Q , E\phi \rangle \mathfrak{Y}$, then $(\mathfrak{X} * \mathfrak{Y})^* = \mathfrak{X}^* * \mathfrak{Y}^*$ (see [34; 7.10d]), or proceed as follows in this special case.

By (1.3.13c), $\mathfrak{F} * \mathcal{S}_{\pi} \subseteq \mathfrak{F}^* * \mathcal{S}_{\pi}$, and so $(\mathfrak{F} * \mathcal{S}_{\pi})^* \subseteq \mathfrak{F}^* * \mathcal{S}_{\pi}$ since the latter is a Lockett class by (2.4.13).

Now for a contradiction, let G be a group of minimal order in $\mathfrak{F}^* * \mathcal{S}_{\pi} \setminus (\mathfrak{F} * \mathcal{S}_{\pi})^*$. By (1.3.9/10) and (1.5.6), $G_{\mathfrak{F} * \mathcal{S}_{\pi}} \geq G' = O^p(G)$, where $p \in \mathbb{P}$. Since $G \notin \mathfrak{F} * \mathcal{S}_{\pi}$, then $p \notin \pi$, and $G = O^{\pi}(G)$. As $G \in \mathfrak{F}^* * \mathcal{S}_{\pi}$, then $G \in \mathfrak{F}^* \subseteq (\mathfrak{F} * \mathcal{S}_{\pi})^*$, a contradiction. q.e.d.

Let σ denote the set of prime divisors of $|G : V|$. Then V contains a Hall σ'-subgroup of G , and so $G \in \mathcal{L}_{\sigma'}(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\sigma}$, by our hypothesis on $\mathfrak{F}$. Thus, $G/G_{\mathfrak{F}} \in \mathcal{S}_{\sigma}$, and so $p = |G : M|$ divides $|G : V| = |M : G_{\mathfrak{F}}|$. Since $G/G_{\mathfrak{F}} \notin \mathcal{N}$, from above, there must exist $q \in \mathbb{P} \setminus \{p\}$ such that $q \mid |M : G_{\mathfrak{F}}|$.

If now $F = M$, then $M/G_{\mathfrak{F}}$ is nilpotent and has order divisible by at least the two distinct primes p and q ; in particular, there exist normal subgroups J and K of G with $G_{\mathfrak{F}} \leq J$, $K \lneqq M$ and $M = JK$, contrary to the fact that $J , K \leq M \cap W \lneqq M$ from above. Thus, since $G_{\mathfrak{F}} \leq F \lneqq M$ and $F \trianglelefteq G$, we have $F \leq M \cap W$. But, since $W = N_G(V)$, then $[M \cap W , V] \leq M \cap W \cap V = G_{\mathfrak{F}}$, and so the non-trivial group $V/G_{\mathfrak{F}}$ is disjoint from and centralizes $F(G/G_{\mathfrak{F}})$, which is impossible in a soluble group. This completes the proof.

<u>2.5.3 Lemma (Hauck. [34; Satz 7.10d])</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $(\mathfrak{F} * \mathcal{S}_{\pi})^* = \mathfrak{F}^* * \mathcal{S}_{\pi}$.

<u>Proof</u>. We may either appeal to Hauck's result, which states that if $\mathfrak{X}$ and $\mathfrak{Y}$ are Fitting classes with $\mathfrak{Y} = \langle Q , E\phi \rangle \mathfrak{Y}$, then $(\mathfrak{X} * \mathfrak{Y})^* = \mathfrak{X}^* * \mathfrak{Y}^*$ (see [34; 7.10d]), or proceed as follows in this special case.

By (1.3.13c), $\mathfrak{F} * \mathcal{S}_{\pi} \subseteq \mathfrak{F}^* * \mathcal{S}_{\pi}$, and so $(\mathfrak{F} * \mathcal{S}_{\pi})^* \subseteq \mathfrak{F}^* * \mathcal{S}_{\pi}$ since the latter is a Lockett class by (2.4.13).

Now for a contradiction, let G be a group of minimal order in $\mathfrak{F}^* * \mathcal{S}_{\pi} \setminus (\mathfrak{F} * \mathcal{S}_{\pi})^*$. By (1.3.9/10) and (1.5.6), $G_{\mathfrak{F} * \mathcal{S}_{\pi}} \geq G' = O^p(G)$, where $p \in \mathbb{P}$. Since $G \notin \mathfrak{F} * \mathcal{S}_{\pi}$, then $p \notin \pi$, and $G = O^{\pi}(G)$. As $G \in \mathfrak{F}^* * \mathcal{S}_{\pi}$, then $G \in \mathfrak{F}^* \subseteq (\mathfrak{F} * \mathcal{S}_{\pi})^*$, a contradiction. q.e.d.

<u>2.5.4 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$ if and only if $\mathcal{L}_\pi(\mathfrak{F}^*) = \mathfrak{F}^* * \mathcal{S}_{\pi'}$.

<u>Proof</u>. Suppose that $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$. By (2.4.3) and (2.5.3), $\mathcal{L}_\pi(\mathfrak{F}^*) = (\mathcal{L}_\pi(\mathfrak{F}))^* = (\mathfrak{F} * \mathcal{S}_{\pi'})^* = \mathfrak{F}^* * \mathcal{S}_{\pi'}$.

Now suppose that $\mathcal{L}_\pi(\mathfrak{F}^*) = \mathfrak{F}^* * \mathcal{S}_{\pi'}$. By (2.1.2b), we always have $\mathfrak{F} * \mathcal{S}_{\pi'} \subseteq \mathcal{L}_\pi(\mathfrak{F})$, and it remains to prove the opposite inclusion. Suppose for a contradiction that G is a group of minimal order in $\mathcal{L}_\pi(\mathfrak{F}) \setminus \mathfrak{F} * \mathcal{S}_{\pi'}$. By (1.3.9/10), $G/G' \in \mathcal{S}_p$ for some $p \in \mathbb{P}$, and $G_{\mathfrak{F} * \mathcal{S}_{\pi'}} \geqslant G'$. Since $G \notin \mathfrak{F} * \mathcal{S}_{\pi'}$, it follows that $p \in \pi$, and so $G = O^{\pi'}(G)$. But $G \in \mathcal{L}_\pi(\mathfrak{F}) \subseteq (\mathcal{L}_\pi(\mathfrak{F}))^* = \mathcal{L}_\pi(\mathfrak{F}^*) = \mathfrak{F}^* * \mathcal{S}_{\pi'}$, by (2.4.3), and $G = O^{\pi'}(G) \in \mathfrak{F}^*$. But then by (1.5.3c), $G_{\mathfrak{F}} \geqslant G'$, and so $G_{\mathfrak{F}}$ is an $\mathfrak{F}$-injector of G. Since $G \in \mathcal{L}_\pi(\mathfrak{F})$, then $G_{\mathfrak{F}}$ must contain a Hall π-subgroup of G. Since $|G : G_{\mathfrak{F}}|$ is a power of p, and $p \in \pi$, then $G = G_{\mathfrak{F}} \in \mathfrak{F} * \mathcal{S}_{\pi'}$, a contradiction. This completes the proof.

It follows that we need answer (2.5.1) only for Lockett classes.

<u>2.5.5 Proposition</u>. Let $\mathfrak{F}$ be a Lockett class and π be a set of primes. Suppose that $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$, and that there exists $q \in \pi$ such that $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_q$. Then $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$.

<u>Proof</u>. Suppose for a contradiction that the assertion is false, and let G be a group of minimal order in $\mathfrak{F} * \mathcal{S}_{\pi'} \setminus \mathfrak{F}$. Then G has a unique maximal normal subgroup $M = G_{\mathfrak{F}}$ of prime index p, and $p \in \pi'$.

Let W denote the regular wreath product $W = G \text{ wr } C_q$. We shall use the notation of (1.7.1); then $W = G^*] \langle \theta \rangle$, where G is the base group and $\langle \theta \rangle$ is the "standard complement", of order q .

Since $\mathfrak{F}$ is a Lockett class, then $W_{\mathfrak{F}} = (G^*)_{\mathfrak{F}} = (G_{\mathfrak{F}})^* = M^*$, by (1.7.3). Since $|W : G^*| = q \in \pi$, and $W_{\mathfrak{F}} \leq G^*$, then $W \notin \mathfrak{F} * \mathcal{S}_{\pi'}$.

Now, $M^* \langle \theta \rangle \in \mathfrak{F} * \mathcal{S}_q$, since $M \in \mathfrak{F}$ and $\langle \theta \rangle \in \mathcal{S}_q$, and so $M^* \langle \theta \rangle \in \mathfrak{F}$ by hypothesis. Since $M = G_{\mathfrak{F}} \triangleleft\cdot G$, then M must be an $\mathfrak{F}$-injector of G , and so M^* is an $\mathfrak{F}$-injector of G^* by (1.5.3j), since $\mathfrak{F}$ is a Lockett class. Suppose that $M^* \langle \theta \rangle \lneqq W_0 < W$, where $W_0 \in \mathfrak{F}$. Then $M^* \lneqq W_0 \cap G^* \in \mathfrak{F}$, contrary to the fact that M^* is an $\mathfrak{F}$-injector of G^* . Thus, $M^* \langle \theta \rangle$ is $\mathfrak{F}$-maximal in W , and so by (1.3.6a), $M^* \langle \theta \rangle$ is an $\mathfrak{F}$-injector of W .

Now, $|W : M^* \langle \theta \rangle| = |G^* : M^*| = p^q$, which is a π'-number. Thus, $M^* \langle \theta \rangle$ contains a Hall π-subgroup of W , and it follows that $W \in \mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$, contrary to the third paragraph of the proof. This completes the proof.

If $\mathfrak{F}$ is any Fitting class with $\mathfrak{F} \subseteq \mathcal{S}_{\pi'}$, where $\pi \subseteq \mathbb{P}$, then $\mathcal{L}_\pi(\mathfrak{F}) = \mathcal{S}_{\pi'} = \mathfrak{F} * \mathcal{S}_{\pi'}$, and so $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$ is not a necessary condition for the Lockett class $\mathfrak{F}$ to satisfy $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$.

However, we ask the following.

<u>2.5.6 Question</u>. Let $\mathfrak{F}$ be a Lockett class and π be a set of primes. Suppose that $\text{char}(\mathfrak{F}) \cap \pi \neq \emptyset$ and $\mathcal{L}_\pi(\mathfrak{F}) = \mathfrak{F} * \mathcal{S}_{\pi'}$. Does it then follow that $\mathfrak{F} = \mathfrak{F} * \mathcal{S}_{\pi'}$?

2.6 Modification of an example due to Cossey.

We shall give a construction leading to (2.6.1) below, and shall draw several conclusions relevant to our work. The construction may be considered as an extension to more general sets of primes of the subgroup of $GL(2,3) \times Sym(4)$ considered by Cossey in [18; 3.7] .

2.6.1 Proposition. Suppose that p, t and r are primes with $p \neq t$ and $r \mid (t-1)$. Then there exists a group $C = C(p,t,r) \in \mathcal{S}_p * \mathcal{S}_t * \mathcal{S}_r$ such that

(a) $C \in (\mathcal{S}_p * \mathcal{H} \cap \mathcal{H} * \mathcal{S}_r) \setminus \mathcal{H}$, where $\mathcal{H} = \mathcal{S}_*$ as usual ; and

(b) $F(C) = O_p(C)$.

Proof. The proof is effected by a construction which falls naturally into two parts. Let p, t and r be primes as in the hypothesis.

(I) We firstly construct an extra-special p-group P on which a non-abelian group H of order tr acts faithfully, centralizing $Z(P)$.

Let D denote an extra-special group of order p^3 , of exponent p if p is odd, or the dihedral group of order 8 if $p = 2$ (see sections 5.4 and 5.5 of [28] , or section III.13 of [41]).

Let H denote a non-abelian group of order tr ; it is well-known that there exists a unique such H , up to isomorphism. We will write $H = \langle\tau\rangle\langle\rho\rangle$, where $\langle\tau\rangle \cong C_t$ and $\langle\rho\rangle \cong C_r$. Then $\langle\tau\rangle \trianglelefteq H$.

Now, H has a faithful, transitive permutation representation, θ say, on the cosets of $\langle\rho\rangle$. With respect to this representation, form

$$W = D \operatorname{wr}_{\theta} H .$$

Then $W = D^*] H$, where D^* is the base group and H is identified

with the "standard complement" (see (1.7.1)).

We may write $D^* = D_1 \times \cdots \times D_t$, where $D_i \cong D$ with isomorphisms $\psi_i : D \to D_i$, $i = 1,\ldots,t$, such that

$$\tau : D_i \to D_{i+1(\mathrm{mod}(t))} \text{ , with } (d\,\psi_i)^\tau = d\,\psi_{i+1(\mathrm{mod}(t))} \text{ .}$$

where $d \in D$,

since τ permutes the cosets of $\langle\rho\rangle$ transitively. We will assume without further comment that our suffices will, as here, be taken "mod(t)", so as to lie in the range $1,\ldots,t$.

Suppose that $Z(D) = \langle z \rangle$, and let $z_i = z\psi_i \in D_i$, so that $\langle z_i \rangle = Z(D_i)$. Then $Z(D^*) = \langle z_1 \rangle \times \cdots \times \langle z_t \rangle$, and $Z(D^*)$ has order p^t . Since $z_i^\tau = z_{i+1}$, then

$$\begin{aligned}[Z(D^*) , \langle\tau\rangle] &= \langle z_1 z_2^{-1} , \ldots , z_{t-1} z_t^{-1} \rangle \\ &= \langle z_1 z_2^{-1} \rangle \times \cdots \times \langle z_{t-1} z_t^{-1} \rangle \quad ,\end{aligned}$$

which has order p^{t-1} .

It follows that $D^*/[Z(D^*) , \langle\tau\rangle]$ can be considered as a central product of t copies of D , with centres identified, and so is itself (see [28; § 5.5]) an extra-special group of order p^{2t+1} , with centre $Z(D^*)/[Z(D^*) , \langle\tau\rangle]$. We note that for all relevant i , j ,

$$z_i \equiv z_j \ (\mathrm{mod}\ [Z(D^*) , \langle\tau\rangle] \ ) \qquad \cdots\cdot(1)$$

Let E denote $D^*/[Z(D^*) , \langle\tau\rangle]$. Since H normalizes each of of D^* , $Z(D^*)$ and $\langle\tau\rangle$, by the structures of H and W , then E admits the action of H . With respect to this action, we form the (abstract) semi-direct product $E]H$, which is naturally isomorphic to $D^*H/[Z(D^*) , \langle\tau\rangle]$; indeed, we shall identify these groups.

Let $P = [E, \langle\tau\rangle]$; by (1.1.3a), $P \trianglelefteq E\langle\tau\rangle$, while since $\langle\rho\rangle$ normalizes E and $\langle\tau\rangle$, it follows that $P \trianglelefteq EH$.

We now prove that P is extra-special. Let us suppose that D is generated by elements a and b of order p, subject to the appropriate relations, such that $[a, b] = z$. Then in D^*, D_i is generated by $a_i = a\psi_i$ and $b_i = b\psi_i$, while $[a_i, b_i] = z_i$. We denote images of a_i and b_i in $E = D^*/[Z(D^*).\langle\tau\rangle]$ by α_i and β_i, respectively ; thus $\alpha_i = a_i[Z(D^*), \langle\tau\rangle]$. Then E is generated by $\alpha_1, \beta_1, \ldots, \alpha_t, \beta_t$ such that $[\alpha_i, \beta_i] = \zeta$, where $\zeta = z_i[Z(D^*), \langle\tau\rangle]$ is independent of i (because of (1)). We note that $[\alpha_i, \alpha_j] = [\beta_i, \beta_j] = [\alpha_i, \beta_j] = 1$ if $i \neq j$, and that $\langle\zeta\rangle = Z(E)$. Further, for each i,

$$\tau : a_i \mapsto a_{i+1},\ b_i \mapsto b_{i+1},\ \alpha_i \mapsto \alpha_{i+1},\ \beta_i \mapsto \beta_{i+1}.$$

The elementary abelian group $E/Z(E)$ has order p^{2t}, and has a basis consisting of the images of α_i, β_i. Because of the action of τ on the α_i, β_i, we have

$$[E/Z(E), \langle\tau\rangle] = \langle \alpha_i\alpha_{i+1}^{-1}Z(E),\ \beta_i\beta_{i+1}^{-1}Z(E),\ i = 1,\ldots,t-1\rangle,$$

which has order p^{2t-2}.

Since now $[E, \langle\tau\rangle]$ must be a non-trivial normal subgroup of E, then $Z(E) \leqslant [E, \langle\tau\rangle]$. It follows that $|[E, \langle\tau\rangle]| = p^{2t-1}$, and

$$P = [E, \langle\tau\rangle] \geqslant \langle \alpha_i\alpha_{i+1}^{-1},\ \beta_i\beta_{i+1}^{-1},\ i = 1,\ldots,t-1\rangle.$$

There are now two cases to consider.

(i) $p \neq 2$. We check by calculation that

$$[\alpha_i\alpha_{i+1}^{-1}, \beta_i\beta_{i+1}^{-1}] = [\alpha_{i+1}^{-1}, \beta_{i+1}^{-1}]\,[\alpha_i, \beta_i]$$

$$= [\alpha_{i+1}, \beta_{i+1}] [\alpha_i, \beta_i]$$

(by [28; 2.2.2 i])

$$= \zeta^2 .$$

Since $p \neq 2$, then $\langle \zeta^2 \rangle = \langle \zeta \rangle$, and so the group $\langle \alpha_i \alpha_{i+1}^{-1}, \beta_i \beta_{i+1}^{-1} \rangle$ is extra-special of order p^3, having centre $\langle \zeta \rangle = Z(E)$. It follows that $P = [E, \langle \tau \rangle]$ is extra-special of order p^{2t-1}, having centre $Z(P) = \langle \zeta \rangle = Z(E)$.

(ii) $p = 2$. In this case, $t \geqslant 3$. We check by calculation that

$$[\alpha_i \alpha_{i+1}^{-1}, \beta_{i+1} \beta_{i+2}^{-1}] = [\alpha_{i+1}^{-1}, \beta_{i+1}]$$

$$= \zeta^{-1} = \zeta \qquad \text{(by [28; 2.2.2 i])} .$$

Thus $\langle \alpha_i \alpha_{i+1}^{-1}, \beta_{i+1} \beta_{i+2}^{-1} \rangle$ is extra-special of order 2^3. Since $\beta_t \beta_1^{-1} = (\beta_{t-1} \beta_t^{-1})^{-1} \cdots (\beta_1 \beta_2^{-1})^{-1}$, then

$$P = [E, \langle \tau \rangle] = \langle \alpha_1 \alpha_2^{-1}, \beta_2 \beta_3^{-1} \rangle \cdots \langle \alpha_{t-1} \alpha_t^{-1}, \beta_t \beta_1^{-1} \rangle ,$$

and so P is extra-special of order 2^{2t-1}, with centre $Z(P) = \langle \zeta \rangle = Z(E)$.

Thus, P is extra-special as claimed.

Since $Z(P) = Z(E) = Z(D^*)/[Z(D^*), \langle \tau \rangle]$, then $Z(P)$ is centralized by $\langle \tau \rangle$; we will show that $H = \langle \tau \rangle \langle \rho \rangle$ centralizes $Z(P)$.

By construction of W and the nature of the permutation representation of H on the cosets of $\langle \rho \rangle$, we may suppose notation chosen so that $\langle \rho \rangle$ stabilizes $D_1 \leqslant D^*$; then ρ permutes the other D_i in orbits of length r. But now, by construction of the wreath product, ρ centralizes D_1, whence $[z_1, \rho] = 1$. But we saw that $\zeta = z_1 [Z(D^*), \langle \tau \rangle]$, and so $[Z(P), \rho] = [Z(P), H] = 1$, as required.

We have thus constructed an extra-special group P, of order p^{2t-1}, admitting the non-abelian group H of order tr, such that $[Z(P), H] = 1$. Since $P = [E, \langle\tau\rangle]$, and $(p,t) = 1$, then by (1.1.3b ii), $P = [P, \langle\tau\rangle]$, and also $P/Z(P) = [P/Z(P), \langle\tau\rangle]$. In particular, $(PH)' = P\langle\tau\rangle \leqslant P]H$.

(II) The second part of the proof consists of using the group PH constructed above to construct the group C of the proposition.

Thus let P_1H_1 and P_2H_2 denote isomorphic copies of the group PH, with $\varphi_i : PH \to P_iH_i$ an isomorphism for $i = 1, 2$, such that

$$\varphi_i : P \mapsto P_i, \quad H \mapsto H_i, \quad \tau \mapsto \tau_i, \quad \rho \mapsto \rho_i .$$

Let A denote P_1H_1 and let B denote $(P_2/Z(P_2))]H_2$, which is isomorphic to $P_2H_2/Z(P_2)$. Let $\overline{P}_2$ denote $P_2/Z(P_2)$.

We have the following facts.

(i) $A' = P_1\langle\tau_1\rangle$ and $B' = \overline{P}_2\langle\tau_2\rangle$.

For $A' \geqslant H_1' = \langle\tau_1\rangle$ and $[P_1, \langle\tau_1\rangle] = P_1$, while, from the final paragraph of part (I), $[\overline{P}_2, \langle\tau_1\rangle] = \overline{P}_2$. Since also $B' = \langle\tau_2\rangle$, the assertion follows.

(ii) $F(A) = P_1$ and $F(B) = \overline{P}_2$.

For certainly $P_1 \leqslant F(A)$ and $\overline{P}_2 \leqslant F(B)$. If, say, $P_1 \nleqslant F(A)$, then $P_1\langle\tau_1\rangle \leqslant F(A)$ since $\langle\tau_1\rangle$ is the unique minimal normal subgroup of H_1. Since $[P_1, \langle\tau_1\rangle] = P_1 \neq 1$, this is impossible. Similarly, $F(B) = \overline{P}_2$.

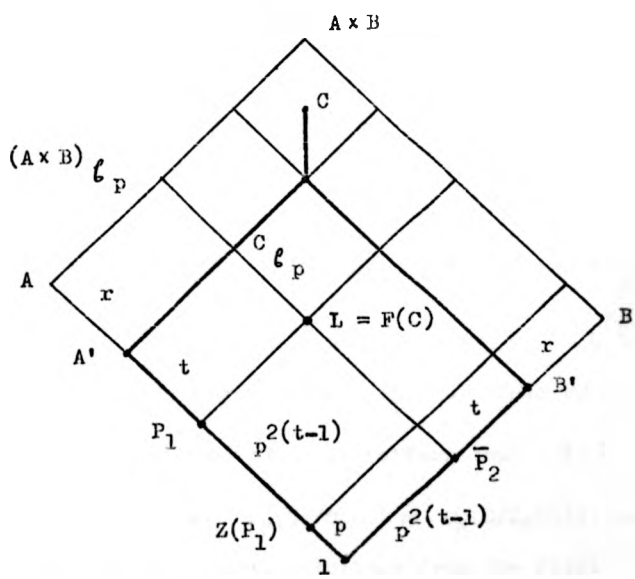
(iii) $A \in \mathcal{G}_p$, while $B_{\mathcal{G}_p} = \overline{P}_2$, where $\mathcal{G}_p$ is as in (1.3.16).

In the case of A, we must have $p\text{-soc}(A) \leq F(A) = P_1$. Now, $Z(P_1)$ is the unique minimal normal subgroup of P_1 and so $p\text{-soc}(A) = Z(P_1) \leq Z(P_1H_1)$ by construction of PH in (I). Thus, $A \in \mathcal{C}_p$.

In the case of B, certainly $B_{\mathcal{C}_p} \geq \overline{P}_2$; suppose that $B_{\mathcal{C}_p} \gneq \overline{P}_2$. Then $B_{\mathcal{C}_p} \geq \overline{P}_2 \langle \tau_2 \rangle$. But by Maschke's Theorem, [28; 3.3.1], $\overline{P}_2$ is a completely reducible $\langle \tau_2 \rangle$-module. Thus, $\overline{P}_2 = p\text{-soc}(\overline{P}_2 \langle \tau_2 \rangle)$, and so τ_2 centralizes $\overline{P}_2$, contrary to the fact that $[\overline{P}_2, \langle \tau_2 \rangle] = \overline{P}_2 \neq 1$. This proves the assertion.

We now form the external direct product $A \times B$ of A and B. Then $(A \times B)' = A' \times B'$ has index r^2 in $A \times B$, by (i), and, being a derived group, belongs to $\mathcal{H} = \mathcal{S}_*$ by (1.4.2).

fig. 2.



Let C be the subgroup of $A \times B$ such that

$$C/(A' \times B') = \{ (\rho_1^i, \rho_2^{-i})(A' \times B') : i \in \mathbb{N} \} .$$

It is not hard to check that C is indeed a subgroup of $A \times B$, and that $|C/(A' \times B')| = r$. (We recall that $H = \langle\tau\rangle\langle\rho\rangle$).

Let L denote $F(A \times B) = F(A) \times F(B) = P_1 \times \overline{P}_2$; we note that, in addition, $L = (A \times B)_{\mathcal{S}_p}$. By (i) above, $L \quad A' \times B' \leqslant C$ and so $L = C_{\mathcal{N}} = F(C)$ and $L = C_{\mathcal{S}_p} = O_p(C)$, proving (2.6.1b).

Now, $C/L \cong (H \text{ wr } C_2)'$, whence $C/L \in \mathcal{H}$ and $C \in \mathcal{S}_p * \mathcal{H}$. Since $A' \times B' \leq C_{\mathcal{H}}$ and $|C/(A' \times B')| = r$, then $C \in \mathcal{H} * \mathcal{S}_r$.

Since $\mathcal{C}_p$ is R_o-closed (see (1.3.17)), it is a Lockett class, by((1.5.3h). Thus, $(A \times B)_{\mathcal{C}_p} = A_{\mathcal{C}_p} \times B_{\mathcal{C}_p} = A \times \overline{P}_2$, by (iii). Thus, $C_{\mathcal{C}_p} = C \cap (A \times \overline{P}_2) = A' \times \overline{P}_2$. But $C/(C_{\mathcal{C}_p}) \cong H$, and by Berger's theorem (2.4.10), $H \notin \mathcal{H}$. Thus $C \notin \mathcal{H}$, by (1.4.8). Taking $C(p,t,r) = C$, we have completed the proof.

<u>2.6.2 Lemma</u>. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are Fitting classes with $\mathcal{F} \subseteq \mathcal{G}$, and suppose that in every group G, we have $G_{\mathcal{G}}/G_{\mathcal{F}} = (G/G_{\mathcal{F}})_{\mathcal{G}}$. Then $\mathcal{F} * \mathcal{H} \subseteq \mathcal{G} * \mathcal{H}$.

In particular, if $\alpha \subseteq \beta \subseteq \mathbb{P}$, then $\mathcal{S}_\alpha * \mathcal{H} \subseteq \mathcal{S}_\beta * \mathcal{H}$.

<u>Proof</u>. Let G be a group with $G \notin \mathcal{G} * \mathcal{H}$. Then $G/G_{\mathcal{G}} \notin \mathcal{H}$. But $G/G_{\mathcal{G}} \cong (G/G_{\mathcal{F}})/(G/G_{\mathcal{F}})_{\mathcal{G}} = (G/G_{\mathcal{F}})/(G/G_{\mathcal{F}})_{\mathcal{G}}$. Thus by (1.4.8), we have $G/G_{\mathcal{F}} \notin \mathcal{H}$, and so $G \notin \mathcal{F} * \mathcal{H}$. It follows that $\mathcal{F} * \mathcal{H} \subseteq \mathcal{G} * \mathcal{H}$.

If $\alpha \subseteq \beta \subseteq \mathbb{P}$, then $O_\beta(G)/O_\alpha(G) = O_\beta(G/O_\alpha(G))$ for all groups G, and the second assertion follows from the first.

<u>2.6.3 Proposition.</u> Let α , β , λ and μ be sets of primes. Then $\mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda \subseteq \mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu$

if and only if either (a) α or λ is empty

or (b) $\phi \neq \alpha \subseteq \beta$ and $\phi \neq \lambda \subseteq \mu$.

<u>Proof.</u> We firstly prove the sufficiency of the conditions (a) , (b) .

If either $\alpha = \phi$ or $\lambda = \phi$, then $\mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda$ (which contains $\mathfrak{H}$ by (1.4.7)) coincides with $\mathfrak{H}$, and so is contained in the class $\mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu$, as asserted.

Now suppose that $\phi \neq \alpha \subseteq \beta$ and $\phi \neq \lambda \subseteq \mu$. Then $\mathcal{S}_\alpha * \mathfrak{H} \subseteq \mathcal{S}_\beta * \mathfrak{H}$ by (2.6.3), while $\mathfrak{H} * \mathcal{S}_\lambda \subseteq \mathfrak{H} * \mathcal{S}_\mu$ by (1.3.13c), and the result follows.

We next prove the necessity of the condition ((a) or (b)) . Thus suppose that $\mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda \subseteq \mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu$ and that neither α nor λ is empty.

Suppose for a contradiction that there exists a prime $p \in \alpha \setminus \beta$. Choose $r \in \lambda$, and let t be any prime such that $r \mid t-1$ and $t \neq p$; such a prime t exists by Dirichlet's theorem, [32; Theorem 15] . Consider the group $C = C(p, t, r)$ of (2.6.1). Then $C \in \mathfrak{H} * \mathcal{S}_\lambda \setminus \mathfrak{H}$. Further, $C_{\mathcal{S}_\alpha} \geqslant L$, whence $C_{\mathcal{S}_\alpha}$ must be either L , $A' \times B'$ or C , depending on whether t , r lie in α . In any case, since C/L is isomorphic to $(H \text{ wr } C_2)' \in \mathfrak{H}$, we have $C \in \mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda \setminus \mathfrak{H}$. Thus $C \in \mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu$ by assumption. But $F(C) = O_p(C)$ by (2.6.1) and so $O_\beta(C) = 1$ since $p \notin \beta$. Since $C \notin \mathfrak{H}$, then $C \notin \mathcal{S}_\beta * \mathfrak{H}$, a contradiction. Thus, $\alpha \subseteq \beta$.

Next, suppose that there exists a prime $r \in \lambda \setminus \mu$. Choose $p \in \alpha$ and $t \in \mathbb{P}$ such that $r \mid t-1$ and $t \neq p$, again possible

by Dirichlet's Theorem. Let $C = C(p, t, r)$. Then by (2.6.1), the sufficiency of this result and our hypothesis, we have

$$C \in \mathcal{S}_p * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_r \subseteq \mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda \subseteq \mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu .$$

In particular, $C \in \mathfrak{H} * \mathcal{S}_\mu$. However, since $C \in \mathfrak{H} * \mathcal{S}_r \setminus \mathfrak{H}$ and $r \notin \mu$, this is impossible. Thus $\lambda \subseteq \mu$, completing the proof.

<u>2.6.4 Corollary</u>. Let α, β, λ and μ be sets of primes. Then $\mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda = \mathcal{S}_\beta * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\mu$ if and only if

either (a) α or λ is empty, and β or μ is empty

or (b) $\phi \neq \alpha = \beta$ and $\phi \neq \lambda = \mu$

<u>2.6.5 Corollary</u>. Let p be a prime. Then there exist uncountably many distinct Fitting classes lying between $\mathfrak{H}$ and $\mathfrak{H} * \mathcal{S}_p$, and also between $\mathfrak{H}$ and $\mathcal{S}_p * \mathfrak{H}$.

<u>Proof</u>. Since there are uncountably many distinct sets of primes, the assertion follows from (2.6.4), if we consider the various classes $\mathcal{S}_\alpha * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_p$ and $\mathcal{S}_p * \mathfrak{H} \cap \mathfrak{H} * \mathcal{S}_\lambda$, respectively.

<u>2.6.6 Proposition</u>. Suppose that $\phi \neq \pi \subsetneqq \mathbb{P}$. Then $\mathcal{L}_\pi(\mathfrak{H}) \subsetneqq \mathcal{K}_\pi(\mathfrak{H})$.

<u>Proof</u>. We saw in (2.4.8) that $\mathcal{L}_\sigma(\mathfrak{H}) \subseteq \mathcal{K}_\sigma(\mathfrak{H})$ for all $\sigma \subseteq \mathbb{P}$.

By (2.4.9b), $\mathcal{L}_\pi(\mathfrak{H}) = \mathfrak{H} * \mathcal{S}_{\pi'}$, while by (2.4.7), (1.3.13) and (2.1.4a), we have $\mathcal{S}_{\pi'} * \mathfrak{H} \subseteq \mathcal{S}_{\pi'} * \mathcal{K}_\pi(\mathfrak{H}) = \mathcal{K}_\pi(\mathfrak{H})$.

Suppose that $\mathcal{S}_{\pi'} * \mathfrak{H} \subseteq \mathfrak{H} * \mathcal{S}_{\pi'}$. Then taking $\lambda = \beta = \mathbb{P}$ in (2.6.4), we conclude, since $\pi' \neq \phi$, that $\mathbb{P} \subseteq \pi'$ and $\pi = \phi$, contrary to supposition. Thus, $\mathcal{K}_\pi(\mathfrak{H}) \supseteq \mathcal{S}_{\pi'} * \mathfrak{H} \nsubseteq \mathfrak{H} * \mathcal{S}_{\pi'} = \mathcal{L}_\pi(\mathfrak{H})$, and the

result follows. We note that a similar argument shows that we cannot have $\mathcal{H} * \mathcal{S}_{\pi'} \subseteq \mathcal{S}_{\pi'} * \mathcal{H}$.

2.6.7 Proposition. The class $\mathcal{C}_{\mathbb{P}}$ (see (1.3.16)) is not Hall-closed.

Proof. Let p , t and r be distinct primes with $r \mid t-1$. Consider the group $P] H \in \mathcal{S}_p * \mathcal{S}_t * \mathcal{S}_r$ as constructed in the proof of (2.6.1). We saw (in (ii), page 85) that $F(PH) = P$, which implies that $soc(PH) = p\text{-}soc(PH)$. We also saw (in (i)) that $PH \in \mathcal{C}_p$. It follows that $PH \in \mathcal{C}_{\mathbb{P}}$. Now, $H \in Hall_{p'}(PH)$, while $soc(PH) \nsubseteq Z(H)$, since $Z(H) = 1$. The assertion follows.

2.6.8 Remark. The above example is of no avail in showing that the class $\mathcal{C}_2$ is not Hall-closed. We will show in (2.8.5) that if $\emptyset \neq \pi \subsetneq \mathbb{P}$ then $\mathcal{C}_\pi$ is indeed not Hall-closed.

2.7 The determination of $\langle \mathcal{Q}_\pi \rangle_{Fitt}$.

If $\mathfrak{X}$ is a class of groups, we recall from (1.3.15) the definition of $\langle \mathfrak{X} \rangle_{Fitt}$. We also recall the following familiar definition.

2.7.1 Definition. Let π be a set of primes. Define

$$\mathcal{Q}_\pi = (G \in \mathcal{S} : G = O^\pi(G)) ,$$

the class of so-called "π-perfect" soluble groups.

2.7.2 Remark. Except in the extremal cases $\pi = \emptyset, \mathbb{P}$, the class $\mathcal{Q}_\pi$ is not S_n-closed ; for example, if $q \in \pi$ and $p \in \mathbb{P} \setminus \pi$, then a group G of type $M(q, p)$ as in (1.7.6) lies in $\mathcal{Q}_\pi$, while $O_q(G)$ certainly does not. Thus, $\mathcal{Q}_\pi$ is not a Fitting class (although it is N_0-closed). In this section, we find that $\langle \mathcal{Q}_\pi \rangle_{Fitt}$ is none other than $\mathcal{L}_\pi(\mathcal{H})$, provided that $\pi \subsetneq \mathbb{P}$; of course, $\mathcal{Q}_{\mathbb{P}} = (1)$.

2.7.3 Proposition. Let π be a set of primes and suppose that $\pi \neq \mathbb{P}$. Then the Fitting class $\langle \mathcal{Q}_\pi \rangle_{Fitt}$ is normal.

Proof. It follows easily from (1.4.2) that $\mathcal{H} = \langle (G' : G \in \mathcal{S}) \rangle_{Fitt}$; thus, it will suffice to show that if $G \in \mathcal{S}$, then $G' \in \langle \mathcal{Q}_\pi \rangle_{Fitt}$.

Let $G \in \mathcal{S}$ and let $p \in \mathbb{P} \setminus \pi$. Let W denote the regular wreath product $W = G \text{ wr } C_p$. Adhering to our notation of (1.7.1), we have $W = G^*] \langle \theta \rangle$, where G^* is the base group and $\langle \theta \rangle \cong C_p$ is the standard complement.

We claim that $(G^*)' \leq [G^*, \langle \theta \rangle]$. To prove this, we may either appeal to [54; Theorem 4.1], or proceed as follows in this case.

As in (1.7.1), we may suppose notation chosen so that

$$G^* = G_1 \times \cdots \times G_p \quad \text{and} \quad \theta : G_i \mapsto G_{i+1(\mathrm{mod}(p))},$$

where the G_i are the coordinate subgroups.

Now, $(G^*)' = G_1' \times \cdots \times G_p'$, and so it will suffice to show that $G_1' \leq [G^*, \langle \theta \rangle]$, say.

Let a_1 and b_1 be arbitrary elements of G_1. Suppose that

$$a_1^{\theta} = a_2 \quad \text{and} \quad b_1^{\theta} = b_2 \quad ,$$

where a_2 , b_2 G_2 . Then $b_2^{-1} = (b_1^{-1})^{\theta}$ and so

$$b_1 b_2^{-1} = [b_1^{-1} , \langle\theta\rangle] \in [G^* , \langle\theta\rangle] \trianglelefteq \langle G^* , \langle\theta\rangle\rangle = W \quad ,$$

in view of (1.1.3a). Thus, remembering that $[G_1 , G_2] = 1$, we have

$$[G^* , \langle\theta\rangle] \ni [a_1 , b_1 b_2^{-1}] = [a_1 , b_2^{-1}] [a_1 , b_1]^{b_2^{-1}} = [a_1 , b_1] \quad .$$

But now $[G^* , \langle\theta\rangle]$ contains an arbitrary generator of G_1' , and so contains G_1' . Thus, $(G^*)' \leq [G^* , \langle\theta\rangle]$, as claimed.

Let W_o denote $[G^* , \langle\theta\rangle]\langle\theta\rangle$; since $[G^* , \langle\theta\rangle] \trianglelefteq W$, then $W_o = [G^* , \langle\theta\rangle]\langle\theta\rangle \trianglelefteq G^*\langle\theta\rangle = W$.

We claim that $W_o = O^{\pi}(W_o)$. For let $L = O^{\pi}(W_o) \trianglelefteq W$. Now, L must contain all π'-subgroups of W_o and so contains $\langle\theta\rangle \cong C_p$. But then $[G^* , \langle\theta\rangle] \leq L$, since $L \trianglelefteq W$, and it follows that $W_o \leq L$. Thus, $W_o = O^{\pi}(W_o) \in \mathcal{Q}_{\pi}$.

Now, $(G')^* = (G^*)' \trianglelefteq [G^* , \langle\theta\rangle] \leq [G^* , \langle\theta\rangle]\langle\theta\rangle = W_o$, and since $(G')^*$ is a direct product of p copies of G' , it follows that $G' \in S_n(\mathcal{Q}_{\pi}) \subseteq \langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}}$, as required. The result follows.

<u>2.7.4 Lemma</u>. If π is a proper subset of $\mathbb{P}$, then

$$\langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}} = \langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}} * \mathcal{S}_{\pi'} \quad .$$

<u>Proof</u>. Let $\mathfrak{F}$ denote $\langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}}$; then $\mathfrak{F} \subseteq \mathfrak{F} * \mathcal{S}_{\pi'}$. Suppose for a contradiction that G is a group of minimal order belonging to $\mathfrak{F} * \mathcal{S}_{\pi'} \setminus \mathfrak{F}$. Then G has a unique maximal normal subgroup $H = G_{\mathfrak{F}}$,

and $|G : M| = p \in \mathbb{P}$; since $G \in \mathfrak{F} * \mathcal{S}_{\pi'}$ then $p \in \pi'$. But now $G' = O^p(G) = G^{\mathcal{N}}$ (see (1.3.10)) and so $G = O^{p'}(G) = O^{\pi}(G) \in \mathcal{Q}_{\pi} \subseteq \mathfrak{F}$, a contradiction. This completes the proof.

<u>2.7.5 Theorem.</u> Let π be a set of primes such that $\pi \neq \mathbb{P}$. Then

$$\langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}} = \mathcal{L}_{\pi}(\mathcal{H}) = \mathcal{H} * \mathcal{S}_{\pi'} .$$

<u>Proof.</u> Let $\mathfrak{F}$ denote $\langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}}$. By (2.7.3), $\mathcal{H} \subseteq \mathfrak{F}$. By (2.4.9b), (1.3.13c) and (2.7.4), we have

$$\mathcal{L}_{\pi}(\mathcal{H}) = \mathcal{H} * \mathcal{S}_{\pi'} \subseteq \mathfrak{F} * \mathcal{S}_{\pi'} = \mathfrak{F} = \langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}} .$$

Now let $G \in \mathcal{Q}_{\pi}$. Since $\mathcal{L}_{\pi}(\mathcal{H}) = \mathcal{L}_{\pi}(\mathcal{H}) * \mathcal{S}_{\pi'}$, and $G' \leq G_{\mathcal{L}_{\pi}(\mathcal{H})}$, then $G/(G_{\mathcal{L}_{\pi}(\mathcal{H})})$ is an abelian π-group. Since $G = O^{\pi}(G)$, then $G = G_{\mathcal{L}_{\pi}(\mathcal{H})} \in \mathcal{L}_{\pi}(\mathcal{H})$. The result follows.

We remark that Doerk has conjectured that $\langle \mathcal{Q}_{\pi} \rangle_{\text{Fitt}} = S_n(\mathcal{Q}_{\pi})$. The proof of (2.6.1) shows that if $G \in \mathcal{S}$, then $G' \in S_n(\mathcal{Q}_{\pi})$.

<u>2.8 The $H_{\mathcal{N}}$-closure of $\mathcal{C}_{\pi}$ and $e_{\pi}(\mathcal{N})$</u>.

In this section, we show that if π is a non-empty set of primes, then the classes $\mathcal{C}_{\pi}$ and $e_{\pi}(\mathcal{N})$ (see (1.3.16)) are both $H_{\mathcal{N}}$-closed, while, if $\pi \neq \mathbb{P}$, neither is Hall-closed. It will be recalled from (1.3.17) that $e_{\pi}(\mathcal{N})$ is a Fischer class (while $\mathcal{C}_{\pi}$ is not), thus fore-stalling any conjecture that an $H_{\mathcal{N}}$-closed Fischer class might be Hall-closed.

<u>2.8.1 Proposition.</u> Let p be a prime. Then the class $\mathcal{C}_p$ is $H_{\mathcal{N}}$-closed (see (2.1.5)).

<u>Proof.</u> Suppose for a contradiction that G is a group of minimal order subject to

(i) $G \in \mathcal{C}_p$; and

(ii) there exists a Hall subgroup H of G with $H \geqslant F(G)$ and $H \notin \mathcal{C}_p$.

Suppose that $\mathrm{char}(H) = \pi$; then $H \in \mathrm{Hall}_\pi(G)$, $F(G) \in \mathcal{S}_\pi$ and $O_{\pi'}(G) = 1$. Since $\mathcal{S}_{p'} \subseteq \mathcal{C}_p$, then $p \in \pi$.

Certainly $H \nleqslant G$, and so we may choose $M \lhd\cdot G$ with $F(G) \leqslant M$. Then $F(M) = F(G) \leqslant M \cap H \in \mathrm{Hall}_\pi(M)$ (by (1.1.2a)), whence $M \cap H$ belongs to $\mathcal{C}_p$, by minimality. In particular, $H \nleqslant M$ and $G = MH$, while $|G : M| = q \in \pi$.

Since $H \notin \mathcal{C}_p$, there exists $L \cdot\!\lhd H$ with $L \in \mathcal{S}_p$ and $L \nleqslant Z(H)$. Since $F(G) \lhd H$ then $[F(G), L] \leqslant F(G) \cap L \lhd H$. By (1.1.1 f), $C_G(F(G)) \leqslant F(G)$ and so $F(G) \cap L \gneqq 1$. Since $L \cdot\!\lhd H$, it follows that $L \leqslant F(G)$. In particular, $L \leqslant M \cap H$.

We may regard L as an irreducible H/L-module (since $L \in \mathcal{A}$). Since $(M \cap H)/L \lhd H/L$, then by Clifford's Theorem ([41; V.17.3] or [23; 3.4.1]), we have

$$L\Big|_{(M \cap H)/L} = U_1 \oplus \cdots \oplus U_n \quad , \text{ for some } n \in \mathbb{N} ,$$

where each U_i is a direct sum of isomorphic irreducible $(M \cap H)/L$-modules. But this means that, as a normal subgroup of $M \cap H$, L can written as a direct product of subgroups, each of which is minimal normal in $M \cap H$. It follows that $L \leqslant p\text{-soc}(M \cap H)$, since $L \in \mathcal{S}_p$.

Since $M \cap H \in \mathcal{L}_p$, we now have

$$L \leqslant Z(M \cap H) \quad . \qquad \cdots (1)$$

Since $L \cdot \trianglelefteq H$ and $L \nleqslant Z(H)$, it follows that

$$H/(M \cap H) \cong C_q \text{ acts faithfully and irreducibly on } L \in \mathcal{S}_p \ . \qquad \cdots (2)$$

It follows that $p \neq q$.

Let J denote the normal closure, $\langle L^g : g \in G \rangle$ of L in G. Since $L \leqslant F(G)$, then $J \leqslant F(G) \leqslant M \cap H$. Since $L \leqslant Z(M \cap H)$ by (1), then $L \leqslant Z(J)$ char $J \trianglelefteq G$, and so $J \leqslant Z(J)$ since J is the normal closure of L in G . It follows that J is an abelian p-group.

Let $S_1 \in \mathrm{Hall}_{\pi'}(G)$. By an easy argument on orders, we have $G = H S_1$, and $M \geqslant S_1$, whence, remembering that $L \trianglelefteq H$, we have

$$J = \langle L^{hs} : h \in H , s \in S_1 \rangle = \langle L^s : s \quad S_1 \rangle \quad .$$

By the Frattini argument (using the conjugacy of Hall-subgroups), we have $G = M N_G(S_1)$, and so there exists a q-element $\tau_1 \in N_G(S_1)$ such that $G = M \langle \tau_1 \rangle$ (since $|G : M| = q$) . By Hall's Theorem, [41; VI.1.8] , there exists $a \in G$ such that $\tau_1^a \in H$. Let $\tau = \tau_1^a$ and $S = S_1^a$; then $\tau \in N_H(S)$, $G = H S$ and $J = \langle L^s : s \in S \rangle$. It follows that

$$L \text{ is contained in no proper } S\text{-invariant subgroup of } J \ . \qquad \cdots (3)$$

Since $\tau \in N_H(S)$, then $S\langle\tau\rangle \leqslant G$. Now J is an abelian p-group, normal in G . Since $p \in \pi$, $S \in \mathcal{S}_{\pi'}$ and $|\tau| = q^{\alpha} \neq p$, then $S\langle\tau\rangle \in \mathcal{S}_{p'}$, and so by (1.1.3 b i) ,

$$J = [J , S\langle\tau\rangle] \times C_J(S\langle\tau\rangle) \quad . \qquad \cdots (4)$$

Since $J \trianglelefteq G$, there exists $J^o \cdot \trianglelefteq G$ with $J^o \leqslant J$. But then

$J^0 \leqslant$ p-soc$(G) \leqslant Z(G)$ and so $C_J(S\langle\tau\rangle) \geqslant J^0 \gneqq 1$, and by (4) we have $[J , S\langle\tau\rangle] \lneqq J$. By (1.1.3b), $[J , S\langle\tau\rangle]$ is $S\langle\tau\rangle$-invariant, and so $S\langle\tau\rangle$ centralizes the group $J/[J , S\langle\tau\rangle]$. But then any subgroup lying between $[J , S\langle\tau\rangle]$ and J must be S-invariant. By statement (3) above, it follows that $[J , S\langle\tau\rangle] L = J$. But then $1 \neq J/[J , S\langle\tau\rangle] = [J , S\langle\tau\rangle] L /[J , S\langle\tau\rangle] \cong L/(L \cap [J , S\langle\tau\rangle])$, and since all the relevant subgroups are $\langle\tau\rangle$-invariant, the isomorphism is in fact a $\langle\tau\rangle$-isomorphism. But $\langle\tau\rangle$ centralizes $J/[J , S\langle\tau\rangle]$, and so $\langle\tau\rangle$ centralizes a non-trivial factor group of L .

But $\tau \in H \setminus M$ and so $H = (M \cap H)\langle\tau\rangle$. Thus, by statement (2) above, L must be a faithful, irreducible module for $\langle\tau\rangle / \langle\tau^q\rangle$, contrary to what we saw in the previous paragraph. This completes the proof.

<u>2.8.2 Theorem</u>. Let π be a non-empty set of primes. Then the class $\mathcal{E}_\pi$ is $H_{\mathcal{N}}$-closed.

<u>Proof</u>. It is not difficult to verify that $\mathcal{E}_\pi = \bigcap_{p \in \pi} \mathcal{E}_p$. The result now follows from (2.8.1) and (1.2.4).

Next comes a similar argument to show that $e_\pi(\mathcal{N})$ is $H_{\mathcal{N}}$-closed. We note that from (2.1.8c), $e_p(\mathcal{N}^2)$ is not $H_{\mathcal{N}}$-closed.

<u>2.8.3 Proposition</u>. Let p be a prime. Then the class $e_p(\mathcal{N})$ is $H_{\mathcal{N}}$-closed .

<u>Proof</u>. Suppose for a contradiction that G is a group of minimal order subject to

(i) $G \in e_p(\mathcal{N})$; and

(ii) there exists a Hall subgroup H of G with $H \geqslant F(G)$ and $H \notin e_p(\mathcal{N})$.

Suppose that $\mathrm{char}(H) = \pi$; then $H \in \mathrm{Hall}_\pi(G)$, $F(G) \in \mathcal{S}_\pi$ and $O_{\pi'}(G) = 1$. Since we certainly have $\mathcal{S}_{p'} \subseteq e_p(\mathcal{N})$, then $p \in \pi$.

Let $M \lhd\cdot G$ be such that $F(G) \leqslant M$. Then $M \in e_p(\mathcal{N})$, $M \cap H \in \mathrm{Hall}_\pi(M)$, by (1.1.2a), and it is easy to check that $F(M) = F(G)$. Since then $F(M) \leqslant M \cap H$, then by minimality we have $M \cap H \in e_p(\mathcal{N})$. In particular, $H \nleqslant M$ and $G = MH$.

Suppose that $F(H) \nleqslant M$. Then $G = M\,F(H)$ and $H = (M \cap H)\,F(H)$. But $M \cap H \in e_p(\mathcal{N})$ and $F(H) \in \mathcal{N} \subseteq e_p(\mathcal{N})$, and so $H \in N_0(e_p(\mathcal{N}))$. Thus $H \in e_p(\mathcal{N})$, contrary to choice, and it follows that $F(H) \leqslant M$.

Since $F(H) \leqslant M$, then $F(H) \leqslant M \cap H \leqslant H$ and so $F(H) = F(M \cap H)$. Since $F(G) \in \mathcal{S}_\pi$ and $H \in \mathrm{Hall}_\pi(G)$, then $F(G) \leqslant O_\pi(G) \leqslant H$, and so $F(G) \leqslant O_\pi(G) \cap F(H)$. But $O_\pi(G) \cap F(H) \leqslant F(O_\pi(G)) \leqslant F(G)$, since $O_\pi(G) \trianglelefteq H$, and it follows that $F(G) = O_\pi(G) \cap F(H)$.

Let $P \in \mathrm{Syl}_p(F(H))$, and let $J = \langle P^g : g \in G \rangle$, the normal closure of P in G . Since $P \leqslant M \trianglelefteq G$, then $J \leqslant M$.

Suppose that $J \in \mathcal{S}_\pi$. Then $J \leqslant O_\pi(G)$ and $P \leqslant O_\pi(G) \cap F(H)$, whence $P \leqslant F(G)$ by the previous paragraph. But then $P \in \mathrm{Syl}_p(F(G))$, and so $P \trianglelefteq G$. If we refine $G \trianglerighteq F(G) \trianglerighteq P \trianglerighteq 1$ to a chief series of G , the chief factors below P are G-central, as $G \in e_p(\mathcal{N})$, and so are H-central, and in fact H-chief since they have order p . But then $H \in e_p(\mathcal{N})$, contrary to choice.

Thus,

J cannot be a π-group ... (1)

Since $M \cap H \in e_p(\mathcal{N})$, the $(M \cap H)$-p-chief factors below $F(M \cap H)$ are $(M \cap H)$-central. An application of Clifford's Theorem, as in the proof of (2.8.1), now shows that the H-p-chief factors inside $F(H) = F(M \cap H)$, being direct sums of $(M \cap H)$-irreducibles, are $(M \cap H)$-central. (We saw that $F(H) = F(M \cap H)$ above). Since $P \leq F(H)$, it follows by [28; 5.3.2] that

P is centralized by any p'-subgroup of $M \cap H$. ... (2)

Let $\underline{\underline{C}}$ be a G-chief series refining $G \trianglerighteq J \trianglerighteq 1$, and suppose that X/Y is a $(\pi \setminus \{p\})$-chief factor appearing in $\underline{\underline{C}}$ with $|X|$ maximal subject to this. Let $L \in \mathrm{Hall}_{(\pi \setminus \{p\})}(J \cap H)$, so that L is a p'-subgroup of $J \cap H \leq M \cap H$. Thus by (2), $[P, L] = 1$. But $X/Y \leq LY/Y$, by an easy argument on orders, (or (1.1.2a)), and so $[P, X] \leq [P, LY] \leq Y \trianglelefteq G$, whence $P \leq C_G(X/Y) \trianglelefteq G$. But then $J = \langle P^g : g \in G \rangle \leq C_G(X/Y)$, and X/Y is J-central. Since $|J/X|$ must be a $(\pi' \cup \{p\})$-number, by the maximal choice of $|X|$, then by the Schur-Zassenhaus Theorem, [28; 6.2.1], we may complement X/Y in J/Y by K/Y (say), and then $J/Y = X/Y \times K/Y$. Thus as $X/Y \neq 1$, then $O^{p'}(J) \leq K \lneq J$, contrary to the fact that J is generated by p-groups. It follows that there can be no such factor X/Y, and so J must be a $(\pi' \cup \{p\})$-group. Now $O_{\pi'}(J) = 1$, since we saw that $O_{\pi'}(G) = 1$, and so $1 \neq O_p(J) = F(J) \leq P$. Since $J \trianglelefteq G$, then $F(J) \leq F(G)$, and since $G \in e_p(\mathcal{N})$, it follows that the p-group $F(J)$ is centralized by any p'-subgroup of G (again by [28; 5.3.2]). But then if $O_p(J) \lneq O_{p,p'}(J)$ $(= J_{\mathcal{S}_{p,p'}})$, we would have a non-trivial

Hall p'-subgroup of $O_{p,p'}(J)$ disjoint from and centralizing $F(G)$, which is impossible.

It follows that $O_p(J) = O_{p,p'}(J) = J$, and $J \in \mathcal{S}_p$, contrary to statement (1) above, since $p \in \pi$. This completes the proof.

<u>2.8.4 Theorem</u>. Let π be a non-empty set of primes. Then the class $e_\pi(\mathcal{N})$ is $H_{\mathcal{N}}$-closed.

<u>Proof</u>. It is not hard to see that $e_\pi(\mathcal{N}) = \bigcap_{p \in \pi} e_p(\mathcal{N})$, and the result follows from (2.8.3) and (1.2.4).

<u>2.8.5 Remarks</u>. (a) Let π be a non-empty, proper, subset of $\mathbb{P}$. Choose $p \in \pi$, $q \in \mathbb{P} \setminus \pi$ and $r \in \mathbb{P} \setminus \{p, q\}$. Let G be a group of type $M(q^\beta, p^\alpha, r)$ (see (1.7.6)). Then $|F(G)| = q^\beta$, and so $F(G) \in \mathcal{S}_{\pi'}$. Thus, $\pi\text{-soc}(G) = 1$, while there are no π-chief factors of G in $F(G) \trianglerighteq 1$. Thus, $G \in \mathcal{E}_\pi \cap e_\pi(\mathcal{N})$.

Let $H \in \mathrm{Hall}_{\{r,p\}}(G)$. Then $H \cong G/F(G)$ is of type $M(p^\alpha, r)$, and $F(H) = \pi\text{-soc}(H)$ has order p^α and is not central in H. Thus, $H \notin \mathcal{E}_\pi$ and $H \notin e_\pi(\mathcal{N})$. It follows that $\mathcal{E}_\pi$ and $e_\pi(\mathcal{N})$ are not Hall-closed, under the assumption that $\emptyset \neq \pi \subsetneq \mathbb{P}$.

(b) We saw in (2.6.7) that the class $\mathcal{E}_{\mathbb{P}}$ is not Hall-closed.

However, suppose that $G \in e_{\mathbb{P}}(\mathcal{N})$. By a theorem of Gaschütz (see [41; III.4.5]), $F(G)/\Phi(G)$ is a direct sum of minimal normal subgroups of $G/\Phi(G)$, and so is G-central. By the same theorem, $F(G)/\Phi(G) = F(G/\Phi(G))$, and by (1.1.1f), $G = F(G)$. Since always $\mathcal{N} \subseteq e_\pi(\mathfrak{F})$, then $e_{\mathbb{P}}(\mathcal{N}) = \mathcal{N}$, which is certainly Hall-closed.

2.9 Hall-determined classes.

2.9.1 Definition. Let n be a natural number. The class $\mathfrak{X}$ of (finite, soluble) groups is said to be n-Hall-determined if $\mathfrak{X}$ has the property that the group $G \in \mathcal{S}$ belongs to $\mathfrak{X}$ if and only if all Hall π-subgroups of G for $\pi \subseteq \mathbb{P}$, $|\pi| \leq n$, belong to $\mathfrak{X}$.

2.9.2 Lemma. Let $\mathfrak{X}$ be a class of groups and n be a natural number. Then $\mathfrak{X}$ is n-Hall-determined if and only if

$$\mathfrak{X} = \bigcap_{\substack{\pi \subseteq \mathbb{P} \\ |\pi| \leq n}} \mathcal{K}_\pi(\mathfrak{X}) .$$

Proof. Let $\mathcal{Y}$ denote $\bigcap_{\substack{\pi \subseteq \mathbb{P} \\ |\pi| \leq n}} \mathcal{K}_\pi(\mathfrak{X})$. Then the condition $\mathfrak{X} \subseteq \mathcal{Y}$ is clearly equivalent to the condition that if $G \in \mathfrak{X}$, then all Hall π-subgroups of G for $|\pi| \leq n$ belong to $\mathfrak{X}$, while the condition $\mathcal{Y} \subseteq \mathfrak{X}$ is equivalent to the condition that $G \in \mathfrak{X}$ whenever all Hall π-subgroups of G for $|\pi| \leq n$ belong to $\mathfrak{X}$; the assertion follows.

For brevity, we shall use $\mathcal{J}_n(\mathfrak{X})$ to denote $\bigcap_{\substack{\pi \subseteq \mathbb{P} \\ |\pi| \leq n}} \mathcal{K}_\pi(\mathfrak{X})$.

2.9.3 Lemma. Suppose that the class $\mathfrak{X}$ is n-Hall-determined for some $n \in \mathbb{N}$. Then $\mathfrak{X}$ is Hall-closed and is, further, m-Hall-determined for all $m \geqslant n$.

Proof. Let $G \in \mathfrak{X}$ and let $H \in \mathrm{Hall}_\pi(G)$ for some $\pi \subseteq \mathbb{P}$.

If $|\pi| \leq n$, then $H \in \mathfrak{X}$ by definition. If $|\pi| > n$, then for all $\sigma \subseteq \mathbb{P}$ with $\sigma| \leq n$, we have $\mathrm{Hall}_\sigma(H) \subseteq \mathrm{Hall}_\sigma(G) \subseteq \mathfrak{X}$, and so $H \in \mathfrak{X}$ since $\mathfrak{X}$ is n-Hall-determined. Letting $m \geq n$, we now have $\mathfrak{X} \subseteq \mathcal{J}_m(\mathfrak{X})$, and equality holds since $\mathfrak{X} = \mathcal{J}_n(\mathfrak{X})$, a "lesser" intersection, completing the proof.

<u>2.9.4 Lemma</u>. (a) If $\{\mathfrak{X}_\alpha\}_{\alpha \in A}$ is a family of n-Hall-determined classes, for $n \in \mathbb{N}$, then $\bigcap_{\alpha \in A} \mathfrak{X}_\alpha$ is n-Hall-determined.

(b) If $\mathfrak{X}$ is any Hall-closed class, then $\mathcal{J}_n(\mathfrak{X})$ is the smallest n-Hall-determined class containing $\mathfrak{X}$, where $n \in \mathbb{N}$.

<u>Proof</u>. (a) A group G belongs to $\bigcap_{\alpha \in A} \mathfrak{X}_\alpha$ if and only if for all $\pi \subseteq \mathbb{P}$ with $|\pi| \leq n$, we have $\mathrm{Hall}_\pi(G) \subseteq \bigcap_{\alpha \in A} \mathfrak{X}_\alpha$, and the result follows.

(b) Since $\mathfrak{X}$ is Hall-closed, then $\mathfrak{X} \subseteq \mathcal{J}_n(\mathfrak{X})$. Now,

$$\bigcap_{\substack{\sigma \subseteq \mathbb{P} \\ |\sigma| \leq n}} \mathcal{K}_\sigma(\mathcal{J}_n(\mathfrak{X})) = \bigcap_{\substack{\sigma \subseteq \mathbb{P} \\ |\sigma| \leq n}} \bigcap_{\substack{\pi \subseteq \mathbb{P} \\ |\pi| \leq n}} \mathcal{K}_\sigma(\mathcal{K}_\pi(\mathfrak{X})) \quad \text{(by (2.1.4f))}$$

$$= \bigcap_{\substack{\sigma \subseteq \mathbb{P} \\ |\sigma| \leq n}} \bigcap_{\substack{\pi \subseteq \mathbb{P} \\ |\pi| \leq n}} \mathcal{K}_{\sigma \cap \pi}(\mathfrak{X}) \quad \text{(by (2.1.4g))}$$

$$= \mathcal{J}_n(\mathfrak{X}) .$$

Thus, $\mathcal{J}_n(\mathfrak{X})$ is an n-Hall-determined class containing $\mathfrak{X}$; on the other hand, if $\mathfrak{X} \subseteq \mathcal{Y}$ and $\mathcal{Y}$ is n-Hall-determined, then by (2.1.4d), $\mathcal{K}_\pi(\mathfrak{X}) \subseteq \mathcal{K}_\pi(\mathcal{Y})$ for all $\pi \subseteq \mathbb{P}$, and so $\mathcal{J}_n(\mathfrak{X}) \subseteq \mathcal{J}_n(\mathcal{Y}) = \mathcal{Y}$, completing the proof.

As an example of the "n-Hall-determination" concept, we prove a result of possible independent interest, namely, that if $\mathfrak{F}$ is a primitive saturated formation of finite defect n (see (1.6.1)), then $\mathfrak{F}$ is n-Hall-determined. A special case of this result, for the class of (finite, soluble) groups of p-length at most $n-1$ (which is a primitive saturated formation of defect n), can be obtained by an easy inductive argument from [42; Satz 2.6] .

We recall from (1.3.11) that " $\circ$ " denotes the formation product, and (see (1.3.12g)) that this coincides with the Fitting class product, " $*$ " if the classes concerned are both Fitting classes and formations (as are primitive saturated formations).

<u>2.9.5 Lemma</u>. Suppose that $\mathfrak{F}$ is a subgroup-closed formation which is n-Hall-determined, where n is a natural number, and let q be a prime. Then $\mathcal{S}_q \circ \mathfrak{F}$ is $(n+1)$-Hall-determined.

<u>Proof</u>. By (1.3.14c), $\mathcal{S}_q \circ \mathfrak{F}$ is S-closed, and so Hall-closed. Thus $\mathcal{S}_q \circ \mathfrak{F} \subseteq \mathcal{J}_{n+1}(\mathcal{S}_q \circ \mathfrak{F})$ (where $\mathcal{J}_n(\ )$ is as defined prior to (2.9.3)), since $\mathcal{S}_q \circ \mathfrak{F} \subseteq \mathcal{K}_\pi(\mathcal{S}_q \circ \mathfrak{F})$ for all $\pi \subseteq \mathbb{P}$.

Let $\mathcal{G}$ denote $\mathcal{J}_{n+1}(\mathcal{S}_q \circ \mathfrak{F})$, and suppose for a contradiction that G is a group of minimal order in $\mathcal{G} \setminus (\mathcal{S}_q \circ \mathfrak{F})$. Since $\mathcal{S}_q \circ \mathfrak{F}$ is a formation, then by (2.1.4e), so is $\mathcal{K}_\pi(\mathcal{S}_q \circ \mathfrak{F})$ for all $\pi \subseteq \mathbb{P}$, and thus $\mathcal{G}$ is a formation. By (1.3.9), G must now have a unique minimal normal subgroup, N say. Further, N is an r-group for some $r \in \mathbb{P}$, and $G/N \in \mathcal{S}_q \circ \mathfrak{F}$. Since $G \notin \mathcal{S}_q \circ \mathfrak{F}$, then we must have $r \neq q$.

Let $H \in \mathrm{Hall}_\rho(G)$ where $r \in \rho \subseteq \mathbb{P}$ and $|\rho| = n+1$.

Since $G \in \mathcal{G} = \bigcap\{\mathcal{K}_\pi(\mathcal{S}_q \circ \mathfrak{F}) : \pi \subseteq \mathbb{P} \text{ and } |\pi| \leq n+1\}$, then $H \in \mathcal{S}_q \circ \mathfrak{F}$.

We claim that in fact $H \in \mathfrak{F}$. If $q \notin \rho$, then certainly $H \in \mathfrak{F}$ as $H \in \mathcal{S}_q \circ \mathfrak{F}$. Suppose that $q \in \rho$. Now, it is not difficult to check that $\mathcal{K}_\rho(\mathcal{S}_q) = \mathcal{S}_{\rho' \cup q}$. Then by (2.2.1), we have

$$O_q(H) = H_{\mathcal{S}_q} = H \cap G_{\mathcal{K}_\rho(\mathcal{S}_q)} = H \cap G_{\mathcal{S}_{\rho' \cup q}} = H \cap 1 = 1 ,$$

since the unique minimal normal subgroup of G is an r-group, and $r \in \rho \setminus \{q\}$. But now $O_q(H) = 1$, and since $H \in \mathcal{S}_q \circ \mathfrak{F}$, then $H \in \mathfrak{F}$, as claimed.

Since $\mathfrak{F}$ is n-Hall-determined, it is Hall-closed by (2.9.3). Thus, all Hall subgroups of H lie in $\mathfrak{F}$. However, if $K \in \mathrm{Hall}_\sigma(G)$ with $|\sigma| \leq n$, then K must be a Hall subgroup of some Hall ρ-subgroup of G with $|\rho| = n+1$ and $r \in \rho$. But then by the above, $K \in \mathfrak{F}$. It follows that $G \in \bigcap\{\mathcal{K}_\sigma(\mathfrak{F}) : \sigma \subseteq \mathbb{P} \text{ and } |\sigma| \leq n\}$, and so $G \in \mathfrak{F}$ since $\mathfrak{F}$ is n-Hall-determined. But $N \in \mathcal{S}_r$ is the unique minimal normal subgroup of G, whence $O_q(G) = 1$, and $G \in \mathcal{S}_q \circ \mathfrak{F}$, as required.

<u>2.9.6 Proposition</u>. Suppose that $\mathfrak{F}$ is a saturated and subgroup-closed formation, and that $\mathfrak{F}$ can be locally defined by the formations $\{f(q)\}_{q \in \mathbb{P}}$, where, for each $q \in \mathbb{P}$, $f(q)$ is a subgroup-closed formation. Suppose that for some $n \in \mathbb{N}$, each $f(q)$ is n-Hall-determined. Then $\mathfrak{F}$ is $(n+1)$-Hall-determined.

<u>Proof</u>. Let $\mathcal{G}$ denote $\mathcal{J}_{n+1}(\mathfrak{F}) = \bigcap\{\mathcal{K}_\pi(\mathfrak{F}) : \pi \subseteq \mathbb{P} \text{ and } |\pi| \leq n+1\}$. Since $\mathfrak{F}$ is S-closed, it is Hall-closed, and so $\mathfrak{F} \subseteq \mathcal{G}$. Our aim is to show that $\mathfrak{F} = \mathcal{G}$; it thus remains to prove that $\mathcal{G} \subseteq \mathfrak{F}$.

Suppose for a contradiction that G is a group of minimal order

belonging to $\mathcal{G} \setminus \mathfrak{F}$. Since $\mathfrak{F}$ is a subgroup-closed formation, then so also are $\mathcal{K}_\pi(\mathfrak{F})$, for all $\pi \subseteq \mathbb{P}$, and $\mathcal{G}$, by (2.1.4e) . It follows that all proper subgroups and factor groups of G belong to $\mathfrak{F}$, and that G has a unique minimal normal subgroup N ; further, $N = G^{\mathfrak{F}} = \mathcal{S}_r$ for some $r \in \mathbb{P}$ (see (1.3.9)). Let F denote $F(G)$; then by the uniqueness of N , it follows that $N \leq F \in \mathcal{S}_r$ and that $N \leq Z(F)$, since $1 \neq Z(F) \trianglelefteq G$.

By [16; Theorem 5.15] (which appears as VI.7.15 of [41]), there exists $T \leq G$ such that $G = N]T$ (see (1.1.1 i)). By Dedekind's identity, we have $F = F \cap NT = N(F \cap T)$. Now $(F \cap T) \trianglelefteq T$, while $[N , F \cap T] = 1$ since $N \leq Z(F)$; it follows that $F \cap T \trianglelefteq NT = G$, and since $T \cap N = 1$, then $F \cap T = 1$ by the uniqueness of N . But now $F = N = O_r(G)$.

Suppose that $t \in \mathbb{P} \setminus \{r\}$. Then $N = O_r(G) \leq O_{t',t}(G)$ and $O_{t',t}(G)/O_r(G) = O_{t',t}(G/O_r(G))$. Since $G/N = G/O_r(G) \in \mathfrak{F}$ and the $f(p)$ locally define $\mathfrak{F}$, it follows that $G/O_{t',t}(G) \in f(t)$. Because $G \notin \mathfrak{F}$, we are forced to conclude that $G/O_{r',r}(G) \notin f(r)$. But $N = O_{r',r}(G)$ since $N \in \mathcal{S}_r$ is the unique minimal normal subgroup of G . Thus, $G/N \notin f(r)$.

Since $G \in \mathcal{G} \setminus \mathfrak{F}$, then G must be divisible by at least $n+2$ distinct primes, by definition of $\mathcal{G}$. Suppose that $\sigma \subseteq \mathbb{P}$ with $|\sigma| \leq n+1$, and let $H \in \mathrm{Hall}_\sigma(G)$. Since $H \lneq G$ then $H \in \mathfrak{F}$, since all proper subgroups of G lie in $\mathfrak{F}$, and so $H/O_{r',r}(H) \in f(r)$.

We claim that $H \in \mathcal{S}_r \circ f(r)$. Suppose firstly that $r \in \sigma$. Then $N \leq H$ and $[O_{r'}(H) , N] \in \mathcal{S}_{r'} \cap \mathcal{S}_r = (1)$, whence $O_{r'}(H) = 1$,

by (1.1.1 f), and so $O_{r',r}(H) = O_r(H)$ and $H/O_r(H) \in f(r)$, giving $H \in \mathcal{S}_r \circ f(r)$ in this case. Next suppose that $r \notin \sigma$. Choose $J \in \mathrm{Hall}_\tau(H)$ where $\tau \subseteq \mathbb{P}$ with $|\tau| \leq n$. Since $|NJ|$ is divisible by at most $n+1$ distinct primes, then $NJ \lneq G$ and so $NJ \in \mathfrak{F}$; thus $NJ/O_{r',r}(NJ) \in f(r)$. Now, $[O_{r'}(NJ), N] \in \mathcal{S}_{r'} \cap \mathcal{S}_r = (1)$, and so $O_{r'}(NJ) = 1$, since $N = F(G)$. It follows that $NJ/O_r(NJ) \in f(r)$, and so $NJ \in \mathcal{S}_r \circ f(r)$. But $\mathcal{S}_r$ and $f(r)$ are subgroup-closed formations, and so $\mathcal{S}_r \circ f(r)$ is subgroup-closed, by (1.3.14 c). It follows that $J \in \mathcal{S}_r \circ f(r)$. Now $J \leq H \in \mathcal{S}_{r'}$ (as $r \notin \sigma$), so that $O_r(J) = 1$ and $J \in f(r)$. But J was an arbitrary Hall τ-subgroup of H for $|\tau| \leq n$, and so $H \in f(r)$, since $f(r)$ is n-Hall-determined. Since $r \notin \sigma$ then in fact $H \in \mathcal{S}_r \circ f(r)$, as claimed.

We have proved that if $H \in \mathrm{Hall}_\sigma(G)$ with $|\sigma| \leq n+1$, then $H \in \mathcal{S}_r \circ f(r)$. But by (2.9.5), $\mathcal{S}_r \circ f(r)$ is $(n+1)$-Hall-determined; it follows that $G \in \mathcal{S}_r \circ f(r)$. But then $G^{f(r)} \leq N$ and $G/N \in f(r)$, contrary to the conclusion of the fourth paragraph of the proof. This completes the proof.

<u>2.9.7 Theorem</u>. Let $\mathfrak{F}$ be a primitive saturated formation of finite defect $n \in \mathbb{N}$. Then $\mathfrak{F}$ is n-Hall-determined.

<u>Proof</u>. If $n = 1$, then $\mathfrak{F}$ can be locally defined by formations $\{f(q) : q \in \mathbb{P}\}$, where for each $q \in \mathbb{P}$, $f(q) = \emptyset$ or $\mathcal{S}$. Thus, $\mathfrak{F} = \mathcal{S}_\pi$ for $\pi = \{q : f(q) = \mathcal{S}\}$. If $G \in \mathcal{S}$, then clearly G belongs to $\mathcal{S}_\pi$ if and only if all Sylow subgroups of G are

π-groups, and so $\mathcal{S}_\pi$ is certainly 1-Hall-determined.

The general result now follows by induction, using (2.9.6), remembering from (1.6.1) that $\mathfrak{F}$ can be locally defined by $\{f(q) : q \in \mathbb{P}\}$, where, for each q, $f(q)$ is a primitive saturated formation (and so a subgroup-closed formation) of defect at most $n-1$; we make use of the second assertion of (2.9.3) if any of the $f(q)$ is of defect $k \lneq n-1$, in which case that $f(q)$ is k-Hall-determined by the induction, and $(n-1)$-Hall-determined by (2.9.3).

<u>2.9.8 Remarks.</u> (a) We thus obtain, for example, the result of Kramer [42] mentioned above that $G \in \mathcal{S}$ has p-length at most n if and only if all Hall π-subgroups of G for $|\pi| \leq n+1$ have p-length at most n.

(b) In section 3.6, it will be shown that the smallest normal Fitting class, $\mathfrak{N}$, is not n-Hall-determined for any $n \in \mathbb{N}$.

CHAPTER THREE.

Products and Normal Fitting Classes.

In this chapter, we shall be chiefly concerned with products of the form $\mathfrak{F} * \mathfrak{H}$, where $\mathfrak{F} \subseteq \mathfrak{S}$ is a Fitting class and $\mathfrak{H}$ is the smallest normal Fitting class ; it will be recalled from (1.4.6) that such a product is again normal. The study of such products seems to have been initiated by Cossey, [18] . As in Chapter Two, we shall be dealing exclusively with classes of soluble groups.

Our two main themes will be to investigate when products $\mathfrak{F} * \mathfrak{H}$ are Hall-closed and when products $\mathfrak{F} * \mathfrak{H}$ and $\mathfrak{G} * \mathfrak{H}$ are equal.

The first two sections are introductory; in section 3.1, we use our knowledge of $\mathfrak{K}_{\pi}(\ )$ to exhibit certain Hall-closed classes $\mathfrak{F} * \mathfrak{H}$.

Section 3.3 contains one of the main theorems of the chapter ; namely, that if $\mathfrak{F}$ is a Fitting class of characteristic π , where $\pi \subsetneqq \mathbb{P}$, then $\mathfrak{F} * \mathfrak{H}$ is Hall-closed if and only if $\mathfrak{F}^* = \mathfrak{S}_{\pi}$. The situation when $\mathfrak{F}$ has characteristic $\mathbb{P}$ is more complicated, and not fully resolved ; various aspects of this question are discussed in sections 3.3, 3.4 and 3.8 .

In Section 3.5 , it is shown that the class $\mathfrak{N}_{\pi} * \mathfrak{H}$ is not $\Xi_{\mathfrak{N}}$-closed if $|\pi| \geq 2$, as promised after (2.3.9).

Section 3.6 contains a diversion from products; we use a lemma from 3.2 to show that $\mathfrak{H}$ is not n-Hall-determined for any $n \in \mathbb{N}$.

In Section 3.7, we discuss a recent important result, (3.7.1) , stemming from work of Laue, Lausch and Pain, [44] , and Bryant and Kovács, [7] ; none of the material in this section is therefore original. The result in question will be used several times in later sections.

In section 3.8, we use the result (3.7.1) in a construction which will have two applications, one in Section 3.9. In 3.8, we use the construction to show that if $\mathfrak{F}$ is a Lockett class (of characteristic $\mathbb{P}$) such that $\mathfrak{F} * \mathfrak{H}$ is Hall-closed, then $\mathfrak{F}$ must itself be Hall-closed. The fact that we seem to need (3.7.1) to prove even this result when $\mathfrak{F}$ is of "full characteristic" contrasts with the result of section 3.3 when $\mathfrak{F}$ is of less-than-full characteristic.

Section 3.9 is devoted to a conjecture of Cossey, [18], on the equality of products of the form $\mathfrak{F} * \mathfrak{H}$. With some help from (3.7.1), plus certain of Cossey's results, the result of section 3.3 on Hall-closed classes $\mathfrak{F} * \mathfrak{H}$ is used to show that $\mathfrak{F} * \mathfrak{H} = \mathfrak{S}_\pi * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{S}_\pi$. It is also shown, using Berger's lemma (1.9.3) that if $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes each of odd characteristic such that $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$, then $\mathfrak{F}^* = \mathfrak{G}^*$. As a corollary, it is shown that if $\mathfrak{G}$ is a Fitting class of odd characteristic for which the Lockett conjecture is true, then $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{G}^*$.

3.1 Products which are Hall-closed.

3.1.1 Proposition. Suppose that $\mathfrak{G}$ is a Hall-closed normal Fitting class. Then $\mathfrak{G} * \mathfrak{F}$ is Hall-closed for all Fitting classes $\mathfrak{F}$. In particular, $\mathfrak{H} * \mathfrak{F}$ is always Hall-closed.

Proof. Let $\mathfrak{F}$ be a Fitting class, and let $G \in \mathfrak{G} * \mathfrak{F}$, so that $G' \geqslant G_{\mathfrak{G}}$ by (1.4.2). Let $\pi \subseteq \mathbb{P}$, and let $H \in \mathrm{Hall}_{\pi}(G)$. Then $H \cap G_{\mathfrak{G}} \in \mathrm{Hall}_{\pi}(G_{\mathfrak{G}})$, and so $H \cap G_{\mathfrak{G}} \in \mathfrak{G}$ since $\mathfrak{G}$ is Hall-closed. But $H \cap G_{\mathfrak{G}} \trianglelefteq H$, and so $H \cap G_{\mathfrak{G}} \leqslant H_{\mathfrak{G}}$. But $H/(H \cap G_{\mathfrak{G}}) \cong H G_{\mathfrak{G}}/G_{\mathfrak{G}}$, which is an abelian $(\pi \cap \mathrm{char}(\mathfrak{F}))$-group, since $G/G_{\mathfrak{G}} \in \mathfrak{F} \cap \mathfrak{A}$. Thus, $H/(H \cap G_{\mathfrak{G}}) \in \mathfrak{F} \cap \mathfrak{A}$, and so $H/H_{\mathfrak{G}} \in \mathfrak{F}$ since $H \cap G_{\mathfrak{G}} \leqslant H_{\mathfrak{G}}$. But now $H \in \mathfrak{G} * \mathfrak{F}$, and the result follows since π was arbitrary.

The above proposition leads us to ask the following. Suppose that $\mathfrak{G}$ is a Fitting class such that for all Fitting classes $\mathfrak{F}$ we have that $\mathfrak{G} * \mathfrak{F}$ is Hall-closed. Must then $\mathfrak{G}$ be normal ?

We will see in (3.4.3) that any such class $\mathfrak{G}$ must at least have characteristic $\mathbb{P}$.

We sketch the proof of the following lemma of Cossey.

3.1.2 Lemma (Cossey, [18; 4.2]). Let $\mathfrak{F}$ be a Fitting class such that $\mathfrak{F}^* \cap \mathfrak{H} \leqslant \mathfrak{F}$. Then $\mathfrak{F} * \mathfrak{H} = \mathfrak{F}^* * \mathfrak{H}$.

Proof. Let $G \in \mathfrak{S}$. Then $G_{\mathfrak{F}} \leqslant G_{\mathfrak{F}^*}$. Since $G' \in \mathfrak{H}$ and $G_{\mathfrak{F}^*} \trianglelefteq G$, then $G' \cap G_{\mathfrak{F}^*} \in \mathfrak{H}$, and $G' \cap G_{\mathfrak{F}^*} \leqslant G_{\mathfrak{F}^* \cap \mathfrak{H}} \leqslant G_{\mathfrak{F}} \leqslant G_{\mathfrak{F}^*}$. Thus, $G_{\mathfrak{F}} = G_{\mathfrak{F}}(G' \cap G_{\mathfrak{F}^*}) = G_{\mathfrak{F}} G' \cap G_{\mathfrak{F}^*}$, by Dedekind's identity.

Let $\overline{}$ denote images in $G/G_{\mathfrak{F}}$.

By the above, $\overline{1} = \overline{G'} \cap \overline{(G_{\mathfrak{F}^*})}$.

Now, $\overline{G}/\overline{G'} \in \mathcal{A} \subseteq \mathcal{N} \subseteq \mathcal{H}$.

Applying the quasi-R_0-lemma, (1.9.2 a), in $\overline{G}$, we find that $\overline{G} \in \mathcal{H}$ if and only if $\overline{G}/\overline{(G_{\mathfrak{F}^*})} \in \mathcal{H}$.

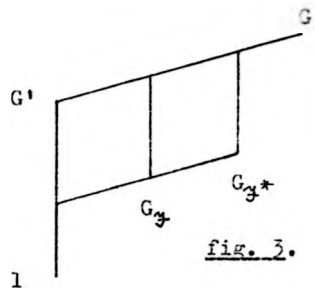


fig. 3.

But $\overline{G}/\overline{(G_{\mathfrak{F}^*})} \cong G/(G_{\mathfrak{F}^*})$, and so $G \in \mathfrak{F} * \mathcal{H}$ if and only if $G \in \mathfrak{F}^* * \mathcal{H}$, completing the proof.

Had the Lockett conjecture, (1.5.14), been true, then the hypothesis $\mathfrak{F}^* \cap \mathcal{H} \subseteq \mathfrak{F}$ in (3.1.2) would have automatically been satisfied by all Fitting classes. To the best of our knowledge, the question as to whether the conclusion of (3.1.2) remains true for arbitrary $\mathfrak{F}$ is still open ; we have had no success in trying to incorporate the Berger-Cossey counter-example to the Lockett conjecture into an example denying (3.1.2). More will be said on this in section 3.9 .

<u>3.1.3 Proposition</u>. Let π be a set of primes, and suppose that $\mathcal{D}$ is a Fitting class with $\mathcal{D}^* = \mathcal{S}_\pi$. Then $\mathcal{D} * \mathcal{H}$ is Hall-closed.

<u>Proof</u>. Since $\mathcal{S}_\pi$ satisfies the Lockett conjecture by (2.4.7 b), then $\mathcal{S}_\pi \cap \mathcal{H} \subseteq \mathcal{D}$ and so $\mathcal{D} * \mathcal{H} = \mathcal{S}_\pi * \mathcal{H}$ by (3.1.2) ; thus it will suffice to show that $\mathcal{S}_\pi * \mathcal{H}$ is Hall-closed.

Let $\tau \subseteq \mathbb{P}$ be arbitrary. By (2.4.7 a), $\mathcal{H} \subseteq \mathcal{K}_\tau(\mathcal{H})$ and so by (2.6.2) and (1.3.13 b), we have $\mathcal{S}_\pi * \mathcal{H} \subseteq \mathcal{S}_{\pi \cup \tau'} * \mathcal{H} \subseteq \mathcal{S}_{\pi \cup \tau'} * \mathcal{K}_\pi(\mathcal{H})$.

It is not hard to check that $\mathcal{K}_\tau(\mathcal{S}_\pi) = \mathcal{S}_{\pi \cup \tau'}$, and so with (2.3.1) we obtain $\mathcal{S}_\tau * \mathcal{H} \subseteq \mathcal{K}_\tau(\mathcal{S}_\pi) * \mathcal{K}_\tau(\mathcal{H}) = \mathcal{K}_\tau(\mathcal{S}_\pi * \mathcal{H})$. Since $\tau \subseteq \mathbb{P}$ was arbitrary, the result follows.

<u>3.1.4 Proposition</u>. Suppose that the Fitting class $\mathcal{Y}$ and set of primes σ satisfy the following condition.

(£) $\mathrm{Char}(\mathcal{Y}) \subseteq \sigma$, and whenever $\hat{\sigma} \subseteq \sigma$ is such that $\sigma = \hat{\sigma} \dot{\cup} \{p\}$ (disjoint union) for some $p \in \sigma$, then $\mathcal{S}_{\hat{\sigma}} \subseteq \mathcal{Y}$.

Then if $\mathcal{G}$ is any Hall-closed Fitting class, it follows that $\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G}$ is also Hall-closed .

<u>Proof</u>. Let $\tau \subseteq \mathbb{P}$ be arbitrary; we show that $\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G} \subseteq \mathcal{K}_\tau(\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G})$.

Now, by (2.3.1) and (2.1.4 g), we have

$$\mathcal{K}_\tau(\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G}) = \mathcal{K}_\tau(\mathcal{K}_\sigma(\mathcal{Y})) * \mathcal{K}_\tau(\mathcal{G}) = \mathcal{K}_{\sigma\cap\tau}(\mathcal{G}) * \mathcal{K}_\tau(\mathcal{G}) . \quad \cdots(1)$$

Suppose firstly that $\sigma \subseteq \tau$. Then $\mathcal{K}_{\sigma\cap\tau}(\mathcal{Y}) = \mathcal{K}_\sigma(\mathcal{Y})$, and so, since $\mathcal{G}$ is Hall-closed, we have using (1),

$$\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G} \subseteq \mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{K}_\tau(\mathcal{G}) = \mathcal{K}_{\sigma\cap\tau}(\mathcal{Y}) * \mathcal{K}_\tau(\mathcal{G}) = \mathcal{K}_\tau(\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G}) ,$$

as required.

Next suppose that $\sigma \not\subseteq \tau$. Then for some $p \in \sigma$ we have $\tau \cap \sigma \subseteq \hat{\sigma}$, where $\sigma = \hat{\sigma} \dot{\cup} \{p\}$. But then by hypothesis, $\mathcal{S}_{\sigma\cap\tau} \subseteq \mathcal{S}_{\hat{\sigma}} \subseteq \mathcal{Y}$ and so $\mathcal{S} = \mathcal{K}_{\sigma\cap\tau}(\mathcal{S}_{\sigma\cap\tau}) \subseteq \mathcal{K}_{\sigma\cap\tau}(\mathcal{Y})$, by (2.1.4d). But now we have

$$\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G} \subseteq \mathcal{S} = \mathcal{S} * \mathcal{K}_\tau(\mathcal{G}) = \mathcal{K}_{\sigma\cap\tau}(\mathcal{Y}) * \mathcal{K}_\tau(\mathcal{G}) = \mathcal{K}_\tau(\mathcal{K}_\sigma(\mathcal{Y}) * \mathcal{G}) ,$$

by (1), concluding the proof .

<u>3.1.5 Remarks</u>. Suppose that $\mathcal{Y}$ and σ satisfy (£) in (3.1.4).

(a) If $|\sigma| \gneq 1$, it follows that $\mathrm{char}(\mathcal{Y}) = \sigma$. However, if $\sigma = \{p\}$, we may take $\mathcal{Y} = (1)$ in (3.1.4), whence $\mathcal{K}_p((1)) * \mathcal{G}$ $(= \mathcal{S}_{p'} * \mathcal{G})$

is Hall-closed whenever $\mathcal{G}$ is. In particular, if p, q and r are primes, then such classes as $\mathcal{S}_{p'} * \mathcal{A}$, $\mathcal{S}_{r'} * \mathcal{S}_{q'} * \mathcal{S}_{p'} * \mathcal{A}$ and $\mathcal{S}_{p'} * \mathcal{S}_{p} * \mathcal{A}$, are Hall-closed, the latter by (3.1.3).

(b) By (2.1.4 j), $\mathrm{char}(\mathcal{K}_\sigma(\mathcal{Y})) = \sigma' \cup \mathrm{char}(\mathcal{Y})$, and so either $\mathrm{char}(\mathcal{Y}) = \sigma$ and $\mathrm{char}(\mathcal{K}_\sigma(\mathcal{Y})) = \mathbb{P}$, or else $\mathcal{Y} = (1)$, $\sigma = \{p\}$ for some $p \in \mathbb{P}$, and $\mathrm{char}(\mathcal{K}_\sigma(\mathcal{Y})) = \{p\}'$.

(c) Suppose that there exist $A, B \in \mathcal{S} \setminus \mathcal{K}_\sigma(\mathcal{Y})$ with $(|A|, |B|) = 1$. Choose $H_A \in \mathrm{Hall}_\sigma(A)$ and $H_B \in \mathrm{Hall}_\sigma(B)$. Since $(|A|, |B|) = 1$, there must exist distinct primes p and q with $H_A \in \mathcal{S}_{\sigma \setminus \{p\}}$ and $H_B \in \mathcal{S}_{\sigma \setminus \{q\}}$. But then $H_A, H_B \in \mathcal{Y}$, by (£), contrary to the fact that $A, B \notin \mathcal{K}_\sigma(\mathcal{Y})$. It follows that if $A, B \in \mathcal{S} \setminus \mathcal{K}_\sigma(\mathcal{Y})$, then $(|A|, |B|) \gneqq 1$.

(e) If $(\mathcal{Y}_1, \sigma_1), \ldots, (\mathcal{Y}_n, \sigma_n)$ all satisfy (£), then by (3.1.2) applied inductively, $\mathcal{K}_{\sigma_n}(\mathcal{Y}_n) * \cdots * \mathcal{K}_{\sigma_1}(\mathcal{Y}_1) * \mathcal{G}$ is Hall-closed whenever $\mathcal{G}$ is.

(f) Examples of pairs $(\mathcal{Y}_i, \sigma_i)$ satisfying (£) include for example $\mathcal{Y}_1 = \mathcal{S}_p * \mathcal{S}_q$, $\sigma_1 = \{p,q\}$ and $\mathcal{Y}_2 = \mathcal{S}_{\{p,q\}} * \mathcal{S}_{\{q,r\}} * \mathcal{S}_{\{r,p\}}$, with $\sigma_2 = \{p,q,r\} \subseteq \mathbb{P}$. In the latter case, $\mathcal{Y}_2$ contains no group of type $M(r,p,q)$, as in (1.7.6), and so $\mathcal{Y}_2 \cap \mathcal{S}_{\sigma_2} \neq \mathcal{S}_{\sigma_2}$, and $\mathcal{K}_{\sigma_2}(\mathcal{Y}_2) \neq \mathcal{S}$.

<u>3.1.6 Definition</u>. The class $\mathcal{X}$ of finite soluble groups is said to be <u>replete</u> if whenever $A, B \in \mathcal{S} \setminus \mathcal{X}$, then $(|A|, |B|) \gneqq 1$; we regard $\mathcal{S}$ as an "honorary" replete class.

Examples of replete classes, apart from the classes $\mathcal{K}_\sigma(\mathcal{Y})$ of (3.1.4/5), include all classes $\mathcal{X}$ such that $\mathcal{X} \supseteq \mathcal{S}_{p'}$ for some $p \in \mathbb{P}$, since then if $A \in \mathcal{S} \setminus \mathcal{X}$, we have $p \mid |A|$; the classes $\mathcal{C}_p$ and $e_p(\mathcal{F})$ fall into this category (see (1.3.16/17)).

3.2 Preliminary results .

3.2.1 Lemma. Suppose that U is a group admitting the cyclic group $P = \langle \phi \rangle$ as a group of operators, where $|\phi| = n = p^\alpha$ ($p \in \mathbb{P}$, $\alpha \in \mathbb{N}$) . Suppose that in the semi-direct product $U] P$ we have $U = (UP)'$.

Let $D = V_1 \times \cdots \times V_n$ ($n \in \mathbb{N}$ as above) be a group which is the (internal) direct product of groups V_i such that for each i , $\mu_i : U \to V_i$ is an isomorphism.

Now D admits $\langle \phi \rangle$ as a group of operators via

$$(v_1 \cdots v_n)^\phi = (v_1 \mu_1^{-1})^\phi \mu_1 \cdots (v_n \mu_n^{-1})^\phi \mu_n \quad : v_i \in V_i \ .$$

Then $D] \langle \phi \rangle \in \mathcal{H}$.

Proof. It is straightforward to check that $\langle \phi \rangle$ does indeed act as a group of operators on D .

Let X denote the regular wreath product $X = (UP) \text{ wr } C_{p^\alpha}$; then $X = (UP)^*] \langle \theta \rangle$, where $(UP)^*$ is the base group and $\langle \theta \rangle$ is the "standard complement", as in (1.7.1) . Let U_iP_i denote the $i^{\text{th.}}$ coordinate subgroup

$$U_iP_i = \{(1,\ldots,\underset{i}{y},\ldots,1) \in (UP)^* : y \in UP\} \ ,$$

so that U_iP_i is naturally isomorphic with UP, via the isomorphism

$$\nu_i : y \mapsto (1,\ldots,\underset{i}{y},\ldots 1) \quad ; \quad y \in UP \ .$$

We may suppose notation chosen so that $\theta : U_iP_i \mapsto U_{i+1}P_{i+1}$ (indices (mod(n)), lying in range $1,\ldots,n$); specifically, we may assume that $\theta : y\nu_i \mapsto y\nu_{i+1}$ for $y \in UP$.

Let ϕ_i denote $\phi\nu_i = (1,\ldots,\phi,\ldots,1)$ P_i . Then

$$\phi_1\phi_2 \cdots \phi_n = \phi_1\phi_2^{-1}(\phi_2\phi_3^{-1})^2 \cdots (\phi_{n-1}\phi_n^{-1})^{n-1} \quad , = \psi \text{ say.}$$

But $\phi_i\phi_{i+1}^{-1} = \phi_i(\phi_i^{\theta})^{-1} \in X'$, and so $\psi \in X'$. By hypothesis, $U = (UP)'$ and so $U_1 \times \cdots \times U_n = (U_1P_1 \times \cdots \times U_nP_n)' \leqslant X'$.

It follows that $(U_1 \times \cdots \times U_n)\langle\psi\rangle \leqslant X'$. But $X = (UP)^*\langle\theta\rangle$, and so $X/(U_1 \times \cdots \times U_n) \in \mathcal{S}_p$. Thus $(U_1 \times \cdots \times U_n)\langle\psi\rangle$ on X' , and so $(U_1 \times \cdots \times U_n)\langle\psi\rangle \in \mathfrak{H}$, since $X' \in \mathfrak{H}$ by (1.4.2).

But we may check that the map

$$\lambda : (u_1 \cdots u_n)\psi^j \mapsto (u_1\mu_1 \cdots u_n\mu_n)\phi^j \quad , u_i \in U_i \ ,$$

is an isomorphism from $(U_1 \times \cdots \times U_n)\langle\psi\rangle$ to $(V_1 \times \cdots \times V_n)\langle\phi\rangle$, and the result follows.

The next lemma is basically a result quoted by P. Hauck during a talk at Warwick in 1977, recast in a form suitable for our needs.

<u>3.2.2 Lemma (Hauck).</u> (a) Suppose that $\mathfrak{F}$ is a Lockett class and that $\mathfrak{G}$ is a Fitting class with $\mathfrak{G} \subseteq \mathfrak{F}$. Suppose that there exists a prime $t \in \mathrm{char}(\mathfrak{F})$ such that the following conditon is satisfied.

(ϕ) Whenever $N \in \mathfrak{G}$ and τ is an automorphism of N of order t , then $N\langle\tau\rangle \in \mathfrak{F}$

Then, it follows that $\mathfrak{G} * \mathcal{S}_t \subseteq \mathfrak{F}$.

(b) Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are Lockett classes and that there exists a prime t such that $\mathfrak{G} * \mathfrak{S}_t \notin \mathfrak{F}$. Let $\mathfrak{D}$ be a Fitting class such that $\mathfrak{D}^* = \mathfrak{G}$. Then, it follows that $\mathfrak{D} * \mathfrak{S}_t \notin \mathfrak{F}$.

<u>Proof</u>. (a) Suppose for a contradiction that $\mathfrak{G} * \mathfrak{S}_t \notin \mathfrak{F}$, and let G be a group of minimal order in $\mathfrak{G} * \mathfrak{S}_t \setminus \mathfrak{F}$. Then G has a unique maximal normal subgroup M, and $M = G_{\mathfrak{F}}$. Since $\mathfrak{G} \subseteq \mathfrak{F}$ and $G \in \mathfrak{G} * \mathfrak{S}_t$, then $|G : G_{\mathfrak{F}}| = t$.

Let $W = G \text{ wr } C_t$, so that $W = G^*] \langle\tau\rangle$, where G^* is the base group and $\langle\tau\rangle$ is the standard complement. Now, $G = G_1 \times \cdots \times G_t$, where the coordinate subgroups G_i are permuted by $\langle\tau\rangle$ in an orbit of length t. But then τ induces an automorphism of order t on $(G_{\mathfrak{G}})^* = (G_{\mathfrak{G}})_1 \times \cdots \times (G_{\mathfrak{G}})_t$. (We note that $G_{\mathfrak{G}}$ is non-trivial, since $G \notin \mathfrak{S}_t \subseteq \mathfrak{F}$). Thus by $(\S)$, $(G_{\mathfrak{G}})^* \langle\tau\rangle \in \mathfrak{F}$. Since $G/G_{\mathfrak{G}} \in \mathfrak{S}_t$, then $W/(G_{\mathfrak{G}})^* \in \mathfrak{S}_t$ and so $(G_{\mathfrak{G}})^* \langle\tau\rangle$ sn W. But then by (1.3.2c), $(G_{\mathfrak{G}})^* \langle\tau\rangle \leq W_{\mathfrak{G}}$. But $W_{\mathfrak{G}} \leq W_{\mathfrak{F}}$ since $\mathfrak{G} \subseteq \mathfrak{F}$, and so $\langle\tau\rangle \leq W_{\mathfrak{F}}$. But $\mathfrak{F}$ is a Lockett class, and so by (1.7.3) we have $W_{\mathfrak{F}} \leq (G_{\mathfrak{F}})^* \leq G^*$, whence $\langle\tau\rangle \nleq W_{\mathfrak{F}}$ contradicting what we just saw. This completes the proof of part (a).

(b) If $\mathfrak{D} * \mathfrak{S}_t \subseteq \mathfrak{F}$, then $\mathfrak{G} * \mathfrak{S}_t = \mathfrak{D}^* * \mathfrak{S}_t = (\mathfrak{D} * \mathfrak{S}_t)^* \subseteq \mathfrak{F}^* = \mathfrak{F}$, by (2.5.3) and (1.5.3e), contrary to hypothesis, completing the proof.

<u>3.2.3 Lemma</u>. Let p and q be distinct primes, and let G be a group of minimal order in $\mathfrak{S}_p * \mathfrak{S}_q \setminus \mathfrak{H}$. Then $G = P] Q$, where $G' = P \in \mathrm{Syl}_p(G)$, Q is a cyclic q-group and $O_q(G) = 1$.

Proof. By (1.3.9/10), G/G' is a cyclic r-group for some $r \in \mathbb{P}$, and $G' = G^{\mathcal{N}}$. Since $G \in \mathcal{S}_p * \mathcal{S}_q$ then $G^{\mathcal{N}} \leq O_p(G)$. If $r = p$, it follows that $G \in \mathcal{S}_p \subseteq \mathcal{H}$, contrary to hypothesis. Thus $r = q$, and then $G^{\mathcal{N}} = O_p(G) = P \in \mathrm{Syl}_p(G)$, while $G/G^{\mathcal{N}} \cong Q \in \mathrm{Syl}_q(G)$.

Both P and $O_q(G)$ are normal in G, and intersect trivially. Further, $G/(P\,O_q(G)) \in \mathcal{S}_q \subseteq \mathcal{N}$. Suppose that $G/O_q(G) \in \mathcal{H}$. Then, since $G/P \cong Q \in \mathcal{S}_q \subseteq \mathcal{H}$, we conclude from the quasi-R_0-lemma, (1.9.2 a), that $G \in \mathcal{H}$, a contradiction. Thus $G/O_q(G) \in \mathcal{S}_p * \mathcal{S}_q \setminus \mathcal{H}$, and so $O_q(G) = 1$, by the minimality of $|G|$, as required.

3.3 Hall-closure of $\mathcal{D} * \mathcal{H}$ where $\mathcal{D}^*$ is not replete.

In this section we shall prove the following result, Theorem 3.3.1, by a sequence of lemmas and constructions. Numbering of statements and equations in the text will run consecutively through the section.

3.3.1 Theorem. Let $\mathcal{D}$ be a Fitting class of characteristic $\pi \subseteq \mathbb{P}$, and let $\mathcal{F} = \mathcal{D}^*$. Suppose that one of the following conditions is satisfied.

(1) $\pi \neq \mathbb{P}$ and $\mathcal{F} \subsetneqq \mathcal{S}_\pi$.

(2) $\pi = \mathbb{P}$ and there exists a pair of groups of co-prime orders neither of which belongs to $\mathcal{F}$. (That is, $\mathcal{F}$ is not replete).

Then, it follows that $\mathcal{D} * \mathcal{H}$ is not Hall-closed.

In view of (3.1.3), we immediately obtain the following.

<u>3.3.2 Theorem</u>. Let $\mathcal{D}$ be a Fitting class with $\mathrm{char}(\mathcal{D}) = \pi \subsetneqq \mathbb{P}$. Then $\mathcal{D} * \mathfrak{N}$ is Hall-closed if, and only if, $\mathcal{D}^* = \mathcal{S}_\pi$.

We now fix for the remainder of this section a Lockett class $\mathfrak{F}$ of characteristic π satisfying either condition (1) or condition (2) of (3.3.1), and a Fitting class $\mathcal{D}$ with $\mathcal{D}^* = \mathfrak{F}$. Conditions (1) and (2) being mutually exclusive, we shall say that $\mathfrak{F}$ is of <u>Type 1</u> or <u>Type 2</u> according as it satisfies condition (1) or (2) , respectively, of (3.3.1). We now come to the first step in the proof of (3.3.1).

<u>3.3.3 Lemma</u>. There exist groups R and N of co-prime orders, possessing automorphisms τ_1 and τ_2 , respectively, each of order $t \in \pi \setminus \mathrm{char}(|R|)$, such that the following hold.

(a) If $\mathfrak{F}$ is of Type 1 , then

(i) R is an elementary abelian r-group, where $r \in \mathbb{P} \setminus \mathrm{char}(\mathfrak{F})$, with $R = [R , \langle\tau_1\rangle]$ and $(R]\langle\tau_1\rangle)_{\mathfrak{F}} = (R]\langle\tau_1\rangle)_{\mathcal{D}} = 1$; and

(ii) $(N]\langle\tau_2\rangle)_{\mathfrak{F}} = (N]\langle\tau_2\rangle)_{\mathcal{D}} = N$.

(b) If $\mathfrak{F}$ is of Type 2 , so that $\mathrm{char}(\mathfrak{F}) = \mathbb{P}$, then

(i) $R_{\mathfrak{F}} = R_{\mathcal{D}} = (R]\langle\tau_1\rangle)_{\mathfrak{F}} = (R]\langle\tau_1\rangle)_{\mathcal{D}} \neq R$, and

$R/R_{\mathfrak{F}} = [R/R_{\mathfrak{F}} , \langle\tau_1\rangle]$ is elementary abelian ; and

(ii) $(N]\langle\tau_2\rangle)_{\mathfrak{F}} = (N]\langle\tau_2\rangle)_{\mathcal{D}} = N$.

<u>Proof</u>. (a) Since in this case $\mathfrak{F} \subsetneqq \mathcal{S}_\pi$, there must exist $t \in \pi$ with $\mathfrak{F} \subsetneqq \mathfrak{F} * \mathcal{S}_t$. Choose $r \in \mathbb{P} \setminus \pi$, and let R be a faithful irreducible GF(r)-module for the cyclic group $\langle\tau_1\rangle$ of order t . Now form the semi-direct product $R]\langle\tau_1\rangle$; since R is the unique

minimal normal subgroup of $R\langle\tau_1\rangle$ and $r \notin \mathrm{char}(\mathfrak{F})$, then $(R]\langle\tau_1\rangle)_{\mathfrak{F}} = (R]\langle\tau_1\rangle)_{\mathfrak{D}} = 1$, as required.

By (3.2.2 b), $\mathfrak{D} * \mathfrak{S}_t \notin \mathfrak{F}$, and so by (3.2.2 a) there exists $N \in \mathfrak{D}$ possessing an automorphism τ_2 of order t such that $N]\langle\tau_2\rangle \notin \mathfrak{F}$. But then $(N]\langle\tau_2\rangle)_{\mathfrak{F}} = (N]\langle\tau_2\rangle)_{\mathfrak{D}} = N$. Since $r \notin \pi$ then $(|R|, |N|t) = 1$, completing the proof of (a) .

(b) Here $\mathfrak{F}$ is of Type 2 , with $\mathrm{char}(\mathfrak{F}) = \mathbb{P}$. Let $\overline{X}$ and $\overline{Y}$ be groups of co-prime orders belonging to $\mathfrak{S} \setminus \mathfrak{F}$. Let $\alpha = \mathrm{char}(\overline{X})$ and $\beta = \mathrm{char}(\overline{Y})$, so that $\alpha \cap \beta = \emptyset$. Let X and Y be subnormal subgroups of $\overline{X}$ and $\overline{Y}$, respectively, each of minimal order subject to not belonging to $\mathfrak{F}$. Then $X_{\mathfrak{F}}$ and $Y_{\mathfrak{F}}$ are the unique maximal normal subgroups of X and Y , respectively, of prime indices $u \in \alpha$ and $v \in \beta$, respectively. Since $\mathfrak{D}^* = \mathfrak{F}$, then, as in (1.5.6), we have $X_{\mathfrak{D}} \geqslant X'$ and $Y_{\mathfrak{D}} \geqslant Y'$, and it follows that $X \in \mathfrak{D} * \mathfrak{S}_u$ and $Y \in \mathfrak{D} * \mathfrak{S}_t$. In fact, since $X \in \mathfrak{S}_\alpha$, then $X \in (\mathfrak{D} \cap \mathfrak{S}_\alpha) * \mathfrak{S}_u \setminus \mathfrak{F}$ and so $(\mathfrak{D} \cap \mathfrak{S}_\alpha) \notin \mathfrak{F}$ while $\mathfrak{D} \cap \mathfrak{S}_\alpha \subseteq \mathfrak{F}$. But now by (3.2.2 a), there exists $A \in \mathfrak{D} \cap \mathfrak{S}_\alpha$ possessing an automorphism ν of order u such that $A]\langle\nu\rangle \notin \mathfrak{F}$. Thus $(A]\langle\nu\rangle)_{\mathfrak{F}} = (A]\langle\nu\rangle)_{\mathfrak{D}} = A$.

Let $W = (A\langle\nu\rangle) \mathrm{wr}\, C_t$, where $t = |Y : Y_{\mathfrak{F}}|$ as above ; then $W = R_0]\langle\tau_1\rangle$, where $R_0 = (A\langle\nu\rangle)^*$ is the base group of W and $\langle\tau_1\rangle$ is the standard complement, of order t . Since $|\nu| = u \in \mathfrak{S}_\alpha$ and $A \in \mathfrak{S}_\alpha$, then $R_0 \in \mathfrak{S}_\alpha$.

Since $\mathfrak{F}$ is a Lockett class and $(A\langle\nu\rangle)_{\mathfrak{F}} = A$, then by (1.7.3),

$$W_{\mathfrak{F}} = (R_0\langle\tau_1\rangle)_{\mathfrak{F}} = (R_0)_{\mathfrak{F}} = ((A\langle\nu\rangle)_{\mathfrak{F}})^* = A^* . \qquad \cdots(1)$$

Now $A \in \mathcal{D}$, whence $A^* \in \mathcal{D}$. Since $\mathcal{D} \subseteq \mathcal{F}$, then (1) implies

$$W_{\mathcal{D}} = (R_o \langle \tau_1 \rangle)_{\mathcal{D}} = (R_o)_{\mathcal{D}} = ((A \langle \nu \rangle)^*)_{\mathcal{D}} = A^* . \qquad \ldots(2)$$

We note that $R_o/(R_o)_{\mathcal{F}} = (A \langle \nu \rangle)^*/A^* \cong \langle \nu \rangle \times \quad \times \langle \nu \rangle$ is is elementary abelian of order u^t. (Notation as in (1.7.1)).

Let $R = (R_o)_{\mathcal{F}} [R_o , \langle \tau_1 \rangle]$. By (1.1.3 a), $R \trianglelefteq R_o \langle \tau_1 \rangle$; we may also check that $R \langle \tau_1 \rangle \trianglelefteq R_o \langle \tau_1 \rangle$. From these observations, together with equations (1) and (2), we obtain

$$R_{\mathcal{F}} = (R_o)_{\mathcal{F}} = (R_o \langle \tau_1 \rangle)_{\mathcal{F}} = (R \langle \tau_1 \rangle)_{\mathcal{F}} = A^* , \qquad \ldots(3)$$

and

$$R_{\mathcal{D}} = (R_o)_{\mathcal{D}} = (R_o \langle \tau_1 \rangle)_{\mathcal{D}} = (R \langle \tau_1 \rangle)_{\mathcal{D}} = A^* . \qquad \ldots(4)$$

Now, $R/R_{\mathcal{F}} = R_{\mathcal{F}} [R_o , \langle \tau_1 \rangle] / R_{\mathcal{F}} = [R_o/R_{\mathcal{F}} , \langle \tau_1 \rangle]$, where τ_1 is regarded as being also an operator on $R_o/R_{\mathcal{F}}$. Since $R_o \in \mathcal{S}_\alpha$ and $t \in \beta$ with $\alpha \cap \beta = \emptyset$, then by (1.1.3 b), we have $[R/R_{\mathcal{F}} , \langle \tau_1 \rangle] = [[R_o/R_{\mathcal{F}} , \langle \tau_1 \rangle] , \langle \tau_1 \rangle] = [R_o/R_{\mathcal{F}} , \langle \tau_1 \rangle]$, and so

$$[R/R_{\mathcal{F}} , \langle \tau_1 \rangle] = R/R_{\mathcal{F}} . \qquad \ldots(5)$$

Now, $R/R_{\mathcal{F}} = [(A \langle \nu \rangle)^*/A^* , \langle \tau_1 \rangle]$, which has order u^{t-1} since $(A \langle \nu \rangle)^*/A^*$ is $\langle \tau_1 \rangle$-isomorphic to the regular $GF(u) \langle \tau_1 \rangle$-module, by the structure of the wreath product (see (1.7.2 a)). Thus, $R_{\mathcal{F}} \lneqq R$, and combining equations (3), (4) and (5), we obtain the assertions of (b)(i).

Now, $Y \in \mathcal{D} * \mathcal{S}_t \setminus \mathcal{F}$ and in fact $Y \in (\mathcal{D} \cap \mathcal{S}_\beta) * \mathcal{S}_t \setminus \mathcal{F}$, since $Y \in \mathcal{S}_\beta$, by choice of Y above. Thus by (3.2.2 a), there exists $N \in \mathcal{D} \cap \mathcal{S}_\beta$ possessing an automorphism τ_2 of order t such that $N] \langle \tau_2 \rangle \notin \mathcal{F}$. But since $N \in \mathcal{D}$, then $(N] \langle \tau_2 \rangle)_{\mathcal{F}} = (N] \langle \tau_2 \rangle)_{\mathcal{D}} = N$.

Because $N \in \mathcal{S}_\beta$ and $t \in \beta$, then $(|R|, |N|t) = 1$; we note that $t \in \mathrm{char}(\mathfrak{J}) = \mathbb{P}$. This disposes of (b)(ii), and completes the proof.

5.3.4 Construction.

We take groups R and N with automorphisms τ_1 and τ_2, respectively, each of order $t \in \pi$ with $(|R|, |N|t) = 1$, as in the conclusion of (5.3.3). The class $\mathfrak{J}$ can be of Type 1 or Type 2.

(a) Choose $q \in \mathbb{P}$ such that $q \nmid |R|$ and $t \mid q-1$; such a choice is possible by Dirichlet's theorem, [32; Theorem 15].

Let $\overline{Q}\,\overline{T}$ denote a non-abelian group of order qt, with $|\overline{Q}| = q$, $|\overline{T}| = t$ and $\overline{Q} \trianglelefteq \overline{Q}\,\overline{T}$; it is well-known that such a group exists. Let $\overline{V}$ denote a faithful irreducible $GF(t)$-module for $\overline{Q}\,\overline{T}$, so that $\overline{V}$ is elementary abelian of order t^α, $\alpha \gneqq 1$. Then $\overline{V}]\overline{Q}\,\overline{T}$ has a unique chief series and $(\overline{VQT})' = \overline{VQ}$. We note that $\overline{T}$ normalizes both $\overline{V}$ and $\overline{Q}$. By (2.4.10), $\overline{QT} \notin \mathcal{H}$, and by (1.4.8), $\overline{VQT} \notin \mathcal{H}$, since $\overline{V} = F(\overline{VQT})$.

(b) Let $Q_a T_a$, $a \in \{0, \infty\}$, be two isomorphic copies of $\overline{QT}$, with fixed isomorphisms $i_a : \overline{QT} \to Q_a T_a$, where $i_a : \overline{Q} \mapsto Q_a$, $\overline{T} \mapsto T_a$.

(c) Let B denote either R or N, as in (5.3.3), and let τ denote τ_1 or τ_2 according as B denotes R or N, respectively.

By (5.3.3), we have

$$(B]\langle\tau\rangle)_{\mathfrak{J}} = B_{\mathfrak{J}} \in \mathcal{D} \quad . \qquad \cdots(6)$$

Let $(B\langle\tau\rangle)^\alpha$ denote the external direct product of α copies of $B\langle\tau\rangle$, where $\alpha \gneqq 1$ is as in (a) above.

With $Q_\infty T_\infty$ as in (b), define $W = W(B)$ as

$$W = W(B) \quad ((B<\tau>)^\alpha) \text{ wr } Q_\infty T_\infty \quad . \text{ (Regular wreath product).}$$

We will identify $Q_\infty T_\infty$ with the "standard complement" in W, and will denote the base group by $(B<\tau>)^\#$, which we will identify with the external direct product of αqt copies of $B<\tau>$. If $S \leq B$, then we define $S^* = \{(s_1,\ldots,s_{\alpha qt}) : s_i \in S\} \leq (B<\tau>)^\#$.

We note that $(B<\tau>)^\#$ is a semi-direct product $B^\#]<\tau>^\#$. Since $\mathfrak{J}$ is a Lockett class, then $(B_{\mathfrak{J}})^\# = (B^\#)_{\mathfrak{J}}$; we will write this as $B^\#_{\mathfrak{J}}$. By (3.3.3),

$$B^\#/B^\#_{\mathfrak{J}} \text{ is abelian.} \qquad \ldots(7)$$

(d) Now let E_o denote an elementary abelian group of order t^α, where α is as in (a) (then $E_o \cong V$ as groups). By [41; I.15.9], there exists an injection

$$j : \overline{VQT} \hookrightarrow E_o \text{ wr } (Q_oT_o) \quad , = Y \text{ say} \quad ,$$

such that (by a reading of the proof) $(\overline{QT})j$ covers $(E_o \text{ wr } Q_oT_o)/E_o^*$, and $j : \overline{V} \hookrightarrow E_o^*$, where E_o^* is the base group of Y, and $\overline{VQT}$ is as in (a). We identify Q_oT_o with the standard complement in Y.

Since $E_o \cong <\tau>^\alpha$, there is an epimorphism k with kernel $B^\#$

$$k : W = (B<\tau>)^\alpha \text{ wr } Q_\infty T_\infty \longrightarrow E_o \text{ wr } Q_oT_o \quad ,$$

such that $k : (B<\tau>)^\# \mapsto\!\!\!\rightarrow E_o^*$, $Q_\infty \mapsto\!\!\!\rightarrow Q_o$ and $T_\infty \mapsto\!\!\!\rightarrow T_o$.

In $W = W(B)$, now take the complete pre-image

$$G = G(B) = (\overline{VQT})j\,k^{-1} \rhd B^\# .$$

Since $(\overline{QT})j$ covers $(E_o \text{ wr } Q_oT_o)/E_o$, then $(\overline{QT})j\,k^{-1}$ covers $W/((B<\tau>)^\#)$

Now, k induces an isomorphism $W/B^{\#} \to E_o \text{ wr } Q_o T_o$; since j is an injection, it follows that $j\,k^{-1}$ induces an isomorphism, h say, from $\overline{VQT}$ to $G/B^{\#}$; in particular, $G/B^{\#} \cong \overline{VQT}$.

Let Γ denote $\overline{V} j\, k^{-1}$, so that $B^{\#} \leq \Gamma \trianglelefteq G$ and $\Gamma/B^{\#} \cong \overline{V}$. Since $j : V \hookrightarrow E_o$ by above, then

$$B^{\#} \leq \Gamma \trianglelefteq (B\langle\tau\rangle)^{\#} . \qquad \ldots(8)$$

Since $\mathfrak{F}$ is a Lockett class, then by (6) we obtain

$$\Gamma_{\mathfrak{F}} = \Gamma \cap ((B\langle\tau\rangle)^{\#})_{\mathfrak{F}} = \Gamma \cap B^{\#}_{\mathfrak{F}} = B^{\#}_{\mathfrak{F}} . \qquad \ldots(9)$$

Since $\Gamma \trianglelefteq G$, then $[G_{\mathfrak{F}} , \Gamma] \leq \Gamma_{\mathfrak{F}} \leq B^{\#}$, and so $[G_{\mathfrak{F}} B^{\#}/B^{\#} , \Gamma/B^{\#}] = 1$. But $\Gamma/B^{\#} = F(G/B^{\#})$ since $\overline{V} = F(\overline{VQT})$ and $h : \overline{VQT} \to G/B^{\#}$ is an isomorphism. By (1.1.1 f), $G_{\mathfrak{F}} B^{\#} \leq \Gamma$, and so with (6) and (9) we obtain

$$G_{\mathfrak{F}} = \Gamma_{\mathfrak{F}} = B^{\#}_{\mathfrak{F}} \in \mathcal{D} . \qquad \ldots(10)$$

Now, $W = (B\langle\tau\rangle)^{\#} \rfloor Q_\infty T_\infty = B^{\#} \rfloor (\langle\tau\rangle^{\#} Q_\infty T_\infty)$. Then $G = G \cap W = G \cap B^{\#}(\langle\tau\rangle^{\#} Q_\infty T_\infty) = B^{\#} \rfloor (G \cap \langle\tau\rangle^{\#} Q_\infty T_\infty)$, by Dedekind's identity, and so there is a canonical isomorphism

$$f : G/B^{\#} \to G \cap \langle\tau\rangle^{\#} Q_\infty T_\infty .$$

Since $j\,k^{-1}$ induces the isomorphism $h : \overline{VQT} \to G/B^{\#}$, we obtain an isomorphism $h\,f : \overline{VQT} \to G \cap \langle\tau\rangle^{\#} Q_\infty T_\infty$. Let V , Q and T denote $\overline{V}\,h\,f$, $\overline{Q}\,h\,f$ and $\overline{T}\,h\,f$, respectively. By definition of Γ and (8), we have $V\,j\,k^{-1} = \Gamma \leq B^{\#}\langle\tau\rangle^{\#}$, and so

$$V = \overline{V}\,h\,f \leq \langle\tau\rangle^{\#} . \qquad \ldots(11)$$

We now have $G = G(B) = B^{\#} \rfloor VQT$; T normalizes $B^{\#}$, V and Q , and Q normalizes $B^{\#}$ and V .

(c) Taking B successively equal to R, N, as in (3.3.3), in the above, we obtain groups

$$G(R) = R^{\#}VQT \quad , \text{ and}$$

$$G(N) = N^{\#}VQT \quad ,$$

respectively, where the subgroups VQT in each case are identified, so that VQT is considered as a group of operators on each of $G(R)$ and $G(N)$. By (10) above and (3.3.3 aii, bii), we have

$$(G(N))_{\mathfrak{J}} = (N^{\#}VQT)_{\mathfrak{J}} = N^{\#} \in \mathcal{D} \quad . \qquad \cdots(12)$$

3.3.5 Lemma. In the group $G(R)$ as constructed above, we have $C_V(R^{\#}/(R_{\mathfrak{J}})^{\#}) = 1$.

Proof. In the group $W = W(R)$ constructed in (3.3.4 c), $(R\langle\tau\rangle)^{\#}$ is a direct power of α q copies of $R\langle\tau\rangle$. Now, $(|R|, |\tau|) = 1$ and $[R/R_{\mathfrak{J}}, \langle\tau\rangle] = R/R_{\mathfrak{J}} \neq 1$, by (3.3.3). Thus $C_{\langle\tau\rangle}(R/R_{\mathfrak{J}}) = 1$, by (1.1.5 bi), and so $C_{\langle\tau\rangle^{\#}}(R^{\#}/R^{\#}_{\mathfrak{J}}) = 1$. Since $V \leq \langle\tau\rangle^{\#}$ by (10), the result follows.

3.3.6 Construction.

Let $\langle\theta\rangle = T \leq R^{\#}VQT$ as constructed in (3.3.4); then $\langle\theta\rangle$ acts as a group of operators on $R^{\#}VQ \trianglelefteq R^{\#}VQT$, normalizing $R^{\#}$, V and Q, by (3.3.4 d), and has order t.

Let Φ denote the (external) direct power $(R^{\#}VQ)^t$ of t copies of $R^{\#}VQ$. Then $\langle\theta\rangle$ acts as a group of operators on Φ via

$$\theta \ : (g_1, \ldots, g_t) \mapsto (g_1^{\theta}, \ldots, g_t^{\theta}) \quad : \ g_j \in R^{\#}VQ \ .$$

If $S \leq R^{\#}VQ$ and $1 \leq i \leq t$, define the coordinate subgroup

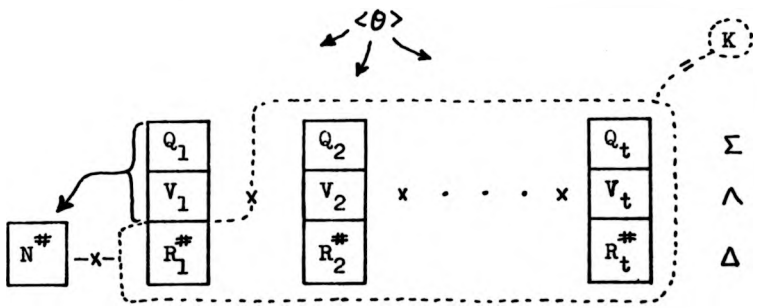
$$S_i = \{(1, \ldots, \underset{i}{x}, \ldots 1) \ : \ x \in S\} \quad .$$

Then $(R^{\#} VQ)_i = R_i^{\#} V_i Q_i$, $(R_i^{\#})_{\mathfrak{z}}$ $(R_{\mathfrak{z}}^{\#})_i$, and $\langle\theta\rangle$ normalizes each of $R_i^{\#}$, $R_{i\mathfrak{z}}^{\#}$, V_i and Q_i ; further, all the groups $R_i^{\#} V_i Q_i \rfloor \langle\theta\rangle$ are isomorphic to $R^{\#} VQT$. We also take $N^{\#} VQT$ as in (3.3.4 e).

Let $K = R_1^{\#} \times R_2^{\#} V_2 Q_2 \times \cdots \times R_t^{\#} V_t Q_t$, so that $K \trianglelefteq \Phi\langle\theta\rangle$, and $\Phi\langle\theta\rangle/K \cong V_1 Q_1 \langle\theta\rangle \cong VQT$, where VQT is as in (3.3.4 e). We may define an action of $\Phi\langle\theta\rangle/K$ on $N^{\#}$ by requiring that $N^{\#} \rfloor (\Phi\langle\theta\rangle/K) \cong N^{\#} VQT$, with $N^{\#} VQT$ as in (3.3.4 e) ; this action may be inflated to an action of $\Phi\langle\theta\rangle$ on $N^{\#}$ with kernel K .

Let Ω denote the semidirect product $N^{\#} \rfloor \Phi\langle\theta\rangle$ thus obtained ; we note that $N^{\#}$ and $R_1^{\#} \times \cdots \times R_t^{\#}$ are both normal in Ω .

fig. 4. The group Ω .



3.3.7 Proposition. The group Ω , as constructed in (3.3.6), belongs to $\mathcal{D} * \mathcal{H}$ and to $\mathfrak{F} * \mathcal{H}$, while the group $\Omega/(R_1^{\#} \times \cdots \times R_t^{\#})$, which is isomorphic to a Hall subgroup of Ω , belongs to neither.

Proof. We retain the notation of (3.3.6); let Δ denote the subgroup $R_1^{\#} \times \cdots \times R_t^{\#}$ of Ω , Λ denote $V_1 \times \cdots \times V_t$ and Σ denote $Q_1 \times \cdots \times Q_t$. Then $\Omega = N^{\#} \Delta \Lambda \Sigma \langle\theta\rangle$; note that $N^{\#}\Delta\Lambda \trianglelefteq \Omega$.

Whether $\mathfrak{F}$ is of Type 1 or Type 2, we have in (3.3.3) that $N \in \mathcal{D} \leq \mathfrak{F}$ and $R_{\mathcal{D}} = R_{\mathfrak{F}}$; thus $\{N^{\#}, R^{\#}_{\mathfrak{F}}\} \subseteq \mathcal{D} \subseteq \mathfrak{F}$. Since, by (3.3.6), $N^{\#} \times \Delta \trianglelefteq \Omega$, and since also $\mathfrak{F}$ is a Lockett class, then

$$\Omega_{\mathfrak{F}} \geq (N^{\#} \times \Delta)_{\mathfrak{F}} = N^{\#} \times \Delta_{\mathfrak{F}} = N^{\#} \times R_{1\mathfrak{F}} \times \cdots \times R_{t\mathfrak{F}} \in \mathcal{D} . \qquad \cdots(13)$$

We will show that $\Omega_{\mathfrak{F}} = N^{\#} \times \Delta_{\mathfrak{F}}$.

Firstly, we show that $(N^{\#}\Delta\Lambda)_{\mathfrak{F}} = N^{\#} \times \Delta_{\mathfrak{F}}$. Let X denote $[(N^{\#}\Delta\Lambda)_{\mathfrak{F}}, \Delta]$. Since $(N^{\#}\Delta\Lambda)_{\mathfrak{F}}$ and Δ are normal in Ω, and since $X \trianglelefteq (N^{\#}\Delta\Lambda)_{\mathfrak{F}}\Delta$ by (1.1.3 a), then $X \leq \Delta_{\mathfrak{F}}$. It follows that $(N^{\#}\Delta\Lambda)_{\mathfrak{F}} \leq C_{N^{\#}\Delta\Lambda}(\Delta/\Delta_{\mathfrak{F}})$, $= C$ say . Since $R^{\#}/R^{\#}_{\mathfrak{F}}$ is abelian, by (7) of (3.3.4), then $\Delta/\Delta_{\mathfrak{F}}$ is abelian and so $N^{\#}\Delta \leq C$. Thus, $C = C \cap N^{\#}\Delta\Lambda = N^{\#}\Delta(C \cap \Lambda) = N^{\#}\Delta\, C_{\Lambda}(\Delta/\Delta_{\mathfrak{F}})$, using Dedekind's identity. Now, $\Delta\Lambda = R^{\#}_1 V_1 \times \cdots \times R^{\#}_t V_t$, and so if there exists $v \in C_{\Lambda}(\Delta/\Delta_{\mathfrak{F}}) \setminus 1$, then for some i there exists $v_i \in C_{V_i}(R^{\#}_i/R^{\#}_{i\mathfrak{F}}) \setminus 1$, contrary to (3.3.5). Thus $C = N^{\#}\Delta$, and so $(N^{\#}\Delta\Lambda)_{\mathfrak{F}} = N^{\#} \times \Delta_{\mathfrak{F}}$.

Let Y denote $[\Omega_{\mathfrak{F}}, N^{\#}\Delta\Lambda]$. Since $\Omega_{\mathfrak{F}}$ and $N^{\#}\Delta\Lambda$ are normal in Ω and $Y \trianglelefteq \Omega_{\mathfrak{F}} N^{\#}\Delta\Lambda$, then $Y \leq (N^{\#}\Delta\Lambda)_{\mathfrak{F}} = N^{\#} \times \Delta_{\mathfrak{F}}$. Thus, $\Omega_{\mathfrak{F}} \leq C_{\Omega}(N^{\#}\Delta\Lambda/N^{\#}\Delta)$, $= L$ say. Since Λ is abelian, then $L \geq N^{\#}\Delta\Lambda$, and so by Dedekind's identity,

$$\Omega_{\mathfrak{F}} \leq L = L \cap \Omega = L \cap (N^{\#}\Delta\Lambda\Sigma\langle\theta\rangle) = N^{\#}\Delta\Lambda(L \cap \Sigma\langle\theta\rangle) \qquad \cdots(14)$$

We claim that $L \cap \Sigma\langle\theta\rangle = C_{\Sigma\langle\theta\rangle}(\Lambda)$. For suppose that $c \in L \cap \Sigma\langle\theta\rangle$. Since $V \trianglelefteq VQT$ in (3.3.4), then by construction of $\Phi\langle\theta\rangle$ in (3.3.6), we have $\Lambda \trianglelefteq \Lambda\Sigma\langle\theta\rangle$. But then $[c, \Lambda] \leq (N^{\#} \times \Delta) \cap \Lambda = 1$. Thus $L \cap \Sigma\langle\theta\rangle \leq C_{\Sigma\langle\theta\rangle}(\Lambda)$.

Since the opposite inclusion is clear, we have

$$L \cap \Sigma\langle\theta\rangle = C_{\Sigma\langle\theta\rangle}(\Lambda) \quad . \qquad \cdots(15)$$

By construction of VQT in (3.3.4), $C_{QT}(V) = 1$. Thus, for each i , $C_{Q_i\langle\theta\rangle}(V_i) = 1$, and so $C_{Q_1\times\cdots\times Q_t}(V_1\times\cdots\times V_t) = 1$, since $[V_i , Q_j] = 1$ if $i \neq j$. Thus $C_{\Sigma}(\Lambda) = 1$, and so $C_{\Sigma\langle\theta\rangle}(\Lambda) = 1$, since any non-trivial normal subgroup of $\Sigma\langle\theta\rangle$ must intersect Σ non-trivially. But now by (14) and (15), we have $\Omega_{\mathfrak{J}} \leq N^{\#}\Delta\Lambda \trianglelefteq \Omega$, and so $\Omega_{\mathfrak{J}} = (N^{\#}\Delta\Lambda)_{\mathfrak{J}} = N^{\#}\times\Lambda_{\mathfrak{J}}$. Since by (13), $N^{\#}\times\Delta_{\mathfrak{J}} \in \mathcal{D}$, then $\Omega_{\mathfrak{J}} \in \mathcal{D} \subseteq \mathfrak{J}$, and so

$$\Omega_{\mathcal{D}} = \Omega_{\mathfrak{J}} = N^{\#}\times\Delta_{\mathfrak{J}} \quad . \qquad \cdots(16)$$

We now have $\Omega/\Omega_{\mathfrak{J}} = N^{\#}\Delta\Lambda\Sigma\langle\theta\rangle/(N^{\#}\times\Delta_{\mathfrak{J}})$

$$\cong \Delta\Lambda\Sigma\langle\theta\rangle/\Delta_{\mathfrak{J}} \qquad \cdots(17)$$

But there is a natural isomorphism

$$\Delta\Lambda\Sigma\langle\theta\rangle/\Delta_{\mathfrak{J}} \cong (\bigtimes_{i=1}^{t}((R_i^{\#}/R_{i\mathfrak{J}}^{\#})V_iQ_i))]\langle\theta\rangle \qquad \cdots(18)$$

where, for each i , $(R_i^{\#}/R_{i\mathfrak{J}}^{\#})V_iQ_i$ admits $\langle\theta\rangle$ in the natural way, so that $(R_i^{\#}/R_{i\mathfrak{J}}^{\#})V_iQ_i\langle\theta\rangle \cong R^{\#}VQT/R_{\mathfrak{J}}^{\#}$ as in (3.3.4).

Let $\hat{R}^{\#}$ denote $R^{\#}/R_{\mathfrak{J}}^{\#}$ as in (3.3.3/4); by equation (7) of (3.3.4), $\hat{R}^{\#}$ is abelian. By (3.3.3) and (3.3.4a), $(tq , |R|) = 1$, and so $(|\hat{R}^{\#}|, |VQ\langle\theta\rangle|) = 1$. By (1.1.3b), we have

$$\hat{R}^{\#} = C_{\hat{R}^{\#}}(VQ\langle\theta\rangle) \times [\hat{R}^{\#} , VQ\langle\theta\rangle] \quad ; \text{ and} \qquad \cdots(19)$$

$$[\hat{R}^{\#} , VQ\langle\theta\rangle] = [[\hat{R}^{\#} , VQ\langle\theta\rangle] , VQ\langle\theta\rangle] \quad . \qquad \cdots(20)$$

By (3.3.4a) and the fact that $VQ\langle\theta\rangle \cong \overline{VQT}$, we have $VQ = (VQ\langle\theta\rangle)'$. With (20) , it follows that

$$[\hat{R}^{\#} , VQ\langle\theta\rangle]\, VQ = ([\hat{R}^{\#} , VQ\langle\theta\rangle]\, VQ\langle\theta\rangle)' \qquad \cdots(21)$$

Since the opposite inclusion is clear, we have

$$L \cap \Sigma\langle\theta\rangle = C_{\Sigma\langle\theta\rangle}(\Lambda) \quad . \qquad \cdots(15)$$

By construction of VQT in (3.3.4), $C_{QT}(V) = 1$. Thus, for each i , $C_{Q_i\langle\theta\rangle}(V_i) = 1$, and so $C_{Q_1\times\cdots\times Q_t}(V_1\times\cdots\times V_t) = 1$, since $[V_i , Q_j] = 1$ if $i \neq j$. Thus $C_{\Sigma}(\Lambda) = 1$, and so $C_{\Sigma\langle\theta\rangle}(\Lambda) = 1$, since any non-trivial normal subgroup of $\Sigma\langle\theta\rangle$ must intersect Σ non-trivially. But now by (14) and (15), we have $\Omega_{\mathfrak{Y}} \leq N^{\#}\Delta\Lambda \trianglelefteq \Omega$, and so $\Omega_{\mathfrak{Y}} = (N^{\#}\Delta\Lambda)_{\mathfrak{Y}} = N^{\#}\times\Lambda_{\mathfrak{Y}}$. Since by (13), $N^{\#}\times\Delta_{\mathfrak{Y}} \in \mathcal{D}$, then $\Omega_{\mathfrak{Y}} \in \mathcal{D} \subseteq \mathfrak{Y}$, and so

$$\Omega_{\mathcal{D}} = \Omega_{\mathfrak{Y}} = N^{\#}\times\Delta_{\mathfrak{Y}} \quad . \qquad \cdots(16)$$

We now have $\Omega/\Omega_{\mathfrak{Y}} = N^{\#}\Delta\Lambda\Sigma\langle\theta\rangle/(N^{\#}\times\Delta_{\mathfrak{Y}})$

$$\cong \Delta\Lambda\Sigma\langle\theta\rangle/\Delta_{\mathfrak{Y}} \qquad \cdots(17)$$

But there is a natural isomorphism

$$\Delta\Lambda\Sigma\langle\theta\rangle/\Delta_{\mathfrak{Y}} \cong (\bigtimes_{i=1}^{t}((R_i^{\#}/R_{i\mathfrak{Y}}^{\#})V_iQ_i))]\langle\theta\rangle \qquad \cdots(18)$$

where, for each i , $(R_i^{\#}/R_{i\mathfrak{Y}}^{\#})V_iQ_i$ admits $\langle\theta\rangle$ in the natural way, so that $(R_i^{\#}/R_{i\mathfrak{Y}}^{\#})V_iQ_i\langle\theta\rangle \cong R^{\#}VQT/R_{\mathfrak{Y}}^{\#}$ as in (3.3.4).

Let $\hat{R}^{\#}$ denote $R^{\#}/R_{\mathfrak{Y}}^{\#}$ as in (3.3.3/4); by equation (7) of (3.3.4), $\hat{R}^{\#}$ is abelian. By (3.3.3) and (3.3.4a), $(tq , |R|) = 1$, and so $(|\hat{R}^{\#}|, |VQ\langle\theta\rangle|) = 1$. By (1.1.3b), we have

$$\hat{R}^{\#} = C_{\hat{R}^{\#}}(VQ\langle\theta\rangle) \times [\hat{R}^{\#} , VQ\langle\theta\rangle] \quad ; \text{ and} \qquad \cdots(19)$$

$$[\hat{R}^{\#} , VQ\langle\theta\rangle] = [[\hat{R}^{\#} , VQ\langle\theta\rangle] , VQ\langle\theta\rangle] \quad . \qquad \cdots(20)$$

By (3.3.4a) and the fact that $VQ\langle\theta\rangle \cong \overline{VQT}$, we have $VQ = (VQ\langle\theta\rangle)'$. With (20) , it follows that

$$[\hat{R}^{\#} , VQ\langle\theta\rangle]\, VQ = ([\hat{R}^{\#} , VQ\langle\theta\rangle]\, VQ\langle\theta\rangle)' \qquad \cdots(21)$$

But now by (3.2.1) and the definition of the action of $\langle\theta\rangle$ from (3.3.6) and at (18) above, it follows that

$$(\bigtimes_i ([\hat{R}_i^{\#}, V_iQ_i\langle\theta\rangle] \; V_iQ_i))\langle\theta\rangle \in \mathcal{H} \quad . \qquad \ldots(22)$$

Let $C_i = C_{\hat{R}_i^{\#}}(V_iQ_i\langle\theta\rangle)$, where $\hat{R}_i^{\#} \quad R_i^{\#}/R_{i\mathfrak{J}}^{\#}$.

By (17), (18) and (19), we have

$$\Omega/\Omega_{\mathfrak{J}} = (\bigtimes_{i=1}^{t} C_i) \times (\bigtimes_{i=1}^{t} ([\hat{R}_i^{\#}, V_iQ_i\langle\theta\rangle] \; V_iQ_i)\langle\theta\rangle) \qquad \ldots(23)$$

However, since $\hat{R}^{\#}$ is abelian, then $\bigtimes_i C_i$ is abelian, and so belongs to $\mathcal{H}$ by (1.4.3). By (22) and (23), $\Omega/\Omega_{\mathfrak{J}} \in \mathcal{H}$, and since $\Omega_{\mathfrak{J}} = \Omega_{\mathcal{D}}$ by (16), we have

$$\Omega \in \mathfrak{J} * \mathcal{H} \ , \quad \text{and} \quad \Omega \in \mathcal{D} * \mathcal{H} \quad . \qquad \ldots(24)$$

We now show that a certain Hall-subgroup of Ω belongs to neither $\mathfrak{J} * \mathcal{H}$ nor $\mathcal{D} * \mathcal{H}$.

By (3.3.3) and (3.3.4a), $(|R|, \text{tq}|N|) = 1$, and so $(|\Delta|, |N^{\#}||\wedge\Sigma\langle\theta\rangle|) = 1$. Since $N^{\#} \trianglelefteq N^{\#}\Delta\wedge\Sigma\langle\theta\rangle = \Omega$, then $N^{\#}\wedge\Sigma\langle\theta\rangle$ is a subgroup of, and thus a Hall-subgroup of, Ω ; we denote $N^{\#}\wedge\Sigma\langle\theta\rangle$ by H , and remark that $H \cong \Omega/\Delta$.

Let $\mathcal{G}$ denote either $\mathcal{D}$ or $\mathfrak{J}$. We will show that $H/H_{\mathcal{G}} \notin \mathcal{H}$.

Since $N^{\#} \in \mathcal{D} \subseteq \mathfrak{J}$ and $N^{\#} \trianglelefteq H$, then $N^{\#} \leq H_{\mathcal{G}}$. Now, by the definition of the action of Φ on N in (3.3.6), we have

$$N^{\#} \times V_2 \times \cdots \times V_t \trianglelefteq H = ((N^{\#}V_1Q_1) \times V_2Q_2 \times \cdots \times V_tQ_t)\langle\theta\rangle \quad . \qquad \ldots(25)$$

Since the V_i are t-groups where $t \in \pi = \text{char}(\mathcal{G})$, by (3.3.4a) , it follows that

$$N^{\#} \times V_2 \times \cdots \times V_t \leq H_{\mathcal{G}} \quad . \qquad \ldots(26)$$

Let Z denote $[H_{\mathcal{G}}, N^{\#}V_1Q_1]$. Since $H_{\mathcal{G}}$ and $N^{\#}V_1Q_1$

are normal subgroups of H, and $Z \trianglelefteq H_{\mathcal{G}} N^{\#} V_1 Q_1$, by (1.1.3), then $Z \leq (N^{\#} V_1 Q_1)_{\mathcal{G}}$. But $(N^{\#} V_1 Q_1)_{\mathcal{G}} \leq (N^{\#} V_1 Q_1 \langle \theta \rangle)_{\mathcal{G}} = N^{\#}$, by (12). Thus,

$$H_{\mathcal{G}} \leq C_H(N^{\#} V_1 Q_1 / N^{\#}), = J \text{ say}. \qquad \cdots(27)$$

Now, we certainly have $J \geqslant N^{\#} \times V_2 Q_2 \times \cdots \times V_t Q_t$, and so

$$J = J \cap H = (N^{\#} \times V_2 Q_2 \times \cdots \times V_t Q_t)(J \cap V_1 Q_1 \langle \theta \rangle) .$$

Let $x \in J \cap V_1 Q_1 \langle \theta \rangle$ and let $g \in V_1 Q_1$. Since $V_1 Q_1 \trianglelefteq V_1 Q_1 \langle \theta \rangle$, then $[x, g] \in N^{\#} \cap V_1 Q_1 = 1$. Thus $J \cap V_1 Q_1 \langle \theta \rangle \leq C_{V_1 Q_1 \langle \theta \rangle}(V_1 Q_1)$, and so $J \cap V_1 Q_1 \langle \theta \rangle = 1$, by construction of $VQ\langle \theta \rangle \cong \overline{VQT}$ in (3.3.4a). It follows that $J = N^{\#} \times V_2 Q_2 \times \cdots \times V_t Q_t$. Since $H_{\mathcal{G}} \leqslant J$ by (27), then we have

$$\begin{aligned} H_{\mathcal{G}} &= H_{\mathcal{G}} = (N^{\#} \times V_2 \times \cdots \times V_t)(Q_2 \times \cdots \times Q_t) \\ &= (N^{\#} \times V_2 \times \cdots \times V_t)(H_{\mathcal{G}} \cap (Q_2 \times \cdots \times Q_t)) . \end{aligned} \qquad \cdots(28)$$

Comparing (28) with the expression for H given in (25), we see that $H/H_{\mathcal{G}}$ is isomorphic to a factor of $(V_1 Q_1 \times Q_2 \times \cdots \times Q_t)\langle \theta \rangle$ by a normal subgroup, M say, contained in $Q_2 \times \cdots \times Q_t$. But then $((V_1 Q_1 \times Q_2 \times \cdots \times Q_t)\langle \theta \rangle / M)_{\mathcal{N}} = (V_1 \times Q_2 \times \cdots \times Q_t)/M$, and so $(H/H_{\mathcal{G}})/(H/H_{\mathcal{G}})_{\mathcal{N}} \cong Q_1 \langle \theta \rangle \cong \overline{QT} \notin \mathcal{H}$, by (3.3.4a). By (1.4.8), it follows that $H/H_{\mathcal{G}} \notin \mathcal{H}$, and since $\mathcal{G}$ denoted either $\mathcal{D}$ or $\mathcal{F}$, we finally obtain $H \notin \mathcal{D} * \mathcal{H}$ and $H \notin \mathcal{F} * \mathcal{H}$.

Since H $(\cong \Omega / \Delta)$ is a Hall subgroup of Ω, and Ω belongs to $\mathcal{D} * \mathcal{H}$ and $\mathcal{F} * \mathcal{H}$ by (24) above, the proof of (3.3.7) is complete, and, with it, the proofs of (3.3.1) and (3.3.2).

3.4 Hall-closure of $\mathfrak{F} * \mathfrak{H}$ for certain replete classes $\mathfrak{F}$.

In this section, we consider the Hall-closure properties of $\mathfrak{F} * \mathfrak{H}$ for certain particular classes $\mathfrak{F}$ (of characteristic $\mathbb{P}$) not covered by (3.3.1). Since the results of this section are of a perhaps less general nature than those of other sections, the proofs given here will be somewhat abbreviated and more in the nature of outlines.

The first case considered is $\mathfrak{F} = \mathfrak{C}_\pi$ for $\phi \neq \pi \subseteq \mathbb{P}$; then $\mathrm{char}(\mathfrak{F}) = \mathbb{P}$ (see (1.3.16)). Suppose that p and q are distinct primes in π. Choose additional primes r and s so that $|\{p,q,r,s\}| = 4$. Then the groups $C_p \text{ wr } C_r$ and $C_q \text{ wr } C_s$ are of co-prime orders, and do not belong to $\mathfrak{C}_\pi$. Thus if $\mathfrak{D} \in \mathrm{Locksec}(\mathfrak{C}_\pi)$ where $|\pi| \geq 2$, then $\mathfrak{D} * \mathfrak{H}$ is not Hall-closed by (3.3.1). However, (3.3.1) cannot be applied to $\mathfrak{C}_p$, which is replete, and we deal with this case now.

<u>3.4.1 Proposition</u>. Let p be a prime, and let $\mathfrak{D} \in \mathrm{Locksec}(\mathfrak{C}_p)$. Then $\mathfrak{D} * \mathfrak{H}$ is not Hall-closed.

<u>Proof</u>. Let q, r and s be additional primes with $|\{p,q,r,s\}| = 4$, $p \mid r-1$ and $q \mid s-1$; such exist by Dirichlet's Theorem. Now form non-abelian groups $\langle\theta\rangle \rceil \langle\varphi\rangle$ and $\langle\alpha\rangle \rceil \langle\beta\rangle$ of orders sq and rp respectively, with $|\theta| = s$, $|\varphi| = q$, $|\alpha| = r$ and $|\beta| = p$.

Let $W = \langle\alpha\rangle\langle\beta\rangle \text{ wr } C_s$. In the notation of (1.7.1), let $\langle\sigma\rangle$ denote the standard complement, $R = \langle\alpha\rangle^*$ and $\tilde{P} = \langle\beta\rangle^*$; then $\tilde{P}$ is normalized by $\langle\sigma\rangle$; further, $R \trianglelefteq W$ by the lemma on page 14 of [36]. Thus $R \leq [R\tilde{P}, \langle\sigma\rangle] \trianglelefteq W = R\tilde{P}\langle\sigma\rangle$. Let $P = [\tilde{P}, \langle\sigma\rangle]$; then $RP = [R\tilde{P}, \langle\sigma\rangle] = [RP, \langle\sigma\rangle]$. Thus, RP admits $\langle\sigma\rangle$.

Now fix a specific isomorphism $\iota : \langle\theta\rangle \to \langle\sigma\rangle$; with respect to ι , RP admits $\langle\theta\rangle$. Let X be the twisted wreath product

$$X = RP \underset{\langle\theta\rangle}{wr} \langle\theta\rangle\langle\varphi\rangle \quad .$$

(We refer to [41; I.15.10] , [53] or [38] for details of the twisted wreath product).

We denote the base group of X by $(RP)^*$, and identify $\langle\theta\rangle\langle\varphi\rangle$ with the standard complement ; then $(RP)^*$ has coordinate subgroups $R_1P_1 , \ldots , R_qP_q$, and $(RP)^* = R_1P_1 \times \cdots \times R_qP_q$. The R_iP_i are permuted transitively by $\langle\varphi\rangle$, and since $\langle\theta\rangle \trianglelefteq \langle\theta\rangle\langle\varphi\rangle$ then $\langle\theta\rangle$ normalizes each R_i , and each P_i ; further, $[R_iP_i , \langle\theta\rangle] = R_iP_i$ (by the action of $\langle\sigma\rangle$ on RP), and so $[(RP)^* , \langle\theta\rangle] = (RP)^*$.

Let V be a faithful irreducible GF(p)-module for $\langle\varphi\rangle$, and inflate V to a module for X with kernel $(RP)^*\langle\theta\rangle$. Let $Y = V] X = V] (RP)^*\langle\theta\rangle\langle\varphi\rangle$. We may check that if $1 \neq N \trianglelefteq (RP)^*$, then $r \mid |N|$; it follows that p-soc(Y) = V . By Lockett's result (1.3.17c), $Y_{\mathcal{C}_p} = C_Y(V) = V \times (RP)^*\langle\theta\rangle$. Now, $Y_{\mathcal{C}_p}/Y_{\mathcal{D}}$ is a central factor of Y by (1.5.3c), and so is centralized by $\langle\varphi\rangle$. Since $\langle\theta\rangle\langle\varphi\rangle$ is non-abelian, it follows that $Y_{\mathcal{D}} \geqslant \langle\theta\rangle$, and, since $[(RP)^* , \langle\theta\rangle] = (RP)^*$, we have $Y_{\mathcal{D}} \geqslant (RP)^*\langle\theta\rangle$. Since $V \in \mathcal{A} \subseteq \mathcal{N}$, then $V \times (RP)^*\langle\theta\rangle = Y_{\mathcal{C}_p} \in \mathcal{L}$, and so $Y_{\mathcal{C}_p} = Y_{\mathcal{D}}$. Since $Y/Y_{\mathcal{D}} \cong \langle\varphi\rangle \cong C_q$, then $Y \in \mathcal{D} * \mathcal{H}$.

Let $H = V P^*\langle\theta\rangle\langle\varphi\rangle \leqslant Y$; then $H \in \mathrm{Hall}_{r'}(Y)$. Now $P^* \trianglelefteq H$, since $[P^* , V] = 1$. Since P^* is elementary abelian and of order co-prime to $\langle\theta\rangle\langle\varphi\rangle$, it follows by Maschke's Theorem, [28; 3.3.1] , that $P^* \leqslant$ p-soc(H) . Thus, p-soc(H) $= V \times P^*$.

Since $P^* = [P^*, \langle\theta\rangle]$, it follows that $C_H(V \times P^*) = V \times P^*$, and so by (1.3.17c), $H_{\mathcal{C}_p} = V \times P^*$. Since $V \times P^* \in \mathcal{D}$, then $H_{\mathcal{D}} = V \times P^*$. But now $H/H_{\mathcal{D}} \cong \langle\theta\rangle\langle\varphi\rangle \notin \mathcal{H}$, by (2.4.10). Thus, $H \notin \mathcal{D} * \mathcal{H}$, completing the proof.

We next consider classes $\mathfrak{F} * \mathcal{H}$ where $\mathfrak{F}$ is an alternating product of the classes $\mathcal{S}_p$ and $\mathcal{S}_{p'}$, for $p \in \mathbb{P}$. It follows from (3.1.3) and (3.1.4) that if $\mathfrak{F}$ is any of the classes $\mathcal{S}_p$, $\mathcal{S}_{p'}$, or $\mathcal{S}_{p'} * \mathcal{S}_p$, then $\mathfrak{F} * \mathcal{H}$ is Hall-closed. These three classes turn out to be the only such alternating products $\mathfrak{F}$ for which $\mathfrak{F} * \mathcal{H}$ is Hall-closed.

<u>3.4.2 Proposition</u>. Let p be a prime and $\mathfrak{F}$ be any of the Fitting classes

(1) $\mathcal{S}_{p'} * \mathcal{S}_p * \cdots * \mathcal{S}_p * \mathcal{S}_{p'}$; (2) $\mathcal{S}_p * \cdots * \mathcal{S}_p * \mathcal{S}_{p'}$;

(3) $\mathcal{S}_{p'} * \mathcal{S}_p * \cdots * \mathcal{S}_{p'} * \mathcal{S}_p$; (4) $\mathcal{S}_p * \cdots * \mathcal{S}_{p'} * \mathcal{S}_p$,

where in each case there are $n \gneq 0$ factors $\mathcal{S}_p$ in the product. Suppose also that $\mathfrak{F}$ is neither of the classes $\mathcal{S}_p$, or $\mathcal{S}_{p'} * \mathcal{S}_p$. Then $\mathfrak{F} * \mathcal{H}$ is not Hall-closed.

<u>Proof</u>. Since $n \gneq 0$, then $\mathfrak{F} \neq \mathcal{S}_{p'}$.

Let q and r be distinct odd primes, neither equal to p. Let K be a group of type $M(p^k, q^1, 2)$ as in (1.7.6), where $k \gneq 1$; then $K/O_p(K) \cong D_{2q}$, the dihedral group of order $2q$.

If $m \in \mathbb{N}$ and $\epsilon \in \{0, 1\}$, construct groups $A = A_m^{\epsilon}$

and $B = B_m^{\varepsilon}$, where

(i) A_m^{ε} is of type $M(q^{\varepsilon c_{m+1}}, p^{a_m}, q^{c_m}, \ldots, p^{a_1}, q^{c_1}, p^k, q, 2)$, and

(ii) B_m^{ε} is of type $M(r^{\varepsilon d_{m+1}}, p^{b_m}, r^{d_m}, \ldots, p^{b_1}, r^{d_1}, p^k, q, 2)$,

where, if $\varepsilon = 0$, we understand that A , B are of types $M(p^{a_m}, \ldots)$, $M(p^{b_m}, \ldots)$, respectively. We further stipulate that $A/A^{\mathcal{W}^3} \cong B/B^{\mathcal{W}^3} \cong K$. There exist involutions $u \in A$ and $v \in B$ such that $A = A'] \langle u \rangle$ and $B = B'] \langle v \rangle$.

Suppose that $\mathfrak{F}$ is of type (1) or (2) in the hypotheses. Take $A = A_n^{\varepsilon}$ and $B = B_n^{\varepsilon}$, where $\varepsilon = 1$ or 0 according as $\mathfrak{F}$ is of type (1) or (2), respectively. Form the external direct product $A \times B$, and let $w = (u,v) \in A \times B$. Let $G = (A' \times B') \langle w \rangle$. By our construction,

$$G/G_{\mathfrak{F}} \cong (A'/A^{\mathcal{W}^3} \times B'/B^{\mathcal{W}^3})] \langle w \rangle \cong (K \text{ wr } C_2)' \in \mathfrak{H} .$$

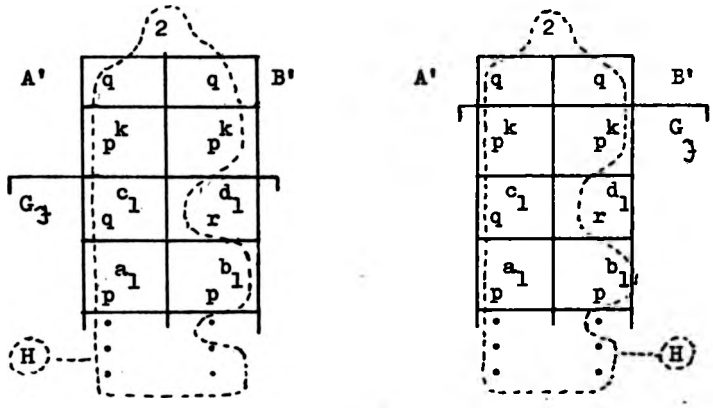
Thus, $G \in \mathfrak{F} * \mathfrak{H}$.

Let $H \in \mathrm{Hall}_{r'}(G)$ be such that $w \in H$. Then $A' \leq H$, while $B' \cap H \in \mathcal{S}_p * \mathcal{S}_q \subseteq \mathfrak{F}$, and $B' \cap H \leq H_{\mathfrak{F}}$. Also, $A' \cap H_{\mathfrak{F}} = A'_{\mathfrak{F}}$, and $A'_{\mathfrak{F}} = A^{\mathcal{W}^3} = (A')^{\mathcal{W}^2}$ by construction. Thus $H/H_{\mathfrak{F}} \cong K$. But $K/O_p(K) \cong D_{2q} \notin \mathfrak{H}$, by (2.4.10), and by (1.4.8), $H/H_{\mathfrak{F}} \notin \mathfrak{H}$; the result follows in these cases.

Suppose next that $\mathfrak{F}$ is of type (3) or (4) of the hypotheses ; in this case, the hypotheses imply that $\mathcal{S}_p * \mathcal{S}_{p'} * \mathcal{S}_p \subseteq \mathfrak{F}$. We take $A = A_{n-1}^{\varepsilon}$ and $B = B_{n-1}^{\varepsilon}$, with $\varepsilon = 1$ or 0 according as $\mathfrak{F}$ is of type (3) or (4), respectively. We define the group G as above. In this case, $G/G_{\mathfrak{F}} \cong ((D_{2q}) \text{ wr } C_2)' \in \mathfrak{H}$ and $G \in \mathfrak{F} * \mathfrak{H}$.

fig. 5.

$\mathfrak{F}$ type (1) or (2). $\mathfrak{F}$ type (3) or (4).



Let $H \in \mathrm{Hall}_{r'}(G)$ be such that $w \in H$. Since $\mathfrak{F}$ contains $\mathcal{S}_p * \mathcal{S}_{p'} * \mathcal{S}_p$, then again $B' \cap H \in \mathfrak{F}$, while $A'_{\mathfrak{F}}$ has index q in A'. Thus (since $\mathfrak{F}$ is a Lockett class), $H/H_{\mathfrak{F}} \cong D_{2q} \notin \mathfrak{H}$, whence $H \notin \mathfrak{F} * \mathfrak{H}$. This completes the proof.

<u>3.4.3 Remark</u>. Let $\mathfrak{G}$ be a Fitting class such that $\mathfrak{G} * \mathfrak{F}$ is Hall-closed for all Fitting classes $\mathfrak{F}$. Then $\mathrm{char}(\mathfrak{G}) = \mathbb{P}$.

<u>Proof</u>. For suppose that there exists $p \in \mathbb{P} \setminus \mathrm{char}(\mathfrak{G})$. Then $\mathfrak{G} \subseteq \mathcal{S}_{p'}$ and $\mathfrak{G} * \mathcal{S}_{p'} = \mathcal{S}_{p'}$. But then

$$\mathfrak{G} * (\mathcal{S}_{p'} * \mathcal{S}_p * \mathcal{S}_{p'} * \mathfrak{H}) = \mathcal{S}_{p'} * \mathcal{S}_p * \mathcal{S}_{p'} * \mathfrak{H}$$

is Hall-closed, contrary to (3.4.2)

The classes $\mathfrak{F}$ of (3.4.2) are primitive saturated formations, and so by (1.6.6) and (3.1.2), if $\mathfrak{D} \in \mathrm{Locksec}(\mathfrak{F})$, then $\mathfrak{D} * \mathfrak{H} = \mathfrak{F} * \mathfrak{H}$ and so $\mathfrak{D} * \mathfrak{H}$ is not Hall-closed, except if $\mathfrak{F} = \mathcal{S}_p$, $\mathcal{S}_{p'}$, or $\mathcal{S}_{p'} * \mathcal{S}_p$.

We next consider classes of the form $e_\pi(\mathcal{G}) * \mathcal{H}$, where $\pi \subseteq \mathbb{P}$ and $\mathcal{G}$ is a Fitting class. We recall that $\mathrm{char}(e_\pi(\mathcal{G})) = \mathbb{P}$, and $e_\phi(\mathcal{G}) = \mathcal{S}$, by convention; then $e_\pi(\mathcal{G}) = e_{(\pi \cap \mathrm{char}(\mathcal{G}))}(\mathcal{G})$. Since $\mathcal{S} * \mathcal{H} = \mathcal{S}$ is Hall-closed, we assume that there exists $p \in \pi$.

(a) Suppose that there exists $q \in \mathbb{P} \setminus \mathrm{char}(\mathcal{G})$ with $q \neq p$. Choose further primes r , s with $|\{p,q,r,s\}| = 4$ and $r \mid s-1$. Let G be a group of type $M(q^\beta, p^\alpha, s, r)$ such that $G/O_{q,p}(G)$ is non-abelian of order sr . Now $G_{\mathcal{G}} = 1$, since $q \notin \mathrm{char}(\mathcal{G})$, and so $G \in e_\pi(\mathcal{G}) \subseteq e_\pi(\mathcal{G}) * \mathcal{H}$. Let $H \in \mathrm{Hall}_{q'}(G)$; we may check that $H/H_{e_\pi(\mathcal{G})}$ has order sr and so $H \notin e_\pi(\mathcal{G}) * \mathcal{H}$. Thus $e_\pi(\mathcal{G}) * \mathcal{H}$ is not Hall-closed in this case.

(b) Next suppose that $\mathrm{char}(\mathcal{G}) = \mathbb{P}$, and that $|\pi| \geq 2$. Choose $q \in \pi \setminus \{p\}$, and let $t \in \mathbb{P} \setminus \{p,q\}$. Then $C_p \mathrm{ wr } C_t$ and $C_q \mathrm{ wr } C_t$ are of co-prime orders and do not belong to $e_\pi(\mathcal{G})$; by (3.3.1), $e_\pi(\mathcal{G}) * \mathcal{H}$ is not Hall-closed.

(c) The remaining case, with $\mathrm{char}(\mathcal{G}) = \mathbb{P}$ and $|\pi| = 1$, has not been fully resolved ; we note, however, that $e_p(\mathcal{S}) = \mathcal{S}_{p'} * \mathcal{S}_p$. By (3.1.5a), $\mathcal{S}_{p'} * \mathcal{S}_p * \mathcal{H}$ is Hall-closed.

We shall have more to say about Hall-closure of classes $\mathcal{F} * \mathcal{H}$ in section 3.8. The following questions remain unresolved (although they are of course answered by (3.3.1/2) for classes of characteristic $\subsetneq \mathbb{P}$).

<u>3.4.4 Questions</u>. (a) For which Fitting classes $\mathcal{D}$ with $\mathrm{char}(\mathcal{D}) = \mathbb{P}$ is $\mathcal{D} * \mathcal{H}$ Hall-closed ?

(b) If $\mathcal{D}$ is a Fitting class with $\mathrm{char}(\mathcal{D}) = \mathbb{P}$, do we have

$$(\mathcal{D} * \mathcal{H} \text{ is Hall-closed}) \Longleftrightarrow (\mathcal{D}^* * \mathcal{H} \text{ is Hall-closed}) \ ?$$

3.5 The $H_{\mathcal{N}}$-closure properties of $\mathcal{N}_\pi * \mathcal{H}$.

3.5.1 Proposition. Let π be a set of primes with $|\pi| \geq 2$. Then $\mathcal{N}_\pi * \mathcal{H}$ is not $H_{\mathcal{N}}$-closed.

Proof. Let q and s be distinct primes belonging to π . Without loss of generality, assume that $q > 2$. Let p be a prime with $p \mid q-1$ (so that possibly $p = s$), and choose $r \in \mathbb{P} \setminus \{p,q,s\}$.

Let A and B be groups with A of type $M(q^\beta, r^\alpha, q^1, p)$ and B of type $M(s^\gamma, r^\alpha, q^1, p)$, as in (1.7.5/6), with the further stipulation that there exist isomorphisms (see (1.7.6c))

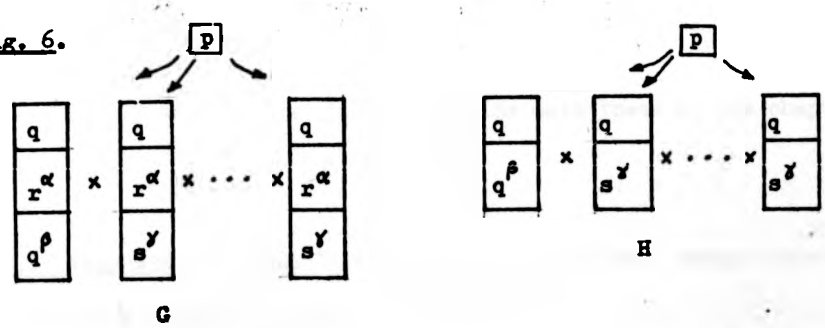
$\theta : A/O_q(A) \to B/O_s(B)$, and

$\varphi : B/O_{s,r}(B) \to D_q^p$, the non-abelian group of order pq .

We may express A and B as semi-direct products $A = A' \rfloor \langle u \rangle$ and $B = B' \rfloor \langle v \rangle$, where $|u| = |v| = p$, and $(u\, O_q(A))\theta = v\, O_s(B)$.

Let D denote the external direct product $A \times (B)^{p-1}$, and let w denote the element $(u,v,\ldots,v)$ of D ; then $|w| = p$. Now, $D' = A' \times (B')^{p-1}$, and $D' \cap \langle w \rangle = 1$. Let $G = D' \rfloor \langle w \rangle \leq D$. Due to the action of u on $O_q(A)$, it follows (since $p \neq q$) that $G_{\mathcal{N}} \leq D'$, and thus that $G_{\mathcal{N}} = (D')_{\mathcal{N}} = A_{\mathcal{N}} \times (B_{\mathcal{N}})^{p-1}$. Further, we have

fig. 6.



$G_{\mathcal{N}_\pi} = G_{\mathcal{N}}$. Because of the existence of θ , we have

$$G/G_{\mathcal{N}_\pi} \cong (B'/B_{\mathcal{N}})^p \rfloor \langle v \rangle \quad ,$$

where the action of v on the direct power $(B'/B_{\mathcal{N}})^p$ is the natural "component-wise" action induced by the action of v on $B'/B_{\mathcal{N}}$ as a section of B . Now, $B'/B_{\mathcal{N}} = ((B'/B_{\mathcal{N}})\langle v \rangle)'$, and so by (3.2.1) we have $(B'/B_{\mathcal{N}})^p \rfloor \langle v \rangle \in \mathcal{H}$, and so $G \in \mathcal{N}_\pi * \mathcal{H}$.

Let $H \in \mathrm{Hall}_{\pi'}(G)$. Certainly $H \geqslant G_{\mathcal{N}}$, and we may check that $H_{\mathcal{N}} = H_{\mathcal{N}_\pi} = Q \times (B_{\mathcal{N}})^{p-1}$ for some suitable $Q \in \mathrm{Syl}_q(A')$. But then $H/H_{\mathcal{N}_\pi} \cong (C_q)^{p-1} \rfloor C_p$, with the natural "component-wise" action of C_p on the direct power $(C_q)^{p-1}$ (recall that $p \mid q-1$). Now, by (3.2.1), the abstract semi-direct product $(C_q)^p \rfloor C_p$, with component-wise action, belongs to $\mathcal{H}$, while by Berger's theorem (2.4.10), $C_q \rfloor C_p \notin \mathcal{H}$. Applying the "Quasi-R_o-Lemma", (1.9.2a) in $(C_q)^p \rfloor C_p$, with normal subgroups $C_q \times (1)^{p-1}$ and $(1) \times (C_q)^{p-1}$, it follows that $(C_q)^{p-1} \rfloor C_p \notin \mathcal{H}$. Thus, $H \notin \mathcal{N}_\pi * \mathcal{H}$, and since $H \geqslant G_{\mathcal{N}}$, it follows that $\mathcal{N}_\pi * \mathcal{H}$ is not $H_{\mathcal{N}}$-closed, as claimed.

3.6 Hall-determination and $\mathcal{H}$.

In this section, we digress from the main theme of the chapter, and and use (3.2.1) to prove the following.

<u>3.6.1 Proposition</u>. Let $n \in \mathbb{N}$; then $\mathcal{H}$ is not n-Hall-determined (see section 2.9).

$G_{\mathcal{N}_\pi} = G_{\mathcal{N}}$. Because of the existence of θ , we have

$$G/G_{\mathcal{N}_\pi} \cong (B'/B_{\mathcal{N}})^p \rfloor \langle v \rangle \quad ,$$

where the action of v on the direct power $(B'/B_{\mathcal{N}})^p$ is the natural "component-wise" action induced by the action of v on $B'/B_{\mathcal{N}}$ as a section of B . Now, $B'/B_{\mathcal{N}} = ((B'/B_{\mathcal{N}})\langle v \rangle)'$, and so by (3.2.1) we have $(B'/B_{\mathcal{N}})^p \rfloor \langle v \rangle \in \mathcal{H}$, and so $G \in \mathcal{N}_\pi * \mathcal{H}$.

Let $H \in \mathrm{Hall}_{\pi'}(G)$. Certainly $H \geqslant G_{\mathcal{N}}$, and we may check that $H_{\mathcal{N}} = H_{\mathcal{N}_\pi} = Q \times (B_{\mathcal{N}})^{p-1}$ for some suitable $Q \in \mathrm{Syl}_q(A')$. But then $H/H_{\mathcal{N}_\pi} \cong (C_q)^{p-1} \rfloor C_p$, with the natural "component-wise" action of C_p on the direct power $(C_q)^{p-1}$ (recall that $p \mid q-1$). Now, by (3.2.1), the abstract semi-direct product $(C_q)^p \rfloor C_p$, with component-wise action, belongs to $\mathcal{H}$, while by Berger's theorem (2.4.10), $C_q \rfloor C_p \notin \mathcal{H}$. Applying the "Quasi-R_0-Lemma", (1.9.2a) in $(C_q)^p \rfloor C_p$, with normal subgroups $C_q \times (1)^{p-1}$ and $(1) \times (C_q)^{p-1}$, it follows that $(C_q)^{p-1} \rfloor C_p \notin \mathcal{H}$. Thus, $H \notin \mathcal{N}_\pi * \mathcal{H}$, and since $H \geqslant G_{\mathcal{N}}$, it follows that $\mathcal{N}_\pi * \mathcal{H}$ is not $H_{\mathcal{N}}$-closed, as claimed.

3.6 Hall-determination and $\mathcal{H}$.

In this section, we digress from the main theme of the chapter, and and use (3.2.1) to prove the following.

3.6.1 Proposition. Let $n \in \mathbb{N}$; then $\mathcal{H}$ is not n-Hall-determined (see section 2.9).

<u>Proof</u>. We note that $11 \equiv 2 \pmod 3$ and $11 \equiv 4 \pmod 7$. By Dirichlet's Theorem, [32; Theorem 15], the set $\{11 + 21m : m \in \mathbb{N}\}$ contains infinitely many distinct primes; all congruent to $2 \pmod 3$ and to $4 \pmod 7$. Let p be such a prime. Since $p \equiv 4 \pmod 7$, then $p^3 \equiv 1 \pmod 7$, while $p, p^2 \not\equiv 1 \pmod 7$.

Let K denote the finite field $GF(p^3)$. We may write the "Extended affine group", A, of K (see [19; page 147]) in the form $A = (K^+)](K^\times)]\langle\tau\rangle)$, where K^+ and $K^\times$ denote the additive and multiplicative groups of K, respectively, and τ is a field automorphism of order 3 of K (so that A is regarded as a subgroup of the group of all permutations of K). We note that K^+ is elementary abelian of order p^3, while $K^\times$ is cyclic of order p^3-1. Since $7 \mid p^3-1$, then $K^\times$ has a subgroup $K_7^\times$ of order 7, which is acted on non-trivially by $\langle\tau\rangle$ since $(K_7^\times \setminus 1) \subseteq (GF(p^3) \setminus GF(p))$. Thus, A has a subgroup $K^+(K_7^\times\langle\tau\rangle)$, where $K_7^\times\langle\tau\rangle$ is non-abelian of order 21. Since p has order 3 (mod 7), then $K_7^\times$ acts faithfully and irreducibly on K^+. Since p has order 2 (mod 3), and since τ acts non-trivially on K^+, then $K^+|_{\langle\tau\rangle}$ must decompose as a direct sum of a non-trivial irreducible two-dimensional submodule, K_2^+ say (see [41; II.2.10]), and a one-dimensional trivial submodule, K_1^+.

By the structure of $K^+K_7^\times\langle\tau\rangle$, we have

$$K^+K_7^\times = (K^+K_7^\times\langle\tau\rangle)' \in \mathcal{H} . \qquad \cdots(1)$$

By [2; Proposition 3] (see [17; page 230]), $K_2^+\langle\tau\rangle \in \mathcal{H}$, and so

$$K^+\langle\tau\rangle = K_2^+\langle\tau\rangle \times K_1^+ \in \mathcal{H} . \qquad \cdots(2)$$

The group $K^+K_7^\times\langle\tau\rangle$ is of type $M(p^3,7,3)$; we write it in the

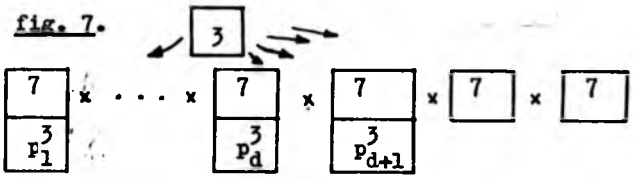
form $G(p)]\langle\tau\rangle$, where $G(p) = K^{+}K_7^{\times}$.

Let d be the smallest multiple of 3 with $d \geqslant n$, and choose primes $p_1, \ldots, p_{d+1}$ with $p_i \equiv 2 \pmod 3$ and $p_i \equiv 4 \pmod 7$. Construct groups $G(p_i)$ as above; each may be considered to admit a (fixed) cyclic group $\langle\tau\rangle$ of order 3 in the obvious way. Let Q_1 and Q_2 be cyclic groups of order 7 , and define D by

$$D = (G(p_1) \times \cdots \times G(p_{d+1}) \times Q_1 \times Q_2)]\langle\tau\rangle \quad ,$$

with "component-wise" action of $\langle\tau\rangle$ (non-trivial on the Q_i).

fig. 7.



We show that $D \notin \mathcal{H}$, while all Hall p-complements of D belong to $\mathcal{H}$. Firstly, $D/F(D) \cong (C_7)^{d-1}] C_3 \notin \mathcal{H}$ (by similar reasoning to that in (3.5.1), using (3.2.1) and (1.9.2a) inductively, and the fact that $3 \mid d$). Thus, by (1.4.8), $D \notin \mathcal{H}$.

Since $K^{+}K_7^{\times} \in \mathcal{H}$ by (1), the Hall 3-complement of D belongs to $\mathcal{H}$.

Since $K^{+}\langle\tau\rangle \in \mathcal{H}$ by (2), then inductive application of (1.9.2a) shows that a Hall 7-complement of D lies in $\mathcal{H}$.

We are left to consider the p_i-complements ; without loss, we consider a p_{d+1} -complement. Let $P \in \mathrm{Syl}_{p_{d+1}}(D)$. Then $P \trianglelefteq D$, and $D/P \cong (G(p_1) \times \cdots \times G(p_d) \times C_7 \times C_7 \times C_7)]\langle\tau\rangle$. Since $3 \mid d$, then by inductive application of (3.2.1) and (1.9.2a), we find that $D/P \in \mathcal{H}$. But now $\mathcal{H}$ is not $d+2$ -Hall-determined, and so not n-Hall-determined by (2.9.3), completing the proof.

3.7 The Laue-Lausch-Pain/ Bryant-Kovács/ Cossey Theorem.

If p and q are primes with $q \mid p-1$, we know that $\mathcal{S}_p * \mathcal{S}_q \not\subseteq \mathcal{H}$ by Berger's result (2.4.10) (and, in certain special cases, by results of Blessenohl and Gaschütz [6]).

The case in which $q \nmid p-1$ is dealt with by the following recent result.

3.7.1 Theorem (Laue-Lausch-Pain [44], Bryant-Kovács [7], Cossey).

Let p and q be distinct primes with $q \nmid p-1$. Then $\mathcal{S}_p * \mathcal{S}_q \not\subseteq \mathcal{H}$.

The genesis of this result lies in the paper [44], where some new normal Fitting classes are constructed by use of the group-theoretical transfer. Bryant and Kovács state in [7] that the main theorem of [7] was provoked by a comment of Lausch and Cossey that such a result, together with the results of [44], would lead to (3.7.1).

Since both [7] and [44] allude to (3.7.1), while neither proves it, we sketch the derivation of (3.7.1) (after a talk at Warwick on the subject in 1977 by H. Laue).

We will consider some transfer-type normal Fitting classes, similar to those of [44], in some detail in Chapter 4. For present purposes, the results of [44] can be put as follows.

3.7.2 Theorem (Laue-Lausch-Pain [44: Satz]). Suppose that X is a group for which the group $K_0 = (\mathrm{Aut}(X))' \langle \alpha \in \mathrm{Aut}(X) : [X, \langle\alpha\rangle] \leqslant X \rangle$ is a proper subgroup of $\mathrm{Aut}(X)$. Then there exists a normal Fitting class $\mathfrak{F}$ such that $X]\langle\gamma\rangle \notin \mathfrak{F}$ if $\gamma \in \mathrm{Aut}(X) \setminus K_0$.

The main theorem of [7] is the following, proved using Lie-theoretical techniques (as in Chapter 5 of [50] , for example).

3.7.3 Theorem (Bryant and Kovács [7]). Let p be a prime. If H is a linear group of finite dimension $d \geqslant 2$ over the field $GF(p)$, there exists a finite p-group P such that the restriction of $Aut(P)$ to $P/\Phi(P)$ is isomorphic, as linear group, to H.

Let now p and q be distinct primes with $q \nmid p-1$, and let $d \in \mathbb{N}$ be minimal such that $q \mid p^d-1$; then $d \geqslant 2$. Let H be a subgroup of $GL(d,p)$ of order q. Take P as ensured by (3.7.3). Now, by a theorem of Burnside, [28; 5.1.4], $C = C_{Aut(P)}(P/\Phi(P))$ is a p-group. By choice, $Aut(P)/C \cong H$ (as linear group). Identify H with a Sylow q-subgroup of $Aut(P)$; let $H = \langle h \rangle$. Now, $K_0 = (Aut(P))' \langle \alpha \in Aut(P) : [P, \langle\alpha\rangle] \lneqq P \rangle = C \lneqq Aut(P)$, since, by choice of d, $P/\Phi(P)$ is a faithful irreducible module for H and so $[P, \langle h\rangle] = P$. But now by (3.7.2), there exists a normal Fitting class $\mathfrak{F}$ with $P \rceil \langle h \rangle \notin \mathfrak{F}$, and (3.7.1) follows.

3.8 A group construction and Hall-closed classes $\mathfrak{F} * \mathfrak{H}$.

We use (3.7.1) and (2.4.10) in a group construction which will be applied in (3.8.2) and (3.9.12).

3.8.1 Construction.

(a) The ingredients for the construction comprise a Fitting class $\mathfrak{X}$, a Lockett class $\mathfrak{Y}$ with $\mathfrak{X} \nsubseteq \mathfrak{Y}$, a group G of minimal order in $\mathfrak{X} \setminus \mathfrak{Y}$, a prime $s \neq t = |G : G_{\mathfrak{Y}}|$, and a set of primes σ with

$\{s,t\} \subseteq \sigma$. Let D denote $G_{\mathcal{Y}}$; then $|G : D| = t$.

(b) Depending on the relationship between s and t , we appeal to (2.4.10) or (3.7.1) to conclude that $\mathcal{S}_t * \mathcal{S}_s \nsubseteq \mathcal{H}$.

Let $\overline{T}]\overline{S}$ be a group of minimal order in $\mathcal{S}_t * \mathcal{S}_s \setminus \mathcal{H}$, where $\overline{T} \in \mathcal{S}_t$ and $\overline{S} \in \mathcal{S}_s$. By (3.2.3), $O_s(\overline{T}\,\overline{S}) = 1$.

We denote $\langle z \in Z(\overline{T}) : z^t = 1 \rangle$ by $\Omega_1(Z(\overline{T}))$ (see [28; p. 17]). Then $\Omega_1(Z(\overline{T}))$ is certainly a non-trivial elementary abelian characteristic subgroup of $\overline{TS}$, of order t^r , $r > 1$, say.

Let $T]S$ denote a group isomorphic with $\overline{TS}/\Omega_1(Z(\overline{T}))$, where $T \in \mathcal{S}_t$ and $S \in \mathcal{S}_s$, and let $n = |TS|$; then $TS \in \mathcal{H}$ by minimality.

(c) Let Γ denote the external direct power G^r of r copies of G , where r is as in (b), and let Δ denote $\{(d_1,\ldots d_r) : d_i \in D\} \trianglelefteq \Gamma$. Since $\mathcal{Y}$ is a Lockett class then $\Delta = \Gamma_{\mathcal{Y}}$ and Γ/Δ is elementary abelian of order t^r , so that $\Gamma/\Delta \cong \Omega_1(Z(\overline{T}))$.

Now form $W = \Gamma \operatorname{wr} (TS)$. We identify TS with the standard complement of W , and adhere to the notation of (1.7.1), so that Γ^* denotes the base group. Since $\mathcal{Y}$ is a Lockett class, then by (1.7.3), $W_{\mathcal{Y}} = \Gamma^*_{\mathcal{Y}} = \Delta^*$.

(d) Suppose that $L \in \mathrm{Hall}_\sigma(G)$, with $L \cap D = K \in \mathrm{Hall}_\sigma(D)$. Define

$$\Lambda = \{(\ell_1,\ldots,\ell_r) : \ell_i \in L\} \leq \Gamma \quad , \text{ and}$$

$$\chi = \{(k_1,\ldots,k_r) : k_i \quad K\} \leq \Delta \quad .$$

As usual, Λ^* and χ^* denote the corresponding subgroups of Γ^* ; thus $\Lambda^* = \{(\lambda_1,\ldots,\lambda_n) : \lambda_i \in \Lambda\}$, where n is as in (b).

Since $|G : D| = t \in \sigma$, then $G = DL$, $\Gamma = \Delta\Lambda$ and $\Gamma^* = \Delta^*\Lambda^*$.

Further, $\Lambda^* \in \mathrm{Hall}_\sigma(\Gamma^*)$ and $\chi^* = \Delta^* \cap \Lambda^* \in \mathrm{Hall}_\sigma(\Delta^*)$; we note that $\chi^* \trianglelefteq \Lambda^*$.

By virtue of the wreath product construction, Δ^* , Λ^* and χ^* are all TS-invariant (see (1.7.1)), so that, in particular, $\chi^* \trianglelefteq \Lambda^* TS$.

Since $\chi^* = \Delta^* \cap \Lambda^*$ and $W = \Gamma^*] TS$ is semi-direct, we have

$$\chi^* = \Delta^* \cap \Lambda^* TS \quad . \qquad \cdots(1)$$

It follows that

$$\begin{aligned} \Lambda^* TS/\chi^* &= \Lambda^* TS/(\Delta^* \cap \Lambda^* TS) \\ &\cong \Lambda^* TS\, \Delta^*/\Delta^* \\ &= W/\Delta^* \qquad \text{(since } W = \Delta^* \Lambda^* TS\text{)} \\ &\cong (\Gamma/\Delta) \text{ wr } TS \qquad \text{(by (1.7.2a))} \quad . \end{aligned}$$

Specifically, let $k : \Lambda^* TS/\chi^* \to (\Gamma/\Delta) \text{ wr } TS$ be the isomorphism obtained by following through this sequence of isomorphisms. We note that

$$k : \Lambda^*/\chi^* \mapsto\!\!\!\to (\Gamma/\Delta)^* \text{ , the base group of } (\Gamma/\Delta) \text{ wr } TS \text{ .} \qquad \cdots(2)$$

We also note that k can be naturally inflated to a homomorphism

$$\hat{k} : \Lambda^* TS \to (\Gamma/\Delta) \text{ wr } TS \text{ , with } \ker(\hat{k}) = \chi^* \text{ .} \qquad \cdots(3)$$

Since $TS = \overline{TS}/\Omega_1(Z(\overline{T}))$ and $\Gamma/\Delta \cong \Omega_1(Z(\overline{T}))$ by part (c), then by [41; I.15.9] , there exists an injection

$$j : \overline{TS} \hookrightarrow (\Gamma/\Delta) \text{ wr } TS \qquad \cdots(4)$$

such that (by a reading of the proof of [41; I,15.9]) we have

$$j : \Omega_1(Z(\overline{T})) \hookrightarrow (\Gamma/\Delta)^* \quad . \qquad \cdots(5)$$

(e) Let Θ and Σ denote the complete pre-images in $\Lambda^* \overline{TS}$ as follows :

$$\Theta = (\overline{T})\, j\, \hat{k}^{-1} \leqslant \Lambda^* \overline{TS} \text{ , and}$$

$$\Sigma = (\overline{S})\, j\, \hat{k}^{-1} \leqslant \Lambda^* \overline{TS} \text{ ,}$$

where j and $\hat{k}$ are as in (4) and (3) respectively.

Then $\chi^* \leqslant \Theta \cap \Sigma$, and, since $j\,\hat{k}^{-1}$ induces the monomorphism

$j\,k^{-1} : \overline{TS} \to \Lambda^* TS/\chi^*$ (and induces isomorphisms $\overline{T} \to \Theta/\chi^*$ and $\overline{S} \to \Sigma/\chi^*$), we have

$$\Theta/\chi^* \cong \overline{T}, \quad \Sigma/\chi^* \cong \overline{S}, \quad \Theta \cap \Sigma = \chi^* \quad \text{and} \quad \Theta\Sigma/\chi^* \cong \overline{TS}. \qquad \ldots(6)$$

Since $\chi^* \in \mathrm{Hall}_{\sigma'}(\Delta^*)$ and $\Theta\Sigma/\chi^* \in \mathcal{S}_\sigma$, then

$$\Theta\Sigma \in \mathrm{Hall}_\sigma(\Delta^* \Theta\Sigma). \qquad \ldots(7)$$

(f) <u>Lemma</u> $\Delta^* \Theta\Sigma \in \mathfrak{X} * \mathfrak{H}$.

<u>Proof</u>. Let $\Phi^* \in \mathrm{Hall}_{\sigma'}(\Delta^*)$, so that $\Delta^* = \chi^* \Phi^*$ by consideration of orders. Thus $\Delta^* \cap \Theta\Sigma = \chi^* \Phi^* \cap \Theta\Sigma = \chi^*(\Phi^* \cap \Theta\Sigma) = \chi^*$, since $\Theta\Sigma \in \mathcal{S}_\sigma$. It follows that

$$\Delta^* \Theta\Sigma/\Delta^* \cong \Theta\Sigma/(\Theta\Sigma \cap \Delta^*) = \Theta\Sigma/\chi^* \cong \overline{TS}, \qquad \ldots(8)$$

the final isomorphism by (6).

Now, Γ^*/Δ^* is an (elementary) abelian t-group, and so $(\Gamma^* \cap \Delta^* \Theta\Sigma)/\Delta^* \trianglelefteq \Gamma^*/\Delta^*$, whence $\Gamma^* \cap \Delta^* \Theta\Sigma \trianglelefteq \Gamma^*$. Since Γ^* is a direct power of $G \in \mathfrak{X}$, it follows that

$$\Gamma^* \cap \Delta^* \Theta\Sigma \leq (\Delta^* \Theta\Sigma)_{\mathfrak{X}} \qquad \ldots(9)$$

We also note (recalling the third statement in (6)) that

$$\begin{aligned}(\Gamma^* \cap \Delta^* \Theta\Sigma)/\Delta^* &= (\Delta^* \Lambda^* \cap \Delta^* \Theta\Sigma)/\Delta^* \\ &= \Delta^*(\Lambda^* \cap \Delta^* \Theta\Sigma)/\Delta^* \\ &\cong (\Lambda^* \cap \Delta^* \Theta\Sigma)/(\Delta^* \cap (\Lambda^* \cap \Delta^* \Theta\Sigma)) \\ &= (\Lambda^* \cap \Delta^* \Theta\Sigma)/\chi^* . \end{aligned} \qquad \ldots(10)$$

Since $j\,k^{-1}$ induces an isomorphism from $\overline{T}$ to Θ/χ^*, then

$$\Omega_1(Z(\Theta/\chi^*)) = (\Omega_1(Z(\overline{T}))\, j\, k^{-1} \leq (\Gamma/\Delta)^*\, k^{-1} = \Lambda^*/\chi^*, \qquad \ldots(11)$$

recalling (5) and (2).

Now $\Omega_1(Z(\Theta/\chi^*)) \leq \Delta^* \Theta\Sigma/\chi^*$, and since $\Theta/\chi^* \cong T \gneq 1$, then

$$1 \neq \Omega_1(Z(\Theta/\chi^*)) \leq (\Delta^* \cap \Lambda^* \Theta\Sigma)/\chi^* . \qquad \ldots(12)$$

Combining (9), (10) and (12), we find that $\Delta^* \lneq (\Delta^* \Theta\Sigma)_{\mathfrak{X}}$. But by (8), $\Delta^*\Theta\Sigma/\Delta^* \cong \overline{TS}$, and so $\Delta^*\Theta\Sigma/(\Delta^*\Theta\Sigma)_{\mathfrak{X}}$, being now isomorphic to a proper factor of $\overline{TS}$, belongs to $\mathfrak{H}$ by minimality of $\overline{TS}$. Thus, $\Delta^*\Theta\Sigma \in \mathfrak{X} * \mathfrak{H}$, as claimed.

(g) <u>Lemma</u>. Suppose that $\mathfrak{G}$ is any Lockett class with $\Lambda^*_{\mathfrak{G}} = \chi^*$. Then $(\Theta\Sigma)_{\mathfrak{G}} = \chi^*$.

<u>Proof</u>. Certainly $\chi^* \trianglelefteq \Theta\Sigma$, and so $\chi^* \leq (\Theta\Sigma)_{\mathfrak{G}}$.

Let $\Xi = (\Theta\Sigma)_{\mathfrak{G}} \cap \Theta$. Suppose that $\Xi \gneq \chi^*$. Then $1 \neq \Xi/\chi^* \trianglelefteq \Theta/\chi^*$, and so, recalling (11) above,

$$1 \neq \Xi/\chi^* \cap \Omega_1(Z(\Theta/\chi^*)) \trianglelefteq \Lambda^*/\chi^* , \qquad \cdots(13)$$

the normality because Λ^*/χ^* is an elementary abelian t-group.

Let Ψ denote the complete preimage in Θ of $\Omega_1(Z(\Theta/\chi^*))$. Then by (13) we have $\chi^* \lneq (\Theta\Sigma)_{\mathfrak{G}} \cap \Psi \trianglelefteq \Lambda^*$. Since $\Omega_1(Z(\Theta/\chi^*)) \trianglelefteq \Theta\Sigma/\chi^*$, then $\Psi \trianglelefteq \Theta\Sigma$, and so $(\Theta\Sigma)_{\mathfrak{G}} \cap \Psi \in \mathfrak{G}$. It follows that $\chi^* \lneq \Lambda^*_{\mathfrak{G}} = \chi^*$, a contradiction.

Thus $(\Theta\Sigma)_{\mathfrak{G}}/\chi^* \cap \Theta/\chi^* = 1$, and so from (6) and (b) we have $\mathcal{S}_s \ni (\Theta\Sigma)_{\mathfrak{G}}/\chi^* \leq O_s(\Theta\Sigma/\chi^*) \cong O_s(\overline{TS}) = 1$, and the assertion follows. This completes our construction.

<u>3.9.2 Proposition</u>. Let $\mathfrak{F}$ be a Lockett class such that $\mathfrak{F} * \mathfrak{H}$ is Hall-closed. Then $\mathfrak{F}$ is itself Hall-closed.

<u>Proof</u>. Suppose for a contradiction that $\mathfrak{F}$ is not Hall-closed. Then there exists $\tau \subseteq \mathbb{P}$ such that $\mathfrak{F} \nsubseteq \mathcal{K}_\tau(\mathfrak{F})$. Let G denote a group of minimal order in $\mathfrak{F} \setminus \mathcal{K}_\tau(\mathfrak{F})$. Then $|G : G_{\mathcal{K}_\tau(\mathfrak{F})}| = t \in \mathbb{P}$;

in fact since $\mathcal{K}_\tau(\mathfrak{F}) = \mathcal{K}_\tau(\mathfrak{F}) * \mathcal{S}_{\tau'}$, then $t \in \tau$. Let s be a prime with $(s, |G|) = 1$, and define $\sigma = \tau \cup \{s\}$.

By (2.4.1b), $\mathcal{K}_\tau(\mathfrak{F})$ is a Lockett class.

We apply construction (3.8.1) with $\mathfrak{F}$ for $\mathcal{X}$, $\mathcal{K}_\tau(\mathfrak{F})$ for $\mathcal{Y}$, and take G , s , t and σ as just defined. We choose, as in (3.8.1b), a group $\overline{T}]\overline{S}$ of minimal order in $\mathcal{S}_t * \mathcal{S}_s \setminus \mathcal{H}$, define $D = G_{\mathcal{K}_\tau(\mathfrak{F})}$, choose $L \in \mathrm{Hall}_\sigma(G)$ and set $L \cap D = K$. We obtain the group $\Delta^* \Theta\Sigma \in \mathcal{X} * \mathcal{H}$.

Since $(s, |G|) = 1$ and $\sigma = \tau \cup \{s\}$, then $L \in \mathrm{Hall}_\tau(G)$, and so $L \notin \mathfrak{F}$, while $K \in \mathrm{Hall}_\tau(D) \subseteq \mathfrak{F}$. Since $K \trianglelefteq L$ and $|L : K| = t$, then $K = L_{\mathfrak{F}}$. Because $\mathfrak{F}$ is a Lockett class, we have $\mathcal{X}^* = \Lambda^*_{\mathfrak{F}}$. But then by (3.8.1g), $(\Theta\Sigma)_{\mathfrak{F}} = \mathcal{X}^*$. However, by (6) of (3.8.1), we have $\Theta\Sigma / \mathcal{X}^* \cong \overline{TS} \notin \mathcal{H}$. Thus $\Theta\Sigma \notin \mathfrak{F} * \mathcal{H}$. But $\Theta\Sigma \in \mathrm{Hall}_\sigma(\Delta^* \Theta\Sigma)$ and $\Delta^* \Theta\Sigma \in \mathfrak{F} * \mathcal{H}$, contradicting the Hall-closure of $\mathfrak{F} * \mathcal{H}$. This completes the proof.

By (3.3.1), $\mathcal{G} = \mathcal{N} * \mathcal{H}$ is not Hall-closed, while $\mathcal{G} * \mathcal{H} = \mathcal{S}$ (c.f. (1.5.3b)) is Hall-closed; the hypothesis that $\mathfrak{F}$ be a Lockett class in (3.8.2) is thus not redundant.

3.9 On a conjecture of Cossey.

In his paper [18] , Cossey proves the following.

3.9.1 Theorem (Cossey, [18; 4.1]). Let $\mathfrak{F}$ and $\mathcal{G}$ be Lockett classes with $\mathfrak{F} * \mathcal{H} = \mathcal{G} * \mathcal{H}$. Then $\mathfrak{F} = \mathcal{G}$.

in fact since $\mathcal{K}_\tau(\mathfrak{J}) = \mathcal{K}_\tau(\mathfrak{J}) * \mathcal{S}_{\tau'}$, then $t \in \tau$. Let s be a prime with $(s , |G|) = 1$, and define $\sigma = \tau \cup \{s\}$.

By (2.4.1b), $\mathcal{K}_\tau(\mathfrak{J})$ is a Lockett class.

We apply construction (3.8.1) with $\mathfrak{J}$ for $\mathcal{X}$, $\mathcal{K}_\tau(\mathfrak{J})$ for $\mathcal{Y}$, and take G , s , t and σ as just defined. We choose, as in (3.8.1b), a group $\overline{T}]\overline{S}$ of minimal order in $\mathcal{S}_t * \mathcal{S}_s \setminus \mathcal{H}$, define $D = G_{\mathcal{K}_\tau(\mathfrak{J})}$, choose $L \in \mathrm{Hall}_\sigma(G)$ and set $L \cap D = K$. We obtain the group $\Delta^* \Theta\Sigma \in \mathcal{X} * \mathcal{H}$.

Since $(s , |G|) = 1$ and $\sigma = \tau \cup \{s\}$, then $L \in \mathrm{Hall}_\tau(G)$, and so $L \notin \mathfrak{J}$, while $K \in \mathrm{Hall}_\tau(D) \subseteq \mathfrak{J}$. Since $K \trianglelefteq L$ and $|L : K| = t$, then $K = L_{\mathfrak{J}}$. Because $\mathfrak{J}$ is a Lockett class, we have $\mathcal{X}^* = \Lambda^*_{\mathfrak{J}}$. But then by (3.8.1g), $(\Theta\Sigma)_{\mathfrak{J}} = \mathcal{X}^*$. However, by (6) of (3.8.1), we have $\Theta\Sigma / \mathcal{X}^* \cong \overline{TS} \notin \mathcal{H}$. Thus $\Theta\Sigma \notin \mathfrak{J} * \mathcal{H}$. But $\Theta\Sigma \in \mathrm{Hall}_\sigma(\Delta^* \Theta\Sigma)$ and $\Delta^* \Theta\Sigma \in \mathfrak{J} * \mathcal{H}$, contradicting the Hall-closure of $\mathfrak{J} * \mathcal{H}$. This completes the proof.

By (3.3.1), $\mathcal{G} = \mathcal{N} * \mathcal{H}$ is not Hall-closed, while $\mathcal{G} * \mathcal{H} = \mathcal{S}$ (c.f. (1.5.3b)) is Hall-closed; the hypothesis that $\mathfrak{J}$ be a Lockett class in (3.8.2) is thus not redundant.

3.9 On a conjecture of Cossey.

In his paper [18] , Cossey proves the following.

<u>3.9.1 Theorem (Cossey. [18; 4.1])</u>. Let $\mathfrak{J}$ and $\mathcal{G}$ be Lockett classes with $\mathfrak{J} * \mathcal{H} = \mathcal{G} * \mathcal{H}$. Then $\mathfrak{J} = \mathcal{G}$.

Cossey observes that his proof of (3.9.1) relies heavily on the hypothesis that $\mathfrak{F}$ and $\mathfrak{G}$ are Lockett classes, and remarks that it is "tempting" to make the following conjecture.

<u>3.9.2 Conjecture (Cossey, [18])</u>. If $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes, then $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{G}^*$.

<u>3.9.3 Remarks</u>. (a) In [18] , Cossey goes on to note that, had the Lockett conjecture been true, then (3.1.2) and (3.9.1) would imply the truth of (3.9.2) (see the remarks following (3.1.2)).

(b) Since the Lockett conjecture is not in general true (see (1.5.14)), then, even if $\mathfrak{F}$ is a class for which it is true, it might be possible to find a Fitting class $\mathfrak{G}$, where $\mathfrak{G} \not\supseteq \mathfrak{G}^* \cap \mathfrak{H}$, with $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ and $\mathfrak{F}^* \neq \mathfrak{G}^*$.

(c) As with the case of (3.1.2), we have had no success in trying to incorporate the Berger-Cossey counterexample to the Lockett conjecture into an example denying (3.9.2) (in either direction).

(d) Theorem (3.9.1) may be regarded as a "cancellation-type" result ; in this section, we prove further cancellation-type results, which lend positive evidence for the "only if" part of (3.9.2). We also prove for certain specific classes $\mathfrak{G}$ that $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{G}^*$.

We first quote another of Cossey's results.

<u>3.9.4 Theorem (Cossey, [18: 4.3])</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes. Then

(a) if $\mathfrak{G}^* = \mathfrak{S}$, then $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{S}$; and

(b) $\mathfrak{F} * \mathfrak{H} = \mathfrak{H}$ ($= (1) * \mathfrak{H}$) if and only if $\mathfrak{F} = (1)$.

The next lemma is crucial; part (a) depends on (2.4.10), while part (b) needs (3.7.1) as well.

<u>3.9.5 Lemma</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes with $\mathfrak{F} * \mathfrak{H} \subseteq \mathfrak{G} * \mathfrak{H}$. Then (a) $\mathrm{char}(\mathfrak{F}) \cap \{2\}' \subseteq \mathrm{char}(\mathfrak{G})$; and

(b) $\mathrm{char}(\mathfrak{F}) \subseteq \mathrm{char}(\mathfrak{G})$.

<u>Proof</u>. (a) For suppose that $p \in (\mathrm{char}(\mathfrak{F}) \cap \{2\}') \setminus \mathrm{char}(\mathfrak{G})$, and let D denote the dihedral group of order $2p$. Then $O_p(D) \leq D_{\mathfrak{F}}$ and since $C_2 \in \mathcal{N} \subseteq \mathfrak{H}$, then $D \in \mathfrak{F} * \mathfrak{H} \subseteq \mathfrak{G} * \mathfrak{H}$. Since $p \notin \mathrm{char}(\mathfrak{G})$, then $D_{\mathfrak{G}} = 1$. Since $D \notin \mathfrak{H}$ by (2.4.10), it follows that $D \notin \mathfrak{G} * \mathfrak{H}$, a contradiction.

(b) Suppose that $2 \in \mathrm{char}(\mathfrak{F}) \setminus \mathrm{char}(\mathfrak{G})$, and let $q \in \mathbb{P} \setminus \{2\}$. By (3.7.1), $\mathcal{S}_2 * \mathcal{S}_q \notin \mathfrak{H}$. Let $T \rceil Q$ $(T \in \mathcal{S}_2 , Q \in \mathcal{S}_q)$ be a group of minimal order in $\mathcal{S}_2 * \mathcal{S}_q \setminus \mathfrak{H}$. By (3.2.3), $O_q(TQ) = 1$, and, as above, $TQ \in \mathfrak{F} * \mathfrak{H} \setminus \mathfrak{G} * \mathfrak{H}$. This is a contradiction, and the assertion follows from (a) .

<u>3.9.6 Remark</u>. Suppose in the situation of (3.9.5) that there exists $p \in \mathrm{char}(\mathfrak{F}) \setminus \mathrm{char}(\mathfrak{G})$, and that there further exists a prime $t \in \mathbb{P} \setminus (\mathrm{char}(\mathfrak{F}) \cup \{2\})$. Then in the group $C = C(p,t,2)$ of (2.6.1), we have $C_{\mathfrak{F}} = F(C)$ and $C \in \mathfrak{F} * \mathfrak{H}$, while $C_{\mathfrak{G}} = 1$ and $C \notin \mathfrak{G} * \mathfrak{H}$. Consequently, (3.7.1) is needed only to eliminate the possibility that $\mathrm{char}(\mathfrak{F}) = \mathbb{P}$ while $\mathrm{char}(\mathfrak{G}) = \{2\}'$, and (2.4.10) suffices in all other cases.

<u>3.9.7 Theorem</u>. Let $\mathfrak{F}$ be a Fitting class with $\mathfrak{F} * \mathfrak{H} = \mathcal{S}_\pi * \mathfrak{H}$, where $\pi \subsetneq \mathbb{P}$. Then $\mathfrak{F}^* = \mathcal{S}_\pi$.

<u>Proof</u>. By (3.9.5), $\mathrm{char}(\mathfrak{F}) = \pi \subsetneqq \mathbb{P}$. By (3.1.3), $\mathfrak{S}_\pi * \mathfrak{H} = \mathfrak{F} * \mathfrak{H}$ is Hall-closed, and so $\mathfrak{F}^* = \mathfrak{S}_\pi$ by (3.3.1), as claimed.

<u>3.9.8 Theorem</u>. Let $\mathfrak{F}$ be a Fitting class and π be a set of primes. Then $\mathfrak{F} * \mathfrak{H} = \mathfrak{S}_\pi * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{S}_\pi$.

<u>Proof</u>. If $\pi = \emptyset$ or $\pi = \mathbb{P}$, this is Cossey's result (3.9.4). Suppose that $\emptyset \neq \pi \subsetneqq \mathbb{P}$. If $\mathfrak{F} * \mathfrak{H} = \mathfrak{S}_\pi * \mathfrak{H}$ then $\mathfrak{F}^* = \mathfrak{S}_\pi$ by (3.9.7). If $\mathfrak{F}^* = \mathfrak{S}_\pi$, then $\mathfrak{F} \supseteq \mathfrak{S}_\pi \cap \mathfrak{H}$ by (2.4.7b), and so by (3.1.2), $\mathfrak{F} * \mathfrak{H} = \mathfrak{S}_\pi * \mathfrak{H}$. This completes the proof.

<u>3.9.9 Theorem</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes such that if $2 \in \mathrm{char}(\mathfrak{F})$ then $\mathfrak{G}^* = \mathfrak{G}^* * \mathfrak{S}_2$. Suppose that $\mathfrak{F} * \mathfrak{H} \subseteq \mathfrak{G} * \mathfrak{H}$. Then $\mathfrak{F}^* \subseteq \mathfrak{G}^*$.

<u>Proof</u>. By (1.5.3e), it will suffice to prove that $\mathfrak{F} \subseteq \mathfrak{G}^*$. Thus for a contradiction, let G be a group of minimal order in $\mathfrak{F} \setminus \mathfrak{G}^*$. By (1.3.9/10) and (1.5.6), G has a unique maximal normal subgroup $M = G_{\mathfrak{G}^*}$, of prime index p say; G/G' is a cyclic p-group, and $G_{\mathfrak{G}} \geqslant G'$. Certainly $p \in \mathrm{char}(\mathfrak{F})$, and by hypothesis on $\mathfrak{F}$ and $\mathfrak{G}$, we must have $p \neq 2$.

Let L denote $G_{\mathfrak{G}}$; then $G' \leqslant L \leqslant M$ and G/L is a cyclic p-group. Let τ be a p-element of G such that $G = L\langle\tau\rangle$ (where possibly $L \cap \langle\tau\rangle \neq 1$).

Let $\tilde{\tau}$ denote the automorphism of L induced by conjugation by τ . If $\tilde{\tau} = 1$, then $[L, \langle\tau\rangle] = 1$, and so $\langle\tau\rangle \trianglelefteq G = L\langle\tau\rangle$, whence $G = \langle\tau\rangle$ by the uniqueness of $M \triangleleft\cdot G$ and the fact that $L \leqslant M$.

But then $G \in \mathcal{S}_p$. By (3.9.5a), $p \in \mathrm{char}(\mathcal{G})$ and so $G \in \mathcal{G}$, a contradiction. Thus $\tilde{\tau} \neq 1$.

Since $L\langle\tau\rangle \in \mathfrak{F}$ and $\langle\tau\rangle \in \mathcal{S}_p \subseteq \mathcal{N} \cap \mathfrak{F}$, then by (1.9.3), it follows that $L]\langle\tilde{\tau}\rangle \in \mathfrak{F}$.

On the other hand, suppose that $L]\langle\tilde{\tau}\rangle \in \mathcal{G}^*$. Since $\langle\tilde{\tau}\rangle$ is a non-trivial p-group, then $p \in \mathrm{char}(\mathcal{G}^*)$, and so $\langle\tau\rangle \in \mathcal{N} \cap \mathcal{G}^*$. But then by (1.9.3), we have $G = L\langle\tau\rangle \in \mathcal{G}^*$, contrary to choice.

Thus,

$$L]\langle\tilde{\tau}\rangle \in \mathfrak{F} \setminus \mathcal{G}^* . \qquad \cdots(1)$$

Now form $W = (L\langle\tilde{\tau}\rangle) \operatorname{wr} C_2$. Let $\langle\sigma\rangle$ denote the standard complement in W; then $W = (L_1\langle\tilde{\tau}_1\rangle \times L_2\langle\tilde{\tau}_2\rangle)]\langle\sigma\rangle$, where $L_i\langle\tilde{\tau}_i\rangle$ is the $i^{\text{th.}}$ coordinate subgroup, as in (1.7.1). Further, $\sigma : L_1 \leftrightarrow L_2$, $\tau_1 \leftrightarrow \tau_2$, because of the notation of (1.7.1). Of course, $L_1 \times L_2 \trianglelefteq W$, and σ normalizes $\langle\tau_1\rangle \times \langle\tau_2\rangle$.

Let $J = (L_1 \times L_2)\langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle$. Then J is a semi-direct product $(L_1 \times L_2)]\langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle$. Further, $J \trianglelefteq L_1\langle\tilde{\tau}_1\rangle \times L_2\langle\tau_2\rangle \in \mathfrak{F}$, and J is normalized by $\langle\sigma\rangle$ since $(\tilde{\tau}_1\tilde{\tau}_2^{-1})^{\sigma} = (\tilde{\tau}_1\tau_2^{-1})^{-1}$. Define $U = J\langle\sigma\rangle$ $(= J]\langle\sigma\rangle) \leq W$. Since $\langle\sigma\rangle \in \mathcal{H}$ and $J \in \mathfrak{F}$, then

$$U \in \mathfrak{F} * \mathcal{H} . \qquad \cdots(2)$$

We next show that $J \notin \mathcal{G}^*$. We note that $L_1 \trianglelefteq J$, $L_2 \trianglelefteq J$, $L_1 \cap L_2 = 1$, and $J/L_1L_2 \cong \langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle \in \mathcal{N}$. Now,

$$J/L_2 = (L_1 \times L_2)\langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle/L_2 \cong L_1/L_1 \cap (L_2\langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle). \qquad \cdots(3)$$

Suppose that $\ell_1 = \tilde{\tau}_1^{m}\tilde{\tau}_2^{-m}\ell_2 \in L_1 \cap (L_2\langle\tilde{\tau}_1\tilde{\tau}_2^{-1}\rangle)$, $\ell_i \in L_i$. Then $\tilde{\tau}_1^{-m}\ell_1 = \tilde{\tau}_2^{-m}\ell_2 \in L_1\langle\tilde{\tau}_1\rangle \cap L_2\langle\tilde{\tau}_2\rangle = 1$.

Since $L_i \cap \langle \tilde{\tau}_i \rangle = 1$, then $\tilde{\tau}_i^m = \ell_i = 1$, and it follows that $L_1 \cap (L_2 \langle \tilde{\tau}_1 \tilde{\tau}_2^{-1} \rangle) = 1$.

It follows from (3) that $J/L_2 \cong L_1$. But now by Hauck's lemma, (1.9.2b), if $J \in \mathfrak{G}^*$ then $J/L_2 \cong L_1 \in \mathfrak{G}^*$, contrary to (1), and so

$$J \notin \mathfrak{G}^* . \qquad \ldots(4)$$

Since σ inverts $\langle \tilde{\tau}_1 \tilde{\tau}_2^{-1} \rangle$, then σ also inverts

$$J/(L_1 \times L_2) = ((L_1 \times L_2)] \langle \tilde{\tau}_1 \tilde{\tau}_2^{-1} \rangle)/(L_1 \times L_2) .$$

Since $L_1 \times L_2 \in \mathfrak{G} \subseteq \mathfrak{G}^*$ then $L_1 \times L_2 \leq J_{\mathfrak{G}} \leq J_{\mathfrak{G}^*} \lneq J$, by (4) and it follows that $U/J_{\mathfrak{G}} = J]\langle\sigma\rangle/J_{\mathfrak{G}} \cong D_{2p^\alpha}$, the dihedral group of order $2p^\alpha$, for some $\alpha \geqslant 1$.

Since $U/J \cong C_2$, then either $U_{\mathfrak{G}} = J_{\mathfrak{G}}$ or $U = J U_{\mathfrak{G}}$. But $[J, U_{\mathfrak{G}}] \leq J_{\mathfrak{G}}$, and so $U_{\mathfrak{G}}$ centralizes $J/J_{\mathfrak{G}}$. Since $U/J_{\mathfrak{G}}$ is non-abelian, it follows that $U_{\mathfrak{G}} = J_{\mathfrak{G}}$. Thus $U/U_{\mathfrak{G}} = U/J_{\mathfrak{G}} \cong D_{2p^\alpha}$, and so $U/U_{\mathfrak{G}} \notin \mathfrak{H}$ by (2.4.10). Since $U \in \mathfrak{F} * \mathfrak{H} \subseteq \mathfrak{G} * \mathfrak{H}$, by (2) , this is a contradiction. The proof is complete.

<u>3.9.10 Theorem</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fitting classes.

(a) If $\mathfrak{F}$ and $\mathfrak{G}$ are each of odd characteristic and $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$, then $\mathfrak{F}^* = \mathfrak{G}^*$.

(b) If $\mathfrak{G}$ has odd characteristic and $\mathfrak{G}$ satisfies the Lockett conjecture, then $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$ if and only if $\mathfrak{F}^* = \mathfrak{G}^*$.

<u>Proof</u>. (a) This follows at once from (3.9.9).

(b) If $\mathfrak{F}^* = \mathfrak{G}^*$, then by (3.1.2), $\mathfrak{F} * \mathfrak{H} = \mathfrak{F}^* * \mathfrak{H} = \mathfrak{G}^* * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$, since the Lockett conjecture holds for $\mathfrak{G}$. If $\mathfrak{F} * \mathfrak{H} = \mathfrak{G} * \mathfrak{H}$, then by (3.9.5), $\mathrm{char}(\mathfrak{F}) = \mathrm{char}(\mathfrak{G}) \subseteq \{2\}'$, and the result follows by (a).

The following consequence of (3.9.9) neither implies, nor is implied by, (3.9.7).

<u>3.9.11 Proposition</u>. Let π be a set of primes, and suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are Fitting classes with $\mathfrak{G}^* = \mathfrak{S}_\pi$ and $\mathfrak{F} * \mathfrak{H} \subseteq \mathfrak{G} * \mathfrak{H}$. Then $\mathfrak{F}^* \subseteq \mathfrak{S}_\pi$.

<u>Proof</u>. Suppose that $2 \in \mathrm{char}(\mathfrak{F})$. By (3.9.5b), it follows that $2 \in \mathrm{char}(\mathfrak{G}) = \pi$. Thus $\mathfrak{G}^* = \mathfrak{S}_\pi = \mathfrak{S}_\pi * \mathfrak{S}_2 = \mathfrak{G}^* * \mathfrak{S}_2$, and the assertion follows from (3.9.9).

The hypothesis that either $\mathfrak{F}$ has odd characteristic or else $\mathfrak{G}^* = \mathfrak{G}^* * \mathfrak{S}_2$ seems crucial to our proof of (3.9.9) : the important point is that any dihedral group of order $2\,p^\alpha$ for $\alpha \geqslant 1$ lies outside $\mathfrak{H}$, by Berger's theorem (2.4.10).

The next result utilises construction (3.8.1).

<u>3.9.12 Proposition</u>. Let π be a set of primes with $|\pi| \geqslant 2$. Suppose that $\mathfrak{F}$ is a Fitting class and that $\mathfrak{G}$ is a Lockett class such that $\mathfrak{F} * (\mathfrak{S}_\pi)_* \subseteq \mathfrak{G} * (\mathfrak{S}_\pi)_*$. Then $\mathfrak{F} \subseteq \mathfrak{G}$.

<u>Proof</u>. Suppose for a contradiction that G is a group of minimal order in $\mathfrak{F} \setminus \mathfrak{G}$. Then $|G : G_{\mathfrak{G}}| = t \in \mathbb{P}$. Since $G \in \mathfrak{F} * \mathfrak{S}_\pi \subseteq \mathfrak{G} * \mathfrak{S}_\pi$, then $t \in \pi$. Choose $s \in \pi \setminus \{t\}$, define $\sigma = \mathbb{P}$, and apply construction (3.8.1) with $\mathfrak{F}$ for $\mathfrak{X}$, $\mathfrak{G}$ for $\mathfrak{Y}$ and G, s, t and σ as just defined. We choose, as in (3.8.1), a group $\overline{T}]\overline{S}$ of minimal order in $\mathfrak{S}_t * \mathfrak{S}_s \setminus \mathfrak{H}$, and obtain $\Delta^* \circledcirc \Sigma \in \mathfrak{F} * \mathfrak{H}$. Since we took $\sigma = \mathbb{P}$, then $G \in \mathrm{Hall}_\sigma(G)$, and so in (3.8.1d) we have $L = G$ and $K = L \cap D = D$, where $D = G_{\mathfrak{G}}$. Thus $\chi^* = \Delta^*$, and so

$\Delta^* \circledast \Sigma = \circledast\Sigma \in \mathfrak{F} * \mathcal{H}$ (and $\Delta^* \lneq (\circledast\Sigma)_{\mathfrak{F}}$ by (3.8.1f)).

Since $\Delta^* \leq (\circledast\Sigma)_{\mathfrak{F}}$ and $\circledast\Sigma / \Delta^* \cong \overline{TS} \in \mathcal{S}_\pi$, by (4) of (3.8.1) and the fact that $\Delta^* = \mathcal{X}^*$, then in fact

$$\circledast\Sigma \in \mathfrak{F} * (\mathcal{H} \cap \mathcal{S}_\pi) = \mathfrak{F} * (\mathcal{S}_\pi)_* , \qquad \cdots(1)$$

by (2.4.7b).

However, $G_{\mathcal{G}} = D$ and so $\Gamma^*_{\mathcal{G}} = \Delta^*$. Since $L = G$ and $K = D$, then $\Lambda^* = \Gamma^*$ and $\mathcal{X}^* = \Delta^*$, whence $\Lambda^*_{\mathcal{G}} = \mathcal{X}^*$. Thus by (3.8.1g), $(\circledast\Sigma)_{\mathcal{G}} = \mathcal{X}^* = \Delta^*$, and $\circledast\Sigma/(\circledast\Sigma)_{\mathcal{G}} \cong \overline{TS} \notin \mathcal{H} \supseteq (\mathcal{S}_\pi)_*$.

Consequently, $\circledast\Sigma \notin \mathcal{G} * (\mathcal{S}_\pi)_*$, a contradiction since from (1) we have $\circledast\Sigma \in \mathfrak{F} * (\mathcal{S}_\pi)_*$. This completes the proof.

The above result is false if $|\pi| = 1$. For suppose that $\pi = \{t\}$ and choose $s \in \mathbb{P} \setminus \{t\}$. Then $(\mathcal{S}_t)_* = \mathcal{S}_t$ and $\mathcal{N}_{\{s,t\}} * \mathcal{S}_t = \mathcal{N}_s * \mathcal{S}_t$, while $\mathcal{N}_{\{s,t\}} \nsubseteq \mathcal{N}_s$.

We note that, taking $\pi = \mathbb{P}$ and $\mathfrak{F}$ as a Lockett class, we obtain Cossey's result (3.9.1) from (3.9.12), at the expense of (3.7.1), which of course Cossey does not use.

CHAPTER FOUR.

A new approach to Berger's Theorem.

In their paper [6] , Blessenohl and Gaschütz introduce the notion of a "normal Fitting pair". A normal Fitting pair (C,d) consists of an abelian group C and a rule d assigning to each $G \in \mathcal{S}$ a homomorphism $d_G : G \to C$ which "respects" restriction to normal subgroups, such that C is covered (set-wise) by the images Gd_C as G runs over $\mathcal{S}$ (see (4.1.6)). Blessenohl and Gaschütz showed that if (C,d) is a normal Fitting pair, then the class $(G \in \mathcal{S} : Gd_C = 1)$ is a normal Fitting class. Lausch, [45] , showed that every normal Fitting class can be viewed in this way ; his method involves taking the restricted direct product of (one representative of each isomorphism class of) all finite soluble groups, Δ say, and factoring Δ by a suitable subgroup containing Δ' to obtain the abelian group A . While of great theoretical interest, Lausch's method does not seem able to give a finite procedure for determining (say) whether or not a given soluble group G belongs to the smallest normal Fitting class $\mathcal{H} = \mathcal{S}_*$.

A number of "concrete" examples of Fitting pairs have been announced in the papers [6] , [13] , [2] and [44] , for example. The maps associated with these pairs can be calculated explicitly , and the existence of these pairs has led to much information concerning $\mathcal{S}_*$; for example the results (2.4.10) and (3.7.1) follow because of the pairs constructed in [2] and [44] , respectively.

In their paper [44] , Laue, Lausch and Pain show how, given a group $U \in \mathcal{S}$, the "transfer" may be used to construct a normal Fitting pair (L^U, ℓ^U), where L^U is a certain abelian factor group of $\mathrm{Aut}(U)$ ..

In his important recent preprint [4] , Berger has shown that if for a given $G \in \mathcal{S}$, we intersect the kernels of the maps t_G^V over all the distinct isomorphism types V of subnormal subgroups of G , we arrive precisely at the $\mathcal{S}_*$-radical of G (in fact, we only need to take certain specified types of V). This means that there is a theoretically finite procedure by which the $\mathcal{S}_*$-radical of any soluble group G may be determined, depending on knowledge of the automorphism groups of certain subnormal subgroups of G . Of course, in concrete examples, this procedure may be rather lengthy.

In their paper [10] , Bryce and Cossey observed that the Blessenohl-Gaschütz-Lausch theory of normal Fitting pairs can be "localised" to " $\mathfrak{F}$-Fitting pairs", which describe the Fitting classes lying in the Lockett section between $\mathfrak{F}_*$ and $\mathfrak{F}$, for an arbitrary Fitting class $\mathfrak{F}$; in this terminology, a normal Fitting pair becomes an $\mathcal{S}$-Fitting pair.

It is natural to ask, as does Berger [4], whether "transfer-type" $\mathfrak{F}$-Fitting pairs can be used to determine the $\mathfrak{F}_*$-radical of a group $G \in \mathfrak{F}$ where $\mathfrak{F}$ is an arbitrary Fitting class. Berger defines some transfer-type Fitting pairs more general than those of [44] , and shows that these "Berger pairs" can in fact be used to determine the $\mathfrak{F}_*$-radical for certain (not-necessarily-soluble) Fitting classes $\mathfrak{F}$ as well as for $\mathcal{S}$; indeed, Berger proves (but does not seem to explicitly state) that his pairs can be so used if $\mathfrak{F}$ is an arbitrary Fischer class.

Berger's proof, once his Fitting pairs have been constructed, proceeds by a complex and abstract induction argument. In this chapter, we give what we think to be a more conceptual, and probably simpler, proof of Berger's theorem. Our proof starts with the construction of some transfer-type $\mathfrak{F}$-Fitting pairs, for certain classes $\mathfrak{F}$, which may be regarded as intermediate in generality between the pairs of Laue,

Lausch and Pain, and of Berger. To show that these pairs "determine" the $\mathfrak{F}_*$-radical, we too proceed by induction; however, we arrange our induction in the context of the so-called "Lausch group" (see (4.1.5)). We believe that this procedure affords a certain amount of insight into both Berger's theorem and the structure of Fitting classes in general; indeed, we are able to use the same general inductive argument to prove a result about the generation of Fitting classes. Our $\mathfrak{F}$-Fitting pairs are defined whenever $\mathfrak{F}$ is a Fischer class, as well as in certain other cases, and it is for such classes $\mathfrak{F}$ that our version of Berger's theorem is thus proved. The inductive part of the proof is arranged to function for general Fitting classes, with the Fitting pairs being brought in at the end, emphasising the dependence of Berger's theorem on the existence of such pairs.

In giving a new proof of a known theorem, it is hard to specify exactly what is original; of course, Berger's theorem was first proved by Berger, and what we claim as new is our approach to its proof in terms of the Lausch group, including our emphasis on the so-called " $\mathfrak{F}$-relevant groups". Material appearing after the proof of Berger's theorem is also claimed as original subject to any provisos we may make.

Section one consists of background material on the Lausch group, and also contains a lemma of Berger. In section two, the concept of an " $\mathfrak{F}$-relevant group" is introduced. In section three, we introduce our transfer-type $\mathfrak{F}$-Fitting pairs, and in section four, Berger's theorem is proved. In section five we discuss the existence of suitable $\mathfrak{F}$-Fitting pairs for arbitrary Fitting classes $\mathfrak{F}$, and in section six we prove a result on the generation of Fitting classes. The remaining sections are devoted to examples of the uses of Berger's theorem.

The finite groups in this chapter need not be soluble.

4.1 The Lausch group.

Lausch's method, [45], for associating a normal Fitting pair to each normal Fitting class can be extended to the Lockett section of an arbitrary Fitting class $\mathfrak{F} \subseteq \mathfrak{E}$. It will be convenient to set up some of the formalism, and to quote certain of the results, here. The subject will be fully covered in the forthcoming Doerk-Hawkes book, [21], and we base our approach in this section partly on a preliminary manuscript for [21] .

4.1.1 Definition (Laue, [43; 3.4]). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes. Then $\mathfrak{X}$ is said to be central under $\mathfrak{Y}$ if $\mathfrak{X} \subseteq \mathfrak{Y}$ and for all $G \in \mathfrak{Y}$ we have $[G , \mathrm{Aut}(G)] \leq G_{\mathfrak{X}}$. (Bryce and Cossey, [10] , call $\mathfrak{X}$ strongly normal in $\mathfrak{Y}$ in this case).

4.1.2 Lemma (Bryce and Cossey, [10; 3.4]). Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Fitting classes. Then $\mathfrak{X}$ is central under $\mathfrak{Y}$ if and only if $\mathfrak{X} \subseteq \mathfrak{Y}$ and $\mathfrak{X}^* = \mathfrak{Y}^*$.

Proof. Suppose that $\mathfrak{X}$ is central under $\mathfrak{Y}$, and let $G \in \mathfrak{Y}$. Then $G^2 = G \times G \in \mathfrak{Y}$ and $[G^2 , \mathrm{Aut}(G^2)] \leq (G^2)_{\mathfrak{X}}$. Let σ denote the automorphism of G^2 permuting the components: $\sigma : (g_1, g_2) \mapsto (g_2, g_1)$. Then if $g \in G$, we have $(g^{-1}, g) = (g,1)^{-1}(g,1)^{\sigma} \in [G^2, \mathrm{Aut}(G^2)] \leq (G^2)_{\mathfrak{X}}$. But then $G \in \mathfrak{X}^*$, and so $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{X}^*$. Thus $\mathfrak{X}^* = \mathfrak{Y}^*$.

On the other hand, if $\mathfrak{X} \subseteq \mathfrak{Y} \subseteq \mathfrak{X}^*$ and if $G \in \mathfrak{Y}$, we have $[G , \mathrm{Aut}(G)] \leq G_{\mathfrak{Y}_*} \leq G_{\mathfrak{X}}$, by (1.5.3c).

4.1.3 Definitions and Notation. (a) Let $\mathfrak{F}$ be a Fitting class. An underlying set for $\mathfrak{F}$ is a set of groups containing exactly one representative of each isomorphism class of groups belonging to $\mathfrak{F}$. We will often denote a given underlying set for $\mathfrak{F}$ by $\mathbb{F}$.

(b) If G and H are groups, a normal embedding of G into H is a monomorphism $\alpha : G \to H$ with $G\alpha \trianglelefteq H$, while a subnormal embedding is a monomorphism $\beta : G \to H$ with $G\beta$ sn H.

The set of all normal embeddings (subnormal embeddings) of G into H will be denoted by Nemb(G,H) (Subnemb(G,H)), respectively.

(c) Let $\{G_i\}_{i \in \underline{I}}$ be a set of groups, where $\underline{I}$ is an index set. The restricted direct product $\prod_{i \in \underline{I}} G_i$ is defined as the set $\Delta_{\underline{I}}$ of maps $f : \underline{I} \to \bigcup_{i \in I} G_i$ of finite support, such that for each $i \in \underline{I}$ we have $(i)f \in G_i$. A group operation is defined on $\Delta_{\underline{I}}$ by "pointwise multiplication" :

$$(i)\, fg = ((i)f)\,((i)g) \in G_i \ .$$

If $\underline{J} \subseteq \underline{I}$, the natural embedding of $\Delta_{\underline{J}}$ into $\Delta_{\underline{I}}$ is the map $\Delta_{\underline{J}} \to \Delta_{\underline{I}}$ which sends $f_o \in \Delta_{\underline{J}}$ to the element $f \in \Delta_{\underline{I}}$, where

$$(i)f = \begin{cases} (i)f_o & \text{if } i \in \underline{J} \\ 1 & \text{if } i \in \underline{I} \setminus \underline{J} \ . \end{cases}$$

For each $i \in \underline{I}$ we will denote the natural embedding of G_i into $\Delta_{\underline{I}}$ by ϵ_{G_i}, or simply by ϵ_i if there is no ambiguity.

4.1.4 Hypotheses. We make the blanket assumption for the rest of this section that $\mathfrak{F}$ is a Fitting class and that $\mathbb{F} = \{G_i\}_{i \in \underline{I}}$ is a fixed underlying set.

4.1.5 Definitions, notation and remarks.

The restricted direct product $\Delta_{\underline{I}} = \prod_{i \in \underline{I}} G_i = \prod_{G \in \mathbb{F}} G$ will be denoted by Δ, and ε_G denotes the natural embedding of $G \in \mathbb{F}$ into Δ. We define the subgroup

$$\Lambda = \langle \{ (g\varepsilon_G)^{-1} (g\alpha)\varepsilon_H : g \in G ; \ G, H \in \mathbb{F} ; \ \alpha \in \mathrm{Nemb}(G,H) \} \rangle$$

of Δ.

We note that if $x \in G$, $G \in \mathbb{F}$, then conjugation by x induces a normal embedding of G into G, and so $(g^{-1} g^{x}) \varepsilon_G \in \Lambda$ for all $g \in G$. Thus $G' \varepsilon_G \leq \Lambda$ for all $G \in \mathbb{F}$, and so $\Delta' \leq \Lambda$.

Define $A = A(\mathbb{F})$ to be the abelian quotient group Δ/Λ, known as the Lausch group of $\mathfrak{F}$ (with respect to $\mathbb{F}$).

4.1.6 Definition (Blessenohl and Gaschütz, [6], Lausch, [45]).

(a) The pair (C,d) is called an $\mathfrak{F}$-Fitting pair if C is an abelian group and $d : \mathbb{F} \to \{\mathrm{Hom}(G,C) : G \in \mathbb{F}\}$ is a map such that to each $G \in \mathbb{F}$, d_G is a homomorphism from G to C satisfying

(1) $d_G = \alpha d_H$ if $G, H \in \mathbb{F}$ and $\alpha \in \mathrm{Nemb}(G,H)$; and

(2) $C = \{ g d_G : g \in G, \ G \in \mathbb{F} \}$.

(b) Suppose that (C,d) is an $\mathfrak{F}$-Fitting pair. If $G_o \in \mathfrak{F}$ and $\sigma : G_o \to G$ is an isomorphism, where $G \in \mathbb{F}$, we may define a homomorphism $d_{G_o} : G_o \to C$ by $g_o d_{G_o} = g_o \sigma d_G \in C$, for $g_o \in G_o$.

If $\tau : G_o \to G$ is a further isomorphism, then the map $\{ g \mapsto g\sigma^{-1}\tau : g \in G \}$ belongs to $\mathrm{Nemb}(G,G)$, and it follows by (1) of part (a) that the definition of d_{G_o} is independent of the choice

of isomorphism σ from $G_o \in \mathfrak{F}$ to $G \in \mathbb{F}$. We may thus consider the "Fitting pair maps" d_G as being defined for all $G \in \mathfrak{F}$ if we do not wish to be restricted to a given underlying set. If $G, H \in \mathfrak{F}$ and $\alpha \in \mathrm{Nemb}(G,H)$, we may check that $g\, d_G = g\alpha\, d_H$ for all $g \in G$. Conversely, in checking that a given " d " is a Fitting pair map, it will not be necessary to actually take groups G, H from $\mathbb{F}$.

<u>4.1.7 Theorem (Blessenohl and Gaschütz. [6; Satz 3.1])</u>. Let (C,d) be an $\mathfrak{F}$-Fitting pair. Then the class $\mathfrak{X} = (G \in \mathfrak{F} : (G)d_G = 1)$ is a Fitting class central under $\mathfrak{F}$. Further, if $G \in \mathfrak{F}$, then $\ker(d_G) = G_{\mathfrak{X}}$. (This is proved only for $\mathfrak{F} = \mathcal{S}$ in [6]; see [21]).

<u>Proof</u>. That $\mathfrak{X}$ is a Fitting class is straightforward and follows as in [6]. Let $K = \ker(d_G)$. Since $K \trianglelefteq G$, then $Kd_K = Kd_G = 1$, and so $K \in \mathfrak{X}$ and $K \leq G_{\mathfrak{X}}$. On the other hand, $1 = G_{\mathfrak{X}}\, d_{G_{\mathfrak{X}}} = G_{\mathfrak{X}}\, d_G$, since $G_{\mathfrak{X}} \trianglelefteq G$, and so $G_{\mathfrak{X}} \leq K$.

Let $H \in \mathfrak{F}$ be arbitrary. Then the natural embeddings $\alpha : H \to H \times 1 \trianglelefteq H \times H$ and $\beta : H \to 1 \times H \trianglelefteq H \times H$ belong to $\mathrm{Nemb}(H, H \times H)$, and so if $h \in H$ then $(h^{-1},h)\, d_{H \times H} = (h^{-1})d_H\, (h)d_H = 1$. But then $(h^{-1},h) \in \ker(d_{H \times H}) = (H \times H)_{\mathfrak{X}}$ and $(H \times H)_{\mathfrak{X}}$ is subdirect in $H \times H$. Thus $H \in \mathfrak{X}^*$ and so $\mathfrak{X}$ is central under $\mathfrak{F}$.

Lausch, [45; 2.5], shows that there is a lattice isomorphism between the lattice of Fitting classes central under $\mathfrak{F}$ and the subgroup lattice of $A = \Delta/\Lambda$; the importance of the Lausch group stems from this fact (in fact, Lausch only proves this for the class $\mathcal{S}$, and Bryce and Cossey, [10;3.3], observe that the proof holds in general).

We need only the following version of Lausch's theorem.

4.1.9 Theorem (Lausch, [45], Bryce and Cossey, [10]).

If $G \in \mathbb{F}$ then $G\varepsilon_G \cap \Lambda = (G_{\mathfrak{F}_*})\varepsilon_G$.

Proof. The proof is basically that of [45; 2.4]. If $G \in \mathbb{F}$, define $d_G : G \to \Delta/\Lambda$ by $g\, d_G = g\varepsilon_G \Lambda \in \Delta/\Lambda$ if $g \in G$. Then $(\Delta/\Lambda, d)$ is an $\mathfrak{F}$-Fitting pair : condition (1) of (4.1.6a) is easily checked, while condition (2) holds by definition of Δ. By (4.1.7), we have $\mathfrak{F}_* \subseteq \mathfrak{X} = (G \in \mathfrak{F} : G\, d_G = 1)$. But now if $G \in \mathbb{F}$, then $1 = (G_{\mathfrak{F}_*})d_{G_{\mathfrak{F}_*}} = (G_{\mathfrak{F}_*})d_G$, and so by definition of d_G, we have $(G_{\mathfrak{F}_*})\varepsilon_G \leq \Lambda \cap G\varepsilon_G$. On the other hand, the argument at the foot of page 69 and the head of page 70 in [45] shows that if $g \in G$ with $g\varepsilon_G \in G\varepsilon_G \cap \Lambda$, then $g \in G_{\mathfrak{F}_*}$, and the assertion follows.

4.1.10 Lemma. Let (C,d) be an $\mathfrak{F}$-Fitting pair. If $f \in \Delta$, define $f\hat{d} \in C$ by $f\hat{d} = \prod_{i \in I} ((i)f)d_{G_i} \in C$.

Then the map $\hat{d} : f \mapsto f\hat{d}$ is a (well-defined) epimorphism from Δ to C such that $\ker(\hat{d}) \geq \Lambda$. Further, if $G \in \mathbb{F}$ and $g \in G$, then $g\, d_G = (g\varepsilon_G)\hat{d} \in C$ and $(\ker(d_G))\varepsilon_G = \ker(\hat{d}) \cap G\varepsilon_G \leq \Delta$.

Proof. Since $f \in \Delta$ is of finite support and C is abelian, then $f\hat{d}$ is a well-defined element of C. If $f, h \in \Delta$, then $(fh)\hat{d} = \prod_i (if)(ih)d_{G_i} = (f\hat{d})(h\hat{d})$, and so $\hat{d}$ is a homomorphism. Since $C = \{g_i d_{G_i} : g_i \in G_i, G_i \in \mathbb{F}\}$, by (4.1.6a(2)), it follows that $\hat{d}$ is an epimorphism.

Let $G_i, G_j \in \mathbb{F}$ and let $\alpha \in \mathrm{Nemb}(G_i, G_j)$. Then if $g \in G_i$,

we have $(g\,\varepsilon_{G_i})^{-1}(g\alpha\,\varepsilon_{G_j})\,\hat{d} = (g\,d_{G_i})^{-1}\,(g\,\alpha\,d_{G_j}) = 1$, since (C,d) is an $\mathfrak{F}$-Fitting pair. But now an arbitrary generator of Λ belongs to $\ker(\hat{d})$, and so $\Lambda \leq \ker(\hat{d})$.

Let $G \in \mathbb{F}$ and $g \in G$. Then $g\,\varepsilon_G \in \Delta$, and by definition of $\hat{d}$, we have $(g\,\varepsilon_G)\,\hat{d} = g\,d_G$; it follows that $(\ker(d_G))\,\varepsilon_G = \ker(\hat{d}) \cap G\,\varepsilon_G$.

<u>4.1.11 Definition</u>. If $G \in \mathfrak{F}$, then a <u>quasi-natural embedding of G into Δ</u> is a monomorphism $e_G : G \to \Delta$ such that there exists an isomorphism $\sigma : G \to G_o$, where $G_o \in \mathbb{F}$, with $e_G = \sigma\,\varepsilon_{G_o}$.

The set of all quasi-natural embeddings of G into Δ will be denoted by <u>$\mathrm{Qnat}(G, \Delta)$</u> .

<u>4.1.12 Notational conventions</u>. (a) In performing calculations in the Lausch group, it will be convenient to work in Δ and to take congruences "modulo Λ " . Thus if a and b are elements of Δ , we shall use $a \equiv b \pmod{\Lambda}$ to signify that $a\,b^{-1} \in \Lambda$.
(b) If G is a group, automorphisms of G will often be denoted by an expression of the form $\{g \mapsto g\alpha\}$, where α is some suitable " operator" on the elements of G .

<u>4.1.13 Lemma</u>. Let $G \in \mathfrak{F}$, and suppose that e_G and e_G^* belong to $\mathrm{Qnat}(G, \Delta)$. Then if $g \in G$, we have $g\,e_G \equiv g\,e_G^* \pmod{\Lambda}$.

<u>Proof</u>. There exist isomorphisms $\sigma, \tau : G \to G_o$, where $G_o \in \mathbb{F}$, such that $e_G = \sigma\,\varepsilon_{G_o}$ and $e_G^* = \tau\,\varepsilon_{G_o}$. Now the map $\{g_o \mapsto g_o\sigma^{-1}\tau : g_o \in G_o\}$

belongs to $\mathrm{Nemb}(G_o,G_o)$ $(= \mathrm{Aut}(G_o))$. Since $g\alpha \in G_o$, then $((g\alpha)\varepsilon_{G_o})^{-1}((g\alpha)\alpha^{-1}\beta)\varepsilon_{G_o} \in \Lambda$, and the result follows.

<u>4.1.14 Lemma</u>. Let G and H be groups in $\mathfrak{F}$, and let $\alpha \in \mathrm{Subnemb}(G,H)$. Let $e_G \in \mathrm{Qnat}(G,\Delta)$ and $e_H \in \mathrm{Qnat}(H,\Delta)$.

Then $g\,e_G \equiv (g\alpha)e_H \pmod{\Lambda}$ for all $g \in G$.

<u>Proof</u>. There exist G_o, $H_o \in \mathbb{F}$ and isomorphisms $\sigma : G \to G_o$ and $\tau : H \to H_o$ such that $e_G = \sigma\,\varepsilon_{G_o}$ and $e_H = \tau\,\varepsilon_{H_o}$.

Suppose firstly that $\alpha \in \mathrm{Nemb}(G,H)$. Then the map $\{g_o \mapsto g_o\sigma^{-1}\alpha\tau : g_o \in G_o\}$ belongs to $\mathrm{Nemb}(G_o,H_o)$. If $g \in G$, then $g\sigma \in G_o$ and so $(g\sigma\,\varepsilon_{G_o})^{-1}((g\sigma)\sigma^{-1}\alpha\tau)\varepsilon_{H_o} \in \Lambda$, and the result follows in this case. The general case now follows by an induction on the length of a suitable subnormal chain in H.

<u>4.1.15 Corollary</u>. Suppose that $\Lambda \leq \Gamma \leq \Delta$ and that G and H are groups in $\mathfrak{F}$ with $\alpha \in \mathrm{Subnemb}(G,H)$. Let $e_G \in \mathrm{Qnat}(G,\Delta)$ and $e_H \in \mathrm{Qnat}(H,\Delta)$. If $X \leq G$, then

$$X\,e_G \leq \Gamma \iff (X\alpha)\,e_H \leq \Gamma .$$

<u>Proof</u>. If $x \in X$ then $(x\,e_G)^{-1}(x\alpha)\,e_H \in \Lambda$ by (4.1.14), and since $\Lambda \leq \Gamma$, the result follows.

The next lemma is due to Berger [4] but with its setting altered to the context of the Lausch group. It forms a second part of (1.9.3), and we will refer to the internal details of the proof of (1.9.3) here.

<u>4.1.16 Lemma (Berger, [4; 3.2])</u>. Let G be a group in $\mathfrak{F}$ such that $G = VM$ where $V \trianglelefteq G$ and $M \in \mathcal{N} \cap \mathfrak{F}$.

Let U be a group isomorphic with V and let $\rho : V \to U$ be an isomorphism. Then ρ induces a homomorphism $\varphi : G \to \mathrm{Aut}(U)$ given by

$$\varphi : g \mapsto \{u \mapsto ((u\rho^{-1})^g)\rho : u \in U\} \in \mathrm{Aut}(U) \quad , \text{ for } g \in G .$$

Now $U]\mathrm{Aut}(U)$ has a subgroup $U](M\varphi)$. Let $e_G \in \mathrm{Qnat}(G, \Delta)$ and $e_{U(M\varphi)} \in \mathrm{Qnat}(U(M\varphi), \Delta)$. Then

$$m\, e_G \equiv (m\varphi)\, e_{U(M\varphi)} \pmod{\Lambda} \quad \text{for all } m \in M .$$

<u>Proof</u>. If $B \in \mathfrak{F}$, then e_B will denote a fixed (but arbitrary) member of $\mathrm{Qnat}(B, \Delta)$.

Let $m \in M$ and let $G_m = V\langle m\rangle$ sn G. Then $G_m \in \mathfrak{F}$ and

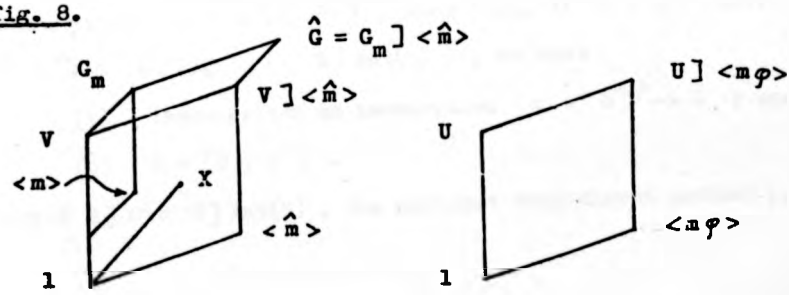
$$m\, e_{G_m} \equiv m\, e_G \pmod{\Lambda} . \qquad \cdots(1)$$

Let $\hat{m}$ denote the inner automorphism of G_m induced by m, let $\hat{G} = G_m]\langle\hat{m}\rangle$ and let $X = \langle m\hat{m}^{-1}\rangle$. We saw in the proof of (1.9.3) that G_m and X are normal in $\hat{G} = G_m X$ and that $X \in \mathcal{N} \cap \mathfrak{F}$, so that $\hat{G} \in \mathfrak{F}$. We thus have

$$m\, e_{G_m} \equiv m\, e_{\hat{G}} \pmod{\Lambda} \text{ ; and} \qquad \cdots(2)$$

$$(m\hat{m}^{-1})\, e_X \equiv (m\, m^{-1})\, e_{\hat{G}} \pmod{\Lambda} . \qquad \cdots(3)$$

<u>fig. 8.</u>



It follows from (1.5.11c) that $\mathcal{N} \cap \mathfrak{F} \subseteq \mathfrak{F}_*$, and so by (4.1.9) we have $(m\,\hat{m}^{-1})\, e_X \in \Lambda$. Thus by (3) we have

$$m\, e_{\hat{G}} \equiv \hat{m}\, e_{\hat{G}} \pmod{\Lambda} \quad . \qquad \cdots(4)$$

Now we saw in the proof of (1.9.3) that $V\langle\hat{m}\rangle \trianglelefteq \hat{G}$ and that $\psi : v\,\hat{m}^i \mapsto (v\,\rho)(m^i\varphi)$ is an isomorphism from $V\langle\hat{m}\rangle$ to $U]\langle m\varphi\rangle$. Of course, $\psi \in \mathrm{Nemb}(V\langle\hat{m}\rangle, U\langle m\varphi\rangle)$, and since $U\langle m\varphi\rangle$ sn $U](M\varphi)$, and $U](M\varphi) \in \mathfrak{F}$ by (1.9.3), we have

$$\hat{m}\, e_{\hat{G}} \equiv \hat{m}\, e_{V\langle\hat{m}\rangle} = (m\varphi)\, e_{U\langle m\varphi\rangle} \equiv (m\varphi)\, e_{U(M\varphi)} \pmod{\Lambda}. \qquad \cdots(5)$$

The result follows by (1) , (2) , (4) and (5) .

<u>4.1.17 Corollary</u>. Let $G = V\,M \in \mathfrak{F}$ and $U \underset{f}{\trianglelefteq} V$ be as in (4.1.16). Suppose that $U](M\varphi)$ sn $K \leq U]\mathrm{Aut}(U)$, where $K \in \mathfrak{F}$, and let $e_K \in \mathrm{Qnat}(K, \Delta)$ and $e_G \in \mathrm{Qnat}(G, \Delta)$. Then we have

$$m\, e_G \equiv (m\varphi)\, e_K \pmod{\Lambda} \quad , \quad \text{for all } m \in M \quad .$$

<u>Proof</u>. This follows from (4.1.16) and (4.1.14) .

4.2 $\mathfrak{F}$-Relevant groups.

<u>4.2.1 Definition</u>. Let $\mathfrak{F}$ be a Fitting class and p be a prime. We say that the group $G \in \mathfrak{F}$ is <u>$(\mathfrak{F}, p)$-relevant</u> if $G \neq 1$ and there exists $U = O^p(U) \in \mathfrak{F}$ such that, if $P \in \mathrm{Syl}_p(\mathrm{Aut}(U))$ and $P^* = P \cap (U]P)_{\mathfrak{F}}$ $(\leq U]\mathrm{Aut}(U))$, we have

(1) there exists an isomorphism $\sigma : U]P^* \to G$; and

(2) $U = [U, P^*]$,

(where $U]P^* \leq U]\mathrm{Aut}(U)$, the abstract semi-direct product).

By Sylow's theorem in $\mathrm{Aut}(U)$, this definition is independent of the particular Sylow p-subgroup, P, of $\mathrm{Aut}(U)$ chosen.

The above definition does not appear in [4], and can be considered to mark the start of our proof of Berger's theorem.

<u>4.2.2 Lemma</u>. Suppose that $\mathfrak{F}$ is a Fitting class, that p is a prime and that G is $(\mathfrak{F},p)$-relevant. Then, in the notation of (4.2.1), we have $U P^* = (U P)_{\mathfrak{F}}$, $P^* \gneqq 1$, $U = (U P^*)^{\mathcal{N}} = O^p(U P^*)$ and $U\sigma = G^{\mathcal{N}} = O^p(G) \lneqq G$. Further, $p \in \mathrm{char}(\mathfrak{F})$.

<u>Proof</u>. Since $U \in \mathfrak{F}$, then $U \leqslant (UP)_{\mathfrak{F}}$, and so by Dedekind's law we have $U P^* = UP \cap (UP)_{\mathfrak{F}} = (UP)_{\mathfrak{F}}$. Since $U = O^p(U)$ and $P^* \in \mathcal{S}_p$ then $(U P^*)^{\mathcal{N}} \leqslant U$. By definition, $G \gneqq 1$. Since $\mathrm{Aut}(1) = 1$, then $U \gneqq 1$. Since $U = [U, P^*]$, then $P^* \gneqq 1$, and also, from above, $(U P^*)^{\mathcal{N}} = U$. Since $U \lneqq U P^* \in \mathfrak{F}$, then $p \in \mathrm{char}(\mathfrak{F})$. The other statements follow because σ is an isomorphism.

<u>4.2.3 Lemma</u>. Suppose that G is $(\mathfrak{F},p)$-relevant and $(\mathfrak{F},q)$-relevant for primes p and $q \in \mathrm{char}(\mathfrak{F})$. Then $p = q$.

<u>Proof</u>. By (4.2.2), we have $O^p(G) = G^{\mathcal{N}} = O^q(G) \lneqq G$, and so $p = q$.

<u>4.2.2 Notation</u>. The group G is said to be <u>$\mathfrak{F}$-relevant</u> if it is $(\mathfrak{F},p)$-relevant for some $p \in \mathrm{char}(\mathfrak{F})$.

The class of $(\mathfrak{F},p)$-relevant ($\mathfrak{F}$-relevant) groups will be denoted by $\mathcal{R}^p_{\mathfrak{F}}$ ($\mathcal{R}_{\mathfrak{F}}$), respectively. Of course, $\mathcal{R}_{\mathfrak{F}} = \bigcup_p \mathcal{R}^p_{\mathfrak{F}}$.

4.3 Transfer-type Fitting pairs.

In this section, we introduce some "transfer-type" Fitting pairs which may be regarded as being intermediate in generality between those of Laue, Lausch and Pain [44] and of Berger [4]: the proofs are similar.

4.3.1 Notation. If A is a group and $B_o \trianglelefteq B \leq A$, where B/B_o is abelian, we shall denote the transfer of A into B/B_o by $V_{A \to B/B_o}$ (see [28; 7.3.2] or [41; IV.1.4]). If $B_o = B'$, we shall simply denote this transfer by $V_{A \to B}$. If $C_o \trianglelefteq C \leq B$ and C/C_o is abelian, we will follow the common practice of writing $V_{A \to C/C_o}$ as the composition $V_{A \to B}\ V_{B \to C/C_o}$: this is justified by [41; IV.1.6], although, strictly speaking, $V_{B \to C/C_o}$ is a homomorphism of B with kernel containing B', while the image of $V_{A \to B}$ is the group B/B'.

4.3.2 Lemma. Let A be a group and suppose that $D \trianglelefteq C \leq B \leq A$, where C/D is abelian.

(a) Let $b \in B$ and $x \in A$. Then $(b\ V_{B \to C/D})^x = b^x\ V_{B^x \to C^x/D^x}$.

(b) Let $b \in B$ and $y \in B$. Then $(b\ V_{B \to C/D})^y = b\ V_{B \to C^y/D^y}$.

(c) Suppose that $E \trianglelefteq B$. If $e \in E$, then there exists $f \in E \cap C$ such that $e\ V_{B \to C/D} = f\,D \in C/D$.

Proof. Let $\{u_1, \ldots, u_k\}$ be a (right) transversal to C in B. If $a \in B$ and $i \in \{1, \ldots, k\}$, define $i(a)$ by $Cu_i a = Cu_{i(a)}$. Let the various elements b, x, y and e be as in the hypotheses.

(a) Let $g_i = u_i b\, u_{i(b)}^{-1}$. Then $b\, V_{B\to C/D} = \prod_i g_i D$. Now, $\{u_1^x,\ldots,u_k^x\}$ is a transversal to C^x in B^x, and $C^x u_i^x b^x = C^x u_{i(b)}$, while $u_i^x b^x (u_{i(b)}^x)^{-1} = g_i^x$. Thus we obtain

$$b^x V_{B^x\to C^x/D^x} = \prod_i g_i^x D^x = (b\, V_{B\to C/D})^x \quad \text{, as required.}$$

(b) Since $V_{B\to C/D}$ is a homomorphism with abelian image, then, since $y \in B$, we have $b\, V_{B\to C^y/D^y} = b^y V_{B^y\to C^y/D^y} = (b\, V_{B\to C/D})^y$, by (a).

(c) By a familiar technique for calculating the transfer (see the proof of [28; 7.3.3] for example), we may renumber the u_i so that for some integers $r_1,\ldots,r_m$, we have $e\, V_{B\to C/D} = \prod_{i=1}^{m} u_i\, e^{r_i}\, u_i^{-1} D$. Since $E \trianglelefteq A$, then $f = \prod_i u_i\, e^{r_i}\, u_i^{-1} \in E \cap C$, as required.

<u>4.3.3 Hypotheses, notation and conventions.</u>

(a) We will assume for the rest of this section that $\mathfrak{F}$ is a fixed, non-trivial, Fitting class and that $R = U]P^* \in \mathfrak{F}$ is a fixed $(\mathfrak{F},p)$-relevant group for some prime $p \in \mathrm{char}(\mathfrak{F})$, where $U = O^p(R)$, and where P is a given, fixed, member of $\mathrm{Syl}_p(\mathrm{Aut}(U))$ and $P^* = P \cap (U]P)_{\mathfrak{F}}$. We further assume that $\mathfrak{F}$ and R satisfy the following condition.

($\mathcal{P}$) Whenever $G \in \mathfrak{F}$ and there exists X sn G with $X \cong U = O^p(R)$, then $XS \in \mathfrak{F}$, where $S \in \mathrm{Syl}_p(\mathrm{Aut}(U))$.

(b) If A is a group and $B \leq A$, then $\{B\,;\,A\}$ will denote the subgroup $\langle \{b^{-1} b^a : b \in B \text{ and } a \in A \text{ with } b^a \in B\}\rangle$ of B. We note that $B' \leq \{B\,;\,A\}$, and that $\{B\,;\,A\} = [B\,,\,A]$ if $B \trianglelefteq A$.

(c) The subgroup $\{P^* ; \mathrm{Aut}(U)\} < \alpha \in P^* : [U , \langle\alpha\rangle] \lneq U >$ of P^* will be denoted by P^*_o . Of course, $P^*_o \geqslant (P^*)'$, by (b).

(d) If $G \in \mathfrak{F}$, then $\mathcal{C}(U,G)$ will denote the set of subnormal subgroups X sn G with $X \cong U$.

(e) If A is a group and $B \leqslant A$ with $a \in N_A(B)$, then $\bar{a}$ will denote the automorphism induced on B by conjugation by a .

<u>4.3.4 Remark</u>. If $\mathfrak{F}_1$ is a Fischer class and R_1 is $(\mathfrak{F}_1 , p_1)$-relevant for $p_1 \in \mathrm{char}(\mathfrak{F}_1)$, then $\mathfrak{F}_1$ and R_1 automatically satisfy condition $(\mathcal{P})$ of (4.3.3a) .

<u>4.3.5 Construction</u>.

Suppose that $G \in \mathfrak{F}$ and that $X \in \mathcal{C}(U,G)$; that is, X sn G with $X \cong U$. Let $\rho = \rho_X : X \to U$ be an isomorphism . Then, as in (1.9.3), ρ induces a homomorphism $\hat{\phi} = \hat{\phi}_X : N_G(X) \to \mathrm{Aut}(U)$ given by

$$n\hat{\phi} = \{u \mapsto ((u\rho^{-1})^n)\rho : u \in U\} = \{u \mapsto u\rho^{-1}\bar{n}\rho\} , \quad n \in N_G(X) .$$

We note that $\ker(\hat{\phi}) = C_G(X)$.

Suppose that $S \in \mathrm{Syl}_p(N_G(X))$. By Sylow's theorem there exists $Q \in \mathrm{Syl}_p(\mathrm{Aut}(U))$ with $S\hat{\phi} \leqslant Q$, and there exists $\lambda \in \mathrm{Aut}(U)$ with $(S\hat{\phi})^\lambda \leqslant P$, where P is our fixed member of $\mathrm{Syl}_p(\mathrm{Aut}(U))$.

By (4.3.3a)$(\mathcal{P})$, $XS \in \mathfrak{F}$, and so by (1.9.3), $U](S\hat{\phi}) \in \mathfrak{F}$, since $p \in \mathrm{char}(\mathfrak{F})$ and $S \in \mathcal{N}\cap\mathfrak{F}$. Since $\lambda \in \mathrm{Aut}(U)$, it follows that $U(S\hat{\phi})^\lambda \in \mathfrak{F}$. But $U(S\hat{\phi})^\lambda$ sn UP , and so $(S\phi)^\lambda \leqslant P^* = P \cap (UP)_{\mathfrak{F}}$.

The homomorphism $n \mapsto (n\hat{\phi})^\lambda : N_G(X) \to \mathrm{Aut}(U)$ will be denoted by ϕ .

Let ν denote the natural homomorphism $P^* \to P^*/P_o^*$, and let $\theta = \phi\nu : S \to P^*/P_o^*$. Let $S_o = \{S ; N_G(X)\}$. Since ϕ is a homomorphism, we may check that $S_o\phi \leq \{P^* ; \mathrm{Aut}(U)\} \leq P_o^*$. Thus $\ker(\theta) \geq S_o$, and we will henceforth regard θ as a homomorphism $S/S_o \to P^*/P_o^*$. We may denote θ by $\theta^{\rho,\lambda}_{(X,S/S_o \to P^*/P_o^*)}$ when we wish to be explicit about the definition.

The next lemma will be used to show that the definition of θ is independent of the choices of ρ and λ, and will also be used later.

<u>4.3.6 Lemma</u>. Let G_1 and G_2 be groups in $\mathfrak{F}$. For $i = 1, 2$, let $X_i \in \mathcal{C}(U_i, G_i)$. Let ρ_i, S_i, λ_i, ϕ_i and $\theta_i = \theta^{\rho_i,\lambda_i}_{(X_i, S_i/S_{i,o} \to P^*/P_o^*)}$ be as in (4.3.5).

Suppose further that there exists an isomorphism $\mu : X_1S_1 \to X_2S_2$. Then, if $s \in S_1$, we have $s\theta_1 = (s\mu)\theta_2$.

<u>Proof</u>. Suppose that $x \in X_1$. Then

$$(x\mu)(\mu^{-1}\bar{s}\mu) = (s^{-1}xs)\mu = (s\mu)^{-1}x\mu(s\mu) = (x\mu)\overline{s\mu} . \qquad \cdots(1)$$

It will suffice to show that $s\phi_1 \equiv (s\mu)\phi_2 \pmod{P_o^*}$. Now,

$$s\phi_1 = \{u \mapsto u\lambda_1^{-1}\rho_1^{-1}\bar{s}\rho_1\lambda_1\} , \text{ and}$$

$$(s\mu)\phi_2 = \{u \mapsto u\lambda_2^{-1}\rho_2^{-1}\overline{s\mu}\rho_2\lambda_2\} .$$

By choice, $s\phi_1$ and $(s\mu)\phi_2$ belong to P^*, and so

$$(s\phi_1)((s\mu)\phi_2)^{-1} \in P^* . \qquad \cdots(2)$$

Suppose that $u \in U$. Then

$u(s\phi_1)\ ((s,\mu)\phi_2)^{-1}$

$= (u\lambda_1^{-1}\rho_1^{-1}\ \bar{s}\ \rho_1\lambda_1)\ \lambda_2^{-1}\rho_2^{-1}\ (\overline{s^{-1}\mu})\ \rho_2\lambda_2$

$= u(\lambda_1^{-1}\rho_1^{-1}\ \mu\ \rho_2\lambda_2)(\lambda_2^{-1}\rho_2^{-1}\ \mu^{-1}\ \bar{s}\ \mu\ \rho_2\lambda_2)$

$(\lambda_2^{-1}\rho_2^{-1}\ \mu^{-1}\ \rho_1\lambda_1)(\lambda_2^{-1}\rho_2^{-1}\ (\overline{s^{-1}\mu})\ \rho_2\lambda_2)$

$\in u^{\{P^*\ ;\ \mathrm{Aut}(U)\}}$, on application of (1) and (2), leading to the result.

<u>4.3.7 Corollary</u>. For a fixed X sn G and a fixed $S \in \mathrm{Syl}_p(N_G(X))$, the definition of the map θ as made in (4.3.5) is independent of the choices there made of ρ and λ ; we thus refer to θ as $\theta_{(X,S/S_o \to P^*/P_o^*)}$.

<u>Proof</u>. In (4.3.6), take $X = X_1 = X_2$, $\mu = 1$, and $\rho = \rho_1 = \rho_2$. Then for a given ρ , the definition of θ is independent of the choice of $\lambda \in \mathrm{Aut}(U)$ such that $(S\hat{\phi})^{\lambda} \leq P$. Next, take ρ_1 and ρ_2 as arbitrary isomorphisms from X to U ; a second application of (4.3.6) yields that the definition of θ is independent of the choice of ρ .

<u>4.3.8 Definition</u>. With G , X and S as in (4.3.5), define

$$\Psi_{X,G} = \Psi_{X,G \to P^*/P_o^*} = V_{G \to S/S_o}\ \theta_{(X,\ S/S_o \to P^*/P_o^*)} : G \to P^*/P_o^* ,$$

where $V_{G \to S/S_o}$ is the transfer homomorphism.

<u>4.3.9 Lemma</u>. The definition of $\Psi_{X,G} : G \to P^*/P_o^*$ is independent of the choice of $S \in \mathrm{Syl}_p(N_G(X))$, and is, further, independent of the choice of the G-conjugate of X used in the definition.

<u>Proof</u>. Let $Y = X^a$, where $a \in G$, and let $T \in \mathrm{Syl}_p(N_Y(Y))$. By

$$u(s\,\phi_1)\,((s,\mu)\,\phi_2)^{-1}$$

$$= (u\,\lambda_1^{-1}\,\rho_1^{-1}\;\bar{s}\;\rho_1\,\lambda_1)\;\lambda_2^{-1}\,\rho_2^{-1}\,(\overline{s^{-1}\mu})\;\rho_2\,\lambda_2$$

$$= u(\lambda_1^{-1}\,\rho_1^{-1}\,\mu\;\rho_2\,\lambda_2)(\lambda_2^{-1}\,\rho_2^{-1}\;\mu^{-1}\,\bar{s}\;\mu\;\rho_2\,\lambda_2)$$

$$(\lambda_2^{-1}\,\rho_2^{-1}\;\mu^{-1}\;\rho_1\,\lambda_1)(\lambda_2^{-1}\,\rho_2^{-1}\,(\overline{s^{-1}\mu})\;\rho_2\,\lambda_2)$$

$\in u^{\{P^*\,;\,\mathrm{Aut}(U)\}}$, on application of (1) and (2), leading to the result.

<u>4.3.7 Corollary</u>. For a fixed X sn G and a fixed $S \in \mathrm{Syl}_p(N_G(X))$, the definition of the map θ as made in (4.3.5) is independent of the choices there made of ρ and λ ; we thus refer to θ as $\theta_{(X,S/S_o \to P^*/P_o^*)}$.

<u>Proof</u>. In (4.3.6), take $X = X_1 = X_2$, $\mu = 1$, and $\rho = \rho_1 = \rho_2$. Then for a given ρ , the definition of θ is independent of the choice of $\lambda \in \mathrm{Aut}(U)$ such that $(S\hat{\phi})^{\lambda} \leq P$. Next, take ρ_1 and ρ_2 as arbitrary isomorphisms from X to U ; a second application of (4.3.6) yields that the definition of θ is independent of the choice of ρ .

<u>4.3.8 Definition</u>. With G , X and S as in (4.3.5), define

$$\Psi_{X,G} = \Psi_{X,G \to P^*/P_o^*} = V_{G \to S/S_o}\;\theta_{(X,\,S/S_o \to P^*/P_o^*)} : G \to P^*/P_o^* ,$$

where $V_{G \to S/S_o}$ is the transfer homomorphism.

<u>4.3.9 Lemma</u>. The definition of $\Psi_{X,G} : G \to P^*/P_o^*$ is independent of the choice of $S \in \mathrm{Syl}_p(N_G(X))$, and is, further, independent of the choice of the G-conjugate of X used in the definition.

<u>Proof</u>. Let $Y = X^a$, where $a \in G$, and let $T \in \mathrm{Syl}_p(N_Y(Y))$. By

Sylow's theorem in $N_G(Y)$, there exists $b \in G$ such that $Y = X^b$ and $T = S^b$. Let $T_o = \{T ; N_G(Y)\}$; then $T_o = S_o^b$. Let $\rho : Y \to U$ be an isomorphism ; then $\bar{b}\rho : X \to U$ is also an isomorphism.

Let $g \in G$ and suppose that $g\, V_{G \to S/S_o} = s\, S_o$. By (4.3.2b), $g\, V_{G \to T/T_o} = s^b\, T_o$, and we have

$$\begin{aligned} g\, \Psi_{Y,G} = t\, T_o\, \theta_{Y,T/T_o \to P^*/P_o^*} &= \{u \mapsto u\, \rho^{-1}(\overline{b^{-1}\, s\, b})\, \rho\} \\ &= \{u \mapsto u\, (\bar{b}\rho)^{-1}\, \bar{s}\, (\bar{b}\rho)\} \\ &= s\, S_o\, \theta_{(X,S/S_o \to P^*/P_o^*)} \\ &= g\, \Psi_{X,G} \quad . \end{aligned}$$

The assertion follows.

4.3.10 Definitions.

Let $G \in \mathfrak{J}$. If $\mathcal{C}(U,G) \neq \emptyset$, let $\{X_1, \ldots, X_k\}$ be a full set of representatives for the G-conjugacy classes of groups in $\mathcal{C}(U,G)$. For each $i \in \{1, \ldots, k\}$, choose $e_i \in \mathbb{N}$ such that

$$e_i\, |N_G(X_i) : S_i| \equiv 1 \pmod{p^\omega} \quad , \qquad \ldots(3)$$

where $S_i \in \mathrm{Syl}_p(N_G(X_i))$, and p^ω is the exponent of P^*/P_o^* : such a choice is possible because $(p\, ,\ |N_G(X_i) : S_i|) = 1$.

Now define a homomorphism $\delta_{X_i,G} = \delta_{X_i,G \to P^*/P_o^*} : G \to P^*/P_o^*$ by

$$g\, \delta_{X_i,G} = (g\, \Psi_{X_i,G \to P^*/P_o^*})^{e_i} \quad , \text{ if } X_i \in \mathcal{C}(U,G) \neq \emptyset \ .$$

Since P^*/P_o^* is abelian, $\delta_{X_i,G}$ is indeed a homomorphism (since Ψ is). Since $(p\, ,\ |N_G(X_i) : S_i|) = 1$, the definition of $\delta_{X_i,G}$ is independent of the choice of e_i satisfying the congruence (3).

Now define a further homomorphism $d_G^R = d_{G\to P^*/P_o^*}^{R,\mathfrak{F}} : G \to P^*/P_o^*$ by

$$g\, d_G^R = 1 \qquad \text{if } \mathcal{C}(U,G) = \phi \ .$$

$$g\, d_G^R = \prod_{i=1}^{k} g\, \delta_{X_i,G} \qquad \text{if } \mathcal{C}(U,G) = \{X_1,\ldots,X_k\}^G \neq \phi \ ,$$

where $g \in G$. The "R" in the designation denotes our chosen $(\mathfrak{F},p)$-relevant group $R = U]P^*$; P^* is our fixed p-subgroup of $\mathrm{Aut}(U)$.

By (4.3.9), the definition of d_G^R is independent of the choice of conjugacy class representatives X_i. It is evident that if $\iota : G \to \hat{G}$ is an isomorphism, then $g\, d_G^R = g\iota\, d_{\hat{G}}^R$ for all $g \in G$.

<u>4.3.11 Proposition</u>. If $G \in \mathfrak{F}$ and $H \trianglelefteq G$, then $d_G^R = d_G^R\big|_H$.

<u>Proof</u>. Let $H \trianglelefteq G \in \mathfrak{F}$. If $X \in \mathcal{C}(U,G)$, we choose $S \in \mathrm{Syl}_p(N_G(X))$ and an isomorphism $\rho : X \to U$ in such a way that ρ induces the homomorphism $\phi : N_G(X) \to \mathrm{Aut}(U)$ so that $S^\phi \leq P^*$ (our fixed p-subgroup of $\mathrm{Aut}(U)$). We also choose $e \in \mathbb{N}$ so that $e\,|N_G(X) : S| \equiv 1 (\mathrm{mod}\ p^\omega)$, where p^ω is the exponent of P^*/P_o^*, as in equation (3) above.

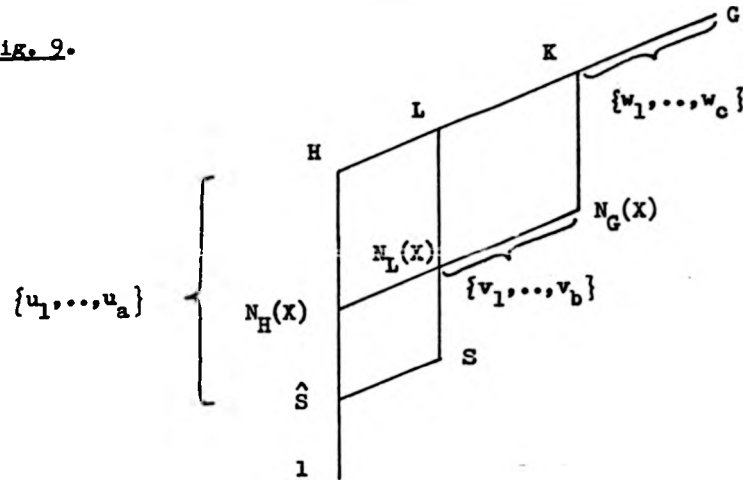
(a) Suppose firstly that $X \not\leq H$; then $X \in \mathcal{C}(U,G) \setminus \mathcal{C}(U,H)$. Let $h \in H$. Then

$$h\, \delta_{X,G} = (h\, \Psi_{X,G})^e = (h\, V_{G\to S/S_o})^e\, \theta(X,\ S/S_o \to P^*/P_o^*) \ .$$

Suppose that $h\, V_{G\to S/S_o} = s\, S_o$. Since $h \in H \trianglelefteq G$, we may assume by (4.3.2c) that $s \in S \cap H$. Thus $[X\ ,\ \langle s\rangle] \leq X \cap H \lneq X$, since $X \not\leq H$. But then $[U\ ,\ \langle s\phi\rangle] \lneq U$, and so $s\phi \in P_o^*$. Thus $h\, \delta_{X,G} = 1 \in P^*/P_o^*$, and so for $X \in \mathcal{C}(U,G) \setminus \mathcal{C}(U,H)$, we obtain trivial contribution towards the calculation of $h\, d_G^R$. Of course, for such an X, there is trivial contribution towards $h\, d_H^R$.

(b) Next suppose that $X \leq H$, so that $X \in \mathcal{C}(U,H)$. Let $L = H\,S$ and $K = H\,N_G(X)$. Let $\{u_1,\ldots,u_a\}$, $\{v_1,\ldots,v_b\}$ and $\{w_1,\ldots,w_c\}$ be (right) transversals to $\hat{S} = S \cap H$ in H, to $N_L(X)$ in $N_G(X)$, and to K in G, respectively.

fig. 9.



The calculation proceeds in steps : from H to L, to K, to G.

Let $f = eb$. Then

$$f\,|N_H(X) : S| = eb\,|N_L(X) : S| = e\,|N_G(X) : S| \equiv 1 \pmod{p^{\omega}}. \quad \cdots(4)$$

Let $h \in H$. Then, if $\hat{S}u_i h = \hat{S}u_{i(h)}$ (so that $Su_i h = Su_{i(h)}$),

$$h\,\psi_{X,H} = h\,V_{H\to\hat{S}/\hat{S}_o}\;\theta_{(X,\,\hat{S}/\hat{S}_o \to P^*/P_o^*)}$$

$$= \prod_{i=1}^{a}(u_i\,h\,u_{i(h)}^{-1})\,\hat{S}_o\;\theta_{(X,\hat{S}/\hat{S}_o \to P^*/P_o^*)}, \text{ and}$$

$$h\,\psi_{X,L} = h\,V_{L\to S/S_o}\;\theta_{(X,S/S_o \to P^*/P_o^*)}$$

$$= \prod_{i=1}^{a}(u_i\,h\,u_{i(h)}^{-1})\,S_o\;\theta_{(X,S/S_o \to P^*/P_o^*)},$$

since $\{u_1,\ldots,u_a\}$ also forms a transversal to S in L. But then $h\,\psi_{X,H}$ and $h\,\psi_{X,L}$ coincide as elements of P^*/P_o^*, because the

elements $u_i\, h\, u_{i(h)}^{-1}$ induce the same automorphisms on U in each case.

If now $g \in N_G(X)$ then, since $h\,\Psi_{X,H}$ is independent of the conjugate of $\hat{S}$ used in its calculation, we have

$$h\,\Psi_{X,H} = h\,\Psi_{X,L^g} \qquad \cdots(5)$$

Next,

$$\begin{aligned}
h\,\delta_{X,K} &= (h\,\Psi_{X,K})^e \\
&= (h\,V_{K\to L}\,V_{L\to S/S_o}\,\theta_{(X,S/S_o\to P^*/P_o^*)})^e \\
&= (\prod_{i=1}^{b}(v_i\,h\,v_i^{-1})\,V_{L\to S/S_o}\,\theta_{(X,S/S_o\to P^*/P_o^*)})^e \\
&\qquad\qquad (\text{since } Lv_ih = Lv_i \text{ as } L \geqslant H \trianglelefteq G\ ) \\
&= (\prod_{i=1}^{b} h\,V_{L^{v_i}\to S^{v_i}/S_o^{v_i}}\,\theta_{(X,S^{v_i}/S_o^{v_i}\to P^*/P_o^*)})^e \\
&\qquad (\text{by } (4.3.2a) \text{ and } (4.3.6), \text{ as } v_i \in N_G(X) \text{ and } (XS)^{v_i} = X\,S^{v_i}). \\
&= (\prod_{i=1}^{b} h\,\Psi_{X,L^{v_i}})^e \\
&= (h\,\Psi_{X,H})^{be} \qquad (\text{by } (5)) \\
&= h\,\delta_{X,H} \quad (\text{by definition, as } f = eb \text{ satisfies } (4))\ . \qquad \cdots(6)
\end{aligned}$$

We now note that X and Y are conjugate in H if and only if they are conjugate in K, since $K = H\,N_G(X)$. We thus have $c = |G : K|$ K-conjugacy classes $(X^{w_1}), \ldots, (X^{w_c})$ of H fusing into a single G-conjugacy class (X). We have

$$\begin{aligned}
h\,\delta_{X,G} &= (h\,V_{G\to K}\,V_{K\to S/S_o}\,\theta_{(X,S/S_o\to P^*/P_o^*)})^e \\
&= (\prod_{i=1}^{c}(w_i h\,w_i^{-1}\,K'\ V_{K\to S/S_o}\,\theta_{(X,S/S_o\to P^*/P_o^*)}))^e \\
&\qquad\qquad (\text{since } Kw_ih = Kw_i \text{ as } K \geqslant H \trianglelefteq G\ )
\end{aligned}$$

$$= \left(\prod_{i=1}^{c} (h\, V_{K^{w_i} \to S^{w_i}/S_o^{w_i}} \quad \theta_{(X^{w_i}, S^{w_i}/S_o^{w_i} \to P^*/P_o^*})\right)^e$$

(by (4.3.2a) and (4.3.6))

$$= \prod_{i=1}^{c} (h\, \delta_{X^{w_i}, K^{w_i}})$$

$$= \prod_{i=1}^{c} h\, \delta_{X^{w_i}, H} \qquad \text{(by (5))} \ . \qquad \cdots(6)$$

Now, $h\, d_G^R$ is the product of $h\, \delta_{X,G}$ over representatives X of G-conjugacy classes in $\mathcal{C}(U,G)$, while $h\, d_H^R$ is the product of $h\, \delta_{Y,H}$ over representatives Y of H-conjugacy classes in $\mathcal{C}(U,H)$. The result follows from (6) and part (a).

<u>4.3.12 Proposition</u>. In the accumulated notation of this section, with our fixed Fitting class $\mathfrak{F}$ and $(\mathfrak{F},p)$-relevant group $R = U \rfloor P^*$, satisfying (4.3.3a)($\mathcal{P}$), we have the following.

(a) The group P^*/P_o^* is a p-group where $p = \mathrm{char}(R/R^{\mathcal{N}})$.

(b) The homomorphism $d^{R,\mathfrak{F}}_{R \to P^*/P_o^*}$ maps R onto P^*/P_o^* and has kernel $U\, P_o^*$.

(c) If $G \in \mathfrak{F}$ is such that $\mathcal{C}(U,G) = \phi$, then

$$G\, d^{R,\mathfrak{F}}_{G \to P^*/P_o^*} = 1 \ .$$

(d) If $\hat{R} \in \mathcal{R}_{\mathfrak{F}}$ is such that $\hat{R} \not\leq R$ and $|\hat{R}^{\mathcal{N}}| \leq |R^{\mathcal{N}}|$, then

$$\hat{R}\, d^{R,\mathfrak{F}}_{\hat{R} \to P^*/P_o^*} = 1.$$

(e) Suppose that $\mathfrak{G}$ is a Fitting class with $\mathfrak{F} \subseteq \mathfrak{G}$, and suppose that R is also $(\mathfrak{G},p)$-relevant and that $\mathfrak{G}$ and R satisfy ($\mathcal{P}$). If $G \in \mathfrak{F}$, then $g\, d^{R,\mathfrak{F}}_{G \to P^*/P_o^*} = g\, d^{R,\mathfrak{G}}_{G \to P^*/P_o^*} \in P^*/P_o^*$, for all $g \in G$.

<u>Proof</u>. (a) This follows from the definitions.

(b) Since $U = O^p(R)$, then $\mathcal{C}(U,R) = \{U\}$, and so

$$d^{R,\mathfrak{F}}_{R\to P^*/P^*_o} = \delta_{U,\ R\to P^*/P^*_o} \qquad \cdots(7)$$

Let $\overline{P} \in \mathrm{Syl}_p(R)$ be such that $P^* \leq \overline{P}$. Let $n = |R : P|$, and let $\overline{P}_o = \{\overline{P} ; R\}$.

Let $x \in P^*$. A reading of the proof of [28; 7.3.4] yields that

$$x\, V_{R\to \overline{P}/\overline{P}_o} = x^n\, \overline{P}_o \quad . \qquad \cdots(8)$$

Let $\rho : U \to U$ be the identity map. Then ρ induces a homomorphism $\hat{\rho} : R = N_R(U) \to \mathrm{Aut}(U)$. Since $P^* \leq \mathrm{Aut}(U)$ and $R = U]P^*$, the semidirect product with respect to the natural action of P^* on U, then $x\hat{\rho} = x \in P^*$. Thus by (4.3.5) we have

$$(x^n\, \overline{P}_o)\, \theta_{(U,\ \overline{P}/\overline{P}_o \to P^*/P^*_o)} = x^n\, P^*_o \quad . \qquad \cdots(9)$$

If $e \in \mathbb{N}$ be chosen so that $en \equiv 1$ (mod(exponent P^*/P^*_o)), then by definition,

$$\begin{aligned} x\, \delta_{U,R\to P^*/P^*_o} &= (x\, V_{R\to \overline{P}/\overline{P}_o}\ \theta_{(U,\ \overline{P}/\overline{P}_o \to P^*/P^*_o)})^e \\ &= (x^n\, P^*_o)^e \qquad \text{(by (8) and (9))} \\ &= x\, P^*_o \ . \end{aligned}$$

Thus, $\delta_{U,R}$ induces the natural homomorphism $P^* \to P^*/P^*_o$, and is, in particular, onto P^*/P^*_o. Since $U = O^p(U)$ and the image of $\delta_{U,R}$ is a p-group, then $U \leq \ker(\delta_{U,R})$ and so $\ker(\delta_{U,R}) = U\, P^*_o$. The assertion now follows by (7).

(c) This is clear by definition of d^R_R.

(d) Since $\hat{R}\, d^R_{\hat{R}} \leq P^*/P^*_o \in \mathcal{A} \cap \mathcal{S}_p$, then $\hat{R}\, d^R_{\hat{R}} = 1$ if $\hat{R} \in \mathcal{R}^q_{\mathfrak{F}}$

where $q \in \mathbb{P} \setminus \{p\}$. Suppose that $R \in \mathcal{R}^{p}_{\mathfrak{F}}$. Since $|\hat{R}^{\mathcal{N}}| \leq |R^{\mathcal{N}}|$, then either $\mathcal{C}(U,\hat{R}) = \phi$, or else $\hat{R}^{\mathcal{N}} = O^{p}(\hat{R}) \cong U = O^{p}(U) = R^{\mathcal{N}}$. Now, if $\hat{R}^{\mathcal{N}} \cong U$, then $\hat{R} \cong (UP)_{\mathfrak{F}} = R$, for $P \in \mathrm{Syl}_{p}(\mathrm{Aut}(U))$, by definition of an $(\mathfrak{F},p)$-relevant group, contrary to hypothesis. Thus $\mathcal{C}(U,\hat{R}) = \phi$, and the result follows by (c) .

(e) Since $\mathcal{G}$ and $R = U P^{*} \in \mathcal{R}^{p}_{\mathcal{G}}$ also satisfy $(\mathcal{P})$, then $d^{R,\mathcal{G}}_{G \to P^{*}/P^{*}_{o}}$ is indeed defined for $G \in \mathfrak{F} \subseteq \mathcal{G}$. If $g \in G$, then the calculations for $g\, d^{R,\mathfrak{F}}_{G \to P^{*}/P^{*}_{o}}$ and $g\, d^{R,\mathcal{G}}_{G \to P^{*}/P^{*}_{o}}$ must give the same element of P^{*}/P^{*}_{o} .

<u>4.3.13 Proposition</u>. With Fitting class $\mathfrak{F}$ and $(\mathfrak{F},p)$-relevant group $R = U]P^{*}$ satisfying (4.3.3a)$(\mathcal{P})$, we have that $(P^{*}/P^{*}_{o}, d^{R,\mathfrak{F}})$ is an $\mathfrak{F}$-Fitting pair.

<u>Proof</u>. If $G \in \mathfrak{F}$, then $d^{R,\mathfrak{F}}_{G \to P^{*}/P^{*}_{o}}$ is a homomorphism from G to P^{*}/P^{*}_{o} . Condition (4.1.6a)(1) is satisfied because of (4.3.11) together with the fact, mentioned in the final sentence of (4.3.10), that $d^{R,\mathfrak{F}}$ is "isomorphism-invariant" . Condition (4.1.6a)(2) is satisfied because $d^{R,\mathfrak{F}}_{R \to P^{*}/P^{*}_{o}}$ maps $R \in \mathcal{R}_{\mathfrak{F}} \subseteq \mathfrak{F}$ onto P^{*}/P^{*}_{o} .

<u>4.3.14 Remarks</u>. (a) The $\mathfrak{F}$-Fitting pair $(P^{*}/P^{*}_{o}, d^{R,\mathfrak{F}})$ is of course dependent on our choice of $R \in \mathcal{R}^{p}_{\mathfrak{F}}$, exhibited in the form $R = U]P^{*}$, where P is a chosen member of $\mathrm{Syl}_{p}(\mathrm{Aut}(U))$, and where $P^{*} = P \cap (UP)_{\mathfrak{F}}$. This dependence need be of no real concern : we

just make such a choice in a given situation and fix it. In any case, we may check that the class $(G \in \mathfrak{F} : G\, d^{R,\mathfrak{F}}_{G \to P^*/P^*_o} = 1)$, which by (4.1.7) is a Fitting class central under $\mathfrak{F}$, is independent of such choices : for example, for different P in $\mathrm{Syl}_p(\mathrm{Aut}(U))$, we obtain maps $d^{R,\mathfrak{F}}$ differing only by an inner automorphism of $\mathrm{Aut}(U)$. It is really a notational convenience that we take an $\mathfrak{F}$-relevant group R exhibited in the form $R = U] P^*$, where P is actually a subgroup of $\mathrm{Aut}(U)$, and then map into P^*/P^*_o ; we could map into R/UP^*_o, which is evidently independent of the choice of P, or we could map into the isomorphic factor group of an arbitrary $(\mathfrak{F}, p)$-relevant group $Q \cong R$.

(b) If $\mathfrak{F}$ is a Fischer class, we may construct such $\mathfrak{F}$-Fitting pairs for arbitrary $\mathfrak{F}$-relevant groups R, by (4.3.4).

4.4 The proof of Berger's theorem.

4.4.1 Hypotheses, notation and conventions.

Throughout this section, $\mathfrak{F}$ will denote a fixed Fitting class, and $\mathbb{F}$ will denote a given underlying set $\mathbb{F} = \{G_i\}_{i \in I}$. We construct the group $\Delta = \prod_{i \in I} G_i$ with subgroup Λ as in (4.1.5) ; if $G \in \mathbb{F}$, then ε_G denotes the natural embedding of G into Δ.

(a) If $w \in \mathbb{N}$, define

$$\Theta^w = \{f \in \Delta : (i)f = 1 \text{ unless } G_i \text{ is } \mathfrak{F}\text{-relevant and } |G_i^{\mathcal{N}}| \leq w\}.$$

We may check that Θ^w is a subgroup of Δ.

(b) For notational convenience, we choose a specific underlying set

for $\mathcal{R}_{\mathfrak{F}}$, the class of $\mathfrak{F}$ -relevant groups. Thus, for each $G \in \mathbb{F}$ which is $\mathfrak{F}$ -relevant, and so $(\mathfrak{F},p)$-relevant for a unique $p \in \mathrm{char}(\mathfrak{F})$, by (4.2.3), we choose a group $U = O^p(U) \cong G^{\mathcal{N}} = O^p(G)$ and a Sylow p-subgroup P of $\mathrm{Aut}(U)$. By definition, we then have $G \cong R$, where $R = (U]P)_{\mathfrak{F}} = U]P^*$, with $P^* = P \cap (UP)_{\mathfrak{F}}$.

The set of all such chosen groups $R = U]P^* \cong G$ as G runs over $\mathbb{F} \cap \mathcal{R}_{\mathfrak{F}}$ will be denoted by $\mathbb{R}_{\mathfrak{F}}$, while $\mathbb{R}^p_{\mathfrak{F}}$ will denote $\mathcal{R}^p_{\mathfrak{F}} \cap \mathbb{R}_{\mathfrak{F}}$.

If we refer to a group $R = U]P^* \in \mathbb{R}_{\mathfrak{F}}$, it will be understood that $U = R^{\mathcal{N}} = O^p(R)$, for the appropriate $p \in \mathrm{char}(\mathfrak{F})$, and that $P^* = P \cap (UP)_{\mathfrak{F}}$, where P is the chosen Sylow p-subgroup of $\mathrm{Aut}(U)$.

(c) We recall that a group G is called <u>single-headed</u> if it has a unique maximal normal subgroup.

<u>4.4.2 Lemma</u>. (a) If $v \leq w \in \mathbb{N}$, then $\Theta^v \leq \Theta^w \leq \Delta$.

(b) If $w \in \mathbb{N}$, then $\{R_i = U_iP_i^* \in \mathbb{R}_{\mathfrak{F}} : |R_i^{\mathcal{N}}| \leq w\}$ is a finite (possibly empty) set.

(c) If $w \in \mathbb{N}$, then $\Theta^w = \prod_i R_ie_i \leq \Delta$, where the (internal, direct) product is taken over those (distinct) $R_i \in \mathbb{R}_{\mathfrak{F}}$ with $|R_i^{\mathcal{N}}| \leq w$, and where $e_i \in \mathrm{Qnat}(R_i , \Delta)$; if there exist no such R_i , then $\Theta^w = 1$. By (b), it follows that Θ^w is always finite.

<u>Proof</u>. (a) This is evident from the definition of Θ^w .

(b) There are only finitely many isomorphism types of groups with order at most w , and each group of order $v \leq w$ has only finitely many prime-power-order subgroups of its automorphism group. There can thus be only finitely many possible isomorphism types of groups which are $\mathfrak{F}$-relevant and have nilpotent residuals of order at most w , as claimed.

(c) This follows from the definitions of Θ^{w} and $\mathrm{Qnat}(R_i, \Delta)$ (see (4.1.11)).

The next lemma is the first step in our proof of Berger's theorem; by part(a), the ascending union of the Θ^{w} supplements Λ in Δ. Part (b) is not needed for Berger's theorem, but will be useful later.

If A and B are groups, then <u>A isn B</u> will mean that A is isomorphic to a subnormal subgroup of B.

<u>4.4.3 Lemma</u>. (a) If $G \in \mathfrak{F}$ and $e_G \in \mathrm{Qnat}(G, \Delta)$, then

$$G\, e_G \leqslant \Lambda\, \Theta^{|G'^{\mathcal{N}}|}.$$

(b) If $G \in \mathfrak{F}$, then $G \in \langle \mathfrak{X} \rangle_{\mathrm{Fitt}}$, where

$$\mathfrak{X} = (A \in \mathfrak{F} : A' = A \text{ and } A \text{ isn } G^{\mathcal{N}}) \cup (C_p : p \in \mathrm{comp}(G))$$
$$\cup\, (R \in \mathcal{R}_{\mathfrak{F}} : R^{\mathcal{N}} \text{ isn } G^{\mathcal{N}} \text{ and } \mathrm{comp}(R) \subseteq \mathrm{comp}(G)),$$

where $\mathrm{comp}(G)$ is as in (1.1.1c).

<u>Proof</u>. Let $G \in \mathfrak{F}$, and let $w = |G^{\mathcal{N}}|$. Since G is finite, it may be expressed as the join of single-headed subnormal subgroups, $A_1, \ldots, A_t$ say. We note that

$$G \in N_0(\{A_1, \ldots, A_t\}) \subseteq \langle \{A_i\}_{i=1,\ldots,t} \rangle_{\mathrm{Fitt}}, \text{ and} \qquad \ldots(1)$$

$$A_i' = A_i^{\mathcal{N}} \leqslant G^{\mathcal{N}}, \text{ for } i = 1, \ldots, t, \qquad \ldots(2)$$

statement (2) by (1.3.10)

For each A_i there are, by (1.3.10), three possibilities : either $A_i' = A_i$, or $1 = A_i' \lneqq A_i$ and A_i is a cyclic p-group, $p \in \mathbb{P}$, or $1 \lneqq A_i' = A_i^{\mathcal{N}} \lneqq A_i$. We consider each case in turn, and show that

$A_i e_G \leq \Lambda\Theta^W$ and that $A_i \in \langle \mathfrak{X} \rangle_{Fitt}$. Part (a) of the lemma will then follow since G is the join of the A_i, while part (b) will follow by (1). In all cases, e_i denotes a fixed but arbitrary member of $Qnat(A_i, \Delta)$: since A_i sn $G \in \mathfrak{F}$, then certainly $A_i \in \mathfrak{F}$.

Suppose firstly that $A_i' = A_i$; then $A_i e_i \leq \Lambda$ by definition of Λ, and so by (4.1.15), $A_i e_G \leq \Lambda \leq \Lambda\Theta^W$ since A_i sn G. Since $A_i = A_i^{\mathcal{N}} \leq G^{\mathcal{N}}$, then A_i sn $G^{\mathcal{N}}$ and so $A_i \in \mathfrak{X}$.

Suppose next that $1 = A_i' \lneq A_i \in \mathcal{S}_p$, $p \in \mathbb{P}$. Since A_i sn G, then $p \in comp(G) \subseteq char(\mathfrak{F})$, and so $A_i \in \mathfrak{F}_*$ by (1.5.11c). By (4.1.9/15), $A_i e_G \leq \Lambda \leq \Lambda\Theta^W$, while $A_i \in \langle C_p \rangle_{Fitt} \subseteq \langle \mathfrak{X} \rangle_{Fitt}$, by the remarks following (1.3.15).

Suppose finally that $1 \lneq A_i' = A_i^{\mathcal{N}} \lneq A_i$, with $A_i/A_i^{\mathcal{N}}$ a cyclic p-group by (1.3.10), $p \in \mathbb{P}$. As above, $p \in comp(G) \subseteq char(\mathfrak{F})$. Let A denote A_i, let N denote $A_i^{\mathcal{N}}$, and let x be a p-element of A such that $A = N\langle x \rangle$. We note that $\langle x \rangle \in \mathfrak{F}$ since $p \in char(\mathfrak{F})$.

By (1.1.3a), $[N, \langle x \rangle] \trianglelefteq A$, whence $\langle x \rangle \leq C_A(N/[N,\langle x \rangle]) \trianglelefteq A$, and since A is single-headed and $A = A'\langle x \rangle$, then $A = C_A(N/[N,\langle x \rangle])$. But then $N = A^{\mathcal{N}} = [A^{\mathcal{N}}, A] = [N, A] \leq [N, \langle x \rangle]$ and so $N = [N, \langle x \rangle]$.

The identity map $N \to N$ induces a homomorphism $\varphi : A \to Aut(N)$. Since $x \in \mathcal{N} \cap \mathfrak{F}$, then by (1.9.3), $N] \langle x\varphi \rangle \in \mathfrak{F}$. Since $N = [N, \langle x \rangle]$, then $N = [N, \langle x\varphi \rangle]$ and

$\langle x\varphi \rangle$ is a non-trivial p-group. ...(3)

Let $S \in Syl_p(Aut(U))$ be such that $\langle x\varphi \rangle \leq S$. Since $N\langle x\varphi \rangle \in \mathfrak{F}$ then $\langle x\varphi \rangle \leq S^* = S \cap (N\,S)_{\mathfrak{F}}$. Since $[N, \langle x\varphi \rangle] = N$, it follows that $[N, S^*] = N$. But $N = A^{\mathcal{N}} = O^p(N)$, and so $N]S^*$ is

$(\mathfrak{F},p)$-relevant ; that is, $N]S^* \in \mathcal{R}^p_{\mathfrak{F}}$.

Let $e^* \in \mathrm{Qnat}(NS^*, \Delta)$. We apply (4.1.17) with the group $A = N\langle x\rangle \in \mathfrak{F}$, $N \cong N$, and $N]\langle x\varphi\rangle$ sn $N]S^* \in \mathfrak{F}$ to conclude that

$$x\, e_A \equiv (x\varphi)\, e^* \pmod{\Lambda} . \qquad \cdots(4)$$

Since $N\,S^* \in \mathcal{R}_{\mathfrak{F}}$, and $N = (NS^*)^{\mathcal{N}}$, then $NS^*\, e^* \leq \Theta^{|N|}$, by definition of $\Theta^{|N|}$. But by (2) and (4.4.2a) we have $\Theta^{|N|} \leq \Theta^{w}$; thus $(x\varphi)\, e^* \in \Theta^{w}$, and so by (4) we have $x\, e_A \in \Lambda\Theta^{w}$. Since $N = A'$ then $N\, e_A \leq \Lambda$,see (4.1.5)), and so $Ae_A = (N\langle x\rangle)e_A \leq \Lambda\Theta^{w}$. This completes the proof of part (a).

By virtue of statement (3), we have $p \in \mathrm{char}(\langle N]\langle x\varphi\rangle\rangle_{\mathrm{Fitt}})$, and so we may apply (1.9.3) to conclude that $A = N\langle x\rangle \in \langle N]\langle x\varphi\rangle\rangle_{\mathrm{Fitt}}$. But $N]\langle x\varphi\rangle$ sn $N\,S^* \in \mathcal{R}_{\mathfrak{F}}$, while by (2), N sn $G^{\mathcal{N}}$. Since A sn G and $A/N = A/A^{\mathcal{N}}$ is a non-trivial p-group, then $p \in \mathrm{comp}(G)$, while since N sn G then $\mathrm{comp}(N) \subseteq \mathrm{comp}(G)$. Thus $\mathrm{comp}(NS^*) \subseteq \mathrm{comp}(G)$. It follows that $A \in \langle \mathfrak{F}\rangle_{\mathrm{Fitt}}$, completing the proof of (b).

The next lemma will provide the induction step for our proof of Berger's theorem.

<u>4.4.4 Lemma</u>. Suppose that $R = U]P^* \in \mathbb{R}_{\mathfrak{F}}$, and let $e_R \in \mathrm{Qnat}(R, \Delta)$. Then $(U]P^*_o)e_R \leq \Lambda\Theta^{|U|-1}$, where, as in (4.3.3c),

$$P^*_o = \{P^*;\ \mathrm{Aut}(U)\}\langle \alpha \in P^* : [U, \langle\alpha\rangle] \nsubseteq U \rangle .$$

<u>Proof</u>. Since $U \leq R'$, then $U\, e_R \leq \Lambda$ (see (4.1.5)).

Let $x \in P^*_o$. Then there exist elements $y_i \in P^*$, $i = 1,\ldots,r$, with $[U, \langle y_i\rangle] \nsubseteq U$, and an element $t \in \{P^*;\ \mathrm{Aut}(U)\}$, such that

$x = t\, y_1 \cdots y_r$, where possibly $r = 0$.

Suppose that $a \in P$, $b \in \mathrm{Aut}(U)$, and that $a^b \in P$. If $G \in \mathfrak{J}$, let e_G be a fixed but arbitrary member of $\mathrm{Qnat}(G, \Delta)$. We note that $U\langle a\rangle$ and $U\langle a^b\rangle$ sn $R \in \mathfrak{J}$. Now, conjugation by b in $U\,\mathrm{Aut}(U)$ induces a normal embedding $U\langle a\rangle \to U\langle a^b\rangle$ with $U \mapsto U^b = U$. Thus $(a\, e_{U\langle a\rangle})^{-1}(a^b\, e_{U\langle a^b\rangle}) \in \Lambda$, and so $(a^{-1}a^b)e_R = a^{-1}e_R\; a^b e_R \in \Lambda$, by (4.1.14). Thus $\{P^*;\ \mathrm{Aut}(U)\}\, e_R \leqslant \Lambda$, and so $t\, e_R \in \Lambda$.

By (1.1.3a), $[U, \langle y_i\rangle] \trianglelefteq U\langle y_i\rangle$. Let $A_i = [U, \langle y_i\rangle]\langle y_i\rangle$; then $A_i \trianglelefteq U\langle y_i\rangle$ sn $UP^* = R$. Now, $A_i^{\mathcal{N}} \leqslant [U, \langle y_i\rangle] \lneqq U$, by choice of y_i , and so by (4.4.3a) and (4.4.2a) we have

$$A_i\, e_i \leqslant \Lambda\Theta^{|A_i^{\mathcal{N}}|} \leqslant \Lambda\Theta^{|U|-1} \quad , \text{ for } e_i \in \mathrm{Qnat}(A_i, \Delta) .$$

Since A_i sn R , it follows that $A_i e_R \leqslant \Lambda\Theta^{|U|-1}$, and so $y_i e_R \in \Lambda\Theta^{|U|-1}$. Thus $(U\, P_o^*)\, e_R \leqslant \Lambda\Theta^{|U|-1}$, as asserted.

4.4.5 Definition and notation.

The class $\mathcal{B}_{\mathfrak{J}}$ of $\mathfrak{J}$-basic groups is defined to be

$$\mathcal{B}_{\mathfrak{J}} = (G \in \mathcal{R}_{\mathfrak{J}} :\ \text{if } G \cong U]P^* \in \mathbb{R}_{\mathfrak{J}} \text{ , then } P_o^* \lneqq P^*) \subseteq \mathcal{R}_{\mathfrak{J}} \quad ,$$

where P_o^* is as in (4.4.4): we will fix this meaning for P_o^* henceforth .

We define $\mathcal{B}_{\mathfrak{J}}^{p} = \mathcal{B}_{\mathfrak{J}} \cap \mathcal{R}_{\mathfrak{J}}^{p}$, for $p \in \mathrm{char}(\mathfrak{J})$, and take the naturally-induced underlying sets $\mathbb{B}_{\mathfrak{J}} = \mathcal{B}_{\mathfrak{J}} \cap \mathbb{R}_{\mathfrak{J}}$ and $\mathbb{B}_{\mathfrak{J}}^{p} = \mathcal{B}_{\mathfrak{J}} \cap \mathbb{R}_{\mathfrak{J}}^{p}$.

The next result gives a factorisation (mod Λ) for certain elements of Δ in terms of elements from $\mathfrak{J}$-basic groups.

<u>4.4.6 Proposition</u>. Suppose that $\xi \in \Lambda\Theta^{w}$, where $w \in \mathbb{N}$. Then either $\xi \in \Lambda$ or else there exist distinct $R_1, \ldots, R_n \in \mathbb{B}_{\mathfrak{J}}$ (where $R_i = U_i]P_i^*$), and elements $x_i \in R_i \setminus U_i]P_{i,o}^*$, such that $|R_i^{\mathcal{N}}| \leq w$ and

$$\xi \equiv x_1e_1 \cdots x_ne_n \pmod{\Lambda}, \quad \text{for } e_i \in \mathrm{Qnat}(R_i, \Delta).$$

<u>Proof</u>. We proceed by induction on w. If $w = 1$, there exist no $\mathfrak{J}$-relevant groups G with $|G^{\mathcal{N}}| \leq w$, and so $\Theta^{w} = 1$ and $\xi \in \Lambda$.

Now suppose that the assertion is true for elements $\xi \in \Lambda\Theta^{v}$ with $v < w$; we may suppose that $\xi \in \Lambda\Theta^{w} \setminus \Lambda\Theta^{w-1}$ where $w > 1$.

By (4.4.2b), there are finitely many $R_k \in \mathbb{R}_{\mathfrak{J}}$ with $|R_k^{\mathcal{N}}| \leq w$; suppose for the time being that these are labelled $R_1, \ldots, R_m$ (we may suppose that these R_i are all distinct and so non-isomorphic).

By (4.4.2c), there exist elements $y_k \in R_k$ $(k = 1, \ldots, m)$, together with $\lambda \in \Lambda$ so that $\xi = \lambda(y_1e_1 \cdots y_me_m)$, for $e_k \in \mathrm{Qnat}(R_k, \Delta)$. But then

$$\xi \equiv y_1e_1 \cdots y_me_m \pmod{\Lambda}. \qquad \cdots(5)$$

Define subsets A and B of $\mathbb{N}$ by

$$A = \{\alpha \leq m : |R_\alpha^{\mathcal{N}}| < w \text{ or } |R_\alpha^{\mathcal{N}}| = w \text{ and } y_\alpha \in U_\alpha P_{\alpha,o}^*\}, \text{ and}$$

$$B = \{\beta \leq m : |R_\beta^{\mathcal{N}}| = w \text{ and } y_\beta \notin U_\beta P_{\beta,o}^*\}.$$

Since $\xi \notin \Lambda$, then $m > 0$ and so $A \cup B \neq \emptyset$.

Since Δ/Λ is abelian, the order of the factors in (5) is immaterial to the congruence, and so after suitable rearrangement of the terms we may write

$$\xi \equiv \Big(\prod_{\alpha \in A} y_\alpha e_\alpha\Big)\Big(\prod_{\beta \in B} y_\beta e_\beta\Big) \pmod{\Lambda}. \qquad \cdots(6)$$

If $\alpha \in A$, then either $|R_\alpha^{\mathcal{N}}| \leq w-1$ and $y_\alpha e_\alpha \in \Theta^{w-1}$, or else $y_\alpha \in U_\alpha P^*_{\alpha,o}$, in which case $y_\alpha e_\alpha \in \Lambda\Theta^{w-1}$ by (4.4.4). Thus

$$\xi_1 = \prod_{\alpha \in A} y_\alpha e_\alpha \in \Lambda\Theta^{w-1} .$$

But now by induction, ξ_1 satisfies a congruence of the form

$$\xi_1 \equiv \prod_{\gamma \in C} z_\gamma e_{R_\gamma} \pmod{\Lambda} , \qquad \cdots(7)$$

for some finite index set C, where the R_γ are distinct members of $\mathbb{B}_{\mathfrak{F}}$ with $|R_\gamma^{\mathcal{N}}| \leq w-1$, and where $z_\gamma \in R_\gamma \setminus U_\gamma P^*_{\gamma,o}$.

Since $\xi \notin \Lambda\Theta^{w-1}$, then $\xi \not\equiv \xi_1 \pmod{\Lambda}$, and so $B \neq \emptyset$. If $\beta \in B$, then $y_\beta \notin U_\beta P^*_{\beta,o}$, and so $R_\beta \in \mathbb{B}_{\mathfrak{F}}$. Since these R_β appear in (5), they are distinct and are, further, distinct from the R_γ for $\gamma \in C$, since $|R_\gamma^{\mathcal{N}}| \leq w-1 < w = |R_\beta^{\mathcal{N}}|$.

Combining (6) and (7) and rationalising notation, we conclude that there exist distinct $R_1, \ldots, R_n \in \mathbb{B}_{\mathfrak{F}}$ with $|R_i^{\mathcal{N}}| \leq w$, together with elements $x_i \in R_i \setminus U_i P^*_{i,o}$, such that $\xi \equiv x_1 e_1 \cdots x_n e_n \pmod{\Lambda}$, where $e_i \in \mathrm{Qnat}(R_i, \Delta)$, completing the induction and, with it, the proof.

The next result shows how, given a suitable supply of $\mathfrak{F}$-Fitting pairs, we may determine the $\mathfrak{F}_*$-radical of an arbitrary group $G \in \mathfrak{F}$.

<u>4.4.7 Proposition</u>. Suppose that for each $R = U]P^* \in \mathbb{B}_{\mathfrak{F}}$ there exists an $\mathfrak{F}$-Fitting pair $(H^R, h^R) = (H^{R,\mathfrak{F}}, h^{R,\mathfrak{F}})$ such that the following conditions are satisfied.

(1) the abelian group H^R is a p-group, where $p = \mathrm{char}(R/R^{\mathcal{N}})$;

(2) in our usual notation, $\ker(h_R^R) = U]P_o^*$; and

(3) if $R_j = U_j P_{j,o}^* \in \mathbb{B}_{\mathfrak{F}}$ with $|U_j| \leq |U|$ and $R_j \not\equiv R$, then $R_j\, d_{R_j}^R = 1$.

For each $R \in \mathbb{B}_{\mathfrak{F}}$, let $\hat{h}^R$ denote the homomorphism $\Delta \to H^R$ constructed from (H^R, h^R) as in (4.1.10), and let $K^R = \ker(h^R) \leq \Delta$.

Then if $G \in \mathfrak{F}$ and $e_G \in \mathrm{Qnat}(G, \Delta)$, we have

(a) $Ge_G \cap \Lambda = Ge_G \cap D$, where

$D = \bigcap \{ K^R \leq \Delta : R \in \mathbb{B}_{\mathfrak{F}}^p$ where $p \mid |G/G^{\mathcal{N}}|$ and $|R^{\mathcal{N}}| \leq |G^{\mathcal{N}}| \} \leq \Delta$;

and

(b) $G_{\mathfrak{F}_*} = \bigcap \{ \ker h_G^{R,\mathfrak{F}} : R \in \mathbb{B}_{\mathfrak{F}}^p$ where $p \mid |G/G^{\mathcal{N}}|$ and $|R^{\mathcal{N}}| \leq |G^{\mathcal{N}}| \}$.

<u>Proof</u>. (a) By (4.1.10), $\Lambda \leq K^R$ for all $R \in \mathbb{B}_{\mathfrak{F}}$, and so $Ge_G \cap \Lambda \leq Ge_G \cap D$.

Suppose that $g \in G$ is such that $g\, e_G \notin Ge_G \cap \Lambda$. By (4.4.3a) , $g\, e_G \in \Lambda \Theta^w$, where $w = |G^{\mathcal{N}}|$. Since $g\, e_G \notin \Lambda$, then by (4.4.6) there exist distinct $R_1, \ldots, R_n \in \mathbb{B}_{\mathfrak{F}}$ (where $n \geq 1$) with $|R_i^{\mathcal{N}}| \leq w$, together with elements $x_i \in R_i \setminus U_i P_{i,o}^*$ such that

$$g\, e_G \equiv x_1 e_1 \cdots x_n e_n \pmod{\Lambda} \quad , \text{ where } e_i \in \mathrm{Qnat}(P_i, \Delta) \ . \quad \cdots (8)$$

Suppose without loss of generality that $|R_n^{\mathcal{N}}| \geq |R_i^{\mathcal{N}}|$ for $i = 1, \ldots, n$. Let $R = U]P_o^*$ denote R_n . Then for all R_j , $j = 1, \ldots, n-1$, we have $R_j\, h_{R_j}^R = 1$, by hypothesis (3), and so $R_j e_j \leq K^R = \ker(\hat{h}^R)$.

Now, $\hat{h}^R$ is a homomorphism of Δ with kernel containing Λ , and so by (8) and the definition of $\hat{h}^R$, we have

$$(g\, e_G)\hat{h}^R = (x_1 e_1)\hat{h}^R \cdots (x_{n-1}\, e_{n-1})\hat{h}^R\, (x_n e_n)\hat{h}^R$$
$$= 1 \quad \cdots \quad 1 \quad (x_n e_n)\hat{h}^R \quad , \qquad \cdots(9)$$

since $R_j e_j \leq K^R$, $j = 1,..,n$.

Since $x_n \in R \setminus U]P_o^*$ and $\ker(h_R^R) = U\, P_o^*$ by hypothesis (2), then $(x_n e_n)\hat{h}^R \neq 1$. It follows that $g\, e_G \notin K^R$, by (9). Now, since $1 \neq (g\, e_G)\hat{h}^R = g\, h_G^R$, by (4.1.10), then h_G^R is a non-trivial homomorphism of G into $H^R \in \mathcal{A} \cap \mathcal{S}_p$, where $p = \mathrm{char}(R/R^{\mathcal{N}})$ by hypothesis (1) . It follows that $p \mid |G/G^{\mathcal{N}}|$. Since $|R^{\mathcal{N}}| \leq w = |G^{\mathcal{N}}|$, by above , it follows that $g\, e_G \notin D$. The result follows.

(b) This follows from (a), since $(G_{\mathfrak{F}_*})e_G = Ge_G \cap \Lambda$ by (4.1.9).

We now come to our version of Berger's theorem.

<u>4.4.8 Theorem (Berger. [4])</u>. Let $\mathfrak{F}$ be a Fischer class and let $G \in \mathfrak{F}$. Then

$$G_{\mathfrak{F}_*} = \bigcap \{ \ker d^{R,\mathfrak{F}}_{G \to P^*/P_o^*} : R = U]P^* \in \mathbb{B}^p_{\mathfrak{F}} \text{ , with } p \mid |G/G^{\mathcal{N}}| \text{ and } R^{\mathcal{N}} \text{ isn } G^{\mathcal{N}} \},$$

where the $(P^*/P_o^*, d^{R,\mathfrak{F}})$ are the $\mathfrak{F}$-Fitting pairs of (4.3.13), and where " $R^{\mathcal{N}}$ isn $G^{\mathcal{N}}$ " means " $R^{\mathcal{N}}$ is isomorphic to a subnormal subgroup of $G^{\mathcal{N}}$ " .

<u>Proof</u>. As remarked in (4.3.14b), we may construct the $\mathfrak{F}$-Fitting pair $(P^*/P_o^*, d^{R,\mathfrak{F}})$ for all $R \in \mathbb{B}_{\mathfrak{F}} \subseteq \mathbb{R}_{\mathfrak{F}}$. But then hypotheses (1) , (2) and (3) of (4.4.7) are satisfied for $(P^*/P_o^*, d^{R,\mathfrak{F}})$ because of parts (a) , (b) and (d) , respectively, of (4.3.12), and by (4.3.12) we have

$$G_{\mathfrak{F}_*} = \bigcap \{ \ker d^{R,\mathfrak{F}}_{G \to P^*/P^*_o} : R \in \mathbb{B}^p_{\mathfrak{F}} \text{ with } p \mid G/G^{\mathcal{N}} | \text{ and } |R^{\mathcal{N}}| \leq |G^{\mathcal{N}}| \}.$$

If $R \in \mathbb{B}_{\mathfrak{F}}$ is such that $\mathcal{C}(R^{\mathcal{N}},G) = \phi$, then by (4.3.12c) we have $G \quad \ker d^{R,\mathfrak{F}}_{G \to P^*/P^*_o}$, and so we need only take the above intersection over those $R \in \mathbb{B}_{\mathfrak{F}}$ with $\mathcal{C}(R^{\mathcal{N}},G) \neq \phi$. Now, if $X \in \mathcal{C}(R^{\mathcal{N}},G)$, where $R \in \mathbb{B}^p_{\mathfrak{F}}$ with $p \mid |G/G^{\mathcal{N}}|$, then, since $X = O^p(X)$, we must have $X \leq G^{\mathcal{N}}$; that is, $X \in \mathcal{C}(R^{\mathcal{N}},G^{\mathcal{N}})$. Thus, we need only take the intersection over those $R \in \mathbb{B}_{\mathfrak{F}}$ for which $\mathcal{C}(R^{\mathcal{N}},G^{\mathcal{N}}) \neq \phi$; that is, for which $R^{\mathcal{N}}$ isn $G^{\mathcal{N}}$. This completes the proof of Berger's theorem.

4.5 Some reflections on Berger's theorem.

It would be of interest to know whether $\mathfrak{F}$-Fitting pairs satisfying the hypotheses of (4.4.7) for all groups $R \in \mathbb{B}_{\mathfrak{F}}$ could be found for an arbitrary Fitting class $\mathfrak{F}$ (or even for an arbitrary Lockett class), and, if so, whether "transfer-type" maps could be used for this. We will see later that there do exist Fitting classes which are not Fischer classes for which our transfer-type maps do indeed suffice : these examples will depend on the fact that the associated classes of $\mathfrak{F}$-basic groups are in some sense "small".

In this section, we offer a first step towards this problem by proving a necessary condition on the internal structure of the group Δ for $\mathfrak{F}$-Fitting pairs satisfying the hypotheses of (4.4.7) for all $R \in \mathbb{B}_{\mathfrak{F}}$ to exist : basically, the condition is that the expression of elements given in (4.4.6) be in a certain sense unique.

Thus, let $\mathfrak{F}$ be a Fitting class with underlying set $\mathbb{F}$, let the group Δ be constructed as in (4.1.3), and let the general notation be

carried over from the preceding sections. If $\xi \in \Delta$, then, since Δ is the restricted direct product, there exists a finite subset $\{G_j\}_{j \in \underline{J}}$ of $\mathbb{F}$ for which $\xi \in \prod_j G_j \varepsilon_j$, and so if we take $w = \max(|G_j|_{j \in \underline{J}})$, then by (4.4.3a) and (4.4.2a) we have $\xi \in \Lambda\Theta^w$. We may thus apply (4.4.6) to obtain an expression (mod Λ) for ξ in terms of elements from $\mathfrak{F}$-basic groups.

<u>4.5.1 Proposition</u>. Suppose that for each $R = U]P^* \in \mathbb{B}_{\mathfrak{F}}$ there exists an $\mathfrak{F}$-Fitting pair $(H^R, h^R) = (H^{R,\mathfrak{F}}, h^{R,\mathfrak{F}})$ satisfying the hypotheses ($\underline{1}$), ($\underline{2}$) and ($\underline{3}$) of (4.4.7). Suppose that $\xi \in \Delta \setminus \Lambda$ and that

$$\begin{aligned} \xi &\equiv x_{11}e_{11} \cdots x_{1n}e_{1n} \pmod{\Lambda} \\ &\equiv x_{21}e_{21} \cdots x_{2m}e_{2m} \pmod{\Lambda}, \end{aligned}$$

where $R_{ij} = U_{ij}]P^*_{ij} \in \mathbb{B}_{\mathfrak{F}}$, $e_{ij} \in \mathrm{Qnat}(R_{ij}, \Delta)$, $x_{ij} \in R_{ij} \setminus U_{ij}]P^*_{ij,o}$ and $R_{ij} \neq R_{ik}$ if $j \neq k$ for fixed i.

Then, $n = m$, $\{R_{1j}\} = \{R_{2k}\}$ and, if $R_{1j} = R_{2k} \in \mathbb{B}_{\mathfrak{F}}$, we have $x_{1j} \equiv x_{2k}$ (mod $U_{1j}P^*_{1j,o}$).

<u>Proof</u>. If $i \in \{1,2\}$ and $j \in \{1, \ldots, \max(m,n)\}$ (as appropriate), let (H^{ij}, h^{ij}) denote the Fitting pair of the hypothesis associated with $R_{ij} \in \mathbb{B}_{\mathfrak{F}}$. Let h^{ij}_{ij} denote $h^{ij}_{R_{ij}\to}$, and let $\hat{h}^{ij}$ denote the corresponding homomorphism $\hat{h}^{ij} : \Delta \to H^{ij}$ as constructed in (4.1.10); we recall that $\ker \hat{h}^{ij} \geqslant \Lambda$.

We proceed by induction on $\ell = \max(m,n)$.

Suppose without loss of generality that $|R^{\mathcal{N}}_{1n}| \geqslant |R^{\mathcal{N}}_{ij}|$ for all appropriate i, j. Then if $R \in \{R_{ij}\}$ and $R \neq R_{1n}$, we have by

hypothesis (4.4.7)(3), together with (4.1.10), that

$$R\, h_R^{1n} = 1 \;, \quad \text{and} \quad R\, e_R \leq \ker \hat{h}^{1n} \;. \qquad \cdots(1)$$

Since $\ker \hat{h}^{1n} \geq \Lambda$, we thus have

$$\begin{aligned}
\xi\, \hat{h}^{1n} &= (x_{11}e_{11})\hat{h}^{1n} \cdots (x_{1,n-1}\, e_{1,n-1})\hat{h}^{1n}\, (x_{1n}e_{1n})\hat{h}^{1n} \\
&= \quad 1 \quad \cdots \quad 1 \quad (x_{1n}e_{1n})\hat{h}^{1n} \\
&= x_{1n}\, h_{1n}^{1n} \qquad \text{(by (4.1.10))} \\
&\neq 1 \qquad \text{(since } \ker h_{1n}^{1n} = U_{1n}\, P^*_{1n,o} \text{ by hypothesis (4.4.7)(2),} \\
&\qquad\qquad \text{and } x_{1n} \notin U_{1n}\, P^*_{1n,o} \text{ by choice)} .
\end{aligned}$$

But also

$$\xi\, \hat{h}^{1n} = (x_{21}e_{21})\hat{h}^{1n} \cdots (x_{2m}e_{2m})\hat{h}^{1n} .$$

Since the R_{2j} are pairwise distinct by hypothesis, then by equation (1) there must exist precisely one value of $j \in \{1,..,m\}$ such that $x_{2j}e_{2j}\, \hat{h}^{1n} = x_{1n}e_{1n}\, \hat{h}^{1n} \neq 1$ and $R_{2j} = R_{1n}$. Without loss of generality, we may assume that $R_{2m} = R_{1n}$, whence

$$x_{2m}e_{2m}\, \hat{h}^{1n} = x_{1n}e_{1n}\, \hat{h}^{1n} \quad \text{and} \quad x_{2m}\, h_{1n}^{1n} = x_{1n}\, h_{1n}^{1n} \;.$$

Since $\ker \hat{h}^{1n} \geq \Lambda$, and $\ker h_{1n}^{1n} = U_{1n}P^*_{1n,o}$ by (4.4.7)(2) , we have

$$x_{2m}e_{2m} \equiv x_{1n}e_{1n} \pmod{\Lambda} \quad \text{and} \quad x_{2m} \equiv x_{1n} \pmod{U_{1n}\, P^*_{1n,o}} \;.$$

We now have

$$\xi_1 = x_{11}e_{11} \cdots x_{1,n-1}\, e_{1,n-1} \equiv x_{21}e_{21} \cdots x_{2,m-1}\, e_{2,m-1} \pmod{\Lambda},$$

and $\xi \equiv \xi_1 x_{1n}e_{1n} \pmod{\Lambda}$.

Now by induction, we either have $\xi_1 \in \Lambda$, or else $n-1 = m-1$, $\{R_{1i}\} = \{R_{2j}\}$ $(i,j \leq n-1)$ and $x_{1i} \equiv x_{2j} \pmod{U_{1i}P^*_{1i,o}}$ if $R_{1i} = R_{2j}$.

Suppose that $\xi_1 \in \Lambda$ and that $n > 1$. Without loss of generality, suppose that $|R_{11}^{\mathcal{N}}| \geq |R_{1i}^{\mathcal{N}}|$ if $1 \leq i \leq n-1$. By (2) and (3) of (4.4.7) and the fact that $x_{11} \notin U_{11} P_{11,o}^*$, then

$$1 \neq x_{11}h_{11}^{11} = x_{11}e_{11}\hat{h}^{11} = \xi_1\hat{h}^{11} = 1 ,$$

since $\xi_1 \in \Lambda$. This is a contradiction, and so if $\xi_1 \in \Lambda$, then $n = m = 1$. The result now follows, the initial case $\ell = n = m = 1$ of the induction being dealt with as in the main analysis.

4.6 Generation of Fitting classes.

In this section we explore consequences of (4.4.3b) concerning the generation of Fitting classes. This will enable us to show in a later section that a certain Fitting class defined by Hawkes [40] is generated by a single group.

Whereas in the proof of Berger's theorem we focussed attention on the $\mathfrak{F}$-basic groups, for the purposes of generation we consider the "$\mathfrak{F}$-constructive groups", another subclass of the $\mathfrak{F}$-relevant groups.

4.6.1 Definition. Let $\mathfrak{F}$ be a Fitting class and $\mathcal{R}_{\mathfrak{F}}$ be the class of $\mathfrak{F}$-relevant groups (see (4.2.1). Let the underlying set $\mathbb{R}_{\mathfrak{F}}$ for $\mathcal{R}_{\mathfrak{F}}$ be chosen as in (4.4.1b), and let notation be as in (4.4.1b).

(a) If $R = U]P^* \in \mathbb{R}_{\mathfrak{F}}$, define

$$P_\infty^* = \langle \alpha \in P^* : [U, \langle\alpha\rangle] \nleq U \rangle \qquad (\leq P_o^* \text{ of } (4.4.4)).$$

(b) Define the class $\mathcal{C}_{\mathfrak{F}}$ of <u>$\mathfrak{F}$-constructive groups</u> as

$$\mathcal{C}_{\mathfrak{F}} = (G \in \mathcal{R}_{\mathfrak{F}} : \text{if } G \cong R = U]P^* \in \mathbb{R}_{\mathfrak{F}}, \text{ then } P_\infty^* \nleq P^*) \subseteq \mathcal{R}_{\mathfrak{F}} .$$

<u>4.6.2 Remarks</u>. (a) We may easily check that P^*_∞ is normal in P^*.

(b) We note that $\mathcal{B}_{\mathfrak{F}} \subseteq \mathcal{C}_{\mathfrak{F}} \subseteq \mathcal{R}_{\mathfrak{F}}$ where $\mathcal{B}_{\mathfrak{F}}$ is as in (4.4.5).

(c) We will take the induced underlying set $\mathbb{C}_{\mathfrak{F}} = \mathcal{C}_{\mathfrak{F}} \cap \mathbb{R}_{\mathfrak{F}}$, where $\mathbb{R}_{\mathfrak{F}}$ is as in (4.2.1b). We will adhere to the notation of (4.2.1b).

<u>4.6.3 Lemma</u>. Suppose that $R \in \mathbb{R}_{\mathfrak{F}}$. Then $R \in \langle \mathcal{Y}_R \rangle_{\mathrm{Fitt}}$, where

$$\mathcal{Y}_R = (B \in \mathfrak{F} : B' = B \text{ and } B \text{ isn } R^{\mathcal{N}}) \cup (C_p : p \in \mathrm{comp}(R))$$
$$\cup (R_\delta \in \mathbb{C}_{\mathfrak{F}} : R_\delta^{\mathcal{N}} \text{ isn } R^{\mathcal{N}} \text{ and } \mathrm{comp}(R_\delta) \subseteq \mathrm{comp}(R)) .$$

<u>Proof</u>. We argue by induction on $|R^{\mathcal{N}}|$, and note that by (4.4.2b) there are only finitely many $\mathfrak{F}$-relevant groups which have nilpotent residuals of lesser order.

We may assume that $R \notin \mathbb{C}_{\mathfrak{F}}$, and that $R = U \rtimes P^*$ in our usual notation. Then since $R \in \mathbb{R}_{\mathfrak{F}} \setminus \mathbb{C}_{\mathfrak{F}}$, we have $P^*_\infty = P^*$; thus $P^* = \langle \alpha \in P^* : [U, \langle\alpha\rangle] \lneqq U \rangle$.

If $\alpha \in P^*$, let A_α denote $[U, \langle\alpha\rangle]\langle\alpha\rangle \leq R$, and let

$$T = \langle A_\alpha : \alpha \in P^* \text{ with } [U, \langle\alpha\rangle] \lneqq U \rangle \leq R .$$

Since $P^* = P^*_\infty$, then $T \geqslant P^*$. By (1.1.3a), $[U, \langle\alpha\rangle] \trianglelefteq U\langle\alpha\rangle$, and so A_α is normalized by U. But then T must be normalized by U, and, since $P^* \leq T$, it follows that $T \trianglelefteq U P^* = R$. Since $R \in \mathbb{R}_{\mathfrak{F}}$ then $U = [U, P^*] \leq T \trianglelefteq R$, and so $T = U P^* = R$. But A_α sn R since $A_\alpha \trianglelefteq U\langle\alpha\rangle$ sn $U P^* = R$, and so

$$R \in N_0(A_\alpha : \alpha \in P^* \text{ with } [U, \langle\alpha\rangle] \lneqq U) . \qquad \ldots(1)$$

Now suppose that $\alpha \in P^*$ with $[U, \langle\alpha\rangle] \lneqq U$. Then we have

$$A_\alpha^{\mathcal{N}} \leq [U, \langle\alpha\rangle] \lneqq U = R^{\mathcal{N}} , \qquad \ldots(2)$$

and by (4.4.3b), we have

$$A_\alpha \in \langle \mathcal{Y}_\alpha \rangle_{\text{Fitt}} \quad , \text{ where}$$

$$\mathcal{Y}_\alpha = (E \in \mathfrak{F} : E' = E \text{ and } E \text{ isn } A_\alpha^{\mathcal{N}}) \cup (C_p : p \in \text{comp}(A_\alpha))$$

$$\cup \bigcup_\beta (R_{\alpha\beta} \in \mathbb{R}_{\mathfrak{F}} : R_{\alpha\beta}^{\mathcal{N}} \text{ isn } A_\alpha^{\mathcal{N}} \text{ and } \text{comp}(R_{\alpha\beta}) \subseteq \text{comp}(A_\alpha)) . \quad \cdots(3)$$

Since $R_{\alpha\beta}^{\mathcal{N}}$ isn $A_\alpha^{\mathcal{N}}$ and $|A_\alpha^{\mathcal{N}}| < |R^{\mathcal{N}}|$ by (2), then $|R_{\alpha\beta}^{\mathcal{N}}| < |R^{\mathcal{N}}|$, and by induction we have for all appropriate α and β,

$$R_{\alpha\beta} \in \langle \mathcal{Y}_{\alpha\beta} \rangle_{\text{Fitt}} \quad , \text{ where}$$

$$\mathcal{Y}_{\alpha\beta} = (D \in \mathfrak{F} : D' = D \text{ and } D \text{ isn } R_{\alpha\beta}^{\mathcal{N}}) \cup (C_p : p \in \text{comp}(R_{\alpha\beta}))$$

$$\cup \bigcup_\gamma (R_{\alpha\beta\gamma} \in \mathbb{C}_{\mathfrak{F}} : R_{\alpha\beta\gamma}^{\mathcal{N}} \text{ isn } R_{\alpha\beta}^{\mathcal{N}} \text{ and } \text{comp}(R_{\alpha\beta\gamma}) \subseteq \text{comp}(R_{\alpha\beta})) . \quad \cdots(4)$$

But $A_\alpha^{\mathcal{N}}$ sn $R^{\mathcal{N}}$, and it follows by (3) and (4) that $R_{\alpha\beta\gamma}^{\mathcal{N}}$ isn $R^{\mathcal{N}}$, for all appropriate α, β and γ. Since A_α sn R (by the remark preceding (1)), then $\text{comp}(A_\alpha) \subseteq \text{comp}(R)$, and so again by (3) and (4), we have $\text{comp}(R_{\alpha\beta\gamma}) \subseteq \text{comp}(R_{\alpha\beta}) \subseteq \text{comp}(A_\alpha) \subseteq \text{comp}(R)$ for all appropriate α, β and γ. Combining (1), (3) and (4) and rationalising notation, we obtain the assertion of the proposition.

<u>4.6.4 Proposition</u>. Let $\mathfrak{F}$ be a Fitting class. Then $\mathfrak{F} = \langle \mathcal{X}_{\mathfrak{F}} \rangle_{\text{Fitt}}$, where $\mathcal{X}_{\mathfrak{F}} = (A \in \mathfrak{F} : A' = A) \cup (C_p : p \in \text{char}(\mathfrak{F})) \cup \mathcal{C}_{\mathfrak{F}}$

<u>Proof</u>. The assertion follows at once from (4.4.3b) and (4.6.3).

<u>4.6.5 Remarks</u>. It is natural to enquire as to the extent of redundancy in the generating class $\mathcal{X}_{\mathfrak{F}}$ of (4.6.4). There is the obvious improvement to be made by requiring that the groups A with $A' = A$ be indecomposable, of course, but we show that, in general, none of the three classes comprising $\mathcal{X}_{\mathfrak{F}}$ can be omitted without destroying the generating property.

(a) Let $\mathfrak{F}_1 = \langle \mathrm{Alt}(5) \rangle_{\mathrm{Fitt}} = D_0(\mathrm{Alt}(5))$ (see (1.3.15)) .
Now, $\mathrm{char}(\mathfrak{F}_1) = \phi$, since no group in $\mathfrak{F}_1$ has an abelian composition factor, while $\mathcal{C}_{\mathfrak{F}_1} = \phi$ since any $\mathfrak{F}_1$-relevant group R must satisfy $1 \lneq R^{\mathcal{N}} \lneq R$. Thus we cannot omit the class $(A \in \mathfrak{F}_1 : A' = A)$ from $\mathcal{X}_{\mathfrak{F}_1}$ without destroying the generating property of $\mathcal{X}_{\mathfrak{F}_1}$.

(b) Let $\mathfrak{F}_2 = \mathcal{N}_\pi$, where $\pi \subseteq \mathbb{P}$. There can be no perfect groups in $\mathfrak{F}_2$, while again $\mathcal{C}_{\mathfrak{F}_2} = \phi$; thus we cannot omit $(C_p : p \in \pi)$ from $\mathcal{X}_{\mathfrak{F}_2}$ without destroying the generating property.

(c) Let $\mathfrak{F}_3 = \langle \mathrm{Sym}(3) \rangle_{\mathrm{Fitt}}$; there are no perfect groups in $\mathfrak{F}_3$, while $\mathfrak{F}_3$ cannot be generated by its cyclic groups, and in this case we cannot omit $\mathcal{C}_{\mathfrak{F}_3}$ from $\mathcal{X}_{\mathfrak{F}_3}$ without destroying the generating property. We note that $\mathrm{Sym}(3) \in \mathcal{C}_{\mathfrak{F}_3}$

(d) Even if $\mathfrak{F} = \langle \mathcal{C}_{\mathfrak{F}} \rangle_{\mathrm{Fitt}}$, it is not necessary that $\mathcal{C}_{\mathfrak{F}}$ be a minimal generating set. Let $\mathfrak{F} = \mathcal{S}_3 * \mathcal{S}_2$; it is evident that $\mathrm{Sym}(3) \in \mathcal{C}_{\mathfrak{F}}$, and so $\mathfrak{F} = \langle \mathcal{C}_{\mathfrak{F}} \rangle_{\mathrm{Fitt}}$ since $2 , 3 \nmid |\mathrm{Sym}(3)|$

Let U be an elementary abelian group of order 3^2 . We identify Aut(U) with GL(2,3). Now, GL(2,3) has Sylow 2-subgroups of order 16, while SL(2,3) has a unique Sylow 2-subgroup Q , of order 8 , and Q is normal in GL(2,3). Let us fix the following notation for elements of GL(2,3).

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \quad L = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

and let $-I$, $-L$, and so on, have the obvious meanings.

Since $J \notin \mathrm{SL}(2,3)$, then $P = \langle Q , J \rangle \in \mathrm{Syl}_2(\mathrm{GL}(2,3))$.

We form the abstract semi-direct product $U]P \leq U]\mathrm{Aut}(U)$; then $UP \in \mathfrak{F}$ and so $P^* = P$ in our customary notation. Since $-I \in Q \leq P$, and $[U, \langle -I \rangle] = U$, then $[U, P] = U$ and UP is $(\mathfrak{F},2)$-relevant.

The element K of Q is inverted by J, and so $D = \langle K, J \rangle$ is dihedral of order 8. Direct calculation shows that D contains the elements L and $-L$, and that J interchanges these two elements. Thus, L and J generate D (in its guise as $C_2 \text{ wr } C_2$). Now, both L and J have an eigenvalue 1, and so if M denotes L or J, then $[U, \langle M \rangle] \lneq U$; it follows that $D \leq \langle \alpha \in P : [U, \langle \alpha \rangle] \lneq U \rangle = P_\infty$. Further calculation reveals that if $\beta \in P \setminus D$, then a suitable power of β coincides with $-I$. It follows that if $\beta \in P \setminus D$ then $[U, \langle \beta \rangle] = U$, and we conclude that $P_\infty = D \lneq P$, and that $UP \in \mathcal{C}_{\mathfrak{F}}$.

Now, $\mathrm{Sym}(3) \times C_3 \cong U\langle L \rangle$ sn UP, and so $\mathrm{Sym}(3)$, which is itself $\mathfrak{F}$-constructive, belongs to $\langle UP \rangle_{\mathrm{Fitt}}$; thus $\mathrm{Sym}(3)$ may be omitted from $\mathcal{C}_{\mathfrak{F}}$ without destroying the generating property of $\mathcal{C}_{\mathfrak{F}}$.

We note that $UP \in \mathcal{C}_{\mathfrak{F}} \setminus \mathcal{B}_{\mathfrak{F}}$. For by the Focal Subgroup Theorem ([28; 7.3.4]), $\{P ; \mathrm{Aut}(U)\} = P \cap (\mathrm{Aut}(U))'$, since $P \in \mathrm{Syl}_2(\mathrm{Aut}(U))$. But $P \cap (\mathrm{Aut}(U))' = P \cap SL(2,3) = Q$. Since $P_\infty = D$ and $P = DQ$ (as $D \notin SL(2,3)$), then $P = P_\infty\{P ; \mathrm{Aut}(U)\} = P_0$, in the notation of (4.4.4), and so UP is not $(\mathfrak{F},2)$-basic (nor, of course, $(\mathfrak{F},q)$-basic for any other $q \in \mathbb{P}$).

4.7 Fischer classes and the Lockett conjecture.

In this section, we show how Berger's theorem (4.4.8) can be used for investigations into the Lockett conjecture (see (1.5.14)). We start by giving a general lemma (4.7.1), and go on to use this lemma to prove

a variation of the crucial lemma [10; 4.7] of the Bryce-Cossey proof of the Lockett conjecture for primitive saturated formations. We note that Berger [3] has given an alternative proof of the theorem of Bryce and Cossey; Berger's proof does not rely on Berger's theorem from [4] (that is, our (4.4.8)), but does use transfer-map techniques.

<u>4.7.1 Lemma</u>. Let $\mathfrak{F}$ and $\mathfrak{G}$ be Fischer classes and let π be a set of primes. Suppose that $\mathfrak{F} \subseteq \mathfrak{G}$ and that $\mathcal{B}^p_{\mathfrak{F}} \subseteq \mathcal{B}^p_{\mathfrak{G}}$ for all $p \in \pi$ (see (4.4.5)). Then $\mathfrak{F}_* \subseteq \mathfrak{F} \cap \mathfrak{G}_* \subseteq \mathfrak{F}_* * \mathcal{S}_{\pi'}$.

<u>Proof</u>. Let $\mathbb{B}_{\mathfrak{G}}$ be an underlying set for $\mathcal{B}_{\mathfrak{G}}$, chosen in the manner of (4.4.1b) and (4.4.5), and let $\mathbb{B}_{\mathfrak{F}} = \mathcal{B}_{\mathfrak{F}} \cap \mathbb{B}_{\mathfrak{G}}$, so that $\mathbb{B}_{\mathfrak{F}}$ is an underlying set for $\mathcal{B}_{\mathfrak{F}} \subseteq \mathcal{B}_{\mathfrak{G}}$.

By (1.5.9), we have $\mathfrak{F}_* \subseteq \mathfrak{F} \cap \mathfrak{G}_*$. Suppose for a contradiction that G is a group of minimal order in $(\mathfrak{F} \cap \mathfrak{G}_*) \setminus (\mathfrak{F}_* * \mathcal{S}_{\pi'})$. Then G has a unique maximal normal subgroup $M = G_{(\mathfrak{F}_* * \mathcal{S}_{\pi'})}$. If G/M is non-abelian, then $G = G' \in \mathfrak{F}_*$ (by (1.5.3c)), and so $G/M \cong C_p$ for some $p \in \mathbb{P}$ and $G/G^{\mathcal{N}} \in \mathcal{S}_p$, by (1.3.10). Since $G/G_{\mathfrak{F}_*}$ is thus a p-group, we must have $p \in \pi$.

By (4.4.8), since $\mathfrak{F}$ is a Fischer class and $G/G^{\mathcal{N}} \in \mathcal{S}_p$, we have

$$G_{\mathfrak{F}_*} = \bigcap \{ \ker d^{R,\mathfrak{F}}_{G \to P^*/P^*_o} : R = U]P^* \in \mathbb{B}^p_{\mathfrak{F}} \text{, with } R^{\mathcal{N}} \text{ isn } G^{\mathcal{N}} \} \lneqq G .$$

Thus there exists $R = U]P^* \in \mathbb{B}^p_{\mathfrak{F}}$ and an element $g \in G$ with $g\, d^{R,\mathfrak{F}}_{G \to P^*/P^*_o} \neq 1$. But then $g\, d^{R,\mathfrak{G}}_{G \to P^*/P^*_o} \neq 1$, by (4.3.12e), since $R \in \mathbb{B}^p_{\mathfrak{F}} \subseteq \mathbb{B}^p_{\mathfrak{G}}$, and $\mathfrak{G}$ is a Fischer class.

But by (4.3.13), $(P^*/P_0^*, d^{R,\mathcal{G}})$ is a $\mathcal{G}$-Fitting pair, and it follows that $G \notin \mathcal{G}_*$, contrary to choice. This completes the proof.

On taking $\pi = \mathbb{P}$ in the above result, we obtain the following.

<u>4.7.2 Corollary</u>. Let $\mathcal{F}$ and $\mathcal{G}$ be Fischer classes with $\mathcal{F} \subseteq \mathcal{G}$. Suppose that $\mathcal{B}_{\mathcal{F}} \subseteq \mathcal{B}_{\mathcal{G}}$. Then $\mathcal{F}_* = \mathcal{F} \cap \mathcal{G}_*$.

<u>4.7.3 Proposition (c.f. Bryce and Cossey, [10; 4.2 and 4.7])</u>.

Let $\mathcal{X}$ and $\mathcal{Y}$ be Fischer classes and let π be a set of primes. If $(\mathcal{X} \cap \mathcal{Y})_* = \mathcal{X} \cap \mathcal{Y}_*$, then $((\mathcal{X} * \mathcal{S}_\pi) \cap \mathcal{Y})_* = (\mathcal{X} * \mathcal{S}_\pi) \cap \mathcal{Y}_*$.

<u>Proof</u>. By (1.3.14a), $\mathcal{X} * \mathcal{S}_\pi$ is a Fischer class, and it follows that $\mathcal{F} = (\mathcal{X} * \mathcal{S}_\pi) \cap \mathcal{Y}$ is also a Fischer class.

Let $p \in \pi$ and let $R \in \mathcal{B}_{\mathcal{F}}^p$, with $R = U]P^*$ in our usual notation, so that $P^* = P \cap (U]P)_{\mathcal{F}}$ for some suitable $P \in \mathrm{Syl}_p(\mathrm{Aut}(U))$. It is evident that $(U]P)_{\mathcal{F}} = (U]P)_{((\mathcal{X} * \mathcal{S}_\pi) \cap \mathcal{Y})} = ((U]P)_{\mathcal{Y}})_{(\mathcal{X} * \mathcal{S}_\pi)}$. Since $p \in \pi$, it follows that $(U\ P)_{\mathcal{F}} = (U\ P)_{\mathcal{Y}}$, and so we have $R = U]P^* \in \mathcal{B}_{\mathcal{Y}}^p$. Since $P_0^* \lneqq P^*$, as $R \in \mathcal{B}_{\mathcal{F}}^p$, then $R \in \mathcal{B}_{\mathcal{Y}}^p$. Thus $\mathcal{B}_{\mathcal{F}}^p \subseteq \mathcal{B}_{\mathcal{Y}}^p$ for all $p \in \pi$, and so by (4.7.1) we have

$$\mathcal{F}_* \subseteq \mathcal{F} \cap \mathcal{Y}_* \subseteq \mathcal{F}_* * \mathcal{S}_{\pi'} . \qquad \cdots(1)$$

Now suppose that G is a group of minimal order in $(\mathcal{F} \cap \mathcal{Y}_*) \setminus \mathcal{F}_*$. Then G has a unique maximal normal subgroup $M = G_{\mathcal{F}_*}$. Since $\mathcal{F} \cap \mathcal{Y}_* \subseteq \mathcal{F}_* * \mathcal{S}_{\pi'}$, by (1), then $|G : G_{\mathcal{F}_*}| = q \in \pi'$. But $\mathcal{F} \subseteq \mathcal{X} * \mathcal{S}_\pi$, and since G is single-headed and $q \notin \pi$, we have $G \in \mathcal{X} \cap \mathcal{Y}_*$.

Thus by hypothesis, $G \in (\mathfrak{X} \cap \mathcal{Y})_* \subseteq ((\mathfrak{X} * \mathcal{S}_\pi) \cap \mathcal{Y})_* = \mathfrak{Z}_*$, contrary to choice. Thus $\mathfrak{Z}_* = \mathfrak{Z} \cap \mathcal{Y}_*$, and the result follows.

The above result can be regarded as a version of [10; 4.7] . In fact, the lemma proved by Bryce and Cossey is somewhat different to the above, and is proved for classes $\mathfrak{X}, \mathcal{Y} \subseteq \mathcal{S}$, where $\mathfrak{X}$ is otherwise arbitrary and where $\mathcal{Y}$ is subgroup-closed (although this latter condition is missing from [10; 4.7] as it stands). Bryce and Cossey use their result to deduce [10; 4.15] that if $\mathcal{Y} \subseteq \mathcal{S}$ is subgroup-closed, then $\mathcal{Y}_*$ is Hall-closed. (We cannot, thus, expect to obtain a word-for-word generalisation of their result to the case that $\mathcal{Y} \subseteq \mathcal{S}$ is a Fischer class, since by (2.4.5) and (2.8.5), $e_\pi(\mathcal{N})_*$ is not Hall-closed if $\emptyset \neq \pi \subsetneq P$, while $e_\pi(\mathcal{N})$ is a Fischer class (see (1.3.17))) .

Using (4.7.3) in place of [10; 4.7] , we may now work through the remaining arguments in the Bryce-Cossey proof of the Lockett conjecture for primitive saturated formations, making only fairly minor alterations, and obtain the following (which is new inasmuch as it refers to Fischer classes $\mathcal{Y}$ instead of subgroup-closed classes). We refer for the details to [10; 2.3, 4.16, 4.17] .

<u>4.7.4 Theorem (Bryce and Cossey, 10; 4.17)</u>. Let $\mathfrak{X}$ be a primitive saturated formation (so that $\mathfrak{X} \subseteq \mathcal{S}$), and let $\mathcal{Y}$ be a Fischer class (of not-necessarily-soluble groups). Then $\mathfrak{X} \cap \mathcal{Y}_* = (\mathfrak{X} \cap \mathcal{Y})_*$.

We will now use (4.7.2) to verify that a Fischer class defined by Hawkes, [39] , satisfies the Lockett conjecture.

<u>4.7.5 Definitions (Hawkes, [39])</u>. Let E denote an extra-special

group of order 5^3 and exponent 5, and let N denote an irreducible $GF(11)\,E$-module, faithful for E. Since $5 \mid (11-1)$, N has degree 5 and is absolutely irreducible (see [41; V.16.14]). Let $B = N]E$.

Define $\mathfrak{X} = \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_3 * \mathcal{S}_2$;

$$\mathcal{Y} = \mathcal{S}_{11} * \mathcal{S}_3 * \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_2 * \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_3 \quad ;$$

$$\mathcal{Z} = \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_2 * \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_3 \quad ; \text{ and}$$

$$\mathfrak{J} = (G \in \mathfrak{X} : \text{if } L = G^{\mathcal{Y}}, \text{ then } L/O_5(L) = \overline{B}_1 \times \cdots \times \overline{B}_t,\ t \geqslant 0,$$

where $\overline{B}_i \cong B$ for each i, and if $\overline{B}_i = B_i/O_5(L)$, then $B_i \trianglelefteq G^{\mathcal{Z}}$) .

<u>4.7.6 Theorem (Hawkes, [39])</u>. (a) The class $\mathfrak{J}$ defined above is a Q-closed Fischer class which is not a formation (and so not S-closed, as any $\langle S, D_o \rangle$-closed class is R_o-closed) (see [39; p.444, line -17]).

(b) We have $\mathcal{N}^3 \cap \mathfrak{X} \subseteq \mathfrak{J}$, in the notation of (4.7.5).

<u>4.7.7 Proposition</u>. Let $\mathfrak{J}$ be as defined above. Then $\mathfrak{J}_* = \mathfrak{J} \cap \mathcal{S}_*$.

<u>Proof</u>. Let $R \in \mathcal{B}^p_{\mathfrak{J}}$ with $R = U]P^*$ in our usual notation, so that $P^* = P \cap (U]P)_{\mathfrak{J}}$ for some suitable $P \in Syl_p(Aut(U))$, and $U = R^{\mathcal{N}}$.

Of course, $p \in char(\mathfrak{J}) = \{2, 3, 5, 11\}$.

Suppose firstly that $p \in \{3,5,11\}$. Since $R \in \mathfrak{J} \subseteq \mathfrak{X}$ and $R/R^{\mathcal{N}}$ is a p-group, we must have $R \in \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_3$, and so any extension of R by a p-group belongs to $\mathcal{N}^3 \cap \mathfrak{X} \subseteq \mathfrak{J}$ (see (4.7.6b)) Thus we have $U]P \in \mathfrak{J}$, and so $P^* = P$ and $R = U]P$. But now $R \in \mathcal{B}^p_{\mathcal{S}}$.

Next suppose that $p = 2$. Define the classes

$$\mathcal{D} = \mathcal{S}_{11} * \mathcal{S}_3 * \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_2 \quad ; \text{ and}$$

$$\mathcal{U} = \mathcal{S}_{11} * \mathcal{S}_5 * \mathcal{S}_2 \quad .$$

Since $U = R^{\mathcal{N}} = (U]P)^{\mathcal{N}}$, and $R/R^{\mathcal{N}}$ and $(U]P)/(U]P)^{\mathcal{N}}$ thus both belong to $\mathcal{S}_2$, we have

$(U]P^*)^{\mathcal{Y}} = R^{\mathcal{Y}} = R^{\mathcal{D}} = U^{\mathcal{D}} = (U]P)^{\mathcal{D}} = (U]P)^{\mathcal{Y}}$, and

$(U]P^*)^{\mathcal{Z}} = R^{\mathcal{Z}} = R^{\mathcal{U}} = U^{\mathcal{U}} = (U\ P)^{\mathcal{U}} = (U]P)^{\mathcal{Z}}$.

But now since $R = U]P^* \in \mathfrak{F}$, it follows from the definition of $\mathfrak{F}$ that $U]P \in \mathfrak{F}$, whence $R = (U]P)_{\mathfrak{F}} = (U]P)$, and again $R \in \mathcal{B}_{\mathcal{S}}^{p}$.

Since $\mathcal{B}_{\mathfrak{F}} = \cup(\mathcal{B}_{\mathfrak{F}}^{p} : p \in \{2,3,5,11\})$, then $\mathcal{B}_{\mathfrak{F}} \subseteq \mathcal{B}_{\mathcal{S}}$, and the assertion follows by (4.7.2).

It would be of interest if the Lockett conjecture could be shown to fail for a Fischer class ; one candidate for the class might be $e_p(\mathcal{N})$, for $p \in \mathbb{P}$, but we have been unable to either confirm or deny the Lockett conjecture in this case.

We note in passing that since the class $\mathfrak{F}$ of (4.7.5/6) satisfies $\mathfrak{F} \subseteq \mathfrak{X} \subseteq \mathcal{N}^4$ and $\mathfrak{X} \cap \mathcal{N}^3 \subseteq \mathfrak{F}$, then $\mathfrak{F}$ must be Hall-closed, thus providing an example of a Hall-closed Fischer class which is not S-closed.

The proof of (4.7.6a) is similar in nature to that of Dark's theorem (1.8.1), and we refer to [39] for the details.

4.8 A Fitting class due to Hawkes.

In this section, we investigate a meta-nilpotent Fitting class defined by Hawkes, [40], and show that it is generated by a single group.

We start by giving Hawkes' definitions and by quoting certain of his results.; we refer to [40] for the proofs.

4.8.1 Hypothesis (Hawkes, [40; 5.1]). Let p and q be distinct primes with $q \neq 2$. Let P be a p-group, and let $Q \in \mathrm{Syl}_q(\mathrm{Aut}(P))$. Assume that each of the following conditions holds :

(a) $P/Z(P)$ is directly indecomposable ;

(b) P has class $c \geqslant 3$;

(c) $\zeta_i(P) = \gamma_{c+1-i}(P)$ for $i = 1$ and 2 ;

(d) $|Q| = q$;

(e) Q acts fixed-point-freely on $P/Z(P)$; and

(f) $[Z(P), Q] = 1$,

where the ζ_i (γ_j) are the terms of the ascending (descending), respectively, central series, in the usual numbering system : $\zeta_0 = 1$, and $\gamma_1 = P$.

For the rest of this section, P and Q will denote fixed groups as above, satisfying the above conditions.

Hawkes, [40], remarks that one can indeed find concrete examples of such groups P and Q ; we will discuss this later.

4.8.2 Remark (Hawkes, [40; 5.2]). Let S be a q-group of operators for P . If $[P, S] \neq 1$, then $[P, S] = P$ and $|S/C_S(P)| = q$; in any case, $[Z(P), S] = 1$.

4.8.3 Definitions (Hawkes, [40; 5.3, 4.3, 5.5, 5.6]).

(a) The class $\mathfrak{X}$ consists of groups of order 1 together with all groups of the form $X = K]A$, where

(i) A is a q-group ;

(ii) K is a central product of A-invariant subgroups $P_1, \ldots, P_s$,

each isomorphic with P and satisfying $[P_i, A] = P_i$; and

(iii) $O_q(X) = 1$.

(b) With $\mathfrak{X}$ as in (a) , define

$$\mathfrak{F}_1 = (G \in \mathcal{S}_p * \mathcal{S}_q : O^p(G/O_q(G)) \in \mathfrak{X}) .$$

Whenever $X \in \mathfrak{X}$ and we write $X = K]A$, we shall implicitly assume that K and A have the meanings described in this definition,.

<u>4.8.4 Proposition (Hawkes, [40: 5.4])</u>. If $X = K]A \in \mathfrak{X}$, with $K = P_1 \cdots P_s$, as in (4.8.4), then A is an elementary abelian q-group and $C_K(A) = Z(K)$. Further (by a reading of the proof), $|A| \mid q^s$.

<u>4.8.5 Theorem (Hawkes, [40: 5.5, 5.6])</u>. The class $\mathfrak{F}_1$ of (4.8.3b) is a Fitting class.

<u>4.8.6 Lemma</u>. Let C be a central product of groups $B_1, \ldots, B_s$ (that is, $C = B_1 \cdot \cdots \cdot B_s$ with $[B_i, B_j] = 1$ if $i \neq j$).

(a) The group C is isomorphic to a factor group of the external direct product $D = B_1 \times \cdots \times B_s$ by a central subgroup.

(b) If $\hat{\alpha}_i \in \mathrm{Aut}(B_i)$ is such that $[Z(B_i), \langle\hat{\alpha}_i\rangle] = 1$, then $\hat{\alpha}_i$ can be "extended" to a (well-defined) automorphism α_i of C by defining

$$c\,\alpha_i = (b_1 \cdots b_s)\,\alpha_i := b_1 \cdots (b_i\,\hat{\alpha}_i) \cdots b_s \quad \text{if } c = b_1 \cdots b_s \in C,\ b_i \in B_i .$$

If $i, j \in \{1, \ldots, s\}$, then $B_j \leq C$ is $\langle\alpha_i\rangle$-invariant. Further, $[B_j, \langle\alpha_i\rangle] = 1$ if $i \neq j$, $[Z(C), \langle\alpha_i\rangle] = 1$, and α_i induces the automorphism $\hat{\alpha}_i$ on $B_i \leq C$; in particular, if $\hat{\alpha}_i \neq 1$, then $\alpha_i \neq 1$.

Proof. These facts are well-known. In (a), the map which takes the element $(b_1,\ldots,b_s) \in B_1 \times \cdots \times B_s$ to the element $b_1 \cdots b_s \in C$ turns out to be a homomorphism whose kernel is contained in $Z(B_1 \times \cdots \times B_s)$. Part (b) is a consequence of straightforward calculations, which we omit.

4.8.7 Determination of the $\mathfrak{F}_1$-relevant groups.

Suppose that $R = V]T^*$ is $(\mathfrak{F}_1, t)$-relevant for $t \in \mathrm{char}(\mathfrak{F}_1)$, where, in our usual notation, $V = O^t(V) = O^t(R) = R^{\mathcal{N}} = [V, T^*]$ and $T^* = T \cap (V]T)_{\mathfrak{F}_1}$ where $T \in \mathrm{Syl}_t(\mathrm{Aut}(V))$.

Since $\mathfrak{F}_1 \subseteq \mathcal{S}_p * \mathcal{S}_q$, we must in fact have $t = q$, $V = O^q(V) \in \mathrm{Syl}_p(R)$, and $T^* \in \mathrm{Syl}_q(R)$. Thus, since $T^* \leq \mathrm{Aut}(V)$, we have $O_q(R) = 1$, while since $V = R^{\mathcal{N}}$ we have $O^p(R) = R$.

It follows that $R \in \mathcal{X}$, by definition of $\mathfrak{F}_1$ and $\mathcal{X}$. Thus V is a central product of T^*-invariant subgroups $P_1, \ldots, P_s$, each isomorphic with P, where $s \geq 1$ and $[P_i, T^*] = P_i$, $i = 1,\ldots,s$. By (4.8.4), we have

$$T^* \text{ is elementary abelian with } |T^*| \mid q^s, \qquad \ldots(1)$$

while by (4.8.6a) and (4.8.1b) we have

$$P_i \lneq V \text{ if } s > 1. \qquad \ldots(2)$$

For each $i \in \{1,\ldots,s\}$, let $\hat{\alpha}_i$ be an automorphism of P_i of order q. By (4.8.2) we have $[P_i, \langle\hat{\alpha}_i\rangle] = P_i$. By (4.8.6), we may extend $\hat{\alpha}_i$ to an automorphism α_i of $V = P_1 \cdots P_s$, where $[P_j, \langle\alpha_i\rangle] = 1$ if $j \neq i$ and

$$[V, \langle\alpha_i\rangle] = [P_i, \langle\alpha_i\rangle] = P_i. \qquad \ldots(3)$$

Since $\alpha_i \neq 1$, by (4.8.6), we must have $|\alpha_i| = q$. It is evident from the definition of the $\{\alpha_i\}$ that if $i \neq j$ $(i,j \in \{1,..,s\})$ then $\alpha_i \neq \alpha_j$ and $[\alpha_i, \alpha_j] = 1$, and it follows that

$$S^* := \langle \alpha_i : i = 1,..,s \rangle \text{ is an elementary abelian subgroup of order } q^s \text{ of } \mathrm{Aut}(V). \qquad \cdots(4)$$

Since $S^* \leq \mathrm{Aut}(V)$, then $O_q(V]S^*) = 1$, and it follows from equation (3) that $V]S^* \in \mathcal{X}$. But $[V, S] \geq P_1 \cdots P_s = V$ (again because of (3)), and since $S^* \leq O^p(V]S^*) \trianglelefteq V]S^*$, it follows that $O^p(V]S^*) = V]S^*$. Thus $O^p(V]S^*/O_q(V]S^*)) = V]S^* \in \mathcal{X}$ and $V]S^* \in \mathfrak{F}_1$.

Let $S \in \mathrm{Syl}_q(\mathrm{Aut}(V))$ be chosen so that $S^* \leq S$. Since $V]S^*$ sn $V]S$ and $V]S^* \in \mathfrak{F}_1$, then

$$V]S^* \leq (V]S)_{\mathfrak{F}_1} \quad \text{and} \quad S^* \leq S \cap (V]S)_{\mathfrak{F}_1}. \qquad \cdots(5)$$

By Sylow's theorem in $\mathrm{Aut}(V)$, we may without loss of generality assume that our $(\mathfrak{F}_1,q)$-relevant group $R = V]T^*$ is such that $T = S$, where $T^* = T \cap (V]T)_{\mathfrak{F}_1}$. But then $T^* = S \cap (V]S)_{\mathfrak{F}_1}$ and so by (5), (1) and (4) we have $T^* = S^* = \langle \alpha_i : i = 1,..,s \rangle$.

If $s > 1$, then $V = P_1 \cdots P_s$ and $[V, \langle \alpha_i \rangle] = P_i \lneq V$ by (2) and (3), for each $i = 1,..,s$, and so $T^* = \langle \alpha_1,.., \alpha_s \rangle = T^*_\infty$ in the notation of (4.6.1a). In particular, $R = V]T^* \notin \mathcal{C}_{\mathfrak{F}_1}$ (see (4.6.1b)).

If $s = 1$, then $V = P_1$ and $T = \langle \alpha_1 \rangle \cong C_q$. By (3), we have $[V, T] = V$. Thus $T^*_\infty = 1$ and $V]T^*$ is $\mathfrak{F}_1$-constructive. Since $P_1 \cong P$, then by (4.8.1d), we must have $T^* = T \in \mathrm{Syl}_q(\mathrm{Aut}(V))$.

It follows that if $R = V]T^*$ is $\mathfrak{F}_1$-constructive, then $V \cong P$,

and $C_q \cong T^* \in \mathrm{Syl}_q(\mathrm{Aut}(V))$. It follows by Sylow's theorem in $\mathrm{Aut}(V)$ that R is then unique up to isomorphism ; indeed, $R \cong P]Q$.

Since $\mathrm{char}(\mathfrak{F}_1) = \{p,q\}$ and $pq \mid |P]Q|$, we obtain the following as a consequence of (4.6.3).

<u>4.8.8 Theorem</u>. In the notation of (4.8.1/3), we have $\mathfrak{F}_1 = \langle P]Q \rangle_{\mathrm{Fitt}}$.

<u>4.8.9 Remarks</u>.

(a) By (4.8.7), there is, up to isomorphism, just one $\mathfrak{F}_1$-constructive group, which we may take as $P]Q$.

We recall the definition of $\{Q\ ;\ \mathrm{Aut}(P)\}$ from (4.3.3); from the definitions of Q_o and Q_∞ in (4.4.4) and (4.6.1a), we notice that $Q_o = Q_\infty\{Q\ ;\ \mathrm{Aut}(P)\} = \{Q\ ;\ \mathrm{Aut}(P)\}$ since $Q_\infty = 1$ by (4.8.7).

Comparing (4.4.5) and (4.6.1b), we see that any $\mathfrak{F}_1$-basic group is $\mathfrak{F}_1$-constructive. We conclude that if $\{Q\ ;\ \mathrm{Aut}(P)\} = Q$ then $Q_o = Q$ and there are no $\mathfrak{F}_1$-basic groups, while if $\{Q\ ;\ \mathrm{Aut}(P)\} = 1$ then $Q_o = 1$ and $P]Q$ is, up to isomorphism, the only $\mathfrak{F}_1$-basic group.

<u>Case I : $\{Q\ ;\ \mathrm{Aut}(P)\} = Q$</u>. In the notation of (4.4.1), let $\mathfrak{S} \in \Delta$. By the remarks preceding (4.5.1), $\mathfrak{S} \in \Lambda\Theta^w$ for some $w \in \mathbb{N}$, and so by (4.4.6) we have $\mathfrak{S} \in \Lambda$ since $\mathbb{B}_{\mathfrak{F}_1} = \emptyset$. Thus $\Delta = \Lambda$ and so $\mathfrak{F}_1 = (\mathfrak{F}_1)_*$.

<u>Case II : $\{Q\ ;\ \mathrm{Aut}(P)\} = 1$</u>. Suppose that $P \cong X$ sn $G \in \mathfrak{F}_1$, and let $S \in \mathrm{Syl}_q(X)$. By (4.8.1d), we have $|S/C_S(X)| \mid q$ and either $XS = X \times S$ or $XS/O_q(XS) \cong P]Q$. Thus $XS \in \mathfrak{F}_1$, whence $\mathfrak{F}_1$ and $P]Q$ satisfy (4.3.3a)($\mathcal{P}$). Thus by (4.3.12) we may define a transfer-type Fitting pair $(Q\ ,\ d^{PQ},\ \mathfrak{F}_1)$, and by (4.4.7) we have $G_{\mathfrak{F}_{1*}} = \ker(d^{PQ}_{G\to Q})$ if $G \in \mathfrak{F}_1$.

(b) The class $\mathfrak{F}_1$ is not a Fischer class ; for example, the subgroup $\gamma_1(P)Q$ of $P]Q$ is not contained in $\mathfrak{F}_1$, as we may easily check.

(c) Hawkes, [40], remarks that we may modify an example of Dark, [20], to obtain a concrete example of groups P and Q satisfying the conditions of (4.8.1). In [20], Dark exhibits a 3-generator metabelian group of exponent 7 and nilpotency class 4 , and gives a basis for the derived group, due to [52]. Dark calls this group P , but we will call it D to avoid confusion. Hawkes observes that there exists a proper quotient P of D and a Sylow 19-subgroup Q of order 19 of $\mathrm{Aut}(P)$ such that the conditions of (4.8.1) are satisfied (with $c = 3$). In fact, a certain amount of calculation reveals that there are a number of choices for P : we may take $|P| = 7^8$ and have $\{Q ; \mathrm{Aut}(P)\} = Q$, or take various choices of P with $|P| = 7^7$ and have either $\{Q ; \mathrm{Aut}(P)\} = Q$ or $\{Q ; \mathrm{Aut}(P)\} = 1$.

BIBLIOGRAPHY.

[1] J. C. Beidleman, "On products and normal Fitting classes", Arch. der Math. 28 (1977), 347-356.

[2] Thomas R. Berger, "More normal Fitting classes of finite solvable groups", Math. Z. 151 (1976), 1-3.

[3] Thomas R. Berger, "Normal Fitting classes and Lockett's conjecture", (preprint; to appear).

[4] Thomas R. Berger, "The smallest normal Fitting class revealed", (preprint; to appear).

[5] Thomas R. Berger and John Cossey, "An example in the theory of normal Fitting classes", Math. Z. 154 (1977), 287-293.

[6] Dieter Blessenohl und Wolfgang Gaschütz, "Über normale Schunk- und Fitting-klassen", Math. Z. 118 (1970), 1-8.

[7] R. M. Bryant and L. G. Kovács, "Lie representations and groups of prime power order", J. L. M. S. (2), 17 (1978), 415-421.

[8] R. A. Bryce and John Cossey, "Fitting formations of finite soluble groups", Math. Z. 127 (1972), 217-223.

[9] R. A. Bryce and John Cossey, "Metanilpotent Fitting classes", J. Austral. Math. Soc. 17 (1974), 285-304.

[10] R. A. Bryce and John Cossey, "A problem in the theory of normal Fitting classes", Math. Z. 141 (1975), 99-110.

[11] R. A. Bryce and John Cossey, "Strong containment of Fitting classes", Proc. Miniconf. Theory of Groups, Canberra 1975, Lecture notes in Mathematics 573, Springer-Verlag, Berlin-Heidleberg-New York, (1974), 6-16.

[12] R. A. Bryce and John Cossey, "Subgroup closed Fitting classes", Math. Proc. Camb. Phil. Soc. 83 (1978), 195-204.

[13] Alan R. Camina, "A note on Fitting classes", Math. Z. 136 (1974), 351-352.

[14] Roger W. Carter, "Nilpotent self-normalizing subgroups of soluble groups", Math. Z. 75 (1961), 136-139.

[15] Roger Carter, Bernd Fischer and Trevor Hawkes, "Extreme classes of finite soluble groups", J. Algebra 9 (1968), 285-313.

[16] Roger Carter and Trevor Hawkes, "The $\mathfrak{F}$-normalizers of a finite soluble group", J. Algebra 5 (1967), 175-202.

[17] John Cossey, "Classes of finite soluble groups", Proc. Second Internat. Conf. Theory of Groups, Canberra 1973, ed. M. F. Newman, Lecture Notes in Mathematics, 372, Springer-Verlag, Berlin-Heidleberg-New York, (1974), 226-237.

[18] John Cossey, "Products of Fitting classes", Math. Z. 141 (1975), 289-295.

[19] Rex S. Dark, "Some examples in the theory of injectors of finite soluble groups", Math. Z. 127 (1972), 145-156.

[20] R. S. Dark, "A complete group of odd order", Math. Proc. Camb. Phil. Soc. 77 (1975), 21-28.

[21] Klaus Doerk and Trevor Hawkes, book on classes of finite groups; in preparation.

[22] Bernd Fischer, Classes of conjugate subgroups in finite solvable groups (Lecture Notes, Yale University, 1966).

[23] B. Fischer, W. Gaschütz und B. Hartley, "Injectoren endlicher auflösbarer Gruppen", Math. Z. 102 (1967), 337-339.

[24] Hans Fitting, "Beiträge zur Theorie der Gruppen endlicher Ordnung", Jber. Deutsche Math.-Verein. 48 (1938), 77-141.

[25] Wolfgang Gaschütz, "Zur Theorie der endlichen auflösbaren Gruppen", Math. Z. 80 (1963), 300-305.

[26] W. Gaschütz, Selected topics in the theory of soluble groups (Lectures given at the Ninth Summer Research Institute of the Australian Mathematical Society, Canberra, 1969. Notes by J. Looker).

[27] Wolfgang Gaschütz, "Zwei Bemerkungen über normale Fittingklassen", J. Algebra 30 (1974), 277-278.

[28] Daniel Gorenstein, Finite Groups (Harper's series in Modern Mathematics, Harper and Row, New York, Evanston and London, 1968).

[29] Marshall Hall, Jr. The Theory of Groups (Macmillan, New York, 1959).

[30] P. Hall, "A note on soluble groups", J. London Math. Soc. 3 (1928), 98-105.

[31] P. Hall, "On non-strictly simple groups", Proc. Camb. Phil. Soc. 59 (1963), 531-553.

[32] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers. 4th. ed. (Clarendon Press, Oxford, 1960; reprinted 1971).

[33] B. Hartley, "On Fischer's dualization of formation theory", Proc. London Math. Soc. (3) 19 (1969), 193-207.

[34] Peter Hauck, Habilitationsschrift, Johannes Gutenberg-Universität, Mainz, 1977.

[35] Trevor O. Hawkes, "On Fitting Formations", Math. Z. 117 (1970), 177-182.

[36] T. O. Hawkes, "An example in the theory of soluble groups", Proc. Camb. Phil. Soc. 67 (1970), 13-16.

[37] Trevor Hawkes, "Skeletal classes of soluble groups", Arch. der Math. 22 (1971), 577-589.

[38] Trevor O. Hawkes, "Two applications of twisted wreath products to finite soluble groups", Trans. Amer. Math. Soc. 214 (1975), 325-335.

[39] Trevor Hawkes, "A Fitting class construction", Math. Proc. Camb. Phil. Soc. 80 (1976), 437-446.

[40] Trevor Hawkes, "On metanilpotent Fitting classes", Warwick preprint, 1978.

[41] B. Huppert, Endliche Gruppen I (Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[42] Otto-Uwe Kramer, "Endliche Gruppen mit Untergruppen mit paarweise teilerfremden Indizes", Math. Z. 139 (1974), 63-68.

[43] Hartmut Laue, "Uber nichtauflösbare normale Fittingklassen", J. Algebra 45 (1977), 274-283.

[44] Hartmut Laue, Hans Lausch und Garry R. Pain, "Verlagerung und normale Fittingklassen endliche Gruppen", Math. Z. 154 (1977), 257-260.

[45] Hans Lausch, "On normal Fitting classes", Math. Z. 130 (1973), 67-72.

[35] Trevor O. Hawkes, "On Fitting Formations", Math. Z. 117 (1970), 177-182.

[36] T. O. Hawkes, "An example in the theory of soluble groups", Proc. Camb. Phil. Soc. 67 (1970), 13-16.

[37] Trevor Hawkes, "Skeletal classes of soluble groups", Arch. der Math. 22 (1971), 577-589.

[38] Trevor O. Hawkes, "Two applications of twisted wreath products to finite soluble groups", Trans. Amer. Math. Soc. 214 (1975), 325-335.

[39] Trevor Hawkes, "A Fitting class construction", Math. Proc. Camb. Phil. Soc. 80 (1976), 437-446.

[40] Trevor Hawkes, "On metanilpotent Fitting classes", Warwick preprint, 1978.

[41] B. Huppert, Endliche Gruppen I (Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin, Heidlebecg, New York, 1967).

[42] Otto-Uwe Kramer, "Endliche Gruppen mit Untergruppen mit paarweise teilerfremden Indizes", Math. Z. 139 (1974), 63-65.

[43] Hartmut Laue, "Uber nichtauflösbare normale Fittingklassen", J. Algebra 45 (1977), 274-283.

[44] Hartmut Laue, Hans Lausch und Garry R. Pain, "Verlagerung und normale Fittingklassen endliche Gruppen", Math. Z. 154 (1977), 257-260.

[45] Hans Lausch, "On normal Fitting classes", Math. Z. 130 (1973), 67-72.

[46] Hans Lausch and Garry R. Pain, "Transfer and normal Fitting classes", Monash University preprint, unpublished (c.f. [44]).

[47] F. P. Lockett, On the theory of Fitting classes of finite soluble groups (Ph.D. Thesis, University of Warwick, Coventry, 1971).

[48] F. Peter Lockett, "On the theory of Fitting classes of finite soluble groups", Math. Z. 131 (1973), 103-115.

[49] F. Peter Lockett, "The Fitting class $\mathfrak{F}^*$", Math. Z. 137 (1974), 131-136.

[50] Wilhelm Magnus, Abraham Karrass and Donald Solitar, Combinatorial Group Theory (reprinted Dover Publications Inc., New York, 1976).

[51] A. K. Makan, "Fitting classes with the wreath product property are normal", J. London Math. Soc. (2) 8 (1974), 245-246.

[52] H. Meier-Wunderli, "Metabelsche Gruppen", Comment. Math. Helv. 25 (1951), 1-10.

[53] B. H. Neumann, "Twisted wreath products of groups", Arch. der Math. 14 (1963), 1-6.

[54] Peter M. Neumann, "On the structure of standard wreath products of groups", Math. Z. 84 (1964), 343-373.